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*Research Article*

# Asymptotic Behavior of Certain Integrodifferential Equations

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This paper deals with asymptotic behavior of nonoscillatory solutions of certain forced integrodifferential equations of the form: $(a(t)x'(t))' = e(t) + \int_c^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) ds$, $c > 1$, $0 < \alpha < 1$. From the obtained results, we derive a technique which can be applied to some related integrodifferential as well as integral equations.

## 1. Introduction

In this paper, we consider the integrodifferential equation

$$\left(a(t)\,x'(t)\right)' = e(t)$$
$$+ \int_c^t (t-s)^{\alpha-1} k(t,s)\, f(s, x(s))\, ds, \qquad (1)$$
$$c > 1, \; 0 < \alpha < 1.$$

In the sequel, we assume that

(i) $a, e \in C([c, \infty), \mathbb{R}^+)$;

(ii) $k \in C([c, \infty) \times [c, \infty), \mathbb{R})$ and also there exists $b \in C([c, \infty), \mathbb{R}^+)$ such that $|k(t,s)| \le b(t)$ for all $t \ge s \ge c$;

(iii) $f \in C([c, \infty) \times \mathbb{R}, \mathbb{R})$ and also there exist $h \in C([c, \infty), \mathbb{R}^+)$ and real numbers $\lambda$, $0 < \lambda \le 1$, and $\gamma$ such that

$$0 \le xf(t,x) \le t^{\gamma-1} h(t)\, |x|^{\lambda+1} \qquad (2)$$

for $x \ne 0$ and $t \ge c$.

We only consider solutions of (1) which are continuable and nontrivial in any neighborhood of $\infty$. Such a solution is said to be *oscillatory* if there exists a sequence $\{t_n\} \subset [c, \infty)$, $t_n \to \infty$, such that $x(t_n) = 0$, and it is *nonoscillatory* otherwise.

In the last few decades, integral, integrodifferential, and fractional differential equations have gained considerable attention due to their applications in many engineering and scientific disciplines as the mathematical models for systems and processes in fields such as physics, mechanics, chemistry, aerodynamics, and the electrodynamics of complex media. For more details one can refer to [1–8].

Oscillation and asymptotic results for integral and integrodifferential equations are scarce; some results can be found in [5, 9–13]. It seems that there are no such results for integral equations of type (1). The main objective of this paper is to establish some new criteria on the oscillatory and the asymptotic behavior of all solutions of (1). From the obtained results, we derive a technique which can be applied to some related integrodifferential as well as integral equations.

## 2. Main Results

To obtain our main results of this paper, we need the following two lemmas.

**Lemma 1** (see [5, 7]). *Let $\beta$, $\gamma$, and $p$ be positive constants such that $p(\beta - 1) + 1 > 0$ and $p(\gamma - 1) + 1 > 0$. Then*

$$\int_0^t (t-s)^{p(\beta-1)}\, s^{p(\gamma-1)}\, ds = t^\theta B, \quad t \ge 0, \qquad (3)$$

*where $B := B[p(\gamma-1)+1, p(\beta-1)+1]$, $B[\zeta, \eta] = \int_0^1 s^{\zeta-1}(1-s)^{\eta-1} ds$, $\zeta, \eta > 0$, and $\theta = p(\beta + \gamma - 2) + 1$.*

**Lemma 2** (see [14]). *If $X$ and $Y$ are nonnegative, then*

$$X^\lambda - (1-\lambda)Y^\lambda - \lambda XY^{\lambda-1} \leq 0, \quad 0 < \lambda < 1, \quad (4)$$

*where equality holds if and only if $X = Y$.*

In what follows, we let

$$g_\pm(t) = e(t) \pm (1-\lambda)\lambda^{\lambda/(1-\lambda)}b(t)$$
$$\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\, ds \quad (5)$$

and $0 < \lambda < 1$, $t \geq t_1$ for some $t_1 \geq c$, where $m \in C([c,\infty),\mathbb{R}^+)$.

Now we give sufficient conditions under which any solution $x$ of (1) satisfies $|x(t)| = O(t^2)$ as $t \to \infty$.

**Theorem 3.** *Let $0 < \lambda < 1$ and conditions (i)–(iii) hold and suppose that $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\gamma = 2 - \alpha - 1/p$, $p(\alpha-1)+1 > 0$, $p(\gamma-1)+1 > 0$, and*

$$\frac{t}{a(t)} \text{ and } b(t) \text{ are bounded on } [c,\infty), \quad (6)$$

$$\int_{t_1}^\infty \frac{s}{a(s)}\, ds < \infty, \quad (7)$$

$$\int_{t_1}^\infty \left(s^2 m(s)\right)^q ds < \infty. \quad (8)$$

*If*

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_1}^t \frac{1}{a(u)} \int_{t_0}^u g_-(s)\, ds\, du < \infty,$$
$$\liminf_{t\to\infty} \frac{1}{t^2} \int_{t_1}^t \frac{1}{a(u)} \int_{t_0}^u g_+(s)\, ds\, du > -\infty \quad (9)$$

*for any $t_1 \geq c$, then every nonoscillatory solution $x(t)$ of (1) satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t^2} < \infty. \quad (10)$$

*Proof.* Let $x$ be a nonoscillatory solution of (1). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (i)–(iii) we may then write

$$\left(a(t) x'(t)\right)' \leq e(t) + b(t) \int_c^t (t-s)^{\alpha-1} f(s, x(s))\, ds \quad (11)$$

and so

$$\left(a(t) x'(t)\right)' \leq e(t) + b(t) \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds$$
$$+ b(t)$$
$$\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \left[h(s) x^\lambda(s) - m(s) x(s)\right] ds \quad (12)$$
$$+ b(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds.$$

Applying (4) of Lemma 2 to $h(s)x^\lambda(s) - m(s)x(s)$ with $X = h^{1/\lambda}x$ and $Y = ((1/\lambda)mh^{-1/\lambda})^{1/(\lambda-1)}$ we have

$$h(s) x^\lambda(s) - m(s) x(s)$$
$$\leq (1-\lambda)\lambda^{\lambda/(1-\lambda)} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s), \quad (13)$$

and hence we obtain

$$\left(a(t) x'(t)\right)' \leq e(t) + b(t) \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds$$
$$+ (1-\lambda)\lambda^{\lambda/(1-\lambda)} b(t)$$
$$\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\, ds + b(t) \quad (14)$$
$$\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds$$

or

$$\left(a(t) x'(t)\right)' \leq b(t) \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)|\, ds + g_+(t)$$
$$+ b(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds$$
$$\leq C_1 + g_+(t) \quad (15)$$
$$+ k_1 \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds,$$

where $C_1$ and $k_1$ are the upper bounds of the functions $b(t)\int_a^{t_1}(t_1-s)^{\alpha-1}|F(s)|ds$ and $b(t)$, respectively. Integrating inequality (15) from $t_1$ to $t$ we have

$$x'(t) \leq \frac{a(t_1) x'(t_1)}{a(t)} + \frac{C_1(t-t_1)}{a(t)}$$
$$+ \frac{1}{a(t)} \int_{t_1}^t g_+(s)\, ds \quad (16)$$
$$+ \frac{k_1}{a(t)} \int_{t_1}^t \int_{t_1}^u (u-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds\, du.$$

Interchanging the order of integration in the last integral, we have

$$x'(t) \leq \frac{a(t_1) x'(t_1)}{a(t)} + \frac{C_1(t-t_1)}{a(t)}$$
$$+ \frac{1}{a(t)} \int_{t_1}^t g_+(s)\, ds \quad (17)$$
$$+ k_2 \int_{t_1}^t (t-s)^\alpha s^{\gamma-1} m(s) x(s)\, ds,$$

where $k_2$ is the upper bound of the function $k_1/\alpha a(t)$. Integrating (17) from $t_1$ to $t$ and interchanging the order of integration in the last integral we find

$$x(t) \leq x(t_1) + a(t_1) x'(t_1) \int_{t_1}^{t} \frac{1}{a(s)} ds$$
$$+ \int_{t_1}^{t} \frac{C_1(s - t_1)}{a(s)} ds + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s) ds\, du \quad (18)$$
$$+ \frac{k_2}{\alpha + 1} \int_{t_1}^{t} (t - s)^{\alpha+1} m(s) x(s) ds.$$

Now, one can easily see that

$$x(t) \leq x(t_1) + a(t_1) x'(t_1) \int_{t_1}^{t} \frac{1}{a(s)} ds$$
$$+ \int_{t_1}^{t} \frac{C_1(s - t_1)}{a(s)} ds + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s) ds\, du \quad (19)$$
$$+ \frac{k_2}{\alpha + 1} t^2 \int_{t_1}^{t} (t - s)^{\alpha-1} m(s) x(s) ds$$

or

$$z(t) := \frac{x(t)}{t^2} \leq 1 + C + k \int_{t_1}^{t} (t - s)^{\alpha-1} m(s) x(s) ds, \quad (20)$$

where $C$ is the upper bound of the function

$$\frac{1}{t^2} \left[ x(t_1) + a(t_1) x'(t_1) \int_{t_1}^{t} \frac{1}{a(s)} ds \right.$$
$$\left. + \int_{t_1}^{t} \frac{C_1(s - t_1)}{a(s)} ds + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s) ds\, du \right] \quad (21)$$

and $k = k_2/(\alpha+1)$. Applying Holder's inequality and Lemma 1 we obtain

$$\int_{t_1}^{t} (t - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds$$
$$\leq \left( \int_{t_1}^{t} (t - s)^{p(\alpha-1)} s^{p(\gamma-1)} ds \right)^{1/p}$$
$$\cdot \left( \int_{t_1}^{t} m^q(s) x^q(s) ds \right)^{1/q}$$
$$\leq \left( \int_{0}^{t} (t - s)^{p(\alpha-1)} s^{p(\gamma-1)} ds \right)^{1/p} \quad (22)$$
$$\cdot \left( \int_{t_1}^{t} m^q(s) x^q(s) ds \right)^{1/q} \leq \left( Bt^\theta \right)^{1/p}$$
$$\cdot \left( \int_{t_1}^{t} m^q(s) x^q(s) ds \right)^{1/q},$$

where $B = B[p(\gamma-1)+1, p(\alpha-1)+1]$, and $\theta = p(\alpha+\gamma-2)+1 = 0$ and so

$$\int_{t_1}^{t} (t - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds$$
$$\leq B^{1/p} \left( \int_{t_1}^{t} m^q(s) x^q(s) ds \right)^{1/q}. \quad (23)$$

Thus, inequality (20) becomes

$$z(t) := \frac{x(t)}{t^2} \leq C + kB^{1/p} \left( \int_{t_1}^{t} m^q(s) x^q(s) ds \right)^{1/q}. \quad (24)$$

Using (24) and the elementary inequality

$$(x + y)^q \leq 2^{q-1}(x^q + y^q), \quad x, y \geq 0, \ q > 1, \quad (25)$$

we obtain from (24)

$$z^q(t)$$
$$\leq 2^{q-1}\left( (1 + C)^q + k^q B^{q/p} \int_{t_1}^{t} s^{2q} m^q(s) z^q(s) ds \right). \quad (26)$$

If we denote $u(t) = z^q(t)$, that is, $z(t) = u^{1/q}(t)$, $P = 2^{q-1}(1 + C)^q$, and $Q = 2^{q-1} k^q B^{q/p}$, then

$$u(t) \leq P + Q \int_{t_1}^{t} s^{2q} m^q(s) u(s) ds, \quad t \geq t_1 \geq c. \quad (27)$$

The conclusion follows from Gronwall's inequality and we conclude that

$$\limsup_{t \to \infty} \frac{x(t)}{t^2} < \infty. \quad (28)$$

If $x$ is eventually negative, we can set $y = -x$ to see that $y$ satisfies (1) with $e(t)$ being replaced by $-e(t)$ and $f(t, x)$ by $-f(t, -y)$. It follows in a similar manner that

$$\limsup_{t \to \infty} \frac{-x(t)}{t^2} < \infty. \quad (29)$$

From (28) and (29) we get (10). This completes the proof. $\square$

Next, by employing Theorem 3 we present the following oscillation result for (1).

**Theorem 4.** *Let $0 < \lambda < 1$ and conditions (i)–(iii), (6)–(9) hold and suppose that $p > 1$, $q = p/(p - 1)$, $\alpha > 0$, $\gamma = 2 - \alpha - 1/p$, $p(\alpha - 1) + 1 > 0$, and $p(\gamma - 1) + 1 > 0$. If for every $M$, $0 < M < 1$,*

$$\limsup_{t \to \infty} \left[ Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_-(s) ds\, du \right] = \infty,$$
$$\liminf_{t \to \infty} \left[ Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s) ds\, du \right] = -\infty \quad (30)$$

*for all $t_1 \geq c$, then (1) is oscillatory.*

*Proof.* Let $x$ be a nonoscillatory solution of (1), say $x(t) > 0$, for $t \geq t_1$ for some $t_1 \geq 0$. The proof when $x$ is eventually negative is similar. Proceeding as in the proof of Theorem 3 we arrive at (19). Therefore,

$$x(t) \leq x(t_1) + a(t_1) x'(t_1) \int_{t_1}^{\infty} \frac{1}{a(s)} ds$$
$$+ \int_{t_1}^{\infty} \frac{C_1(s - t_1)}{a(s)} ds$$
$$+ \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s) \, ds \, du \tag{31}$$
$$+ kt^2 \left( \int_{t_1}^{\infty} s^{2q} m^q(s) \left( \frac{x(s)}{s^2} \right)^q ds \right)^{1/q}.$$

Clearly, the conclusion of Theorem 3 holds. This together with (7) and (8) implies that the first, second, and fourth integrals on the above inequality are bounded and hence one can easily see that

$$x(t) \leq M_1 + Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s) \, du, \tag{32}$$

where $M_1$ and $M$ are positive constants. Note that we make $M < 1$ possible by increasing the size of $t_1$. Finally, taking $\liminf$ in (32) as $t \to \infty$ as well as using (30) result is a contradiction with the fact that $x$ is eventually positive. □

The following corollary is immediate.

**Corollary 5.** *Let* $0 < \lambda < 1$ *and conditions (i)–(iii), (6)–(9) hold for some* $t_1 \geq c$. *In addition, assume that*

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s) \, ds \, du > \infty,$$

$$\liminf_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s) \, ds \, du > -\infty,$$

$$\lim_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} (t - s)^{\alpha-1} s^{\gamma-1} m^{\lambda/(\lambda-1)}(s) \tag{33}$$
$$\cdot h^{1/(1-\lambda)}(s) \, ds \, du < \infty.$$

*If for every* $M$, $0 < M < 1$,

$$\limsup_{t\to\infty} \left[ Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s) \, ds \, du \right] = \infty,$$

$$\liminf_{t\to\infty} \left[ Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s) \, ds \, du \right] = -\infty, \tag{34}$$

*for all* $t_1 > c$, *then (1) is oscillatory.*

The following example is illustrative.

*Example 6.* Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly,

$$p(\alpha - 1) + 1 = p(\gamma - 1) + 1 = p \left( 1 - \frac{1}{2p} - 1 \right) + 1$$
$$= \frac{1}{2} > 0, \tag{35}$$
$$\theta = p(\alpha + \gamma - 2) + 1 = 0.$$

Let the functions $a(t)$ and $b(t)$ be as in (i) and (ii) with $b(t)$ being a bounded function and let $a(t) = e^t$, $e(t) = te^t \sin t$, and $f(t, x) = t^{\gamma-1} h(t) x^{\lambda}$, where $0 < \lambda < 1$, $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $h(t) = m(t)$, $\int^{\infty} s^{2q} h^q(s) ds < \infty$, and

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} e^{-u} \int_{t_1}^{u} (u - s)^{\alpha-1} s^{\gamma-1} h(s) \, ds \, du < \infty. \tag{36}$$

Condition (34) is also fulfilled. Thus, all conditions of Theorem 3 are satisfied and hence every nonoscillatory solution $x$ of (1) satisfies $\limsup_{t\to\infty} (|x(t)|/t^2) < \infty$.

Now if $e(t) = t^{\delta} e^t \sin t$, $\delta \geq 2$, we see that all the hypotheses of Corollary 5 are satisfied and hence (1) is oscillatory.

Similar reasoning to that in the sublinear case guarantees the following theorems for the integrodifferential equation (1) when $\lambda = 1$.

**Theorem 7.** *Let* $\lambda = 1$ *and the hypotheses of Theorems 3 and 4 hold with* $m(t) = h(t)$ *and* $g_{\pm} = e(t)$. *Then the conclusion of Theorems 3 and 4 holds, respectively.*

From the obtained results, we apply the employed technique to some related integrodifferential equations.

Now, we consider the integrodifferential equation

$$x'(t) = e(t) + \int_{c}^{t} (t - s)^{\alpha-1} k(t, s) f(s, x(s)) \, ds, \tag{37}$$
$$c > 1, \quad \alpha \in (0, 1).$$

We will give sufficient conditions under which any nonoscillatory solution $x$ of (37) satisfies $|x(t)| = O(t)$ as $t \to \infty$.

**Theorem 8.** *Let* $0 < \lambda < 1$ *and let condition (ii) hold and suppose that* $p > 1$, $q = p/(p-1)$, $0 < \alpha < 1$, *and* $\gamma = 2 - \alpha - 1/p$, $p(\alpha - 1) + 1 > 0$, *and* $p(\gamma - 1) + 1 > 0$,

$$\int_{t_1}^{\infty} s^q m^q(s) \, ds < \infty, \tag{38}$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_{c}^{t} g_-(s) \, ds < \infty,$$

$$\liminf_{t\to\infty} \frac{1}{t} \int_{c}^{t} g_+(s) \, ds > -\infty \tag{39}$$

*for any* $t_1 \geq c$. *If* $x$ *is a nonoscillatory solution of (37), then*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty. \tag{40}$$

*Proof.* Let $x$ be a nonoscillatory solution of (37). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (ii) we may then write

$$
\begin{aligned}
x'(t) \\
\leq e(t) + \int_c^{t_1} (t - s)^{\alpha-1} |F(s)| \, ds \\
+ \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} \left[ h(s) x^\lambda(s) - m(s) x(s) \right] ds \\
+ \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) \, ds.
\end{aligned}
\tag{41}
$$

Proceeding as in the proof of Theorem 3, we obtain

$$
\begin{aligned}
x'(t) \leq \int_c^{t_1} (t - s)^{\alpha-1} |F(s)| \, ds + e(t) + (1 - \lambda) \\
\cdot \lambda^{\lambda/(1-\lambda)} \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} m^{\lambda/(\lambda-1)}(s) \\
\cdot h^{1/(1-\lambda)}(s) \, ds + \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) \, ds.
\end{aligned}
\tag{42}
$$

Integrating inequality (42) from $t_1$ to $t$ and interchanging the order of integration one can easily obtain

$$
\begin{aligned}
x(t) \leq x(t_1) + \int_{t_1}^t g_+(s) \, ds \\
+ \int_{t_1}^t \int_{t_1}^u (u - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) \, ds \, du \\
+ \int_{t_1}^t \int_c^{t_1} (t_1 - s)^{\alpha-1} |F(s)| \, ds \, du.
\end{aligned}
\tag{43}
$$

Interchanging the order of integration in second integral we have

$$
\begin{aligned}
x(t) \leq x(t_1) + \int_{t_1}^t g_+(s) \, ds \\
+ \frac{t}{\alpha} \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) \, ds \\
+ \int_{t_1}^t \int_c^{t_1} (t_1 - s)^{\alpha-1} |F(s)| \, ds \, du.
\end{aligned}
\tag{44}
$$

The rest of the proof is similar to that of Theorem 3 and hence is omitted. □

*Example 9.* Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly,

$$
\begin{aligned}
p(\alpha - 1) + 1 &= p(\gamma - 1) + 1 = p\left(1 - \frac{1}{2p} - 1\right) + 1 \\
&= \frac{1}{2} > 0, \\
\theta &= p(\alpha + \gamma - 2) + 1 = 0.
\end{aligned}
\tag{45}
$$

Let the functions $e(t) = t \sin t$ and $f(t, x) = t^{\gamma-1} h(t) x^\lambda$, where $0 < \lambda < 1$, $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $h(t) = m(t)$, $\int^\infty s^q h^q(s) ds < \infty$, and

$$
\limsup_{t \to \infty} \frac{1}{t} \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} h(s) \, ds < \infty.
\tag{46}
$$

Condition (39) is also fulfilled. Thus, all conditions of Theorem 8 are satisfied and hence every nonoscillatory solution $x$ of (37) satisfies $\limsup_{t \to \infty}(|x(t)|/t) < \infty$.

Finally, we consider the integral equation

$$
\begin{aligned}
x(t) = e(t) + \int_c^t (t - s)^{\alpha-1} k(t, s) f(s, x(s)) \, ds, \\
c > 1, \quad \alpha \in (0, 1).
\end{aligned}
\tag{47}
$$

Now we give sufficient conditions for the boundedness of any nonoscillatory solution of (47).

**Theorem 10.** *Let $0 < \lambda < 1$ and let condition (ii) hold and suppose that $p > 1$, $q = p/(p-1)$, $0 < \alpha < 1$, and $\gamma = 2 - \alpha - 1/p$, $p(\alpha - 1) + 1 > 0$, and $p(\gamma - 1) + 1 > 0$,*

$$
\int_{t_1}^\infty m^q(s) \, ds < \infty,
\tag{48}
$$

$$
\begin{aligned}
\limsup_{t \to \infty} g_-(t) < \infty, \\
\liminf_{t \to \infty} g_+(t) > -\infty,
\end{aligned}
\tag{49}
$$

*where $g(t)$ is defined as in (5) for any $t_1 \geq c$. If $x$ is a nonoscillatory solution of (47), then $x$ is bounded.*

*Proof.* Let $x$ be an eventually positive solution of (47). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (ii) we may then write

$$
\begin{aligned}
x(t) \\
\leq \int_c^{t_1} (t - s)^{\alpha-1} |F(s)| \, ds + e(t) \\
+ \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} \left[ h(s) x^\lambda(s) - m(s) x(s) \right] ds \\
+ \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) \, ds
\end{aligned}
\tag{50}
$$

or

$$
\begin{aligned}
x(t) \leq \int_c^t (t - s)^{\alpha-1} |F(s)| \, ds + g_+(t) \\
+ \int_{t_1}^t (t - s)^{\alpha-1} s^{\gamma-1} m(s) x(s) \, ds.
\end{aligned}
\tag{51}
$$

The rest of the proof is similar to that of Theorem 3 and hence is omitted. □

*Example 11.* Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly,

$$
\begin{aligned}
p(\alpha - 1) + 1 = p(\gamma - 1) + 1 &= p\left(1 - \frac{1}{2p} - 1\right) + 1 \\
&= \frac{1}{2} > 0, \\
\theta = p(\alpha + \gamma - 2) + 1 &= 0.
\end{aligned}
\tag{52}
$$

Let the functions $e(t) = \sin t$ and $f(t, x) = t^{\gamma-1} h(t) x^{\lambda}$, where $0 < \lambda < 1$, $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $h(t) = m(t)$, $\int^{\infty} h^q(s) ds < \infty$, and

$$
\limsup_{t \to \infty} \int_{t_1}^{t} (t - s)^{\alpha-1} s^{\gamma-1} h(s) \, ds < \infty.
\tag{53}
$$

Condition (49) is also fulfilled. Thus, all conditions of Theorem 10 are satisfied and hence every nonoscillatory solution $x$ of (37) is bounded.

Similar reasoning to that in the sublinear case guarantees the following theorems for the integrodifferential equations (37) and (47) when $\lambda = 1$.

**Theorem 12.** *Let $\lambda = 1$ and the hypotheses of Theorems 8 and 10 hold with $m(t) = h(t)$. Then the conclusion of Theorems 8 and 10 holds.*

We may note that results similar to Theorem 4 can be obtained for (37) and (47). The details are left to the reader.

## 3. General Remarks

(i) The results of this paper are presented in a form which is essentially new and it can also be employed to investigate the asymptotic and oscillatory behavior of certain integrodifferential equations of higher order $\alpha \in (n-1, n)$, $n \geq 1$. The details are left to the reader.

(ii) It would be of interest to study (1) when $f$ satisfies condition (iii) with $\lambda > 1$.

## Competing Interests

The authors declare that they have no competing interests.

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