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Oscillation Criteria for Third-Order Nonlinear Functional Difference Equations with Damping

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Abstract: In this paper, we obtain some new criteria for the oscillation of certain third-order difference equations using comparison principles with a suitable couple of first-order difference equations. The presented results improve and extend the earlier ones. Examples are provided to illustrate the main results.

Keywords: Third-order delay difference equation, damping term, oscillation.

1 Introduction

Consider the third-order nonlinear delay difference equation of the form

$$\Delta(a_n\Delta(b_n(\Delta x_n)^\alpha)) + p_n(\Delta x_{n+1})^\alpha + q_n f(x_{\sigma(n)}) = 0, \quad n \geq n_0, \tag{1}$$

where $n_0 \in \mathbb{N}$ is a fixed integer and $\alpha \geq 1$ is a quotient of odd positive integers. Throughout this paper, we assume that the following hypotheses hold:

(H_1) $\{a_n\}$, $\{b_n\}$ and $\{q_n\}$ are real positive sequences for all $n \geq n_0$;
(H_2) $\{p_n\}$ is a nonnegative real sequence for all $n \geq n_0$;
(H_3) $\{\sigma(n)\}$ is a real nondecreasing sequence of integers with

$$\sigma(n) \leq n \quad \text{and} \quad \sigma(n) \to \infty \quad \text{as} \quad n \to \infty;$$

(H_4) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$uf(u) > 0 \quad \text{and} \quad \frac{f(u)}{u^\beta} \geq M > 0 \quad \text{for all} \quad u \neq 0,$$

where $\beta \leq \alpha$ is a ratio of odd positive integers.

By a solution of (1), we mean a nontrivial sequence $\{x_n\}$ defined for all $n \geq n_0 - \sigma(n_0)$ that satisfies (1) for all $n \geq n_0$. A solution of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. A difference equation is called nonoscillatory (oscillatory) if all its solutions are nonoscillatory (oscillatory).

Oscillation problems for third-order difference equations have been investigated in recent years, see, for example, [2–6, 8–18] and the references contained therein. However, compared to second-order difference equations, the study of third-order difference equations has received considerably less attention even though such equations have applications in economics, mathematical biology and other areas of mathematics [1, 7].

The aim of this paper is to complement the very recent studies [6, 12, 14, 17] on asymptotic and oscillatory properties of (1). The methods and arguments used in the present paper are different than those in [6, 14, 17]. We rely on the assumption that the related second-order difference equation

$$\Delta(a_n\Delta z_n) + \frac{p_n}{b_{n+1}} z_{n+1} = 0 \tag{2}$$

is nonoscillatory, and we obtain that all solutions of (1) are oscillatory.

It is interesting to note how the asymptotic behavior of (1) changes when the middle term is inserted. As an example, we consider the following difference equation for demonstration.

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Example 1. The difference equation

$$\Delta^3 x_n + 3\Delta x_{n+1} + \frac{1}{8} x_n = 0$$

admits three oscillatory solutions. But the corresponding equation without damping

$$\Delta^3 x_n + \frac{1}{8} x_n = 0$$

has one nonoscillatory solution and two oscillatory solutions.

Because of the middle term $p_n(\Delta x_{n+1})^\alpha$, the problem of nonexistence of a nonoscillatory solution $\{y_n\}$ with $y_n \Delta y_n < 0$ seems to be crucial and challenging. We recall the related existing result for the case $\alpha = \beta = 1$.

Lemma 1(see [6, Lemma 2.4]). *Let $\{\mu_n\}$ be a positive real sequence defined for $n \geq n_0$ and set*

$$\phi_n = b_{n+2}\Delta(a_{n+1}\Delta\mu_n) + \mu_n p_n.$$

Furthermore, assume that

$$\Delta\mu_n \geq 0, \quad \phi_n \geq 0,$$
$$\Delta(b_{n+2}\Delta(a_{n+1}\Delta\mu_n)) \geq 0 \quad (or\ \Delta(\mu_n p_n) \leq 0)$$
$$for \quad n \geq n_0$$

and

$$\sum_{n=n_0}^{\infty} (k\mu_n q_n - \Delta\phi_n) = \infty,$$

where

$$k\mu_n q_n - \Delta\phi_n \geq 0 \quad for \quad n \geq n_0.$$

If $\sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty$ and $\{x_n\}$ is a nonoscillatory solution of (1) which satisfies $x_n(a_n\Delta x_n) \leq 0$ for n sufficiently large, then $\lim_{n\to\infty} x_n = 0$.

However, since the proof of Lemma 1 uses the summation by parts formula, it cannot be generalized for $\alpha \neq 1$. In this paper, we will take this problem into account and use a different method to obtain oscillation results for (1). On the other hand, in [14], the authors offered a partial result for (1) in the sense that either every solution $\{x_n\}$ of (1) is oscillatory or $\{a_n\Delta(b_n(\Delta x_n)^\alpha)\}$ is oscillatory, and the oscillation of all solutions of (1) is left as an interesting open problem.

In view of the above observations, in this paper, we obtain sufficient conditions for the oscillation of all solutions of (1) by using Riccati-type transformations and comparison theorems.

2 Preliminary Results

As in [14], we define

$$L_0(x_n) = x_n,$$
$$L_1(x_n) = b_n((\Delta x_n)^\alpha),$$
$$L_2(x_n) = a_n\Delta(L_1(x_n)),$$

and

$$L_3(x_n) = \Delta(L_2(x_n))$$

for all $n \geq n_0$. With this notation, (1) can be rewritten as

$$L_3(x_n) + \frac{p_n}{b_{n+1}} L_1(x_n) + q_n f(x_{\sigma(n)}) = 0, \quad n \geq n_0. \qquad (3)$$

Following [14], we define the functions

$$R_1(n,N) = \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}},$$
$$R_2(n,N) = \sum_{s=N}^{n-1} \frac{1}{a_s},$$
$$R_3(n,N) = \sum_{s=N}^{n-1} \left(\frac{R_2(s,N)}{b_s}\right)^{\frac{1}{\alpha}},$$

and

$$R(\sigma(n),n) = \frac{R_3(\sigma(n),N)}{R_3(n+1,N)}$$

for all $n \geq N \geq n_0$. Throughout and without further mentioning, it will be assumed that

$$R_1(n,n_0) \to \infty \quad \text{and} \quad R_2(n,n_0) \to \infty \quad \text{as} \quad n \to \infty.$$

All the functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all n large enough.

In the sequel, we present several auxiliary results which will be used to prove our main results.

Lemma 2. *Let $\{z_n\}$ be a solution of (2) which is positive for all $n \geq N$. Then*

$$\Delta z_n > 0 \qquad (4)$$

and

$$\Delta\left(\frac{z_n}{R_2(n,N)}\right) \leq 0 \qquad (5)$$

for all $n \geq N$.

Proof. Let $\{z_n\}$ be a solution of (2) with $z_n > 0$ for all $n \geq N$. Then $\Delta(a_n\Delta z_n) < 0$ for all $n \geq N$, so that $\{a_n\Delta z_n\}$ is decreasing for $n \geq N$. First assume that $a_{N_1}\Delta z_{N_1} < 0$ for some $N_1 \geq N$. Then $a_n\Delta z_n \leq a_{N_1}\Delta z_{N_1} = c < 0$ for all $n \geq N_1$, and thus

$$z_n = z_{N_1} + \sum_{s=N_1}^{n-1} \Delta z_s \leq z_{N_1} + c\sum_{s=N_1}^{n-1} \frac{1}{a_s}$$
$$= z_{N_1} - c\sum_{s=N}^{N_1-1} \frac{1}{a_s} + cR_2(n,N) \to \infty \quad \text{as} \quad n \to \infty,$$

a contradiction. Thus (4) holds. Next, let $n \geq N$. Then

$$z_n \geq z_n - z_N = \sum_{s=N}^{n-1} \frac{1}{a_s} a_s\Delta z_s \geq a_n\Delta z_n R_2(n,N),$$

and we see that

$$\Delta\left(\frac{z_n}{R_2(n,N)}\right)=\frac{R_2(n,N)\Delta z_n - z_n\frac{1}{a_n}}{R_2(n+1,N)R_2(n,N)}\le 0.$$

Hence $\{z_n/R_2(n,N)\}$ is nonincreasing for all $n\ge N$. This completes the proof.

Lemma 3(see [17, Theorem 2.1]). *Assume that $\{z_n\}$ is a positive solution of (2) for $n\ge n_0$. Then*

$$\begin{aligned}\Delta(a_n\Delta(b_n(\Delta x_n)^\alpha))+p_n(\Delta x_{n+1})^\alpha \\ =\frac{1}{z_{n+1}}\Delta\left(a_nz_nz_{n+1}\Delta\left(\frac{b_n}{z_n}(\Delta x_n)^\alpha\right)\right)\end{aligned} \tag{6}$$

for all $n\ge n_0$.

If (2) is nonoscillatory, then a nontrivial solution $\{z_n\}$ of (2) is called principal solution (unique up to a constant multiple) provided

$$\sum_{n=n_0}^{\infty}\frac{1}{a_nz_nz_{n+1}}=\infty.$$

Since every eventually positive solution of (2) is increasing, the principal solution of (2) satisfies

$$\sum_{n=n_0}^{\infty}\frac{1}{a_nz_nz_{n+1}}=\infty \quad\text{and}\quad \sum_{n=n_0}^{\infty}\left(\frac{z_n}{b_n}\right)^{1/\alpha}=\infty. \tag{7}$$

In the proofs of our theorems, an equivalent form of (1) without damping term will be used repeatedly. This will allow us to take into account the possible case of $L_2(x_n)$ being oscillatory, which was missing in the previous results.

Lemma 4(see [14, Lemma 2.1]). *Suppose that (2) is nonoscillatory. If $\{x_n\}$ is a nonoscillatory solution of (1) for all $n\ge n_0$, then there exists an integer $N\ge n_0$ such that*

$$x_nL_1(x_n)>0 \tag{8}$$

or

$$x_nL_1(x_n)<0 \tag{9}$$

for all $n\ge N$.

Lemma 5. *If $\{x_n\}$ is a nonoscillatory solution of (1) with $x_nL_1(x_n)>0$ for all $n\ge N\ge n_0$, then*

$$x_nL_2(x_n)\ge 0 \quad \textit{and} \quad x_nL_3(x_n)<0$$

for all $n\ge N$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1), say $x_n>0$, $x_{\sigma(n)}>0$ and $L_1(x_n)>0$, for all $n\ge N$. By (3), we see that $L_3(x_n)<0$ for all $n\ge N$, so $L_2(x_n)$ is strictly decreasing for all $n\ge N$. Now assume that there exists $N_1\ge N$ with $L_2(x_{N_1})<0$. Then, for $n\ge N_1$, we have

$$\begin{aligned}L_1(x_n)&=L_1(x_{N_1})+\sum_{s=N_1}^{n-1}\Delta(L_1(x_s))\\ &=L_1(x_{N_1})+\sum_{s=N_1}^{n-1}\frac{L_2(x_s)}{a_s}\\ &\le L_1(x_{N_1})+L_2(x_{N_1})R_2(n,N_1)\to\infty \quad \text{as} \quad n\to\infty,\end{aligned}$$

a contradiction. This completes the proof.

Lemma 6(see [14, Lemma 2.2]). *Let $\{x_n\}$ be a nonoscillatory solution of (1) with $x_nL_1(x_n)>0$ for all $n\ge N\ge n_0$. Then*

$$L_1(x_n)\ge R_2(n,N)L_2(x_n), \quad n\ge N \tag{10}$$

and

$$x_n\ge R_3(n,N)L_2^{1/\alpha}(x_n), \quad n\ge N. \tag{11}$$

Lemma 7. *Let $\{x_n\}$ be a nonoscillatory solution of (1) with $x_nL_1(x_n)>0$ for all $n\ge N\ge n_0$. If, for every $k>0$,*

$$\sum_{n=N}^{\infty}\frac{1}{a_n}\sum_{s=n}^{\infty}\left(\frac{p_s}{b_{s+1}}+kq_sR_1^\beta(\sigma(s),N)\right)=\infty, \tag{12}$$

then $\lim_{n\to\infty}L_1(x_n)=\infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1). Without loss of generality, we may assume $x_n>0$, $x_{\sigma(n)}>0$ and $L_1(x_n)>0$ for all $n\ge N\ge n_0$. Then, by Lemma 5, $L_2(x_n)\ge 0$ and $L_1(x_n)$ is increasing, so $L_1(x_n)\ge L_1(x_N)=d>0$. Clearly,

$$x_{\sigma(n)}\ge d^{1/\alpha}R_1(\sigma(n),N) \quad \text{for} \quad n\ge N.$$

Using both estimates in (3) and summing from n to ∞, one obtains

$$L_2(x_n)\ge d\sum_{s=n}^{\infty}\frac{p_s}{b_{s+1}}+Md^{\beta/\alpha}\sum_{s=n}^{\infty}q_sR_1^\beta(\sigma(s),N).$$

Summing again the last inequality from N to ∞, we obtain the desired result using (12). This completes the proof.

Lemma 8. *Assume (12) holds. Let $\{x_n\}$ be a nonoscillatory solution of (1) with $x_nL_1(x_n)>0$ for all $n\ge N\ge n_0$. Then there exists an integer $N_1>N$ such that*

$$x_{\sigma(n)}\ge R(\sigma(n),N)x_{n+1} \quad \textit{for all} \quad n\ge N_1. \tag{13}$$

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1), say $x_n>0$, $x_{\sigma(n)}>0$ and $L_1(x_n)>0$ for all $n\ge N$. From (10), we have

$$\Delta\left(\frac{L_1(x_n)}{R_2(n,N)}\right)=\frac{R_2(n,N)L_2(x_n)-L_1(x_n)}{R_2(n,N)R_2(n+1,N)a_n}\le 0$$

for $n \geq N_1$. Thus, $\{\frac{L_1(x_n)}{R_2(n,N)}\}$ is nonincreasing for $n \geq N_1$, and, moreover, this fact yields

$$\begin{aligned} x_n =& x_N + \sum_{s=N}^{n-1} \frac{R_2^{1/\alpha}(s,N)L_1^{1/\alpha}(x_s)}{b_s^{1/\alpha}R_2^{1/\alpha}(s,N)} \\ \geq& \frac{L_1^{1/\alpha}(x_n)}{R_2^{1/\alpha}(n,N)} \sum_{s=N}^{n-1} \frac{R_2^{1/\alpha}(s,N)}{b_s^{1/\alpha}} \\ =& \frac{R_3(n,N)L_1^{1/\alpha}(x_n)}{R_2^{1/\alpha}(n,N)} \end{aligned} \tag{14}$$

for $n \geq N$. Hence,

$$\Delta\left(\frac{x_n}{R_3(n,N)}\right) = \frac{L_1^{1/\alpha}(x_n)R_3(n,N) - x_nR_2^{1/\alpha}(n,N)}{b_n^{1/\alpha}R_3(n,N)R_3(n+1,N)} \leq 0$$

for $n \geq N_1$, which implies that $\{\frac{x_n}{R_3(n,N)}\}$ is nonincreasing for all $n \geq N_1$. Thus, if $\sigma(n) \geq N_1$, then

$$x_{\sigma(n)} \geq \frac{R_3(\sigma(n),N)}{R_3(n,N)}x_n \geq R(\sigma(n),N)x_{n+1}$$

for $n \geq N_1$. This completes the proof.

Lemma 9. *Let $\{x_n\}$ be a nonoscillatory solution of (1) with $x_nL_1(x_n) > 0$ for all $n \geq N \geq n_0$. If, for every $k > 0$,*

$$\sum_{n=N}^{\infty}\left(\frac{p_s}{b_{s+1}}R_2(s,N) + kq_sR_3^{\beta}(\sigma(s),N)\right) = \infty, \tag{15}$$

then $\lim_{n\to\infty} \frac{x_n}{R_3(n,N)} = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1). Without loss of generality, we may assume $x_n > 0$, $x_{\sigma(n)} > 0$ and $L_1(x_n) > 0$ for $n \geq N$. By the discrete L'Hôpital rule [1], it is easy to see that

$$\lim_{n\to\infty} \frac{x_n}{R_3 n,N} = \lim_{n\to\infty} L_2(x_n).$$

Assume to the contrary that $L_2(x_n) \geq d > 0$ for all $n \geq N$. Summing (3) from N to $n-1$ and then using (10) and (11), we find

$$\begin{aligned} L_2(x_n) \geq& \sum_{s=N}^{n-1} \frac{p_s}{b_{s+1}}L_1(x_s) + \sum_{s=N}^{n-1} q_s f(x_{\sigma(n)}) \\ \geq& d\sum_{s=N}^{n-1} \frac{p_s}{b_{s+1}}R_2(s,N) + d^{\beta/\alpha}\sum_{s=N}^{n-1} q_sR_3^{\beta}(\sigma(s),N). \end{aligned}$$

Letting $n \to \infty$, one obtains a contradiction with (15), and so $d = 0$. This completes the proof.

3 Main Results

In this section, we present the main results of the paper. We begin with the following lemma.

Lemma 10. *Assume (2) is nonoscillatory. If*

$$\sum_{n=N}^{\infty} \frac{R_2^{1/\alpha}(n,N)}{b_n^{1/\alpha}}\left(\sum_{s=n}^{\infty}\frac{\sum_{t=s}^{\infty} q_t}{a_sR_2(s,N)}\right)^{1/\alpha} = \infty, \tag{16}$$

then any solution $\{x_n\}$ of (1) with $x_nL_1(x_n) < 0$ converges to zero as $n \to \infty$.

Proof. Assume to the contrary that $\{x_n\}$ is a nonoscillatory solution of (1), say $x_n > 0$, $x_{\sigma(n)} > 0$ and $L_1(x_n) < 0$ for $n \geq N \geq n_0$, such that

$$\lim_{n\to\infty} x_n = d \geq 0.$$

Using (H_4) and (6) in (1), we have

$$\Delta\left(a_nz_nz_{n+1}\Delta\left(\frac{b_n}{z_n}(\Delta x_n)^{\alpha}\right)\right) + Mq_nz_{n+1}x_{\sigma(n)}^{\beta} \leq 0 \tag{17}$$

for $n \geq N$. Then, by [17], x_n satisfies

$$\begin{aligned} &\Delta x_n < 0, \\ &\Delta\left(\frac{b_n}{z_n}(\Delta x_n)^{\alpha}\right) > 0, \\ &\Delta\left(a_nz_nz_{n+1}\Delta\left(\frac{b_n}{z_n}(\Delta x_n)^{\alpha}\right)\right) < 0 \end{aligned} \tag{18}$$

for all $n \geq N$. Summing (17) from n to ∞ and using $x_{\sigma(n)} \geq d$, we obtain

$$\Delta\left(\frac{b_n}{z_n}(\Delta x_n)^{\alpha}\right) \geq \frac{Md^{\beta}}{a_nz_nz_{n+1}}\sum_{s=n}^{\infty} q_sz_{s+1}. \tag{19}$$

Since $\{z_n\}$ is increasing by (4), we have from (19) that

$$\Delta\left(\frac{b_n}{z_n}(\Delta x_n)^{\alpha}\right) \geq \frac{d_1}{a_nz_n}\sum_{s=n}^{\infty} q_s,$$

where $d_1 = Md^{\beta} > 0$. Summing the last inequality from n to ∞ and using (5) from Lemma 2, we find

$$\begin{aligned} -(\Delta x_n)^{\alpha} \geq& d_1\frac{z_n}{b_n}\sum_{s=n}^{\infty}\frac{\sum_{t=s}^{\infty} q_t}{a_sz_s} \\ \geq& d_1\frac{R_2(n,N)}{b_n}\sum_{s=n}^{\infty}\frac{\sum_{t=s}^{\infty} q_t}{a_sR_2(s,N)}, \quad n \geq N. \end{aligned}$$

Finally, by summing the last inequality from N to $n-1$, we have

$$x_N \geq d_1^{1/\alpha}\sum_{s=N}^{n-1}\frac{R_2^{1/\alpha}(s,N)}{b_s^{1/\alpha}}\left(\sum_{t=s}^{\infty}\frac{\sum_{j=t}^{\infty} q_j}{a_tR_2(t,N)}\right)^{1/\alpha}.$$

Letting $n \to \infty$, we obtain a contradiction with (16). Hence, $d = 0$, and the proof is complete.

Theorem 1. *Assume that (2) is nonoscillatory. Suppose conditions (12), (15), and (16) hold. If there exists a constant $c > 0$ and a positive real sequence $\{\rho_n\}$ such that*

$$\limsup_{n\to\infty} \sum_{s=N}^{n-1} \left[M\rho_s q_s R^\beta(\sigma(s),N) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A_s^{\alpha+1}}{B_s^\alpha} \right] = \infty, \quad (20)$$

where, for $n \geq N$,

$$A_n = \frac{\Delta\rho_n}{\rho_{n+1}} - \frac{\rho_n}{\rho_{n+1}} \frac{p_n}{b_{n+1}} R_2(n,N)$$

and

$$B_n = \beta c^{\beta/\alpha-1} \frac{\rho_n}{\rho_{n+1}} \left(\frac{R_2(n,N)}{b_n \rho_{n+1}} \right)^{\frac{1}{\alpha}} R_3^{\beta/\alpha-1}(n+1,N),$$

then every solution $\{x_n\}$ of (1) is either oscillatory or converges to zero as $n \to \infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1) for all $n \geq N$. Without loss of generality, we may assume that $x_n > 0$ and $x_{\sigma(n)} > 0$ for $n \geq N \geq n_0$. From Lemma 4, it follows that $L_1(x_n) > 0$ or $L_1(x_n) < 0$ for all $n \geq N$.

First, we assume $L_1(x_n) > 0$ for $n \geq N$. By Lemma 5, $L_2(x_n) \geq 0$ for $n \geq N$. Using the estimate (13) in (3) and (H_4), we obtain

$$L_3(x_n) + \frac{p_n}{b_{n+1}} L_1(x_n) + MR^\beta(\sigma(n),N) q_n x_{n+1}^\beta \leq 0 \quad (21)$$

for all $n \geq N_1 \geq N$. Define

$$w_n = \rho_n \frac{L_2(x_n)}{x_n^\beta} > 0 \quad \text{for} \quad n \geq N_1. \quad (22)$$

From (22), we have

$$\Delta w_n = \rho_n \frac{\Delta(L_2(x_n))}{x_{n+1}^\beta} + \frac{\Delta\rho_n L_2(x_{n+1})}{x_{n+1}^\beta} - \frac{\rho_n L_2(x_n)}{x_n^\beta x_{n+1}^\beta} \Delta x_n^\beta,$$

and using (21) and (10), we obtain

$$\Delta w_n \leq -M\rho_n q_n R^\beta(\sigma(n),N) + A_n w_{n+1} - \beta \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta x_n}{x_{n+1}}. \quad (23)$$

From the definition of $L_1(x_n)$ and (10), we obtain

$$\Delta x_n = \left(\frac{L_1(x_n)}{b_n} \right)^{1/\alpha} \geq \left(\frac{R_2(n,N)}{b_n} \right)^{1/\alpha} L_2^{1/\alpha}(x_n).$$

Thus,

$$\frac{\Delta x_n}{x_{n+1}} \geq \left(\frac{R_2(n,N)}{b_n \rho_{n+1}} \right)^{1/\alpha} w_{n+1}^{1/\alpha} x_{n+1}^{\beta/\alpha-1},$$

and the inequality (23) becomes

$$\Delta w_n \leq -M\rho_n q_n R^\beta(\sigma(n),N) + A_n w_{n+1} - \beta \frac{\rho_n}{\rho_{n+1}} \left(\frac{R_2(n,N)}{b_n \rho_{n+1}} \right)^{1/\alpha} w_{n+1}^{1/\alpha} x_{n+1}^{\beta/\alpha-1}. \quad (24)$$

By Lemma 9, it follows from (15) that

$$0 < \frac{x_{n+1}}{R_3(n+1,N)} \leq L_2(x_{N_1}) = c \quad \text{for all} \quad n \geq N.$$

Hence,

$$x_{n+1}^{\beta/\alpha-1} \geq c^{\beta/\alpha-1} (R_3(n+1,N))^{\beta/\alpha-1}. \quad (25)$$

Using (25) in (24), we obtain

$$\Delta w_n \leq -M\rho_n q_n R^\beta(\sigma(n),N) + A_n w_{n+1} - B_n w_{n+1}^{1/\alpha} \quad (26)$$

for $n \geq N_1$. Using the inequality

$$Cu - Du^{1+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{C^{\alpha+1}}{D^\alpha} \quad \text{for} \quad D > 0,$$

we obtain from (26) that

$$\Delta w_n \leq -M\rho_n q_n R^\beta(\sigma(n),N) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A_n^{\alpha+1}}{B_n^\alpha}$$

holds for all $n \geq N_1$. Summing the last inequality from N_1 to n, we get

$$\sum_{s=N_1}^{n} \left(M\rho_s q_s R^\beta(\sigma(s),N) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A_s^{\alpha+1}}{B_s^\alpha} \right) \leq w_{N_1},$$

which contradicts (20).

Next, assume that $L_1(x_n) < 0$ for $n \geq N$. By Lemma 10, (16) ensures that any solution of (1) tends to zero as $n \to \infty$. This completes the proof.

Remark. Note that Lemma 10 and Theorem 1 extend the results in [6].

In the following, we obtain sufficient conditions for the oscillation of all solutions of (1).

Theorem 2. *Assume $\sigma(n) < n$ for all $n \geq n_0$. Let the hypotheses of Theorem 1 hold except (16). If there exists a constant $c_* > 0$ such that*

$$\limsup_{n\to\infty} \sum_{s=\sigma(n)}^{n-1} \frac{R_2^{1/\alpha}(s,N)}{b_s^{1/\alpha}} \left(\sum_{t=s}^{n-1} \frac{\sum_{j=t}^{n-1} q_j}{a_t R_2(t,N)} \right)^{1/\alpha} = c_*, \quad (27)$$

then every solution of (1) is oscillatory.

Proof. Assume to the contrary that $\{x_n\}$ is a nonoscillatory solution of (1), say $x_n > 0$, $x_{\sigma(n)} > 0$ and $L_1(x_n) < 0$ for $n \geq N \geq n_0$. As in the proof of Lemma 10, we obtain that $\{x_n\}$ is a solution of (17) satisfying (18) for all $n \geq N$. Since $\alpha \geq \beta$, there exists an integer $N_1 \geq N$ such that

$$x_{\sigma(n)}^{\beta-\alpha} \geq c^{\beta-\alpha} \tag{28}$$

for all $n \geq N_1$ and every $c > 0$. Using (28) in (17), we have

$$\Delta\left(a_n z_n z_{n+1} \Delta\left(\frac{b_n}{z_n}(\Delta x_n)^\alpha\right)\right) + Mc^{\beta-\alpha} q_n z_{n+1} x_{\sigma(n)}^\alpha \leq 0, \tag{29}$$

$n \geq N_1$. Summing (29) twice from s to $n-1$, $n > s+1$, one obtains

$$-\Delta x_s \geq Mc^{\beta-\alpha}\left(\frac{z_s}{b_s}\right)^{1/\alpha}\left(\sum_{t=s}^{n-1}\frac{\sum_{j=t}^{n-1} q_j z_{j+1} x_{\sigma(j)}^\alpha}{a_t z_t z_{t+1}}\right)^{1/\alpha}. \tag{30}$$

Using the property (5) of $\{z_n\}$, the inequality (30) becomes

$$-\Delta x_s \geq Mc^{\beta-\alpha}\left(\frac{R_2(s,N)}{b_s}\right)^{1/\alpha}\left(\sum_{t=s}^{n-1}\frac{\sum_{j=t}^{n-1} q_j x_{\sigma(j)}^\alpha}{a_t R_2(t,N)}\right)^{1/\alpha}.$$

Summing the above inequality from $\sigma(n)$ to $n-1$, we obtain

$$x_{\sigma(n)} \geq Mc^{\beta-\alpha} x_{\sigma(n)} \sum_{s=\sigma(n)}^{n-1}\frac{R_2^{1/\alpha}(s,N)}{b_s^{1/\alpha}}\left(\sum_{t=s}^{n-1}\frac{\sum_{j=t}^{n-1} q_j}{a_t R_2(t,N)}\right)^{1/\alpha},$$

which is a contradiction with (27). This completes the proof.

Next, we present another condition in which the function $\{p_n\}$ is directly included.

Theorem 3. *Assume that $\sigma(n) < n$ for all $n \geq n_0$. Let the hypotheses of Theorem 1 hold except (16). If there exists a constant $c_* > 0$ such that*

$$\limsup_{n\to\infty}\left\{\sum_{s=\sigma(n)}^{n-1}\frac{1}{b_s^{1/\alpha}}\left(\sum_{t=s}^{n-1}\frac{1}{a_t}\sum_{j=t}^{n-1} Q_j\right)^{1/\alpha}\right\} > 1, \tag{31}$$

where

$$Q_n = \left(Mc_*^{\beta-\alpha} q_n - \frac{p_n R_2(n,N)}{b_{n+1} R_3^\alpha(n,\sigma(n))}\right) > 0, \quad n \geq N_1,$$

then every solution of (1) *is oscillatory.*

Proof. Assume to the contrary that $\{x_n\}$ is a nonoscillatory solution of (1), say $x_n > 0$, $x_{\sigma(n)} > 0$ and $L_1(x_n) < 0$ for $n \geq N \geq n_0$. Consider $L_2(x_n)$. The case $L_2(x_n) \leq 0$ cannot hold for all $n \geq N_1 \geq N$ since by summing this inequality, we see that

$$\Delta x_n = \left(\frac{L_1(x_n)}{b_n}\right)^{1/\alpha} \leq \left(\frac{L_1(x_{N_1})}{b_n}\right)^{1/\alpha}, \quad n \geq N_1,$$

which contradicts the positivity of $\{x_n\}$. Therefore, either $L_2(x_n) > 0$ or $L_2(x_n)$ changes sign for all $n \geq N_1$. From the proof of Lemma 10, we obtain that $\{x_n\}$ is a positive solution of (17) satisfying (18) for all $n \geq N$. Now, for $s \geq j \geq N$, we obtain

$$\begin{aligned} x_j - x_s &= -\sum_{t=j}^{s-1}\left(\frac{z_t}{b_t}\right)^{1/\alpha}\left(\frac{b_t}{z_t}(\Delta x_t)^\alpha\right)^{1/\alpha} \\ &\geq -\Delta x_s\left(\frac{b_s}{z_s}\right)^{1/\alpha}\sum_{t=j}^{s-1}\left(\frac{z_t}{b_t}\right)^{1/\alpha} \\ &\geq \frac{-L_1^{1/\alpha}(x_s)}{R_2^{1/\alpha}(s,N)}\sum_{t=j}^{s-1}\left(\frac{R_2(t,N)}{b_t}\right)^{1/\alpha} \\ &= \frac{-L_1^{1/\alpha}(x_s) R_3(s,j)}{R_2^{1/\alpha}(s,N)}. \end{aligned} \tag{32}$$

Using $s = n$, $j = \sigma(n)$ and $-L_1(x_n) > 0$ in (32), we obtain

$$x_{\sigma(n)} \geq \frac{R_3(n,\sigma(n))}{R_2^{1/\alpha}(n,N)}(-L_1^{1/\alpha}(x_n)) \quad \text{for all} \quad n \geq N,$$

i.e.,

$$L_1(x_n) \geq \frac{-R_2(n,N)}{R_3^\alpha(n,\sigma(n))} x_{\sigma(n)}^\alpha.$$

Using this inequality in (3), we obtain

$$-L_3(x_n) \geq \left(Mq_n x_{\sigma(n)}^{\beta-\alpha} - \frac{p_n R_2(n,N)}{b_{n+1} R_3^\alpha(n,\sigma(n))}\right) x_{\sigma(n)}^\alpha,$$

$n \geq N$. Since $\{x_n\}$ is decreasing and $\alpha \geq \beta$, there exists an integer $N_1 \geq N$ such that

$$x_{\sigma(n)}^{\beta-\alpha} \geq c^{\beta-\alpha} \tag{33}$$

for every $c > 0$ and for all $n \geq N_1$. Thus, we have

$$\begin{aligned} -L_3(x_n) &\geq \left(Mc^{\beta-\alpha} q_n - \frac{p_n R_2(n,N)}{b_{n+1} R_3^\alpha(n,\sigma(n))}\right) x_{\sigma(n)}^\alpha \\ &= Q_n x_{\sigma(n)}^\alpha > 0 \quad \text{for} \quad n \geq N_1. \end{aligned} \tag{34}$$

Hence, $L_3(x_n) < 0$, and similarly as in the proof of Lemma 5, we see that $L_2(x_n) \geq 0$ for all $n \geq N_1$. Summing (34) from s to $n-1$, $n > s+1$, we obtain

$$L_2(x_s) \geq \sum_{t=s}^{n-1} Q_t x_{\sigma(t)}^\alpha.$$

Summing again from s to $n-1$, we get

$$-L_1^{1/\alpha}(x_s) \geq \left(\sum_{t=s}^{n-1}\frac{1}{a_t}\sum_{j=t}^{n-1} Q_j x_{\sigma(j)}^\alpha\right)^{1/\alpha}.$$

Finally, summing the last inequality form $\sigma(n)$ to $n-1$, we find

$$x_{\sigma(n)} \geq x_{\sigma(n)} \sum_{s=\sigma(n)}^{n-1} \frac{1}{b^{1/\alpha}} \left(\sum_{t=s}^{n-1} \frac{1}{a_t} \sum_{j=t}^{n-1} Q_j \right)^{1/\alpha},$$

which in view (31) results in contradiction. This completes the proof.

From the above theorems, we obtain the following corollary.

Corollary 1. *Assume that $\sigma(n) < n$ for all $n \geq n_0$. Let the hypotheses of Theorem 1 hold except (16). If there exists a constant $c_* > 0$ such that (27) or (31) holds, then every solution of (1) is oscillatory.*

Remark. The condition (31) slightly differs from the one used in [14] but this correctly takes into account the class of nonoscillatory solutions such that $x_n L_2(x_n)$ is oscillatory.

4 Examples

In this section, we provide two examples to illustrate the importance of the main results.

Example 2. Consider the third-order delay difference equation of the form

$$\Delta^3 x_n + \frac{1}{6n^2} \Delta x_{n+1} + \left(1 - \frac{1}{6n^2}\right) x_{n-3} = 0, \quad n \in \mathbb{N}. \tag{35}$$

Note that $\Delta^2 z_n + \frac{1}{5n^2} z_{n+1} = 0$ is nonoscillatory by [2, Theorem 1.14]. Here, $R_1(n,1) \backsim n$, $R_2(n,1) \backsim n$, $R_3(n,1) \backsim \frac{n^2}{2}$. By a simple calculation, we can show that all conditions of Theorem 1 are satisfied. Hence, every solution of (35) is oscillatory. In fact, $\{x_n\} = \{\cos \frac{n\pi}{3}\}$ is one such solution of (35). We believe that the conclusion is not deducible from the oscillation criteria in [6, 14, 17] or other known results.

Example 3. Consider the difference equation

$$\Delta^2 (n^{1/4} (\Delta x_n)^{1/3}) + \frac{3}{16n^{7/4}} (\Delta x_{n+1})^{1/3} + \frac{10}{n^{25/12}} x_{n-2}^{1/3} = 0, \quad n \in \mathbb{N}. \tag{36}$$

Here, $a_n = 1$, $b_n = n^{1/4}$, $p_n = \frac{3}{16n^{7/4}}$, $q_n = \frac{10}{n^{25/12}}$, $\alpha = \beta = \frac{1}{3}$ and $\sigma(n) = n-2$. By a simple calculation, one can show that all conditions of Theorem 2 are satisfied. Hence, every solution of (36) is oscillatory. Again, it is not possible that the conclusion is deducible from the results in [6, 14, 17].

5 Conclusion

The results presented in this paper are new and of high degree of generality. From the results in [6, 12, 15, 16], one can conclude that every solution of (1) is either oscillatory or tends to zero as $n \to \infty$ when $\alpha = \beta = 1$. Further, from the results obtained in [14], one can conclude that every solution $\{x_n\}$ of (1) is either oscillatory or $\{L_2(x_n)\}$ is oscillatory. Also note that to apply the results in [17], one should know explicitly at least one nonoscillatory solution of (2), but that is not required in this paper. Therefore, the results presented in this paper improve and complement those in [5, 6, 8, 9, 11–18].

It might be also interesting to extend the results of this paper to higher-order difference equation of the form

$$\Delta(a_n \Delta(b_n (\Delta^{m-2} x_n)^\alpha)) + p_n (\Delta^{m-2} x_{n+1})^\alpha + q_n f(x_{\sigma(n)}) = 0,$$

where $m \in \mathbb{N}$ is odd. This would be left to further research.

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