# Oscillation Criteria for Fourth Order Nonlinear Positive Delay Differential Equations with a Middle Term 

Said R. Grace<br>Elvan Akin<br>Missouri University of Science and Technology, akine@mst.edu

Follow this and additional works at: https://scholarsmine.mst.edu/math_stat_facwork
Part of the Mathematics Commons, and the Statistics and Probability Commons

## Recommended Citation

S. R. Grace and E. Akin, "Oscillation Criteria for Fourth Order Nonlinear Positive Delay Differential Equations with a Middle Term," Dynamic Systems and Applications, vol. 25, no. 3, pp. 431-438, Dynamic Publishers, Jan 2016.

This Article - Journal is brought to you for free and open access by Scholars' Mine. It has been accepted for inclusion in Mathematics and Statistics Faculty Research \& Creative Works by an authorized administrator of Scholars' Mine. This work is protected by U. S. Copyright Law. Unauthorized use including reproduction for redistribution requires the permission of the copyright holder. For more information, please contact scholarsmine@mst.edu.

# OSCILLATION CRITERIA FOR FOURTH ORDER NONLINEAR POSITIVE DELAY DIFFERENTIAL EQUATIONS WITH A MIDDLE TERM 

SAID GRACE AND ELVAN AKIN<br>Department of Engineering Mathematics, Cairo University, Orman, Giza 12221, Egypt<br>Missouri University of Science Technology, 310 Rolla Building, MO, 65409-0020


#### Abstract

In this article, we establish some new criteria for the oscillation of fourth order nonlinear delay differential equations of the form $$
x^{(4)}(t)+p(t) x^{(2)}(t)+q(t) f(x(g(t)))=0
$$ provided that the second order equation $$
z^{(2)}(t)+p(t) z(t)=0
$$ is nonoscillatiory or oscillatory. This equation with $g(t)=t$ is considered in [8] and some oscillation criteria for this equation via certain energy functions are established. Here, we continue the study on the oscillatory behavior of this equation via some inequalities.


Key words. oscillation, differential equations, higher order, delay.
AMS (MOS) Subject Classification. 34C10, 39A10.

## 1. INTRODUCTION

In this article, we consider nonlinear fourth order functional differential equations of the form

$$
\begin{equation*}
x^{(4)}(t)+p(t) x^{(2)}(t)+q(t) f(x(g(t)))=0, \quad t \geq t_{0}>0 \tag{1.1}
\end{equation*}
$$

together with the associated second order equation

$$
\begin{equation*}
z^{(2)}(t)+p(t) z(t)=0 \tag{1.2}
\end{equation*}
$$

We assume that

1. $p, q \in C\left(\left[t_{0}, \infty\right), \mathbb{R}^{+}\right)$;
2. $g \in C^{1}\left(\left[t_{0}, \infty\right), \mathbb{R}^{+}\right)$such that $g(t)<t, g^{\prime}(t) \geq 0$ and $\lim _{t \rightarrow \infty} g(t)=\infty$;
3. $f \in C(\mathbb{R}, \mathbb{R})$ such that $x f(x)>0$ and $\frac{f(x)}{x^{\beta}} \geq k>0$ for $x \neq 0$, where $k$ is a constant and $\beta$ is the ratio of positive odd integers.

We restrict our attention to those solutions of equation (1.1) which exist on $I=\left[t_{0}, \infty\right)$ and satisfy the condition

$$
\sup \left\{|x(t)|: t_{1} \leq t<\infty\right\}>0 \text { for } t_{1} \in\left[t_{0}, \infty\right)
$$

Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if it has an oscillatory solution. The oscillatory behavior of fourth order differential equations with middle term enjoys a great deal of interest, see [1]- [4] and [6]- [17] references contained therein. The important role in the investigation of equation (1.1) is played by the fact whether the associated second order linear equation (1.2) is oscillatory or nonoscillatory.

In [8, they considered (1.1) with $g(t)=t$ and employed an approach based on a suitable energy function for equation (1.1) and a comparison method for equation (1.1) and obtained the following result, see [8], Theorem 3.1].

Theorem 1.1. Assume that $\beta=1$, equation (1.2) is nonoscillatory,

$$
\lim _{t \rightarrow \infty} \frac{q(t)}{p(t)}=\infty, \quad p^{2}(t) \leq 4 q(t) \text { for all large } t
$$

and

$$
\int^{\infty} s^{2} q(s) d s=\infty
$$

Then (1.1) with $g(t)=t$ is oscillatory.

If $\beta<1$ and equation (1.2) is oscillatory, the following oscillation criterion for equation (1.1) has been proved in [8, Theorem 3.4].

Theorem 1.2. Let $\beta<1$ and equation (1.2) be oscillatory. Assume that $p(t) \geq p>$ $0, p^{\prime}(t) \leq 0$ and $p^{\prime \prime}(t)>0$ and

$$
\lim _{t \rightarrow \infty} t^{2(\beta-1)} q(t)=\infty
$$

Then (1.1) with $g(t)=t$ is oscillatory.
Motivated by these results in [8] which are applicable to equation (1.1) with $g(t)=t$, we study the oscillation of equation (1.1) with delay. We allow that the function $p$ can tend to a real number or to infinity as $t \rightarrow \infty$ and both cases that the corresponding second order equation (1.2) is nonoscillatory (oscillatory) are considered.

## 2. MAIN RESULTS

To obtain our results, we need the following lemmas.
Lemma 2.1 ( $1, ~ 2]$ ). Every eventually positive solution $x(t)$ of equation (1.1) is one of the following types:

- Type $(a) . x(t)>0, x^{\prime}(t)>0$ and $x^{(2)}(t)<0$ for large $t$,
- Type (b). $x(t)>0, x^{\prime}(t)>0, x^{(2)}(t)>0$ and $x^{(3)}(t)>0$ for large $t$,
- Type $(c) . x^{(2)}(t)$ changes sign eventually.

Moreover, if equation (1.2) is nonoscillatory, then $x$ is of Type (a) or Type (b) and if equation (1.2) is oscillatory, then $x$ is of Type ( $a$ ) or Type (c).

Lemma 2.2. Let $\beta \leq 1$ and equation (1.2) be nonoscillatory. If

$$
\begin{equation*}
\int^{\infty}\left(p(s)+g^{2 \beta}(s) q(s)\right) d s=\infty \tag{2.1}
\end{equation*}
$$

then equation (1.1) has no solution of Type (b), i.e., every eventually positive solution of (1.1) is of Type (a).

Proof. Let $x$ be an eventually positive solution of equation (1.1) of Type (b). There exist two positive constants $c_{1}$ and $c_{2}$ and $t_{1} \geq t_{0}$ such that $x^{(2)}(t) \geq c_{1}$ and so we get $x(g(t)) \geq c_{2} g^{2}(t)$ for all $t \geq t_{1}$. Integrating equation (1.1) from $t_{1}$ to $t$, we have

$$
\begin{aligned}
\infty & >-x^{(3)}(t)+x^{(3)}\left(t_{1}\right) \\
& \geq \int_{t_{1}}^{t}\left(c_{1} p(s)+k c_{2}^{\beta} g^{2 \beta}(s) q(s)\right) d s \\
& \geq C \int_{t_{1}}^{t}\left(p(s)+g^{2 \beta}(s) q(s)\right) d s \rightarrow \infty \text { as } t \rightarrow \infty
\end{aligned}
$$

where $C=\max \left\{c_{1}, k c_{2}^{\beta}\right\}$ which contradicts the fact that $x^{(3)}(t)$ is bounded. This completes the proof.

Lemma 2.3. Let $\beta \leq 1$ and equation (1.2) be nonoscillatory. If for every positive constant $c$, the first order delay equation

$$
\begin{equation*}
y^{\prime}(t)+c k q(t) g^{3 \beta}(t) y^{\beta}(g(t))=0 \tag{2.2}
\end{equation*}
$$

is oscillatory, then equation (1.1) has no solution of Type (b), i.e., every eventually positive solution of (1.1) is of Type (a).

Proof. Let $x$ be an eventually positive solution of equation (1.1) of Type (b). It is easy to see that there exist a constant $c^{*}, 0<c^{*}<1$ and $t_{1} \geq t_{0}$ such that

$$
\begin{equation*}
x^{(2)}(t) \geq c^{*} x^{(3)}(t) \text { for } t \geq t_{1} \tag{2.3}
\end{equation*}
$$

Integrating (2.3) twice from $t_{1}$ to $t$, we see that there exist a constant $c>0$ and a $t_{2} \geq t_{1}$ such that

$$
\begin{equation*}
x(t) \geq c t^{3} x^{(3)}(t) \text { for } t \geq t_{2} \tag{2.4}
\end{equation*}
$$

Using the inequalities (2.3) and (2.4) in equation (1.1), we get

$$
y^{\prime}(t)+c^{*} p(t) t y(t)+k c^{\beta} q(t) g^{3 \beta}(t) y^{\beta}(g(t)) \leq 0
$$

or

$$
y^{\prime}(t)+k c^{\beta} q(t) g^{3 \beta}(t) y^{\beta}(g(t)) \leq 0
$$

where $y(t)=x^{(3)}(t)>0$ for $t \geq t_{2}$. It follows from Theorem 1 in 3] that the corresponding equation (2.2) also has a positive solution. This gives us a contradiction.

The following corollary is an immediate consequence of Lemma 2.3
Corollary 2.4. Let $\beta \leq 1$ and equation (1.2) be nonoscillatory. If for every positive constant $c$,

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{g(t)}^{t} q(s) g^{3 \beta}(s) d s>\frac{1}{c^{\beta} k e} \tag{2.5}
\end{equation*}
$$

then equation (1.1) has no solution of Type (b).
Lemma 2.5. Let $\beta \leq 1$ and equation (1.2) be (non)oscillatory. If there exist a function $h \in C^{1}(I, \mathbb{R})$ such that $g(t) \leq h(t) \leq t, h^{\prime}(t) \geq 0$ for $t \geq t_{0}$ such that the second order inequality

$$
\begin{equation*}
w^{\prime \prime}(t) \geq P(t) w(h(t)) \tag{2.6}
\end{equation*}
$$

where $P(t)=c q(t) g^{\beta}(t)(h(t)-g(t))^{\beta}-p(t)>0$ for some constant $c>0$, has no positive bounded solutions, then equation (1.1) has no solution of Type (a).

Proof. Let $x$ be an eventually positive solution of equation (1.1) of Type (a). It is easy to see that there exist a constant $c^{*}$ such that $0<c^{*}<1$ and $t_{1} \geq t_{0}$ such that

$$
\begin{equation*}
x(t) \geq c^{*} t x^{\prime}(t) \text { for } t \geq t_{1} . \tag{2.7}
\end{equation*}
$$

Using (2.7) in equation (1.1), one can easily find that

$$
\begin{equation*}
y^{(3)}(t)+p(t) y^{\prime}(t)+\left(c^{*}\right)^{\beta} k q(t) g^{\beta}(t) y^{\beta}(g(t)) \leq 0 \text { for } t \geq t_{1} \tag{2.8}
\end{equation*}
$$

where $y(t)=x^{\prime}(t)$. Clearly, we see that $y(t)>0, y^{\prime}(t)<0$ and $y^{\prime \prime}(t)>0$ for $t \geq t_{1}$.
Now for $v \geq u \geq t_{1}$ we have

$$
\begin{equation*}
y(u) \geq y(u)-y(v)=-\int_{u}^{v} y^{\prime}(s) d s \geq(v-u)\left(-y^{\prime}(v)\right) . \tag{2.9}
\end{equation*}
$$

For $t \geq t_{1}$ setting $u=g(t)$ and $v=h(t)$ in (2.9), we get

$$
\begin{equation*}
\left.y(g(t)) \geq(h(t)-g(t))\left(-y^{\prime}(h(t))\right)\right) \tag{2.10}
\end{equation*}
$$

Using (2.10) in (2.8), we get

$$
\begin{align*}
w^{\prime \prime}(t)+p(t) w(t) & \geq k\left(c^{*}\right)^{\beta} q(t) g^{\beta}(t)(h(t)-g(t))^{\beta} w^{\beta}(h(t))  \tag{2.11}\\
& =k\left(c^{*}\right)^{\beta} q(t) g^{\beta}(t)(h(t)-g(t))^{\beta} w^{\beta-1}(h(t)) w(h(t)) \tag{2.12}
\end{align*}
$$

where $w(t)=-y^{\prime}(t)>0$ for $t \geq t_{1}$. Using the fact that $g(t) \leq h(t) \leq t, \beta \leq 1$ and $w(t)$ is decreasing, we obtain

$$
\begin{equation*}
w^{\prime \prime}(t)+p(t) w(h(t)) \geq\left(c^{*}\right)^{\beta} C q(t) g^{\beta}(t)(h(t)-g(t))^{\beta} w(h(t)) \tag{2.13}
\end{equation*}
$$

for some constant $C>0$. It is easy to see that inequality (2.13) has a positive bounded solution, which is a contradiction.

The following two lemmas are concerned with the bounded solutions of second order delay differential inequality (2.6).

Lemma 2.6. If

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{h(t)}^{t}(h(t)-h(s)) P(s) d s>1 \tag{2.14}
\end{equation*}
$$

for positive $P$, then inequality (2.6) has no positive bounded solutions.
Proof. Let $w(t)$ be a bounded nonoscillatory solution of inequality (2.6), say $w(t)>0$ and $w(h(t))>0$ for $t \geq t_{1} \geq t_{0}$. Then we obtain

$$
\begin{equation*}
w(t)>0, w^{\prime}(t)<0 \text { and } w^{\prime \prime}(t) \geq 0 \text { for } t \geq t_{1} \geq t_{0} \tag{2.15}
\end{equation*}
$$

Now, for $v \geq u \geq t_{1}$ we have

$$
\begin{equation*}
w(u) \geq w(u)-w(v)=-\int_{u}^{v} w^{\prime}(s) d s \geq(v-u)\left(-w^{\prime}(v)\right) \tag{2.16}
\end{equation*}
$$

For $t \geq s \geq t_{1}$ setting $u=h(s)$ and $v=h(t)$ in (2.16), we get

$$
\begin{equation*}
w(h(s)) \geq(h(t)-h(s))\left(-w^{\prime}(h(t))\right) \tag{2.17}
\end{equation*}
$$

Integrating equation (2.6) from $h(t) \geq t_{2}$ to $t$, we have

$$
\begin{equation*}
-w^{\prime}(h(t)) \geq w^{\prime}(t)-w^{\prime}(h(t)) \geq \int_{h(t)}^{t} P(s) w(h(s)) d s \tag{2.18}
\end{equation*}
$$

Using (2.17) in (2.18), we have

$$
-w^{\prime}(h(t)) \geq\left(\int_{h(t)}^{t}(h(t)-h(s)) P(s) d s\right)\left(-w^{\prime}(h(t))\right)
$$

or

$$
\begin{equation*}
1 \geq \int_{h(t)}^{t}(h(t)-h(s)) P(s) d s \tag{2.19}
\end{equation*}
$$

We take limsup as $t \rightarrow \infty$ of both sides of (2.19), we have a contradiction to condition (2.14) and completes the proof of the lemma.

Lemma 2.7. If

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{h(t)}^{t}\left(\int_{u}^{t} P(s)\right) d u>1 \tag{2.20}
\end{equation*}
$$

then inequality (2.6) has no positive bounded solutions.

Proof. Let $x$ be a bounded nonoscillatory solution of inequality (2.6), say $x(t)>0$ and $x(h(t))>0$ for $t \geq t_{1} \geq t_{0}$. As in Lemma 2.6] we obtain (2.15). Integrating (2.6) from $u$ to $t$

$$
w^{\prime}(t)-w^{\prime}(u) \geq \int_{u}^{t} P(s) w(h(s)) d s
$$

or

$$
-w^{\prime}(u) \geq\left(\int_{u}^{t} P(s) d s\right) w(h(t))
$$

Integrating this inequality from $h(t)$ to $t$, we get

$$
w(h(t)) \geq\left[\int_{h(t)}^{t}\left(\int_{u}^{t} P(s) d s\right) d u\right] w(h(t))
$$

or

$$
1 \geq\left[\int_{h(t)}^{t}\left(\int_{u}^{t} P(s) d s\right) d u\right]
$$

The rest of the proof is similar to that of Lemma 2.6 and hence is omitted. This completes the proof.

Theorem 2.8. Let $\beta \leq 1$ and equation (1.2) be nonoscillatory. If condition (2.1) (or for every constant $c>0$, then equation (2.2) is oscillatory) holds and either condition (2.14) or (2.20) hold, then equation (1.1) is oscillatory.

Proof. Let $x$ be an eventually positive solution of equation (1.1). Since equation (1.2) is nonoscillatory, then $x$ is of Type $(a)$ or of Type ( $b$ ) by Lemma 2.1. It follows from Lemma 2.2 or 2.3 that equation (1.1) has no solution of Type (b) and by Lemmas 2.5] 2.7 equation (1.1) has no solution of Type (a). This completes the proof.

Theorem 2.9. Let $\beta \leq 1$ and equation (1.2) be oscillatory. If condition (2.14) (or (2.20)) holds, then every solution $x$ of equation (1.1) is oscillatory or $x^{\prime \prime}(t)$ is oscillatory.

Proof. Let $x$ be an eventually positive solution of equation (1.1). Since equation (1.2) is oscillatory, then $x$ is of Type $(a)$ or of Type $(c)$ by Lemma 2.1. By Lemmas 2.5 [2.7 equation (1.1) has no solution of Type $(a)$. This completes the proof.

Example 2.10. Consider the fourth order delay equation

$$
\begin{equation*}
x^{(4)}(t)+\frac{1}{4 t^{2}} x^{(2)}(t)+\left(1-\frac{1}{4 t^{2}}\right) x(t-\pi)=0 \tag{2.21}
\end{equation*}
$$

Here we let $g(t)=t-\pi$ and $h(t)=t-\frac{\pi}{2}$. All conditions of Theorem 2.8 are satisfied and hence all solutions of equation (2.21) are oscillatory. One such solution is $x(t)=\sin t$. We also note that Theorem 1.1 is applicable to this equation with $g(t)=t$.

Example 2.11. Consider the fourth order delay equation

$$
\begin{equation*}
x^{(4)}(t)+2 x^{(2)}(t)+x(t-2 \pi)=0 \tag{2.22}
\end{equation*}
$$

Here we let $g(t)=t-2 \pi$ and $h(t)=t-\pi$. All conditions of Theorem 2.9 are satisfied and hence all solutions of equation (2.22) are oscillatory. One such solution is $x(t)=\sin t$. We note that Theorem 1.2 is applicable to this equation with $g(t)=t$, i.e.,

$$
x^{(4)}(t)+2 x^{(2)}(t)+x(t)=0
$$

where its solution set is $\{\sin t, \cos t, t \sin t, t \cos t\}$ while

$$
x^{(4)}(t)-2 x^{(2)}(t)+x(t)=0
$$

has solution set $\left\{e^{-t}, e^{t}, t e^{-t}, t e^{t}\right\}$. Clearly, the associated second order equation

$$
x^{(2)}(t)-2 x(t)=0
$$

is nonoscillatory and Theorem 2.8 fails to apply to this equation because $p(t)=-2<$ 0 .

## 3. GENERAL REMARKS

1. The results of this article are presented in a form which is essentially new and of high degree of generality.
2. It will be of interest to extend the results of this paper to higher order ( $>4$ ) equations.
3. It is also of interest to study equation (1.1) with $f(x)=x^{\gamma}, \gamma$ is the ratio of positive odd integers and $1<\gamma$.

## REFERENCES

[1] R. P. Agarwal, and S. R. Grace. The oscillation of higher order differential equations with deviating arguments, Comput. Math. Appl. 38 (1999), 185-199.
[2] R. P. Agarwal, S. R. Grace, I. T. Kguradze, and D. O'Regan. Oscillation of functional differential equations, Math. Comput. Modelling, 41 (2005), 417-461.
[3] R. P. Agarwal, S. R. Grace, and D. O'Regan. Oscillation of certain fourth order functional differential equations, Ukran. Mat. Zh. 59 (2007), 291-313.
[4] R. P. Agarwal, S. R. Grace, and P. J. Wong. On the bounded oscillation of certain fourth order functional differential equations, Nonlinear Dyn. Syst. Theory 5 (2005), 215-227.
[5] R. P. Agarwal, S. R. Grace, and D. O'Regan. Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 2000.
[6] R. P. Agarwal, S. R. Grace, and D. O'Regan. Oscillation criteria for certain nth order differential equations with deviating arguments, J. Math. Anal. Appl. 262 (2001), 601-622.
[7] R. P. Agarwal, S. R. Grace, and D. O'Regan. Oscillation Theory for Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations, Kluwer Academic Publishers, Dordrecht, 2002.
[8] M. Bartusek and Z. Dosla. Oscillatory solutions of nonlinear fourth order differential equations with middle term, EJQTDE, 55 (2014), 1-9.
[9] M. Bartusek and Z. Dosla. Asymptotic problems for fourth order nonlinear differential equations, $B V P, 89$ (2013), 10 pages.
[10] M. Bartusek, and Z. Dosla. Asymptotic problems for fourth-order nonlinear differential equations, Bound. Value Probl. 2013, No. 89, 15 pp.
[11] M. Bartusek, Z. Dosla. Oscillation of fourth order sub-linear differential equations, Appl. Math. Lett. 36 (2014), 36-39.
[12] M. Bartusek, M. Cecchi, Z. Dosla, and M.Marini. Asymptotics for higher order differential equations with a middle term, J. Math. Anal. Appl. 388 (2012), 1130-1140.
[13] M. Bartusek, M. Cecchi, Z. Dosla, and M. Marini. Fourth-order differential equation with deviating argument, Abstr. Appl. Anal. 2012, Art. ID 185242, 17 pp.
[14] E. Berchio, A. Ferrero, F. Gazzola, and P. Karageorgis. Qualitative behavior of global solutions to some nonlinear fourth order differential equations, J. Differential Equations 251 (2011), 2696-2727.
[15] I. Gyori, and G. Ladas. Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.
[16] I. Kiguradze, An oscillation criterion for a class of ordinary differential equations, Differ. Uravn. 28 (1992), 201-214.
[17] T. Kusano, M. Naito, F. Wu, On the oscillation of solutions of 4-dimensional Emden Fowler differential systems, Adv. Math. Sci. Appl. 11 (2001), 685-71.
[18] C. G. Philos, On the existence of nonoscillatory solutions tending to zero at for differential equations with positive delays, Arch. Math. (Basel) 36 (1981), 168-178.

