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Nonoscillation Criteria for Two-Dimensional Time-Scale Systems

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Abstract: We study the existence and nonexistence of nonoscillatory solutions of a two-dimensional system of first-order dynamic equations on time scales. Our approach is based on the Knaster and Schauder fixed point theorems and some certain integral conditions. Examples are given to illustrate some of our main results.

Keywords: Time-scale systems; Nonoscillation; Dynamic Equations

1 Introduction

In this paper, we study on the asymptotic behavior of solutions of the nonlinear system of the first-order dynamic equations

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(t)), \end{cases} \tag{1.1}$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing such that $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$ and $a, b \in \mathrm{C_{rd}}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$. Whenever we write $t \geq t_1$, we mean that $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. A time scale, denoted by $\mathbb{T}$, is a closed subset of real numbers. An excellent introduction of time scales calculus can be found in [2, 3] by Bohner and Peterson. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. We call (x, y) a *proper solution* if it is defined on $[t_0, \infty)_{\mathbb{T}}$ and $\sup\{|x(s)|, |y(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. A solution (x, y) of (1.1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise, it is said to be oscillatory. Throughout this paper, without loss of generality, we assume that x is eventually positive. Our results can be shown for that x is eventually negative similarly.

If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, equation (1.1) turns out to be system of first-order differential equations and difference equations

$$\begin{cases} x' = a(t)f(y(t)) \\ y' = -b(t)g(x(t)) \end{cases}$$

see [9],

$$\begin{cases} \Delta x_n = a_n f(y_n) \\ \Delta y_n = -b_n g(x_n) \end{cases}$$

see [10], respectively. Oscillation and nonoscillation criteria for two-dimensional time scale systems have been studied by [1], [5], [8], [11, 12].

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One can easily show that any nonoscillatory solution (x, y) of system (1.1) belongs to one of the following classes:

$$M^+ := \{(x, y) \in M : \; x(t)y(t) > 0 \; \text{ eventually}\}$$
$$M^- := \{(x, y) \in M : \; x(t)y(t) < 0 \; \text{ eventually}\},$$

where M is the set of all nonoscillatory solutions of system (1.1). In this paper, we only focus on the existence and nonexistence of solutions of system (1.1) in M^-.

The set up of this paper is as follows. In Section 1, we give preliminary lemmas that are used in the proofs of our main theorems. In Section 2, we introduce the subclasses that are obtained by using system (1.1) and show the existence of nonoscillatory solutions of system (1.1) by using the Knaster and Schauder fixed point theorems and certain improper integrals. In Section 3, we show the nonexistence of such solutions by relaxing the monotonicity condition on the functions f and g. We finalize the paper by giving some examples and a conclusion.

The following lemma is shown in [1].

Lemma 1.1. *If (x, y) is a nonoscillatory solution of system (1.1), then the component functions x and y are themselves nonoscillatory.*

For convenience, let us set

$$Y(t) = \int_t^\infty a(t)\Delta t \quad \text{and} \quad Z(t) = \int_t^\infty b(t)\Delta t.$$

The following lemma shows the existence and nonexistence of nonoscillatory solutions of system (1.1) by using convergence/divergence of $Y(t)$ and $Z(t)$.

Lemma 1.2. *Let $t_0 \in \mathbb{T}$. Then we have the following:*

(a) *[1, Lemma 2.3] If $Y(t_0) < \infty$ and $Z(t_0) < \infty$, then system (1.1) is nonoscillatory.*

(b) *[1, Lemma 2.2] If $Y(t_0) = \infty$ and $Z(t_0) = \infty$, then system (1.1) is oscillatory.*

(c) *If $Y(t_0) < \infty$ and $Z(t_0) = \infty$, then any nonoscillatory solution (x, y) of system (1.1) belongs to M^-, i.e., $M^+ = \emptyset$.*

(d) *If $Y(t_0) = \infty$ and $Z(t_0) < \infty$, then any nonoscillatory solution (x, y) of system (1.1) belongs to M^+, i.e., $M^- = \emptyset$.*

Proof. Here we only prove part (c) because (d) can be shown similarly. Suppose that $Y(t_0) < \infty$ and $Z(t_0) = \infty$. So assume that there exists a nonoscillatory solution (x, y) of system (1.1) in M^+ such that $xy > 0$ eventually. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_1$. Then by monotonicity of x and g, there exists a number $k > 0$ such that $g(x(t)) \geq k$ for $t \geq t_1$. Integrating the second equation of system (1.1) from t_1 to t gives us

$$y(t) \leq y(t_1) - k \int_{t_1}^{t} b(s)\Delta s.$$

As $t \to \infty$, it follows that $y(t) \to -\infty$. But this contradicts that y is eventually positive. Proof is by contradiction. □

The following two lemmas are related with the first component function of any nonoscillatory solutions of (1.1) when $Y(t_0) < \infty$.

Lemma 1.3. *Let (x, y) be a nonoscillatory solution of system (1.1) and $Y(t_0) < \infty$. Then the component function x has a finite limit.*

Proof. Suppose that $Y(t_0) < \infty$ and (x, y) is a nonoscillatory solution of system (1.1). Then by Lemma 1.1, x and y are themselves nonoscillatory. Without loss of generality, assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. If $(x, y) \in M^-$, then by the first equation of system (1.1), $x^\Delta(t) < 0$ for $t \geq t_1$. Therefore, the limit of x exists. So let us show that the assertion follows if $(x, y) \in M^+$. Suppose $(x, y) \in M^+$. Then from the first equation of system (1.1), we have $x^\Delta(t) > 0$ for $t \geq t_1$. Hence two possibilities might happen: The limit of the component function x exists or blows up. Now let us show that $\lim_{t\to\infty} x(t) = \infty$ cannot happen. Integrating the first equation of system (1.1) from t_1 to t and using the monotonicity of y and f yield

$$x(t) \leq x(t_1) + f(y(t_1)) \int_{t_1}^{t} a(s)\Delta s.$$

Taking the limit as $t \to \infty$, it follows that x has a finite limit. This completes the proof. □

Lemma 1.4. *Let $Y(t_0) < \infty$. If (x, y) is a nonoscillatory solution of system (1.1), then there exist $c, d > 0$ and $t_1 \geq t_0$ such that*

$$c \int_{t}^{\infty} a(s)\Delta s \leq x(t) \leq d$$

or

$$-d \leq x(t) \leq -c \int_{t}^{\infty} a(s)\Delta s$$

for $t \geq t_1$.

Proof. Suppose that $Y(t_0) < \infty$ and (x, y) is a nonoscillatory solution of system (1.1). Without loss of generality, let us assume that x is eventually positive. Then by Lemma 1.3, we have $x(t) \leq d$ for $t \geq t_1$ and for some $d > 0$. If $y(t) > 0$ for $t \geq t_1$, then x is eventually increasing by the first equation of system (1.1). So for large t, the assertion follows. If $y(t) < 0$ for $t \geq t_1$, then integrating the first equation of system (1.1) from t to ∞ and the monotonicity of f and y give

$$\begin{aligned} x(t) &= x(\infty) - \int_{t}^{\infty} a(s)f(y(s))\Delta s \\ &\geq -f(y(t_1)) \int_{t}^{\infty} a(s)\Delta s. \end{aligned}$$

Setting $c = -f(y(t_1)) > 0$ on the last inequality proves the assertion. □

According to Lemma 1.2 (c), we assume $Y(t_0) < \infty$ and $Z(t_0) = \infty$ from now on. Let (x, y) be a nonoscillatory solution of system (1.1) such that the component function x of the solution (x, y) is eventually positive. Then by the second equation of system (1.1), we have $y < 0$ and eventually decreasing. Then for $d < 0$, we have $y \to d$ or $y \to -\infty$. In view of Lemma 1.3, x has a finite limit. So in light of this information, we obtain the following lemma.

Lemma 1.5. *Any nonoscillatory solution of system (1.1) in M^- belongs to one of the following subclasses:*

$$\begin{aligned} M^-_{0,B} &= \left\{(x, y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \quad \lim_{t\to\infty} |y(t)| = d\right\}, \\ M^-_{B,B} &= \left\{(x, y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d\right\}, \\ M^-_{0,\infty} &= \left\{(x, y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \quad \lim_{t\to\infty} |y(t)| = \infty\right\}, \\ M^-_{B,\infty} &= \left\{(x, y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = \infty\right\}, \end{aligned}$$

where $0 < c < \infty$ and $0 < d < \infty$.

2 Existence of Nonoscillatory Solutions in M^-

The following theorems show the existence of nonoscillatory solutions in subclasses of M^- given in Lemma 1.5.

Theorem 2.1. *$M^-_{0,B} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} b(t)g\left(c_1 \int_{t}^{\infty} a(s)\Delta s\right)\Delta t < \infty \tag{2.1}$$

for some $c_1 \neq 0$.

Proof. Suppose that there exists a solution $(x, y) \in M^-_{0,B}$ such that $x(t) > 0$ for $t \geq t_0$, $x(t) \to 0$ and $y(t) \to -d$ as $t \to \infty$, where $d > 0$. By Lemma 1.4, there exists $c > 0$ such that

$$x(t) \geq c \int_{t}^{\infty} a(s)\Delta(s), \quad t \geq t_0. \tag{2.2}$$

By integrating the second equation from t_0 to t, using inequality (2.2) with $c = c_1$ and the monotonicity of g, we have

$$y(t) = y(t_0) - \int_{t_0}^{t} b(s)g(x(s))\Delta s \leq -\int_{t_0}^{t} b(s)g\left(c_1 \int_{s}^{\infty} a(\tau)\Delta\tau\right)\Delta s.$$

So as $t \to \infty$, the assertion follows since y has a finite limit. (For the case $x < 0$ eventually, the proof can be shown similarly with $c_1 < 0$.)

Conversely, suppose that (2.1) holds for some $c_1 > 0$. (For the case $c_1 < 0$ can be shown similarly.) Then there exist $t_1 \geq t_0$ and $d > 0$ such that

$$\int_{t_1}^{\infty} b(t)g\left(c_1 \int_{t}^{\infty} a(s)\Delta s\right)\Delta t < d, \quad t \geq t_1, \tag{2.3}$$

where $c_1 = -f(-3d)$. Let X be the space of all continuous and bounded functions on $[t_1, \infty)_{\mathbb{T}}$ with the norm $\|y\| = \sup\limits_{t \in [t_1,\infty)_{\mathbb{T}}} |y(t)|$. Let Ω be the subset of X such that

$$\Omega := \{y \in X : \quad -3d \leq y(t) \leq -2d, \quad t \geq t_1\}$$

and define an operator $T : \Omega \to X$ such that

$$(Ty)(t) = -3d + \int_{t}^{\infty} b(s)g\left(-\int_{s}^{\infty} a(\tau)f(y(\tau))\Delta\tau\right)\Delta s.$$

It is easy to see that T maps into itself. Indeed, we have

$$-3d \leq (Ty)(t) \leq -3d + \int_{t}^{\infty} b(s)g\left(-\int_{s}^{\infty} a(\tau)f(-3d)\Delta\tau\right)\Delta s \leq -2d$$

by (2.3). Let us show that T is continuous on Ω. Let x_n be a sequence in Ω such that $x_n \to x \in \Omega = \bar{\Omega}$. Then

$$|(Ty_n)(t) - (Ty)(t)| \\ \leq \int_{t_1}^{\infty} b(s)\left|\left[g\left(-\int_{s}^{\infty} a(\tau)f(y_n(\tau))\Delta\tau\right) - g\left(-\int_{s}^{\infty} a(\tau)f(y(\tau))\Delta\tau\right)\right]\right|\Delta s.$$

Then the Lebesque dominated convergence theorem and the continuity of g give $\left\|(Ty_n) - (Ty)\right\| \to 0$ as $n \to \infty$, i.e., T is continuous. Also since

$$0 < -(Ty)^{\Delta}(t) = b(t)g\left(-\int_t^{\infty} a(\tau)f(y(\tau))\Delta\tau\right) < \infty,$$

it follows that $T(\Omega)$ is relatively compact. Then by the Schauder Fixed point theorem, there exists $\bar{y} \in \Omega$ such that $\bar{y} = T\bar{y}$. So as $t \to \infty$, we have $\bar{y}(t) \to -3d < 0$. Setting

$$\bar{x}(t) = -\int_t^{\infty} a(\tau)f(\bar{y}(\tau))\Delta\tau > 0$$

gives that $\bar{x}(t) \to 0$ ans $t \to \infty$, i.e., $M^-_{0,B} \neq \emptyset$. □

Theorem 2.2. *$M^-_{B,B} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} b(t)g\left(d_1 - f(c_1)\int_t^{\infty} a(s)\Delta s\right)\Delta t < \infty$$

for some $c_1 < 0$ and $d_1 > 0$. (Or $c_1 > 0$ and $d_1 < 0$.)

Proof. Suppose that there exists a nonoscillatory solution $(x, y) \in M^-_{B,B}$ such that $x > 0$ eventually, $\lim_{t\to\infty} x(t_1) = c_2 > 0$ and $\lim_{t\to\infty} y(t) = d_2 < 0$. Since x and y have finite limits, there exist $t_1 \geq t_0$, $c_3 > 0$ and $d_3 < 0$ such that $c_2 \leq x(t) \leq c_3$ and $d_2 \leq y(t) \leq d_3$ for $t \geq t_1$. Integrating the first equation from t to ∞ gives

$$x(t) = c_2 - \int_t^{\infty} a(s)f(y(s))\Delta s \geq c_2 - f(d_3)\int_t^{\infty} a(s)\Delta s. \tag{2.4}$$

By integrating the second equation from t_1 to t and using (2.4,) we get

$$y(t) \leq -\int_{t_1}^{t} b(s)g(x(s))\Delta s \leq -\int_{t_1}^{t} b(s)g\left(c_2 - f(d_3)\int_s^{\infty} a(\tau)\Delta\tau\right)\Delta s.$$

By setting $c_2 = d_1 > 0$ and $d_3 = c_1 < 0$ and taking the limit of the last inequality as $t \to \infty$, the assertion follows. (The case $x < 0$ eventually can be done similarly with $c_1 > 0$ and $d_1 < 0$.)

Conversely, choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} b(t)g\left(d_1 - f(c_1)\int_t^{\infty} a(s)\Delta s\right)\Delta t < \frac{-c_1}{2}, \tag{2.5}$$

where $c_1 < 0$ and $d_1 > 0$. (The case $c_1 > 0$ and $d_1 < 0$ can be done similarly.) Let X be the set of all all bounded and continuous functions endowed with the norm $\|y\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |y(t)|$. Clearly $(X, \|\cdots\|)$ is a Banach space, see [4]. Define a subset Ω of X such that

$$\Omega =: \left\{y \in X : \quad c_1 \leq y(t) \leq \frac{c_1}{2}, \quad t \geq t_1\right\}.$$

Define an operator $F : \Omega \to X$ such that

$$(Fy)(t) = c_1 + \int_t^{\infty} b(s)g\left(d_1 - \int_s^{\infty} a(\tau)f(y(\tau))\Delta\tau\right)\Delta s.$$

First, we show that $F:\Omega\to\Omega$.

$$c_1 \le (Fy)(t) \le c_1 + \int_t^\infty b(s)g\left(d_1 - \int_s^\infty a(\tau)f(c_1)\Delta\tau\right)\Delta s \le \frac{c_1}{2}.$$

Second, we show that F is continuous on Ω. Let y_n be a sequence in Ω such that $y_n \to y \in \Omega = \bar{\Omega}$. Then

$$\|Fy_n - Fy\| \le \int_{t_1}^\infty b(s)\left(\left|g\left(d - \int_s^\infty a(\tau)f(y_n(\tau))\Delta\tau\right)\right| - \left|g\left(d - \int_s^\infty a(\tau)f(y(\tau))\Delta\tau\right)\right|\right)\Delta s.$$

By the Lebesque dominated convergence theorem and the continuity of f and g, it follows that F is continuous.

Third, we show that $F(\Omega)$ is relatively compact. Since $Y(t_0) < \infty$, we have

$$0 < -(Fy)^\Delta(t) = b(t)g\left(d_1 - \int_t^\infty a(\tau)f(y(\tau))\Delta\tau\right) < \infty$$

and therefore F is equibounded and equicontinuous, i.e., relatively compact. So by the Schauder fixed point theorem, there exists $\bar{y} \in X$ such that

$$\bar{y}(t) = F\bar{y}(t) = c_1 + \int_t^\infty b(s)g\left(d_1 - \int_s^\infty a(\tau)f(\bar{y}(\tau))\Delta\tau\right)\Delta s.$$

Setting $\bar{x}(t) = d_1 - \int_t^\infty a(\tau)f(y(\tau))\Delta\tau$ and taking limit as $t\to\infty$, we have that there exists a nonoscillatory solution in M^- such that $\bar{x}(t) \to d_1 > 0$ and $\bar{y}(t) \to c_1 < 0$, i.e., $M^-_{B,B} \neq \emptyset$. □

Theorem 2.3. *$M^-_{B,\infty} \neq \emptyset$ if and only if*

$$\int_{t_0}^\infty a(s)f\left(g(c_1)\int_{t_0}^s b(\tau)\Delta\tau\right)\Delta s < \infty$$

for some $c_1 \neq 0$, where f is an odd function.

Proof. Suppose that there exists a nonoscillatory solution $(x,y) \in M^-_{B,\infty}$ such that $x > 0$ eventually, $x(t) \to c_2$ and $y(t) \to -\infty$ as $t \to \infty$, where $0 < c_2 < \infty$. Because of the monotonicity of x and the fact that x has a finite limit, there exist $t_1 \ge t_0$ and $c_3 > 0$ such that

$$c_2 \le x(t) \le c_3 \quad \text{for} \quad t \ge t_1. \tag{2.6}$$

Integrating the first equation from t_1 to t gives us

$$c_2 \le x(t) = x(t_1) + \int_{t_1}^t a(s)f(y(s))\Delta s \le c_3, \quad t \ge t_1.$$

So by taking the limit as $t \to \infty$, we have

$$\int_{t_1}^\infty a(s)|f(y(s))|\Delta s < \infty. \tag{2.7}$$

The monotonicity of g, (2.6) and integrating the second equation from t_1 to t yield

$$y(t) \le y(t_1) - g(c_2)\int_{t_1}^t b(s)\Delta s \le -g(c_2)\int_{t_1}^t b(s)\Delta s.$$

Since $f(-u) = -f(u)$ for $u \neq 0$ and by the monotonicity of f, we have

$$|f(y(t))| \geq f\left(g(c_2)\int_{t_1}^{t} b(s)\Delta s\right), \quad t \geq t_1. \tag{2.8}$$

By (2.7) and (2.8), we have

$$\int_{t_1}^{t} a(s)|f(y(s))|\Delta s \geq \int_{t_1}^{t} a(s)f\left(g(c_2)\int_{t_1}^{s} b(\tau)\Delta\tau\right)\Delta s.$$

As $t \to \infty$, the proof is finished. (The case x<0 eventually can be proved similarly with $c_1 < 0$.) □

Conversely, suppose that $\int_{t_0}^{\infty} a(s)f\left(g(c_1)\int_{t_0}^{s} b(\tau)\Delta\tau\right)\Delta s < \infty$ for some $c_1 \neq 0$. Without loss of generality, assume that $c_1 > 0$. (The case $c_1 < 0$ can be done similarly.) Then we can choose $t_1 \geq t_0$ and $d > 0$ such that

$$\int_{t_1}^{\infty} a(s)f\left(g(c_1)\int_{t_1}^{s} b(\tau)\Delta\tau\right)\Delta s < d, \quad t \geq t_1, \tag{2.9}$$

where $c_1 = 2d > 0$. Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \{x \in X : \quad d \leq x(t) \leq 2d, \quad t \geq t_1\}. \tag{2.10}$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = d + \int_{t}^{\infty} a(s)f\left(\int_{t_1}^{s} b(\tau)g(x(\tau))\Delta\tau\right)\Delta s, \quad t \geq t_1. \tag{2.11}$$

First, we need to show that $F : \Omega \to \Omega$ is an increasing mapping into itself. It is obvious that it is an increasing mapping and since

$$d \leq (Fx)(t) = d + \int_{t}^{\infty} a(s)f\left(\int_{t_1}^{s} b(\tau)g(x(\tau))\Delta\tau\right)\Delta s \leq 2d$$

by (2.9), it follows that $F : \Omega \to \Omega$. Then by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = d + \int_{t}^{\infty} a(s)f\left(\int_{t_1}^{s} b(\tau)g(\bar{x}(\tau))\Delta\tau\right)\Delta s, \quad t \geq t_1. \tag{2.12}$$

By taking the derivative of (2.12) and the fact that f is an odd function, we have

$$\bar{x}^{\Delta}(t) = a(t)f\left(-\int_{t_1}^{t} b(\tau)g(\bar{x}(\tau))\Delta\tau\right), \quad t \geq t_1.$$

Setting $\bar{y} = -\int_{t_1}^{t} b(\tau)g(\bar{x}(\tau))\Delta\tau$ and using the monotonicity of g give

$$\bar{y}(t) \leq -g(d)\int_{t_1}^{t} b(\tau)\Delta\tau, \quad t \geq t_1.$$

So we have that $\bar{x}(t) > 0$ and $\bar{y}(t) < 0$ for $t \geq t_1$, and $\bar{x}(t) \to d$ and $\bar{y}(t) \to -\infty$ as $t \to \infty$. This completes the proof.

Theorem 2.4. *If*

$$\int_{t_0}^{\infty} a(t) f\left(\int_t^{\infty} b(s) g(c_1) \Delta s\right) \Delta t < \infty$$

and

$$\int_{t_0}^{\infty} b(t) g\left(d_1 \int_t^{\infty} a(s) \Delta s\right) \Delta t = \infty \quad (-\infty)$$

for some $c_1 > 0$ *and any* $d_1 > 0$ ($c_1 < 0$ *and* $d_1 < 0$), *where* f *is an odd function, then* $M_{0,\infty}^- \neq \emptyset$.

Proof. Choose $t_1 \geq t_0$ and $c_1 > 0$ such that

$$\int_{t_1}^{\infty} a(t) f\left(g(c_1) \int_t^{\infty} b(s) \Delta s\right) \Delta t < \frac{c_1}{2}, \quad t \geq t_1. \tag{2.13}$$

Let X be the partially ordered Banach space of all real-valued continuous functions endowed with the norm $\|x\| = \sup_{t \in [t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \{x \in X : \quad f(1) \int_t^{\infty} a(s) \Delta s \leq x(t) \leq \frac{c_1}{2}, \quad t \geq t_1\}.$$

It is clear that $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = \int_t^{\infty} a(s) f\left(\int_{t_1}^{s} b(\tau) g(x(\tau)) \Delta \tau\right) \Delta s.$$

It is clear that F is an increasing mapping. We also need to show that $F : \Omega \to \Omega$. By (2.13), the monotonicity of g and the fact that $x \in \Omega$, we have

$$(Fx)(t) \leq \int_t^{\infty} a(s) f\left(g(c_1) \int_{t_1}^{s} b(\tau) \Delta \tau\right) \Delta s \leq \frac{c_1}{2}.$$

Also since

$$\int_{t_0}^{\infty} b(t) g\left(d_1 \int_t^{\infty} a(s) \Delta s\right) \Delta t = \infty,$$

we can choose $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} b(s) g\left(d_1 \int_s^{\infty} a(\tau) \Delta \tau\right) \Delta s > 1$$

for $t \geq t_2$ and any $d_1 > 0$. So by setting $f(1) = d_1$, we have

$$(Fx)(t) \geq \int_t^{\infty} a(s) f\left(\int_{t_1}^{s} b(\tau) g\left(f(1) \int_{\tau}^{\infty} a(\lambda) \Delta \lambda\right) \Delta \tau\right) \Delta s \geq f(1) \int_t^{\infty} a(s) \Delta s.$$

Then by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. Setting

$$\bar{y}(t) = -\int_{t_1}^{t} b(\tau) g(\bar{x}(\tau)) \Delta \tau,$$

using the fact that $\bar{x} \in \Omega$ and taking the limit of $\bar{x}$ and $\bar{y}$ as $t \to \infty$, the proof is complete. (The case $c_1 < 0$ and $d_1 < 0$ can be shown similarly.)

□

3 Nonexistence of Nonoscillatory Solutions in M^-

In the previous section, we used the monotonicity of the functions f and g in order to show the existence of nonoscillatory solutions of system (1.1). Nonexistence of such solutions in $M^-_{0,B}$, $M^-_{B,B}$, and $M^-_{B,\infty}$ directly follows from Theorems 2.1 - 2.3. In this section, we relax this condition by assuming that there exist positive constants F and G such that

$$\frac{f(u)}{u} \geq F \quad \text{and} \; \frac{g(u)}{u} \geq G \quad \text{for} \quad u \neq 0 \tag{3.1}$$

in order to get the emptiness of those subclasses. The following theorems show the nonexistence of such solutions in the subclasses of M^- given in Lemma 1.5.

Theorem 3.1. *Suppose that (3.1) holds. If*

$$\int_{t_1}^{\infty} a(s) \left(\int_{t_1}^{s} b(\tau) \left(\int_{\tau}^{\infty} a(\lambda)\Delta\lambda \right) \Delta\tau \right) \Delta s = \infty, \tag{3.2}$$

then $M^-_{0,\infty} = \emptyset$.

Proof. Assume that there exists a solution $(x, y) \in M^-$ such that $x > 0$ eventually, $x \to 0$ and $y \to -\infty$ as $t \to \infty$. By Lemma 1.4, there exist $c_1 > 0$ and $t_1 \geq t_0$ such that

$$c_1 \int_{t}^{\infty} a(s)\Delta s \leq x(t), \quad t \geq t_1. \tag{3.3}$$

By integrating the second equation from t_1 to t, and using (3.1) and (3.3), there exist $t_2 \geq t_1$ and $G > 0$ such that

$$y(t) \leq -c_1 G \int_{t_1}^{t} b(s) \left(\int_{s}^{\infty} a(\tau)\Delta\tau \right) \Delta s, \quad t \geq t_2. \tag{3.4}$$

By integrating the first equation from t_2 to t, and using (3.4) and (3.1), there exist $t_3 \geq t_2$ and $F > 0$ such that

$$x(t_2) \geq c_1 FG \int_{t_2}^{t} a(s) \left(\int_{t_1}^{s} b(\tau) \left(\int_{\tau}^{\infty} a(\lambda)\Delta\lambda \right) \Delta\tau \right) \Delta s, \quad t \geq t_3. \tag{3.5}$$

As $t \to \infty$, it contradicts to (3.2). So the assertion follows. Proof is by contradiction. □

Theorem 3.2. *Suppose that (3.1) holds. If*

$$\int_{t_0}^{\infty} b(t) \left(\int_{t}^{\infty} a(s)\Delta s \right) \Delta t = \infty, \tag{3.6}$$

then $M^-_{0,B} = \emptyset$ *and* $M^-_{B,B} = \emptyset$.

Proof. We only show the emptiness of $M^-_{0,B}$ since $M^-_{B,B} = \emptyset$ can be shown similarly. So assume that there exists a nonoscillatory solution (x, y) in $M^-_{0,B}$ such that $x > 0$ eventually, $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = d_1 < 0$. By Lemma 1.4, we have that there exist $c_2 > 0$ and $t_1 \geq t_0$ such that

$$c_2 \int_{t}^{\infty} a(s)\Delta s \leq x(t), \quad t \geq t_1. \tag{3.7}$$

Integrating the first equation from t to ∞ gives

$$x(t) = -\int_t^\infty a(s)f(y(s))\Delta s, \quad t \geq t_1. \tag{3.8}$$

By integrating the second equation from t_1 to t, and by using (3.1) and (3.8), we have that there exists $G > 0$ such that

$$y(t) \leq -Gc_2 \int_{t_1}^t b(s) \left(\int_s^\infty a(\tau)\Delta\tau \right) \Delta s.$$

So as $t \to \infty$, it contradicts to (3.6). Proof is by contradiction. □

Theorem 3.3. *Suppose that (3.1) holds and f is an odd function. If*

$$\int_{t_0}^\infty a(s) \left(\int_{t_0}^s b(\tau)\Delta\tau \right) \Delta s = \infty, \tag{3.9}$$

then $M_{B,\infty}^- = \emptyset$.

Proof. Suppose (3.9) holds and that there exists a nonoscillatory (x, y) solution of (1.1) in $M_{B,\infty}^-$ such that $x > 0$ eventually, $x(t) \to c_1 > 0$ and $y(t) \to -\infty$ as $t \to \infty$. Since x has a finite limit, there exist $t_1 \geq t_0$ such that $c_1 \leq x(t)$ for $t \geq t_1$. Integrating the first equation from t_1 to t gives

$$x(t) = x(t_1) + \int_{t_1}^t a(s)f(y(s))\Delta s. \tag{3.10}$$

By taking the limit of (3.10) as $t \to \infty$, we have

$$\int_{t_1}^\infty a(s)|f(y(s))|\Delta s < \infty. \tag{3.11}$$

By integrating the second equation from t_1 to t, using (3.1) and the fact that $x(t) \geq c_1$ for $t \geq t_1$, we have that there exist $t_2 \geq t_1$ and $G > 0$ such that

$$y(t) = y(t_1) - \int_{t_1}^t b(s)g(x(s))\Delta s \leq -Gc_1 \int_{t_1}^t b(s)\Delta s, \quad t \geq t_2. \tag{3.12}$$

By (3.12) and the fact that f is an odd function, there exist $t_3 \geq t_2$ and $F > 0$ such that

$$|f(y(t))| \geq f\left(Gc_1 \int_{t_1}^t b(s)\Delta s \right) \geq FGc_1 \int_{t_1}^t b(s)\Delta s, \quad t \geq t_3. \tag{3.13}$$

Multiplying (3.13) by $a(t)$ and integrating the resulting inequality from t_3 to t give us

$$\int_{t_3}^t a(s)|f(y(s))|\Delta s \geq FGc_1 \int_{t_3}^t a(s) \left(\int_{t_3}^s b(\tau)\Delta\tau \right) \Delta s.$$

By taking the limit of the last inequality as $t \to \infty$ and by (3.11), we obtain a contradiction. So the assertion follows. □

4 Examples

In this section, we give some examples in order to highlight our main results.

Example 4.1. *Let $\mathbb{T} = q^{\mathbb{N}_0}$, $t_0 = 1$, $q > 1$, $a(t) = \frac{t^{\frac{1}{3}}}{(t+1)(tq+1)(2t-1)^{\frac{1}{3}}}$, $b(t) = \frac{(t+1)^{\frac{5}{3}}}{qt^2}$, $f(u) = u^{\frac{1}{3}}$, $c_1 = 1$, $g(u) = u^{\frac{5}{3}}$, $t = q^n$ and $s = tq^m$, where $n, m \in \mathbb{N}_0$ in system (1.1). First we need to show $Y(1) < \infty$ and $Z(1) = \infty$.*

One can easily show that

$$\int_1^T a(s)\Delta s = (q-1) \sum_{s\in[1,T)_{q^{\mathbb{N}_0}}} \frac{s^{\frac{4}{3}}}{(s+1)(sq+1)(2s-1)^{\frac{1}{3}}} \le (q-1) \sum_{s\in[1,T)_{q^{\mathbb{N}_0}}} \frac{1}{t^{\frac{2}{3}}}. \tag{4.1}$$

So as $s \to \infty$, we have that

$$Y(1) \le \sum_{n=0}^{\infty} \left(\frac{1}{q^{\frac{2}{3}}}\right)^n < \infty.$$

One can also show

$$\int_1^T b(s)\Delta s = \sum_{s\in[1,T)_{q^{\mathbb{N}_0}}} \frac{(s+1)^{\frac{5}{3}}}{qs^2}(q-1)s \ge \frac{q-1}{q} \sum_{s\in[1,T)_{q^{\mathbb{N}_0}}} s^{\frac{2}{3}}.$$

So as $T \to \infty$, we have

$$Z(1) = \int_1^\infty b(s)\Delta s \ge \frac{q-1}{q} \sum_{m=0}^{\infty} (q^{\frac{2}{3}})^m = \infty.$$

Now let us show that (2.1) holds. First we have

$$\int_t^T a(s)\Delta s \le \sum_{s\in[t,T)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}}$$

by (4.1). So taking the limit as $T \to \infty$, we have

$$\int_t^\infty a(s)\Delta s \le \sum_{s\in[t,\infty)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}} = \frac{q^{\frac{2}{3}}}{(q^{\frac{2}{3}}-1)t^{\frac{2}{3}}}.$$

Therefore,

$$\int_1^T b(t) g\left(c_1 \int_t^\infty a(s)\Delta s\right) \Delta t \le \alpha \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} \frac{(t+1)^{\frac{5}{3}}}{t^{\frac{19}{10}}},$$

where $\alpha = \frac{(q-1)q^{\frac{1}{9}}}{(q^{\frac{2}{3}}-1)^{\frac{5}{3}}}$. So as $T \to \infty$, we have that (2.1) holds by using the ratio test. One can also show that $\left(\frac{1}{t+1}, -2+\frac{1}{t}\right)$ is a solution of

$$\begin{cases} \Delta_q x(t) = \frac{t^{\frac{1}{3}}}{(t+1)(tq+1)(2t-1)^{\frac{1}{3}}} \left(-2+\frac{1}{t}\right)^{\frac{1}{3}} \\ \Delta_q y(t) = -\frac{(t+1)^{\frac{5}{3}}}{qt^2} \left(\frac{1}{t+1}\right)^{\frac{5}{3}} \end{cases}$$

such that $x(t) \to 0$ and $y(t) \to -2$, i.e., $M_{0,B}^- \neq \emptyset$ by Theorem 2.1.

Example 4.2. *Let $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $a_n = 2^{\frac{-6n}{5}-1}$, $b_n = \frac{4^n}{1+2^n}$, $c_1 = 1$, $f(u) = u^{\frac{1}{5}}$ and $g(u) = u$. It is clear that $Y(0) < \infty$ and $Z(0) = \infty$. Also note that*

$$\int_0^T a(s)f\left(\int_0^s b(\tau)g(c_1)\Delta\tau\right)\Delta s = \sum_{s=0}^{T-1} 2^{\frac{-6s}{5}-1}\left(\sum_{\tau=0}^{s-1}\frac{4^\tau}{1+2^\tau}\right)^{\frac{1}{5}} \leq \frac{1}{2}\sum_{s=0}^{T-1}\left(\frac{1}{2}\right)^s.$$

So as $T \to \infty$, it follows that

$$\int_0^\infty a(s)f\left(\int_0^s b(\tau)g(c_1)\Delta\tau\right)\Delta s < \infty$$

by the geometric series. It can also be shown that $(x_n, y_n) = (1 + 2^{-n}, -2^n)$ is a nonosicllatory solution of

$$\begin{cases} \Delta x_n = 2^{\frac{-6n}{5}-1}(y_n)^{\frac{1}{5}} \\ \Delta y_n = -\frac{4^n}{1+2^n}(x_n) \end{cases}$$

such that $x_n \to 1$ and $y_n \to -\infty$ as $n \to \infty$, i.e., $M^-_{B,\infty} \neq \emptyset$ by Theorem 2.3 (or Theorem 10 in [10]).

5 Conclusions

In this paper, we consider the case $Y(t_0) < \infty$ and $Z(t_0) = \infty$ in order to show the existence and nonexistence of nonoscillatory solutions in M^-. When we have the case

$$Y(t_0) = \infty \quad \text{and} \quad Z(t_0) < \infty, \tag{5.1}$$

we know from Lemma 1.2(d) that all nonoscillatory solutions belong to M^+. So as a future work, we will consider the case (5.1) in order to show the existence and nonexistence of nonoscillatory solutions in M^+.

Another open problem is to extend our main results to the delay equation

$$\begin{cases} x^\Delta(t) = a(t)f(y(t)) \\ y^\Delta(t) = -b(t)g(x(\tau(t))), \end{cases} \tag{5.2}$$

where $\tau : \mathbb{T} \to \mathbb{T}$ is an increasing function such that $\tau(t) < t$ and $\tau(t) \to \infty$ as $t \to \infty$. Even though the system

$$\begin{cases} x^\Delta(t) = a(t)f(y(t)) \\ y^\Delta(t) = -b(t)g(x(t-\tau)), \end{cases} \tag{5.3}$$

where $\tau > 0$, is considered in [11], it is not valid for all time scales, such as $\mathbb{T} = q_0^{\mathbb{N}}$, where $q > 1$.

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