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The Generalized Hypergeometric Difference Equation

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Abstract: A difference equation analogue of the generalized hypergeometric differential equation is defined, its contiguous relations are developed, and its relation to numerous well-known classical special functions are demonstrated.

Keywords: special functions, discrete hypergeometric series, delay difference equations, contiguous relation, generalized hypergeometric functions

MSC: 33C20, 39A12

1 Introduction

The Pochhammer symbol $(a)_k$ is defined for $k \in \mathbb{N}$ by $(a)_k = a(a+1)\dots(a+k-1)$ and for $k=0$ by $(a)_0 = 1$. We define the product notation

$$\prod_{s=1,(k)}^{m} A_s = \left(\prod_{s=1}^{k-1} A_s\right)\left(\prod_{s=k+1}^{m} A_s\right).$$

We use the forward difference operator Δ defined by $\Delta f(t) = f(t+1) - f(t)$. From the definition of Δ it is easy to see that $\Delta[(-1)^n(-t)_n] = n(-1)^{n-1}(-t)_{n-1}$ is the discrete analogue of the power rule of differentiation. The generalized hypergeometric series ${}_p\mathcal{F}_q$ is defined by

$${}_p\mathcal{F}_q(a_1,\dots,a_p;b_1,\dots,b_q;t) = \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{t^k}{k!}. \tag{1}$$

The derivative of (1) is known [1, 5.2.2. (1)]:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_p\mathcal{F}_q(a_1,\dots,a_p;b_q,\dots,b_q;t) = \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}{}_p\mathcal{F}_q(a_1,\dots,a_p;b_1,\dots,b_q;t). \tag{2}$$

Let θ denote the operator $\theta = t\frac{\mathrm{d}}{\mathrm{d}t}$. It is known [2, §46 (3)] that if $y(t) = {}_p\mathcal{F}_q(a_1,\dots,a_p;b_1,\dots,b_q;t)$, then y satisfies the differential equation

$$\left[\theta\prod_{j=1}^{q}(\theta + b_j - 1) - z\prod_{i=1}^{q}(\theta + a_i)\right]y(t) = 0. \tag{3}$$

Two hypergeometric functions are called contiguous if one of their parameters a_i or b_i differs in one by ± 1. We adopt the following notation:

$$\mathcal{F} = {}_p\mathcal{F}_q(a_1,\dots,a_p;b_1,\dots,b_q;t),$$

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$$\mathcal{F}(a_i\pm) = {}_p\mathcal{F}_q(a_1, \dots, a_i \pm 1, \dots, a_p; b_q, \dots, b_q; t),$$

and

$$\mathcal{F}(b_i\pm) = {}_p\mathcal{F}_q(a_1, \dots, a_p; b_1, \dots, b_i \pm 1, \dots, b_q; t).$$

Following [3, p. 81] we adopt the following notations:

$$S_n = \frac{(a_1+n)(a_2+n)\dots(a_p+n)}{(b_1+n)\dots(b_q+n)}, \tag{4}$$

$$\tau_{n,k} = \frac{S_n}{a_k+n}, \tag{5}$$

$$\frac{(a_1)_{n+1}\dots(a_p)_{n+1}}{(b_1)_{n+1}\dots(b_q)_{n+1}} = S_n \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}, \tag{6}$$

and

$$U_j = \frac{\prod_{s=1}^{p}(a_s - b_j)}{b_j \prod_{s=1,(j)}^{q}(b_s - b_j)}.$$

A total of $2p+q$ contiguous relations exist for ${}_p\mathcal{F}_q$, split into three classes [3, p. 85]: if $p<q$, then

$$\begin{cases} (a_1-a_k)\mathcal{F} = a_1\mathcal{F}(a_1+) - a_k\mathcal{F}(a_k+), & k=2,3,\dots,p \\ (a_1-b_k+1)\mathcal{F} = a_1\mathcal{F}(a_1+) - (b_k-1)\mathcal{F}(b_k-), & k=1,2,\dots,q \\ \mathcal{F} = \mathcal{F}(a_k-) + t\sum_{j=1}^{q} W_{j,k}\mathcal{F}(b_j+), & k=1,2,\dots,p \\ a_1\mathcal{F} = a_1\mathcal{F}(a_1+) - t\sum_{j=1}^{q} U_j\mathcal{F}(b_j+). \end{cases} \tag{7}$$

If $p=q$, then

$$\begin{cases} (a_1-a_k)\mathcal{F} = a_1\mathcal{F}(a_1+) - a_k\mathcal{F}(a_k+), & k=2,3,\dots,p \\ (a_1-b_k+1)\mathcal{F} = a_1\mathcal{F}(a_1+) - (b_k-1)\mathcal{F}(b_k-), & k=1,2,\dots,q \\ \mathcal{F} = \mathcal{F}(a_k-) + t\sum_{j=1}^{q} W_{j,k}\mathcal{F}(b_j+), & k=1,2,\dots,p \\ (a_1+t)\mathcal{F} = a_1\mathcal{F}(a_1+) - t\sum_{j=1}^{q} U_j\mathcal{F}(b_j+). \end{cases} \tag{8}$$

If $p=q+1$, then

$$\begin{cases} (a_1-a_k)\mathcal{F} = a_1\mathcal{F}(a_1+) - a_k\mathcal{F}(a_k+), \quad k=2,3,\dots,p \\ (a_1-b_k+1)\mathcal{F} = a_1\mathcal{F}(a_1+) - (b_k-1)\mathcal{F}(b_k-), \quad k=1,2,\dots,q \\ (1-t)\mathcal{F} = \mathcal{F}(a_k-) + t\sum_{j=1}^{q} W_{j,k}\mathcal{F}(b_j+), \quad k=1,2,\dots,p \\ \left[(1-t)a_1 + (A-B)t\right]\mathcal{F} = (1-t)a_1\mathcal{F}(a_1+) - t\sum_{j=1}^{q} U_j\mathcal{F}(b_j+), \end{cases} \tag{9}$$

where $A = \sum_{s=1}^{p} a_s$ and $B = \sum_{s=1}^{q} b_s$.

Many classical special functions can be expressed in terms of ${}_p\mathcal{F}_q$. The classical exponential function $\exp\left(\int_0^t p(\tau)\mathrm{d}\tau\right)$ satisfies the initial value problem

$$y'(t) = p(t)y(t), \quad y(0) = 1. \tag{10}$$

Substituting αt for t in the relationship given by [4, (2.1.9)] yields

$$e^{\alpha t} = {}_0\mathcal{F}_0(;;\alpha t). \tag{11}$$

The functions $\sin(at)$ and $\cos(at)$ satisfy the differential equation

$$y''(t) = -a^2 y(t).$$

The relationship between $\sin(at)$ and ${}_0\mathcal{F}_1$ is known [4, (2.1.7)] to be

$$\sin(at) = at{}_0\mathcal{F}_1\left(;\frac{3}{2};-\frac{a^2t^2}{4}\right). \tag{12}$$

The relationship between $\cos(at)$ and ${}_0\mathcal{F}_1$ is known [4, (2.1.8)] to be

$$\cos(at) = {}_0\mathcal{F}_1\left(;\frac{1}{2};-\frac{a^2t^2}{4}\right). \tag{13}$$

Classical Bessel functions $\mathcal{J}_n$ are defined by the series

$$\mathcal{J}_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+n}}{k!\Gamma(k+n+1)2^{2k+n}}.$$

It is known [2, p. 108 (1)] that

$$\mathcal{J}_n(t) = \frac{t^n}{2^n\Gamma(n+1)}{}_0\mathcal{F}_1\left(;n+1;-\frac{t^2}{4}\right), \tag{14}$$

and consequently, $y(t) = {}_0\mathcal{F}_1\left(;n+1;-\frac{t^2}{4}\right) = \frac{2^n n! J_n(t)}{t^n}$ satisfies the differential equation [2, p. 109 (4)]

$$t^2y''(t) + ty'(t) + (t^2-n^2)y(t) = 0. \tag{15}$$

A “hypergeometric difference equation”, originally investigated in [5], is defined in [6] by

$$(a_2x+b_2)y(x+2) + (a_1x+b_1)y(x+1) + (a_0x+b_0)y(x) = 0,$$

and it is named this “because . . . its solutions can be expressed in terms of the hypergeometric series”, which is referring specifically to ${}_2\mathcal{F}_1$. The “basic hypergeometric series” (or “q-hypergeometric series”) is defined in [7] by the notation

$${}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k\dots(a_r;q)_k}{(q;q)_k(b_1;q)_k\dots(b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k.$$

The paper [8] defines a “discrete hypergeometric function” by

$${}_rM_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z) = \sum_{k=0}^{\infty} \frac{(q^{a_1})_k\dots(q^{a_2})_k z^{(k)}}{(q)_k(q^{b_1})_k\dots(q^{b_s})_k},$$

where $z^{(k)}$ is a q-analogue of z^k. The paper [9] speaks of “discrete analogues” of theorems related to classical hypergeometric functions, but the authors simply mean restricting the otherwise complex parameters and variables to nonnegative integers. Difference equations “of hypergeometric type” are discussed in [10] and are defined by the mixed forward and backwards difference equation

$$\sigma(t)\nabla\Delta y(t) + \tau(t)\Delta y(t) + \lambda y(t) = 0,$$

where σ and τ are first- or second-order polynomials.

A Taylor theorem for the difference operator is known [11, Theorem 1.113]:

$$f(t) = \sum_{k=0}^{\infty} \frac{\Delta^k f(0)(-1)^k(-t)_k}{k!}. \tag{16}$$

Let $p : \mathbb{Z} \to \mathbb{C}$ be so that for all $t \in \mathbb{Z}$, $1 + p(t) \neq 0$. The discrete exponential function $e_p : \mathbb{Z} \to \mathbb{C}$ is defined to be the unique solution of the initial value problem

$$\Delta y(t) = p(t)y(t), \quad y(0) = 1. \tag{17}$$

In particular, if $p \equiv \alpha$ is constant and $\alpha \in \mathbb{C} \setminus \{-1\}$, then (16) shows

$$e_\alpha(t) = \sum_{k=0}^{\infty} \alpha^k \frac{(-1)^k(-t)_k}{k!}. \tag{18}$$

The discrete trigonometric functions $\sin_\alpha$ and $\cos_\alpha$ are often defined for $\alpha \in \mathbb{C} \setminus \{1, -1\}$ [11, Definition 3.25] by

$$\sin_\alpha(t) = \frac{e_{ip}(t) - e_{-ip}(t)}{2i} \quad \text{and} \quad \cos_\alpha(t) = \frac{e_{ip}(t) + e_{-ip}(t)}{2}.$$

Both functions are solutions of the difference equation

$$\Delta^2 y(t) + \alpha^2 y(t) = 0. \tag{19}$$

It is easy to see that the function $\cos_\alpha$ obeys the discrete Taylor series

$$\cos_\alpha(t) = \sum_{k=0}^{\infty} \frac{(-1)^k(-t)_{2k}}{(2k)!}, \tag{20}$$

and that the function $\sin_\alpha$ obeys the discrete Taylor series

$$\sin_\alpha(t) = \sum_{k=0}^{\infty} \frac{(-1)^k(-t)_{2k+1}}{(2k+1)!}. \tag{21}$$

Discrete Bessel functions, J_n, were defined in [12] by

$$J_n(t) = \frac{(-1)^n(-t)_n}{2^n n!} {}_2\mathcal{F}_1\left(\frac{n-t}{2}, \frac{n+1-t}{2}; n+1; -1\right). \tag{22}$$

It was shown that this function obeys the difference equation

$$t(t-1)\Delta^2 y(t-2) + t\Delta y(t-1) + t(t-1)y(t-2) - n^2 y(t) = 0.$$

For further results on J_n, see also the recent paper [13] by Antonín Slavík.

2 Discrete hypergeometric series

We define the discrete generalized hypergeometric series for $p, q \in \mathbb{N}_0$ by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; t, n, \xi) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{\xi^k(-1)^{nk}(-t)_{nk}}{k!}, \tag{23}$$

where $a_j \in \mathbb{C}$ for $j = 1, 2, \ldots, q$ and $b_j \in \mathbb{C} \setminus (-\mathbb{N}_0)$ for $j \in \mathbb{N}$. If $n = 0$ in (23), then ${}_pF_q$ becomes a constant value of ${}_p\mathcal{F}_q$ independent of t:

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; t, 0, \xi) = {}_p\mathcal{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; (-1)^n \xi).$$

If $n = 1$ in (23), then

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; t, 1, \xi) = {}_{p+1}\mathcal{F}_q\left(a_1, \ldots, a_p, -t; b_1, \ldots, b_q; (-1)^n \xi\right).$$

To get the relationship between ${}_pF_q$ and ${}_p\mathcal{F}_q$ for $n \in \mathbb{N} \setminus \{1\}$, we begin with the following lemma.

Lemma 1. *We have*

$$(-t)_{nk} = n^{nk} \prod_{j=0}^{n-1} \left(\frac{-t+j}{n} \right)_k$$

for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Proof. We compute

$$\begin{aligned}
(-t)_{nk} &= (-t)(-t+1)\dots(-t+n)(-t+n+1)\dots(-t+nk-1) \\
&= \left[\prod_{j=0}^{k-1}(-t+jn)\right]\left[\prod_{j=0}^{k-1}(-t+jn+1)\right]\dots\left[\prod_{j=0}^{k-1}(-t+jn+(n-1))\right] \\
&= n^{nk}\frac{\left[\prod_{j=0}^{k-1}(-t+jn)\right]}{n^k}\frac{\left[\prod_{j=0}^{k-1}(-t+jn+1)\right]}{n^k}\dots\frac{\left[\prod_{j=0}^{k-1}(-t+jn+(n-1))\right]}{n^k} \\
&= n^{nk}\left[\prod_{j=0}^{k-1}\left(\frac{-t}{n}+j\right)\right]\left[\prod_{j=0}^{k-1}\left(\frac{-t+1}{n}+j\right)\right]\dots\left[\prod_{j=0}^{k-1}\left(\frac{-t+(n-1)}{n}+j\right)\right] \\
&= n^{nk}\left(\frac{-t}{n}\right)_k\dots\left(\frac{-t+(n-1)}{m}\right)_k \\
&= n^{nk}\prod_{j=0}^{n-1}\left(\frac{-t+j}{n}\right)_k,
\end{aligned}$$

as was to be shown. □

Now we may express ${}_pF_q$ in terms of ${}_{p+n}\mathcal{F}_q$ for $n \in \mathbb{N}$.

Proposition 2. *Let $y(t) = {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t, n, \xi)$. Then,*

$$y(t) = {}_{p+n}\mathcal{F}_q\left(a_1, \dots, a_p, \frac{-t}{n}, \frac{-t+1}{n}, \dots, \frac{-t+n-1}{n}; b_1, \dots, b_q; (-n\xi)^n\right). \tag{24}$$

We now consider the convergence of the series defining ${}_pF_q$.

Theorem 3. *The series* (23) *converges for all $t \in \mathbb{N}$. Now suppose $t \in \mathbb{C} \setminus \mathbb{N}$, then* (23) *converges when:*
i. $p + n < q + 1$,
ii. $p + n = q + 1$ and $|\xi|n^n < 1$, or
iii. $p + n = q + 1$, $|\xi|n^n = 1$, and

$$\operatorname{Re}\left(\sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k - \frac{1}{n}\sum_{k=0}^{n-1} -t + k\right) > 0.$$

Also, (23) *diverges provided that*
iv. $p + n > q + 1$, or
v. $p + n = q + 1$ and $|\xi|n^n > 1$.

Proof. If $t \in \mathbb{N}$, then $(-t)_{nk}$ eventually terminates by definition. Assume $t \in \mathbb{C} \setminus \mathbb{N}$. For $j = 1, \dots, p$, we see $\frac{(a_j)_{k+1}}{(a_j)_k} = a_j + k$. Similarly, for $j = 1, \dots, q$, we see $\frac{(b_j)_k}{(b_j)_{k+1}} = \frac{1}{b_j + k}$. Now we compute

$$\frac{(-t)_{n(k+1)}}{(-t)_{nk}} = \frac{(-t)_{nk}(-t+nk)_n}{(-t)_{nk}} = (-t+nk)(-t+nk+1)\dots(-t+nk+n-1).$$

Applying the ratio test, we take the limit as $k \to \infty$ of

$$\left|\xi \frac{(a_1+k)\dots(a_p+k)(nk-t)\dots(nk-1+n-t)}{(b_1+k)\dots(b_q+k)k}\right| = \left|\xi \frac{n^n k^{n+p}+\dots}{k^{q+1}+\dots}\right|.$$

The limit is zero when $p+n<q+1$, yielding convergence of ${}_pF_q$ in that case. The limit is infinity when $p+n>q+1$, yielding divergence in that case. If $p+n=q+1$, then the limit is $|\xi|n^n$ and so when this quantity is less than 1 we get convergence and when it is greater than one we get divergence. The case $|\xi|n^n=1$ can be resolved by noting (24) and the well-known convergence properties of ${}_{q+1}\mathcal{F}_q$ [2, p. 74]. □

We now provide an analogue of (2).

Theorem 4. *We have*

$$\Delta^n\left[{}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};t,1,\xi\right)\right] = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{\ell=1}^{q}(b_\ell)_n}\xi^n {}_pF_q\left(\begin{matrix}a_1+n,\dots,a_p+n\\ b_1+n,\dots,b_q+n\end{matrix};t,1,\xi\right).$$

Proof. We compute

$$\begin{aligned}\Delta^n\left[{}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};t,1,\xi\right)\right] &= \Delta^n\left[\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{\xi^k(-1)^k(-t)_k}{k!}\right]\\ &= \sum_{k=n}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{\xi^k(-1)^{k-n}(-t)_{k-n}}{(k-n)!}\\ &= \sum_{k=0}^{\infty}\frac{(a_1)_{k+n}\dots(a_p)_{k+n}}{(b_1)_{k+n}\dots(b_q)_{k+n}}\frac{\xi^{k+n}(-1)^k(-t)_k}{k!}\\ &= \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{\ell=1}^{q}(b_\ell)_n}\xi^n {}_pF_q\left(\begin{matrix}a_1+n,\dots,a_p+n\\ b_1+n,\dots,b_q+n\end{matrix};t,1,\xi\right),\end{aligned}$$

which is exactly the desired relation. □

Corollary 5. *We have*

$$\Delta\left[{}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};t,1,\xi\right)\right] = \frac{a_1a_2\dots a_p}{b_1b_2\dots b_p}\xi {}_pF_q\left(\begin{matrix}a_1+1,\dots,a_p+1\\ b_1+1,\dots,b_q+1\end{matrix};t,1,\xi\right).$$

Define the function shift operator ρ by $(\rho f)(t)=f(t-1)$, and define the operator Θ by $\Theta=t\rho\Delta$, which is a discrete analogue of θ.

Lemma 6. *We have*

$$\Theta(-1)^k(-t)_k = k(-1)^k(-t)_k$$

for $k\in\mathbb{N}$.

Proof. We calculate

$$\begin{aligned}\Theta(-1)^k(-t)_k &= t\rho\Delta[(-1)^k(-t)_k]\\ &= t\rho[k(-1)^{k-1}(-t)_{k-1}]\\ &= t[k(-1)^{k-1}(-t+1)_{k-1}]\\ &= k(-1)^k(-t)_k,\end{aligned}$$

which is exactly the desired relation. □

We now prove that ${}_pF_q$ satisfies a difference equation that resembles a generalized form of (3).

Theorem 7. *Define $y(t) = {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t, n, \xi)$. Then y satisfies the difference equation*

$$\left[\Theta \prod_{j=1}^{q} \left(\frac{1}{n}\Theta + b_j - 1\right) - n\xi(-1)^n(-t)_n\rho^n \prod_{i=1}^{p} \left(\frac{1}{n}\Theta + a_i\right)\right] y(t) = 0.$$

Proof. We first compute

$$\begin{aligned}
n\xi(-1)^n(-t)_n\rho^n \prod_{i=1}^{p}\left(\frac{1}{n}\Theta + a_i\right) y(t) &= n\xi(-1)^n(-t)_n\rho^n \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}}{k!} \prod_{i=1}^{p}\left(\frac{1}{n}\Theta + a_i\right)(-t)_{nk} \\
&= n\xi(-1)^n(-t)_n\rho^n \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}(-t)_{nk}}{k!} \prod_{i=1}^{p}(k + a_i) \\
&= n\xi(-1)^n \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}(-t)_{nk+n}}{k!} \prod_{i=1}^{p}(k + a_i).
\end{aligned}$$

Now we compute

$$\begin{aligned}
\Theta \prod_{j=1}^{q}\left(\frac{1}{n}\Theta + b_j - 1\right) y &= \Theta \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}}{k!} \prod_{j=1}^{q}\left(\frac{1}{n}\Theta + b_j - 1\right)(-t)_{nk} \\
&= \Theta \sum_{k=0}^{\infty} (a_1)_k \dots (a_p)_k \frac{\prod_{j=1}^{q}(k + b_j - 1)}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}(-t)_{nk}}{k!} \\
&= \Theta \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_{k-1} \dots (b_q)_{k-1}} \frac{\xi^k(-1)^{nk}(-t)_{nk}}{k!} \\
&= nt\rho \sum_{k=1}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_{k-1} \dots (b_q)_{k-1}} \frac{\xi^k(-1)^{nk-1}(-t)_{nk-1}}{(k-1)!} \\
&= nt\rho \sum_{k=0}^{\infty} \frac{(a_1)_{k+1} \dots (a_p)_{k+1}}{(b_1)_k \dots (b_q)_k} \frac{\xi^{k+1}(-1)^{nk+n-1}(-t)_{nk+n-1}}{k!} \\
&= n\xi(-t)\rho \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk+n}(-t)_{nk+n-1}}{k!} \prod_{i=1}^{p}(k + a_i) \\
&= n\xi(-t) \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk+n}(-t+1)_{nk+n-1}}{k!} \prod_{i=1}^{p}(k + a_i) \\
&= n\xi(-1)^n \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}(-t)_{nk+n}}{k!} \prod_{i=1}^{p}(k + a_i),
\end{aligned}$$

which is the same series as above, completing the proof. □

If $n = 1$ in Theorem 7, then we obtain a direct analogue of (3).

Corollary 8. *Let $y(t) = {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t, 1, \xi)$. Then y satisfies the difference equation*

$$\left[\Theta \prod_{j=1}^{q}(\Theta + b_j - 1) - \xi t\rho \prod_{i=1}^{p}(\Theta + a_i)\right] y(t) = 0.$$

3 Contiguous relations

We define F, $F(a_i\pm)$, and $F(b_i\pm)$ in the same way as $\mathcal{F}$, $\mathcal{F}(a_i\pm)$, and $\mathcal{F}(b_j\pm)$, but we use ${}_pF_q$ in place of ${}_p\mathcal{F}_q$. The following with $n = 1$ is an analogue of [2, p. 82 (12)].

Lemma 9. *The following recurrence holds for $j \in \{1, 2, \dots, p\}$:*

$$\left(\frac{1}{n}\Theta + a_j\right) F = a_j F(a_j+).$$

Proof. We calculate

$$\left(\frac{1}{n}\Theta + a_j\right) F = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k (-1)^{nk}}{k!} \left(\frac{1}{n}\Theta + a_j\right) (-t)_{nk} = F(a_j+),$$

as was to be shown. □

The following with $n = 1$ is an analogue of [2, p. 82 (13)].

Lemma 10. *The following recurrence holds for $j \in \{1, 2, \dots, q\}$:*

$$\left(\frac{1}{n}\Theta + b_j - 1\right) F = (b_j - 1) F(b_j-).$$

Proof. We calculate

$$\left(\frac{1}{n}\Theta + b_j - 1\right) F = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_j)_{k-1} \dots (b_q)_k} \frac{\xi^k (-1)^{nk} (-t)_{nk}}{k!} = (b_j - 1) F(b_j-),$$

as was to be shown. □

The following with $n = 1$ is an analogue of [2, p. 82 (14)].

Lemma 11. *The following formula holds for $j \in \{2, 3, \dots, p\}$:*

$$(a_1 - a_j) F = a_1 F(a_1+) - a_j F(a_j+).$$

Proof. Applying Lemma 9 with $j = 1$ and $j \in \{2, \dots, p\}$ yields

$$(a_1 - a_j) F = (a_1 F(a_1+) - \Theta F) - (a_j F(a_j+) - \Theta F) = a_1 F(a_1+) - a_j F(a_j+),$$

as was to be shown. □

The following with $n = 1$ is an analogue of [2, p. 82 (15)].

Lemma 12. *The following formula holds for $j \in \{1, 2, \dots, q\}$:*

$$(a_1 - b_j + 1) F = a_1 F(a_1+) - (b_j - 1) F(b_j-).$$

Proof. Using Lemma 9 and Lemma 10, we get

$$(a_1 - b_j + 1) F = a_1 F(a_1+) - (b_j - 1) F(b_j-),$$

as was to be shown. □

The following with $n = 1$ is an analogue of [2, p. 83 (16)].

Lemma 13. *If $p < q$ and the b_i are pairwise different, then we have*

$$\frac{1}{n}\Theta F = \xi (-1)^n (-t)_n \rho^n \sum_{j=1}^{q} U_j F(b_j+).$$

Proof. Using (6), we compute

$$\begin{aligned}\frac{1}{n}\Theta F &= \sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{\xi^k(-1)^{nk}}{k!}\left[\frac{1}{n}\Theta(-t)_{nk}\right]\\ &= \sum_{k=1}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{\xi^k(-1)^{nk}}{(k-1)!}(-t)_{nk}\\ &= \sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_p)_{k+1}}{(b_1)_{k+1}\dots(b_q)_{k+1}}\frac{\xi^{k+1}(-1)^{nk+n}}{k!}(-t)_{nk+n}\\ &= \sum_{k=0}^{\infty}S_k\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{\xi^{k+1}(-1)^{nk+n}}{k!}(-t)_{nk+n}.\end{aligned} \tag{25}$$

If $p < q$, then it is known [2, p. 82] that $S_n = \sum_{j=1}^{q}\frac{b_jU_j}{b_j+n}$. Therefore,

$$\frac{1}{n}\Theta F = \sum_{k=0}^{\infty}\left(\sum_{j=1}^{q}\frac{b_jU_j}{b_j+k}\right)\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{\xi^{k+1}(-1)^{nk+n}}{k!}(-t)_{nk+n}.$$

Note that $(-t)_{nk+n} = (-t)_n(-t+n)_{nk}$ and

$$\frac{1}{(b_j)_k}\frac{b_j}{b_j+k} = \frac{1}{(b_j+1)_k}, \tag{26}$$

so we obtain after interchanging sums,

$$\frac{1}{n}\Theta F = \xi(-1)^n(-t)_n\rho^n\sum_{j=1}^{q}U_jF(b_j+),$$

which is exactly the desired relation. □

The following lemma with $n = 1$ is an analogue of [2, p. 83 (17)].

Lemma 14. *If $p < q$ and the b_j are pairwise different, then*

$$a_1F = a_1F(a_1+) - \xi(-1)^n(-t)_n\rho^n\sum_{j=1}^{q}U_jF(b_j+).$$

Proof. Lemma 9 with $j = 1$ says

$$\frac{1}{n}\Theta F + a_1F = a_1F(a_1+).$$

Combining this with Lemma 13 yields

$$a_1F = a_1F(a_1+) - \xi(-1)^n(-t)_n\rho^n\sum_{j=1}^{q}U_jF(b_j+),$$

as was to be shown. □

The following with $n = 1$ is an analogue of [2, p. 83 (18)].

Lemma 15. *When $p = q$, we have*

$$a_1F + \xi(-1)^n(-t)_n\rho^nF = a_1F(a_1+) - \xi(-1)^n(-t)_n\rho^n\sum_{j=1}^{q}U_jF(b_j+).$$

Proof. By (25), we get

$$\frac{1}{n}\Theta F = \sum_{k=0}^{\infty} S_k \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^{k+1}(-1)^{nk+n}}{k!}(-t)_{nk+n}.$$

If $p = q$, then it is known [2, p. 83] that $S_n = 1 + \sum_{j=1}^{q} \frac{b_j U_j}{b_j + n}$. Interchanging sums and taking into account (26), we find

$$\frac{1}{n}\Theta F = \xi(-1)^n(-t)_n\rho^n F + \xi(-1)^n(-t)_n\rho^n \sum_{j=1}^{q} U_j F(b_j+).$$

Lemma 9 with $j = 1$ says

$$\frac{1}{n}\Theta F = a_1 F(a_1+) - a_1 F.$$

Combining the previous two formulas to eliminate $\frac{1}{n}\Theta F$ yields

$$a_1 F + \xi(-1)^n(-t)_n\rho^n F = a_1 F(a_1+) - \xi(-1)^n(-t)_n\rho^n \sum_{j=1}^{q} U_j F(b_j+),$$

completing the proof. □

The following lemma with $n = 1$ is an analogue of [2, p. 84 (20)].

Lemma 16. *The following formula holds for $p \le q$ and $i \in \{1, 2, \dots, p\}$:*

$$F = F(a_i-) + (-1)^n(-t)_n\rho^n \sum_{j=1}^{q} W_{j,i} F(b_j+).$$

Proof. We use Lemma 6 to compute

$$\begin{aligned}
\frac{1}{n}\Theta F(a_i-) &= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_i-1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}}{k!}\left[\frac{1}{n}\Theta(-t)_{nk}\right] \\
&= \sum_{k=1}^{\infty} \frac{a_i - 1}{a_i + n} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k(-1)^{nk}(-t)_{nk}}{(k-1)!} \\
&= \sum_{k=0}^{\infty} \frac{a_i - 1}{a_i + k} \frac{(a_1)_{k+1} \dots (a_p)_{k+1}}{(b_1)_{k+1} \dots (b_q)_{k+1}} \frac{\xi^{k+1}(-1)^{nk+n}(-t)_{nk+n}}{k!} \\
&= \xi(-1)^n(-t)_n\rho^n \sum_{k=0}^{\infty} \frac{a_i - 1}{a_i + k} \frac{(a_1)_{k+1} \dots (a_p)_{k+1}}{(b_1)_{k+1} \dots (b_q)_{k+1}} \frac{\xi^k(-1)^{nk}(-t)_{nk}}{k!}.
\end{aligned}$$

Formula (6) shows

$$\frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{1}{a_k + n} = \tau_{n,k} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k}.$$

Since $p \le q$, it is known [2, p. 84] that

$$\tau_{n,k} = \sum_{j=1}^{q} \frac{b_j W_{j,k}}{b_j + n}.$$

Therefore, we have

$$\frac{1}{n}\Theta F(a_i-) = (a_i - 1)(-1)^n(-t)_n\rho^n \sum_{j=1}^{q} W_{j,i} F(b_j+).$$

By Lemma 9,

$$\frac{1}{n}\Theta F(a_i-) = (a_i - 1)(F - F(a_i-)).$$

Therefore, using the preceding two formulas to eliminate $\frac{1}{n}\Theta F(a_i-)$, we complete the proof. □

The following theorem presents the contiguous relations for ${}_pF_q$ analogous to (7), (8), and (9).

Theorem 17. *If $p \le q$, then there are a total of $2p+q$ contiguity relations for ${}_pF_q$. If $p<q$, then*

$$\begin{cases} (a_1-a_k)F = a_1F(a_1+) - a_kF(a_k+), & k=2,3,\dots,p \\ (a_1-b_k+1)F = a_1F(a_1+) - (b_k-1)F(b_k-), & k=1,2,\dots,q \\ F = F(a_k-) + (-1)^n(-t)_n\rho^n \sum_{j=1}^{q} W_{j,k}F(b_j+), & k=1,2,\dots,p \\ a_1F = a_1F(a_1+) - \xi(-1)^n(-t)_n\rho^n \sum_{j=1}^{q} U_jF(b_j+). \end{cases} \tag{27}$$

If $p=q$, then

$$\begin{cases} (a_1-a_k)F = a_1F(a_1+) - a_kF(a_k+), & k=2,3,\dots,p \\ (a_1-b_k+1)F = a_1F(a_1+) - (b_k-1)F(b_k-), & k=1,2,\dots,q \\ \mathcal{F} = \mathcal{F}(a_k+) + t\sum_{j=1}^{q} W_{j,k}\mathcal{F}(b_j+), & k=1,2,\dots,p \\ (a_1+t)F = a_1\mathcal{F}(a_1+) - t\sum_{j=1}^{q} U_j\mathcal{F}(b_j+). \end{cases} \tag{28}$$

If $p=q+1$, then F becomes a constant independent of t whose contiguous relations are exactly the first two of those in (9).

Proof. Considering (24), there are a total of $2(p+n)+q+1 = 2p+2n+q+1$ contiguous relations of ${}_{p+n}\mathcal{F}_q$ that may be relevant. The n relations involving the parameters $\frac{-t+j}{n}$ for $j=0,\dots,n-1$ are not relevant under the definition of contiguous relations for ${}_pF_q$. The first two relations in (27) and (28) are those of Lemma 11 and Lemma 12. Lemma 14 and Lemma 16 complete the proof for $p<q$. Lemma 15 and Lemma 16 complete the proof for $p=q$. If $p=q+1$, then Theorem 3 guarantees that $n=0$ and (2) shows that it is independent of t, completing the proof. □

4 Discrete special functions in terms of discrete hypergeometric series

The following proposition is immediate from (18) and is an analogue of (11).

Proposition 18. *For $\alpha \in \mathbb{C}\setminus\{-1\}$, we have*

$$e_\alpha(t) = {}_0F_0(;;t,1,\alpha).$$

Now we show that ${}_0F_0$ satisfies numerous difference equations, generalizing (17).

Theorem 19. *The function $y(t) = {}_0F_0(;;t,n,\alpha)$ satisfies the difference equation*

$$t\Delta y(t-1) - n\alpha(-1)^n(-t)_n y(t-n) = 0.$$

Proof. By Theorem 7, y satisfies

$$\Theta y(t) - n\alpha(-1)^n(-t)_n\rho^n y(t) = 0.$$

Substituting the definition of Θ yields

$$t\rho\Delta y(t) - n\alpha(-1)^n(-t)_n\rho^n y(t) = 0,$$

and simplifying completes the proof. □

The following formula is an analogue of (12) and follows immediately from (21).

Proposition 20. *For $\alpha \in \mathbb{C} \setminus \{-1, 1\}$, we have*

$$\sin_\alpha(t) = \alpha t \rho_0 F_1\left(; \frac{3}{2}; t, 2, -\frac{\alpha^2}{4}\right).$$

The following theorem is a modification and generalization of (19).

Theorem 21. *The function $y(t) = {}_0F_1\left(; \frac{3}{2}; t, n, -\frac{\alpha^2}{4}\right)$ satisfies the difference equation*

$$\frac{t(t-1)}{n}\Delta^2 y(t-2) + \left(\frac{1}{n} + \frac{1}{2}\right) t\Delta y(t-1) + \frac{n\alpha^2(-1)^n}{4}(-t)_n y(t-n) = 0.$$

Proof. By Theorem 7, y satisfies

$$\Theta\left(\frac{1}{n}\Theta + \frac{1}{2}\right) y(t) + \frac{n\alpha^2(-1)^n}{4}(-t)_n \rho^n y(t) = 0.$$

Substituting in the definition of Θ yields

$$t\rho\Delta\left[\frac{1}{n}t\Delta y(t-1) + \frac{1}{2}\right] + \frac{n\alpha^2(-1)^n}{4}(-t)_n \rho^n y(t) = 0,$$

hence

$$\frac{t(t-1)}{n}\Delta^2 y(t-2) + \left(\frac{1}{n} + \frac{1}{2}\right) t\Delta y(t-1) + \frac{n\alpha^2(-1)^n}{4}(-t)_n y(t-n) = 0.$$

This completes the proof. □

The following formula is an analogue of (13) and follows immediately from (20).

Proposition 22. *For all $\alpha \in \mathbb{C} \setminus \{-1, 1\}$, we have*

$$\cos_\alpha(t) = {}_0F_1\left(; \frac{1}{2}; t, 2, -\frac{\alpha^2}{4}\right).$$

The following theorem generalizes (19), which holds for $n = 2$.

Theorem 23. *The function $y(t) = {}_0F_1\left(; \frac{1}{2}; t, n, -\frac{\alpha^2}{4}\right)$ satisfies the difference equation*

$$\frac{t(t-1)}{n}\Delta^2 y(t-2) + \left(\frac{1}{n} - \frac{1}{2}\right) t\Delta y(t-1) + \frac{n\alpha^2(-1)^n}{4}(-t)_n y(t-n) = 0.$$

Proof. By Theorem 7, y satisfies

$$\Theta\left(\frac{1}{n}\Theta - \frac{1}{2}\right) y(t) + \frac{n\alpha^2}{4}(-1)^n(-t)_n \rho^n y(t) = 0.$$

Substituting the definition of Θ yields

$$t\rho\Delta\left[\frac{1}{n}t\Delta y(t-1) - \frac{1}{2}y(t)\right] + \frac{n\alpha^2(-1)^n}{4}(-t)_n y(t-n) = 0,$$

or equivalently,

$$\frac{t(t-1)}{n}\Delta^2 y(t-2) + \left(\frac{1}{n} - \frac{1}{2}\right) t\Delta y(t-1) + \frac{n\alpha^2(-1)^n}{4}(-t)_n y(t-n) = 0,$$

as was to be shown. □

We now relate the discrete Bessel function to the discrete hypergeometric ${}_0F_1$ by finding an analogue of (14).

Lemma 24. *The discrete Bessel function satisfies*

$$J_n(t) = \frac{(-1)^n(-t)_n}{2^n n!}\,{}_0F_1\left(;n+1;t-n,2,-\frac{1}{4}\right).$$

Proof. By direct computation, we get

$$\left(\frac{n-t}{2}\right)_k\left(\frac{n+1-t}{2}\right)_k = \frac{(n-t)_{2k}}{2^{2k}} = \frac{(t-n)_{2k}}{2^{2k}}.$$

Therefore, (22) shows

$$\begin{aligned} J_n(t) &= \frac{(-1)^n(-t)_n}{2^n n!}\sum_{k=0}^{\infty}\frac{\left(\frac{n-t}{2}\right)_k\left(\frac{n+1-t}{2}\right)_k(-1)^k}{(n+1)_k k!} \\ &= \frac{(-1)^n(-t)_n}{2^n n!}\sum_{k=0}^{\infty}\frac{(n-t)_{2k}(-1)^k}{(n+1)_k 2^{2k} k!} \\ &= \frac{(-1)^n(-t)_n}{2^n n!}\sum_{k=0}^{\infty}\frac{(t-n)_{2k}(-\frac{1}{4})^k}{(n+1)_k k!} \\ &= \frac{(-1)^n(-t)_n}{2^n n!}\,{}_0F_1\left(;n+1;t-n,2,-\frac{1}{4}\right), \end{aligned}$$

which is exactly the desired relation. □

Now we prove an analogue of (15).

Theorem 25. *The function* $y(t) = \dfrac{(-1)^n J_n(t)2^n n!}{(-t)_n}$ *satisfies the difference equation*

$$t\Delta^2 y(t-1) + (2n+1)\Delta y(t) + ty(t-1) = 0.$$

Proof. By Lemma 24 and Theorem 7, y solves

$$\left[\Theta\left(\frac{1}{2}\Theta + (n+1) - 1\right) - 2\left(-\frac{1}{4}\right)(-1)^2(-t)_2\rho^2\right]y(t) = 0,$$

which becomes

$$\frac{t\rho}{2}\Delta(t\rho\Delta y(t)) + nt\rho\Delta y(t) + \frac{t(t-1)}{2}\rho^2 y(t) = 0.$$

The product rule for Δ shows

$$\Delta(t\rho\Delta y(t)) = \Delta(t\Delta y(t-1)) = \Delta y(t) + t\Delta^2 y(t-1).$$

Therefore, y satisfies

$$\frac{t(t-1)}{2}\Delta^2 y(t-2) + \left(n+\frac{1}{2}\right)t\Delta y(t-1) + \frac{t(t-1)}{2}y(t-2) = 0.$$

Dividing by t, replacing t with $t+1$, and multiplying by 2 complete the proof. □

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