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A Course in Harmonic Analysis

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Contents

Chapter 1

Introduction

These notes were written to accompany the courses Math 6461 and Math 6462 (Harmonic Analysis I and II) at Missouri University of Science & Technology.

The goal of these notes is to provide an introduction into a range of topics and techniques in harmonic analysis, covering material that is interesting not only to students of pure mathematics, but also to those interested in applications in computer science, engineering, physics, and so on. We will focus on giving an overall sense of the available results and the analytic techniques used to prove them; in particular, complete generality or completely optimal results may not always be pursued. Technical details will sometimes be left to the reader to work out as exercises; solving these exercises is an important part of solidifying the reader's understanding of the material. At times we will not develop the full theory but rather give a survey of results, along with citations to references containing full details.

These notes are organized as follows:

- In Chapter 2, we introduce Fourier series, motivating their development through an application to solving PDE (a common theme for us). We then develop the Fourier transform, also providing some applications to PDE. Other topics are discussed, including questions of pointwise convergence, the Fourier transform on distributions, and the Paley–Wiener theorem.

- In Chapter 3 we discuss the question of sampling of signals (e.g. the Shannon–Nyquist theorem), as well as the discrete and fast Fourier transform. We close this chapter with a discussion of compressed sensing, providing a relatively complete presentation of the result of Candes, Romberg, and Tao [4] on reconstruction of signals using randomly sampled Fourier coefficients.

- In Chapter 4, we present a survey of results in abstract Fourier transform, relying primarily on the textbook of Folland [9]. In particular, we demonstrate how many of the preceding topics may be viewed under the same umbrella (i.e. Fourier analysis on locally compact abelian groups). Most results are stated without proof. We then briefly discuss the case of Fourier analysis on compact groups and present a few important examples in detail (namely, $SU(2)$ and $SO(n)$ for $n \in \{3, 4\}$).

- In Chapter 5, we discuss the continuous and discrete wavelet transforms, as well as the notion of multiresolution analysis. In addition to wavelet transforms, we also frequently discuss the 'windowed' Fourier transform. Our primary reference is the book of Daubechies [7]. This chapter provides a relatively brief introduction into a very rich topic with a wide range of applications.

- In Chapter 6, we begin discussing what I have called 'classical' harmonic analysis (although this distinction of 'classical' versus 'modern' should not be taken too seriously). This includes the theory of interpolation of linear operators, some 'classical inequalities' (like convolution inequalities and Sobolev embedding), the Hardy–Littlewood maximal function (and vector maximal function), and finally the Calderón–Zygmund theory for singular integral operators.

- In Chapter 7, we continue the study of 'classical' topics in harmonic analysis. We firstly prove the Mihlin multiplier theorem. We then develop Littlewood–Paley theory (including the Littlewood–Paley square function estimate and some fractional calculus estimates). Finally, we study oscillatory integrals (proving, for example, the stationary phase theorem and providing some applications to PDE).

- In Chapter 8, we begin our study of more 'modern' topics in harmonic analysis. We begin with a study of semiclassical analysis. This is not actually a subfield of harmonic analysis; however, it is closely related due to the frequent analysis of oscillatory integrals. We give a brief introduction based on the textbook of Martinez [19]; we get as far as the proof of L^2 boundedness for pseudodifferential operators. In the rest of this chapter, we prove the non-endpoint cases of the Coifman–Meyer multiplier theorem.

- In Chapter 9, we continue our study of ‘modern’ topics and turn to the question of sharp inequalities and existence of optimizers. We consider two examples, namely, the Gagliardo–Nirenberg inequality and Sobolev embedding. For Gagliardo–Nirenberg, we present a proof based on radial decreasing rearrangements and the compactness of the radial Sobolev embedding. For Sobolev embedding, we present a proof based on profile decompositions, thus giving a short introduction into ‘concentration-compactness’ techniques (which have come to play an important role in the setting of nonlinear PDE).

- In Chapter 10, we prove some basic results in ‘restriction theory’. This refers to the question of when it makes sense to restrict a function’s Fourier transform to a surface. We begin with a result due to Strichartz for the paraboloid. This result can be interpreted as a space-time estimate for solutions to the linear Schrödinger equation. We take a slight detour to prove a wider range of such estimates (which now go by the name of Strichartz estimates). We then return to restriction theory and prove the ‘Tomas–Stein’ result (up to the endpoint) for the case of the sphere.

- Finally, in the appendix, we have collected some prerequisite material for the reader’s reference.

The material from these notes has been drawn from many different sources. In addition to the references listed in the bibliography, this includes the author’s personal notes from a harmonic analysis course given by M. Visan at UCLA.

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Chapter 2

Fourier analysis, part I

2.1 Separation of variables

Consider the following partial differential equation (PDE):

$$\begin{cases} \partial_t u = \partial_x^2 u & (t,x) \in (0,\infty) \times (0,1) \\ u(0,x) = f(x) & x \in (0,1) \\ u(t,0) = u(t,1) = 0 & t \in [0,\infty), \end{cases} \tag{2.1}$$

where $f : (0,1) \to \mathbb{R}$ is some given function. This is the well-known **heat equation**. This is an example of an initial-value problem (the solution is specified at $t = 0$), as well as a boundary-value problem (the values of the solution are prescribed at the boundary of the spatial domain $(0,1)$).

One approach to solving PDEs like this is the method of **separation of variables**, which entails looking for separated solutions of the form

$$u(t,x) = p(t)q(x).$$

Using (2.1) and rearranging, we find that for u to be a solution we must have

$$\frac{-p'(t)}{p(t)} = \frac{-q''(x)}{q(x)}.$$

As the left-hand side depends only on t and the right-hand side depends only on x, we are led to the problem

$$p'(t) = -\lambda p(t) \quad \text{and} \quad -q''(x) = \lambda q(x) \quad \text{for some constant} \quad \lambda.$$

The equation for p is solvable for any λ; indeed, $p(t) = e^{-\lambda t}p(0)$ does the job. The problem for q is more interesting, since in addition to the ordinary differential equation (ODE) it must also satisfy the boundary conditions. One finds that there are solutions only for special choices of λ, namely, $\lambda = (n\pi)^2$ for some integer $n > 0$. A corresponding solution is then given by $q(x) = \sin(n\pi x)$.

What we have therefore discovered is that the method of separation of variables yields a countable family of solutions to the heat equation satisfying the boundary conditions, namely

$$e^{-(n\pi)^2 t} \sin(n\pi x) c_n \quad \text{for any} \quad n > 0 \quad \text{and} \quad c_n \in \mathbb{R}.$$

Furthermore, any linear combination of these solutions solves the PDE and satisfies the boundary condition. Therefore, we can solve the initial-value problem in (2.1) provided we can find c_n such that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x).$$

The possibility of finding such expansions and understanding the sense in which they hold are precisely the questions addressed by the study of Fourier series.

Remark 2.1.1. Separation of variables has led us to the 'eigenvalue problem for the Dirichlet Laplacian'. That is, we were led to look for eigenvalues of $-\partial_x^2$ on the domain $(0,1)$ with eigenfunctions q satisfying the 'Dirichlet' boundary conditions $q(0) = q(1) = 0$. This is a good perspective to keep in mind if one wishes to generalize the discussion above to a wider class of equations and boundary conditions.

In Section 2.3 we will prove the following theorem.

Theorem 2.1.2. *Let $F : [-L, L] \to \mathbb{C}$ satisfy $F(-L) = F(L)$ and assume $F \in L^2([-L, L])$. There exist $c_n \in \mathbb{C}$ such that F can be expanded in the* ***Fourier series***

$$F(x) = \sum_{n \in \mathbb{Z}} c_n e_n(x) \quad \textit{as functions in} \quad L^2([-L, L]),$$

where

$$e_n(x) := e^{\frac{in\pi x}{L}}.$$

The ***Fourier coefficients*** c_n *are given by*

$$c_n = \langle F, e_n \rangle = \tfrac{1}{2L} \int_{-L}^{L} F(x) \bar{e}_n(x)\, dx.$$

We may also write

$$c_n = \hat{F}(n).$$

Using this result, one can recover sine series and cosine series, which are often used in the solution of PDEs via separation of variables (see the exercises).

Note that we may identify periodic functions on $[-L, L]$ with functions on the torus or on the circle, which we may denote by $\mathbb{T}_L$.

2.2 Fourier series in general

Instead of proving Theorem 2.1.2 directly, let us begin with a more general perspective, inspired largely by the presentation in [14].

Consider the space $L^2(E)$ for some measurable $E \subset \mathbb{R}^d$. Recall that L^2 admits an inner product given by

$$\langle f, g \rangle = \int_E f(x)\bar{g}(x)\, dx, \quad \|f\|_{L^2(E)} = \|f\| = \sqrt{\langle f, f \rangle}.$$

If $\langle f, g \rangle = 0$, then we call f and g **orthogonal**. A set $\{\phi_\alpha\}_{\alpha \in A}$ is **orthogonal** if any two of its elements are orthogonal and **orthonormal** if it is orthogonal and $\|\phi_\alpha\| = 1$ for all $\alpha \in A$. By convention, we always assume that orthogonal sets consist only of nonzero elements. Using separability, one can show that any orthogonal set in L^2 is necessarily countable (see the exercises). An orthogonal set $\{\phi_k\}$ is **complete** if $\langle f, \phi_k \rangle = 0$ for all k implies $f = 0$.

Suppose now that $\{\phi_k\}$ is an orthonormal set in L^2. For $f \in L^2$, we define the **Fourier coefficients** of f (with respect to $\{\phi_k\}$) by

$$c_k = \langle f, \phi_k \rangle = \int_E f\bar{\phi}_k.$$

We define the **Fourier series** of f (with respect to $\{\phi_k\}$) by

$$S[f] = \sum_k c_k \phi_k.$$

We define the partial Fourier series by

$$s_N = \sum_{k=1}^{N} c_k \phi_k.$$

We will prove the following:

Theorem 2.2.1. *Suppose $\{\phi_k\}$ is a complete orthonormal set in L^2. Then for every $f \in L^2$, the Fourier series $S[f]$ converges to f in L^2.*

Remark 2.2.2. If one supposes that there is a decomposition of the form $f = \sum_k c_k\phi_k$, one can already see that we should have $c_k = \langle f, \phi_k\rangle$ by taking the inner product of both sides with ϕ_k and using orthonormality.

Remark 2.2.3. The space L^2 admits complete orthonormal sets. To see this, first take a countable dense subset (L^2 is separable), and then apply the Gram–Schmidt algorithm from linear algebra to find an orthonormal basis for L^2. This means an orthonormal set whose span is dense in L^2 (where span means the collection of all finite linear combinations). Using Cauchy–Schwarz, one can then prove that any orthogonal basis for L^2 must be complete (see the exercises).

The key property of Fourier series is that they give the best L^2 approximation using linear combinations of the $\{\phi_k\}$.

Lemma 2.2.4. *Let $\{\phi_k\}$ be an orthonormal set in L^2 and $f \in L^2$.*

(i) Given N, the best L^2 approximation to f using the ϕ_k is given by the partial Fourier series.

(ii) (Bessel's inequality) We have $c := \{c_k\} \in \ell^2$ and

$$\|c\|_{\ell^2} \leq \|f\|_{L^2},$$

where $\{c_k\}$ are the Fourier coefficients of f.

Proof. Fix N and $\gamma := (\gamma_1, \cdots, \gamma_N)$ and consider linear combinations of the form

$$F = F(\gamma) = \sum_{k=1}^{N} \gamma_k\phi_k.$$

By orthonormality,

$$\|F\|^2 = \sum_{k=1}^{N} |\gamma_k|^2.$$

Thus, recalling $c_k := \langle f, \phi_k\rangle$, we can write

$$\begin{aligned}
\|f - F\|^2 &= \langle f - \sum \gamma_k\phi_k, f - \sum \gamma_k\phi_k\rangle \\
&= \|f\|^2 - \sum_{k=1}^{N} [\bar{\gamma}_k c_k + \gamma_k\bar{c}_k] + \sum_{k=1}^{N} |\gamma_k|^2 \\
&= \|f\|^2 + \sum_{k=1}^{N} |c_k - \gamma_k|^2 - \sum_{k=1}^{N} |c_k|^2.
\end{aligned}$$

It follows that

$$\min_{\gamma} \|f - F(\gamma)\|^2 = \|f\|^2 - \sum_{k=1}^{N} |c_k|^2$$

and

$$\text{argmin}_{\gamma} \|f - F(\gamma)\|^2 = (c_1, \cdots, c_N).$$

This proves (i). Furthermore (evaluating at $\gamma = (c_1, \ldots, c_N)$) we can deduce

$$\sum_{k=1}^{N} |c_k|^2 = \|f\|^2 - \|f - s_N\|^2,$$

which yields Bessel's inequality upon sending $N \to \infty$. □

If equality holds in Bessel's inequality (i.e. $\|c\|_{\ell^2} = \|f\|_{L^2}$), we say f satisfies **Parseval's formula**. From the proof of Bessel's inequality, we deduce the following:

Proposition 2.2.5. *Parseval's formula holds if and only if $S[f]$ converges to f in L^2.*

In particular, we have reduced the proof of Theorem 2.2.1 to proving that Parseval's formula always holds whenever $\{\phi_k\}$ is a complete orthonormal set.

Before proving this, we need a result that allows us to use Fourier coefficients to define L^2 functions.

Proposition 2.2.6 (Riesz–Fischer)**.** *Let $\{\phi_k\}$ be an orthonormal set in L^2 and $\{c_k\} \in \ell^2$. There exists an $f \in L^2$ such that $S[f] = \sum c_k \phi_k$ and f satisfies Parseval's formula.*

Proof. Write $t_N = \sum_{k=1}^{N} c_k \phi_k$. For $M < N$, orthonormality implies

$$\|t_N - t_M\|^2 = \sum_{k=M+1}^{N} |c_k|^2.$$

Thus $\{c_k\} \in L^2$ implies $\{t_N\}$ is Cauchy and hence converges to some $f \in L^2$. Now observe for $N \geq k$

$$\int f\bar{\phi}_k = \int (f - t_N)\bar{\phi}_k + \int t_N \bar{\phi}_k = \int (f - t_N)\bar{\phi}_k + c_k$$

which tends to c_k as $N \to \infty$ by Cauchy–Schwarz and the fact that $t_N \to f$ in L^2. Thus $S[f] = \sum c_k \phi_k$ and $t_N = s_N(f)$. In particular, Parseval's formula follows from the fact that $t_N \to f$ in L^2. □

This result does not guarantee uniqueness. However, one does have uniqueness if the set $\{\phi_k\}$ is *complete*. Indeed, if f and g have the same Fourier coefficients then $f-g$ is perpendicular to each ϕ_k.

Finally, we turn to the following:

Proposition 2.2.7. *An orthonormal set $\{\phi_k\}$ is complete if and only if Parseval's formula holds for every $f \in L^2$.*

Proof. If $\{\phi_k\}$ is complete and $f \in L^2$, then Bessel's inequality implies that the Fourier coefficients $\{c_k\}$ are in ℓ^2. Thus (by Riesz–Fischer) there exists $g \in L^2$ with $S[g] = \sum c_k\phi_k$ and $\|g\|^2 = \sum |c_k|^2$. Because f, g have the same Fourier coefficients and $\{\phi_k\}$ is complete, we get $f = g$ a.e. Thus $\|f\|^2 = \|g\|^2 = \sum |c_k|^2$.

Conversely, if $\langle f, \phi_k\rangle = 0$ for all k and $\|f\|^2 = \sum |\langle f, \phi_k\rangle|^2$, then $\|f\| = 0$ which shows that the $\{\phi_k\}$ are complete. □

Proof of Theorem 2.2.1. Theorem 2.2.1 now follows from the combination of Proposition 2.2.5 and Proposition 2.2.7. □

One can consider even more general settings and prove similar results in the setting of abstract Hilbert spaces. However, at this point we will return to the more specific setting of Fourier series for periodic functions on $[-L, L]$.

2.3 Fourier series, revisited

In light of Theorem 2.2.1, to prove Theorem 2.1.2, we need only to verify that the set $\{\frac{1}{\sqrt{2L}}e_n : n \in \mathbb{Z}\}$ is orthonormal and complete in $L^2([-L, L])$, where we recall

$$e_n(x) := e^{\frac{in\pi x}{L}}.$$

To simplify formulas, let us fix $L = \pi$ in what follows; in particular, $e_n(x) = e^{inx}$.

A direct computation shows that

$$\tfrac{1}{2\pi}\langle e_n, e_m\rangle = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\pi}\,dx = \delta_{nm},$$

where

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & \text{otherwise.} \end{cases}$$

One calls δ_{nm} the **Kronecker delta**. Thus the family $\{\frac{1}{\sqrt{2\pi}}e_n\}$ is orthonormal.

It remains to prove that this set is complete.

Lemma 2.3.1. *Let $f \in L^2([-\pi,\pi])$. If $\langle f, e_n\rangle = 0$ for all n, then $f = 0$.*

Proof. By assumption, we have

$$S_N f \equiv 0, \quad \text{where} \quad S_N f(x) = \tfrac{1}{2\pi}\sum_{n=-N}^{N} \langle f, e_n\rangle e_n(x).$$

We can rewrite

$$S_N f(x) = f * D_N(x),$$

where D_N is the **Dirichlet kernel** given by

$$D_N(x) = \tfrac{1}{2\pi}\sum_{n=-N}^{N} e_n(x).$$

If we could prove that $S_N f \to f$ (in some sense), we would be finished. However, this is difficult because the kernels D_N do not form a family of good kernels. In fact, we can compute the Dirichlet kernel explicitly:

$$D_N(x) = \tfrac{1}{2\pi}\frac{\sin([N+\frac{1}{2}]x)}{\sin(\frac{1}{2}x)} \tag{2.2}$$

(see the exercises). One can verify, for example, that the L^1-norm of D_N grows like $\log N$. (Again, see the exercises.)

As we will see, averaging improves the situation. In particular, if we define the **Cesáro means** by

$$\sigma_N f = \tfrac{1}{N}\sum_{n=0}^{N-1} S_n f,$$

then we may write

$$\sigma_N f(x) = f * F_N(x),$$

where F_N is the **Fejér kernel** given by

$$F_N(x) = \tfrac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \tfrac{1}{2\pi N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{ikx}.$$

By assumption, we have $\sigma_N f \equiv 0$. On the other hand, we will prove that F_N are a family of good kernels, so that $\sigma_N f \to f$ in L^2 as $N \to \infty$ (see Lemma A.3.2). From this we can conclude $f = 0$, as desired.

First, a direct computation shows

$$\int_{-\pi}^{\pi} F_N(x)\,dx = \tfrac{1}{2\pi N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} 2\pi\delta_{k0} = 1.$$

For the next property, we use the identity

$$F_N(x) = \tfrac{1}{2\pi N}\frac{[\sin(\frac{N}{2}x)]^2}{[\sin(\frac{1}{2}x)]^2},$$

which we also leave as an exercise. In particular, $F_N(x) \geq 0$, so that

$$\int_{-\pi}^{\pi} |F_N(x)|\,dx = 1$$

as well. Finally, we fix $\delta > 0$ and observe that $|\sin(\frac{1}{2}x)| \gtrsim \delta$ for $|x| > \delta$. Thus, using the identity above for example, we find

$$\int_{|x|>\delta} |F_N(x)|\,dx \lesssim \tfrac{1}{N\delta^2} \to 0 \quad \text{as} \quad N \to \infty.$$

The result follows. □

2.4 Convergence of Fourier series

We have now seen that Fourier series for periodic functions in $L^2([-L, L])$ (or equivalently, for functions on the torus/circle) converge in the sense of L^2. It is a natural question to ask in what other senses the Fourier series of a function converge.

The arguments in the proof of Lemma 2.3.1 show that if $f \in L^p$, then the Cesáro means $\sigma_N f$ converge to f in L^p. Indeed, the Fejér kernels are good kernels. Similarly, if f is (uniformly) continuous, then $\sigma_N f$ converges to f uniformly. One can further show that for $f \in L^1$ and a point x,

$$\lim_{N\to\infty} \sigma_N f(x) = \tfrac{1}{2}[f(x+) + f(x-)], \tag{2.3}$$

where $f(x\pm)$ denotes limits from the right/left (see the exercises). In particular, $\sigma_N f(x)$ will converge to $f(x)$ at any point of continuity.

As for the convergence of $S_N f$ to f, we have so far established convergence in L^2-norm. Other notions of convergence are much more subtle. Let us begin with some negative results, which are essentially consequences of the fact that the Dirichlet kernels are unbounded in L^1, along with the uniform boundedness principle. We will continue to work on the interval $[-\pi, \pi]$ for convenience.

Proposition 2.4.1. *The following hold:*

(i) The Fourier series of an L^1 function need not converge in L^1.

(ii) The Fourier series of a continuous function need not converge uniformly.

(iii) There exists a continuous function with Fourier series diverging at a point.

Proof. Recall that the Fourier series of a function f is given by

$$S_N f = f * D_N, \quad D_N(x) = \tfrac{1}{2\pi} \frac{\sin([N + \frac{1}{2}]x)}{\sin(\frac{1}{2}x)},$$

where we recall that

$$\|D_N\|_{L^1} \gtrsim \log N.$$

We also recall the Fejér kernels $F_N = \frac{1}{N} \sum_{n=0}^{N-1} D_n$.

(i) For each n, we may view S_n as a linear operator from L^1 to L^1. We define the operator norm of S_n by

$$\|S_n\|_{L^1 \to L^1} = \sup\{\|S_n f\|_{L^1} : f \in L^1 \quad \text{with} \quad \|f\|_{L^1} = 1\}.$$

Recalling that the Fejér kernels F_N are uniformly bounded in L^1 (by 1), we have

$$\|S_n(F_N)\|_{L^1} \leq \|S_n\|_{L^1 \to L^1} \|F_N\|_{L^1} \leq \|S_n\|_{L^1 \to L^1}.$$

On the other hand, $S_n(F_N) = \sigma_N(D_n)$ (check!), which yields

$$\|S_n(F_N)\|_{L^1} = \|\sigma_N(D_n)\|_{L^1} \to \|D_n\|_{L^1} \quad \text{as} \quad N \to \infty,$$

where we use that the σ_N are good kernels. We conclude that

$$\lim_{n \to \infty} \|S_n\|_{L^1 \to L^1} = \infty. \tag{2.4}$$

This implies that $S_n f$ must not converge to f for every $f \in L^1$. Indeed, if $S_n f \to f$ for every $f \in L^1$ then (2.4) would contradict the uniform boundedness principle.

(ii) Similar to (i), we will show that

$$\lim_{n\to\infty} \|S_n\|_{L^\infty\to L^\infty} = \infty$$

by showing that

$$\|S_n\|_{L^\infty\to L^\infty} \gtrsim \|D_n\|_{L^1}.$$

We let $\psi_n(x) = \text{signum}[D_n(x)]$, except in small intervals around the $2n$ points of discontinuity of $\text{signum}[D_n(x)]$. In particular, we can make ψ_n be continuous, with

$$\|\psi_n\|_{L^\infty} \leq 1 \quad \text{uniformly in} \quad n.$$

Choosing the total length of the small intervals to be smaller than $\varepsilon/2n$ for some small $\varepsilon > 0$, we have

$$\|S_n\psi_n\|_{L^\infty} \geq |S_n\psi_n(0)| \gtrsim \left| \int D_n(x)\psi_n(x)\,dx \right| \gtrsim \|D_n\|_{L^1} - \varepsilon.$$

For example, we can use the fact that $\sup_x |D_n'(x)| \lesssim n$ to get a bound of n for $\int |D_n\psi_n|$ on each small interval and then sum over all intervals. This completes the proof.

(iii) Finally, consider the functionals $\ell_n : C([-\pi,\pi]) \to \mathbb{C}$ defined by $f \mapsto S_n f(0)$. The proof of (ii) shows that $\|\ell_n\| \to \infty$. Thus, there must exist f such that $\ell_n f(0) \to \infty$, for otherwise we would reach a contradiction to the uniform boundedness principle. □

The previous result shows failure of convergence (in general) in L^1 and L^∞, but in fact L^p convergence *does* hold for $1 < p < \infty$. Instead of proving this general fact, let us simply prove the following positive result.

Proposition 2.4.2. *Let f be of bounded variation on $\mathbb{T}$. Then*

$$\lim_{n\to\infty} S_n f(x) = \tfrac{1}{2}[f(x+) + f(x-)],$$

where $f(x\pm)$ denotes limits from the right/left. In particular, $S_n f(x)$ converges to $f(x)$ at any point of continuity of f.

Remark 2.4.3. This result is related to the well-known Gibbs phenomenon. In particular, we see that at a jump discontinuity the Fourier series will converge to the middle of the jump. It turns out that near the jump the Fourier series will 'overshoot' and 'undershoot' the function on either side of the jump in a way that does not diminish as one increases the number of terms in the series.

Example 2.4.1. Let $f(x) = 0$ for $|x| \in (\frac{\pi}{2}, \pi)$ and $f(x) = a$ for $|x| < \frac{\pi}{2}$. Then

$$c_n = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \tfrac{a}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-inx}\, dx = \begin{cases} \frac{a}{2} & n = 0 \\ \frac{a}{\pi n} \sin(\frac{n\pi}{2}) & n \neq 0. \end{cases}$$

As this is an even sequence, the partial Fourier sums are given by

$$S_N f(x) = \sum_{n=-N}^{N} c_n e^{inx} = \tfrac{a}{2} + \tfrac{2a}{\pi} \sum_{n=1}^{N} \tfrac{1}{n} \sin(\tfrac{n\pi}{2}) \cos(nx),$$

which we may rewrite as

$$S_{2K+1} f(x) = \tfrac{a}{2} + \tfrac{2a}{\pi} \sum_{k=1}^{K} \frac{(-1)^k \cos((2k+1)x)}{2k+1}.$$

In particular, note that $S_{2K+1} f(\pm\frac{\pi}{2}) \equiv \frac{a}{2}$, as we expect. However, let us now consider $x = \frac{\pi}{2} + \varepsilon$. Then (employing some trig identities) the series becomes

$$g_K(\varepsilon) := -\sum_{k=1}^{K} \frac{\sin((2k+1)\varepsilon)}{2k+1},$$

with $g_K(0) \equiv 0$ as expected. However we can see that $|g_K'(0)| = K$, which suggests that this series can reach size 1 even over an interval of length $1/K$.

In fact, more detailed analysis (or studying this example in Mathematica) would reveal that (for all large K) $g_K(\varepsilon)$ reaches size about $-.92$ for $\varepsilon = 0+$ and $+.92$ for $\varepsilon = 0-$, and hence we see that the Fourier series will undershoot/overshoot the correct value of the function by a fixed amount, yielding approximate values

$$\tfrac{a}{2} \mp \tfrac{a}{2}\big(\tfrac{4}{\pi} \cdot .92\big)$$

as x approaches $\frac{\pi}{2}$ from the right/left.

This is just a special case of the more general Gibbs phenomenon, in which one sees the Fourier series undershooting/overshooting the correct value at a jump discontinuity by about 9% of the jump on each side.

Proof of Proposition 2.4.2. As we will see, the key is to establish some decay for the Fourier coefficients $\hat{f}(n)$ of f.

In fact, we claim that

$$|\hat{f}(n)| \lesssim \tfrac{1}{|n|} \|f\|_{BV} \quad \text{for} \quad |n| \geq 1. \tag{2.5}$$

Indeed, this follows from the integration by parts formula for Riemann–Stieltjes integrals (see Proposition A.1.3):

$$\begin{aligned} |\hat{f}(n)| &= \left| \tfrac{1}{2\pi} \int e^{-inx} f(x)\, dx \right| \\ &= \left| \tfrac{1}{2\pi i n} \int e^{-inx} df(x) \right| \lesssim \tfrac{1}{|n|} \|f\|_{BV}. \end{aligned}$$

The next step is to restate things in terms of $\sigma_n f$, which are known to satisfy the conclusion of the proposition (cf. (2.3)). First, observe that by expanding the definition of $(n+1)\sigma_{n+1}$, we can deduce the identity

$$\begin{aligned} \sigma_{n+1} f(x) &= \sum_{|k| \leq n} [1 - \tfrac{|k|}{n+1}] \hat{f}(k) e^{ikx} \\ &= S_n f(x) - \sum_{|k| \leq n} \tfrac{|k|}{n+1} \hat{f}(k) e^{ikx}. \end{aligned}$$

Now let $m > n$ be an integer to be determined shortly. Similar to the above, we can write

$$\sigma_{m+1} f(x) = S_n f(x) - \sum_{|k| \leq n} \tfrac{|k|}{m+1} \hat{f}(k) e^{ikx} + \sum_{n < |k| \leq m} (1 - \tfrac{|k|}{m+1}) \hat{f}(k) e^{ikx}.$$

Now we can see that any linear combination of the form

$$\alpha \sigma_{n+1} f + (1 - \alpha) \sigma_{m+1} f$$

will produce a single copy of $S_n f$ plus some 'error terms'. As any such combination converges to the desired limit as $n, m \to \infty$, we can complete the proof if we can find a suitable combination for which we can control the error terms.

The sums involving $|k| \leq n$ are the most problematic, as they do not tend to zero individually; for example, applying (2.5), the best estimate we have for the term appearing in σ_{n+1} yields

$$\left| \sum_{|k| \leq n} \tfrac{|k|}{n+1} |\hat{f}(k)| \right| \lesssim \tfrac{n}{n+1},$$

which does not converge to 0 as $n \to \infty$. Thus, we choose a combination in order to make the two sums over $|k| \leq n$ cancel. In particular (choosing $\alpha = -(\frac{n+1}{m-n})$), we may write

$$S_n f = -\tfrac{n+1}{m-n} \sigma_{n+1} f + \tfrac{m+1}{m-n} \sigma_{m+1} f - \tfrac{m+1}{m-n} \sum_{n < |k| \leq m} (1 - \tfrac{|k|}{m+1}) \hat{f}(k) e^{ikx}.$$

Our final step is to show that for m chosen suitably depending on n, this final term can be made arbitrarily small (uniformly as $n \to \infty$). In fact, using (2.5), we can estimate

$$\tfrac{m+1}{m-n} \sum_{n<|k|\leq m} |(1 - \tfrac{|k|}{m+1})\hat{f}(k)| \lesssim \tfrac{m+1}{m-n} \sum_{n<|k|\leq m} \tfrac{1}{|k|} \lesssim \tfrac{m+1}{m-n} \log(\tfrac{m}{n}).$$

Given $\varepsilon > 0$, the result now follows by choosing m to be the integer part of $(1+\delta)n$ for small $\delta = \delta(\varepsilon) > 0$. $\square$

Remark 2.4.4. In the preceding proof, we used some regularity condition on f to prove that the Fourier coefficients converged to zero quantitatively as $n \to \infty$. In fact, from Bessel's inequality we know that the Fourier coefficients of an L^2 function always tend to zero as $n \to \infty$; this is also sometimes called the **Riemann–Lebesgue lemma**. However, without imposing some regularity conditions, it is possible to have a sequence of Fourier coefficients converging to zero arbitrarily slowly.

The phenomenon that smoothness yields decay of Fourier coefficients (and vice versa!) is an important fact in Fourier analysis. Revisiting the proof of (2.5), for example, one can see that being k-times continuously differentiable would imply decay of the Fourier coefficients like $|n|^{-k}$ (by repeating the integration by parts k times). In fact, more can be said. In the case of the torus, it is a fact that a function is analytic if and only if its Fourier coefficients decay exponentially. We will not pursue this result on the torus, but will prove a related result (the Paley–Wiener theorem) for the Fourier transform in the next chapter. Also see the exercises, where these topics are explored a bit more.

To close the discussion, let us finally mention the deep result of Carleson:

Theorem 2.4.5 (Carleson)**.** *Fourier series of functions in L^2 converge pointwise almost everywhere.*

We will briefly discuss this result below in Section 2.5.1.

One can extend much of what we have done above to the case of higher dimensional tori, although we will not pursue the details here. We also remark that it is possible to study analogues of Fourier series on more general groups than the torus (e.g. compact Lie groups). We will venture briefly in this direction in Chapter 4 below. For now, we turn to the extension of the preceding ideas from $[-L, L]$ to the whole real line.

2.5 The Fourier transform

We have seen that for a periodic L^2 function $f : [-L, L] \to \mathbb{C}$, we can write f as a linear combination of waves of frequencies $\frac{n}{2L}$, namely,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \tfrac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}}\, dx. \tag{2.6}$$

The **Fourier transform** extends this to the case $L \to \infty$. For $f : \mathbb{R} \to \mathbb{C}$, we define $\hat{f} : \mathbb{R} \to \mathbb{C}$ formally by

$$\hat{f}(\xi) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

That is, $\hat{f}(\xi)$ is the 'Fourier coefficient' at frequency $\xi \in \mathbb{R}$. The question is then whether or not we can recover f from $\hat{f}$; i.e. do we have an analogue of (2.6)?

Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfies $f(x) = 0$ for $|x| > M$, and let $L > M$. Then

$$c_n = \tfrac{\pi}{L} \tfrac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-\frac{in\pi x}{L}}\, dx = \tfrac{1}{\sqrt{2\pi}} \tfrac{\pi}{L} \hat{f}(\tfrac{n\pi}{L}),$$

and hence for fixed $x \in \mathbb{R}$ we have

$$f(x) = \tfrac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \tfrac{\pi}{L} \hat{f}(\tfrac{n\pi}{L}) e^{\frac{in\pi x}{L}}.$$

Writing $\varepsilon = \frac{\pi}{L}$ and $G(y) = \hat{f}(y) e^{ixy}$, we can send $L \to \infty$ (i.e. $\varepsilon \to 0$) to formally deduce

$$f(x) = \tfrac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \varepsilon G(\varepsilon n) \to \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} G(\xi)\, d\xi = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, dy$$

Thus we arrive formally at the **Fourier inversion formula**

$$f(x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, d\xi, \quad \text{where} \quad \hat{f}(\xi) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

We call the function $\hat{f}$ the **Fourier transform** of f. This extends naturally to higher dimensions as follows: for $f : \mathbb{R}^d \to \mathbb{C}$ and $\xi \in \mathbb{R}^d$,

$$\hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-ix\xi}\, dx,$$

where $x\xi$ really denotes $x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d$.

Remark 2.5.1. Recall the viewpoint that the Fourier series of a function is an expansion in terms of eigenfunctions of $-\partial_x^2$ (or, in higher dimensions, the Laplacian). Here we see that the Fourier transform of a function is an expansion in terms of generalized eigenfunctions of $-\partial_x^2$ on $\mathbb{R}$ (or, in higher dimensions, the Laplacian). The difference is that the Laplacian has discrete eigenvalues (or spectrum) on a compact domain, while it has continuous spectrum on $\mathbb{R}^d$. Spectral theory allows for a unified interpretation of Fourier series/transform, namely, as a spectral resolution of the Laplacian. We will also see a unification of Fourier series and Fourier transform through the perspective of Fourier analysis on locally compact abelian groups in Chapter 4.

Remark 2.5.2. There are other normalizations for the Fourier transform. A common one is to define

$$\hat{f}(\xi) = \int e^{-2\pi i x\xi} f(x)\, dx.$$

We will use this normalization in the next chapter and at times below.

We turn to the details. Note that $\hat{f}$ is not necessarily well-defined for an arbitrary function $f : \mathbb{R}^d \to \mathbb{C}$, as the integral may not converge. On the other hand, for any $f \in L^1$ we have $\hat{f}$ well-defined as a bounded function on $\mathbb{R}^d$.

We will begin by restricting to a nice function space, which (as we will see) is very compatible with the Fourier transform.

Definition 2.5.3 (Schwartz space)**.** We define

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : x^\alpha \partial^\beta f \in L^\infty \quad \text{for all multi-indices} \quad \alpha, \beta\}.$$

The Schwartz space is a topological vector space, with the topology generated by the open sets

$$\{f \in \mathcal{S}(\mathbb{R}^d) : \|x^\alpha \partial^\beta (f - g)\|_{L^\infty} < \varepsilon\}$$

for some $g \in \mathcal{S}(\mathbb{R}^d)$, $\varepsilon > 0$, and multi-indices α, β. More can be said about the structure of Schwartz space, but it will not be too relevant for our discussions here.

If f is a Schwartz function, then f is absolutely integrable, and hence the Fourier transform of f is well-defined pointwise. In fact, we will show that $\hat{f}$ is also a Schwartz function! We begin with the following lemma.

Lemma 2.5.4. *Let $f \in \mathcal{S}(\mathbb{R}^d)$.*

- *If* $g(x) = \partial^\alpha f(x)$*, then* $\hat{g}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$*.*
- *If* $g(x) = (-ix)^\alpha f(x)$*, then* $\hat{g}(\xi) = \partial^\alpha \hat{f}(\xi)$*.*

Proof. Let us consider the simplest case of $d = 1$ and a single derivative or power of x, leaving the rest as an exercise. First, if $g(x) = f'(x)$, then integration by parts (and the fact that $f \to 0$ as $|x| \to \infty$) yields

$$\begin{aligned}\hat{g}(\xi) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} f'(x)\,dx \\ &= (2\pi)^{-\frac{1}{2}} i\xi \int_{\mathbb{R}} e^{-ix\xi} f(x) = i\xi \hat{f}(\xi),\end{aligned}$$

as desired. Similarly, if $g(x) = -ixf(x)$, then

$$\begin{aligned}\hat{g}(\xi) &= -(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} ixe^{-ix\xi} f(x)\,dx \\ &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(x) \tfrac{d}{d\xi}[e^{-ix\xi}]\,dx = \tfrac{d}{d\xi}\hat{f}(\xi).\end{aligned}$$

This completes the proof. $\square$

This lemma already suggests the connection between the Fourier transform and partial differential equations (PDE): it interchanges taking derivatives and multiplication by x. We leave the following corollary as an exercise:

Corollary 2.5.5. *If* $f \in \mathcal{S}(\mathbb{R}^d)$ *then* $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$*.*

Thus, defining the transformation $\mathcal{F}$ which takes $f \in \mathcal{S}(\mathbb{R}^d)$ and returns $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, we can see that $\mathcal{F}$ (also called the **Fourier transform**) is a well-defined linear transformation on $\mathcal{S}$. Linearity is straightforward to check and is left to the reader. In fact, more is true:

Theorem 2.5.6. *The Fourier transform* $\mathcal{F}$ *is a bijection on Schwartz space* $\mathcal{S}(\mathbb{R}^d)$*, and the Fourier inversion formula holds. That is,*

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\,d\xi \quad \textit{for} \quad f \in \mathcal{S}(\mathbb{R}^d).$$

To prove this theorem, we need a few auxiliary lemmas.

Lemma 2.5.7 (Multiplication formula)**.** *For* $f, g \in \mathcal{S}(\mathbb{R}^d)$*, we have*

$$\int_{\mathbb{R}^d} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^d} \hat{f}(y) g(y)\,dy.$$

Proof. This is a consequence of Fubini's theorem. $\square$

Lemma 2.5.8. *Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then the following hold:*

- *If $g(x) = f(-x)$, then $\hat{g}(\xi) = \hat{f}(-\xi)$.*
- *If $g(x) = f(x-h)$, then $\hat{g}(\xi) = e^{-ih\xi}\hat{f}(\xi)$.*
- *If $g(x) = f(\lambda x)$, then $\hat{g}(\xi) = \frac{1}{\lambda^d}\hat{f}(\frac{\xi}{\lambda})$.*

Proof. Let us verify the third identity and leave the first two as exercises. This follows from the change of variables formula. Indeed,

$$\int_{\mathbb{R}^d} e^{-ix\xi} f(\lambda x)\,dx = \frac{1}{\lambda^d}\int_{\mathbb{R}^d} e^{-\frac{iy\xi}{\lambda}} f(y)\,dy.$$

The result follows. $\square$

As a consequence of the first two identities, we observe that

$$\text{if} \quad g(y) = f(x-y), \quad \text{then} \quad \hat{g}(y) = e^{-ixy}\hat{f}(-y).$$

Lemma 2.5.9. *Define $f(x) = e^{-|x|^2/2}$. Then $f \in \mathcal{S}(\mathbb{R}^d)$ and $\hat{f} = f$.*

Proof. Let us prove this identity for the case $d = 1$. We leave it to the reader to see why this implies the general case.

We make an ODE argument. Using the fact that $f'(x) = -xf(x)$ and Lemma 2.5.4, we find that $\frac{d}{d\xi}\hat{f} = -\xi\hat{f}$. Therefore $\hat{f}(\xi) = e^{-\xi^2/2}\hat{f}(0)$. However,

$$\hat{f}(0) = (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-x^2/2}\,dx = 1.$$

Thus the result follows. $\square$

In the following, we will define $K(x) = (2\pi)^{-\frac{d}{2}}e^{-|x|^2/2}$ and for $\varepsilon > 0$ set

$$K_\varepsilon(x) = \varepsilon^{-d}K(\tfrac{x}{\varepsilon}).$$

It follows that K_ε form a family of good kernels as $\varepsilon \to 0$. Furthermore, by the scaling property of the Fourier transform proven above, we have that

$$\text{if} \quad G_\varepsilon(x) = K(\varepsilon x), \quad \text{then} \quad \hat{G}_\varepsilon = K_\varepsilon.$$

We can now prove Theorem 2.5.6.

Proof of Theorem 2.5.6. We begin with the inversion formula for $f \in \mathcal{S}(\mathbb{R}^d)$. Using the lemmas above, we compute

$$\begin{aligned} f * K_\varepsilon(x) &= \int_{\mathbb{R}^d} f(x-y)K_\varepsilon(y)\,dy \\ &= \int_{\mathbb{R}^d} f(x-y)\hat{G}_\varepsilon(y)\,dy \\ &= \int_{\mathbb{R}^d} e^{-ixy}\hat{f}(-y)K(\varepsilon y)\,dy = \int_{\mathbb{R}^d} \hat{f}(y)e^{ixy}K(-\varepsilon y)\,dy. \end{aligned}$$

Sending ε to zero (applying dominated convergence and noting $K(0) = (2\pi)^{-\frac{d}{2}}$), we deduce the inversion formula

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \hat{f}(y)e^{ixy}\,dy.$$

To see that $\mathcal{F}$ is a bijection, we can define $\tilde{\mathcal{F}} : \mathcal{S} \to \mathcal{S}$ via

$$\tilde{\mathcal{F}}g(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\xi}g(\xi)\,d\xi$$

and observe that the Fourier inversion formula yields $\tilde{\mathcal{F}} \circ \mathcal{F} = 1$ on $\mathcal{S}$. Combining this with the fact that $\tilde{\mathcal{F}}f(y) = \mathcal{F}f(-y)$, it follows that $\mathcal{F}\circ\tilde{\mathcal{F}} = 1$ on $\mathcal{S}$ as well. We conclude that $\tilde{\mathcal{F}} = \mathcal{F}^{-1}$ and $\mathcal{F}$ is a bijection on $\mathcal{S}$. $\square$

While the Schwartz space is clearly well-suited for the Fourier transform, it is not the end of the story. Given what we have learned about Fourier series, it is natural to seek an extension of the Fourier transform to $L^2(\mathbb{R}^d)$. The key to this is the **Plancherel** formula. Before we state and prove it, let us recall the inner product structure on $L^2(\mathbb{R}^d)$, namely,

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\bar{g}(x)\,dx, \quad \text{and} \quad \|f\|_{L^2} = \sqrt{\langle f, f\rangle}.$$

Theorem 2.5.10 (Plancherel)**.** *For $f, g \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle.$$

In particular, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

We begin with a convolution identity that is of more general use.

Lemma 2.5.11. *For $f, g \in \mathcal{S}(\mathbb{R}^d)$,*

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{\frac{d}{2}}\hat{f}(\xi)\hat{g}(\xi).$$

Proof. We compute directly:

$$\begin{aligned}\mathcal{F}(f * g)(\xi) &= (2\pi)^{-\frac{d}{2}} \iint g(y)e^{-iy\xi} f(x-y)e^{-i(x-y)\xi}\,dx\,dy \\ &= (2\pi)^{\frac{d}{2}} \hat{f}(\xi)\hat{g}(\xi),\end{aligned}$$

where we have used Fubini's theorem. □

Now we can prove the Plancherel formula:

Proof. Define $G(x) = \bar{g}(-x)$, so that $\hat{G}(\xi) = \bar{\hat{g}}(\xi)$. Then by the Fourier inversion formula and the convolution identity above,

$$\begin{aligned}\int f(x)\bar{g}(x)\,dx &= f * G(0) \\ &= (2\pi)^{-\frac{d}{2}} \int \mathcal{F}(f * G)(\xi)\,d\xi \\ &= \int \hat{f}(\xi)\hat{G}(\xi)\,d\xi = \int_{\mathbb{R}^d} \hat{f}(\xi)\bar{\hat{g}}(\xi)\,d\xi,\end{aligned}$$

as desired. □

Using Theorem 2.5.10, we can now extend the Fourier transform to a linear operator acting on L^2. In fact, the Fourier transform acts as a *unitary* operator on L^2.

Theorem 2.5.12. *The Fourier transform extends from $\mathcal{S}(\mathbb{R}^d)$ to a unitary map on $L^2(\mathbb{R}^d)$.*

We will use the following lemma, which is left as an exercise.

Lemma 2.5.13. *Schwartz space is dense in L^2.*

Proof of Theorem 2.5.12. We let $f \in L^2$ and choose $f_n \in \mathcal{S}(\mathbb{R}^d)$ such that $f_n \to f$ in L^2-norm. As $\mathcal{F}$ restricted to $\mathcal{S}(\mathbb{R}^d)$ is an isometry, it follows that $\{\hat{f}_n\}$ is Cauchy in L^2. We define $\hat{f}$ to be the L^2 limit of $\hat{f}_n$ and set $\mathcal{F}f = \hat{f}$.

To see that $\mathcal{F}f$ is well-defined, suppose g_n is another sequence in $\mathcal{S}(\mathbb{R}^d)$ that converges to f in L^2. Let h_n be the intertwining of f_n and g_n, so that $h_n \to f$ in L^2. Then $\hat{h}_n$ is Cauchy in L^2 and hence converges. As the subsequence $\hat{f}_n$ converges to $\hat{f}$, it follows that the subsequence $\hat{g}_n$ must also converge to $\hat{f}$.

The fact that $\mathcal{F}$ is an isometry on L^2 follows from the corresponding property on Schwartz space. Let us finally show that $\mathcal{F}$ maps onto L^2 (so that $\mathcal{F}$ is unitary). As the range of $\mathcal{F}$ contains a dense subclass of L^2 (namely, the Schwartz functions), it suffices to show that the range of $\mathcal{F}$ (which is a linear subspace of L^2) is closed. To this end, we let g be in the L^2-closure of the range of $\mathcal{F}$, so that there exist $f_n \in L^2$ so that $\hat{f}_n \to g$. As $\mathcal{F}$ is an isometry, $\{f_n\}$ is Cauchy in L^2. Denoting f by the limit, we apply the isometry property one more time to deduce that $\hat{f}_n \to \hat{f}$. This implies $\hat{f} = g$, which completes the proof. □

In fact, the Fourier transform extends from $\mathcal{S}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for all $1 \leq p \leq 2$, with a corresponding bound (known as the **Hausdorff–Young** inequality):

$$\|\hat{f}\|_{L^{p'}} \lesssim \|f\|_{L^p} \quad \text{for all} \quad 1 \leq p \leq 2, \quad \text{where} \quad \tfrac{1}{p} + \tfrac{1}{p'} = 1.$$

In fact, this is the only range for which this is possible. That is, if the estimate

$$\|\hat{f}\|_{L^q} \lesssim \|f\|_{L^p}$$

holds for all Schwartz functions, then $q = p'$ and $1 \leq p \leq 2$. We will discuss the Hausdorff–Young inequality later in the setting of interpolation; as for the second point, we leave it as an exercise.

2.5.1 Remarks about pointwise convergence

Let us briefly discuss the question of pointwise convergence of the Fourier transform; this is closely related to the problem of convergence of Fourier series (cf. Theorem 2.4.5 above). We will not discuss the full proof of Carleson's theorem; we refer the reader to [13, 12] for a streamlined proof. Instead, let us show how the result is implied by a 'weak type $(2,2)$ bound' for a suitable 'maximal operator' (cf. Definition 6.1.1 below, for example). In particular, this discussion may serve in part to preview some of the topics to be covered later in these notes.

Recall (from Theorem 2.5.6) that we have the Fourier inversion formula

$$f(x) = \lim_{N\to\infty} \tfrac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{ix\xi} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}, \tag{2.7}$$

for all Schwartz functions $f \in \mathcal{S}(\mathbb{R})$. Carleson's theorem [2, 8] states that the same convergence holds almost everywhere for functions f that are merely in $L^2(\mathbb{R})$.

Recall that the integral appearing on the right-hand side of (2.7) can be written in terms of the convolution of f with the Dirichlet kernel $D_N(x) =$

$\frac{\sin Nx}{\pi x}$. Thus, the convolution is a combination of the singular integral part $\frac{1}{x}$ (which would fall under the purview of Calderon–Zygmund theory, cf. Section 6.4 below) and the oscillatory part $\sin Nx$, which requires some additional techniques.

Following [13], we will work with the (equivalent) one-sided inversion formula

$$f(x) = \lim_{N\to\infty} \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{N} e^{ix\xi} \hat{f}(\xi)\, d\xi, \tag{2.8}$$

which again holds on $\mathbb{R}$ for any Schwartz function f. As Schwartz functions are dense in L^2 (Exercise 2.9.13), the problem reduces to proving that the set of functions for which almost everywhere convergence holds is closed.

For such purposes, it is common to introduce a suitable maximal function. In particular, we define the **Carleson operator**

$$\mathcal{C}f(x) = \sup_N \left| \int_{-\infty}^{N} e^{ix\xi} \hat{f}(\xi)\, d\xi \right|, \quad x \in \mathbb{R}.$$

We can then show that a weak type $(2,2)$ bound for $\mathcal{C}$ implies the desired result. (One can compare this to the fact that the weak type $(1,1)$ bound for the Hardy–Littlewood maximal function implies the Lebesgue differentiation theorem, for example; see Proposition 6.3.4.)

Proposition 2.5.14. *Suppose that $\mathcal{C}$ obeys the weak type $(2,2)$ bound*

$$|\{Cf > \lambda\}| \lesssim \lambda^{-2} \|f\|_{L^2}^2.$$

Then (2.8) holds almost everywhere for any $f \in L^2$.

Proof. Let $f \in L^2$ and $\varepsilon > 0$. Choose $g \in \mathcal{S}$ such that

$$\|f - g\|_{L^2} < \varepsilon^{\frac{3}{2}}.$$

Now, setting

$$L_f = \limsup_{N\to\infty} \left| f(x) - \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{N} e^{ix\xi} \hat{f}(\xi)\, d\xi \right|,$$

we note that since (2.8) holds for g (for all x), we may write

$$L_f \le \mathcal{C}(f-g) + |f-g|.$$

Using the weak type $(2,2)$ bound, we find

$$|\{\mathbb{C}(f-g) > \tfrac{1}{2}\varepsilon\}| \lesssim \varepsilon^{-2} \|f-g\|_{L^2}^2 \lesssim \varepsilon.$$

Similarly, by Tchebychev's inequality,

$$\{|f-g|>\tfrac{1}{2}\varepsilon\}|\lesssim \varepsilon^{-2}\|f-g\|_{L^2}^2\lesssim \varepsilon.$$

Thus

$$|\{L_f>\varepsilon\}|\lesssim \varepsilon$$

for any $\varepsilon>0$. It follows that $L_f=0$ almost everywhere, which implies the desired result. □

Actually proving the weak type bound for the Carleson operator is quite an undertaking. We again refer the interested reader to [13, 12] and end our discussion of Carleson's theorem here.

2.6 Applications to PDE

In this section, we discuss a few applications of the Fourier transform to the solution of some linear partial differential equations.

Example 2.6.1. Consider the Poisson/Laplace equation

$$-\Delta u=f,\quad u:\mathbb{R}^3\to\mathbb{R},$$

where $f:\mathbb{R}^3\to\mathbb{R}$ is a given function. Here Δ is the Laplacian,

$$\Delta u=\sum_{j=1}^{3}\tfrac{\partial^2 u}{\partial x_j^2}.$$

Applying the Fourier transform and Lemma 2.5.4, we find that the equation is equivalent to

$$|\xi|^2\hat{u}=\hat{f},\quad \text{so that}\quad \hat{u}=|\xi|^{-2}\hat{f}=(2\pi)^{-\frac{3}{2}}\mathcal{F}(K*f)(\xi),$$

where $K=\mathcal{F}^{-1}(|\xi|^{-2})$. In particular, $u(x)=(2\pi)^{-\frac{3}{2}}K*f(x)$.

Here we reach a bit of a subtle point: the function $|\xi|^{-2}$ is not a Schwartz function, nor is it even an L^2 function! Let us ignore this subtlety for the moment—it can be resolved by the theory of distributions (see below). Instead, let us see if we can compute a formula for K anyway.

There is an elegant way to compute $K(x)$ exactly using the gamma function (see the exercises). Let us instead argue by symmetry to deduce the form of $K(x)$. First, we observe by Lemma 2.5.8 that

$$|\lambda\xi|^{-2}=\lambda^{-2}|\xi|^{-2}\implies K(\lambda x)=\lambda^{-1}K(x).$$

Next, because $|\xi|^{-2}$ is invariant under rotations, so is K (see the exercises). Consequently, K is constant on the unit sphere. It follows that

$$K(x) = |x|^{-1} K(\tfrac{x}{|x|}) = c|x|^{-1} \quad \text{for some} \quad c \in \mathbb{R}.$$

To compute c, let us go back to the PDE. Noting that

$$u(x) = (2\pi)^{-\frac{3}{2}} c|x|^{-1} * f, \quad \text{we want} \quad -|x|^{-1} * \Delta f = c^{-1}(2\pi)^{-\frac{3}{2}} f.$$

Using translation invariance, it is enough to evaluate both sides at $x = 0$. Thus, we are left to find c such that

$$-\int_{\mathbb{R}^3} |x|^{-1} \Delta f(x)\, dx = c(2\pi)^{-\frac{3}{2}} f(0).$$

A computation using integration by parts (see the exercises) yields

$$-\int_{\mathbb{R}^3} |x|^{-1} \Delta f(x)\, dx = 4\pi f(0).$$

Thus $(2\pi)^{-\frac{3}{2}} c^{-1} = 4\pi$, and we conclude

$$u(x) = \tfrac{1}{4\pi|x|} * f \quad \text{solves} \quad -\Delta u = f$$

in three dimensions.

Example 2.6.2 (Heat equation). We next consider the heat equation on $(0, \infty) \times \mathbb{R}^d$:

$$\begin{cases} u_t - \Delta u = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u(0, x) = f(x) & x \in \mathbb{R}^d. \end{cases}$$

We apply the Fourier transform in the x variables only. We find

$$\widehat{u}_t(t, \xi) = \mathcal{F}(\Delta u)(t, \xi) \quad \iff \quad \widehat{u}_t(t, \xi) = -|\xi|^2 \widehat{u}(t, \xi).$$

For each ξ, this is an ODE in t that we can solve:

$$\widehat{u}(t, \xi) = \widehat{u}(0, \xi) e^{-t|\xi|^2} = e^{-t|\xi|^2} \widehat{f}(\xi).$$

Thus

$$u(t, x) = \mathcal{F}^{-1}[\widehat{f}\, e^{-t|\xi|^2}](x) = (2\pi)^{-d/2} [f * \mathcal{F}^{-1}(e^{-t|\xi|^2})](x),$$

and again we need to compute an inverse Fourier transform.

Fortunately, we have already done this computation! Recall that

$$\mathcal{F}(e^{-|x|^2/2})(\xi) = e^{-|\xi|^2/2}$$

Thus (by Lemma 2.5.8), we have

$$\mathcal{F}(e^{-|x|^2/4t})(\xi) = (2t)^{\frac{d}{2}} e^{-t|\xi|^2}.$$

We conclude that the solution to the heat equation is given by

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} e^{-|\cdot|^2/4t} * f(x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y)\, dy.$$

2.7 The Fourier transform of distributions

In computing the solution to the Poisson equation, we quickly ran into the problem of taking Fourier transforms or inverse Fourier transforms of functions that are not even L^2. In fact, one can extend the Fourier transform quite naturally to the setting of 'tempered distributions', which includes a much larger class of functions than Schwartz space or L^2. Without delving too deeply into this topic, let us introduce some of the main points.

A **tempered distribution** u is a continuous linear functional acting on Schwartz space, that is, $u : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. The set of all tempered distributions is denoted $\mathcal{S}'(\mathbb{R}^d)$, or just by $\mathcal{S}'$. In this setting, one often calls the elements of $\mathcal{S}$ (which are the arguments of elements of $\mathcal{S}'$) **test functions**.

Schwartz functions themselves may be embedded in the set of tempered distributions through the mapping $T : \mathcal{S} \to \mathcal{S}'$ given by

$$Tu(f) = \int uf\, dx \quad \text{for} \quad u, f \in \mathcal{S}.$$

As an exercise, one should check that the map T is injective, and hence we can identify a function u with the distribution Tu. In fact, the mapping T makes sense for any function that is integrable against Schwartz functions, hence a very large class of functions may naturally be viewed as distributions (e.g. any L^p function multiplied by any polynomial).

Not all distributions are given by functions. A classical example is the **Dirac delta distribution** $\delta_0 \in \mathcal{S}'$, defined by

$$\delta_0(f) = f(0) \quad \text{for} \quad f \in \mathcal{S}.$$

We remark that if K_n is a family of good kernels, then $K_n \to \delta_0$ 'in the sense of distributions'.

The multiplication formula (Lemma 2.5.7) reveals how to extend the Fourier transform to the space of distributions. Recalling that

$$\int f\hat{g} = \int \hat{f}g \quad \text{for all} \quad f, g \in \mathcal{S},$$

we define the **Fourier transform** of a distribution $u \in \mathcal{S}'$ to be the distribution $\hat{u} \in \mathcal{S}'$ satisying

$$\hat{u}(f) = u(\hat{f}) \quad \text{for all} \quad f \in \mathcal{S}.$$

This definition guarantees that $\hat{u}$ agrees with the usual definition of the Fourier transform in the case that u actually arises from a Schwartz function under the mapping T introduced above. Thus the Fourier transform $\mathcal{F}$ extends to a mapping $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$.

Similarly, if we define $\mathcal{F}^* : \mathcal{S}' \to \mathcal{S}'$ by $\mathcal{F}^*u(f) = u(\mathcal{F}^{-1}f)$, we can deduce that $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = Id$, and hence $\mathcal{F}^* = \mathcal{F}^{-1}$ and $\mathcal{F}$ is a bijection on $\mathcal{S}'$.

To define operations on distributions, one first observes how these operations behave on Schwartz functions. For example, given a multiindex α and $u, f \in \mathcal{S}$, integration by parts yields

$$\int (\partial^\alpha u) f = (-1)^{|\alpha|} \int u \partial^\alpha f.$$

Then for a distribution $u \in \mathcal{S}'$, we define $\partial^\alpha u \in \mathcal{S}'$ by

$$\partial^\alpha u(f) = (-1)^{|\alpha|} u(\partial^\alpha f) \quad \text{for} \quad f \in \mathcal{S}.$$

Note that this allows us to take derivatives of non-differentiable functions!

Similarly, a computation shows that the correct definition of $f * u$ (for $f \in \mathcal{S}$ and $u \in \mathcal{S}'$) should be

$$(f * u)(g) = u(\tilde{f} * g), \quad \text{where} \quad \tilde{f}(x) = f(-x), \quad g \in \mathcal{S}.$$

Alternately, one can define $f * u$ as a function via

$$(f * u)(x) = u(\widetilde{\tau_x f}), \quad \tau_x f(y) = f(y - x). \tag{2.9}$$

One can check that these two definitions agree.

Finally, for $u \in \mathcal{S}'$ and a (moderately well-behaved) function f, we can define $fu \in \mathcal{S}'$ via $[fu](g) = u(fg)$.

Using the definitions introduce above, one can verify all of the nice properties of the Fourier transform continue to hold in the setting of distributions, e.g.

$$\widehat{\partial^\alpha u} = (i\xi)^\alpha \hat{u}, \quad \mathcal{F}(f * u) = (2\pi)^{\frac{d}{2}} \hat{f}\hat{u},$$

and so on. The moral is that one can often perform 'formal' computations with the Fourier transform, even if the functions involved are not Schwartz. The resulting computations will typically be valid, provided they are interpreted in the appropriate sense.

Let us conclude this section with an example.

Example 2.7.1. Consider $\delta_0 \in \mathcal{S}'$. Then

$$\hat{\delta}_0(f) = \delta_0(\hat{f}) = \hat{f}(0) = (2\pi)^{-\frac{d}{2}} \int f.$$

Thus $\hat{\delta}_0 = (2\pi)^{-\frac{d}{2}}$.

2.8 The Paley–Wiener theorem

For the final topic of this section, let us return to the idea that Fourier transformation exchanges decay and smoothness.

We present a classical result on the line known as the Paley–Wiener theorem.

Theorem 2.8.1 (Paley–Wiener theorem). *For $f \in L^2(\mathbb{R})$, the following are equivalent:*

(i) f is the restriction to $\mathbb{R}$ of a function F defined on a strip $\{x + iy : x \in \mathbb{R}, |y| < a\} \subset \mathbb{C}$ that is holomorphic and satisfies

$$\int |F(x+iy)|^2 \, dx \lesssim 1 \quad \text{for all} \quad |y| < a.$$

(ii) $e^{a|\xi|}\hat{f} \in L^2(\mathbb{R})$.

Proof. Suppose (ii) holds. We then define

$$F(z) = \tfrac{1}{\sqrt{2\pi}} \int e^{iz\xi} \hat{f}(\xi) \, d\xi,$$

which satisfies $F|_{\mathbb{R}} = f$ by the Fourier inversion formula. This defines a holomorphic function on the strip $\{|y| < a\}$ due to the exponential decay of $\hat{f}$. Furthermore, by Plancherel,

$$\int |F(x+iy)|^2 \, dx = \tfrac{1}{2\pi} \int |\hat{f}(\xi)|^2 e^{-2y\xi} \, d\xi \lesssim \|\hat{f} e^{a|\xi|}\|_{L^2}^2$$

uniformly for $|y| < a$. This implies (i).

Next suppose (i) holds. Denoting $f_y(x) = F(x+iy)$ (so that $f_0 = f$), we will show $\hat{f}_y(\xi) = \hat{f}(\xi)e^{-\xi y}$. Then by Plancherel's theorem (just as above), we will have

$$\int |\hat{f}(\xi)|^2 e^{2\xi y}\,d\xi \lesssim 1$$

for $|y| < a$, yielding (ii).

We would be able to say $\hat{f}_y(\xi) = \hat{f}(\xi)e^{-\xi y}$ immediately if (ii) already held. We therefore utilize a family of good kernels to introduce compactly supported approximating functions (which, in particular, satisfy (ii)).

We utilize the following family, which form a family of good kernels:

$$K_\lambda(x) = \lambda K(\lambda x),$$

where

$$K(x) = \tfrac{1}{2\pi}\big(\tfrac{\sin(x/2)}{x/2}\big)^2 = \tfrac{1}{2\pi}\int_{-1}^{1}(1-|\xi|)e^{ix\xi}\,d\xi$$

(check!).

We set

$$G_\lambda(z) = K_\lambda * F(z) = \int_{\mathbb{R}} F(z-w)K_\lambda(w)\,dw.$$

Then G_λ is holomorphic in $\{|y| < a\}$. We now define

$$g_{\lambda,y}(x) = G_\lambda(x+iy) = K_\lambda * f_y(x).$$

In particular

$$\hat{g}_{\lambda,y}(\xi) = \hat{K}_\lambda(\xi)\hat{f}_y(\xi) \quad \text{for each} \quad \lambda.$$

Now observe that each $\hat{g}_{\lambda,y}(\xi)$ has compact support, specifically, in $[-\lambda,\lambda]$. In particular, (ii) holds for $\hat{g}_{\lambda,y}$ and hence

$$\hat{g}_{\lambda,y}(\xi) = \hat{g}_{\lambda,0}(\xi)e^{-\xi y}.$$

As $\hat{K}(\frac{\xi}{\lambda})$ is supported in $[-\lambda,\lambda]$, it follows that

$$\hat{f}_{\lambda,y}(\xi) = \hat{f}_{0,y}(\xi)e^{-\xi y} \quad \text{for} \quad |\xi| < \lambda.$$

Sending $\lambda \to \infty$ yields the result. $\square$

Using this theorem, we can prove the following important fact:

Corollary 2.8.2. *Let $f \in L^2(\mathbb{R})$ and let $\hat{f} \in L^2(\mathbb{R})$ denote the Fourier transform of f. Then f and $\hat{f}$ cannot both be compactly supported (unless $f \equiv 0$).*

Sketch of proof. Suppose $\hat{f}$ and f are both compactly supported. Then $e^{a|\xi|}\hat{f} \in L^2$ for any $a > 0$. Thus f is the restriction of an entire function F. However, F vanishes on $\mathbb{R}\backslash[-M, M]$, and hence (by the uniqueness theorem of complex analysis) $F \equiv 0$. This implies $f \equiv 0$. $\square$

2.9 Exercises

Exercise 2.9.1. The 1d Dirichlet Laplacian on $(0,1)$ is the operator $-\partial_x^2$ defined on the set of smooth functions $f : (0,1) \to \mathbb{R}$ satisfying $f(0) = f(1) = 0$. The 1d Neumann Laplacian is also defined to be $-\partial_x^2$ but on the set of smooth functions $f(0,1) \to \mathbb{R}$ satisfying $f'(0) = f'(1) = 0$.

(i) Find the eigenfunctions and eigenvalues for the Dirichlet Laplacian.

(ii) Find the eigenfunctions and eigenvalues for the Neumann Laplacian.

Exercise 2.9.2. Using Fourier series, show the following:

(i) For $f : (0,1) \to \mathbb{R}$ smooth and satisfying $f(0) = f(1) = 0$, we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{for some} \quad a_n \in \mathbb{R}.$$

Moreover, find a formula for the coefficients a_n.

(ii) For $f : (0,1) \to \mathbb{R}$ smooth and satisfying $f'(0) = f'(1) = 0$, we have

$$f(x) = \sum_{n=0}^{\infty} b_n \cos(n\pi x) \quad \text{for some} \quad b_n \in \mathbb{R}.$$

Moreover, find a formula for the coefficients b_n.

Exercise 2.9.3. Suppose $\{\phi_k\}$ is a complete orthonormal set in L^2 and $f, g \in L^2$. Let $\{\hat{f}_k\}$ and $\{\hat{g}_k\}$ be the Fourier coefficients of f, g. Use Parseval's theorem to prove the following:

$$\langle f, g\rangle = \sum_k \hat{f}_k \overline{\hat{g}_k}.$$

Exercise 2.9.4. Show that any orthogonal set in L^2 is at most countably infinite.

Exercise 2.9.5. Show that any orthogonal basis in L^2 is complete. In particular, there exists a complete orthonormal basis for L^2.

Exercise 2.9.6. Compute the formula for the Dirichlet kernel by showing

$$\sum_{n=-N}^{N} e^{inx} = \frac{\sin[(N+\frac{1}{2})x]}{\sin(\frac{1}{2}x)}.$$

Compute the formula for the Fejér kernel by showing

$$\tfrac{1}{2\pi N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{ikx} = \tfrac{1}{2\pi N}\frac{[\sin(\frac{N}{2}x)]^2}{[\sin(\frac{1}{2}x)]^2}$$

Hint. Write $e^{inx} = (e^{ix})^n$ and sum the geometric series.

Exercise 2.9.7. Prove Fejér's theorem: for $f \in L^1(\mathbb{T})$,

$$\lim_{n\to\infty} \sigma_n f(x) = \tfrac{1}{2}[f(x+) + f(x-)],$$

provided these limits exist. *Hint:* Use the facts that the Fejér kernels are good kernels that are positive, even, and decay away from $x = 0$.

Exercise 2.9.8. Let f be a function on the torus with Fourier coefficients $\hat{f}(n)$.

(i) Show that if f is k-times differentiable and $f^{(k)} \in L^1$, then

$$|\hat{f}(n)| \le \min_{0\le j\le k} |n|^{-j}\|f^{(j)}\|_{L^1}.$$

(ii) Show that if f is Hölder continuous of order $\alpha \in (0,1]$ then

$$|\hat{f}(n)| \lesssim |n|^{-\alpha}.$$

Exercise 2.9.9. Let f be a function on the torus. Show that f is analytic if and only if there exist $K > 0$ and $a > 0$ such that $|\hat{f}(n)| \le Ke^{-a|n|}$.

Exercise 2.9.10. This exercise appears as Theorem I.4.1 in [15]: Let $\{a_n\}_{n=-\infty}^{\infty}$ be an even sequence of nonnegative numbers such that $a_n \to 0$ as $|n| \to \infty$. Suppose that for $n > 0$ we have

$$a_{n+1} - a_{n-1} - 2a_n \ge 0.$$

Show that there exists a nonnegative function $f \in L^1(\mathbb{T})$ such that the Fourier coefficients of f are given by $\hat{f}(n) = a_n$.

Exercise 2.9.11. Show that the Dirichlet kernels satisfy $\|D_n\|_{L^1} \gtrsim \log n$ for large n.

Exercise 2.9.12. Let f be a Schwartz function. Show that for any multii-indices α, β and any $1 \leq p \leq \infty$ we have $x^\alpha \partial^\beta f \in L^p$.

Exercise 2.9.13. Show that Schwartz space is dense in L^2.

Exercise 2.9.14. Show that if an estimate of the form

$$\|\hat{f}\|_{L^q} \lesssim \|f\|_{L^p}$$

holds for Schwartz functions, then $q = p'$ and $1 \leq p \leq 2$. *Hint* For the first part, use a scaling argument. For the second, consider $f(x) = e^{-(1+it)|x|^2/2}$ and send $t \to \infty$.

Exercise 2.9.15. Let A be a $d \times d$ invertible matrix with real entries. Show that if $g(x) = f(Ax)$, then

$$\hat{g}(\xi) = |\det A|^{-1} \hat{f}((A^t)^{-1}\xi).$$

Exercise 2.9.16. Show that

$$\mathcal{F}\big[\pi^{-\frac{d-\alpha}{2}}\Gamma(\tfrac{d-\alpha}{2})|x|^{\alpha-d}\big] = \pi^{-\frac{\alpha}{2}}\Gamma(\tfrac{\alpha}{2})|\xi|^{-\alpha}.$$

This appears in [26, Lemma 1, p.117]. It can be computed using the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt,$$

as we now sketch. First, show that

$$\int_0^\infty e^{-\pi t|x|^2} t^{\frac{d-\alpha}{2}} \frac{dt}{t} = \pi^{-(\frac{d-\alpha}{2})}\Gamma(\tfrac{d-\alpha}{2})|x|^{\alpha-d}.$$

Now compute the Fourier transform, using the fact that the Fourier transform of a Gaussian is a Gaussian (which follows from computing the appropriate Gaussian integral):

$$\int e^{-2\pi i x\xi} \int_0^\infty e^{-\pi t|x|^2} t^{\frac{d-\alpha}{2}} \frac{dt}{t}\, dx = \int_0^\infty e^{-\frac{\pi|\xi|^2}{t}} t^{-\frac{\alpha}{2}} \frac{dt}{t} = \pi^{-\frac{\alpha}{2}}|\xi|^{-\alpha}\Gamma(\tfrac{\alpha}{2}),$$

where the last equality comes from a change of variables.

Exercise 2.9.17. Show that for $f \in C_c^\infty(\mathbb{R}^3)$,

$$-\int |x|^{-1}\Delta f(x)\, dx = 4\pi f(0).$$

Exercise 2.9.18. Solve the Poisson equation $-\Delta u = f$ in dimensions $d \geq 4$.

Exercise 2.9.19. Use the Fourier transform to solve the wave equation $\partial_{tt}u - \partial_{xx}u = 0$ with initial condition $(u(0,x), \partial_t u(0,x) = (f(x), g(x))$ in one space dimension.

Exercise 2.9.20. Let $c \in \mathbb{R}^d$. Use the Fourier transform to solve the transport equation

$$\partial_t u + c \cdot \nabla u = 0$$

with initial condition $u(0,x) = f(x)$.

Exercise 2.9.21. Solve the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0, \quad u(0,x) = f(x)$$

on both the torus and $\mathbb{R}^d$.

Exercise 2.9.22. For f a Schwartz function, define the tempered distribution Tf by

$$Tf(g) = \int fg\, dx.$$

Show that the mapping $f \mapsto Tf$ is injective.

Exercise 2.9.23. Let T be the mapping as in Exercise 2.9.22. Show that if f is a Schwartz function then $[Tf]' = T[f']$. (The notation on the left refers to the distributional derivative of Tf.)

Exercise 2.9.24. Compute the second distributional derivative of the function $f(x) = |x|$

Exercise 2.9.25. Compute the Fourier transform of $\partial^\alpha \delta_0$.

Exercise 2.9.26. Give full details for the proof of Corollary 2.8.2.

Chapter 3

Fourier analysis, part II

In this section, we continue our study of the Fourier transform and consider several applied topics. We will use the following normalization for the Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i t\xi} f(t)\, dt,$$

and similarly for Fourier series.

3.1 Sampling of signals

Consider a time-dependent signal (e.g. an audio recording). This is modeled as a function $f : \mathbb{R} \to \mathbb{R}$, which we write as $f = f(t)$ to keep the interpretation of 'time' clear. In practice, we should consider signals that are compactly supported in time, but let us return to this point later.

We will consider the problem of reconstructing a signal f using a collection of samples $\{f(t_k)\}_{k=-\infty}^{\infty}$. We will assume that the sampling rate is constant, e.g. $t_k = kr$ for some $r > 0$. (The sampling rate would then be $1/r$, i.e. the number of samples taken per second.) In practice, one can only take finitely many samples, in which case we would like as good of an approximation of f as possible.

We begin with the observation that this problem is essentially hopeless unless we restrict to **bandlimited** signals, that is, signals with compact Fourier support.

Example 3.1.1. Fix $0 < r < 1$. Let

$$f(x) = \chi_{[-1,1]}(x) \quad \text{and} \quad g(x) = \chi_{[-1,1]}(x) \cos(\tfrac{2\pi x}{r}).$$

Then

$$g(kr) = \chi_{[-1,1]}(kr)\cos(2\pi k) = \chi_{[-1,1]}(kr) = f(kr)$$

for all $k \in \mathbb{Z}$. Thus f and g are indistinguishable by sampling at the points $t_k = kr$.

The previous example hints at the general principle that in order to faithfully represent a signal with sampling rate r^{-1}, the sample should not contain frequencies higher than r^{-1}. (Actually, both signals above contain arbitrarily high frequencies due to the fact that they are compactly supported in time.)

Let us turn to the positive result. We define the function

$$\operatorname{sinc}(x) := \tfrac{\sin \pi x}{\pi x}.$$

Theorem 3.1.1 (Shannon–Nyquist sampling theorem)**.** *Let $r > 0$. Let $f \in \mathcal{S}(\mathbb{R})$ and suppose $\hat{f}(\xi) = 0$ for all $|\xi| \geq \frac{1}{2r}$. Then*

$$f(t) = \sum_{k\in\mathbb{Z}} f(kr)\operatorname{sinc}[\tfrac{1}{r}(t-kr)].$$

In particular, to reconstruct a function that is bandlimited to frequencies $|\xi| \leq \frac{1}{2r}$, one must use a sampling rate at least twice the highest frequency (i.e. $\frac{1}{r}$). This is called the **Nyquist frequency** associated to the sampling rate r^{-1}.

Note that a function with compactly supported Fourier transform is analytic (cf. the Paley–Wiener theorem); thinking of such functions as 'infinite degree polynomials', it is perhaps not surprising that the function may be reconstructed using only countably many sample points.

Remark 3.1.2. The appearance of the sinc function in Theorem 3.1.1 may seem a bit mysterious. One approach for *deriving* the formula appearing in the Shannon–Nyquist theorem is to use the fact that

$$\hat{f} = \chi_{[-\frac{1}{2r},\frac{1}{2r}]}[\Pi_{\frac{1}{r}} * \hat{f}]$$

and take the inverse Fourier transform. Here Π_N denotes the distribution

$$\Pi_N(\varphi) = \sum_{k\in\mathbb{Z}} \varphi(kN),$$

and the effect of convolution with Π_N is to periodize the function, cf.

$$\Pi_N * \varphi(x) = \sum_{k\in\mathbb{Z}} \varphi(x-kN).$$

Computing the inverse Fourier transform of $\chi_{[-\frac{1}{2r},\frac{1}{2r}]}$ (see Lemma 3.1.3 below) and $\Pi_{\frac{1}{r}}$ (see Exercise 3.5.1), one can deduce the formula appearing in Theorem 3.1.1. See Exercise 3.5.2.

We turn to Theorem 3.1.1. Let us begin by recording the key property of the sinc function.

Lemma 3.1.3. *The function* sinc *belongs to* $L^2(\mathbb{R})$. *Its Fourier transform is given by*

$$\mathcal{F}[\mathrm{sinc}](\xi) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi).$$

In particular, for any $r > 0$ *and* $k \in \mathbb{Z}$,

$$\mathcal{F}[\mathrm{sinc}(\tfrac{1}{r}(\cdot - kr))](\xi) = r\chi_{[-\frac{1}{2r},\frac{1}{2r}]}(\xi)e^{-2\pi i\xi kr}.$$

Proof. That $\mathrm{sinc} \in L^2$ follows from the fact that it is bounded near $x = 0$ and decays like $|x|^{-1}$ for large x.

Next, we compute

$$\mathcal{F}^{-1}[\chi_{[-\frac{1}{2},\frac{1}{2}]}](x) = \int_{-1/2}^{1/2} e^{2\pi i x\xi}\,d\xi = \frac{\sin(\pi x)}{\pi x} = \mathrm{sinc}(x),$$

which yields the desired identity. The final identity then follows from applying the scaling and translation identities for the Fourier transform, obtained by a change of variables. $\square$

Corollary 3.1.4. *For* $r > 0$, *the family* $\{\mathrm{sinc}(\frac{1}{r}(\cdot - kr))\}_{k\in\mathbb{Z}}$ *forms an orthonormal basis for the subspace of* L^2 *defined by*

$$H = \{g \in L^2 : \hat{g} \quad \textit{is supported in} \quad [-\tfrac{1}{2r}, \tfrac{1}{2r}]\}.$$

Proof. By Plancherel's theorem and Lemma 3.1.3, the family is orthonormal. Now if $g \in H$ is orthogonal to every element of this family, then then again applying Plancherel's theorem we see that the Fourier series of $\hat{g}$ is identically zero, and hence g is zero. This shows that the family is complete, as needed. $\square$

The next lemma is an important result of independent interest. It is known as the **Poisson summation formula**.

Lemma 3.1.5 (Poisson summation formula). *For* $\varphi \in \mathcal{S}$, *we have*

$$\sum_{k\in\mathbb{Z}} \varphi(k) = \sum_{k\in\mathbb{Z}} \hat{\varphi}(k).$$

Proof. Let

$$\Phi(t) = \sum_{k\in\mathbb{Z}} \varphi(t-k).$$

Note that $\Phi(0)$ gives the left-hand side of the Poisson summation formula.

Next, observe that $\Phi(t)$ is periodic of period one. Thus it has a Fourier series expansion

$$\Phi(t) = \sum_{m\in\mathbb{Z}} \hat{\Phi}(m) e^{2\pi imt}.$$

The Fourier coefficients $\hat{\Phi}(m)$ as computed as follows:

$$\begin{aligned}
\hat{\Phi}(m) &= \int_0^1 e^{-2\pi imt}\Phi(t)\,dt \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 e^{-2\pi imt}\varphi(t-k)\,dt \\
&= \sum_{k\in\mathbb{Z}} \int_{-k}^{-k+1} e^{-2\pi imt}\varphi(t)\,dt = \hat{\varphi}(m),
\end{aligned}$$

where we have changed variables and used $e^{-2\pi imk} \equiv 1$. In particular, we now see that $\Phi(0)$ also equals the right-hand side of the Poisson summation formula, and thus the result follows. □

We next prove a lemma from which Theorem 3.1.1 will follow directly.

Lemma 3.1.6. *Let $r > 0$ and $f \in \mathcal{S}(\mathbb{R})$. Define*

$$g(t) = \sum_{k\in\mathbb{Z}} f(kr)\,\mathrm{sinc}[\tfrac{1}{r}(t-kr)].$$

Then $g \in L^2$, $g(kr) = f(kr)$ for all $k \in \mathbb{Z}$, and

$$\hat{g}(\xi) = \chi_{[-\frac{1}{2r},\frac{1}{2r}]}(\xi) \sum_{k\in\mathbb{Z}} \hat{f}(\xi - \tfrac{k}{r}).$$

Proof. That $g \in L^2$ follows from the fact that it is a linear combination of orthogonal functions with rapidly decaying coefficients; this latter fact follows from the assumption that $f \in \mathcal{S}$. Next,

$$g(\ell r) = \sum_{k\in\mathbb{Z}} f(kr)\,\mathrm{sinc}[\ell - k] = \sum_{k\in\mathbb{Z}} f(kr)\frac{\sin \pi(\ell-k)}{\pi(\ell-k)} = f(\ell r).$$

Now we compute the Fourier transform. By Lemma 3.1.3, we have

$$\hat{g}(\xi) = r\chi_{[-\frac{1}{2r},\frac{1}{2r}]}(\xi)\sum_{k\in\mathbb{Z}} f(kr)e^{-2\pi i\xi kr}.$$

Applying the Poisson summation formula to the function

$$h(x) = f(xr)e^{-2\pi i\xi xr}$$

(for fixed $r > 0$ and $\xi \in \mathbb{R}$) now yields

$$\hat{g}(\xi) = \chi_{[-\frac{1}{2r},\frac{1}{2r}]}(\xi)\sum_{k\in\mathbb{Z}} \hat{f}(\xi - \tfrac{k}{r}),$$

as was needed to show. □

Proof of Theorem 3.1.1. Define $g(t)$ as in Lemma 3.1.6. In particular,

$$\hat{g}(\xi) = \chi_{[-\frac{1}{2r},\frac{1}{2r}]}(\xi)\sum_{k\in\mathbb{Z}} \hat{f}(\xi - \tfrac{k}{r}).$$

Thus, if $\hat{f}$ is supported in $[-\frac{1}{2r},\frac{1}{2r}]$, it follows that $\hat{g} = \hat{f}$. This yields the result. □

We have seen that to reconstruct a signal using a discrete set of samples, we need two things: (i) we need the signal to be bandlimited, and (ii) we need (countably) infinitely many samples. In practice, neither of these will be satisfied. Indeed, since our signals will be compactly supported in time, they *cannot* be bandlimited. This is the content of Corollary 2.8.2. Furthermore, clearly we can only take finitely many samples of any signal.

Thus, in practice one should either decide upon a feasible sampling rate (or a feasible maximum frequency that one hopes to capture) and then take as many samples as possible with the appropriate sampling rate.

If one uses too low of a sampling rate, one can run into the problem of '**aliasing**'. This can be understood by considering Lemma 3.1.6. In particular, for a Schwartz function f we take samples at the points kr and define the function

$$g(t) = \sum_{k\in\mathbb{Z}} f(kr)\,\mathrm{sinc}[\tfrac{1}{r}(t - kr)]$$

as a possible candidate for reconstructing f. In particular, note that $g(kr) = f(kr)$.

Suppose that $\hat{f}$ is supported in $[-\frac{N}{2}, \frac{N}{2}]$. If $r^{-1} > N$ then we see that $\hat{g}(\xi)$ will contain one single copy of $\hat{f}(\xi)$, confirming what we already know: for a sufficiently high sample rate of a bandlimited signal, we can reproduce f by a discrete set of samples. On the other hand, if $r^{-1} < N$, then $\hat{g}$ will contain extra (shifted) copies of $\hat{f}$. In particular, certain frequencies will be 'counted twice' (or more than twice); this is called **aliasing**. Thus, if the sample rate is too low, the function g will *not* be a faithful reproduction of the signal. Indeed, in this case we will not have $\hat{g} = \hat{f}$.

3.2 Discrete Fourier transform

Recall that given a continuous signal $f : \mathbb{R} \to \mathbb{R}$, we will generally only be able to take finitely many samples, say $\{f(t_n)\}_{n=0}^{N-1}$. In this section we describe how to define a corresponding discrete version of the Fourier transform of f.

Let us write the sampled signal as the following distribution on $\mathbb{R}$:

$$f_d(t) := \sum_{n=0}^{N-1} f(t_n)\delta(t - t_n),$$

where $t_n = nr$ for some $r > 0$. In order for this to faithfully reproduce f, we should assume that $\hat{f}$ is supported on $[0, r^{-1}]$, say. (Previously it was $[-\frac{1}{2r}, \frac{1}{2r}]$, but let us shift here for convenience.) We should also assume that the bulk of the support of f is the interval $[0, Nr]$.

Now note that

$$\hat{f}_d(\xi) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i \xi t_n}.$$

We would like to use finite samples of $\hat{f}_d$ to approximate $\hat{f}_d$. Since f is supported mostly in $[0, Nr]$, we should take a sample rate of Nr. Thus we set $s_m = \frac{m}{Nr}$ (for $m = 0, \ldots, N-1$, which should cover the entire support of $\hat{f}_d$) and define

$$F_d(s_m) = \hat{f}_d(s_m) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_m t_n},$$

i.e.

$$F_d(\frac{m}{Nr}) = \sum_{n=0}^{N-1} f(nr)e^{-2\pi i mn/N}.$$

We regard $\{F_d(s_m)\}$ as an approximate discrete version of $\hat{f}$. In fact, by considering the Riemann sum approximation to the continuous integral, we expect

$$F_d(\tfrac{m}{Nr}) \sim r^{-1} \int_0^{Nr} f(x) e^{-2\pi i x m/Nr}\, dx = r^{-1} \hat{f}(\tfrac{m}{Nr}).$$

With this motivating example in mind, we proceed to the definition of the discrete Fourier transform.

Definition 3.2.1. Let $f \in \mathbb{C}^N$ have entries $f[n]$, $n = 0, \dots, N-1$. The **discrete Fourier transform** of f is the vector

$$\hat{f} = \mathcal{F} f \in \mathbb{C}^N$$

with entries

$$\hat{f}[m] = \sum_{n=0}^{N-1} e^{-2\pi i m n/N} f[n].$$

Note that $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ is a linear transformation and hence is represented by an $N \times N$ matrix $\mathcal{F}$ with entries $\mathcal{F}_{mn} = e^{-2\pi i m n/N}$ (where we index $m, n \in \{0, 1, \dots, N-1\}$). Writing

$$\omega = \omega_N = e^{2\pi i/N},$$

we can also write $\mathcal{F}_{mn} = \omega^{-mn}$.

It is also convenient to introduce the vector $\underline{\omega}$ with entries $\underline{\omega}[k] = \omega^k$, where $k \in \{0, \dots, N-1\}$. Then the n^{th} column of $\mathcal{F}$ is given by $\underline{\omega}^{-n}$, where this notation refers to component-wise operation. In particular,

$$\mathcal{F} f = \sum_{n=0}^{N-1} \underline{\omega}^{-n} f[n].$$

To make sure the notation is clear, here is $\mathcal{F}$ in the case $N = 2$:

$$\mathcal{F} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} \\ 1 & \omega^{-2} & \omega^{-4} \end{pmatrix}.$$

Just as in the continuous case, the discrete Fourier transform is invertible. To invert the matrix $\mathcal{F}$, it suffices to solve the equations

$$\mathcal{F} \underline{x} = \delta_k \quad \text{for each} \quad k = 0, \dots, N-1,$$

where δ_k has a 1 in the k^{th} position and zero elsewhere. (The columns of $\mathcal{F}^{-1}$ are then the solutions $\vec{u}_k$.) By analogy with the continuous case, we might expect that we should take $\underline{x} = \underline{\omega}^k$. Let us check:

$$\mathcal{F}\underline{\omega}^k = \sum_{n=0}^{N-1} \underline{\omega}^{-n}\underline{\omega}^k[n] = \sum_{n=0}^{N-1} \underline{\omega}^{-n}\omega^{kn}.$$

Now $\underline{\omega}^{-n}[m] = \omega^{-nm}$, and so

$$\underline{\omega}^{-n}[m]\omega^{kn} = \omega^{n(k-m)}.$$

Now we claim that

$$\sum_{n=0}^{N-1} \omega^{n(k-m)} = N\delta_{km}, \tag{3.1}$$

which will imply

$$\mathcal{F}\underline{\omega}^k = N\delta_k,$$

whence

$$\mathcal{F}^{-1} = \tfrac{1}{N}\overline{\mathcal{F}}.$$

For (3.1), we just need to check that the sum is zero when $k \neq m$, i.e.

$$\sum_{n=0}^{N-1} (\omega^\ell)^n = 0 \quad \text{for any} \quad \ell \in \{-N-1, \dots, -1, 1, \dots, N-1\}.$$

In fact, this is a geometric series with $\omega^\ell \neq 1$, and so equals

$$\sum_{n=0}^{N-1} (\omega^\ell)^n = \tfrac{1-\omega^{\ell N}}{1-\omega^\ell}.$$

Since $\omega^N = 1$, the numerator is zero, as desired.

Given $f \in \mathbb{C}^N$, its Fourier transform $\hat{f}$ can naturally be extended to a periodic sequence of period N, i.e. $\hat{f}(m + \ell N) = \hat{f}(m)$ for any ℓ. This follows from the fact that $\omega^{-k(m+N)} = \omega^{-km}$. In particular we can view

$$\hat{f}[m] = \sum_{k=0}^{N-1} \omega^{-km} f[k] \quad \text{for all} \quad m \in \mathbb{Z}.$$

Because the inverse Fourier transform has the same general structure as the Fourier transform, it is also natural to assume that the original discrete signal f is also periodic of period N. This will be helpful below.

We now turn to some properties of the discrete Fourier transform that parallel what we already know for the continuous Fourier transform.

Lemma 3.2.2 (Plancherel)**.** *We have the following:*

$$\mathcal{F}f \cdot \mathcal{F}g = N(f \cdot g),$$

where $f \cdot g = \sum_k f[k]\bar{g}[k]$ *denotes the standard complex inner product on* $\mathbb{C}^N$*.*

Proof. We have

$$\mathcal{F}f \cdot \mathcal{F}g = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} f[j]\bar{g}[k]\underline{\omega}^{-j} \cdot \underline{\omega}^{-k}.$$

By the computations above,

$$\underline{\omega}^{-j} \cdot \underline{\omega}^{-k} = \sum_{\ell=0}^{N-1} \omega^{\ell(k-j)} = N\delta_{kj},$$

where δ_{kj} is the Kronecker delta. Thus

$$\mathcal{F}f \cdot \mathcal{F}g = N \sum_{k=0}^{N-1} f[k]\bar{g}[k] = Nf \cdot g,$$

as desired. $\square$

We also note the following identities (left as an exercise):

$$\begin{aligned} \mathcal{F}\big(f[\cdot - a]\big) &= \underline{\omega}^{-a}\mathcal{F}f, \\ \mathcal{F}\big[\underline{\omega}^{a}f\big] &= \mathcal{F}f[\cdot - a]. \end{aligned} \tag{3.2}$$

This requires the interpretation of $f, \mathcal{F}f$ as periodic sequences. Furthermore, the product of two vectors should be interpreted as component-wise product, i.e. $(vw)[k] = v[k]w[k]$.

We next turn to convolution of sequences. This also requires the interpretation of $f \in \mathbb{C}^N$ as an infinite periodized sequence. The **convolution** of f and g and is then defined by

$$f * g[m] = \sum_{k=0}^{N-1} f[k]g[m-k], \quad m \in \{0, \dots, N-1\}.$$

The convolution of two periodized elements of $\mathbb{C}^N$ is then another periodized element of $\mathbb{C}^N$.

Lemma 3.2.3. *The following identities hold. First,*

$$\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g),$$

where as usual the product on the right is component-wise product. Next,

$$\mathcal{F}(fg) = N^{-1}\mathcal{F}f * \mathcal{F}g,$$

where again fg denotes component-wise product.

Proof. We have

$$\begin{aligned} N\mathcal{F}^{-1}(\hat{f}\hat{g})[m] &= \sum_{n=0}^{N-1} \hat{f}[n]\hat{g}[n]\omega^{mn} \\ &= \sum_{n,k,\ell} f[k]g[\ell]\omega^{n(m-\ell-k)} \\ &= N\sum_{k,\ell} f[k]g[\ell]\delta_{\ell(m-k)} \\ &= N\sum_{k} f[k]g[m-k] = Nf * g[m], \end{aligned}$$

and the desired identity follows from applying $\mathcal{F}$. Now, an analogous computation shows that

$$\mathcal{F}^{-1}(a * b) = N\mathcal{F}^{-1}a\mathcal{F}^{-1}b.$$

Thus

$$\mathcal{F}^{-1}[N^{-1}\mathcal{F}f * \mathcal{F}g] = fg,$$

which implies the second identity upon applying $\mathcal{F}$. □

Applying convolutions to signals (or, equivalently, multiplying their discrete Fourier transforms by functions) allows us to perform various 'filters' on our signal (e.g. low-pass, high-pass, band-pass filters, and many other variations). We will not pursue this topic here, but would like to remark that this is a starting point for many important applications.

We would also like to make the following observation, which is useful to keep in mind in applications. For real signals (i.e. vectors in $\mathbb{R}^N$), the Fourier transform has some symmetry properties. In particular, the discrete Fourier transform splits at index $N/2$. Note that

$$\mathcal{F}f[N/2] = \sum_{k=0}^{N-1}(-1)^k f[k],$$

and in particular is real. One finds in general that

$$\mathcal{F}f[\tfrac{N}{2}+k] = \overline{\mathcal{F}f}[\tfrac{N}{2}-k]$$

for $k = 0, \dots, \frac{N}{2}-1$. Note $\mathcal{F}f[0]$ is just the sum of the components of f. One calls $\mathcal{F}f[0]$ the 'DC' component; the frequencies $m = 1, \dots, \frac{N}{2}-1$ the 'negative frequencies'; and the frequencies $m = \frac{N}{2}+1, \dots, N-1$ the 'positive frequencies'. All of the information about $\mathcal{F}f$ is contained in the DC component, negative frequencies, and the value $\mathcal{F}f[N/2]$.

3.3 Fast Fourier transform

In the previous section, we saw that the discrete Fourier transform and its inverse are simply linear transformations on $\mathbb{C}^N$ and hence are represented by $N \times N$ matrices. In applications, however, it can be rather costly to perform matrix multiplications. In general, multiplication of an $N \times N$ matrix by an $N \times 1$ vector has computational complexity $\mathcal{O}(N^2)$.

The fast Fourier transform gives an efficient method for computing the discrete Fourier transform. This approach actually appears in work of Gauss (modeling orbits by Fourier series). The algorithm as it will be described here is more commonly associated to Cooley and Tukey.

In the following, we will typically assume that $N = 2^j$ for some $j \in \mathbb{N}$. We will write $\mathcal{F} = \mathcal{F}_p$ for the discrete Fourier transform on sequences of length p. We recall $\omega_a = e^{-2\pi i/a}$.

Lemma 3.3.1. *Let $f \in \mathbb{C}^N$ with N even. Then for $m = 0, 1, \dots, N/2$,*

$$\mathcal{F}_N f[m] = \mathcal{F}_{N/2} f_e[m] + \omega_N^{-m} \mathcal{F}_{N/2} f_0[m],$$
$$\mathcal{F}_N f[m+N/2] = \mathcal{F}_{N/2} f_e[m] - \omega_N^{-m} \mathcal{F}_{N/2} f_0[m],$$

where

$$f_e[n] = f[2n] \quad \textit{and} \quad f_o[n] = f[2n+1] \quad \textit{for} \quad n = 0, \dots, \tfrac{N}{2}-1.$$

Proof. Let $m \in \{0, 1, \dots, N/2\}$. Then, splitting into even and odd parts,

$$\mathcal{F}_N f[m] = \sum_{k=0}^{N/2-1} f[2k] e^{-2\pi i m(2k)/N} + \sum_{k=0}^{N/2-1} f[2k+1] e^{-2\pi i m(2k+1)/N}.$$

The first term equals

$$\sum_{k=0}^{N/2-1} f_e[k] e^{-2\pi i m k/(N/2)} = \mathcal{F}_{N/2} f_e[m].$$

The second term is treated similarly to produce $\mathcal{F}_{N/2} f_o[m]$, except there appears an extra power of $e^{-2\pi i m/N} = \omega_N^{-m}$.

Next consider $\mathcal{F}_N f[m + N/2]$. In this case we only need to observe that

$$e^{2\pi i(m+N/2)(2k)/N} = e^{2\pi i m(2k)/N} e^{2\pi i k} = e^{2\pi i m(2k)/N},$$

which leads to the $\mathcal{F}_{N/2} f_e[m]$ term again. On the other hand,

$$e^{2\pi i(m+N/2)(2k+1)/N} = e^{2\pi i m(2k+1)/N} e^{2\pi i(k+\frac{1}{2})} = -e^{2\pi i m(2k+1)/N},$$

which accounts for the minus sign in the formula above. This completes the proof. □

For $N = 2^j$, this lemma can be iterated until one is reduced to computing $\mathcal{F}_1$, which is trivial. Computing the discrete Fourier transform this way is the **fast Fourier transform** algorithm.

We will look at the fast Fourier transform in more detail below. First, let us see what computational advantage it has.

Proposition 3.3.2. *Let $F(N)$ be the number of operations it takes to compute the discrete Fourier transform with the fast Fourier transform algorithm. Then*

$$F(N) \sim N \log N.$$

Recalling that computing the discrete Fourier transform using matrix multiplication takes $\mathcal{O}(N^2)$ elementary operations (e.g. additions and multiplications), we see that the fast Fourier transform provides a huge computational advantage.

Proof. Using Lemma 3.3.1, we find that

$$F(N) = 2F(N/2) + cN \quad \text{for some} \quad c > 0.$$

Rearranging, this yields

$$\tfrac{F(N)}{N} = \tfrac{F(N/2)}{N/2} + c.$$

Assuming $N = 2^j$ and setting $a_j = \frac{F(2^j)}{2^j}$, this simply reads

$$a_j = a_{j-1} + c, \quad \text{so that} \quad a_j = jc + a_0.$$

But $a_0 = F(1)$, the number of operations needed to compute the discrete Fourier transform of a single point. In particular, $a_0 = 0$, so that $a_j = jc$. Thus

$$F(N) = cN \log N,$$

as claimed. □

Let us now look closer at the fast Fourier transform algorithm. As we will see, this algorithm amounts to a factorization of $\mathcal{F}_N$. Recalling Lemma 3.3.1, we need to define an operation that sorts a vector into its even and odd indices.

Definition 3.3.3. For $f \in \mathbb{C}^N$ with N even, let

$$\pi^0 f[k] = f[2k] \quad \text{and} \quad \pi^1 f[k] = f[2k+1]$$

for $k \in \{0, 1, \ldots, \frac{N}{2} - 1\}$. In particular, $\pi^0 f \in \mathbb{C}^{N/2}$ and $\pi^1 f \in C^{N/2}$. We denote sequential applications in a contravariant fashion, namely

$$\pi^{01} f = \pi^1 \pi^0 f,$$

and so on. In particular, the input will always be a vector in $\mathbb{C}^N$, but the input is a vector in $\mathbb{C}^{N/2^j}$, where j is the number of digits appearing in superscripts.

Next let us write I_N for the $N \times N$ identity matrix and Ω_N for the $N \times N$ diagonal matrix with $\Omega_N[m, m] = \omega_{2N}^m$. Then Lemma 3.3.1 may be written as

$$\mathcal{F}_N f = B_N \begin{pmatrix} \mathcal{F}_{N/2}\pi^0 f \\ \mathcal{F}_{N/2}\pi^1 f \end{pmatrix}, \quad B_N = \begin{pmatrix} I_{N/2} & \Omega_{N/2} \\ I_{N/2} & -\Omega_{N/2} \end{pmatrix}.$$

Now we repeat the process, yielding

$$\mathcal{F}_N f = B_N \cdot \operatorname{diag}(B_{N/2}) \begin{pmatrix} \mathcal{F}_{N/4}\pi^{00} f \\ \\ \mathcal{F}_{N/4}\pi^{01} f \\ \\ \mathcal{F}_{N/4}\pi^{10} f \\ \\ \mathcal{F}_{N/4}\pi^{11} f \end{pmatrix},$$

where $\operatorname{diag}(B_{N/2})$ is the $N \times N$ block diagonal matrix with $B_{N/2}$ in the upper left and lower right blocks.

Now continue this until we reach a vector with N applications of $\mathcal{F}_1 = \mathrm{Id}$. This leads to

$$\mathcal{F}_N = \left[\prod_{k=0}^{\log N - 1} \operatorname{diag}(B_{N/2^k}) \right] (\pi^{c(k)} f)_{k=0}^{N-1}, \tag{3.3}$$

where diag should always be taken with a suitable interpretation, and $c(k)$ denotes the unique sequence of $\log N$ binary digits corresponding to the element $k \in \{0, 1, \ldots, N-1\}$. For example, if $N = 8$ then $c(4) = 100$, $c(5) = 101$, and so on. By uniqueness, we may also define the map c^{-1} taking sequences of binary digits to the corresponding integer (so that $c^{-1}(101) = 5$ in the example just given).

Let us continue from (3.3). The vector $(\pi^{c(k)} f)_{k=1}^N$ consists of some rearrangement of the entries of f. We can understand exactly which rearrangement occurs through the following lemma.

Lemma 3.3.4. *Given a sequence $d = d_1 \cdots d_j$ of binary digits, define $Rd = d_j \cdots d_1$ to be the **reversal** of d. Then*

$$\pi^d f = f[c^{-1}(Rd)]$$

for any vector $f \in \mathbb{C}^{2^j}$, where c^{-1} is the map from sequences to integers introduced above. Equivalently,

$$\pi^{d_j} \cdots \pi^{d_1} f = f[c^{-1}(d_j \cdots d_1)].$$

Proof. We proceed by induction. If $j = 1$ then $d = Rd$ and so the claim boils down to to $\pi^d f = f[d]$ for $d \in \{0, 1\}$ and $f \in \mathbb{C}^2$, which is true by definition.

Now suppose the result holds up to level $j - 1$. Then

$$\pi^{d_j \cdots d_2} \pi^{d_1} f = (\pi^{d_1} f)[c^{-1}(d_j \cdots d_2)].$$

There are two cases, namely $d_1 \in \{0, 1\}$. Let us first assume $d_1 = 0$. In this case,

$$(\pi^0 f)[k] = f[2k], \quad \text{so that} \quad (\pi^0 f)[c^{-1}(d_j \cdots d_2)] = f[2c^{-1}(d_j \cdots d_2)].$$

It therefore remains to show that

$$2c^{-1}(d) = c^{-1}(d0)$$

for any binary sequence d, where we note the slight abuse of notation in the map c^{-1} above. Indeed, multiplying a number by 2 just increases each power of 2 in its binary expansion, which equivalently shifts the binary sequence to the left.

If $d_1 = 1$ then $(\pi^1 f)[k] = f[2k+1]$, so that

$$(\pi^1 f)[c^{-1}(d_j \cdots d_2)] = f[2c^{-1}(d_j \cdots d_2) + 1].$$

Thus we need to check

$$2c^{-1}(d) + 1 = c^{-1}(d1).$$

In fact, multiplying by 2 appends a zero to the sequence (as we just saw), and adding one changes this zero to a one. This completes the proof. $\square$

Returning to the setting of (3.3),

$$\pi^{c(k)} f = f[c^{-1}(Rc(k))].$$

Thus we see that to perform the fast Fourier transform consists of first sorting the indices of f according to the rule above and subsequently multiplying by $\log N$ explicit block diagonal matrices. (This also provides another derivation of the $\mathcal{O}(N \log N)$ computational complexity of the fast Fourier transform.)

Note that the sorting described above defines a linear transformation on $\mathbb{R}^N$ and hence is represented by an $N \times N$ matrix P. In particular, imposing

$$(Pf)[j] = f[c^{-1}(Rc(j))]$$

leads to

$$P[j,k] = \begin{cases} 1 & c(k) = Rc(j) \\ 0 & \text{otherwise.} \end{cases}$$

Thus for each row there will be precisely one nonzero entry (with value 1).

Example 3.3.1. If $N = 8$, the fast Fourier transform is computed by

$$B_8 \begin{pmatrix} B_4 & 0 \\ 0 & B_4 \end{pmatrix} \begin{pmatrix} B_2 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix} \begin{pmatrix} f[0] \\ f[4] \\ f[2] \\ f[6] \\ f[1] \\ f[5] \\ f[3] \\ f[7] \end{pmatrix}.$$

The sorting matrix has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.3.5. If N is not of the form 2^j, a common trick is to 'pad' with zeros until N is of this form. Despite increasing the dimension, this can still result in a computational advantage. We will not pursue this topic here.

3.4 Compressed sensing

We next turn to an introduction to the area of **compressed sensing**. In many applications, signals are in some sense 'sparse', and because of this it is often possible to (i) reconstruct signals using far fewer measurements than expected and (ii) compress these signals significantly without losing information (for purposes of data storage, for example). This is a field that has developed rapidly in recent years and has many important applications. We will primary present the results of [4]; we also use [10] as a reference. At times, certain standard probabilistic estimates may be used without proof. Most of the proof will be presented; however, some very technical elements will be relegated to the exercises.

We focus on the problem of recovering sparse signals from small sets of Fourier coefficients. Sparseness can be measured using the 'ℓ^0 norm' (it is not a norm, nor even a quasi-norm): for $f \in \mathbb{C}^N$,

$$\|f\|_{\ell^0} = |\operatorname{supp}(f)|, \quad \operatorname{supp}(f) = \{j : f[j] \neq 0\}.$$

Here $|\cdot|$ denotes counting measure.

We begin with the following preliminary result.

Theorem 3.4.1. *Let $f \in \mathbb{C}^N$ satisfy $\|f\|_{\ell^0} = s$. If $N \geq 2s$, then f can be reconstructed from its first $2s$ Fourier coefficients $\{\hat{f}[n] : n = 0, 1, \ldots, 2s - 1\}$.*

Remark 3.4.2. More generally, if N is prime, and $\|f\|_{\ell^0} = s$ then f can be reconstructed from *any* collection of $2s$ Fourier coefficients. We will be content to prove Theorem 3.4.1. See [24].

Proof of Theorem 3.4.1. We need to determine $S = \operatorname{supp}\{f\}$ and $\{f[j] : j \in S\}$. The proof will actually demonstrate how one may reconstruct f.

Consider

$$p(t) = \tfrac{1}{N} \prod_{k \in S} (1 - e^{-2\pi i k/N} e^{2\pi i t/N}).$$

This is a **trigonometric polynomial**, i.e. a polynomial in $e^{2\pi i t/N}$. Note that $p(t) = 0$ for $t \in S$.

Furthermore, since $f[j] = 0$ for $t \notin S$, we have

$$p(t)f(t) = 0 \quad \text{for all} \quad 0 \leq t \leq N - 1.$$

In particular,

$$\hat{p} * \hat{f} = \widehat{pf} = 0,$$

where pf denotes component-wise product. We may rewrite this as

$$\sum_{k=0}^{N-1} \hat{p}(k)\hat{f}(j-k) = 0 \tag{3.4}$$

for $j = 0, \dots, N-1$.

Now observe (by considering the inverse Fourier transform) that $\hat{p}(k)$ is the coefficient of $p(t)$ appearing with $e^{2\pi ikt/N}$. In particular, $\hat{p}(0) = 1$ and (since p is a trigonometric polynomial of degree $\leq s$) $\hat{p}(k) = 0$ for $k > s$.

Rewriting (3.4) for $s \leq j \leq 2s-1$ leads to

$$\begin{aligned}
\hat{f}[s] + \hat{p}[1]\hat{f}[s-1] + \cdots + \hat{p}[s]\hat{f}[0] &= 0,\\
\hat{f}[s+1] + \hat{p}[1]\hat{f}[s] + \cdots + \hat{p}[s]\hat{f}[1] &= 0,\\
&\vdots\\
\hat{f}[2s-1] + \hat{p}[1]\hat{f}[2s-2] + \cdots + \hat{p}[s]\hat{f}[s-1] &= 0.
\end{aligned}$$

We rewrite this as a linear system

$$\begin{bmatrix} \hat{f}[s-1] & \hat{f}[s-2] & \cdots & \hat{f}[0] \\ \vdots & \cdots & & \vdots \\ \hat{f}[2s-2] & \cdots & \cdots & \hat{f}[s-1] \end{bmatrix} \begin{bmatrix} \hat{p}[1] \\ \vdots \\ \hat{p}[s] \end{bmatrix} = - \begin{bmatrix} \hat{f}[s] \\ \vdots \\ \hat{f}[2s-1] \end{bmatrix}. \tag{3.5}$$

Now, the matrix on the left and the vector on the right are known quantities. In particular, given $f \in \mathbb{C}^N$ and knowledge of its first $2s$ Fourier coefficients, we can write down the system (3.5), which must have at least one solution (namely $\{\hat{p}[k]\}_{k=1}^{s}$). However, this solution may not be unique. In the following, we find some solution $\hat{q}$ to (3.5), which we extend to $\mathbb{C}^N$ by setting $\hat{q}[0] = 1$ and $\hat{q}[k] = 0$ for $k > s$.

Then, reversing the steps above, we have $\hat{q} * \hat{f}[j] = 0$ for $s \leq j \leq 2s-1$. In particular qf (component-wise product) has Fourier transform vanishing on s consecutive indices. We claim that this implies $qf \equiv 0$. To see this, we first note that since $f = 0$ outside of S, to compute the vector of Fourier coefficients $\widehat{qf}[j]$ for $s \leq j \leq 2s-1$ it suffices to multiply the vector $\{qf[j]\}_{j \in S}$ by the $s \times s$ submatrix A of $\mathcal{F}_N$ defined by choosing the rows s through $2s-1$ of $\mathcal{F}_N$ and the columns defined by indices in S. Thus it remains to check that A is invertible.

To see this, note that A is of the form $A[i,j] = \omega^{-(s+i)n_j}$, where $i = 0, \dots, s-1$ and $S = \{n_j : j = 0, \dots, s-1\}$. In particular, after factoring out ω^{-sn_j} from each column, A is the transpose of a Vandermonde matrix (i.e. a matrix whose rows are geometric progressions) corresponding to the parameters $\omega^{-n_0}, \dots \omega^{-n_{s-1}}$. Thus $\det A$ is a nonzero multiple of the product over $0 < j < k \leq s-1$ of $\omega^{-n_j} - \omega^{-n_k}$, and so the result follows provided $\omega^{n_j} \neq \omega^{n_k}$ for any j, k. In fact, this follows from the fact that $0 \leq n_j, n_k \leq N-1$ and $n_j \neq n_k$.

We conclude that A is invertible, and hence $qf[j] = 0$ for $j \in S$. In particular, $q[j] = 0$ for $j \in S$. Now, recalling that $\hat{q}[k] = 0$ for $k > s$ (so that q is a trigonometric polynomial of degree $\leq s$), we see that the fact that $q[j] = 0$ for $j \in S$ actually identifies S. That is, S is given precisely by the zeros of q.

Finally, to find $f[j]$ for $j \in S$, note that $\hat{f}[k]$ for $k = 0, \dots, 2s-1$ are given by $2s$ linear equations involving the unknowns $f[j]$. Solving this system yields $f[j]$. □

Remark 3.4.3. Let us summarize the proof above. To reconstruct a signal f with $\|f\|_{\ell^0} = s$ from its first $2s$ Fourier coefficients, proceed as follows:

- Find a solution $\hat{q}$ to the linear system (3.5). Extend to $\mathbb{C}^N$ by setting $\hat{q}[0] = 1$ and $\hat{q}[k] = 0$ for $k > s$.
- Find the zeros of $q = \mathcal{F}^{-1}\hat{q}$. This identifies S.
- Solve the linear system that produces the first $2s$ Fourier coefficients of f from the unknowns $f[j]$, $j \in S$.

The results just discussed suggest that in general, one should expect that knowledge of a signal's Fourier coefficients on a set Ω should suffice to construct signals with support of around the same size as Ω. Strictly speaking, this is not true.

Example 3.4.1. Suppose N is a perfect square. Define f by $f[j\sqrt{N}] = 1$ for $j = 0, 1, \dots \sqrt{N} - 1$ and $f[k] = 0$ otherwise. Then $\|f\|_{\ell^0} = \sqrt{N}$. Let us compute the Fourier transform. We have

$$\hat{f}[k] = \sum_{\ell=0}^{N-1} e^{-2\pi i \ell k/N} f[\ell] = \sum_{j=0}^{\sqrt{N}-1} e^{-2\pi i jk/\sqrt{N}}.$$

Now, if $k = p\sqrt{N}$ for some p, then the sum yields $\sqrt{N}$. Otherwise, summing the geometric series (as we did when computing the inverse Fourier transform in general) yields zero. Thus

$$\hat{f} = \sqrt{N} f.$$

In particular, we may choose Ω to be the set of frequencies precisely avoiding $\{p\sqrt{N} : p = 0, \ldots, \sqrt{N} - 1\}$. Then $|\Omega| = N - \sqrt{N}$, but knowing the Fourier coefficients on Ω cannot distinguish f (supported on a set of much smaller size $\sqrt{N}$) from the zero signal.

The result that we will present essentially restores the intuition introduced above (which was just shown to be wrong, strictly speaking). The key is that one must incorporate a probabilistic viewpoint.

Before stating the result, let us first note that the reconstruction problem under consideration is equivalent to the following ℓ^0 minimization problem:

$$\text{minimize } \|g\|_{\ell^0} \quad \text{subject to} \quad \hat{g}|_\Omega = \hat{f}|_\Omega. \tag{P$_0$}$$

This turns out to be computationally expensive and not particularly robust (e.g. if one is dealing with noisy measurements). In practice, one instead may consider the following ℓ^1 minimization problem:

$$\text{minimize } \|g\|_{\ell^1} \quad \text{subject to} \quad \hat{g}|_\Omega = \hat{f}|_\Omega. \tag{P$_1$}$$

The result we will prove will ultimately construct solutions to (P$_1$).

Before moving on to the main result, let us quickly show that solving (P$_1$) yields 'sparse' signals (in the real-valued case, at least).

Proposition 3.4.4. *Suppose there exists a unique minimizer g to the problem* (P$_1$) *over $\mathbb{R}^N$. Then $\|g\|_{\ell^0} \le |\Omega|$.*

Proof. The problem consists of minimizing the ℓ^1 norm subject to a constraint of the form $Ag = \hat{f}$ for $A \in \mathbb{C}^{|\Omega| \times N}$. In fact, A is just a submatrix of $\mathcal{F}_N$.

Let g be the unique minimizer and $S = \operatorname{supp}(g)$. Writing a_j for the columns of A, we will show that $\{a_j : j \in \operatorname{supp}(g)\}$ is independent, which implies $|\operatorname{supp}(g)| \le |\Omega|$, as desired.

Suppose $Av = 0$ for some v with $\operatorname{supp} v \subset S$. Suppose toward a contradiction that $v \neq 0$.

As g is the unique minimizer of (P$_1$) and $A(g + v) = Ag$, we have

$$\|g\|_{\ell^1} < \|g + tv\|_{\ell^1} = \sum_{j \in S} \operatorname{sign}(g[j] + tv[j])(g[j] + tv[j])$$

for any $t \neq 0$. Choosing

$$|t| < \min_{j \in S} |g[j]| \|v\|_{\ell^\infty}^{-1},$$

we are guaranteed that $\operatorname{sign}(g[j] + tv[j]) = \operatorname{sign}(g[j])$ for each j. Thus

$$\begin{aligned}\|g\|_{\ell^1} &< \sum_{j\in S} \operatorname{sign}(g[j])g[j] + t\sum_{j\in S}\operatorname{sign}(g[j])v[j] \\ &= \|g\|_{\ell^1} + t\sum_{j\in S}\operatorname{sign}(g[j])v[j].\end{aligned}$$

This gives a contradiction upon sending $|t| \to 0$. $\square$

Let us finally state the main result of this section.

Theorem 3.4.5 (Candès–Romberg–Tao)**.** *Let $f \in \mathbb{C}^N$ and $M \geq 1$. There exists $C_M > 0$ such that the following holds:*

Suppose f is supported on some set S. Choose Ω of size $|\Omega| = N_\omega$ uniformly at random. If

$$|\Omega| = N_\omega \geq C_M |S| \log N,$$

then with probability at least $1 - \mathcal{O}(N^{-M})$ the minimizer of (P_1) *is unique and equals f.*

Remark 3.4.6. In Theorem 3.4.5, the Fourier coefficients are randomly sampled. In particular, given N_ω, we choose Ω uniformly at random from all sets of this size. Thus each of the $\binom{N}{N_\omega}$ possible subsets are equally likely. The result says that the fraction of such subsets from which we can reconstruct f is at least $1 - \mathcal{O}(N^{-M})$, provided $N_\omega \geq C_M|S|\log N$.

Remark 3.4.7. This theorem is optimal. Consider again the function in Example 3.4.1. To have a chance of recovering f, the set Ω must overlap $W = \operatorname{supp} \hat{f}$ in at least one point. Now, choosing Ω uniformly at random, we have

$$\mathbb{P}(\Omega \cap W = \emptyset) = \binom{N - \sqrt{N}}{|\Omega|} \div \binom{N}{|\Omega|}.$$

Now, we should already be assuming that $|\Omega| > |S| = \sqrt{N}$. Under this assumption, we get the following lower bound:

$$\mathbb{P}(\Omega \cap W = \emptyset) \geq (1 - \tfrac{2|\Omega|}{N})^{\sqrt{N}}$$

See Exercise 3.5.4. Therefore, if we hope for $\mathbb{P}(\Omega \cap W = \emptyset) \leq N^{-M}$, we need

$$\sqrt{N}\log(1 - \tfrac{2|\Omega|}{N}) \leq -M\log N.$$

Supposing we also want to avoid the case when $|\Omega|$ is comparable to N (so that $|\Omega| \ll \frac{1}{2}N$), we can view $\log(1-\frac{2|\Omega|}{N})$ as comparable to $-\frac{2|\Omega|}{N}$, whence

$$|\Omega| \gtrsim_M \sqrt{N}\log N \sim_M \log N|S|.$$

In particular, the $\log N$ appearing in Theorem 3.4.5 cannot be avoided.

We turn to the proof of Theorem 3.4.5. In the following, we denote the restricted Fourier transform by $\mathcal{F}_{S\to\Omega}$; that is, if $e : \ell^2(S) \to \ell^2(\mathbb{C}^N)$ is extension by zero, then $\mathcal{F}_{S\to\Omega} : \ell^2(S) \to \ell^2(\Omega)$ is given by

$$\mathcal{F}_{S\to\Omega} f = \mathcal{F}_N(ef)|_\Omega \quad \text{for all} \quad f \in \ell^2(S).$$

For complex vectors $f \in \mathbb{C}^N$ supported on a set S, we let the sign vector $\operatorname{sgn}(f)$ be defined by

$$\operatorname{sgn}(f)[n] = \frac{f[n]}{|f[n]|} \quad \text{for} \quad n \in S,$$

with $\operatorname{sgn}(f) = 0$ off S.

We will prove the following proposition, which we will then use in the proof of Theorem 3.4.5.

Proposition 3.4.8. *Let $\Omega \subset \{0, \dots, N-1\}$. Suppose $f \in \mathbb{C}^N$ and $S = \operatorname{supp}(f)$. Suppose that there exists $P \in \mathbb{C}^N$ such that*

- $\operatorname{supp} \hat{P} \subset \Omega$,
- $P[t] = \operatorname{sgn} f[t]$ *on* S,
- $|P[t]| < 1$ *for* $t \notin S$.

Then if $\mathcal{F}_{S\to\Omega}$ is injective, and the minimizer to (P_1) *is unique and equals f. (Conversely, if f is the unique minimizer of* (P_1) *then there exists P as above.)*

Proof of Proposition 3.4.8. Let us prove only the forward direction, which is most directly useful for us.

Suppose such a vector P exists. Suppose that g satisfies $\hat{g}|_\Omega = \hat{f}|_\Omega$. Set $h = g - f$. Then on S we have

$$|g| = |f+h| \geq |f| + \operatorname{Re}[h\overline{\operatorname{sgn}(f)}] = |f| + \operatorname{Re}[h\bar{P}].$$

To establish this inequality, write $f = |f|e^{i\theta}$ (i.e. $e^{i\theta} = \operatorname{sgn}(f)$) and $h = |h|e^{i\alpha}$; then the inequality is equivalent to

$$||f| + |h|\cos\beta + i|h|\sin\beta| \geq |f| + |h|\cos\beta, \quad \beta = \alpha - \theta.$$

Outside of S, we have

$$|g| = |h| \geq \operatorname{Re}[h\bar{P}],$$

since $|P| < 1$.

It follows that

$$\|g\|_{\ell^1} \geq \|f\|_{\ell^1} + \sum_{n=0}^{N-1} \operatorname{Re}\bigl(h[n]\bar{P}[n]\bigr).$$

Applying Plancherel, using the properties of P, and recalling $\hat{h} = 0$ on Ω, we deduce that

$$\sum_{n=0}^{N-1} \operatorname{Re}\bigl(h[n]\bar{P}[n]\bigr) = \tfrac{1}{N} \sum_{n=0}^{N-1} \operatorname{Re}\bigl(\hat{h}[n]\overline{\hat{P}}[n]\bigr) = 0.$$

It follows that $\|g\|_{\ell^1} \geq \|f\|_{\ell^1}$, so that f is a minimizer of (P_1).

Now suppose that $\|g\|_{\ell_1} = \|f\|_{\ell^1}$. Then (considering the argument above) we must have

$$|h[n]| = \operatorname{Re}(h[n]\bar{P}[n]) \quad \text{for} \quad n \notin S.$$

However, since $|P[n]| < 1$ for $n \notin S$, this implies $h \equiv 0$ off of S. By the assumption that $\mathcal{F}_{S\to\Omega}$ is injective, we also have that $h = 0$ on S. In particular, $f = g$, and hence f is the unique minimizer of (P_1). □

The strategy will now be to construct a suitable polynomial satisfying the first two properties in Proposition 3.4.8 and to show that it satisfies the desired upper bound with high probability. We would like to choose

$$P := \mathcal{F}^*_{\mathbb{C}^N\to\Omega}\mathcal{F}_{S\to\Omega}\bigl(\mathcal{F}^*_{S\to\Omega}\mathcal{F}_{S\to\Omega}\bigr)^{-1} e^* \operatorname{sgn}(f), \tag{3.6}$$

where the notation means the following. First, * denotes adjoint (i.e. conjugate transpose). In particular, recalling that e is extension by zero, we have that $e^* : \ell^2(\mathbb{C}^N) \to \ell^2(S)$ is simply restriction to S.

If we can define such P (given S and Ω), then P automatically has Fourier support in Ω. Furthermore, we claim that $e^*P = e^* \operatorname{sgn}(f)$. In fact, this follows from

$$e^*\mathcal{F}^*_{\mathbb{C}^N\to\Omega} = (\mathcal{F}_{\mathbb{C}^N\to\Omega}e)^* = \mathcal{F}^*_{S\to\Omega}.$$

Thus, fixing f and its support S, the proof of Theorem 3.4.5 boils down to proving that if Ω is chosen uniformly at random from sets of size $\gtrsim_M |S| \log N$, then

1. The operator $\mathcal{F}_{S\to\Omega}$ is injective with probability $1 - \mathcal{O}(N^{-M})$, and

2. The function P defined by (3.6) satisfies $|P| < 1$ off of S with probability $1 - \mathcal{O}(N^{-M})$.

Indeed, if item (1) is satisfied then $\mathcal{F}^*_{S\to\Omega}\mathcal{F}_{S\to\Omega}$ is necessarily invertible. (Both are equivalent to $\mathcal{F}_{S\to\Omega}$ having full column rank.)

It turns out to be simpler to prove these when one uses a different probabilistic model than simply selecting Ω uniformly at random. In particular, let us first consider the **Bernoulli** model. Given $0 < \tau < 1$, we create the random sequence

$$I_\omega = \begin{cases} 1 & \text{with probability} \quad \tau \\ 0 & \text{with probability} \quad 1-\tau. \end{cases} \tag{3.7}$$

We then can then define a random set of Fourier coefficients by $\Omega = \{\omega : I_\omega = 1\}$. The size $|\Omega|$ is random and follows a binomial distribution with $\mathbb{E}(|\Omega|) = \tau N$. In fact, when N is large, one has $|\Omega| \sim \tau N$ with high probability (by large deviation estimates).

We will prove the following two propositions.

Proposition 3.4.9 (Invertibility). *Let $S \subset \mathbb{C}^N$ and $M \geq 1$. Choose Ω according to the Bernoulli model with parameter τ. Suppose*

$$\tau N \gtrsim_M |S| \log N.$$

Then $\mathcal{F}^*_{S\to\Omega}\mathcal{F}_{S\to\Omega}$ is invertible with probability at least $1 - \mathcal{O}(N^{-M})$.

Proposition 3.4.10 (Bounds). *Under the assumptions of Proposition 3.4.9, the function P defined by* (3.6) *satisfies $|P| < 1$ off the set S with probability at least $1 - \mathcal{O}(N^{-M})$.*

Assuming these two results, let us complete the proof of Theorem 3.4.5.

Proof of Theorem 3.4.5, assuming Propositions 3.4.9 and 3.4.10. Let $F(\Omega)$ be the event that no polynomial P exists as in Proposition 3.4.8 if we choose the set Ω of Fourier coefficients.

Let Ω be of size N_ω drawn uniformly, and Ω' be drawn according to the Bernoulli model with $\tau = \frac{N_\omega}{N}$. Writing Ω_k for a set of frequencies chosen uniformly at random with $|\Omega_k| = k$, we have

$$\mathbb{P}(F(\Omega')) = \sum_{k=0}^{N} \mathbb{P}(F(\Omega') : |\Omega'| = k)\, \mathbb{P}(|\Omega'| = k) = \sum_{k=0}^{N} \mathbb{P}(F(\Omega_k))\, \mathbb{P}(|\Omega'| = k).$$

Now note that $\mathbb{P}(F(\Omega_k))$ is decreasing in k (it only gets easier to reconstruct using larger sets). We also claim that

$$\mathbb{P}(|\Omega'| \leq \tau N) > \tfrac{1}{2},$$

which follows from the fact that τN is an integer and hence the median of the random variable $|\Omega'|$. Thus

$$\mathbb{P}(F(\Omega')) \geq \sum_{k=1}^{N_\omega} \mathbb{P}(F(\Omega_k))\, \mathbb{P}(|\Omega'| = k) \geq \mathbb{P}(F(\Omega)) \sum_{k=1}^{N_\omega} \mathbb{P}(|\Omega'| = k) \geq \tfrac{1}{2}\, \mathbb{P}(F(\Omega)).$$

In particular, if we can bound the probability of failure for the Bernoulli model, then the probability of failure for the uniform model will be no more than twice as large. □

The key to proving both Proposition 3.4.9 and 3.4.10 is to establish certain probabilistic estimates for random matrices. From this point on, we assume the Bernoulli model holds and also assume $|\tau N| > M \log N$.

Define

$$Hf[t] = -\sum_{\omega \in \Omega} \sum_{S \ni s \neq t} e^{2\pi i \omega (t-s)/N} f[s] \tag{3.8}$$

and set $H_0 = e^* H$. Writing I_S for the identity operator on $\ell^2(S)$ (so that $e^* e = I_S$), we have

$$\begin{aligned} e - \tfrac{1}{|\Omega|} H &= \tfrac{1}{|\Omega|} \mathcal{F}^*_{\mathbb{C}^N \to \Omega} \mathcal{F}_{S \to \Omega}, \\ I_S - \tfrac{1}{|\Omega|} H_0 &= \tfrac{1}{|\Omega|} \mathcal{F}^*_{S \to \Omega} \mathcal{F}_{S \to \Omega}. \end{aligned}$$

In particular, introducing H_0 separates the diagonal term of $\mathcal{F}^*_{S\to\Omega}\mathcal{F}_{S\to\Omega}$ (which equals $|\Omega|$ identically) from the oscillatory off-diagonal. To define P as in (3.6), then we wish to have

$$P = (e - \tfrac{1}{|\Omega|} H)(I_S - \tfrac{1}{|\Omega|} H_0)^{-1} e^* \operatorname{sgn}(f). \tag{3.9}$$

To prove invertibility (cf. Proposition 3.4.9), we need to estimate the operator norm of H_0. We will prove the following below:

Lemma 3.4.11 (Moment bounds)**.** *Let* $\tau \leq (1+e)^{-1}$ *and* $n_0 = \frac{\tau N}{4|S|(1-\tau)}$. *If* $n \leq n_0$, *then*

$$\mathbb{E}\big\{\operatorname{tr}(H_0^{2n})\big\} \leq 2\big(\tfrac{4}{e(1-\tau)}\big)^n n^{n+1} |\tau N|^n |S|^{n+1}.$$

To prove the upper bound in Proposition 3.4.10, we will also need estimates on H. The crucial estimate will be the following:

Lemma 3.4.12 (Moment bounds, II)**.** *Let* $\tau \leq (1+e)^{-1}$ *and* $n_0 = \frac{\tau N}{4|S|(1-\tau)}$. *For* $n = km \leq n_0$,

$$\mathbb{E}\big\{|H_0^m \operatorname{sgn}(f)|^{2k}\big\} \leq 2\big(\tfrac{4}{e(1-\tau)}\big)^n n^{n+1} |\tau N|^n |S|^n$$

uniformly on S.

Assuming these moment bounds for now, let us complete the proof of Proposition 3.4.9 and (3.4.10). Then, finally, we will prove the moment bounds and thereby complete the proof of Theorem 3.4.5.

Proof of Proposition 3.4.9. We fix $M > 0$. We need to prove invertibility of the matrix

$$I_S - \tfrac{1}{|\Omega|} H_0,$$

where $H_0 = e^* H$ and H is as in (3.8). For this, we essentially need to show that we have the following bound on the operator norm:

$$\|H_0\| < c|\Omega|$$

for some $c < 1$ (see Exercise 3.5.5). Recall that $\mathbb{E}\{|\Omega|\} = \tau N$. We first deal with the probability that $|\Omega|$ is far from its expectation.

We will use a standard large deviation estimate, namely

$$\mathbb{P}\{|\Omega| < \mathbb{E}(|\Omega|) - t\} \leq \exp\{-\tfrac{t^2}{2\,\mathbb{E}(|\Omega|)}\}$$

for any $t > 0$. Applying this with the choice

$$t = \tau N \varepsilon_M, \quad \text{where} \quad \varepsilon_M := \sqrt{\tfrac{2M \log N}{\tau N}} \tag{3.10}$$

yields

$$\mathbb{P}\{B_M\} \leq N^{-M}, \quad \text{where} \quad B_M = \{|\Omega| < (1-\varepsilon_M)\tau N\}.$$

Next, let A_M denote the event $\{\|H_0\| \geq \frac{\tau N}{\sqrt{2}}\}$. We would like to bound $\mathbb{P}\{A_M\}$. For this we will rely on Lemma 3.4.11 and the fact that (since H_0 is self-adjoint)

$$\|H_0\|^{2n} = \|H_0^n\|^2 \leq \operatorname{tr}(H_0^{2n}).$$

See Exercise (3.5.6). In particular, for any $n \leq \frac{\tau N}{4|S|(1-\tau)}$, we use Tchebychev and Lemma 3.4.11 to estimate

$$\begin{aligned}
\mathbb{P}\{\|H_0\| \geq \tfrac{\tau N}{\sqrt{2}}\} &= \mathbb{P}\{\|H_0\|^{2n} \geq (\tfrac{\tau N}{\sqrt{2}})^{2n}\} \\
&\leq \mathbb{P}\{\mathrm{tr}(H_0^{2n}) \geq (\tfrac{\tau N}{\sqrt{2}})^{2n}\} \\
&\leq \tfrac{2^n}{(\tau N)^{2n}}\, \mathbb{E}\{\mathrm{tr}(H_0^{2n})\} \\
&\leq \tfrac{2^n}{(\tau N)^{2n}}\, 2(\tfrac{4}{e(1-\tau)})^n n^{n+1} (\tau N)^n |S|^{n+1}.
\end{aligned}$$

Recalling the assumed lower bound $\tau N \gtrsim_M |S| \log N$, we see that we may choose $n \sim_M \log N$, say $n = (M+1)\log N$. Choosing constants appropriately, the upper bound above is of the form

$$2n\big(\tfrac{8n|S|}{\tau N(1-\tau)}\big)^n |S| e^{-n} \lesssim |\tau N| N^{-(M+1)} \lesssim N^{-M}.$$

In conclusion, we have shown that on $A_M^c \cap B_M^c$, we have

$$\|H_0\| \leq \tfrac{\tau N}{\sqrt{2}} \leq \frac{|\Omega|}{\sqrt{2}(1-\varepsilon_M)} < c|\Omega|$$

for some uniform $0 < c < 1$. In fact, this holds with the Frobenius norm of H_0. This shows the desired invertibility with the desired probability and completes the proof. $\square$

So far we have established that we may define the function (3.6) with high probability. We turn to the proof of the upper bounds in Proposition 3.4.10, which will again rely on Lemma 3.4.11 and Lemma 3.4.12.

Proof of Proposition 3.4.10. We need to prove bounds for

$$P = (e - \tfrac{1}{|\Omega|} H)(I_S - \tfrac{1}{|\Omega|} H_0)^{-1} e^* \,\mathrm{sgn}(f).$$

on the complement of S. We begin by writing

$$\begin{aligned}
(I_S - \tfrac{1}{|\Omega|} H_0)^{-1} &= (I_S - \tfrac{1}{|\Omega|^n} H_0^n)^{-1} \left(\sum_{m=0}^{n-1} \tfrac{1}{|\Omega|^m} H_0^m \right) \\
&= \left(I_S + \sum_{p=1}^{\infty} \tfrac{1}{|\Omega|^{np}} H_0^{np} \right) \left(\sum_{m=0}^{n-1} \tfrac{1}{|\Omega|^m} H_0^m \right),
\end{aligned}$$

where we have used the identity $(1-M)^{-1} = (1-M^n)^{-1}(1+M+\cdots+M^{n-1})$. In the following, we denote

$$R = \sum_{p=1}^{\infty} \tfrac{1}{|\Omega|^{np}} H_0^{np}$$

and regard this as a remainder term. In fact, writing $\|H_0\|_F = \sqrt{\operatorname{tr} H_0 H_0^*}$ for the Frobenius norm, we have the following implication:

$$\|H_0\|_F \le \alpha|\Omega| \implies \|R\|_F \le \tfrac{\alpha^n}{1-\alpha^n}.$$

Using the Cauchy–Schwarz inequality, we can write

$$\|R\|_\infty \le |S|^{\frac{1}{2}}\|R\|_F,$$

where

$$\|R\|_\infty := \sup_i \sum_j |R[i,j]| = \sup_{\|x\|_{\ell^\infty}\le 1} \|Rx\|_{\ell^\infty}.$$

Indeed here S is the number of columns of R. In particular, we have the following implication:

$$\|H_0\|_F \le \alpha|\Omega| \implies \|R\|_\infty \le |S|^{\frac{1}{2}}\tfrac{\alpha^n}{1-\alpha^n}. \tag{3.11}$$

As we will see, this will deal with the contribution of R in the formulas above. Thus we will focus on proving estimates for the truncated series

$$\sum_{m=0}^{n-1} \tfrac{1}{|\Omega|^m} H_0^m.$$

We claim that we may write

$$P = P_0 + P_1 \quad \text{off of} \quad S,$$

where

$$P_0 = D_n \operatorname{sgn}(f), \quad P_1 = \tfrac{1}{|\Omega|} HRe^*(I + D_{n-1}) \operatorname{sgn}(f),$$
$$D_n = \sum_{m=1}^{n} \tfrac{1}{|\Omega|^m} H_0^m.$$

To this end, note that by (3.9) we have

$$P = -\tfrac{1}{|\Omega|} H (I_S - \tfrac{1}{|\Omega|} H_0)^{-1} e^* \operatorname{sgn}(f) \quad \text{off of} \quad S.$$

Continuing from above, some rearrangement shows that the claim boils down to the identity

$$e^* \sum_{m=0}^{n-1} |\Omega|^{-m} (He^*)^m \operatorname{sgn}(f) = \sum_{m=0}^{n-1} |\Omega|^{-m} (e^* H)^m \operatorname{sgn}(f).$$

We prove this by induction. In particular, if $m = 0$ then $e^* \operatorname{sgn}(f) = \operatorname{sgn}(f)$ since f is supported on S. Next, if $e^*(He^*)^m \operatorname{sgn}(f) = (e^*H)^m \operatorname{sgn}(f)$, then

$$(e^*H)^{m+1} \operatorname{sgn} f = e^*H[e^*(He^*)^m \operatorname{sgn} f] = e^*(He^*)^{m+1} \operatorname{sgn} f,$$

as desired.

Now, choosing any $a_0, a_1 > 0$ with $a_0 + a_1 = 1$, we begin with the bound

$$\mathbb{P}\big(\sup_{t \notin S} |P(t)| > 1\big) \leq \mathbb{P}(\|P_0\|_\infty > a_0) + \mathbb{P}(\|P_1\|_\infty > a_1).$$

Let us first focus on proving bounds for P_0; we will return to P_1 below. For the P_0 term, we will use the moment bounds in Lemma 3.4.12. Recalling the proof of Proposition 3.4.9, we have the set B_M on which $|\Omega| < (1 - \varepsilon_M)\tau N$, where ε_M is as in (3.10).

On the complement of B_M, we have

$$|P_0| \leq \sum_{m=1}^{n} Y_m, \quad Y_m = \tfrac{1}{(1-\varepsilon_M)^m (\tau N)^m} |H_0^m \operatorname{sgn}(f)|.$$

We suppose $n = 2^J - 1$ for some J and let β_j be positive numbers such that

$$\sum_{j=0}^{J-1} 2^j \beta_j \leq \alpha_0.$$

Then, by Tchebychev,

$$\mathbb{P}\big(\sum_{m=1}^{n} Y_m > a_0\big) \leq \sum_{j=0}^{J-1} \sum_{m=2^j}^{2^{j+1}-1} \mathbb{P}(Y_m > \beta_j) \leq \sum_{j=0}^{J-1} \sum_{m=2^j}^{2^{j+1}-1} \beta_j^{-2K_j}\, \mathbb{E}\{|Y_m|^{2K_j}\},$$

where $K_j := 2^{J-j}$. Now, for $2^j \leq m < 2^{j+1}$, we have $n \leq K_j m < 2n$. Thus, recalling $|S| \lesssim \tau N/n$ (with $n \sim_M \log N$), we can apply Lemma 3.4.12 to get a bound like

$$\mathbb{E}\{|Y_m|^{2K_j}\} \lesssim (1 - \varepsilon_M)^{-2n} n e^{-n} \alpha^n$$

for some $0 < \alpha < 1$. If we choose $\beta_j^{-K_j} \equiv \beta_0^{-n}$, then summing the above gives

$$\mathbb{P}\{|P_0(t)| > a_0\} \leq 2(1 - \varepsilon_M)^{-2n} n^2 e^{-n} \alpha^n \beta_0^{-2n}.$$

With $\beta_0 \sim .42$, one has $\sum_j 2^j \beta_j \leq .91$ and hence one can conclude

$$\mathbb{P}\{|P_0(t)| > a_0\} \leq \varepsilon_n := 2(1 - \varepsilon_M)^{-2n} n^2 e^{-n} \alpha^{2n} (.42)^{-2n},$$

where $a_0 \sim .91$. In particular, we have a set $A(t)$ with $\mathbb{P}\{A(t)\} > 1 - \varepsilon_n$ and $|P_0(t)| < .91$ on $A(t) \cap B_M^c$. As a consequence,

$$\mathbb{P}\big\{\sup_t |P_0(t)| > a_0\big\} \leq N^{-M} + N\varepsilon_n. \tag{3.12}$$

We now need to deal with P_1. For this, we observe

$$P_1 = \tfrac{1}{|\Omega|} HRe^*\big(\operatorname{sgn}(f) + Q_0\big), \quad Q_0 := D_{n-1}\operatorname{sgn}(f),$$

so that

$$\|P_1\|_\infty \leq \tfrac{1}{|\Omega|}\|HR\|_\infty\big(1 + \|Q_0\|_\infty\big).$$

Note the argument just given for P_0 applies equally well to Q_0.

Consider the event $E = \{\|H_0\|_F \leq \alpha|\Omega|\}$ for some $\alpha > 0$. As we saw in the proof of Proposition 3.4.9, the probability of E exceeds $1 - \mathcal{O}(N^{-M})$. Using the crude bound $\|H\|_\infty \leq |S||\Omega|$ (since H has $|S|$ columns and each entry is bounded by $|\Omega|$) and (3.11), we have

$$\tfrac{1}{|\Omega|}\|R\|_\infty\|H\|_\infty \leq |S|^{\frac{3}{2}}\tfrac{\alpha^n}{1-\alpha^n} \quad \text{on} \quad E.$$

Putting together the pieces and recalling the bound (3.12) (for Q_0), we see that

$$\|P_1\|_\infty \leq 2|S|^{\frac{3}{2}}\tfrac{\alpha^n}{1-\alpha^n} \leq a_1 \tag{3.13}$$

with probability $1 - \mathcal{O}(N^{-M})$, provided that the second inequality holds. Recall that $a_1 \sim .09$ is just a fixed constant here.

It remains to put together the pieces and complete the proof of Proposition 3.4.10. Recall that we choose $n \sim (M+1)\log N$ and that we have a free parameter α appearing in the definition of ε_n, which needs to be chosen small. The choice is ultimately dictated by the fact that we want the probability in (3.12) to be $\mathcal{O}(N^{-M})$. In particular, we should take

$$\alpha = .42(1 - \varepsilon_M).$$

It remains to check that the final inequality in (3.13) holds. Using the crude bound $|S| \leq N$, we need

$$2N^{\frac{3}{2}}\frac{[.42(1-\varepsilon_M)]^n}{1 - [.42(1-\varepsilon_M)]^n} \leq .09,$$

where we recall

$$\varepsilon_M = \sqrt{\tfrac{2M\log N}{\tau N}} \quad \text{and} \quad n \sim (M+1)\log N.$$

In particular, the inequality above holds for N, M reasonably large. This completes the proof. $\square$

Finally, we turn to the probabilistic estimates in Lemma 3.4.11 and Lemma 3.4.12. The proofs are similar, so let us focus on Lemma 3.4.11.

Proof of Lemma 3.4.11. Let us write the matrix elements of the $|S| \times |S|$ matrix H_0 as follows:

$$H_0(t,t') = \begin{cases} 0 & t = t', \\ c(t-t') & t \neq t', \end{cases} \quad \text{where} \quad c(u) := \sum_{\omega \in \Omega} e^{2\pi i \omega u/N}.$$

In particular, the diagonal entries of H_0^{2n} are given by

$$H_0^{2n}(t_1,t_1) = \sum_{t_2,\ldots,t_{2n}:t_j \neq t_{j+1}} c(t_1-t_2)\cdots c(t_{2n}-t_1),$$

where we write $t_{2n+1} = t_1$. It follows that

$$\begin{aligned}
&\mathbb{E}\{\mathrm{tr}(H_0^{2n})\} \\
&\quad = \sum_{t_1,\ldots,t_{2n}:t_j \neq t_{j+1}} \mathbb{E}\Big[\sum_{\omega_1,\ldots,\omega_{2n}\in\Omega} \exp\{\tfrac{2\pi i}{N}\sum_{j=1}^{2n}\omega_j(t_j - t_{j+1})\}\Big] \\
&\quad = \sum_{t_1,\ldots,t_{2n}:t_j \neq t_{j+1}} \sum_{0\le\omega_1,\ldots,\omega_{2n}\le N-1} \exp\{\tfrac{2\pi i}{N}\sum_{j=1}^{2n}\omega_j(t_j - t_{j+1})\}\,\mathbb{E}\Big\{\prod_{j=1}^{2n} I_{\omega_j}\Big\},
\end{aligned}$$

where I_{ω_j} is the random variable defined in (3.7) and we have used the linearity of expectation.

Now, for any $\underline{\omega} = \{\omega_1,\ldots,\omega_{2n}\}$, we may define an equivalence relation $R(\omega)$ on $A = \{1,\ldots,2n\}$ by imposing that

$$jR(\omega)k \quad \text{iff} \quad \omega_j = \omega_k.$$

We claim that the expectation above depends only on the equivalence class of $R(\omega)$, denoted $A/R(\omega)$. This is because the I_{ω_j} are independent and identically distributed. In particular,

$$\mathbb{E}\Big\{\prod_{j=1}^{2n} I_{\omega_j}\Big\} = \tau^{|A/R(\omega)|}.$$

(The number of elements in the equivalence class $A/R(\omega)$ tells you how many times you should really multiply the probability τ to compute the expected value.) With this in mind, we rewrite

$$\mathbb{E}\{\mathrm{tr}(H_0^{2n})\} = \sum_{t_1,\dots,t_{2n}:t_j\neq t_{j+1}} \sum_{R\in\mathcal{P}(A)} \tau^{|A/R|} \sum_{\underline{\omega}\in\Omega(R)} \exp\left\{\tfrac{2\pi i}{N}\sum_{j=1}^{2n}\omega_j(t_j - t_{j+1})\right\}, \tag{3.14}$$

where $\mathcal{P}(A)$ is the set of all equivalence relations on A and

$$\Omega(R) = \{\underline{\omega} : R(\omega) = R\} = \{\underline{\omega} : \omega_a = \omega_b \iff aRb\}.$$

In particular, the equation (3.14) implicitly contains sums defined by imposing $\omega_a \neq \omega_b$ for some a, b. The next step will be to rewrite the sums in a way that avoids such 'exclusions', so that we can end up writing sums as products that are easily understood. Here is the relevant identity:

Lemma 3.4.13 (Inclusion-exclusion formula)**.**

$$\sum_{\underline{\omega}\in\Omega(R)} f(\underline{\omega}) = \sum_{R'\leq R} (-1)^{|A/R|-|A/R'|} \prod_{A'\in A/R'} (|A'/R|-1)! \sum_{\underline{\omega}\in\Omega_{\leq}(R')} f(\underline{\omega}),$$

where

$$R' \leq R \quad \textit{if} \quad aRb \implies aR'b$$

and

$$\Omega_{\leq}(R) = \{\underline{\omega} : R(\omega) \leq R\} = \{\underline{\omega} : aRb \implies \omega_a = \omega_b\}.$$

The proof of this lemma is outlined in Exercise 3.5.7.

Let us continue from (3.14), writing the inner sum as

$$\sum_{R\in\mathcal{P}(A)} \tau^{|A/R|} \sum_{\underline{\omega}\in\Omega(R)} f(\underline{\omega}),$$

where

$$f(\underline{\omega}) = \exp\{\tfrac{2\pi i}{N}\sum_{j=1}^{2n}\omega_j(t_j - t_{j+1})\}.$$

We apply Lemma 3.4.13 to this expression. By rearranging the sums over $R \in \mathcal{P}(A)$ and over $R' \leq R$, we may rewrite the expression above as

$$\sum_{R'\in\mathcal{P}(A)} T(R') \sum_{\underline{\omega}\in\Omega_{\leq}(R')} f(\underline{\omega}),$$

where

$$T(R') := \sum_{R\geq R'} \tau^{|A/R|}(-1)^{|A/R|-|A/R'|} \prod_{A'\in A/R'} (|A'/R|-1)!.$$

We now claim that (by splitting A into equivalence classes of A/R' and further splitting the relations R'' on $A' \in A/R'$ by number of equivalence classes), we may rewrite this as

$$T(R') = \prod_{A' \in A/R'} \sum_{k=1}^{|A'|} S(|A'|, k)\tau^k (-1)^{|A'|-k}(k-1)!, \tag{3.15}$$

where S denotes the Stirling number of the second kind, i.e.

$$S(n,k) = \#\{R \in \mathcal{P}(A) : |A/R| = k\}, \quad \text{with} \quad \#A = n.$$

See Exercise 3.5.9. We denote the sum appearing above by $F_{|A'|}(\tau)$, i.e.

$$F_n(\tau) := \sum_{k=1}^{n} (k-1)! S(n,k)(-1)^{n-k}\tau^k. \tag{3.16}$$

Using this notation, let us continue from above to finally express the desired expected value in a way that is amenable to estimation. So far, we have arrived at

$$\mathbb{E}\{\mathrm{tr}(H_0^{2n})\} = \sum_{R \in \mathcal{P}(A)} \sum_{t_1,\ldots,t_{2n}: t_j \neq t_{j+1}} \prod_{A' \in A/R} F_{|A'|}(\tau) \sum_{\underline{\omega} \in \Omega_{\leq}(R)} f(\underline{\omega}),$$

with f as above. Now let us work on this final sum. Note that for any equivalence class $A' \in A/R$ and $\underline{\omega} \in \Omega_{\leq}(R)$, we have $\omega_a = \omega_b$ for any $a, b \in A'$. Denote this common value by $w_{A'}$. Denote also $t_{A'} = \sum_{a \in A'}(t_a - t_{a+1})$. Then we can write

$$\exp\{\tfrac{2\pi i}{N} \sum_j \omega_j (t_j - t_{j+1})\} = \prod_{A' \in A/R} \exp\{\tfrac{2\pi i}{N} \omega_{A'} t_{A'}\}.$$

Using this,

$$\sum_{\underline{\omega} \in \Omega_{\leq}(R)} f(\underline{\omega}) = \prod_{A' \in A/R} \sum_{\omega_{A'}} \exp\{\tfrac{2\pi i}{N} \omega_{A'} t_{A'}\},$$

where the sum is over all possible $\omega_{A'} \in \{0, \ldots, N-1\}$. But now we observe that the inner sum equals N when $t_{A'} = 0$ and equals zero otherwise. In conclusion:

$$\mathbb{E}\{\mathrm{tr}(H_0^{2n})\} = \sum_{R \in \mathcal{P}(A)} \sum_{\mathcal{T}(R)} N^{|A/R|} \prod_{A' \in A/R} F_{|A'|}(\tau), \tag{3.17}$$

where

$$\mathcal{T}(R) = \{t_1, \ldots, t_{2n} \quad \text{s.t.} \quad t_j \neq t_{j+1} \quad \text{and} \quad t_{A'} = 0 \quad \text{for all} \quad A' \in A/R\}.$$

This formula implies that we may disregard any R such that some equivalence class in A/R is a singleton. Indeed, if $A' \in A/R$ equals $\{j\}$, then $t_{A'} = t_j - t_{j+1} \neq 0$ (because of the constraint on the set $\mathcal{T}(R)$). Disregarding such relations, we can get the bound

$$\#\mathcal{T}(R) \leq |S|^{2n-|A/R|+1}.$$

This follows from the fact that there are $|A/R|$ many constraints on the t_j coming from the condition $t_{A'} = 0$, and one more coming from $\sum_{j=1}^{2n}(t_j - t_{j+1}) = 0$. Thus, continuing from (3.17),

$$\mathbb{E}\{\mathrm{tr}(H_0^{2n})\} \leq \sum_{k=1}^{n} N^k |S|^{2n-k+1} \sum_{R \in \mathcal{P}(A,k)} \prod_{A' \in A/R} F_{|A'|}(\tau), \tag{3.18}$$

where $\mathcal{P}(A,k)$ contains equivalence relations on A with k equivalence classes and no singleton classes. We will now estimate $F_n(\tau)$ and then the final inner sum and product.

We claim

$$F_n(\tau) \leq G(n) := \begin{cases} \frac{\tau}{1-\tau} & \log \frac{\tau}{1-\tau} \leq 1-n, \\ e^{(n-1)(\log(n-1) - \log\log \frac{1-\tau}{\tau} - 1)} & \log \frac{\tau}{1-\tau} > 1-n. \end{cases} \tag{3.19}$$

Sketch of proof. Recall the definition of F_n in (3.16). Now, note that rhe Stirling numbers satisfy the recurrence relation

$$S(n+1,k) = S(n,k-1) + kS(n,k).$$

Indeed, if $a \in A$ and $R \in \mathcal{P}(A)$ has k equivalence classes, then either a not equivalent to any other element of A (so R has $k-1$ equivalence classes on $A \backslash \{a\}$) or A is equal to one of the k equivalence classes of $A \backslash \{a\}$. Using this recurrence and induction, one can prove the identity

$$F_n(\tau) = \sum_{k=1}^{\infty} (-1)^{n+k} g(k), \quad \text{where} \quad g(x) = \frac{\tau^x x^{n-1}}{(1-\tau)^x} \tag{3.20}$$

for $0 \leq \tau \leq \frac{1}{2}$, say. We leave this as an exercise (see also [4]). Now g is increasing for $0 < x < x_*$ and decreasing for $x > x^*$, with

$$x^* = \frac{n-1}{\log(\frac{1-\tau}{\tau})}.$$

The different ranges of τ correspond to $x^* \leq 1$ or $x^* > 1$; in either case one gets the appropriate bound by looking at the alternating series. $\square$

Continuing from (3.18), we replace $F_{|A'|}(\tau)$ with $G(|A'|)$. We are then faced with estimating

$$Q(2n,k) := \sum_{R\in\mathcal{P}(A,k)} \prod_{A'\in A/R} G(|A'|).$$

We claim

$$Q(n,k) \le G(2)^k (2n)^{n-k}, \tag{3.21}$$

where we note that $G(2) = \frac{\tau}{1-\tau}$. This will complete the proof as follows. Noting

$$\tfrac{NG(2)}{4n|S|} \ge 1,$$

we can apply (3.21) to see

$$\begin{aligned}
\mathbb{E}\{\mathrm{tr}(H_0^{2n})\} &\le \sum_{k=1}^{n} N^k |S|^{2n-k+1} G(2)^k (4n)^{2n-k} \\
&\le |S|^{2n+1}(4n)^{2n} \sum_{k=1}^{n} (\tfrac{NG(2)}{4n|S|})^k \\
&\le n|S|^{2n+1}(4n)^{2n}(\tfrac{NG(2)}{4n|S|})^n \\
&\le n|S|^{n+1} N^n G(2)^n (4n)^n,
\end{aligned}$$

which is the desired estimate (recalling $G(2) = \frac{\tau}{1-\tau}$ and $\tau \le \frac{1}{1+e}$).[I]

It remains to verify (3.21), which we only sketch (and leave the details as an exercise). The key is to establish the recursive estimate

$$Q(n,k) \le (n-1)[Q(n-1,k) + G(2)Q(n-2,k-1)]$$

for $n \ge 3$, $k \ge 1$, for then (3.21) can be deduced by induction (in $n \ge 3$, for fixed k). For the recursive estimate, fix any $\alpha \in \{1,\dots,n\}$ and let $R \in \mathcal{P}(\{1,\dots,n\})$. Two situations are possible:

(i) $[\alpha]_R$ contains only one other element (note that there are $n-1$ choices). Removing $[\alpha]_R$ from the product gives the $(n-1)G(2)Q(n-2,k-1)$ term.

(ii) $[\alpha]_R$ has more than two elements, so that removing α from $\{1,\dots,n\}$ yields an equivalence class in $\mathcal{P}' = \mathcal{P}(\{1,\dots,n\}\backslash\{\alpha\},k)$. Now let $R' \in \mathcal{P}'$ and write $A_1,\dots,A_k$ for the corresponding classes. Then α is attached to

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[I] Actually, we will miss by the factor e^{-n} if n is not too large. However, in the application above, n was of size $\sim \log N$, which may be expected to be large. Recovering this factor in general requires an additional argument; see [4].

one of these classes A_i, and we claim that $G([\alpha]_R) \le |A_i| G(|A_i|)$. Indeed, this follows from $G(n+1) \le nG(n)$ (a consequence of log convexity of G). Thus the total contribution to $Q(n,k)$ is bounded by

$$\sum_{R' \in \mathcal{P}'} \sum_{i=1}^{k} |A_i| \prod_{A' \in \{1,\dots,n\} \setminus \{\alpha\}/R} G(|A'|).$$

As $\sum_{k=1}^{n} |A_i| = n-1$, the contribution becomes $(n-1)Q(n-1,k)$. This completes the proof. □

3.5 Exercises

Exercise 3.5.1. Prove that

$$\mathcal{F}^{-1}[\Pi_N] = N^{-1} \Pi_{N^{-1}}. \tag{3.22}$$

Hint. Use the Poisson summation formula to treat the case $N = 1$. Then compute the general case by scaling.

Exercise 3.5.2. Derive the formula appearing in Theorem 3.1.1 by following the scheme outlined in Remark 3.1.2.

Exercise 3.5.3. Prove the identities (3.2).

Exercise 3.5.4. Suppose $n \ge b > a \ge 0$ are integers with $n - b - a > 0$. Show that

$$\frac{\binom{n-a}{b}}{\binom{n}{b}} \ge (1 - \tfrac{2b}{n})^a.$$

(*Hint:* You can use induction on a.)

Exercise 3.5.5. Show that a matrix $I - A$ is invertible if $\|A\| < \frac{1}{2}$, where $\|\cdot\|$ denotes operator norm.

Exercise 3.5.6. Let $H \in \mathbb{C}^{N \times N}$ and let H^* denote the adjoint (i.e. conjugate transpose) of H. Let $\|H\|$ denote the operator norm of H. Show that $\|H\|$ equals the largest (in magnitude) eigenvalue of H.

Show also that

$$\|H\| = \|H^*\| = \sqrt{\|HH^*\|}.$$

and that

$$\|H\| \le \|H\|_F := \sqrt{\mathrm{tr}(HH^*)}.$$

Exercise 3.5.7. Prove exercise Lemma 3.4.13 by completing the following argument. The details are found in Section IV B of [4].

One can pass from A to A/R to assume that R is simply equality $=$. After relabeling A as $A = \{1, \ldots, n\}$, the formula reduces to

$$\sum_{\omega_1,\ldots\omega_n \text{ distinct}} f(\underline{\omega}) = \sum_R (-1)^{n-|A/R|} \prod_{A' \in A/R} (|A|-1)! \sum_{\underline{\omega} \in \Omega_{\leq}(R)} f(\underline{\omega}), \quad (3.23)$$

where the sum is over all equivalence relations R on A. This formula may be proved by induction. The base case $n = 1$ follows because both sides equal $\sum f(\underline{\omega})$. Suppose the formula has been proven up to level $n-1$. Then rewrite the left-hand side as

$$\sum_{\omega_1,\ldots,\omega_{n-1} \text{ distinct}} \left[\sum_{\omega_n} f(\underline{\omega}', \omega_n) - \sum_{j=1}^{n-1} f(\underline{\omega}', \omega_j) \right], \quad (3.24)$$

and apply the inductive hypothesis to get a new expression for the left-hand side. Here $\underline{\omega}' = (\omega_1, \ldots, \omega_{n-1})$.

Now work on the right-hand side of the formula above that will eventually lead to the same formula just derived. Note that any equivalence class R on A can be restricted to an equivalence class R_0 on $A_0 = \{1, \ldots, n-1\}$. Then R can be formed from R_0 either by adding $\{n\}$ as a new equivalence class (in which case we write $R = \{R_0, \{n\}\}$), or by having nRj for some $j \in A_0$, in which case we write $R = \{R_0, \{n\}\}/(n = j)$. In the latter case, there may be multiple ways to recover R; in particular there are $|[j]_{R_0}|$ ways, where $[\cdot]$ denotes equivalence class. It follows that for any function F defined on equivalence classes R of A, we can write

$$\sum_R F(R) = \sum_{R_0} F(\{R_0, \{n\}\}) + \sum_{R_0} \sum_{j=1}^{n-1} \tfrac{1}{|[j]_{R_0}|} F(\{R_0, \{n\}/(n = j)\}).$$

Apply this identity to the right-hand side of (3.23). This produces two terms that can be shown to match the two terms arising from (3.24). To make the second terms match, one must utilize the identity

$$\tfrac{1}{|[j]_{R_0}|} \prod_{A' \in A/(\{R_0, \{n\}\}/(n=j))} (|A'|-1)! = \prod_{A' \in \{1,\ldots,n-1\}/R_0} (|A'|-1)!.$$

Using the above as a guide, complete the proof.

Exercise 3.5.8. Prove (3.20).

Exercise 3.5.9. Prove equation (3.15).

Chapter 4

Abstract Fourier analysis

In this section, we will take a tour through some topics in abstract harmonic analysis. Our goal will not be to present a thorough theoretical presentation, but rather to show how many of the preceding topics can be understood as special cases of a more general theory. In particular, many preliminary results will simply be quoted as needed; the interested reader is encouraged to pick up [9] to find complete details. We will also explore related topics in some new settings (e.g. in the setting of compact Lie groups).

4.1 Preliminaries

Definition 4.1.1. A **topological group** is a group G with a topology such that the group operation and inverse operation are continuous (from $G \times G \to G$ and from $G \to G$, respectively).

We will restrict our attention to groups whose topology is **Hausdorff** (i.e. around any two distinct points one can find disjoint neighborhoods). We will typically consider either compact or locally compact groups. Here **locally compact** means that every point has a compact neighborhood.

Definition 4.1.2. A **left Haar measure** on G is a nonzero Radon measure μ (i.e. a Borel measure, finite on compact sets, outer regular on Borel sets, and inner regular on open sets) on G that satisfies $\mu(xE) = \mu(E)$ for all Borel sets $E \subset G$ and every $x \in G$. A **right Haar measure** instead satisfies $\mu(Ex) = \mu(E)$.

Example 4.1.1. If $G = \mathbb{R}\backslash\{0\}$ (with multiplication), then $\frac{dx}{|x|}$ is a Haar measure on G.

Example 4.1.2. If $G = GL(n, \mathbb{R})$ (the group of invertible $n \times n$ matrices), then $|\det T|^{-n} dT$ is a left and right Haar measure on G (where dT is Lebesgue measure on the space of $n \times n$ matrices).

Example 4.1.3. If G is the $ax+b$ group of all affine transformations $x \mapsto ax+b$ on $\mathbb{R}$ (with $a > 0$ and $b \in \mathbb{R}$), then $a^{-2}\, da\, db$ is a left Haar measure and $a^{-1}\, da\, db$ is a right Haar measure on G. This measure will appear in the setting of wavelets.

The basic facts we need about Haar measure are the following:

- Every locally compact group possesses a left Haar measure ([9, Theorem 2.10]). Left Haar measure is unique up to a multiplicative constant ([9, Theorem 2.220]).

- If λ is a left Haar measure and $x \in E$, then $\lambda_x(E) := \lambda(Ex)$ is again a left Haar measure. Thus there exists $\Delta(x)$ so that $\lambda_x = \Delta(x)\lambda$. This defines a function (the **modular function**) $\Delta : G \to (0, \infty)$.

We need some facts about Banach algebras as well.

Definition 4.1.3. A **Banach algebra** refers to a Banach space with a product $*$ such that $\|x * y\| \leq \|x\|\, \|y\|$. An involution is a map $x \mapsto x^*$ such that

$$(x+y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda} x^*, \quad (xy)^* = y^* x^*, \quad (x^*)^* = x.$$

A Banach algebra equipped with an involution is called a $*$-algebra. An algebra is called **unital** if it contains a unit element. If A and B are $*$-algebras, a **$*$-homomorphism** from A to B is a homomorphism ϕ such that $\phi(x^*) = \phi(x)^*$.

Example 4.1.4. If H is a Hilbert space, then $L(H)$ (the space of bounded operators on H) is a unital Banach algebra using the operator norm and composition of operators. The involution is given by $T \mapsto T^*$ (the adjoint of T). In fact, this makes $L(H)$ a C^* **algebra**, which means $\|x^* x\| = \|x\|^2$ for all x.

We need a few facts about Banach algebras as well.

- The **spectrum** of a commutative Banach algebra is the set of all nonzero homomorphisms from the algebra to $\mathbb{C}$. For unital Banach algebras, the spectrum is compact (in the weak star topology).

- If G is a locally compact group, then the space $L^1(G)$ forms a Banach $*$ algebra under the product given by convolution, defined by

$$f * g(x) = \int f(y)g(y^{-1}x)\,dy,$$

where dy denotes Haar measure, and involution given by $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$, where Δ is the modular function.

Fourier analysis on groups is closely connected to the topic of **representation theory**.

We first define the notion of $*$-representation.

Definition 4.1.4. A $*$-representation of an algebra A on a Hilbert space H is a $*$-homomorphism ϕ from A to $L(H)$ (the space of bounded operators on H). We call ϕ **nondegenerate** if there is no nonzero $v \in H$ such that $\phi(x)v = 0$ for all $x \in A$.

Next, we have the notion of a unitary representation of a group.

Definition 4.1.5. A **unitary representation** of a group G is a homomorphism π from G into the group $U(H_\pi)$ of unitary operators on some nonzero Hilbert space H_π that is continuous with respect to the strong operator topology. The dimension of the representation space H_π is called the degree of π.

The definition above means $\pi(xy) = \pi(x)\pi(y)$ with $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, with

$$x \mapsto \pi(x)u$$

continuous from G to H_π for any $u \in H_\pi$.

If π_1 and π_2 are unitary representations of the same group G, we define an **intertwining operator** for π_1 and π_2 to a bounded linear map $T : H_{\pi_1} \to H_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. The set of such operators is denoted by $C(\pi_1, \pi_2)$. We call π_1 and π_2 **unitarily equivalent** if $C(\pi_1, \pi_2)$ contains a unitary operator U, so that $\pi_2(x) = U\pi_1(x)U^{-1}$.

Group representations essentially allow group elements to be represented by matrices, with the group operation replaced by matrix multiplication. This has applications in algebraic problems, as it can reduce questions about group theory to problems in linear algebra. Group representations are also widely found in modern physics, where the groups in question are typically symmetry groups for some physical model.

The two notions of representation above are related, in the sense that any unitary representation π of G corresponds to a $*$-representation of $L^1(G)$, which we may still denote by π. In particular, for $f \in L^1(G)$ we define the bounded operator $\pi(f) \in L(H_\pi)$ by

$$\pi(f) = \int_G f(x)\pi(x)\,dx,$$

where dx denotes Haar measure. We interpret this operator in the weak sense, namely

$$\langle \pi(f)u, v\rangle_{H_\pi} = \int_G f(x)\langle \pi(x)u, v\rangle_{H_\pi}\,dx.$$

A closed subspace M of H_π is called **invariant** for the representation π if $\pi(x)M \subset M$ for all $x \in G$. If π admits a nontrivial invariant subspace, then π is called **reducible**. Otherwise, π is **irreducible**.

If G is an abelian group (i.e. $xy = yx$ for all $x, y \in G$), then every irreducible representation of G is one-dimensional (see [9, Corollary 3.6]). This still leaves the question of the existence of such representations (besides the trivial representation $\pi_0(x) \equiv Id$). For this, we quote the following theorem, known as the Gelfand–Raikov theorem (see [9, Theorem 3.34]).

Theorem 4.1.6 (Gelfand–Raikov)**.** *Let G be a locally compact group. Then for any distinct $x, y \in G$, there exists an irreducible representation π such that $\pi(x) \neq \pi(y)$.*

4.2 Locally compact abelian groups

Let G be a locally compact abelian group and suppose π is an irreducible representation of G. In particular, π is one-dimensional and hence we may take $H_\pi = \mathbb{C}$. In this case we may write

$$\pi(x)z = \langle x, \xi\rangle z, \quad z \in \mathbb{C},$$

where $x \mapsto \langle x, \xi\rangle$ denotes a continuous homomorphism from G into $\mathbb{T}$ (the circle group). We call ξ a **character** of G, and denote the set of all characters by $\hat{G}$, which we call the **dual group** of G. Indeed, $\hat{G}$ forms an abelian group. The group operation is given by

$$\langle x, \xi_1\xi_2\rangle := \langle x, \xi_1\rangle\langle x, \xi_2\rangle \quad (x \in G, \xi_j \in \hat{G}),$$

with

$$\langle x, \xi^{-1} \rangle = \langle x^{-1}, \xi \rangle = \overline{\langle x, \xi \rangle}.$$

We give $\hat{G}$ the weak* topology (inherited as a subset of $L^\infty(G)$). It turns out (cf. [9, Theorem 3.31]) that this coincides with the topology of 'compact convergence' on G, under which the group operations are continuous. This also guarantees (via [9, Proposition 1.10(c)] and Alaoglu's theorem) that $\hat{G}$ is locally compact.

In fact, $\hat{G}$ can be identified with the spectrum of $L^1(G)$. Indeed, ξ gives a nondegenerate $*$-representation of $L^1(G)$ on $\mathbb{C}$ via

$$\xi(f) = \int_G \overline{\langle x, \xi \rangle} f(x)\, dx. \tag{4.1}$$

Proposition 4.2.1. *If G is compact and its Haar measure is normalized so that $|G| = 1$, then $\hat{G}$ is an orthonormal set in $L^2(G)$.*

Proof. Let $\xi \in \hat{G}$. Then $|\xi|^2 \equiv 1$. As $|G| = 1$, this yields

$$\|\xi\|_{L^2(G)} = 1.$$

Now, if $\xi \neq \eta$, then there exists $x_0 \in G$ so that $\langle x_0, \xi\eta^{-1} \rangle \neq 1$. Writing dx for Haar measure, we then have

$$\begin{aligned}
\int \langle x, \xi \rangle \overline{\langle x, \eta \rangle}\, dx &= \int \langle x, \xi\eta^{-1} \rangle\, dx \\
&= \langle x_0, \xi\eta^{-1} \rangle \int \langle x_0^{-1} x, \xi\eta^{-1} \rangle\, dx \\
&= \langle x_0, \xi\eta^{-1} \rangle \int \langle x, \xi\eta^{-1} \rangle\, dx,
\end{aligned}$$

where we have used the translation invariance of Haar measure. This implies

$$\int \langle x, \xi \rangle \overline{\langle x, \eta \rangle}\, dx = 0,$$

as desired. □

We will next prove the following result.

Proposition 4.2.2. *If G is discrete then $\hat{G}$ is compact. If G is compact then $\hat{G}$ is discrete.*

Proof. If G is discrete, then $L^1(G)$ has a unit element. Indeed, we take $\delta(1) = 1$ and $\delta = 0$ otherwise. Thus the spectrum of $L^1(G)$ (which is identified with $\hat{G}$) is compact.

Next, suppose G is compact. Using Proposition 4.2.1, we observe that

$$\int \xi = \begin{cases} 1 & \xi = 1 \\ 0 & \xi \neq 1. \end{cases}$$

In particular,

$$\{f \in L^\infty(G) : \left|\int f\right| > \tfrac{1}{2}\} \cap \hat{G} = \{1\}.$$

We claim that this implies that $\{1\}$ is open in $\hat{G}$. Indeed, because G is compact, we have that the constant function 1 is in $L^1(G)$ (so that $\int f$ may be viewed as $1(f)$ through the identification of $(L^1)^*$ with L^∞). This in turn implies that every singleton set in $\hat{G}$ is open, i.e. $\hat{G}$ is discrete. □

The next result puts the Fourier transform, Fourier series, and the discrete Fourier transform under the same umbrella.

Theorem 4.2.3. *We have the following:*

- $\hat{\mathbb{R}} = \mathbb{R}$ *with* $\langle x, \xi \rangle = e^{2\pi i x \xi}$.
- $\hat{\mathbb{T}} = \mathbb{Z}$ *with* $\langle \alpha, n \rangle = \alpha^n$.
- $\hat{\mathbb{Z}} = \mathbb{T}$ *with* $\langle n, \alpha \rangle = \alpha^n$.
- *If* Z_k *is the additive group of integers modulo* k*, then* $\hat{Z}_k = Z_k$ *with* $\langle m, n \rangle = e^{2\pi i m n/k}$.

Remark 4.2.4. If we write $\alpha \in \mathbb{T}$ as $\alpha = e^{2\pi i x}$ for some $x \in [-1, 1]$ then we recover the familiar pairing $\langle \alpha, n \rangle = e^{2\pi i x n}$.

Proof. If $\phi \in \hat{\mathbb{R}}$, then $\phi(0) = 1$ (these represent the identity elements in $\mathbb{R}$ and $\mathbb{T}$, respectively). Thus there exists $a > 0$ so that

$$A := \int_0^a \phi(t)\, dt \neq 0.$$

Now (as ϕ is a homomorphism)

$$A\phi(x) = \int_0^a \phi(x + t)\, dt = \int_x^{a+x} \phi(t)\, dt,$$

which implies (by the fundamental theorem of calculus) that ϕ is differentiable, with

$$\phi'(x) = A^{-1}[\phi(a+x) - \phi(x)] = c\phi(x), \quad c := A^{-1}[\phi(a) - 1].$$

Thus $\phi(t) = e^{ct}$; however, since $|\phi| = 1$ we may write $c = 2\pi i\xi$ for some $\xi \in \mathbb{R}$.

Next, since $\mathbb{T}$ can be identified with $\mathbb{R}/\mathbb{Z}$ (via the identification of $x \in \mathbb{R}/\mathbb{Z}$ with $\alpha = e^{2\pi i x}$), the characters of $\mathbb{T}$ are the characters of $\mathbb{R}$ that are trivial on $\mathbb{Z}$, so the result follows from above.

Now if $\phi \in \hat{\mathbb{Z}}$ then $\alpha := \phi(1) \in \mathbb{T}$ and $\phi(n) = [\phi(1)]^n = \alpha^n$ (by the homomorphism property).

Finally, the characters of Z_k are the characters of $\mathbb{Z}$ that are trivial on $k\mathbb{Z}$. Thus they are of the form $\phi(n) = \alpha^n$ where α is a k^{th} root of unity. $\square$

One can also check that if $G_1, \ldots, G_n$ are locally compact Abelian groups, then

$$(G_1 \times \cdots \times G_n)\hat{} = \hat{G}_1 \times \cdots \times \hat{G}_n.$$

This allows us to extend the previous result to see $\hat{\mathbb{R}}^n = \mathbb{R}^n$, $\hat{\mathbb{T}}^n = \mathbb{Z}^n$, $\hat{\mathbb{Z}}^n = \mathbb{T}^n$, and finally $\hat{G} = G$ for any finite Abelian group G.

To define the **Fourier transform** on G, we make use of (4.1). In particular, the Fourier transform is the map from $L^1(G)$ to $C(\hat{G})$ given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_G \overline{\langle x, \xi\rangle} f(x)\, dx.$$

Even in this generality, the Fourier transform enjoys many of the familiar properties that we are used to. For example, it defines a norm-decreasing $*$-homomorphism from $L^1(G)$ to $C_0(\hat{G})$. It can also be extended to complex Radon measures on G via

$$\hat{\mu}(\xi) = \int_G \overline{\langle x, \xi\rangle} d\mu(x).$$

This defines a bounded continuous function on $\hat{G}$. The reverse works as well: if μ is a complex Radon measure on $\hat{G}$, then

$$\phi_\mu(x) = \int_{\hat{G}} \langle x, \xi\rangle d\mu(\xi)$$

defines a bounded continuous function on G; furthermore, the mapping $\mu \mapsto \phi_\mu$ is linear and injective. A fundamental result in the theory (known as

Bochner's theorem, cf. [9, Theorem 4.19]) states that if ϕ is a continuous function of positive type on G (i.e. $\int_G [f^* * f]\phi \geq 0$ for all $f \in L^1(G)$), then there exists a unique positive measure μ on $\hat{G}$ such that $\phi = \phi_\mu$.

One also has suitable notions of **Fourier inversion formulas** in this generality. For example, if $f = \phi_\mu$ for some complex Radon measure μ on $\hat{G}$ and additionally $f \in L^1(G)$, then $\hat{f} \in L^1(\hat{G})$ and

$$f(x) = \int \langle x, \xi \rangle \hat{f}(\xi)\, d\xi$$

provided Haar measure $d\xi$ on $\hat{G}$ is suitably normalized relative to the given Haar measure on G. We can also write $d\mu_f(\xi) = \hat{f}(\xi)\, d\xi$. One calls $d\xi$ the **dual measure** of the given Haar measure on G.

Example 4.2.1. If we identify $\hat{\mathbb{R}}$ with $\mathbb{R}$ via $\langle x, \xi \rangle = e^{2\pi i \xi x}$, then Lebesgue measure is its own dual. Indeed, the inversion formula holds with both dx and $d\xi$ given by Lebesgue measure. If we instead identify $\hat{\mathbb{R}}$ with $\mathbb{R}$ with $\langle x, \xi \rangle = e^{i\xi x}$, then the dual of dx is $\frac{1}{2\pi}\, d\xi$. If we use the Haar measure $\frac{1}{\sqrt{2\pi}}\, dx$, then the measure is again its own dual.

We also have the following result related to Proposition 4.2.2.

Proposition 4.2.5. *If G is compact and Haar measure is chosen so that $|G| = 1$, then the dual measure on $\hat{G}$ is counting measure. If G is discrete and Haar measure is taken to be counting measure, then the dual measure on $\hat{G}$ satisfies $|\hat{G}| = 1$.*

Proof. Suppose G is compact. Let $g \equiv 1$. Then (using Proposition 4.2.1) we have $\hat{g} = \chi_{\{1\}}$. It follows that

$$g(x) = \sum_{\xi \in \hat{G}} \langle x, \xi \rangle \hat{g}(\xi).$$

where we have used that $\langle x, 1 \rangle \equiv 1$. This shows that the dual measure on $\hat{G}$ must be counting measure.

On the other hand, if G is discrete then we let $g = \chi_{\{1\}}$. Then $\hat{g} \equiv 1$ and

$$g(x) = \int_{\hat{G}} \langle x, \xi \rangle\, d\xi,$$

provided $d\xi$ is chosen so that $|\hat{G}| = 1$. Here we are using Proposition 4.2.1 again, together with the fact that $\xi \mapsto \langle x, \xi \rangle$ is a character on $\hat{G}$ for each $x \in G$. □

Example 4.2.2. The groups $\mathbb{T}$ and $\mathbb{Z}$ are dual. The dual measures can be taken to be normalized Lebesgue measure $\frac{d\theta}{2\pi}$ and counting measure. Then Fourier inversion becomes

$$\hat{f}(n) = \int_0^{2\pi} f(\theta)e^{-in\theta}\,\tfrac{d\theta}{2\pi}, \quad f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}.$$

Example 4.2.3. If $G = Z_k$ then the dual of counting measure is counting measure divided by k (so that $|Z_k| = 1$). Fourier inversion reads

$$\hat{f}(m) = \sum_{n=0}^{k} f(n)e^{-2\pi imn/k}, \quad f(n) = \tfrac{1}{k}\sum_{m=0}^{k} \hat{f}(m)e^{2\pi imn/k}.$$

The general form of the Plancherel theorem is given by the following (see [9, Theorem 4.26]).

Theorem 4.2.6 (Plancherel)**.** *The Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$. Consequently, if G is compact and $|G| = 1$, then $\hat{G}$ is an orthonormal basis for $L^2(G)$.*

We turn to our final main result concerning Fourier analysis on locally compact Abelian groups, namely, the Pontrjagin duality theorem. Recall that by definition, elements of $\hat{G}$ are characters on G. We can also view elements of G as characters on $\hat{G}$. Indeed, for $x \in G$ we can define a character $\Phi(x)$ on $\hat{G}$ via

$$\langle \xi, \Phi(x)\rangle = \langle x, \xi\rangle.$$

It follows that Φ defines a group homomorphism from G to $\hat{\hat{G}}$. The Pontrjagin duality theorem (see [9, Theorem 4.32]) states the following:

Theorem 4.2.7 (Pontrjagin duality)**.** *If G is a locally compact Abelian group, then Φ is an isomorphism of topological groups.*

According to this theorem, we may freely write $\langle x, \xi\rangle$ or $\langle \xi, x\rangle$ for the pairing between G and $\hat{G}$.

One consequence of this theorem is the other form of Fourier inversion: if $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$, then

$$f(x) = \int_{\hat{G}} \langle x, \xi\rangle \hat{f}(\xi)\, d\xi$$

almost everywhere. We also have the dual form of Proposition 4.2.2; that is, if $\hat{G}$ is compact then G is discrete, and if $\hat{G}$ is discrete then G is compact.

Fourier analysis in the setting of groups can be applied to express an arbitrary unitary representation of a locally compact Abelian group in terms of irreducible representations (i.e. characters). We close this section by stating the following result (see [9, Theorem 4.45]).

Theorem 4.2.8. *Suppose π is a unitary representation of a locally compact Abelian group G. There exists a unique regular H_π-projection-valued measure P on $\hat{G}$ so that*

$$\pi(x) = \int_{\hat{G}} \langle x, \xi \rangle dP(\xi) \quad \text{for} \quad x \in G,$$
$$\pi(f) = \int_{\hat{G}} \xi(f) dP(\xi) \quad \text{for} \quad f \in L^1(G),$$

where $\xi(f) = \int_G \langle x, \xi \rangle f(x)\, dx$.

4.3 Compact groups

We next discuss some of the basic results of representation theory and Fourier analysis for compact (but not necessarily Abelian) groups. We will focus on introducing the relevant terms and stating the necessary results; we will then work through some specific examples.

We begin with the following (see [9, Theorem 5.2]).

Theorem 4.3.1. *If G is compact, then any irreducible representation of G is finite-dimensional. Every unitary representation of G is a direct sum of irreducible representations.*

When G is an abelian group, we saw that $\hat{G}$ is a set of continuous functions on G. The general definition of $\hat{G}$ is the set of unitary equivalence classes of irreducible representations of G. In the abelian case, we considered characters of G. In the general case the corresponding set of functions is the set of matrix elements of the irreducible representations of G.

Definition 4.3.2. Suppose π is a unitary representation of G. The functions

$$\phi_{u,v}(x) = \langle \pi(x)u, v \rangle_{H_\pi}$$

for $u, v \in H_\pi$ are called the **matrix elements** of π.

Note that if u, v belong to an orthonormal basis $\{e_j\}$, then $\phi_{u,v}(x)$ is one of the entries of the matrix for $\pi(x)$ in that basis, cf.

$$\pi_{ij}(x) = \langle \pi(x)e_j, e_i \rangle.$$

We denote the span of the matrix elements of π by $\mathcal{E}_\pi$. This defines a subspace of $C(G)$ and depends only on the unitary equivalence class of π.

The matrix elements of irreducible representions can be used to build an orthonormal basis for $L^2(G)$. This relies on two main results.

First, we have the Schur orthogonality relations (see [9, Theorem 5.8]):

Theorem 4.3.3. *Let π, π' be irreducible unitary representations of G. Consider $\mathcal{E}_\pi, \mathcal{E}_{\pi'}$ as subspaces of $L^2(G)$.*

- *If $[\pi] \neq [\pi']$ then $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$.*
- *If $\{e_j\}$ is an orthonormal basis for H_π and π_{ij} is defined as above, then*
$$\{\sqrt{\dim H_\pi}\pi_{ij} : i, j = 1, \ldots, \dim H_\pi\}$$
is an orthonormal basis for $\mathcal{E}_\pi$.

The next result we need is the following theorem (see [9, Theorem 5.11]).

Theorem 4.3.4. *Let $\mathcal{E}$ denote the linear span of*
$$\bigcup_{[\pi]\in\hat{G}} \mathcal{E}_\pi.$$
Then $\mathcal{E}$ is dense in $C(G)$ in the uniform norm and in $L^p(G)$ for all $p < \infty$.

We now state the main result (called the Peter–Weyl theorem [9, Theorem 5.12]). In the following, given an equivalence class $[\pi]$ we assume we have chosen one fixed representative π.

Theorem 4.3.5 (Peter–Weyl theorem)**.** *Let G be a compact group. Then*
$$L^2(G) = \bigoplus_{[\pi]\in\hat{G}} \mathcal{E}_\pi,$$
and if
$$\pi_{ij}(x) = \langle \pi(x)e_j, e_i\rangle,$$
then the set
$$\left\{\sqrt{\dim H_\pi}\pi_{ij} : i, j = 1, \ldots \dim H_\pi,\ [\pi] \in \hat{G}\right\}$$
is an orthonormal basis for $L^2(G)$.

This is the starting point for Fourier analysis on compact groups. In particular, for $f \in L^2(G)$ we get the representation

$$f = \sum_{[\pi]\in\hat{G}} \sum_{i,j=1}^{\dim H_\pi} c^\pi_{i,j}\pi_{ij}, \quad c^\pi_{ij} = \dim H_\pi \int_G f(x)\overline{\pi_{ij}(x)}\, dx.$$

As stated, this requires that we choose an orthonormal basis for each H_π. Alternately, we can define the **Fourier transform** of $f \in L^1(G)$ at π to be the operator $\hat{f} : H_\pi \to H_\pi$ given by

$$\hat{f}(\pi) = \int f(x)\pi(x^{-1})\, dx = \int f(x)\pi(x)^*\, dx$$

In particular, given an orthonormal basis for H_π, then $\hat{f}(\pi)$ is represented by the matrix

$$\hat{f}(\pi)_{ij} = \int f(x)\overline{\pi_{ji}(x)}\, dx = \tfrac{1}{\dim H_\pi} c^\pi_{ji},$$

where the coefficients are as above. In this case, we get

$$\sum_{i,j} c^\pi_{ij}\pi_{ij}(x) = \dim H_\pi \sum_{i,j} \hat{f}(\pi)_{ji}\pi_{ij}(x) = \dim H_\pi \operatorname{tr}\big[\hat{f}(\pi)\pi(x)\big].$$

Thus we arrive at the Fourier inversion formula

$$f(x) = \sum_{[\pi]\in\hat{G}} \dim H_\pi \operatorname{tr}\big[\hat{f}(\pi)\pi(x)\big],$$

where convergence should be understood in the L^2 sense. The Parseval formula now reads

$$\|f\|^2_{L^2(G)} = \sum_{[\pi]\in\hat{G}} \dim H_\pi \operatorname{tr}[\hat{f}(\pi)^*\hat{f}(\pi)].$$

We turn to one more formulation. If π is a finite-dimensional unitary representation of G, we define the **character** χ_π of π by the function

$$\chi_\pi(x) = \operatorname{tr}\pi(x).$$

In fact, this depends only on the equivalence class of π. A direct computation shows

$$\operatorname{tr}[\hat{f}(\pi)\pi(x)] = \textstyle\int f(y)\operatorname{tr}[\pi(y^{-1})\pi(x)]\, dy = \int f(y)\operatorname{tr}\pi(y^{-1}x)\, dy = f * \chi_\pi(x),$$

so that the Fourier inversion formula may be written

$$f = \sum_{[\pi]\in\hat{G}} \dim H_\pi \, f * \chi_\pi.$$

In particular, $\dim H_\pi f * \chi_\pi$ is the orthogonal projection of f onto $\mathcal{E}_\pi$.

We introduce one final notion before working through some examples. A function f on G is called **central** if f is constant on conjugacy classes, i.e. $f(yxy^{-1}) = f(x)$ for all $x, y \in G$. For example, the character of any finite-dimensional representation is central, as

$$\mathrm{tr}[\pi(x)\pi(y)] = \mathrm{tr}[\pi(y)\pi(x)].$$

We denote the set of central functions with the prefix Z, e.g. $ZC(G)$ and $ZL^p(G)$. The linear span of $\{\chi_\pi : [\pi] \in \hat{G}\}$ is dense in $ZC(G)$ as well as $ZL^p(G)$ (see [9, Proposition 5.25]). One has that $L^p(G)$ and $C(G)$ form Banach algebras under convolution, with $ZL^p(G)$ and $ZC(G)$ their centers.

Our final result (appearing as [9, Proposition 5.23]), states:

Proposition 4.3.6. *We have that*

$$\{\chi_\pi : [\pi] \in \hat{G}\}$$

forms an orthonormal basis for $ZL^2(G)$.

4.4 Examples

We work through the details of some special examples, namely $SU(2)$ and $SO(n)$ for $n \in \{3, 4\}$.

Example 4.4.1 ($SU(2)$). Let $U(n)$ denote the group of unitary transformations of $\mathbb{C}^n$, that is, the set of $n \times n$ matrices T satisfying $T^*T = I$. We let $SU(n)$ be the subgroup consisting of $T \in U(n)$ with $\det T = 1$. Note that $T \in U(n)$ if and only if $TT^* = I$, so that the rows of T are an orthonormal set.

When $n = 2$, we can write

$$T := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$$

if and only if $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$ and $a\bar{c} + b\bar{d} = 0$. In particular, (a, b) is a unit vector and $(c, d) = e^{i\theta}(-\bar{b}, \bar{a})$ for some $\theta \in \mathbb{R}$. It follows that $\det A = e^{i\theta}$, and so $T \in SU(2)$ if and only if $e^{i\theta} = 1$. Writing

$$U_{a,b} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

we have

$$SU(2) = \{U_{a,b} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1\}.$$

The correspondence $U_{a,b}$ with $(a, b) = (1, 0)U_{a,b}$ identifies $SU(2)$ with a subset of the unit sphere $S^3 \subset \mathbb{C}^2$ (where the identity element is identified with $(1, 0)$.

Three one-parameter subgroups of $SU(2)$ are of particular interest:

$$F(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$
$$G(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},$$
$$H(\psi) = \begin{pmatrix} \cos\psi & i\sin\psi \\ i\sin\psi & \cos\psi \end{pmatrix}.$$

These are three mutually orthogonal great circles in the sphere that intersect at $\pm Id$.

Proposition 4.4.1. *Every $T \in SU(2)$ is conjugate to precisely one matrix $F(\theta)$ as above, with $0 \le \theta \le \pi$.*

Proof. Unitary matrices are normal, so by the spectral theorem there exists $V \in U(2)$ with

$$VTV^{-1} = \mathrm{diag}(\alpha, \beta).$$

For $T \in SU(2)$ we have $\beta = \bar{\alpha} = e^{-i\theta}$ for some $\theta \in [-\pi, \pi]$. In particular, $VTV^{-1} = F(\theta)$. By replacing V with $[\det V]^{-\frac{1}{2}}V$, we may assume that $V \in SU(2)$ as well. Furthermore, using

$$F(-\theta) = H(\tfrac{1}{2}\pi)F(\theta)H(-\tfrac{1}{2}\pi),$$

we may reduce to $\theta \in [0, \pi]$. To conclude, note that if $\theta_1, \theta_2 \in [0, \pi]$ then $F(\theta_1)$ and $F(\theta_2)$ have different eigenvalues (and hence are not conjugate) unless $\theta_1 = \theta_2$. □

This implies the following corollary.

Corollary 4.4.2. *Let g be a continuous function on $SU(2)$ and set*

$$g^0(\theta) := g(F(\theta)).$$

Then $g \mapsto g^0$ is an isomorphism from the algebra of continuous central functions on $SU(2)$ to $C([0, \pi])$.

We now describe a family of unitary representations of $SU(2)$. We let $\mathcal{P}$ denote the space of all polynomials

$$P(z,w) = \sum c_{jk} z^j w^k$$

in two complex variables, and $\mathcal{P}_m \subset \mathcal{P}$ be the space of homogeneous polynomials of degree m, i.e.

$$\mathcal{P}_m = \Big\{ \sum_{j=0}^{m} c_j z^j w^{m-j} : c_j \in \mathbb{C} \Big\}.$$

Now let σ denote normalized surface measure on S^3. Then we can view $\mathcal{P}$ as a subset of $L^2(\sigma)$ with

$$\langle P, Q \rangle = \int_{S^3} P\overline{Q}\, d\sigma.$$

We will show that the monomials $z^j w^k$ are orthogonal in $\mathcal{P}$.

To this end, given $(z,w) \in \mathbb{C}^2$, we introduce polar coordinates

$$(z,w) = Z = rZ', \quad r = |Z| = \sqrt{|z|^2 + |w|^2}, \quad Z' \in S^3.$$

Denote Lebesgue measure on $\mathbb{C}^2$ by d^4Z and Lebesgue measure on $\mathbb{C}$ by d^2z or d^2w. Then

$$d^4Z = d^2z d^2w = cr^3\, dr\, d\sigma(Z'),$$

where $c = 2\pi^2$ is the Euclidean surface measure of S^3 (cf. the following lemma).

Lemma 4.4.3. *Suppose $f : \mathbb{C}^2 \to \mathbb{C}$ and $f(aZ) = a^m f(Z)$ for $a > 0$. Then*

$$\int_{S^3} f(Z')\, d\sigma(Z') = \tfrac{1}{\pi^2 \Gamma(\frac{m}{2}+2)} \int_{\mathbb{C}^2} f(Z) e^{-|Z|^2} d^4Z,$$

where $\Gamma(\cdot)$ is the Gamma function.

Proof. We compute

$$\begin{aligned}
\int_{\mathbb{C}^2} f(Z) e^{-|Z|^2} d^4Z &= c \int_0^\infty \int_{S^3} f(rZ') e^{-r^2} r^3\, d\sigma(Z')\, dr \\
&= c \int_0^\infty r^{m+3} e^{-r^2} \int_{S^3} f(Z')\, d\sigma(Z')\, dr \\
&= \tfrac{c}{2} \Gamma(\tfrac{m}{2} + 2) \int_{S^3} f(Z')\, d\sigma(Z').
\end{aligned}$$

To complete the proof, take $f = 1$, so that

$$\tfrac{c}{2} = \int_{\mathbb{C}^2} e^{-|Z|^2}\, d^4 Z = \left[\int_{\mathbb{R}} e^{-t^2}\, dt\right]^4 = \pi^2.$$

□

We now can prove the following.

Proposition 4.4.4. *Let p, q, r, s denote nonnegative integers. Then*

$$\int z^p \bar{z}^q w^r \bar{w}^s d\sigma(z,w) = \begin{cases} 0 & q \neq p \quad \text{or} \quad s \neq r, \\ \frac{p!r!}{(p+r+1)!} & q = p \quad \text{and} \quad s = r. \end{cases}$$

Consequently, the spaces $\mathcal{P}_m$ are mutually orthogonal in $L^2(\sigma)$, and

$$\left\{ \sqrt{\tfrac{(m+1)!}{j!(m-j)!}} z^j w^{m-j} : 0 \leq j \leq m \right\}$$

forms an orthonormal basis for $\mathcal{P}_m$.

Proof. By the previous lemma, the integral equals

$$\frac{1}{\pi^2 \Gamma(\frac{1}{2}(p+q+r+s)+2)} \int z^p \bar{z}^q e^{-|z|^2}\, d^2 z \int w^r \bar{w}^s e^{-|w|^2}\, d^2 w.$$

These integrals can be computed using polar coordinates, viz.

$$\begin{aligned} \int z^p \bar{z}^q e^{-|z|^2}\, d^2 z &= \int_0^\infty \int_0^{2\pi} e^{i(p-q)\theta} r^{p+q+1} e^{-r^2}\, dr\, d\theta \\ &= \begin{cases} 0 & p \neq q \\ 2\pi \cdot \frac{1}{2}\Gamma(p+1) & p = q. \end{cases} \end{aligned}$$

This implies the result. □

We can now describe a representation π of $SU(2)$ on $\mathcal{P}$ and $\mathcal{P}_m$. Given $U_{a,b}$, we will define $\pi(U_{a,b}) : \mathcal{P} \to \mathcal{P}$ via

$$[\pi(U_{a,b})P](z,w) = P(U_{a,b}^{-1}(z,w)) = P(\bar{a}z - bw, \bar{b}z + aw), \tag{4.2}$$

where we use the natural action of $SU(2)$ on $\mathbb{C}^2$. Note that $\mathcal{P}_m$ is invariant under π. We denote the subrepresentation of π on $\mathcal{P}_m$ by π_m. Then each π_m is a unitary representation of $SU(2)$ on $\mathcal{P}_m$ (with respect to the inner product in $L^2(\sigma)$).

Our next goal is to show that each π_m is irreducible. We begin with a lemma.

Lemma 4.4.5. *If M is a π-invariant subspace of $\mathcal{P}_m$ and $P \in M$, then*

$$z\tfrac{\partial P}{\partial w} \in M \quad \textit{and} \quad w\tfrac{\partial P}{\partial z} \in M.$$

Proof. Let $G(\phi)$ be as defined above. By assumption, for any $\phi \neq 0$, we have

$$\tfrac{1}{\phi}\big[\pi(G(\phi))P - P\big] \in M.$$

As $\phi \to 0$, the coefficients of this polynomial approach those of

$$\tilde{P} := \tfrac{d}{d\phi}\pi(G(\phi))P|_{\phi=0}.$$

As $\mathcal{P}_m$ is finite dimensional, we have that M is closed in $\mathcal{P}_m$. Thus, $\tilde{P} \in M$. Now we compute

$$\begin{aligned}\tilde{P} &= \tfrac{d}{d\phi}\big[P(z\cos\phi - w\sin\phi, z\sin\phi + w\cos\phi)\big]|_{\phi=0} \\ &= z\tfrac{\partial P}{\partial w} - w\tfrac{\partial P}{\partial z}.\end{aligned}$$

A similar argument using $H(\psi)$ yields

$$z\tfrac{\partial P}{\partial w} + w\tfrac{\partial P}{\partial z} = \tfrac{1}{i}\tfrac{d}{d\psi}\pi(H(\psi))P|_{\psi=0} \in M.$$

Adding and subtracting yields the result. $\square$

Theorem 4.4.6. *For $m \geq 0$, each π_m is irreducible.*

Proof. Suppose M is an invariant subspace of $\mathcal{P}_m$. We need to show $M = \mathcal{P}_m$.

Let $0 \neq P \in M$. Write

$$P(z,w) = \sum_{j=0}^{m} c_j z^j w^{m-j}$$

and let J denote the largest J such that $c_j \neq 0$. Then we have

$$(w\tfrac{\partial}{\partial z})^J P(z,w) = c_J J! w^m.$$

By the previous lemma, this implies $w^m \in M$. Now, by applying $z\frac{\partial}{\partial w}$ (and the lemma) successively, we can deduce

$$zw^{m-1} \in M, \quad z^2\omega^{m-2} \in M, \quad \text{and finally} \quad z^m \in M.$$

It follows that $M = \mathcal{P}_m$, as desired. $\square$

Our next goal is to show that the π_m's give a complete list of the irreducible representations of $SU(2)$:

Theorem 4.4.7. $[SU(2)]\hat{} = \{[\pi_m] : m \geq 0\}$.

Proof. First note that none of the π_m's are equivalent representations, because they all have different dimensions (and different characters, as we will see).

Now, let χ_m be the character of π_m, and define

$$\chi_m^0(\theta) = \chi_m(F(\theta))$$

as in the corollary above. Note that the orthogonal basis vectors $z^j w^{m-j}$ for $\mathcal{P}_m$ are eigenvectors for $\pi_m F(\theta)$; indeed, using (4.2) with $a = e^{i\theta}$ and $b = 0$, we get

$$\pi_m(F(\theta))(z^j w^{m-j}) = e^{i(2j-m)\theta} z^j w^{m-j}.$$

Thus

$$\chi_m^0(\theta) = \sum_{j=0}^{m} e^{i(2j-m)\theta} = \tfrac{\sin((m+1)\theta)}{\sin\theta} \tag{4.3}$$

(see Exercise 4.5.3). It follows that $\chi_0^0(\theta) \equiv 1$, $\chi_1^0(\theta) = 2\cos\theta$, and more generally

$$\chi_m^0(\theta) - \chi_{m-2}^0(\theta) = 2\cos m\theta \quad \text{for} \quad m \geq 2. \tag{4.4}$$

(see Exercise 4.5.4). Thus the span of $\{\chi_m^0\}$ equals the span of $\{\cos m\theta\}$. The latter is uniformly dense in $C([0,\pi])$; thus, by Corollary 4.4.2, the span of $\{\chi_m^0\}$ is uniformly dense in the space of continuous central functions on $SU(2)$.

This means that the only function orthogonal to all χ_m is the zero function. By Proposition 4.3.6, this shows that χ_m must include all possible irreducible characters. This completes the proof. $\square$

To do 'Fourier analysis' on $SU(2)$ (i.e. to write down a decomposition of $L^2(SU(2))$, we now need to compute the matrix elements of the representations π_m relative to the orthonormal bases given in Proposition 4.4.4. We set

$$e_j(z,w) = \sqrt{\tfrac{(m+1)!}{j!(m-j)!}}\, z^j w^{m-j}.$$

In what follows, we reparametrize $SU(2)$ by replacing b with $\bar{b}$. So we set

$$\pi_m(a,b) = \pi_m(U_{a,\bar{b}}), \quad \pi_m^{jk}(a,b) = \langle \pi_m(a,b)e_k, e_j\rangle.$$

Now, by definition of the representations, we have

$$\begin{aligned}\sqrt{\tfrac{(m+1)!}{k!(m-k)!}}(\bar{a}z-\bar{b}w)^k(bz+aw)^{m-k} &= [\pi_m(a,b)e_k](z,w)\\ &= \sum_j \pi_m^{jk}(a,b)e_j(z,w)\\ &= \sum_j \sqrt{\tfrac{(m+1)!}{j!(m-j)!}}\pi_m^{jk}(a,b)z^j w^{m-j}.\end{aligned}$$

If we set $z = e^{2\pi it}$ and $w = 1$, then the sum becomes a Fourier series and $\pi_m^{jk}(a,b)$ are computed like Fourier coefficients:

$$\pi_m^{jk}(a,b) = \sqrt{\tfrac{j!(m-j)!}{k!(m-k)!}}\int_0^1 (\bar{a}e^{2\pi it}-\bar{b})^k(be^{2\pi it}+a)^{m-k}e^{-2\pi ijt}\,dt.$$

When $k = 0$, one can compute

$$\pi_m^{j0}(a,b) = \tfrac{1}{\sqrt{m+1}}e_j(b,a) \tag{4.5}$$

(see Exercise 4.5.5). In particular $\{\pi_m^{j0} : 0 \le j \le m\}$ span $\mathcal{P}_m$. Here $\sqrt{m+1} = \dim H_{\pi_m}$ is needed to normalize the matrix elements.

Let us also discuss the span of $\{\pi_m^{jk} : 0 \le j \le m\}$. The identities above show that $\pi_m^{jk}(a,b)$ is a polynomial in the variables $(a,b,\bar{a},\bar{b})$ that is homogeneous of degree $m-k$ in (a,b) and of degree k in $(\bar{a},\bar{b})$. That is, it has **bidegree** $(m-k,k)$.

Furthermore, as a function on $\mathbb{C}^2$, each π_m^{jk} is harmonic:

$$\sum_{n=1}^{4}\tfrac{\partial^2\pi_m^{jk}}{\partial x_n^2} = 0, \quad a = x_1+ix_2, \quad b = x_3+ix_4$$

(check!).

We conclude this example with the following: We identify $SU(2)$ with the unit sphere in $\mathbb{C}^3$ by identifying $U_{a,b}$ with $(a,\bar{b})$. Then the Peter–Weyl decomposition

$$L^2 = \bigoplus_{m=0}^{\infty}\mathcal{E}_{\pi_m}$$

agrees with the decomposition of functions on the sphere into 'spherical harmonics'. We then have the further decomposition

$$\mathcal{E}_{\pi_m} = \bigoplus_{p+q=m}\mathcal{H}_{p,q},$$

where $\mathcal{H}_{p,q}$ is the span of π^{jq}_{p+q} for $0 \leq j \leq p+q$. This is a grouping of the spherical harmonics of degree m according to their bidegree.

Example 4.4.2 ($SO(3)$). We write $SO(n)$ for the group of rotations on $\mathbb{R}^n$. To describe these groups (and their connections to $SU(2)$) we will make use of the **quaternions**, denoted by $\mathbb{H}$.

As a real vector space, we may identify $\mathbb{H}$ with $\mathbb{R} \times \mathbb{R}^3$, with elements denoted by (a, x). Multiplication in $\mathbb{H}$ is given by

$$(a, x)(b, y) = (ab - x \cdot y, bx + ay + x \times y),$$

where $\cdot$ denotes dot product and $\times$ denotes cross product. One can verify that this product is associative and that $|\xi\eta| = |\xi|\,|\eta|$ (where $\xi, \eta \in \mathbb{R}^4$ and $|\cdot|$ denotes euclidean length in $\mathbb{R}^4$).

The subspace $\mathbb{R} \times \{0\}$ is the center of $\mathbb{H}$. We identify this with $\mathbb{R}$ and view it as the 'real axis', and we identify $\{0\} \times \mathbb{R}^3$ with $\mathbb{R}^3$. Then instead of writing (a, x), we may write $a + x$. Denoting the standard basis of $\mathbb{R}$ by i, j, k, we then have

$$(a, x) = a + x = a + x_1 i + x_2 j + x_3 k.$$

The multiplication law is determined by giving the products of these vectors:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The **conjugate** of $\xi = (a, x)$ is $\bar{\xi} = (a, -x)$. Note that

$$\xi\bar{\xi} = \bar{\xi}\xi = a^2 + |x|^2 = |\xi|^2.$$

Defining

$$U(\mathbb{H}) = \{\xi \in \mathbb{H} : |\xi| = 1\},$$

we find that $U(\mathbb{H})$ forms a group. For $\xi \in U(\mathbb{H})$, the map

$$\eta \mapsto \xi\eta\xi^{-1}$$

is an isometric linear map from $\mathbb{H}$ to $\mathbb{H}$ that leaves the center $\mathbb{R}$ (and hence the subspace $\mathbb{R}^3$) invariant. The restriction of this map to $\mathbb{R}^3$ is thus an element of $SO(3)$ denoted by $\kappa(\xi)$, i.e.

$$\kappa(\xi)x = \xi x \xi^{-1}.$$

This belongs to $SO(3)$ (and not just $O(3)$) because κ is a continuous map on the connected set $U(\mathbb{H})$.

Theorem 4.4.8. *The map κ is a 2-to-1 homomorphism from $U(\mathbb{H})$ onto $SO(3)$. In particular,*

$$SO(3) \cong U(\mathbb{H})/\{\pm 1\}.$$

Proof. Let $\xi \in U(\mathbb{H})$. Write

$$\xi = \cos\theta + [\sin\theta]u,$$

where $\theta \in [0, \pi]$ and $u \in \mathbb{R}^3$ is a unit vector. The angle θ is unique, as is u (except when $\sin\theta = 0$, which corresponds to $\xi = \pm 1$ and $\kappa(\pm 1) = I$). In particular, we may assume $\theta \in (0, \pi)$.

Now let $v \perp u$ be another unit vector in $\mathbb{R}^3$ and define $w = v \times u$. Then $\{u, v, w\}$ forms an orthonormal basis for $\mathbb{R}^3$ satisfying

$$uv = -uv = w, \quad wu = -uw = v.$$

An explicit computation yields

$$\xi(au + bv + cw)\xi^{-1} = au + (b\cos 2\theta - c\sin 2\theta)v + (c\cos 2\theta + b\sin 2\theta)w, \tag{4.6}$$

so that $\kappa(\xi)$ is a rotation of angle 2θ about the u-axis.

Let us now show that every rotation of $\mathbb{R}^3$ is of this form. To see this, note that if $T \in SO(3)$ then the eigenvalues of T have absolute value 1, have product equal to 1, and the non-real eigenvalues come in conjugate pairs. Thus (ignoring the case that $T = I$) we have that 1 is an eigenvalue of multiplicity one, and T is then a rotation about the u-axis (where u is the corresponding unit eigenvector). □

Let us now connect $\mathbb{H}$ with $SU(2)$. We write

$$a + bi + cj + dk = (a + bi) + (c + di)j$$

and observe that the algebra structure on $\mathbb{H}$ restricted to elements of the form $a + bi$ coincides with that of $\mathbb{C}$, with

$$j(a + bi) = (a - bi)j.$$

In particular, we can identify $\mathbb{H}$ with $\mathbb{C}^2$, where multiplication corresponds to

$$(z + wj)(u + vj) = (zu - w\bar{v}) + (zv + w\bar{u})j,$$

with $z, w, u, v \in \mathbb{C}$. This identity shows that we may define an isomorphism by identifying $z + wj$ with the complex matrices

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = U_{z,w}.$$

In particular, this gives an identification of $U(\mathbb{H})$ with $SU(2)$. We therefore have the following:

Corollary 4.4.9. *We have*

$$SO(3) \cong SU(2)/\{\pm I\}.$$

This shows that the representations of $SO(3)$ are given by representations of $SU(2)$ that are trivial on $\pm I$. Since the irreducible representation π_m of $SU(2)$ on $\mathcal{P}_m$ satisfies $\pi_m(-I) = (-1)^m I$, one can deduce the following:

Corollary 4.4.10. *We have*

$$[SO(3)]\hat{} = \{[\rho_k] : k = 0, 1, 2, \dots\},$$

where

$$\rho_k(Ad(U)) = \pi_{2k}(U).$$

Here $Ad(U)A := UAU^{-1}$.

Example 4.4.3 ($SO(4)$). We define $L, R : SU(2) \to L(U(\mathbb{H}))$ via

$$L(\xi)\zeta = \xi\zeta \quad \text{and} \quad R(\xi)\zeta = \zeta\xi^{-1},$$

where $\xi, \zeta \in U(\mathbb{H}) \cong SU(2)$. Because the Euclidean norm on $\mathbb{H}$ is multiplicative, it follows that the images of $SU(2)$ under L, R are closed subgroups of $SO(4)$ if we identify $\mathbb{H}$ with $\mathbb{R}^4$.

These subgroups commute (this is the associative law) and intersect only at ± 1. To see this, suppose $L(\xi) = R(\eta)$. Then

$$\xi = L(\xi)1 = R(\eta)1 = \eta^{-1}, \quad \text{so that} \quad L(\xi) = R(\xi^{-1}).$$

Thus ξ belongs to the center of $\mathbb{H}$, i.e. $\mathbb{R}$. Then $\xi = \pm 1$ and $\eta = \xi^{-1} = \pm 1$.

Next, we have the following.

Theorem 4.4.11. *If $T \in SO(4)$, then there exist $\xi, \eta \in SU(2)$ (unique up to a common factor of ± 1) such that $T = L(\xi)R(\eta)$. Thus*

$$SO(4) \cong [SU(2) \times SU(2)]/\{\pm(1, 1)\}.$$

Proof. Let $T \in SO(4)$ and $\zeta = T(1)$ (where we view $1 = (1, 0) \in \mathbb{R} \times \mathbb{R}^3$ as above). Write $S = L(\zeta)^{-1}T$, so that $S \in SO(4)$ and $S(1) = 1$. In particular, S leaves the real axis pointwise fixed, and hence can be viewed as a rotation of the orthogonal subspace $\mathbb{R}^3$. By the preceding example, we saw that S

must be given by conjugation by some $\eta \in U(\mathbb{H})$. In the present notation, that means

$$S = L(\eta)R(\eta).$$

Thus $T = L(\zeta)S = L(\xi)R(\eta)$, where $\xi := \zeta\eta$. Uniqueness and the fact that $(\xi, \eta) \mapsto L(\xi)R(\eta)$ is a homomorphism follow from the remarks above. □

Using this result, one can describe the irreducible representations of $SO(4)$ in terms of those of $SU(2) \times SU(2)$, which may ultimately be described in terms of the irreducible representations of $SU(2)$. Without delving into the theory, one defines the representation π_{mn} of $SU(2) \times SU(2)$ on $\mathcal{P}_m \otimes \mathcal{P}_n$ via

$$\pi_{mn}(\xi, \eta) = \pi_m(\xi) \otimes \pi_n(\eta)$$

(where $\otimes$ denotes tensor product). The conclusion is the following:

Corollary 4.4.12. *We have*

$$[SO(4)]^{\wedge} = \{[\rho_{mn}] : m, n \geq 0, \quad m \equiv n \ mod\ 2\},$$

where

$$\rho_{mn}(L(\xi)R(\eta)) := \pi_{mn}(\xi, \eta).$$

The restriction that m, n have the same parity comes from the fact that $\pi_m(-1) = (-1)^m I$.

4.5 Exercises

Exercise 4.5.1. Recover the theory of Fourier series using the Peter–Weyl theorem and the discussion thereafter.

Exercise 4.5.2. Verify that the maps defined in (4.2) are unitary representations of $SU(2)$.

Exercise 4.5.3. Compute the sum in (4.3).

Exercise 4.5.4. Prove the identity (4.4).

Exercise 4.5.5. Prove (4.5).

Exercise 4.5.6. Verify (4.6).

Chapter 5

Wavelet transforms

In this section we give an introduction to wavelet transforms, which are mathematical tools that have a wide range of applications within mathematics as well as in physics, engineering, and so on. Our primary reference is [7], which covers much more ground than we can hope to cover here.

5.1 Continuous wavelet transforms

Given $\psi : \mathbb{R} \to \mathbb{C}$, we define a family of 'wavelets' by rescaling and translating ψ:

$$\psi^{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\tfrac{x-b}{a}\right),$$

where $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$. These are L^2 normalized, i.e.

$$\|\psi^{a,b}\|_{L^2} \equiv \|\psi\|_{L^2},$$

and we will assume that $\|\psi\|_{L^2} = 1$.

We assume $\int \psi = 0$, and more generally we will assume the admissibility condition

$$C_\psi := 2\pi \int |\xi|^{-1}|\hat{\psi}(\xi)|^2 \, d\xi < \infty$$

(i.e. $\psi \in \dot{H}^{-\frac{1}{2}}$). The necessity of this condition will become apparent below.

The **continuous wavelet transform** with respect to this wavelet family is defined by

$$Tf(a,b) = \langle f, \psi^{a,b} \rangle.$$

The wavelet transform yields a resolution of the identity in the following sense.

Proposition 5.1.1. *For $\psi \in \dot{H}^{-\frac{1}{2}}$, we have*

$$f = C_\psi^{-1} \iint Tf(a,b)\psi^{a,b} a^{-2}\, da\, db$$

in the weak sense; i.e. for all $f, g \in L^2$,

$$\langle f, g\rangle = C_\psi^{-1} \iint Tf(a,b)\overline{Tg(a,b)} a^{-2}\, da\, db.$$

Proof. The proof is a direct computation. We begin by using Plancherel's theorem and the definition of the wavelet transform to write

$$\begin{aligned} &\iint Tf(a,b)\overline{Tg(a,b)} a^{-2}\, da\, db \\ &\quad = \iint \left[\int \hat{f}(\xi)|a|^{\frac{1}{2}} e^{-ib\xi} \overline{\hat{\psi}(a\xi)}\, d\xi \right] \\ &\qquad \times \left[\int \overline{\hat{g}(\eta)} |a|^{\frac{1}{2}} e^{ib\eta} \hat{\psi}(a\eta)\, d\eta \right] a^{-2}\, da\, db. \end{aligned}$$

Now consider the functions

$$F_a(\xi) := |a|^{\frac{1}{2}} \hat{f}(\xi)\overline{\hat{\psi}(a\xi)} \quad \text{and} \quad G_a(\xi) = |a|^{\frac{1}{2}} \hat{g}(\xi)\overline{\hat{\psi}(a\xi)}.$$

Then

$$\int \hat{f}(\xi)|a|^{\frac{1}{2}} e^{-ib\xi}\overline{\hat{\psi}(a\xi)}\, d\xi = (2\pi)^{\frac{1}{2}} \hat{F}_a(b),$$

with a similar expression for the remaining integral. Applying Plancherel (in the db integral), we continue from above to write

$$\begin{aligned} \iint Tf(a,b)\overline{Tg(a,b)} a^{-2}\, da\, db &= 2\pi \iint F_a(\xi)\overline{G_a(\xi)} a^{-2}\, d\xi\, da \\ &= 2\pi \int \hat{f}(\xi)\overline{\hat{g}(\xi)} \left[\int |\hat{\psi}(a\xi)|^2 |a|^{-1} da \right] d\xi \\ &= C_\psi \langle f, g\rangle, \end{aligned}$$

where we have changed variables in the da integral above and then applied Plancherel once more. □

Remark 5.1.2. It is possible to be a bit more quantitative about the sense in which convergence holds, but we will not concern ourselves with all of the details here. See [7] for more details.

There are several variations of the situation descibed above. For example, if we use a real-valued ψ, then we get

$$\hat{\psi}(-\xi) = \overline{\hat{\psi}(\xi)},$$

and we can write

$$\tilde{C}_\psi = 2\pi \int_0^\infty |\xi|^{-1} |\hat{\psi}(\xi)|^2 \, d\xi = 2\pi \int_{-\infty}^0 |\xi|^{-1} |\hat{\psi}(\xi)|^2 \, d\xi.$$

Then we have

$$f = \tilde{C}_\psi^{-1} \int_0^\infty \int_{\mathbb{R}} Tf(a,b) \psi^{a,b} a^{-2} \, db \, da.$$

One can also investigate using bandlimited functions, wavelets, or using complex-valued wavelets with real-valued signals, and so on.

One important variation involves using different wavelets for the decomposition and the reconstruction of the signal f. In particular, we have:

Proposition 5.1.3. *If ψ_1, ψ_2 satisfy*

$$\int |\hat{\psi}_1(\xi)| \, |\hat{\psi}_2(\xi)| \, |\xi|^{-1} \, d\xi < \infty,$$

then

$$f = C_{\psi_1,\psi_2}^{-1} \iint \langle f, \psi_1^{a,b} \rangle \psi_2^{a,b} a^{-2} \, da \, db$$

in the weak sense, where

$$C_{\psi_1,\psi_2} = 2\pi \int \overline{\hat{\psi}(\xi_1)} \hat{\psi}_2(\xi) |\xi|^{-1} \, d\xi.$$

Proof. The proof proceeds exactly as in Proposition 5.1.1. $\square$

As before, under some reasonable conditions on f, the convergence above can be shown to hold in stronger senses (e.g. pointwise at points of continuity for f).

Reproducing kernel Hilbert spaces.

The continuous wavelet transform is related to the notion of a **reproducing kernel Hilbert space**, as we briefly explain here.

Definition 5.1.4. Let H be a Hilbert space of real-valued functions on some set X. For $x \in X$, define the evaluation functional $L_x : H \to \mathbb{R}$ by $L_x f = f(x)$. We call H a reproducing kernel Hilbert space (rkHs) if for all $x \in X$, L_x defines a bounded operator on H.

By the Riesz representation theorem (see Section A.2), if H is a reproducing kernel Hilbert space, then for every $x \in X$, there exists unique $g_x \in H$ such that

$$f(x) = L_x f = \langle f, g_x \rangle \quad \text{for all} \quad f \in H.$$

If the inner product is given by integration, this becomes

$$f(x) = \int f(y) \overline{g(x,y)} \, dy \quad \text{for all} \quad f \in H.$$

The function g is called the reproducing kernel for the Hilbert space.

Now let us return to the continuous wavelet transform. For $f \in L^2(\mathbb{R})$, we have

$$C_\psi^{-1} \iint |Tf(a,b)|^2 a^{-2} \, da \, db = \int |f(x)|^2 \, dx,$$

so that T maps $L^2(\mathbb{R})$ isometrically into the Hilbert space

$$L^2(\mathbb{R}^2; C_\psi^{-1} a^{-2} \, da \, db).$$

Let H denote the closed subspace given by the image of L^2 under T. Then H is a reproducing kernel Hilbert space. In fact,

$$F(a,b) = \langle f, \psi^{a,b} \rangle = C_\psi^{-1} \iint K(a,b;\alpha,\beta) F(\alpha,\beta) \alpha^{-2} \, d\alpha \, d\beta,$$

where

$$K(a,b;\alpha,\beta) = \langle \psi^{\alpha,\beta}, \psi^{a,b} \rangle.$$

That is, point evaluation is given by integration against a kernel.

The windowed Fourier transform.

We turn to a comparison of the wavelet transform with the so-called **windowed Fourier transform**. Given a smooth function g supported near the origin, we define the windowed Fourier transform by

$$\mathcal{F}_g f(\omega, t) = \langle f, g^{\omega,t} \rangle, \quad \text{where} \quad g^{\omega,t}(x) = e^{i\omega x} g(x-t).$$

That is, $\mathcal{F}_g f(\omega, t)$ represents the Fourier transform of the portion of $f(x)$ around the point $x = t$. This is a natural quantity to consider in the setting of signal processing.

Arguing as in the proof of Proposition 5.1.1, we have the following:

Proposition 5.1.5. *We have*

$$f = \frac{1}{2\pi\|g\|_{L^2}^2} \iint \mathcal{F}_g f(\omega, t) g^{\omega,t} \, d\omega \, dt$$

in the weak sense. That is, for all $f_1, f_2 \in L^2$,

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi\|g\|_{L^2}^2} \iint \mathcal{F}_g f_1(\omega, t) \overline{\mathcal{F}_g f_2(\omega, t)} \, d\omega \, dt.$$

One can use any $g \in L^2$ for the windowing function. Typically, one normalizes $\|g\|_{L^2} = 1$. The windowed Fourier transform also maps L^2 to a reproducing kernel Hilbert space, viz.

$$F(\omega, t) = \tfrac{1}{2\pi} \iint K(\omega, t; \omega', t') F(\omega', t') \, d\omega' \, dt',$$

where

$$F(\omega, t) = \mathcal{F}_g f(\omega, t), \quad K(\omega, t; \omega', t') = \langle g^{\omega',t'}, g^{\omega,t} \rangle.$$

Construction of operators and time-frequency localization.

We saw in Proposition 5.1.1 the following reconstruction formula:

$$f = C_\psi^{-1} \iint \langle f, \psi^{a,b} \rangle \psi^{a,b} a^{-2} \, da \, db.$$

This can also be viewed as a 'resolution of the identity', i.e.

$$Id = C_\psi^{-1} \iint \langle \cdot, \psi^{a,b} \rangle \psi^{a,b} a^{-2} \, da \, db$$

for an admissible 'mother wavelet' ψ. Similarly, we have a resolution of the identity using the windowed Fourier transform:

$$Id = \tfrac{1}{2\pi} \iint \langle \cdot, g^{\omega,t} \rangle g^{\omega,t} \, d\omega \, dt$$

for an L^2-normalized windowing function.

Note that in both cases, we are reconstructing the identity operator as a linear combination of rank one projections. That is, the operator

$$f \mapsto \langle f, \phi \rangle \phi$$

is simply the rank one projection onto the span of ϕ.

For the identity matrix, we give each projection equal weight. However, by varying the weight given to each projection operator, we can construct a wide variety of operators.

Let us consider this first in the setting of the windowed Fourier transform. Inspired by applications in quantum mechanics, we switch to the position/momentum variables (p, q) (instead of (t, ω)). Given a weight function $w(p, q)$ and an L^2-normalized window function, we may define the operator

$$W = \tfrac{1}{2\pi} \iint w(p, q)\langle \cdot, g^{p,q}\rangle g^{p,q}\, dp\, dq.$$

If w is unbounded, then W may be an unbounded operator. However, for reasonably chosen w and g, one can get a densely defined operator.

Example 5.1.1. With $w(p, q) = p^2$, one gets

$$W = -\tfrac{d^2}{dx^2} + C_g \mathrm{Id},$$

where

$$C_g = \int \xi^2 |\hat{g}(\xi)|^2\, d\xi.$$

With $w(p, q) = v(q)$, one gets

$$Wf(x) = V_g(x)f(x), \quad V_g(x) = v * |g|^2.$$

These correspond to the "quantized versions" of the phase space function $w(p, q)$, up to the additional g-dependent parts. Ideas related to these appeared in work of Lieb establishing the validity of the Thomas–Fermi model in the limit as the nuclear charge tends to infinity (i.e. very heavy atoms).

Let us consider the operator corresponding to **time-frequency localization**. Recall from Corollary 2.8.2 that a non-trivial function and its Fourier transform cannot both be compactly supported. Nonetheless, in practice it is of great utility to localize signals in both space and frequency as much as possible. We will consider two notions of time-frequency localization.

Example 5.1.2. First, we make use of the orthogonal projection operators Q_T and P_Ω, given by

$$Q_T f(x) = 1_{[-T,T]}(x)f(x) \quad \text{and} \quad \mathcal{F}[P_\Omega f](\xi) = 1_{[-\Omega,\Omega]}(\xi)\hat{f}(\xi).$$

Suppose f is time-limited, i.e. $f = Q_T f$. If we transmit f over a bandlimited signal, the result is of the form $P_\Omega Q_T f$. To measure how faithfully the transmitted signal represents the original signal, we may compute

$$\frac{\|P_\Omega Q_T f\|_{L^2}^2}{\|f\|_{L^2}} = \frac{\langle Q_T P_\Omega Q_T f, f\rangle}{\|f\|_{L^2}^2}.$$

The largest value of this ratio corresponds to the largest eigenvalue of the symmetric operator $Q_T P_\Omega Q_T$, which is an integral operator with integral kernel

$$(Q_T P_\Omega Q_T)(x,y) = 1_{[-T,T]}(x) 1_{[-T,T]}(y) \tfrac{\sin[\Omega(x-y)]}{\pi(x-y)}$$

(exercise). The eigenvalues and eigenvectors of this operator are known, due to the fact that this operator commutes with the operator

$$A = \tfrac{d}{dx}(T^2 - x^2)\tfrac{d}{dx} - \tfrac{\Omega^2}{\pi^2}x^2,$$

which was previously studied in the context of solving the Helmholtz equation via separation of variables. The eigenfunctions are called 'prolate spheroidal wave functions'.

The eigenvalues may be put in decreasing order. One finds (by a rescaling argument) that the eigenvalues of $Q_T P_\Omega Q_T$ depend only on the product $T\Omega$. It turns out that (for fixed $T\Omega$) the eigenvalues begin close to 1 before suddenly plunging down to essentially zero. There are about $\frac{2T\Omega}{\pi}$ eigenvalues near one before a plunge region of width about $\log(T\Omega)$.

This gives a rigorous version of something empirically observed long ago. In particular, in a time and band-limited region, there are $2T\Omega/\pi$ 'degrees of freedom' (i.e. independent functions that are essentially time and band-limited in this way). This is proportional to the area of $[-T,T]\times[-\Omega,\Omega]$; it also corresponds to the number of sampling times within $[-T,T]$ dictated by Shannon's sampling theorem (Theorem 3.1.1) for a function with Fourier support in $[-\Omega,\Omega]$.

Example 5.1.3. Next, returning to the discussion above, let us use the windowed Fourier transform to define an operator that corresponds to time-frequency localization to a set $S \subset \mathbb{R}^2$. We define

$$L_S = \tfrac{1}{2\pi}\iint_{(\omega,t)\in S} \langle \cdot, g^{\omega,t}\rangle g^{\omega,t}\,d\omega\,dt.$$

Note that $0 \le L_S \le Id$ in the sense of operators. The lower bound is immediate, while the upper bound follows from the resolution of the identity property:

$$|\langle L_S f, f\rangle| \le \tfrac{1}{2\pi}\iint |\langle f, g^{\omega,t}\rangle|^2\,d\omega\,dt = \|f\|_{L^2}^2.$$

Suppose S is a bounded subset of $\mathbb{R}^2$ and φ_n is an orthonormal basis for $L^2(\mathbb{R})$. Then

$$\begin{aligned}&\sum \langle L_S\varphi_n, \varphi_n\rangle \\ &\quad = \tfrac{1}{2\pi} \iint_S \sum_n \left|\langle \varphi_n, g^{\omega,t}\rangle\right|^2 \, d\omega \, dt = \tfrac{1}{2\pi} \iint_S \|g^{\omega,t}\|_{L^2}^2 \, dw \, dt = |S|.\end{aligned}$$

This shows that L_S is a trace class operator (see Section A.2).

By the spectral theorem, we may find a complete set of eigenvectors ϕ_n forming an orthonormal basis for L^2 with nonnegative eigenvalues λ_n decreasing to zero.

The interpretation of $L_S f$ is that we build up f out of time-frequency localized pieces $\langle f, g^{\omega,t}\rangle g^{\omega,t}$, only using $(\omega, t) \in S$.

Even if $S = [-\Omega, \Omega] \times [-T, T]$, the operator L_S will not be the same as $Q_T P_\Omega Q_T$. In this construction, we are free to choose more general sets S.

In general, it is difficult to compute the eigenvalues and eigenvectors of the operator L_S constructed above, in which case the construction is not particularly useful. However, there is a special case in which things can be computed explicitly.

Example 5.1.4. Consider the operator $L_R = L_{S_R}$ defined as above, with

$$S = S_R = \{(\omega, t) : \omega^2 + t^2 \leq R^2\}$$

and

$$g(x) = g_0(x) := \pi^{-\frac{1}{4}} e^{-x^2/2}.$$

In this case, things can be computed fairly explicitly. We only mention some of the results; more details may be found in [7, pp38–40].

With these choices of g_0 and S_R, one finds by direct computation that L_R commutes with the harmonic oscillator Hamiltonian, defined by

$$H = -\tfrac{d^2}{dx^2} + x^2 - 1.$$

In fact, one can compute the action of the unitary group e^{-isH} on g_0 explicitly and prove that

$$L_R e^{-isH} = e^{-isH} L_R,$$

which implies the result. The proof boils down to showing that the action of e^{-isH} on $g_0^{\omega,t}$ simply produces (up to a phase factor) some other $g_0^{\omega_s,t_s}$, where (ω_s, t_s) are given by a rotation matrix applied to (ω, t). One then applies change of variables (given by a rotation, which leaves the domain S_R invariant) to prove the identity above.

As L_R commutes with H, these operators are simultaneously diagonalizable. Fortunately, it is a well-known exercise in quantum mechanics to compute the eigenvalues and eigenvectors of the harmonic oscillator (see e.g. [11]). The eigenvalues are simply $\{2n\}$ and the eigenfunctions are the Hermite functions

$$\phi_n = 2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}(x - \tfrac{d}{dx})^n g_0(x).$$

The eigenvalues for L_R can be found (after some computation), using

$$\lambda_n(R) = \langle L_R\phi_n, \phi_n\rangle = \tfrac{1}{n!}\int_0^{\frac{1}{2}R^2} s^n e^{-s}\, ds.$$

This is called an incomplete Γ-function and the behavior as a function of n and R is understood:

For each R, the $\lambda_n(R)$ decrease monotonically as n increases. They start close to 1 and then 'plunge' down to zero. One finds

$$\max\{n : \lambda_n(R) \geq \tfrac{1}{2}\} \sim \tfrac{1}{2}R^2,$$

which (writing it as $\pi R^2/2\pi$) is again the area of the time-frequency localization region divided by 2π. The width of the 'plunge region' is larger this time, but still essentially negligible compared to $\frac{1}{2}R^2$. The eigenfunctions turn out to be independent of the size of S_R; that is, the R-dependence is completely represented through the eigenvalues.

So far, we have focused on using the windowed Fourier transform to build operators. Of course, a similar construction is possible using the wavelet transform. That is, given a weight $w(a, b)$, we may define

$$W = C_\psi^{-1} \iint w(a, b)\langle\cdot, \psi^{a,b}\rangle\psi^{a,b}a^{-2}\, da\, db.$$

Once again, one may interested in using a weight $w(a, b)$ that simply cuts off to a subset S. Once again this leads to operators L_S satisfying $0 \leq L_S \leq 1$, which (provided S is compact and does not contain $a = 0$) are trace class. As before, the eigenvalues/eigenvectors can be difficult to analyze except in some special cases. One such case involves taking $\hat{\psi}(\xi) = 2\xi e^{-\xi}\chi_{[0,\infty)}(\xi)$ and using the identity

$$1 = C_\psi^{-1} \int_0^\infty \int [\langle\cdot, \psi_+^{a,b}\rangle\psi_+^{a,b} + \langle\cdot, \psi_-^{a,b}\rangle\psi_-^{a,b}]a^{-2}\, db\, da,$$

where $\psi_+ = \psi$ and $\hat{\psi}_-(\xi) = \hat{\psi}(\xi)$. One then considers localization to regions

$$S_C = \{(a,b) \in (0,\infty) \times \mathbb{R} : a^2 + b^2 + 1 \leq 2aC\}$$

corresponding to the disks $|z - iC|^2 \leq C^2 - 1$ in the upper half of the complex plane. The role of the harmonic oscillator Hamiltonian is played by a different operator that commutes with L_C and is diagonalized by Laguerre polynomials. We refer the interested reader to [7] and the references therein.

Let us now consider one final 'purely mathematical' application of the continuous wavelet transform.

Characterization of local regularity.

We will prove two theorems relating regularity of functions to their wavelet transforms. The first is a global result; we have seen similar results in the setting of the Fourier transform or Fourier series. The second is a local result; this type of result is typically not possible using the Fourier transform or windowed Fourier transform.

Theorem 5.1.6. *Suppose*

$$\int (1 + |x|)|\psi(x)|\, dx < \infty \quad \textit{and} \quad \hat{\psi}(0) = 0.$$

If f is bounded and Hölder continuous of order α, then

$$|\langle f, \psi^{a,b}\rangle| \lesssim |a|^{\alpha + \frac{1}{2}}. \tag{5.1}$$

Conversely, suppose ψ is compactly supported and $f \in L^2(\mathbb{R})$ is bounded and continuous. If (5.1) holds, then f is Hölder continuous of order α.

Proof. The first statement follows from a direct computation. Using $\int \psi = 0$,

$$\begin{aligned} |\langle \psi^{a,b}, f\rangle| &\lesssim \int |a|^{-\frac{1}{2}} \big|\psi(\tfrac{x-b}{a})\big| |f(x) - f(b)|\, dx \\ &\lesssim \int |a|^{-\frac{1}{2}} |\psi(\tfrac{x-b}{a})| |x - b|^{\alpha}\, dx \\ &\lesssim |a|^{\alpha + \frac{1}{2}} \int |\psi(y)|\, |y|^{\alpha}\, dy, \end{aligned}$$

as claimed.

We turn to the converse statement. Let $\psi_2 \in C_c^\infty$ with $\int \psi_2 = 0$. We can choose ψ_2 so that the constant $C_{\psi,\psi_2} = 1$. Using the resolution of the identity,

$$f(x) = \iint \langle f, \psi^{a,b}\rangle \psi_2^{a,b}(x) a^{-2}\, da\, db.$$

We split the integral into the part where $|a| \leq 1$ and $|a| > 1$. We call these two terms f_S and f_L (for small and large scales, respectively).

First consider the large scale piece, which is actually already guaranteed to be Lipschitz. We begin by showing f_L is uniformly bounded. In fact,

$$\begin{aligned} |f_L(x)| &\lesssim \iint_{|a|\geq 1} \|f\|_{L^2}\|\psi\|_{L^2} a^{-\frac{5}{2}} |\psi_2(\tfrac{x-b}{a})|\, da\, db \\ &\lesssim \|f\|_{L^2}\|\psi\|_{L^2}\|\psi_2\|_{L^1} \int_{|a|\geq 1} |a|^{-\frac{3}{2}}\, da \lesssim 1. \end{aligned}$$

Now let $|h| \leq 1$ and write

$$\begin{aligned} &|f_L(x+h) - f_L(x)| \\ &\leq \iiint_{|a|\geq 1} a^{-3}|f(y)||\psi(\tfrac{y-b}{a})|\big|\psi_2(\tfrac{x+h-b}{a}) - \psi_2(\tfrac{x-a}{b})\big|\, dy\, db\, da \\ &\lesssim |h| \iiint_S a^{-4}|f(y)|\, da\, db\, dy, \end{aligned}$$

where (using the compact support of ψ and ψ_2)

$$S = \{(a,b,y) : |a| \geq 1, \quad |y-b| \leq |a|R, \quad |x-b| \leq |a|R\}$$

for some $R > 0$. In particular, using Hölder's inequality,

$$|f_L(x+h) - f_L(x)| \lesssim_R |h| \|f\|_{L^2} \int_{|a|\geq 1} a^{-4} \cdot a^{\frac{3}{2}}\, da \lesssim |h|.$$

We turn to the small scale piece, beginning again with uniform boundedness:

$$\begin{aligned} |f_S(x)| &\lesssim \iint_{|a|\leq 1} |a|^{\alpha+\frac{1}{2}}|a|^{-\frac{1}{2}}|\psi_2(\tfrac{x-b}{a})|a^{-2}\, da\, db \\ &\lesssim \|\psi_2\|_{L^1} \int_{|a|\leq 1} a^{-1+\alpha}\, da \lesssim 1. \end{aligned}$$

Next, for $|h| \leq 1$, we consider $f_S(x+h) - f_S(x)$ and split the integral into two regions. Firåst, we get the contribution

$$\iint_{|a|\leq|h|} |a|^{\alpha}\big(|\psi_2(\tfrac{x-b}{a})| + |\psi_2(\tfrac{x+h-b)}{a})|\big)a^{-2}\,da\,db$$
$$\lesssim \iint_S a^{\alpha-2}\,da\,db \lesssim \int_{|a|\leq h} |a|^{-1+\alpha}\,da \lesssim |h|^{\alpha},$$

where

$$S = \{(a,b) : |a| \leq |h| \quad \text{and} \quad |x-b| \leq |a|R\}$$

for some $R > 0$. Next, using the fact that ψ_2 is Lipschitz, the contribution of $|h| \leq |a| \leq 1$ is controlled by

$$\iint_{\tilde{S}} |a|^{\alpha}|h||a|^{-1}a^{-2}\,da\,db \lesssim |h| \int_{|h|\leq|a|\leq 1} |a|^{-2+\alpha} \lesssim |h|^{\alpha},$$

where we have denoted

$$\tilde{S} = \{(a,b) : |h| \leq |a| \leq 1 \quad \text{and} \quad |x-b| \leq |a|R\}$$

for some $R > 0$. This completes the proof. $\square$

Next, we turn to the following characterization of *local* Hölder regularity.

Theorem 5.1.7. *Suppose*

$$\int |\psi(x)|(1+|x|)\,dx < \infty \quad \textit{and} \quad \hat{\psi}(0) = 0.$$

If f is bounded and Hölder continuous of order α at x_0, then

$$|\langle f, \psi^{a,x_0+b}\rangle| \lesssim |a|^{\frac{1}{2}}(|a|^{\alpha} + |b|^{\alpha}).$$

Conversely, suppose ψ is compactly supported and $f \in L^2$ is bounded and continuous. If there exists $\gamma > 0$ such that

$$|\langle f, \psi^{a,b}\rangle| \lesssim |a|^{\gamma+\frac{1}{2}} \quad \textit{and} \quad |\langle f, \psi^{a,b+x_0}\rangle| \lesssim |a|^{\frac{1}{2}}\big(|a|^{\alpha} + \tfrac{|b|^{\alpha}}{|\log|b||}\big) \tag{5.2}$$

(uniformly in a, b), then f is Hölder continuous of order α at x_0.

Proof. By applying a translation, we may assume $x_0 = 0$.

Again, the first part follows from a computation: using $\int \psi = 0$,

$$\begin{aligned}
|\langle f, \psi^{a,b}\rangle| &\leq \int |f(x) - f(0)||a|^{-\frac{1}{2}}|\psi(\tfrac{x-b}{a})|\,dx \\
&\lesssim \int |x|^{\alpha}|a|^{-\frac{1}{2}}|\psi(\tfrac{x-b}{a})|\,dx \\
&\lesssim |a|^{\alpha+\frac{1}{2}} \int |\psi(y)|\,|y + \tfrac{b}{a}|^{\alpha}\,dy \lesssim |a|^{\frac{1}{2}+\alpha} + |a|^{\frac{1}{2}}|b|^{\alpha},
\end{aligned}$$

as desired.

We turn to the converse statement. In fact, the argument proceeds similar to the proof of Theorem 5.1.6. We actually only need to establish a suitable bound on $|f_S(h) - f_S(0)|$ for $|h| \leq 1$. This time we split into four pieces:

$$\iint_{|a|\leq|h|^{\alpha/\gamma}} |a|^{\gamma}|\psi_2(\tfrac{h-b}{a})|a^{-2}\,da\,db, \tag{5.3}$$

$$\iint_{|h|^{\alpha/\gamma}\leq|a|\leq|h|} (|a|^{\alpha} + \tfrac{|b|^{\alpha}}{|\log|b||})|\psi_2(\tfrac{h-b}{a})|a^{-2}\,da\,db, \tag{5.4}$$

$$\iint_{|a|\leq|h|} (|a|^{\alpha} + \tfrac{|b|^{\alpha}}{|\log|b||})|\psi_2(-\tfrac{b}{a})|a^{-2}\,da\,db, \tag{5.5}$$

$$\iint_{|h|\leq|a|\leq 1} (|a|^{\alpha} + \tfrac{|b|^{\alpha}}{|\log|b||})|\psi_2(\tfrac{h-b}{a}) - \psi_2(\tfrac{-b}{a})|a^{-2}\,da\,db. \tag{5.6}$$

We first estimate

$$(5.3) \lesssim \int_{|a|\leq|h|^{\alpha/\gamma}} |a|^{-1+\gamma}\|\psi_2\|_{L^1} \lesssim |h|^{\alpha},$$

which is acceptable.

Next, supposing the support of ψ_2 is contained in $[-R, R]$, we have

$$\begin{aligned}
(5.4) \lesssim_{\|\psi_2\|_{L^1}} &\int_{|a|\leq|h|} |a|^{-1+\alpha}\,da \\
&+ \int_{|h|^{\alpha/\gamma}\leq|a|\leq|h|} |a|^{-1}\frac{(|a|R + |h|)^{\alpha}}{|\log(|a|R + |h|)|}\,da \\
&\lesssim |h|^{\alpha}\Big[1 + \tfrac{1}{|\log|h||}\int_{|h|^{\alpha/\gamma}\leq|\alpha|\leq|h|} |a|^{-1}\,da\Big] \lesssim |h|^{\alpha}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(5.5) &\lesssim \int_{|a|\leq|h|} |a|^{-1+\alpha}\,da + \int_{|a|\leq|h|} |a|^{-1}\frac{(|a|R)^{\alpha}}{|\log(|a|R)|}\,da \\
&\lesssim |h|^{\alpha}.
\end{aligned}$$

[To solve the latter integral, make the substitution $u = \log(|a|)$.]

Finally, using the Lipschitz bound for ψ_2,

$$
\begin{aligned}
(5.6) &\lesssim |h| \int_{|h|\le|a|\le 1} |a|^{-3}\left[|a|^{\alpha} + \frac{(|a|R+|h|)^{\alpha}}{|\log(|a|R+|h|)|}\right](|a|R+|h|)\,da \\
&\lesssim |h| \int_{|h|\le|a|\le 1} |a|^{-3+\alpha}(|a|R+|h|) + |a|^{-3}\frac{(|a|R+|h|)^{1+\alpha}}{|\log(|a|R)|}\,da \\
&\lesssim |h|^{\alpha}.
\end{aligned}
$$

This completes the proof. □

5.2 Discrete wavelet transforms

We turn to the discrete wavelet transform. We let ψ be an admissible function. We will restrict to positive scales $a > 0$, so that the admissibility condition becomes

$$
C_\psi = \int_0^\infty |\xi|^{-1}|\hat{\psi}(\xi)|^2\,d\xi = \int_{-\infty}^0 |\xi|^{-1}|\hat{\psi}(\xi)|^2\,d\xi < \infty.
$$

For a fixed dilation step $a_0 > 1$, we will restrict to the discrete set of scales $a = a_0^m$ where $m \in \mathbb{Z}$. We next fix a translation parameter $b_0 > 0$ and define the rescaled and translated wavelets

$$
\psi_{m,n}(x) = a_0^{-\frac{m}{2}}\psi(a_0^{-m}x - nb_0) = a_0^{-\frac{m}{2}}\psi(\tfrac{x-nb_0a_0^m}{a_0^m}),
$$

where $m, n \in \mathbb{Z}$.

The basic questions we are interested in are whether functions are uniquely determined by their wavelet coefficients $\langle f, \psi_{m,n}\rangle$, as well as how functions may be reconstructed using wavelets and wavelet coefficients.

We will need the notion of a frame.

Frames and reconstruction.

Definition 5.2.1. A family of functions $\{\varphi_j\}_{j\in J}$ in a Hilbert space H is called a **frame** if there exist $0 < A \le B < \infty$ such that for all $f \in H$,

$$
A\|f\|^2 \le \sum_{j\in J} |\langle f, \varphi_j\rangle|^2 \le B\|f\|^2.
$$

We call A, B the frame constants. If $A = B$ then we call $\{\varphi_j\}$ a **tight frame**.

If $\{\varphi_j\}$ is a tight frame, then for any $f \in H$ we have

$$\|f\|^2 = \tfrac{1}{A} \sum_j |\langle f, \varphi_j \rangle|^2,$$

which implies (via the polarization identity) that

$$f = A^{-1} \sum_j \langle f, \varphi_j \rangle \varphi_j \tag{5.7}$$

in the weak sense.

Frames (even tight frames) need not be orthonormal bases.

Example 5.2.1. In $H = \mathbb{C}^2$, $\{\varphi_j\}_{j=1}^3$ defined by

$$\varphi_1 = (0,1), \quad \varphi_2 = (-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}), \quad \varphi_3 = (\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2})$$

form a tight frame with $A = \frac{3}{2}$.

In this case, $A = \frac{3}{2}$ gives the redundancy ratio (i.e. we are using three vectors in a two-dimensional space). If $A = 1$ then a tight frame is an orthonormal basis.

Proposition 5.2.2. *If $\{\varphi_j\}$ is a tight frame with bound $A = 1$ and $\|\varphi_j\| \equiv 1$, then $\{\varphi_j\}$ is an orthonormal basis.*

Proof. By definition, if $\langle f, \varphi_j \rangle \equiv 0$ then $f = 0$. Thus, it suffices to verify orthonormality. To this end, we observe

$$\|\varphi_j\| = \sum_k |\langle \varphi_j, \varphi_k \rangle|^2 = \|\varphi_j\|^4 + \sum_{j \neq k} |\langle \varphi_j, \varphi_k \rangle|^2.$$

As $\|\varphi_j\| = 1$, this implies $\langle \varphi_j, \varphi_k \rangle = 0$ for any $j \neq k$. $\square$

We turn to the question of reconstruction. In the case of a tight frame, one has (5.7). To deal with more general frames, we first introduce the notion of the frame operator.

Definition 5.2.3. If $\{\varphi_j\}_{j \in J}$ is a frame in H, the **frame operator** F is the linear operator $F : H \to \ell^2(J)$ defined by $(Ff)_j = \langle f, \varphi_j \rangle$.

The frame operator is bounded, as $\|Ff\|^2 \leq B\|f\|^2$. The adjoint of F is computed via

$$\langle F^* c, f \rangle = \langle c, Ff \rangle = \sum_j c_j \overline{\langle f, \varphi_j \rangle} = \sum_j c_j \langle \varphi_j, f \rangle,$$

which yields

$$F^*c = \sum_{j\in J} c_j \varphi_j.$$

Note that $\|F^*\| = \|F\| \leq B^{\frac{1}{2}}$ and that

$$\sum_j |\langle f, \varphi_j \rangle| = \|Ff\|^2 = \langle F^*Ff, f\rangle.$$

Thus, the frame condition may be written

$$A\, Id \leq F^*F \leq B\, Id.$$

As a consequence, we find that F^*F is invertible (cf. the lemma below). In fact, one has $B^{-1}Id \leq (F^*F)^{-1} \leq A^{-1}Id$.

Lemma 5.2.4. *Let S be a positive bounded linear operator on a Hilbert space H satisfying $S \geq \alpha\, Id$. Then S is invertible, with S^{-1} bounded by α^{-1}.*

Proof. Exercise. □

Proposition 5.2.5. *Suppose $\{\varphi_j\}$ is a frame and F is the frame operator. Define*

$$\tilde{\varphi}_j = (F^*F)^{-1}\varphi_j.$$

Then $\{\tilde{\varphi}_j\}$ is a frame with frame constants B^{-1} and A^{-1}. The associated frame operator $\tilde{F}$ satisfies

$$\tilde{F} = F(F^*F)^{-1},$$

as well as

$$\tilde{F}^*\tilde{F} = (F^*F)^{-1}, \quad \tilde{F}^*F = F^*\tilde{F} = Id.$$

Finally, $\tilde{F}F^* = F\tilde{F}^*$ is the orthogonal projection in ℓ^2 onto $R(F) = R(\tilde{F})$.

Proof. Exercise. □

One calls $\{\tilde{\varphi}_j\}$ the **dual frame** of φ_j. (The dual frame of $\tilde{\varphi}_j$ is simply the φ_j again.) The conclusions of Proposition 5.2.5 may be succinctly written as

$$\sum_j \langle f, \varphi_j \rangle \tilde{\varphi}_j = f = \sum_j \langle f, \tilde{\varphi}_j \rangle \varphi_j.$$

This yields a reconstruction formula for f using $\langle f, \varphi_j \rangle$, while simultaneously writing f as a linear combination of the φ_j.

Note that typically the φ_j are not even linearly independent, and thus there may be many linear combinations of the φ_j that yield f. The particular linear combination given above has a minimality property.

Proposition 5.2.6. *If $f = \sum_j c_j \varphi_j$ but we do not have $c_j \equiv \langle f, \tilde{\varphi}_j \rangle$, then*

$$\sum_j |c_j|^2 > \sum_j |\langle f, \tilde{\varphi} \rangle|^2.$$

Proof. First note that $f = \sum_j c_j \varphi_j$ is equivalent to $f = F^* c$. Now decompose $c = \tilde{F} g + b$ where $\tilde{F} g \in R(\tilde{F}) = R(F)$ and $b \in [R(F)]^\perp = \mathrm{nul}(F^*)$. Then

$$\|c\|^2 = \|Fg\|^2 + \|b\|^2.$$

Now (since $F^* \tilde{F} = Id$)

$$f = F^* c = F^* \tilde{F} g + F^* b \implies f = g.$$

Then

$$\sum |c_j|^2 = \|c\|^2 = \|\tilde{F} f\|^2 + \|b\|^2 = \sum |\langle f, \tilde{\varphi}_j \rangle|^2 + \|b\|^2,$$

which implies the result. □

Example 5.2.2. In the example above we had

$$v = \tfrac{2}{3} \sum_{j=1}^{3} \langle v, \varphi_j \rangle \varphi_j.$$

However, since $\sum \varphi_j = 0$, we also have

$$v = \tfrac{2}{3} \sum_{j=1}^{3} [\langle v, e_j \rangle + \alpha] \varphi_j \quad \text{for any} \quad \alpha \in \mathbb{C}.$$

However, the minimal length representation occurs when we choose $\alpha = 0$.

The $\{\tilde{\varphi}_j\}$ also play a special role in the decomposition for f, in the sense that if $f = \sum_j \langle f, \varphi_j \rangle u_j$ then

$$\sum_j |\langle u_j, g \rangle|^2 \geq \sum_j |\langle \tilde{\varphi}_j, g \rangle|^2$$

for all $g \in H$.

We return to the question of reconstruction. Given a frame $\{\varphi_j\}$, the problem boils down to computing the inverse of F^*F. Suppose $r = \frac{B}{A} - 1 \ll 1$. Then we may expect $F^*F \sim \frac{1}{2}(A+B)Id$, and hence

$$(F^*F)^{-1} \sim \tfrac{2}{A+B} Id, \quad \text{and so} \quad \tilde{\varphi} \sim \tfrac{2}{A+B}\varphi_j.$$

With this in mind, we set $R = Id - \frac{2}{A+B}F^*F$ and write

$$f = \tfrac{2}{A+B}\sum_j \langle f, \varphi_j\rangle \varphi_j + Rf.$$

By construction, we have

$$-\tfrac{B-A}{B+A}\, Id \le R \le \tfrac{B-A}{B+A}\, Id, \quad \text{and so} \quad \|R\| \le \tfrac{B-A}{B+A} = \tfrac{r}{2+r}.$$

This shows that simply writing

$$f \sim \tfrac{2}{A+B}\sum_j \langle f, \varphi_j\rangle \varphi_j$$

yields an approximate reconstruction of f with error of size $\sim r\|f\|$.

Pushing this further, we can describe an algorithm for reconstruction based on the fact that

$$F^*F = \tfrac{A+B}{2}(Id - R), \quad \text{so that} \quad (F^*F)^{-1} = \tfrac{2}{A+B}(Id - R)^{-1}.$$

As $\|R\| \le \frac{B-A}{B+1} < 1$, we can write

$$(Id - R)^{-1} = \sum_{k=0}^{\infty} R^k.$$

Then

$$\tilde{\varphi}_j = \tfrac{2}{A+B}\sum_{k=0}^{\infty} R^k \varphi_j.$$

The approximation above corresponds to keeping only the $k = 0$ term. More generally, we can define

$$\tilde{\varphi}_j^N = \tfrac{2}{A+B}\sum_{k=0}^{N} R^k \varphi_j = \tilde{\varphi}_j - \tfrac{2}{A+B}\sum_{k=N+1}^{\infty} R^k \varphi_j = [Id - R^{N+1}]\tilde{\varphi}_j.$$

It follows that

$$\begin{aligned}
\big\|f - \sum_j \langle f, \varphi_j\rangle \tilde{\varphi}_j^N\big\| &= \sup_{\|g\|=1} \big|\langle f - \sum_j \langle f, \varphi_j\rangle \tilde{\varphi}_j^N, g\rangle\big| \\
&= \sup_g \big|\sum_j \langle f, \varphi_j\rangle \langle \tilde{\varphi}_j - \tilde{\varphi}_j^N, g\rangle\big| \\
&= \sup_j \big|\sum_j \langle f, \varphi_j\rangle \langle R^{N+1}\tilde{\varphi}_j, g\rangle\big| \\
&\le \|R\|^{N+1}\|f\| \le \big(\tfrac{r}{2+r}\big)^{N+1}\|f\|.
\end{aligned}$$

As for actually computing the $\tilde{\varphi}_j^N$, one can use an iterative algorithm (that actually is relatively practical to implement) and write

$$\tilde{\varphi}_j^N = \tfrac{2}{A+B}\varphi_j + R\tilde{\varphi}_j^{N-1}.$$

See [7] and the references therein for more details.

Frames of wavelets.

We return to the setting of the discretized wavelet transform, with

$$\psi_{m,n}(x) = a_0^{-m/2}\psi(a_0^{-m}x - nb_0).$$

We will next discuss necessary and sufficient conditions for this family to form a frame in $L^2(\mathbb{R})$. The following appear as Theorem 3.3.1 and Proposition 3.3.2 in [7].

Theorem 5.2.7. *Suppose $\psi_{m,n}$ form a frame for L^2 with frame bounds A, B. Then*

$$\begin{aligned}
\tfrac{b_0\log a_0}{2\pi}A &\le \int_0^\infty \xi^{-1}|\hat{\psi}(\xi)|^2\,d\xi \le \tfrac{b_0\log a_0}{2\pi}B, \\
\tfrac{b_0\log a_0}{2\pi}A &\le \int_{-\infty}^0 |\xi|^{-1}|\hat{\psi}(\xi)|^2\,d\xi \le \tfrac{b_0\log a_0}{2\pi}B.
\end{aligned}$$

Conversely, suppose the following hold:

$$\inf_{1\le|\xi|\le a_0}\sum_{m\in\mathbb{Z}}|\hat{\psi}(a_0^m\xi)|^2 > 0, \qquad \sup_{1\le|\xi|\le a_0}\sum_{m\in\mathbb{Z}}|\hat{\psi}(a_0^m\xi)|^2 < \infty,$$

and

$$\beta(s) := \sup_\xi \sum_m |\hat{\psi}(a_0^m\xi)|\,|\hat{\psi}(a_0^m\xi + s)|$$

decays at least as fast as $(1+|s|)^{-(1+\varepsilon)}$ *for some* $\varepsilon > 0$*. Then there exists* $b_0^* > 0$ *such that the* $\psi_{m,n}$ *form a frame for any* $b_0 < b_0^*$*, with the following frame bounds:*

$$A = \tfrac{2\pi}{b_0}\Big\{\inf_{1\le|\xi|\le a_0}\sum|\hat\psi(a_0^m\xi)|^2 - \sum_{k\neq 0}\big[\beta(\tfrac{2\pi}{b_0}k)\beta(-\tfrac{2\pi}{b_0}k)\big]^{\frac12}\Big\},$$
$$B = \tfrac{2\pi}{b_0}\Big\{\sup_{1\le|\xi|\le a_0}\sum|\hat\psi(a_0^m\xi)|^2 + \sum_{k\neq 0}\big[\beta(\tfrac{2\pi}{b_0}k)\beta(-\tfrac{2\pi}{b_0}k)\big]^{\frac12}\Big\}.$$

Instead of going through the technical details of the proof of Theorem 5.2.7, we will focus on discussing a few examples below. At this point, we only remark that the conditions on ψ are satisfied, if (for example)

$$|\hat\psi(\xi)| \lesssim |\xi|^\alpha(1+|\xi|)^{-\gamma}, \quad \text{with} \quad \alpha > 0 \quad \text{and} \quad \gamma > \alpha+1.$$

Roughly speaking, if ψ is an admissible mother wavelet (in the sense of the continuous wavelet transform), then we can expect that the discretized $\psi_{m,n}$ will form a frame for (a_0,b_0) close to $(1,0)$. In fact, for some examples we can use (a_0,b_0) rather far from this special value.

It is convenient to consider the value $a_0 = 2$. If we hope to have an (almost) tight frame, then the bounds

$$A \le \tfrac{2\pi}{b_0}\sum_m |\hat\psi(a_0^m\xi)|^2 \le B$$

imply that $\sum_m |\hat\psi(2^m\xi)|^2$ should be roughly constant (for $\xi \neq 0$). This is a rather strong condition. One way to remedy this is to use multiple wavelets $\psi^1,\dots,\psi^N$, and to consider the frame obtained from $\{\psi^\nu_{m,n}\}$ (setting $a_0 = 2$ for each).

The analogue of Theorem 5.2.7 for the windowed Fourier transform is given by the following. In this case, we consider the functions

$$g_{m,n}(x) = e^{im\omega_0 x}g(x-nt_0)$$

for $m,n\in\mathbb{Z}$.

Theorem 5.2.8. *If* $g_{m,n}$ *form a frame for* L^2 *with frame bounds* A, B*, then*

$$A \le \tfrac{2\pi}{\omega_0 t_0}\|g\|^2_{L^2} \le B.$$

Conversely, suppose

$$\inf_{0\le x\le t_0}\sum_{n\in\mathbb{Z}}|g(x-nt_0)|^2 > 0, \qquad \sup_{0\le x\le t_0}\sum_{n\in\mathbb{Z}}|g(x-nt_0)|^2 < \infty,$$

and

$$\beta(s) := \sup_{0\le x\le t_0} \sum_n |g(x-nt_0)|\,|g(x-nt_0+s)|$$

decays at least as fast as $(1+|s|)^{-(1+\varepsilon)}$ *for some* $\varepsilon>0$. *Then there exists* $\omega_0^*>0$ *so that the* $g_{m,n}$ *form a frame whenever* $\omega_0<\omega_0^*$, *with the following frame bounds:*

$$A = \tfrac{2\pi}{\omega_0}\big\{\inf_x \sum_n |g(x-nt_0)|^2 - \sum_{k\neq 0}\big[\beta(\tfrac{2\pi}{\omega_0}k)\beta(-\tfrac{2\pi}{\omega_0}k)\big]^{\frac12}\big\},$$
$$B = \tfrac{2\pi}{\omega_0}\big\{\sup_x \sum_n |g(x-nt_0)|^2 + \sum_{k\neq 0}\big[\beta(\tfrac{2\pi}{\omega_0}k)\beta(-\tfrac{2\pi}{\omega_0}k)\big]^{\frac12}\big\}.$$

Example 5.2.3 (The Mexican hat function). The Mexican hat function ψ is the second derivative of the Gaussian $e^{-x^2/2}$. Normalizing the L^2 norm and imposing $\psi(0)>0$, one finds

$$\psi(x) = \tfrac{2}{\sqrt{3}}\pi^{-\frac14}(1-x^2)e^{-x^2/2}.$$

If one uses at least two voices, this yields an essentially tight frame for $b_0\le .75$. This wavelet is often used in computer vision applications.

Example 5.2.4. For the windowed Fourier transform, the Gaussian $g(x) = \pi^{-\frac14}e^{-x^2/2}$ is commonly used. It turns out that the ratio $\omega_0 t_0 \div 2\pi$ is relevant. In particular, the $g_{m,n}$ form a frame whenever $\omega_0 t_0 < 2\pi$. This is related to the notion of time-frequency density, which is discussed in detail in [7, Chapter 4].

Time-frequency localization in frames.

We have the following result regarding time-frequency localization.

Theorem 5.2.9. *Suppose* $\psi_{m,n}(x) = a_0^{-m/2}\psi(a_0^{-m}x-nb_0)$ *forms a frame with bounds* A,B. *Suppose that*

$$|\psi(x)| \lesssim \langle x\rangle^{-\alpha} \quad \textit{and} \quad |\hat\psi(\xi)| \lesssim |\xi|^{\beta}\langle\xi\rangle^{-(\beta+\gamma)}$$

for some $\alpha>1$, $\beta>0$, *and* $\gamma>0$. *Then for any* $\varepsilon>0$, $0<\Omega_0<\Omega_1$, *and* $T>0$, *there exists a finite set* B *such that for any* $f\in L^2$,

$$\begin{aligned}\|f - &\sum_B \langle f,\psi_{m,n}\rangle\widetilde{\psi_{m,n}}\|_{L^2}\\ &\le \sqrt{\tfrac{B}{A}}\big[\varepsilon\|f\|_{L^2} + \|f\|_{L^2(|x|>T)} + \|\hat f\|_{L^2(|\xi|\le\Omega_0)} + \|\hat f\|_{L^2(|\xi|>\Omega_1)}\big].\end{aligned}$$

The set B is defined by

$$B = \{(m,n) : m_0 \le m \le m_1, \quad |nb_0| \le a_0^{-m}T + t\}, \tag{5.8}$$

where m_0, m_1, t are chosen depending on $\Omega_0, \Omega_1, T, \varepsilon$.

The analogous result for the windowed Fourier transform is the following.

Theorem 5.2.10. *Suppose $g_{n,m}(x) = e^{im\omega_0 x}g(x - nt_0)$ form a frame with bounds A, B. Suppose that*

$$|g(x) \lesssim \langle x\rangle^{-\alpha} \quad \textit{and} \quad |\hat{g}(\xi)| \lesssim \langle \xi\rangle^{-\alpha}$$

with $\alpha > 1$. Then for any $\varepsilon > 0$, there exists $t_\varepsilon, \omega_\varepsilon$ such that for any $f \in L^2$ and $T, \Omega > 0$, we have

$$\begin{aligned}&\|f - \sum_B \langle f, g_{m,n}\rangle \tilde{g}_{m,n}\|_{L^2}\\ &\sqrt{\tfrac{B}{A}}\big[\varepsilon\|f\|_{L^2} + \|f\|_{L^2(|x|>T)} + \|\hat{f}\|_{L^2(|\xi|>\Omega)}\big],\end{aligned}$$

where $B = \{(m,n) : |m\omega_0| \le \Omega + \omega_\varepsilon, \quad |nt_0| \le T + t_\varepsilon\}$.

We will only sketch a proof of Theorem 5.2.9. Similar ideas suffice to establish Theorem 5.2.10.

Sketch of the proof of Theorem 5.2.9. We define B as in (5.8). We will estimate the norm by duality. Fix $h \in L^2$ with $\|h\|_{L^2} = 1$. Then

$$\langle f, h\rangle - \sum_B \langle f, \psi_{m,n}\rangle\langle \widetilde{\psi_{m,n}}, h\rangle = \sum_{B^c} \langle f, \psi_{m,n}\rangle\langle \widetilde{\psi_{m,n}}, h\rangle.$$

This can be controlled by two sums. First, we have

$$\sum_{B_1} [|\langle P_\Omega f, \psi_{m,n}\rangle| + |\langle (1 - P_\Omega)f, \psi_{m,n}]|\langle \widetilde{\psi_{m,n}}, h\rangle|$$

where

$$B_1 = \{(m,n) : m \le m_0 \quad \text{or} \quad m > m_0\}.$$

Here we write P_Ω to be the Fourier multiplier operator cutting off the frequency support to $\Omega_0 \le |\xi| \le \Omega_1$. Next, we have

$$\sum_{B_2} [|\langle Q_T f, \psi_{m,n}\rangle| + |\langle (1 - Q_T)f, \psi_{m,n}\rangle|]|\langle \widetilde{\psi_{m,n}}, h\rangle|,$$

where

$$B_2 = \{(m,n) : m_0 \le m \le m_1 \quad \text{and} \quad |nb_0| > a_0^{-m}T + t\}.$$

Using the fact that the $\widetilde{\psi_{m,n}}$ are a frame with frame bounds B^{-1}, A^{-1}, we can estimate

$$\begin{aligned}
&\sum_{B_1} |\langle (1-P_\Omega)f, \psi_{m,n}\rangle|\,|\langle\widetilde{\psi_{m,n}}, h\rangle| \\
&\qquad \le \Big(\sum_{m,n} |\langle (1-P_\Omega)f, \psi_{m,n}\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{m,n} |\langle\widetilde{\psi_{m,n}}, h\rangle|^2\Big)^{\frac{1}{2}} \\
&\qquad \le B^{\frac{1}{2}} \|(1-P_\Omega)f\|_{L^2} A^{-\frac{1}{2}} \|h\|_{L^2} \\
&\qquad \le \sqrt{\tfrac{B}{A}}\big[\|\hat f\|_{L^2(|\xi|\le\Omega_0)} + \|\hat f\|_{L^2(|\xi|>\Omega_1)}\big].
\end{aligned}$$

Similarly, the contribution of $(1-Q_T)f$ to the B_2 sum is controlled by the $\|f\|_{L^2(|x|>T)}$ term. It remains to show that the remaining terms can be controlled by $\varepsilon\sqrt{\frac{B}{A}}\|f\|_{L^2}$. We deal only with the P_Ω term, leaving the Q_T term as an exercise.

Applying Cauchy–Schwarz and writing $g = P_\Omega f$, we are led to estimate

$$\begin{aligned}
\sum_{B_1} |\langle g, \psi_{m,n}\rangle|^2 &= \sum_{B_1} \Big| \int \hat g(\xi) a_0^{\frac{m}{2}} \hat\psi(a_0^m \xi) e^{ib_0 a_0^m n\xi}\, d\xi \Big|^2 \\
&= \sum_{B_1} \Big| \int_0^{2\pi b_0^{-1} a_0^{-m}} e^{ib_0 a_0^m n\xi} \sum_{\ell\in\mathbb{Z}} \hat g(\xi + 2\pi\ell a_0^{-m} b_0^{-1}) \hat\psi(a_0^m\xi + 2\pi\ell b_0^{-1})\, d\xi \Big|^2 \\
&= \tfrac{2\pi}{b_0} \sum_{m\in B_1'} \int_0^{2\pi b_0^{-1} a_0^{-m}} \Big| \sum_{\ell} \hat g(\xi + 2\pi\ell a_0^{-m} b_0^{-1}) \hat\psi(a_0^m\xi + 2\pi\ell b_0^{-1}) \Big|^2 d\xi,
\end{aligned}$$

where we use Plancherel (on the torus) in the final step and

$$B_1' = \{m : m \le m_0 \quad \text{or} \quad m > m_0\}.$$

Expanding this out, this becomes

$$\tfrac{2\pi}{b_0} \sum_{\ell\in\mathbb{Z}, m\in B_1'} \int_{S_-} |\hat f(\xi)|\,|\hat f(\xi - 2\pi\ell a_0^{-m} b_0^{-1})|\,|\hat\psi(\xi)|\,|\hat\psi(\xi - 2\pi\ell b_0^{-1})|\, d\xi,$$

where

$$S_\pm = \{\Omega_0 \le |\xi|, |\xi \pm 2\pi\ell a_0^{-m} b_0^{-1}| \le \Omega_1\}.$$

We now apply Cauchy–Schwarz and a change of variables. We are led to estimate

$$b_0^{-1}\sum_{\ell}\prod_{\sigma\in\pm}\left[\int_{S_\sigma}|\hat{f}(\xi)|^2F_\sigma(\xi)\,d\xi\right]^{\frac{1}{2}},$$

where

$$F_\pm=\sum_{m\in B_1'}|\hat{\psi}(a_0^m\xi)|^{2-\lambda}\,|\hat{\psi}(a_0^m\xi\pm 2\pi b_0^{-1}\ell)|^{\lambda}$$

for some $0<\lambda<1$ to be chosen below. Now, by assumption,

$$\begin{aligned}|\hat{\psi}(a_0^m\xi)|\big|\hat{\psi}(a_0^m\xi\pm 2\pi b_0^{-1}\ell)\big| &\lesssim \langle a_0^m\xi\rangle^{-\gamma}\langle a_0^m\xi\pm 2\pi b_0^{-1}\ell\rangle^{-\gamma}\\ &\lesssim \langle\ell\rangle^{-\gamma}.\end{aligned}$$

Continuing from above,

$$\begin{aligned}&\sum_{B_1}|\langle P_\Omega f,\psi_{m,n}\rangle|^2\\ &\qquad\lesssim b_0^{-1}\|P_\Omega f\|_{L^2}^2\sum_{\ell}\langle\ell\rangle^{-\gamma\lambda}\sup_{\Omega_0\le|\xi|\le\Omega_1}\sum_{m\in B_1'}|\hat{\psi}(a_0^m\xi)|^{2(1-\lambda)}.\end{aligned}$$

The sum in ℓ converges provided $\gamma\lambda>1$, while

$$\begin{aligned}&\sup_{\Omega_0\le|\xi|\le\Omega_1}\sum_{m\in B_1'}|\hat{\psi}(a_0^m\xi)|^{2(1-\lambda)}\\ &\qquad\lesssim\sum_{m>m_1}\langle a_0^m\Omega_0\rangle^{-2\gamma(1-\lambda)}+\sum_{m<m_0}(a_0^m\Omega_1)^{2\beta(1-\lambda)}\\ &\qquad\lesssim(\Omega_0a_0^{m_1})^{-2\gamma(1-\lambda)}+(\Omega_1a_0^{m_0})^{2\beta(1-\lambda)}.\end{aligned}$$

Choosing $\lambda=\frac{1}{2}(1+\gamma^{-1})$ (say) and then choosing m_0 sufficiently negative and m_1 sufficiently positive, we can arrange

$$\sum_{B_1}|\langle P_\Omega f,\psi_{m,n}\rangle|\le B\varepsilon^2\|f\|_{L^2}^2,$$

as desired. The Q_Tf term is left as an exercise. This completes the proof. □

Redundancy in frames

We close this section by making a brief comment about redundancy in frames. For frames that are tight or close to tight, this can be measured by the size of $\frac{1}{2}(A+B)$.

Suppose $\{\varphi_j\}_{j\in J}$ is a frame. If additionally $\{\varphi_j\}$ is an orthonormal basis then the map $f \mapsto \langle f, \varphi_j\rangle$ is a unitary map from H onto $\ell^2(J)$. If the frame is redundant, then the range is a strict subset of $\ell^2(J)$.

Recall that the reconstruction formula

$$f = \sum_{j\in J} \langle f, \varphi_j\rangle \tilde{\varphi}_j$$

involves a projection onto the range of this map, written $f = \tilde{F}^* F f$.

Now, if the coefficients $\langle f, \varphi_j\rangle$ were modified by adding some α_j (e.g. because of a roundoff error in a numerical computation) then the reconstructed function would be given by

$$f_{app} = \tilde{F}^*(Ff + \alpha).$$

However, since $\tilde{F}^*$ includes a projection onto the range of F, the part of the sequence α that is orthogonal to the range of F does not contribute. Thus, we expect

$$\|f - f_{app}\| = \|\tilde{F}^*\alpha\|$$

to be smaller than $\|\alpha\|$. This effect should become even more pronounced if the frame is more redundant, since in this case the range of F becomes even 'smaller'.

Example 5.2.5. Let $u_1 = (1, 0)$ and $u_2 = (0, 1)$, giving an orthonormal basis for $\mathbb{C}^2$. Let

$$e_1 = u_2, \quad e_2 = -\tfrac{\sqrt{3}}{2}u_1 - \tfrac{1}{2}u_2, \quad e_3 = \tfrac{\sqrt{3}}{2}u_1 - \tfrac{1}{2}u_2.$$

Then e_1, e_2, e_3 is a tight frame with frame bound $\frac{3}{2}$.

Suppose we add $\alpha_j\varepsilon$ to the coefficients $\langle f, u_j\rangle$ or $\langle f, e_j\rangle$, where α_j are independent normal random variables. Then one can compute

$$\mathbb{E}\big\{\|f - \textstyle\sum[\langle f, u_j\rangle + \alpha_j\varepsilon]u_j\|^2\big\} = 2\varepsilon^2,$$

while

$$\mathbb{E}\big\{\|f - \tfrac{2}{3}\textstyle\sum[\langle f, e_j\rangle + \alpha_j\varepsilon]e_j\|^2\big\} = \tfrac{4}{3}\varepsilon^2.$$

5.3 Multiresolution analysis

We turn to the notion of a multiresolution analysis.

Definition 5.3.1. A **multiresolution analysis** is a sequence of closed subspaces $V_j \subset L^2$ satisfying the following conditions:

(i) $V_j \subset V_{j-1}$ for all $j \in \mathbb{Z}$,

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(iv) $f \in V_j \iff f(2^j \cdot) \in V_0$,

(v) $f \in V_0 \implies f(\cdot - n) \in V_0$ for all $n \in \mathbb{Z}$,

(vi) There exists $\phi \in V_0$ (called the 'scaling function') such that

$$\{\phi_{0,n} : n \in \mathbb{Z}\}$$

is an orthonormal basis for V_0, where

$$\phi_{j,n}(x) := 2^{-\frac{j}{2}}\phi(2^{-j}x - n).$$

Thus, all the spaces are scaled versions of a central space V_0, which is invariant under integer translations. The final condition (vi) may be relaxed considerably, but we begin with this simpler setting.

We will write P_j for the orthogonal projection onto V_j.

Example 5.3.1 (Haar multiresolution analysis). Let

$$V_j = \{f \in L^2(\mathbb{R}) : f\big|_{[2^jk, 2^j(k+1))} = \text{constant for all} \quad k \in \mathbb{Z}\}.$$

We may take $\phi(x) = 1_{[0,1]}(x)$.

We will prove the following theorem.

Theorem 5.3.2. *If $\{V_j\}_{j\in\mathbb{Z}}$ forms a multiresolution analysis of L^2, then there exists an orthonormal wavelet basis $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ such that*

$$P_{j-1} = P_j + \sum_k \langle \cdot, \psi_{j,k}\rangle \psi_{j,k}.$$

Proof. Let W_j denote the orthogonal compliment of V_j in V_{j-1}, so that

$$V_{j-1} = V_j \oplus W_j.$$

Thus

$$V_j = V_J \bigoplus_{k=0}^{J-j-1} W_{J-k} \quad \text{for all} \quad j < J.$$

By (ii) and (iii), this implies

$$L^2 = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Next, note that the W_j also inherit the scaling property (iv):

$$f \in W_j \iff f(2^j\cdot) \in W_0.$$

Using the scaling/transation property, the problem therefore reduces to finding $\psi \in W_0$ such that $\psi(\cdot - k)$ produces an orthonormal basis for W_0.

To do this, we will utilize the scaling function ϕ of (vi). In particular, since $\phi \in V_0 \subset V_{-1}$ and $\phi_{-1,n}$ are an orthonormal basis for V_{-1}, we may write

$$\phi = \sum_n h_n \phi_{-1,n}, \quad \text{with} \quad h_n = \langle \phi, \phi_{-1,n}\rangle \quad \text{and} \quad \sum_n |h_n|^2 = 1.$$

We rewrite this as

$$\phi(x) = \sqrt{2}\sum_n h_n \phi(2x-n), \quad \text{so that} \quad \hat{\phi}(\xi) = \tfrac{1}{\sqrt{2}}\sum_n h_n e^{-in\xi/2}\hat{\phi}(\xi/2).$$

Defining

$$m_0(\xi) = \tfrac{1}{\sqrt{2}}\sum_n h_n e^{-in\xi},$$

this becomes $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$, with m_0 a 2π periodic function.

We will now show that with

$$\hat{\psi}(\xi) := e^{i\xi/2}\overline{m_0(\tfrac{\xi}{2}+\pi)}\hat{\phi}(\tfrac{\xi}{2}), \tag{5.9}$$

the functions $\{\psi(\cdot - k)\}$ form an orthonormal basis for W_0 (and so $\{\psi_{j,k}\}$ form a wavelet basis for L^2).

To this end, we let $f \in W_0$, i.e. $f \in V_{-1}$ and $f \perp V_0$. By assumption, we have

$$f = \sum_n f_n \phi_{-1,n}, \quad f_n = \langle f, \phi_{-1,n}\rangle,$$

which becomes (as in the above computation)

$$\hat{f}(\xi) = m_f(\xi/2)\hat{\phi}(\xi/2), \quad m_f(\xi) = \tfrac{1}{\sqrt{2}}\sum_n f_n e^{-in\xi}.$$

The assumption that $f \perp \phi_{0,k}$ for all k becomes

$$0 = \int_0^{2\pi} e^{ik\xi} \sum_\ell \hat{f}(\xi + 2\pi\ell)\overline{\hat{\phi}(\xi + 2\pi\ell)}\, d\xi,$$

so that

$$\sum_\ell \hat{f}(\xi + 2\pi\ell)\overline{\hat{\phi}(\xi + 2\pi\ell)} \equiv 0.$$

Inserting the identities above (evaluating at 2ξ), we get

$$\sum_\ell m_f(\tfrac{\xi}{2} + \pi\ell)\overline{m_0(\tfrac{\xi}{2} + \pi\ell)}|\hat{\phi}(\tfrac{\xi}{2} + \pi\ell)|^2 = 0$$

As m_f and m_0 are 2π periodic, we this becomes

$$\begin{aligned} 0 = m_f(\tfrac{\xi}{2})\overline{m_0(\tfrac{\xi}{2})} &\sum_{\ell \text{ even}} |\hat{\phi}(\tfrac{\xi}{2} + \pi\ell)|^2 \\ + m_f(\tfrac{\xi}{2} + \pi)\overline{m_0(\tfrac{\xi}{2} + \pi)} &\sum_{\ell \text{ odd}} |\hat{\phi}(\tfrac{\xi}{2} + \pi\ell)|^2. \end{aligned}$$

We next claim

$$\sum_{\ell \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi\ell)|^2 \equiv \tfrac{1}{2\pi}, \tag{5.10}$$

which then implies

$$m_f(\cdot)\overline{m_0(\cdot)} + m_f(\cdot + \pi)\overline{m_0(\cdot + \pi)} \equiv 0. \tag{5.11}$$

To prove (5.10), we rely on orthogonality. In particular,

$$\begin{aligned} \delta_{k0} &= \int \phi(x)\overline{\phi(x-k)}\, dx \\ &= \int e^{ik\xi}|\hat{\phi}(\xi)|^2\, d\xi = \int_0^{2\pi} e^{ik\xi} \sum_{\ell\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi\ell)|^2\, d\xi, \end{aligned}$$

which implies (5.10).

Now, arguing as we did to derive (5.11), we have

$$\sum_\ell |m_0(\tfrac{\xi}{2} + \pi\ell)|^2|\hat{\phi}(\tfrac{\xi}{2} + \pi\ell)|^2 \equiv \tfrac{1}{2\pi},$$

which yields

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 \equiv 1.$$

In particular, $m_0(\cdot)$ and $m_0(\cdot+\pi)$ cannot both equal zero on a set of nonzero measure, and hence (5.11) implies that there exists a 2π-periodic function λ so that

$$m_f(\xi) = \lambda(\xi)\overline{m_0(\xi+\pi)} \quad \text{and} \quad \lambda(\xi)+\lambda(\xi+\pi) \equiv 0.$$

We rewrite the final expression as

$$\lambda(\xi) = e^{i\xi}\nu(2\xi)$$

for some 2π-periodic ν. Returning to our formula for $\hat{f}$, we get

$$\hat{f}(\xi) = e^{i\xi/2}\overline{m_0(\tfrac{\xi}{2}+\pi)}\nu(\xi)\hat{\phi}(\tfrac{\xi}{2}),$$

which (by periodicity) allows us to write

$$\hat{f}(\xi) = \big(\sum_k \nu_k e^{-ik\xi}\big)\hat{\psi}(\xi), \quad \text{i.e.} \quad f = \sum_k \nu_k \psi(\cdot - k)$$

for suitable ν_k.

It therefore remains to verify that the $\{\psi(\cdot - k)\}$ belong to $V_{-1} \cap V_0^{\perp}$ and are orthonormal. This we leave as an exercise! □

Remark 5.3.3. The proof above provides an example of a wavelet that one can use. Such examples are not unique—one could modify $\hat{\psi}$ by multiplying by any 2π-periodic function that has magnitude one almost everywhere. Using this freedom, we will take

$$\psi = \sum_n g_n \phi_{-1,n}, \quad g_n = (-1)^n \overline{h_{-n+1}}.$$

Let us mention a few examples.

Example 5.3.2. In the Haar multiresolution analysis, we take $\phi = 1_{[0,1]}$. In this case

$$h_n = \sqrt{2}\int \phi(x)\phi(2x-n)\,dx = \begin{cases} \frac{1}{\sqrt{2}} & n = 0,1 \\ 0 & \text{else.} \end{cases}$$

It follows that

$$\psi = \tfrac{1}{\sqrt{2}}\phi_{-1,0} - \tfrac{1}{\sqrt{2}}\phi_{-1,1} = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{else.} \end{cases}$$

This is the Haar wavelet basis.

Example 5.3.3 (Meyer basis). Define ϕ via its Fourier transform:

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \begin{cases} 1 & |\xi| \le \frac{2\pi}{3} \\ \cos[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\xi| - 1)] & \frac{2\pi}{3} \le |\xi| \le \frac{4\pi}{3}, \\ 0 & \text{else,} \end{cases}$$

where ν satisfies $\nu = 0$ for $x \le 0$, $\nu = 1$ for $x \ge 1$, and $\nu(x) + \nu(1-x) = 1$. We then take V_0 to be the closed subspace spanned by the (orthonormal) set $\{\phi(\cdot - k)\}$. In this case, it turns out that

$$\hat{\psi}(\xi) = \sqrt{2\pi}e^{i\xi/2}[\hat{\phi}(\xi + 2\pi) + \hat{\phi}(x - 2\pi)]\hat{\phi}(\tfrac{\xi}{2}).$$

See [7] (say) for more details.

In practice, orthonormality of the basis $\phi(\cdot - k)$ can be relaxed to requiring that

$$A \sum |c_k|^2 \le \left\| \sum_k c_k \phi(\cdot - k) \right\|_{L^2}^2 \le B \sum |c_k|^2$$

(in which case they are called a **Riesz basis**). In many examples, one starts with the scaling function ϕ and then defines V_0 by taking $\{\phi(\cdot - k)\}$ as an (orthonormal) basis. The spaces V_j are then the closed subspace spanned by

$$\phi_{j,k}(x) = 2^{-j/2}\phi(2^{-j}x - k).$$

This construction will lead to a multiresolution analysis provided

$$\phi(x) = \sum_n c_n \phi(2x - n)$$

for some $\{c_n\} \in \ell^2$, with

$$0 < \alpha \le \sum_\ell |\hat{\phi}(\xi + 2\pi\ell)|^2 \le \beta < \infty.$$

See [7] for more detail.

We close this section by describing the connection between multiresolution analysis and subband filtering schemes, which play an important role in applications.

Suppose we begin with some initial approximation to a function f, say $f^0 = P_0 f$. We can then write

$$f^0 = f^1 + \delta^1 \in V_1 \oplus W_1 = V_0,$$

where $f^1 = P_1 f^0$ is the next coarser approximation to f in the multiresolution analysis, and $\delta^1 = f^0 - f^1 = Q_1 f^0$ represents the information lost in the transition from f^0 to f^1. Here Q_1 denotes orthogonal projection onto W_1.

Recalling that V_j, W_j have orthonormal bases $\phi_{j,k}, \psi_{j,k}$, we may write

$$f^j = \sum_n c_n^j \phi_{j,n}, \quad \delta^1 = \sum_n d_n^1 \psi_{1,n}.$$

In fact, we can describe these coefficients c_k^1 and d_k^1. To see this, first recall that by construction we may write

$$\psi_{j,k} = \sum_n g_n \phi_{j-1,2k+n} = \sum_n g_{n-2k} \phi_{j-1,n},$$

and similarly

$$\phi_{j,k} = \sum_n h_{n-2k} \phi_{j-1,n}.$$

Using this, we deduce

$$c_k^1 = \sum_n \overline{h_{n-2k}} c_n^0 \quad \text{and} \quad d_k^1 = \sum_n \overline{g_{n-2k}} c_n^0.$$

In compact notation, $c^1 = \overline{H} c^0$ and $d^1 = \overline{G} c^0$.

Continuing this, we can write $f^1 = f^2 + \delta^2$ with $c^2 = \overline{H} c^1$ and $d^2 = \overline{G} c^1$.

In practice, one stops this after a finite number of levels. In particular, we write the information in c^0 in the vectors $d^1, \dots, d^J$ and a final coarse approximation c^J.

This process is invertible, and can be summarized via

$$\begin{aligned} c_k^j &= \sum_n \overline{h_{n-2k}} c_n^{j-1}, \quad d_k^j = \sum_n \overline{g_{n-2k}} c_n^{j-1}, \\ c_n^{j-1} &= \sum_k \big[h_{n-2k} c_k^j + g_{n-2k} d_k^j \big]. \end{aligned}$$

In engineering applications, these are the analysis and synthesis steps of a 'subband filtering scheme'. In the analysis part, the incoming sequence is convolved with two filters (one 'low-pass' and one 'high-pass'). In the synthesis part, the resulting sequences are subsampled (i.e. only the even or odd entries are retained).

This type of subband filtering scheme is central to many applications of wavelets (e.g. the JPEG2000 compression algorithm).

5.4 Exercises

Exercise 5.4.1. Work out the details in the discussion of Example 5.1.2.

Exercise 5.4.2. Solve the differential equation discussed in Example 5.1.4.

Exercise 5.4.3. Show that if $\{\varphi_j\}$ is a tight frame with bound A then

$$f = A^{-1} \sum_j \langle f, \varphi_j \rangle \varphi_j.$$

Exercise 5.4.4. Prove Lemma 5.2.4.

Exercise 5.4.5. Prove Proposition 5.2.5.

Exercise 5.4.6. Show that for t chosen sufficiently large in the proof of Theorem 5.2.9, we can arrange

$$\|Q_T f\|_{L^2}^2 \lesssim B\varepsilon \|f\|_{L^2}^2.$$

Exercise 5.4.7. Show that the functions $\{\psi(\cdot - k)\}$ defined in Theorem 5.3.2 are orthonormal. [*Hint:* Use the same trick appearing in the proof of the theorem. That is, apply Plancherel and split the integral into a sum over $\ell \in \mathbb{Z}$. Split the sum into even and odd parts and use the properties of m_0 and $\hat{\phi}$ that were already established.]

Chapter 6

Classical harmonic analysis, part I

Many problems in harmonic analysis are related to the boundedness properties of certain operators on various function spaces. For example, in this section we will study a special example (the Hardy–Littlewood maximal function) and its variants, along with other classes of operators for which we can understand boundedness properties.

An extremely useful tool for proving boundedness properties of operators is that of interpolation, and so we begin there.

6.1 Interpolation

In this section we will consider **sublinear** operators, i.e. those satisfying $|T(cf)| = |c|\,|T(f)|$ and $|T(f+g)| \leq |T(f)| + |T(g)|$. This includes not only linear operators, which include many important examples, but also maximal operators, as well as square function type operators. The latter two may have the form

$$Tf = \sup_n |T_n f| \quad \text{or} \quad Tf = \big(\sum_n |T_n f|^2\big)^{\frac{1}{2}}$$

for some collection of linear or sublinear operators T_n.

We begin by making some notions of boundedness precise.

Definition 6.1.1. Let $1 \leq p, q \leq \infty$ and let T be sublinear. We say T is of **strong type** (p, q) if we have an estimate of the form

$$\|Tf\|_{L^q} \leq C\|f\|_{L^p}$$

for some $C = C(T, p, q)$ and all $f \in L^p$. Here we typically take $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, but one can of course work in $L^p(X)$ and $L^q(Y)$ for more general measure spaces. Also, we may initially only require that T be defined pointwise on a dense subclass of L^p; the bound above allows for a unique extension to all of L^p.

Similarly, for $q < \infty$ we say T is of **weak type** (p, q) if

$$\|Tf\|_{L^{q,\infty}} \leq C\|f\|_{L^p}$$

(cf. Section A.1). When $q = \infty$, we define weak type (p, q) to mean strong type (p, q).

Note that if T is of type (p, q), then T is of weak type (p, q); indeed, by Tchebychev's inequality,

$$\|Tf\|_{L^{q,\infty}} \leq \|Tf\|_{L^q}.$$

As usual, we can define operator norms

$$\|T\|_{L^p \to L^q} = \inf\{C : \|Tf\|_{L^q} \leq C\|f\|_{L^p} \quad \text{for all} \quad f \in L^p\},$$

and so on.

In the following, we will always consider operators with the property that $\langle |Tf|, |g| \rangle$ is finite whenever f, g are taken to be simple functions with finite measure support.

We will study the problem of interpolating various types of bounds for operators T. Let us begin with the following application of Hölder's inequality, which is essentially an interpolation-type result for the identity operator.

Lemma 6.1.2 (Hölder's inequality)**.** *We have the following estimate:*

$$\|f\|_{L^p} \lesssim \|f\|_{L^{p_0}}^{\theta} \|f\|_{L^{p_1}}^{1-\theta}$$

whenever $1 \leq p_0, p_1 \leq \infty$ *and*

$$\tfrac{1}{p} = \tfrac{\theta}{p_0} + \tfrac{1-\theta}{p_1} \quad \textit{for some} \quad \theta \in [0, 1].$$

To prove this, write $|f| = |f|^{\theta}|f|^{1-\theta}$ and apply Hölder's inequality with exponents $\frac{p_0}{\theta}$ and $\frac{p_1}{1-\theta}$.

Let us turn to a slightly more general result, in which we begin with strong type bounds and vary the exponent in either the domain or target space alone.

Proposition 6.1.3 (Warmup version of interpolation)**.** *The following hold:*

(i) *Suppose T is type (p, q_0) and type (p, q_1) for some $1 \leq p, q_0, q_1 \leq \infty$. Then T is type (p, q) for all q such that*

$$\tfrac{1}{q} = \tfrac{\theta}{q_0} + \tfrac{1-\theta}{q_1} \quad \textit{for some} \quad \theta \in [0,1].$$

In fact,

$$\|T\|_{L^p \to L^q} \lesssim \|T\|^{\theta}_{L^p \to L^{q_0}} \|T\|^{1-\theta}_{L^p \to L^{q_1}}.$$

(ii) *Suppose T is type (p_0, q) and type (p_1, q) for some $1 \leq p_0, p_1, q \leq \infty$. Then T is type (p, q) for all p such that*

$$\tfrac{1}{p} = \tfrac{\theta}{p_0} + \tfrac{1-\theta}{p_1} \quad \textit{for some} \quad \theta \in [0,1].$$

In fact,

$$\|T\|_{L^p \to L^q} \lesssim \|T\|^{\theta}_{L^{p_0} \to L^q} \|T\|^{1-\theta}_{L^{p_1} \to L^q}.$$

Proof. For (i), we apply Hölder's inequality (in the form of the lemma above). We find

$$\|Tf\|_{L^q} \lesssim \|Tf\|^{\theta}_{L^{q_0}} \|Tf\|^{1-\theta}_{L^{q_1}} \lesssim \|T\|^{\theta}_{L^p \to L^{q_0}} \|T\|^{1-\theta}_{L^p \to L^{q_1}} \|f\|_{L^p},$$

which yields the result.

We turn to (ii). We let $\lambda > 0$ and estimate

$$\begin{aligned} \|Tf\|_{L^q} &\leq \|T\big(f\chi_{\{|f| \leq \lambda\}}\big)\|_{L^q} + \|T\big(f\chi_{\{|f|>\lambda\}}\big)\|_{L^q} \\ &\lesssim \|T\|_{L^{p_0} \to L^q} \|f\chi_{\{|f|\leq\lambda\}}\|_{L^{p_0}} + \|T\|_{L^{p_1} \to L^q} \|f\chi_{\{|f|>\lambda\}}\|_{L^{p_1}} \\ &\lesssim \sum_{j=0}^{1} \lambda^{1-\frac{p}{p_j}} \|T\|_{L^{p_j} \to L^q} \|f\|^{\frac{p}{p_j}}_{L^p}. \end{aligned}$$

Optimizing in λ now yields the result. □

In the following we will extend Proposition 6.1.3 to the setting where we vary both of the parameters p and q. Furthermore, we will see that even if we begin with weak type bounds, we can obtain strong type bounds through interpolation. There are two standard results, known as 'real interpolation' and 'complex interpolation' (in reference to the techniques used in the proofs).

Our first main goal is the following:

Theorem 6.1.4 (Marcinkiewicz interpolation theorem)**.** *Let T be a sublinear operator and*

$$1 \leq p_0, q_0, p_1, q_1 \leq \infty, \quad p_0 \neq p_1, \quad q_0 \neq q_1.$$

If T is of weak type (p_0, q_0) and weak type (p_1, q_1), then T is of type (p, q) for all (p, q) such that $p \leq q$ and

$$(\tfrac{1}{p}, \tfrac{1}{q}) = (\tfrac{\theta}{p_0} + \tfrac{1-\theta}{p_1}, \tfrac{\theta}{q_0} + \tfrac{1-\theta}{q_1}) \quad \textit{for some} \quad \theta \in (0,1). \tag{6.1}$$

Remark 6.1.5. We have not pursued optimal hypotheses here. For example, it suffices to assume 'restricted weak type' bounds on T rather than weak type bounds, and the conclusion of the theorem can be stated in terms of more general Lorentz spaces (rather than just Lebesgue spaces). We will not pursue such generality in these notes; however, we would like to point out that although we will work in the context of Lebesgue measure, this result applies to operators mapping $L^p(d\mu_1)$ to $L^q(d\mu_2)$ for more general measures.

Remark 6.1.6. We would also like to point out that to obtain strong type bounds, the restriction to $p \leq q$ is necessary. To see this consider for example

$$Tf = |x|^{-\frac{1}{2}} f$$

in dimension $d = 1$. Let us check that T maps L^2 to $L^{1,\infty}$ and L^∞ to $L^{2,\infty}$. However, T does not map L^p into $L^{\frac{2p}{p+2}}$ for any $2 < p < \infty$.

First, since Tf is a.e. bounded by the function $|x|^{-\frac{1}{2}} \|f\|_{L^\infty}$, it is not difficult to verify that $T : L^\infty \to L^{2,\infty}$ boundedly. Now suppose $f \in L^2$ and let us prove that

$$|\{|x|^{-\frac{1}{2}}|f| > \alpha\}| \lesssim \alpha^{-1} \|f\|_{L^2}$$

uniformly in α; this yields the $L^2 \to L^{1,\infty}$ bound. We write

$$\{|x|^{-\frac{1}{2}}|f| > \alpha\} \subset \bigcup_{N \in 2^{\mathbb{Z}}} \{|x|^{\frac{1}{2}}\alpha \sim N \quad \text{and} \quad |f| > N\}$$

Thus, by Tchebychev's inequality and volume bounds we can estimate

$$|\{|x|^{-\frac{1}{2}}|f| > \alpha\}| \lesssim \sum_{N \in 2^{\mathbb{Z}}} \min\{\alpha^{-2} N^2, N^{-2} \|f\|_{L^2}^2\}.$$

Choosing the optimal N and summing leads to the desired estimate.

Finally, to see that T does not map L^p into $L^{\frac{2p}{p+2}}$, consider the function

$$x \mapsto |x|^{-\frac{1}{p}}[\log(|x|+|x|^{-1})]^{-\frac{p+2}{2p}}.$$

and observe that

$$x \mapsto |x|^{-1}\log(|x|+|x|^{-1})^{-\theta}$$

belongs to L^1 if and only if $\theta > 1$. (See the exercises for more details.)

Before beginning the proof of Marcinkiewicz interpolation, let us collect a few preliminary lemmas. The first lemma is an improvement of Hölder's inequality in a special case.

Lemma 6.1.7. *Let $f \in L^{q,\infty}$ with $q > 1$ and let E be a set of finite measure. Then*

$$|\langle f, \chi_E\rangle| \lesssim \|f\|_{L^{q,\infty}}|E|^{\frac{1}{q'}}$$

Proof. Note that the distribution function of $f\chi_E$ is given by

$$\alpha \mapsto |\{x \in E : |f| > \alpha\}|,$$

so that

$$\int |f\chi_E|\,dx = \int_0^\infty |\{x \in E : |f| > \alpha\}|\,d\alpha.$$

Now, for each α we have the bound

$$|\{x \in E : |f| > \alpha\}| \lesssim \min\{|E|, \alpha^{-q}\|f\|_{L^{q,\infty}}^q\}.$$

Setting

$$\alpha_0 = \|f\|_{L^{q,\infty}}|E|^{-\frac{1}{q}},$$

we therefore have

$$\begin{aligned}\int |f\chi_E|\,dx &\lesssim \int_0^{\alpha_0}|E|\,d\alpha + \int_{\alpha_0}^\infty \alpha^{-q}\|f\|_{L^{q,\infty}}^q\,d\alpha\\ &\lesssim \|f\|_{L^{q,\infty}}|E|^{\frac{1}{q'}},\end{aligned}$$

as desired. □

The next lemma is a weakened version of Marcinkiewicz.

Lemma 6.1.8 (Weak Marcinkiewicz). *Under the assumptions of the Marcinkiewicz theorem, the operator T satisfies*

$$|\langle T\chi_F, \chi_E\rangle| \lesssim |F|^{\frac{1}{p}}|E|^{\frac{1}{q'}}$$

for all (p,q) as in (6.1) and all finite-measure sets F, E.

Remark 6.1.9. The estimate appearing above is an immediate consequence of the refined Hölder inequality and a weak type bound, but only provided $q > 1$. In particular, we do not necessarily know this bound holds for T using the exponents (p_j, q_j). The purpose of this lemma will be to allow us to assume $q_j > 1$ (at the price of replacing weak type bounds with the restricted weak type bounds appearing above).

Proof. Let F be a set of finite measure. By assumption,

$$\alpha|\{|T\chi_F| > \alpha\}|^{\frac{1}{q_j}} \lesssim \|\chi_F\|_{L^{p_j}} \lesssim |F|^{\frac{1}{p_j}}$$

uniformly in α for $j = 0, 1$. Recalling the definition of (p,q), it follows that

$$\alpha|\{|T\chi_F| > \alpha\}|^{\frac{1}{q}} \lesssim |F|^{\frac{\theta}{p_0}+\frac{1-\theta}{p_1}} \lesssim |F|^{\frac{1}{p}},$$

uniformly in α, and thus

$$\|T\chi_F\|_{L^{q,\infty}} \lesssim |F|^{\frac{1}{p}}.$$

for all sets of finite measure. As we may assume $q > 1$ (cf. $q_0 \neq q_1$), we may apply the previous lemma to deduce

$$|\langle T\chi_F, \chi_E\rangle| \lesssim \|T\chi_F\|_{L^{q,\infty}}\|\chi_E\|_{L^{q'}} \lesssim |F|^{\frac{1}{p}}|E|^{\frac{1}{q'}},$$

which implies the desired result. □

With the preliminaries in place, we are now ready to prove the Marcinkiewicz interpolation theorem.

Proof of the Marcinkiewicz interpolation theorem. Our goal is to show that

$$\|Tf\|_{L^q} \lesssim \|f\|_{L^p}$$

for (p,q) as defined above. By duality and density, it suffices to prove that

$$|\langle |Tf|, |g|\rangle| \lesssim \|f\|_{L^p}\|g\|_{L^{q'}} \tag{6.2}$$

for f and g simple functions with finite measure support, say. Furthermore, we may assume f, g are nonnegative.

Using Lemma 6.1.8, we may assume that assume that

$$|\langle |T\chi_F|, |\chi_E|\rangle| \lesssim \min_{j=0,1}\{|F|^{1/p_j}|E|^{1/q_j'}\} \tag{6.3}$$

for any F, E (cf. Remark 6.1.9 above).

We turn to (6.2). In fact, it will be convenient to write down a different decomposition for f and g. First, writing

$$g = \sum_{m\in\mathbb{Z}} g \cdot \chi_{\{2^{m-1}\leq g<2^m\}},$$

we find (by the triangle inequality) that we may assume

$$g = \sum_{m\in\mathbb{Z}} 2^m \chi_{E_m},$$

where E_m are disjoint sets such that

$$\|g\|_{L^{q'}} \sim \left(\sum_m 2^{mq'}|E_m|\right)^{\frac{1}{q'}}.$$

For f, it is not enough to use upper bounds (due to the presence of T). Instead, we write

$$f = \sum_{n\in\mathbb{Z}} f \cdot \chi_{\{2^{n-1}\leq f<2^n\}} = \sum_{n\in\mathbb{Z}} 2^n \sum_{j=1}^{\infty} 2^{-j}\chi_{A_n^j},$$

where A_n^j is the set of x such that $2^{n-1} \leq f(x) < 2^n$ and the coefficient of 2^{-j} in the binary expansion of $2^{-n}f(x)$ is 1. In particular, we may write

$$f = \sum_{j=1}^{\infty} 2^{-j} \sum_{n\in\mathbb{Z}} 2^n \chi_{F_n^j},$$

where F_n^j are disjoint sets and for each j

$$\left(\sum_n 2^{np}|F_n^j|\right)^{\frac{1}{p}} \lesssim \|f\|_{L^p}. \tag{6.4}$$

In particular, by applying sublinearity and noting that $\sum_{j\geq 1} 2^{-j}$ is finite, we find that it also suffices to consider f of the form

$$f = \sum_{n\in\mathbb{Z}} 2^n \chi_{F_n}$$

with the F_n obeying (6.4).

Applying (6.3) and sublinearity, we find

$$\begin{aligned}|\langle |Tf|, |g|\rangle| &\lesssim \sum_{m,n} 2^n 2^m \min_{j=0,1}\{|F_n|^{1/p_j}, |E_m|^{1/q_j'}\}\\ &\lesssim \sum_{m,n} 2^n |F_n|^{\frac{1}{p}} \min_{j=0,1}\big\{|F_n|^{\frac{1}{p_j}-\frac{1}{p}}|E_m|^{\frac{1}{q_j'}-\frac{1}{q'}}\big\}2^m|E_m|^{\frac{1}{q'}}\end{aligned}$$

Recalling the definition of p, q, it follows that we may rewrite the minimum appearing above as

$$\min_{j=0,1}\left[|F_n|^{\frac{1}{p_1}-\frac{1}{p_0}}|E_m|^{\frac{1}{q_1'}-\frac{1}{q_0'}}\right]^{j-\theta}.$$

Applying a dyadic decomposition, we are now led to estimate

$$\sum_{A,B\in 2^{\mathbb{Z}}}\Bigg(\sum_{n:|F_n|\sim A} 2^n A^{\frac{1}{p}}\Bigg)\min_{j=0,1}\left[A^{\frac{1}{p_1}-\frac{1}{p_0}}B^{\frac{1}{q_1'}-\frac{1}{q_0'}}\right]^{j-\theta}\Bigg(\sum_{m:|E_m|\sim B} 2^m B^{\frac{1}{q'}}\Bigg).$$

Now, because $p_0 \neq p_1$ and $q_0 \neq q_1$, we are in a position to apply Hölder's inequality and Schur's test (Lemma A.3.4) and bound this quantity by

$$\begin{aligned}&\Bigg\|\sum_{n:|F_n|\sim A} 2^n A^{\frac{1}{p}}\Bigg\|_{\ell^p_A}\Bigg\|\sum_{m:|E_m|\sim B} 2^m B^{\frac{1}{q'}}\Bigg\|_{\ell^{p'}_B}\\ &\quad\lesssim \Bigg\|\sum_{n:|F_n|\sim A} 2^n A^{\frac{1}{p}}\Bigg\|_{\ell^p_A}\Bigg\|\sum_{m:|E_m|\sim B} 2^m B^{\frac{1}{q'}}\Bigg\|_{\ell^{q'}_B},\end{aligned}$$

where we have used $p \leq q$ and the nesting of sequence spaces in the final step. It therefore remains to verify that

$$\sum_A\Bigg(\sum_{n:|F_n|\sim A} 2^n A^{\frac{1}{p}}\Bigg)^p \lesssim \sum_A \sum_{n:|F_n|\sim A} 2^{np}|A| \sim \sum_n 2^{np}|F_n|,$$

and similarly for the $\ell^{q'}_B$ sum. The desired inequality follows from the fact that

$$\big(\sum_{n\in S} 2^n\big)^p \leq |2\max_{n\in S} 2^n|^p \leq 2^p \sum_{n\in S} 2^{np}$$

for any finite set $S \subset \mathbb{Z}$. This completes the proof that T is type (p,q). $\square$

We turn to another interpolation result. This estimate requires strong type bounds to begin with, but yields strong type bounds with no restriction on the order of p and q; it also easily gives an estimate on the operator norm.

Theorem 6.1.10 (Riesz–Thorin complex interpolation)**.** *Let T be a linear operator and $1 \le p_0, p_1, q_0, q_1 \le \infty$.*

If T is strong type (p_0, q_0) and (p_1, q_1), then T is strong type (p, q) for all (p, q) such that

$$(\tfrac{1}{p}, \tfrac{1}{q}) = (\tfrac{\theta}{p_0} + \tfrac{1-\theta}{p_1}, \tfrac{\theta}{q_0} + \tfrac{1-\theta}{q_1}) \quad \textit{for some} \quad \theta \in (0, 1).$$

In fact,

$$\|T\|_{L^p \to L^q} \lesssim \|T\|^{\theta}_{L^{p_0} \to L^{q_0}} \|T\|^{1-\theta}_{L^{p_1} \to L^{q_1}}.$$

Proof. Clearly we may assume $(p_0, q_0) \neq (p_1, q_1)$; otherwise, there is nothing to prove. Let us prove the estimate

$$\left|\langle Tf, g\rangle\right| \lesssim A_0^{\theta} A_1^{1-\theta} \|f\|_{L^p} \|g\|_{L^{q'}}$$

for all f, g simple functions of finite measure support, where

$$A_j = \|T\|_{L^{p_j} \to L^{q_j}}.$$

Let us first assume that none of the exponents equal infinity. Writing $f = \sum a_k \chi_{F_k}$, we observe that

$$f = \operatorname{sgn}(f) \left|\sum |a_k|^{\frac{p}{p_0}} \chi_{F_k}\right|^{\theta} \left|\sum |a_k|^{\frac{p}{p_1}} \chi_{F_k}\right|^{1-\theta},$$

which follows from the fact that

$$\sum a_k^p \chi_{F_k} = \left(\sum a_k \chi_{F_k}\right)^p$$

for disjoint sets F_k. In other words, we may factor

$$f = c f_0^{\theta} f_1^{1-\theta},$$

where f_j is a simple function in L^{p_j}. Similarly, we may factor

$$g = c' g_0^{\theta} g_1^{1-\theta}, \quad g_j \in L^{q_j'}.$$

Note that under this decomposition, we have

$$\|f_0\|^{\theta}_{L^{p_0}} \|f_1\|^{1-\theta}_{L^{p_1}} \sim \|f\|_{L^p}$$

and similarly for g.

Now we define

$$F(z) = \langle T(cf_0^z f_1^{1-z}), c' g_0^z g_1^{1-z} \rangle.$$

One can verify (by linearity of T and the fact that the functions involved are simple functions) that F is analytic and has at most exponential growth in z. Furthermore, we claim that by hypothesis we have

$$|F(z)| \lesssim \begin{cases} A_0 \|f_0\|_{L^{p_0}} \|g_0\|_{L^{q_0'}} & \operatorname{Re} z = 1 \\ A_1 \|f_1\|_{L^{p_1}} \|g_1\|_{L^{q_1'}} & \operatorname{Re} z = 0. \end{cases}$$

Indeed, let us consider the case $\operatorname{Re} z = 1$. Then we can write

$$f_0^z f_1^{1-z} = f_0 \cdot h \quad \text{where} \quad |h| \equiv 1$$

on the support of f, and similarly for the g term. Thus the result follows from Hölder's inequality and the assumption that T is type (p_0, q_0).

Applying the three lines lemma of complex analysis (Lemma A.3.6), we deduce that

$$\begin{aligned} |F(\theta)| &\lesssim A_0^\theta A_1^{1-\theta} \|f_0\|_{L^{p_0}}^\theta \|f_1\|_{L^{p_1}}^{1-\theta} \|g_0\|_{L^{q_0}}^\theta \|g_1\|_{L^{q_1}}^{1-\theta} \\ &\lesssim A_0^\theta A_1^{1-\theta} \|f\|_{L^p} \|g\|_{L^{q'}} \end{aligned}$$

for $\theta \in (0,1)$, which yields the result.

It remains to consider the case when one or more of the exponents is infinite. In light of Proposition 6.1.3, we may assume $p_0 \neq p_1$ and $q_0 \neq q_1$. Let us give the idea, but leave the details to the interested reader. If, say, $p_1 = \infty$, we use the fact that $\frac{1}{p} = \frac{\theta}{p_0}$ and write

$$\sum |a_k| \chi_{F_k} = \left(\sum |a_k|^{\frac{p}{p_0}} \chi_{F_k} \right)^\theta$$

in place of the decomposition above. We do a similar decomposition if one of the q_j is infinite. Following the same argument as above then yields the result. $\square$

A similar arguments extends this to the case where T also varies analytically in z. We leave the following result due to Stein as an exercise.

Theorem 6.1.11 (Stein interpolation)**.** *Let T_z be a family of linear operators defined on $\{0 \le \operatorname{Re} z \le 1\}$ such that for each pair of simple functions f, g of finite measure support, we have*

$$z \mapsto \langle T_z f, g\rangle$$

defines an analytic function on the strip with at most double-exponential growth. Suppose that for some $1 \le p_0, p_1, q_0, q_1 \le \infty$ we have

$$\|T_z\|_{L^{p_j} \to L^{q_j}} \le A_j \quad \textit{for} \quad \operatorname{Re} z = j.$$

Then

$$\|T_\theta\|_{L^p \to L^q} \le A_0^{1-\theta} A_1^\theta$$

for $\theta \in [0, 1]$ and (p, q) satisfying

$$(\tfrac{1}{p}, \tfrac{1}{q}) = (\tfrac{\theta}{p_0} + \tfrac{1-\theta}{p_1}, \tfrac{\theta}{q_0} + \tfrac{1-\theta}{q_1}).$$

6.2 Some classical inequalities

In this section we discuss some applications of the interpolation results given in the previous section.

The first is an estimate for the Fourier transform.

Theorem 6.2.1 (Hausdorff–Young inequality)**.** *The Fourier transform $\mathcal{F}$ is strong type (p', p) for all $2 \le p \le \infty$. That is,*

$$\|\hat{f}\|_{L^p} \lesssim \|f\|_{L^{p'}} \quad \textit{for all} \quad 2 \le p \le \infty.$$

Here we recall p′ is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. The Fourier transform $\mathcal{F}$ is strong type $(1, \infty)$ and $(2, 2)$. Therefore, by interpolation, $\mathcal{F}$ is strong type (p', p) for all $2 \le p \le \infty$. □

Next we have the Hardy–Littlewood–Sobolev inequality. This is a stronger version of what is commonly known as Young's convolution inequality.

Theorem 6.2.2 (Hardy–Littlewood–Sobolev)**.** *The following estimate holds:*

$$\|f * g\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^{q,\infty}}$$

whenever

$$1 < p < r < \infty, \quad 1 < q < \infty, \quad \textit{and} \quad 1 + \tfrac{1}{r} = \tfrac{1}{p} + \tfrac{1}{q}.$$

In particular,

$$\||x|^{-(d-\alpha)} * f\|_{L^r} \lesssim \|f\|_{L^p}$$

whenever

$$0 < \alpha < d, \quad 1 < p < r < \infty, \qquad \text{and} \quad \tfrac{1}{r} + \tfrac{\alpha}{d} = \tfrac{1}{p}.$$

Before turning to the proof, we recall two basic convolution inequalities.

Lemma 6.2.3. *The following hold:*

$$\|f * g\|_{L^\infty} \lesssim \|f\|_{L^p}\|g\|_{L^{p'}}, \quad \|f * g\|_{L^p} \lesssim \|f\|_{L^p}\|g\|_{L^1},$$

where $1 \leq p \leq \infty$.

Proof. The first estimate follows from Hölder's inequality, while the second follows from duality: for $h \in L^{p'}$, we use Fubini's theorem to write

$$|\langle f * g, h\rangle| = |\langle g, \tilde{f} * h\rangle| \lesssim \|g\|_{L^1}\|f * h\|_{L^\infty} \lesssim \|g\|_{L^1}\|f\|_{L^p}\|h\|_{L^{p'}},$$

where $\tilde{f}(x) = f(-x)$. □

Proof of Theorem 6.2.2. The first estimate implies the second, cf.

$$\||x|^{-(d-\alpha)} * f\|_{L^r} \lesssim \|f\|_{L^p}\||x|^{-(d-\alpha)}\|_{L^{\frac{d}{d-\alpha},\infty}} \lesssim \|f\|_{L^p}$$

provided we define exponents as above.

We turn to the first estimate. We fix $1 < q < \infty$ and $g \in L^{q,\infty}$ and consider the linear operator

$$Tf = f * g.$$

We normalize g so that $\|g\|_{L^{q,\infty}} = 1$. Our goal is then to prove that T is strong type (p, r) for all $1 < p < r < \infty$ satisfying the scaling relation above. In fact, by (Marcinkiewicz) interpolation it suffices to show that T is weak type (p, r) for such exponents. Thus we aim to show

$$|\{|f * g| > \alpha\}| \lesssim \alpha^{-r}\|f\|_{L^p}^r$$

uniformly in $\alpha > 0$. To this end, let us fix $\alpha > 0$.

Let $\lambda > 0$ and split $g = g_1 + g_2$, where $g_1 = g\chi_{\{|g| \leq \lambda\}}$. In particular,

$$|\{|f * g| > \alpha\}| \leq |\{|f * g_1| > \tfrac{1}{2}\alpha\}| + |\{|f * g_2| > \tfrac{1}{2}\alpha\}|.$$

We claim that by choosing λ sufficiently small, the first term on the right-hand side vanishes. To see this, we first need the following bound:

$$\|g \cdot \chi_{|g|\le\lambda}\|_{L^a} \le \|g\|_{L^{q,\infty}}^{\frac{q}{a}} \lambda^{1-\frac{q}{a}} \quad \text{for any} \quad 1 < q < a < \infty, \tag{6.5}$$

which we leave as an exercise. Thus, using (6.5) with $a = p'$ (which is greater than q provided $r < \infty$), we find

$$\|f * g_1\|_{L^\infty} \lesssim \|g_1\|_{L^{p'}} \|f\|_{L^p} \lesssim \lambda^{1-\frac{q}{p'}} \|f\|_{L^p},$$

which implies the claim, provided we choose $\lambda^{1-\frac{q}{p'}} \ll \alpha \|f\|_{L^p}^{-1}$, i.e.

$$\lambda = c\alpha^{\frac{p'}{p'-q}} \|f\|_{L^p}^{-\frac{p'}{p'-q}}$$

for $0 < c \ll 1$.

We are left to consider the contribution of g_2 above. In this case, we need the following:

$$\|g \cdot \chi_{|g|>\lambda}\|_{L^1} \lesssim \|g\|_{L^{q,\infty}}^{q} \lambda^{1-q} \quad \text{for any} \quad 1 < q < \infty, \tag{6.6}$$

which we leave as a further exercise. Now, applying Tchebychev and (6.6), we estimate (using the form of λ above)

$$\begin{aligned}
|\{|f * g_2| > \tfrac{1}{2}\alpha\}| &\lesssim \alpha^{-p} \|f * g_2\|_{L^p}^p \\
&\lesssim \alpha^{-p} \|f\|_{L^p}^p \|g_2\|_{L^1}^p \\
&\lesssim \alpha^{-p} \lambda^{p(1-q)} \|f\|_{L^p}^p \|g_2\|_{L^{q,\infty}}^{pq} \\
&\lesssim \alpha^{-p+p(1-q)\frac{p'}{p'-q}} \|f\|_{L^p}^{p-p(1-q)\frac{p'}{p'-q}}.
\end{aligned}$$

Using the scaling relations, we can simplify this to

$$|\{|f * g_2| > \tfrac{1}{2}\alpha\}| \lesssim \alpha^{-r} \|f\|_{L^p}^r,$$

which finally completes the proof. □

An important application of the Hardy–Littlewood–Sobolev inequality yields a class of so-called **Sobolev embedding** inequalities. To describe them, we need to introduce a family of operators that extend the usual notion of derivatives.

Recall that differentiation operators act as multiplication operators under the Fourier transform. More precisely, we found

$$\mathcal{F}[\partial^\alpha f](\xi) = (i\xi)^\alpha \hat{f}(\xi),$$

which we may write more succinctly as

$$\partial^\alpha = \mathcal{F}^{-1}(i\xi)^\alpha \mathcal{F}$$

as operators. In fact, an important class of operators (known as **Fourier multiplier operators**) arises in this fashion: given a function $m(\xi)$, we may define the operator

$$T_m := \mathcal{F}^{-1} m(\xi) \mathcal{F}.$$

Using Lemma 2.5.11, we find that (as long as m is a reasonably well-behaved function) the operator T_m is given as a convolution operator, namely

$$T_m f = (2\pi)^{-\frac{d}{2}} \check{m} * f,$$

where $\check{m} = \mathcal{F}^{-1} m$. Conversely, a convolution operator (i.e. an operator of the form $Tf = g * f$) may also be viewed as the Fourier multiplier operator with symbol $(2\pi)^{\frac{d}{2}} \hat{g}$.

Remark 6.2.4. We have just observed that convolution operators and Fourier multipliers are one in the same. We remark here that a (bounded) operator T is of this type if and only if it commutes with translations. Indeed, to see that this is sufficient, note that for any test function ψ, the fact that T commutes with translation implies

$$\psi * Tf = T^* \psi * f,$$

where T^* denotes the adjoint of T. Thus, by Lemma 2.5.11,

$$\hat{\psi} \cdot \widehat{Tf} = \widehat{T^* \psi} \cdot \hat{f}.$$

Choosing ψ so that $\hat{\psi}$ is everywhere nonzero (e.g. by choosing $\psi = e^{-|x|^2}$, cf. Lemma 2.5.9), we find that T is a Fourier multiplier operator, as desired. Let us finally note here that all Fourier multipliers commute with one another, as well.

Returning to the above, we can naturally extend the notion of derivative and define the class of operators $|\nabla|^s$ as Fourier multiplier operators, namely,

$$|\nabla|^s = \mathcal{F}^{-1} |\xi|^s \mathcal{F}, \quad s \in \mathbb{R}.$$

For $-d < s < 0$ these operators are instead known as 'fractional integration' operators. In this case we can actually compute the inverse Fourier transform of $|\xi|^s$ (in the sense of distributions) and so compute the corresponding convolution kernel. In fact, we already saw a special case of this in Section 2.6, when we used a symmetry argument to deduce that

$$\mathcal{F}^{-1}|\xi|^{-2} = c|x|^{-1} \quad \text{when} \quad d = 3.$$

Arguing similarly, one finds that

$$\mathcal{F}^{-1}|\xi|^{-s} = c_{s,d}|x|^{s-d} \quad \text{for} \quad 0 < s < d.$$

Exercise 2.9.16 shows how to compute exactly the constants $c_{s,d}$.

Having introduced this notion of fractional derivative (or fractional integration), we can now state an important class of Sobolev embedding inequalities.

Theorem 6.2.5 (Sobolev embedding)**.** *Let*

$$s > 0, \quad 1 < p < q < \infty \quad \textit{and} \quad \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{s}{d}.$$

For any $f \in \mathcal{S}$, we have

$$\|f\|_{L^q} \lesssim \||\nabla|^s f\|_{L^p}.$$

Remark 6.2.6. Let us briefly explain the name Sobolev embedding. Just as one has 'Lebesgue spaces', there is also a notion of 'Sobolev spaces'. In particular, one defines the spaces

$$\dot{W}^{s,p} := \{f \in \mathcal{S}' : \||\nabla|^s f\|_{L^p} < \infty\}$$

for $1 \le p \le \infty$ and $s \in \mathbb{R}$. The theorem above asserts that

$$\dot{W}^{s,p} \hookrightarrow L^q \quad \text{whenever} \quad s > 0, \quad 1 < p < q < \infty \quad \text{and} \quad \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{s}{d}.$$

That is, it yields an embedding result between Sobolev and Lebesgue spaces. Such results have wide application in the field of PDE.

Proof of Theorem 6.2.5. We argue by duality. For $f, g \in \mathcal{S}$ we first observe (by Plancherel and the definition of $|\nabla|^s$) that

$$|\langle f, g\rangle| = |\langle |\nabla|^s f, |\nabla|^{-s} g\rangle| \le \| \, |\nabla|^s f\|_{L^p} \||\nabla|^{-s} g\|_{L^{p'}}.$$

Now, by Hardy–Littlewood–Sobolev (and the scaling relations)

$$\||\nabla|^{-s} g\|_{L^{p'}} \lesssim \||x|^{s-d} * g\|_{L^{p'}} \lesssim \|g\|_{L^{q'}}.$$

The result follows. $\square$

The result just stated only applies to the fractional derivative operators $|\nabla|^s$. What about when $s \in \mathbb{N}$? For example, do we have

$$\|f\|_{L^q} \lesssim \|\nabla f\|_{L^p} \quad \text{whenever} \quad 1 < p < q < \infty \quad \text{and} \quad \tfrac{1}{q} = \tfrac{1}{p} - 1.$$

This would follow if we could establish that

$$\||\nabla|f\|_{L^p} \lesssim \|\nabla f\|_{L^p}$$

for any $1 < p < \infty$. This question, in turn, boils down to a question about certain Fourier multiplier operators. Namely, are the operators defined with multipliers

$$m_j(\xi) = \tfrac{\xi_j}{|\xi|}$$

(known as **Riesz transforms**) bounded on L^p? The answer is quickly seen to be yes for $p = 2$; this is a consequence of Plancherel. For $p \neq 2$ the answer is still yes, but it is not as simple to prove it. We will return to this question below.

6.3 Hardy–Littlewood maximal function

We turn to our next object of study, namely, the Hardy–Littlewood maximal function. This is one of the most fundamental objects of study in harmonic analysis.

For $x \in \mathbb{R}^d$ and $r > 0$, we denote the ball of radius r centered at x by

$$B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}.$$

Given a locally integrable function f on $\mathbb{R}^d$ and $r > 0$, we consider the average

$$\tfrac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy.$$

This is well-defined and is a continuous function of x. The operator

$$Mf(x) := \sup_{r>0} \tfrac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

is called the **Hardy–Littlewood maximal function** of f. The operator M is called the Hardy–Littlewood maximal operator.

It is easy to observe that M is strong type (∞, ∞). Our main goal will be to establish the following.

Theorem 6.3.1. *The Hardy–Littlewood maximal operator is weak type* $(1,1)$*; that is,*

$$\|Mf\|_{L^{1,\infty}} \lesssim \|f\|_{L^1}.$$

Applying the Marcinkiewicz interpolation theorem, we can then conclude the following:

Corollary 6.3.2. *The Hardy–Littlewood maximal operator is strong type* (p,p) *for all* $1 < p \leq \infty$.

Before we begin the proof, let us make a few observations about the operator M. First, suppose f is a nontrivial function. By considering $r \sim |x| \gg 1$, we can observe that

$$|Mf(x)| \gtrsim |x|^{-d}.$$

Thus we should not expect M to be strong type $(1,1)$; instead, mapping into weak L^1 is the natural assertion. Of course, this rate of decay is compatible with mapping into all higher L^p spaces.

We will need the following lemma.

Lemma 6.3.3 (Wiener covering lemma)**.** *Let* $B_j = B_j(x_j, r_j)$ *be a finite collection of balls in* $\mathbb{R}^d$. *There exists a subcollection of balls, denoted* S, *such that*

- *distinct balls in* S *are disjoint,*
- $\cup B_j \subset \cup_S B(x_j, 3r_j)$.

Proof. We run the following algorithm. Begin by setting $S = \emptyset$.

1. Choose a ball of largest radius from the remaining collection and add it to S.
2. Discard from the remaining collection all balls that intersect a ball in S.
3. If no balls remain, stop. Otherwise, return to step 1.

This algorithm terminates in finitely many steps; indeed, we remove at least one ball from the collection in each step. The balls in S are disjoint by construction, so it remains to verify the second point. In fact, if B_j does not belong to S, it must interest a ball in S of larger radius; it then belongs in the three times dilate of that ball. This completes the proof. $\square$

We are now ready to prove Theorem 6.3.1.

Proof of Theorem 6.3.1. Let $\alpha > 0$ and consider an arbitrary compact subset K of the set

$$\{x : Mf(x) > \alpha\}.$$

We need to show that $|K| \lesssim \alpha^{-1}\|f\|_{L^1}$ (with implicit constant independent of K).

Let $x \in K$. Then there exists a radius $r(x) > 0$ such that

$$\tfrac{1}{|B(x,r(x))|} \int_{B(x,r(x))} |f(y)|\, dy \geq \alpha.$$

We therefore have

$$K \subset \bigcup_{x \in K} B(x, r(x)),$$

and hence by compactness there exists a finite subcollection $B_j = B(x_j, r(x_j))$ such that $K \subset \cup_{j=1}^{N} B_j$. By the Wiener covering lemma, we may find a subcollection S of disjoint balls such that $K \subset \cup_S B(x_j, 3r(x_j))$.

We now write

$$|K| \leq \sum_S |3B_j| \leq 3^d \sum_S |B_j|.$$

Now, by the choice of $r(x_j)$, we have

$$|B_j| \lesssim \alpha^{-1} \int_{B_j} |f(y)|\, dy \quad \text{for each} \quad j,$$

and hence (since the balls are disjoint) we have

$$|K| \leq 3^d \alpha^{-1} \sum_S \int_{B_j} |f(y)|\, dy \leq 3^d \alpha^{-1} \int |f(y)|\, dy.$$

As the implicit constant (namely, 3^d) is independent of K, we may now take the supremum over compact $K \subset \{x : Mf > \alpha\}$ to conclude the desired result. □

A standard application of the Hardy–Littlewood maximal inequality is to prove the Lebesgue differentiation theorem:

Proposition 6.3.4. *Let f be locally integrable. Then*

$$\lim_{r \to 0} \tfrac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x)$$

for almost every x.

The details are left as an exercise, but we sketch the idea as follows. First, one should show that it suffices to treat functions in L^1 (not merely locally integrable). Next, one should observe that the result follows for smooth, compactly supported functions. Finally, one should extend this to L^1 by approximation; it is here that the maximal function estimate will come into play.

Let us next discuss a generalization of the result above that we will need later. For a locally integrable function $\omega : \mathbb{R}^d \to [0, \infty)$, we define a measure via

$$\omega(E) = \int_E \omega(x)\, dx.$$

Then we have the following result:

Theorem 6.3.5. *We have $M : L^1(M\omega\, dx) \to L^{1,\infty}(\omega\, dx)$ and $M : L^p(M\omega\, dx) \to L^p(\omega\, dx)$ for $1 < p \leq \infty$. That is,*

$$\omega\big(\{Mf > \alpha\}\big) \lesssim \alpha^{-1} \int |f(x)|\, M\omega(x)\, dx,$$

$$\int |Mf(x)|^p \omega(x)\, dx \lesssim \int |f(x)|^p M\omega(x)\, dx \quad \textit{for} \quad 1 < p < \infty,$$

$$\|Mf\|_{L^\infty(\omega\, dx)} \lesssim \|f\|_{L^\infty(M\omega\, dx)}.$$

Sketch of proof. Again, by Marcinkiewicz interpolation it suffices to establish strong type (∞, ∞) and weak type $(1, 1)$ bounds. For the (∞, ∞) bounds, we note

$$\|Mf\|_{L^\infty(\omega\, dx)} = \inf_{\omega(E)=0} \sup_{x \in E^c} Mf(x) \leq \|f\|_{L^\infty(dx)},$$

and thus it suffices to check that

$$\|f\|_{L^\infty(dx)} = \inf_{|E|=0} \sup_{x \in E^c} |f(x)| \leq \inf_{(M\omega)(E)=0} \sup_{x \in E^c} |f(x)| = \|f\|_{L^\infty(M\omega\, dx)}.$$

(cf. Exercise A.4.3). In fact, this inequality is a consequence of the fact that $(M\omega)(E) = 0$ implies $|E| = 0$.

For the weak type $(1, 1)$ bound, the argument is similar to that appearing in the proof of Theorem 6.3.1. One constructs the set K, the balls B_j, and the subcollection S as in that proof. This time we need to estimate $\omega(3B_j)$ (where $3B_j := B_j(x_j, 3r_j)$, which we do by writing

$$\omega(3B_j) = \int_{3B_j} \omega(x)\, dx \leq \int_{|x-y|<4r_j} \omega(x)\, dx \leq 4^d |B_j| M\omega(y)$$

for any $y \in B_j$, so that

$$\omega(3B_j) \int_{B_j} |f(y)|\, dy \lesssim |B_j| \int_{B_j} |f(y)| M\omega(y)\, dy.$$

This implies (by the choice of B_j)

$$\omega(3B_j) \lesssim \alpha^{-1} \int_{B_j} |f(y)| M\omega(y)\, dy,$$

which then gives

$$\omega(K) \leq \sum_S \omega(3B_j) \lesssim \alpha^{-1} \int_{B_j} |f(y)| M\omega(y)\, dy \lesssim \alpha^{-1} \int |f(y)| M\omega(y)\, dy,$$

yielding the desired bound. □

Remark 6.3.6. It is also a natural question to ask for which weights ω we have that M maps $L^p(\omega\, dx)$ to $L^p(\omega\, dx)$ boundedly for some $1 < p < \infty$. The sharp condition for this is known; in particular, one needs that

$$\sup_B \tfrac{1}{|B|} \int_B \omega(y)\, dy \cdot \left(\tfrac{1}{|B|} \int_B \omega(y)^{-\frac{p'}{p}}\, dy \right)^{\frac{p}{p'}} \lesssim 1.$$

In this case we call ω an A_p weight, and write $\omega \in A_p$. We will not pursue the topic of A_p weights in these notes; we refer the interested reader to [27].

We next turn our attention to so-called **vector-valued maximal functions**.

Definition 6.3.7. For $f : \mathbb{R}^d \to \ell^2(\mathbb{C})$ given by $f(x) = \{f_n(x)\}_{n\geq 1}$, we define

$$\|f\|_{L^p} = \left(\int \|f(x)\|_{\ell^2}^p\, dx \right)^{\frac{1}{p}}.$$

We define the vector maximal function by

$$\bar{M}f(x) = \left(\sum_{n\geq 1} |Mf_n(x)|^2 \right)^{\frac{1}{2}} = \|\{Mf_n(x)\}\|_{\ell^2}.$$

The result we will prove is the following.

Theorem 6.3.8. *The operator $\bar{M}$ is weak type $(1,1)$ and strong type (p,p) for all $1<p<\infty$. That is,*

$$|\{x : \bar{M}f > \alpha\}| \lesssim \alpha^{-1}\|f\|_{L^1}$$

and

$$\|\bar{M}f\|_{L^p} \lesssim \|f\|_{L^p} \quad \textit{for} \quad 1<p<\infty.$$

Remark 6.3.9. In the scalar case, the L^∞ case was trivial. In the vector case, it is false! In particular, one can check that if $f_n(x) = \chi_{[2^{n-1},2^n]}(x)$ then $f \in L^\infty$ but $|\bar{M}f(x)|^2 \equiv \infty$. Instead, the trivial estimate is from L^2 to L^2. Indeed,

$$\begin{aligned}
\|\bar{M}f\|_{L^2} &= \left\| \left(\sum_n |Mf_n|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \\
&\lesssim \left(\int \sum |Mf_n(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\lesssim \left(\int \sum |f_n|^2 \, dx \right)^{\frac{1}{2}} = \|f\|_{L^2},
\end{aligned}$$

where we use that $M : L^2 \to L^2$ boundedly.

We now have enough tools to prove Theorem 6.3.8 when $p \in (2,\infty)$.

Proof of Theorem 6.3.8 for $p \in (2,\infty)$. Recall from Theorem 6.3.5 that we have

$$\int |Mf_n(x)|^2 \omega(x)\, dx \lesssim \int |f_n(x)|^2 M\omega(x)\, dx,$$

so that

$$\int |\bar{M}f(x)|^2 \omega(x)\, dx \lesssim \int \|f(x)\|_{\ell^2}^2 M\omega(x)\, dx$$

for any locally integrable ω. Now let $2<p<\infty$ and set $q = (\frac{p}{2})'$. Then by duality we have

$$\begin{aligned}
\|\bar{M}f\|_{L^p}^2 = \|(\bar{M}f)^2\|_{L^{\frac{p}{2}}} &= \sup_{\|\omega\|_{L^q}=1} \int \left|\bar{M}f\right|^2 \omega \, dx \\
&\lesssim \sup_{\|\omega\|_{L^q}=1} \int \|f(x)\|_{\ell^2}^2 M\omega \, dx \\
&\lesssim \sup_{\|\omega\|_{L^q}=1} \left\| \|f(x)\|_{\ell^2}^2 \right\|_{L^{\frac{p}{2}}} \|M\omega\|_{L^q} \lesssim \|f\|_{L^p}^2,
\end{aligned}$$

where we have used that M maps $L^q \to L^q$ boundedly. This completes the proof. $\square$

It remains to establish the weak type $(1,1)$ bound in Theorem 6.3.8, for then all of the remaining cases follow from Marcinkiewicz interpolation. For this, we introduce the so-called **Calderon–Zygmund decomposition**.

Lemma 6.3.10 (Calderon–Zygmund decomposition). *Let $f \in L^1(\mathbb{R}^d; \ell^2(\mathbb{C}))$ and set $\alpha > 0$. There exists a decomposition $f = g + b$ such that g, b have the following properties:*

- $\|g(x)\|_{\ell^2} \leq \alpha$ *for a.e.* $x \in \mathbb{R}^d$.
- *The support of b is a union of nonoverlapping cubes Q_k, with*

$$\alpha < \tfrac{1}{|Q_k|} \int_{Q_k} \|b(x)\|_{\ell^2} \, dx \leq 2^d \alpha.$$

- *We have $g = f(1 - \sum \chi_{Q_k})$.*

Proof. We begin by decomposing $\mathbb{R}^d$ into a mesh of equal-sized nonoverlapping cubes whose common diameter is large enough that

$$\tfrac{1}{|Q|} \int_Q \|f(x)\|_{\ell^2} \, dx \leq \alpha$$

for all cubes in the mesh.

Let Q be one of the cubes in this mesh. Subdivide Q into 2^d congruent cubes, and let Q' denote one of the resulting cubes. If

$$\tfrac{1}{|Q'|} \int_{Q'} \|f(x)\|_{\ell^2} \, dx > \alpha,$$

stop and select Q' as one of the cubes Q_k. Note that in this case we have

$$\alpha < \tfrac{1}{|Q'|} \int_{Q'} \|f(x)\|_{\ell^2} \, dx \leq \tfrac{2^d}{|Q|} \int_Q \|f(x)\|_{\ell^2} \, dx \leq 2^d \alpha.$$

If instead

$$\tfrac{1}{|Q'|} \int_{Q'} \|f(x)\|_{\ell^2} \, dx \leq \alpha,$$

then we subdivide further into 2^d congruent cubes and repeat the same selection process for each of the resulting cubes. We continue subdividing until (if ever) we are forced into the first case.

We repeat this with all of the cubes in the mesh, leading to the collection of nonoverlapping cubes Q_k. We then define $b = f \cdot \sum \chi_{Q_k}$, which has the desired bounds by construction.

It remains to verify that $\|g(x)\|_{\ell^2} \leq \alpha$ for a.e. $x \in \mathbb{R}^d$, where $g = f - b$. To this end, note that for any $x \notin \cup Q_k$, there exists a sequence of cubes $Q \ni x$ with diameter tending to zero and such that

$$\tfrac{1}{|Q|} \int_Q \|f(y)\|_{\ell^2}\, dy \leq \alpha.$$

Applying the Lebesgue differentiation theorem (to the integrable function $x \mapsto \|f(x)\|_{\ell^2}$), we deduce that

$$\|f(x)\|_{\ell^2} \leq \alpha \quad \text{for almost every} \quad x \notin \cup Q_k.$$

As $f = g$ outside of $\cup Q_k$, the result follows. $\square$

Remark 6.3.11. The same proof and decomposition works for scalar valued f. In this case we can extend the result to get a mean zero condition for b, namely

$$\tfrac{1}{|Q_k|} \int_{Q_k} b\, dx = 0$$

for each k. Indeed, in this case we let

$$g(x) = \begin{cases} f(x) & x \notin \cup Q_k, \\ \frac{1}{|Q_k|} \int_{Q_k} f(y)\, dy & x \in Q_k^\circ \end{cases}$$

and again set $b = f - g$. In this case we have $|g(x)| \leq 2^d \alpha$, while

$$\int_{Q_k} b(x)\, dx = \int_{Q_k} [f(x) - g(x)]\, dx = 0.$$

Note that we still have

$$\tfrac{1}{|Q_k|} \int_{Q_k} |b(x)|\, dx \lesssim \alpha.$$

Proof of Theorem 6.3.8. Let $\alpha > 0$ and $f \in L^1(\mathbb{R}^d; \ell^2(\mathbb{C}))$. We use the Calderon–Zygmund decomposition to write $f = g + b$ as above. In particular,

$$|\{|\bar{M} f| > \alpha\}| \leq |\{|\bar{M} g| > \tfrac{1}{2}\alpha\}| + \{|\bar{M} b| > \tfrac{1}{2}\alpha\}|.$$

We can first estimate by Tchebychev and the strong type $(2,2)$ bound for $\bar{M}$:

$$|\{|\bar{M} g| > \tfrac{1}{2}\alpha\}| \lesssim \alpha^{-2} \|\bar{M} g\|_{L^2}^2 \lesssim \alpha^{-2} \|g\|_{L^2}^2.$$

Now, since $\|g(x)\|_{\ell^2} \le \alpha$ a.e., we have

$$\alpha^{-2}\int \|g(x)\|_{\ell^2}^2\,dx \le \alpha^{-1}\int \|g(x)\|_{\ell^2}\,dx \le \alpha^{-1}\|g\|_{L^1} \le \alpha^{-1}\|f\|_{L^1},$$

which is acceptable. It remains to treat the contribution of b.

Observe that by construction we have

$$\left|\cup Q_k\right| \le \alpha^{-1}\|f\|_{L^1}.$$

In particular, writing $2Q_k$ for the dilate of Q_k with the same center, we have

$$\left|\cup 2Q_k\right| \lesssim 2^d\alpha^{-1}\|f\|_{L^1},$$

and hence it suffices to show that

$$|\{x \in \left(\cup 2Q_k\right)^c : \bar{M}b > \tfrac{1}{2}\alpha\}| \lesssim \alpha^{-1}\|f\|_{L^1}.$$

For this, we introduce an averaged version of b with components

$$\tilde{b}_n = \sum_k \chi_{Q_k}\tfrac{1}{|Q_k|}\int_{Q_k}|f_n(y)|\,dy.$$

We first observe that for $x \in Q_k$,

$$\|\tilde{b}(x)\|_{\ell^2} = \left\|\tfrac{1}{|Q_k|}\int_{Q_k}|f_n(y)|\,dy\right\|_{\ell^2} \le \tfrac{1}{|Q_k|}\int_{Q_k}\|f_n(y)\|_{\ell^2}\,dy \lesssim \alpha.$$

This also shows

$$\|\tilde{b}\|_{L^1} = \sum_k \int_{Q_k}\|\tilde{b}(x)\|_{\ell^2}\,dx \le \sum_k\int_{Q_k}\|f(y)\|_{\ell^2}\,dy \le \|f\|_{L^1}.$$

Finally, note that if $x \notin \cup_k 2Q_k$ and $B(x,r)\cap Q_k \ne \emptyset$, then $Q_k \subset B(x,2r)$. Thus, letting $S = S_{r,x} = \{k : B(x,r)\cap Q_k \ne \emptyset\}$, we can estimate

$$\begin{aligned}
Mb_n(x) &= \sup_{r>0}\tfrac{1}{|B(x,r)|}\sum_{k\in S}\int_{B(x,r)\cap Q_k}|b_n(y)|\,dy \\
&\lesssim \sup_{r>0}\tfrac{1}{|B(x,2r)|}\sum_{k\in S}|Q_k|\cdot\tfrac{1}{|Q_k|}\int_{Q_k}|b_n(y)|\,dy \\
&\lesssim \sup_{r>0}\tfrac{1}{|B(x,2r)|}\int_{B(x,2r)}\sum_k \chi_{Q_k}(z)\tfrac{1}{|Q_k|}\int_{Q_k}|b_n(y)|\,dy\,dz \\
&\lesssim M\tilde{b}_n(x),
\end{aligned}$$

whence $\bar{M}b \lesssim \bar{M}\tilde{b}$.

Using the above and arguing as we did for g,

$$\begin{aligned}
|\{x \notin \cup_k 2Q_k : \bar{M}b > \tfrac{1}{2}\alpha\}| &\lesssim |\{\bar{M}\tilde{b} \gtrsim \alpha\}| \\
&\lesssim \alpha^{-2}\|\bar{M}\tilde{b}\|_{L^2}^2 \\
&\lesssim \alpha^{-2}\|\tilde{b}\|_{L^2}^2 \\
&\lesssim \alpha^{-1}\|\tilde{b}\|_{L^1} \lesssim \alpha^{-1}\|f\|_{L^1},
\end{aligned}$$

which is acceptable. This completes the proof. □

Remark 6.3.12. The theory of vector maximal functions can be extended to ℓ^q instead of ℓ^2, but we will not pursue this here.

6.4 Calderón–Zygmund theory

Recall that (in the setting of Sobolev embedding) we encountered the question of whether the Fourier multiplier operators defined via the symbol $m_j(\xi) = \frac{\xi_j}{|\xi|}$ (known as **Riesz transforms**) are bounded on L^p spaces. To answer this question (as well as to understand some other fundamental operators in harmonic analysis) we will need to develop what is known as Calderón–Zygmund theory. This theory addresses the case of 'singular integral operators'. The precise definition we need is the following.

Definition 6.4.1. A **Calderón–Zygmund convolution kernel** is a function $K : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ that obeys

(a) $|K(x)| \lesssim |x|^{-d}$ uniformly for $x \neq 0$,

(b) For any $0 < R_1 < R_2 < \infty$,

$$\int_{R_1 \leq |x| \leq R_2} K(x)\, dx = 0,$$

(c) The estimate

$$\int_{|x| \geq 2|y|} |K(x+y) - K(x)|\, dx \lesssim 1$$

holds uniformly for $y \in \mathbb{R}^d$.

Given a Calderón–Zygmund kernel K, we will consider the operator defined via $f \mapsto K * f$.

Example 6.4.1 (Riesz transforms). Consider $m_j(\xi) = \frac{-i\xi_j}{|\xi|}$. The Fourier multiplier operator with symbol m_j is a convolution operator with the kernel

$$K_j(x) = \mathcal{F}^{-1}m_j(x) = -\partial_{x_j}\mathcal{F}^{-1}(|\xi|^{-1}).$$

Now recall that $\mathcal{F}^{-1}(|\xi|^{-1}) = c|x|^{1-d}$, and so

$$K_j(x) = cx_j|x|^{-(d+1)}$$

for some c.

Using this, we readily observe that (a) holds. As for (b), let us consider (without loss of generality) the case $j = 1$ and $d \geq 2$. Define $A = \operatorname{diag}(-1, -1, 1, \ldots, 1)$ and consider the change of variables $y = Ax$. As $\det A = 1$, $|y| = |Ax|$, and $y_1 = -x_1$, this shows

$$\int_{R_1 \leq |x| \leq R_2} x_j|x|^{-(d+1)}\, dx = -\int_{R_1 \leq |x| \leq R_2} x_j|x|^{-(d+1)}\, dx,$$

which shows this integral is zero. Finally, for (c) we observe that

$$|\nabla K_j| = \mathcal{O}(|x|^{-(d+1)}),$$

and hence the desired inequality follows from Lemma 6.4.2 below.

Lemma 6.4.2. *A kernel K obeys (c) whenever $|\nabla K(x)| \lesssim |x|^{-(d+1)}$.*

Proof. We use the fundamental theorem of calculus to write

$$K(x+y) - K(x) = \int_0^1 \nabla K(x + \theta y) \cdot y\, d\theta.$$

Thus

$$\begin{aligned}
\int_{|x| \geq 2|y|} |K(x+y) - K(y)|\, dx &\lesssim \int_0^1 \int_{|x|>2|y|} |y||\nabla K(x+\theta y)|\, d\theta\, dx \\
&\lesssim |y| \int_0^1 \int_{|x| \geq 2|y|} |x + \theta y|^{-(d+1)}\, dx\, d\theta \\
&\lesssim |y| \int_{|x| \geq 2|y|} |x|^{-(d+1)}\, dx \lesssim 1.
\end{aligned}$$

In the above, we have used

$$|x| \geq |x + \theta y| - \theta|y| \geq |x + \theta y| - \tfrac{1}{2}\theta|x|,$$

which implies

$$|x+\theta y|^{-(d+1)} \lesssim |x|^{-(d+1)}.$$

We also used

$$\int_{|x|\geq 2|y|} |x|^{-(d+1)}\,dx \lesssim |y|^{-1},$$

which can be seen by changing to spherical coordinates:

$$\int_{|x|\geq 2|y|} |x|^{-(d+1)}\,dx = \int_{\mathbb{S}^{d-1}} \int_{2|y|}^{\infty} r^{-(d+1)} r^{d-1}\,dr\,d\omega \lesssim \int_{2|y|}^{\infty} r^{-2}\,dr \lesssim |y|^{-1}.$$

This completes the proof. □

Example 6.4.2 (Hilbert transform). Define $K : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by $K(x) = \frac{1}{\pi x}$. This kernel defines the **Hilbert transform**

$$f \mapsto \tfrac{1}{\pi} \int \tfrac{f(x-y)}{y}\,dy.$$

This kernel clearly satisfies (a) and (b), while to verify (c) we compute the derivative of K.

We turn to our first main result, which shows that Calderón–Zygmund operators define bounded operators on L^2 (i.e. are type $(2,2)$).

Theorem 6.4.3. *Let K be a Calderón–Zygmund convolution kernel. Given $\varepsilon > 0$, define*

$$K_\varepsilon(x) = \chi_{\{\varepsilon<|x|<\varepsilon^{-1}\}}(x) K(x).$$

Then for all $f \in \mathcal{S}$,

$$\|K_\varepsilon * f\|_{L^2} \lesssim \|f\|_{L^2} \quad \textit{uniformly in} \quad \varepsilon > 0.$$

Consequently, the operator

$$f \mapsto K * f := \lim_{\varepsilon\to 0} K_\varepsilon * f$$

extends from Schwartz space to a bounded operator on L^2.

Proof. Let $f \in \mathcal{S}$. We will show that $\{K_\varepsilon * f\}_{\varepsilon>0}$ is Cauchy in L^2 as $\varepsilon \to 0$. This implies that $K_\varepsilon * f$ converges in L^2. We denote the limit by $K * f$, which then satisfies

$$\|K * f\|_{L^2} \lesssim \|f\|_{L^2} + o(1) \quad \text{as} \quad \varepsilon \to 0.$$

Now let $0 < \varepsilon_1 < \varepsilon_2$. Then we may write

$$\begin{aligned}
&K_{\varepsilon_1} * f(x) - K_{\varepsilon_2} * f(x) \\
&\quad = \int_{\varepsilon_1 \le |y| \le \varepsilon_2} K(y) f(x-y)\,dy - \int_{\varepsilon_2^{-1} \le |y| \le \varepsilon_1^{-1}} K(y) f(x-y)\,dy.
\end{aligned}$$

Consider first the contribution of $\varepsilon_1 \le |y| \le \varepsilon_2$. Using the cancellation condition (b), the fundamental theorem of calculus, and the bound in (a), we may write this term as

$$\begin{aligned}
&\left| \int_{\varepsilon_1 \le |y| \le \varepsilon_2} K(y)[f(x-y) - f(x)]\,dy \right| \\
&\lesssim \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |K(y)||y| \int_0^1 |\nabla f(x+\theta y)|\,d\theta\,dy \\
&\lesssim \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |y|^{1-d} \|\nabla f\|_{L^\infty(\{|y| \sim |x|\})}\,dy.
\end{aligned}$$

Because f is a Schwartz function, we may bound this gradient term by $\langle x \rangle^{-100d}$ (where $\langle x \rangle = \sqrt{1+|x|^2}$). Thus, performing the integral in y, we find

$$\left\| \int_{\varepsilon_1 \le |y| \le \varepsilon_2} K(y) f(x-y)\,dy \right\|_{L^2_x} \lesssim \varepsilon_2 - \varepsilon_1 \to 0 \quad \text{as} \quad \varepsilon_2, \varepsilon_1 \to 0.$$

It remains to treat the range $\varepsilon_2^{-1} \le |y| \le \varepsilon_1^{-1}$. For this, we use the convolution inequality Lemma 6.2.3 and (a) to bound

$$\begin{aligned}
&\left\| \int_{\varepsilon_2^{-1} \le |y| \le \varepsilon_1^{-1}} K(y) f(x-y)\,dy \right\|_{L^2} \\
&\lesssim \|K \chi_{\varepsilon_2^{-1} \le |y| \le \varepsilon_1^{-1}} * f\|_{L^2} \\
&\lesssim \|f\|_{L^1} \|K \chi_{\varepsilon_2^{-1} \le |y| \le \varepsilon_1^{-1}}\|_{L^2} \\
&\lesssim \|f\|_{L^1} \left(\int_{\varepsilon_2^{-1} \le |y| \le \varepsilon_1^{-1}} |x|^{-2d}\,dx \right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^1} \varepsilon_2^{\frac{d}{2}} \to 0 \quad \text{as} \quad \varepsilon_2 \to 0.
\end{aligned}$$

Now we need to establish $L^2 \to L^2$ bounds for K_ε that are uniform in $\varepsilon > 0$. To do this, we will show that K_ε themselves are Calderón–Zygmund convolution kernels. In fact, properties (a) and (b) are straightforward to see, so we will focus on (c). We need to bound

$$\int_{|x|\geq 2|y|} |K_\varepsilon(x+y) - K_\varepsilon(x)|\,dx.$$

This splits into three regions: (i) If $\varepsilon \leq |x|$ and $|x+y| \leq \varepsilon^{-1}$ then this integral is equal to the integral of $|K(x+y) - K(x)|$ and the desired bound follows from the fact that K is a Calderón–Zygmund kernel. (ii) If $\varepsilon \leq |x| \leq \varepsilon^{-1}$ but $|x+y| < \varepsilon$ or $|x+y| > \varepsilon^{-1}$, this integral reduces to an integral of $|K(x)|$ alone. (iii) If $\varepsilon \leq |x+y| \leq \varepsilon^{-1}$ but $|x| \leq \varepsilon$ or $|x| \geq \varepsilon^{-1}$, this integral reduces to an integral of $|K(x+y)|$ alone.

Consider region (ii). If $|x+y| \leq \varepsilon$, then

$$|x| \leq |x+y| + |y| \leq \varepsilon + \tfrac{1}{2}|x| \implies |x| \leq 2\varepsilon.$$

Thus this region can be estimated by

$$\int_{\varepsilon\leq|x|\leq 2\varepsilon} |x|^{-d}\,dx \lesssim 1$$

uniformly in $\varepsilon > 0$. If instead $|x+y| \geq \varepsilon^{-1}$, then we have

$$|x| \geq |x+y| - |y| \geq \varepsilon^{-1} - \tfrac{1}{2}|x| \implies |x| \geq \tfrac{2}{3}\varepsilon^{-1}.$$

Thus this region can be estimated by

$$\int_{\frac{2}{3}\varepsilon^{-1}\leq|x|\leq\varepsilon^{-1}} |x|^{-d}\,dx \lesssim 1$$

uniformly in $\varepsilon > 0$.

The treatment of region (iii) follows along similar lines. If $|x| \leq \varepsilon$ then $|y| \leq \varepsilon/2$, and hence $|x+y| \leq \frac{3\varepsilon}{2}$. Thus this region is controlled by

$$\int_{\varepsilon\leq|x+y|\leq\frac{3\varepsilon}{2}} |x+y|^{-d} \lesssim 1.$$

If $|x| > \varepsilon^{-1}$ then $-|y| > -\frac{1}{2}\varepsilon^{-1}$ and so $|x+y| \geq |x| - |y| \geq \frac{1}{2}\varepsilon^{-1}$, and the region is controlled by

$$\int_{\frac{1}{2}\varepsilon^{-1}\leq|x+y|\leq\varepsilon^{-1}} |x+y|^{-d} \lesssim 1.$$

This completes the proof that K_ε is a Calderon–Zygmund kernel (with implicit bounds independent of ε).

To complete the the proof, we will show that $\|\hat{K}_\varepsilon\|_{L^\infty} \lesssim 1$ (uniformly in $\varepsilon > 0$), which implies that $K_\varepsilon : L^2 \to L^2$ boundedly (uniformly in ε). Indeed,

$$\|K_\varepsilon * f\|_{L^2} \sim \|\hat{K}_\varepsilon \hat{f}\|_{L^2} \lesssim \|\hat{K}_\varepsilon\|_{L^\infty}\|f\|_{L^2}$$

by Plancherel's theorem.

We fix $\xi \in \mathbb{R}^d \backslash \{0\}$. Then (up to constants depending only on π, d),

$$\begin{aligned}\hat{K}_\varepsilon(\xi) &= \int e^{-ix\xi} K_\varepsilon(x)\,dx \\ &= \int_{|x|\le|\xi|^{-1}} e^{-ix\xi} K_\varepsilon(x)\,dx + \int_{|x|>|\xi|^{-1}} e^{-ix\xi} K_\varepsilon(x)\,dx.\end{aligned}$$

Using condition (b) and then (a),

$$\begin{aligned}&\left|\int_{|x|\le|\xi|^{-1}} e^{-ix\xi} K_\varepsilon(x)\,dx\right| \\ &\quad= \left|\int_{|x|\le|\xi|^{-1}} [e^{-ix\xi} - 1] K_\varepsilon(x)\,dx\right| \\ &\quad\lesssim \int_{|x|\le|\xi|^{-1}} |x|^{1-d}|\xi|\,dx \lesssim 1\end{aligned}$$

uniformly in ξ. The remaining region will take a bit more effort.

We begin by writing

$$\begin{aligned}2&\int_{|x|>|\xi|^{-1}} e^{-ix\xi} K_\varepsilon(x)\,dx \\ &= \int_{|x|>|\xi|^{-1}} e^{-ix\xi} K_\varepsilon(x)\,dx - \int_{|x|>|\xi|^{-1}} e^{-ix\xi} e^{i\pi} K_\varepsilon(x)\,dx \\ &= \int_{|x|>|\xi|^{-1}} e^{-ix\xi} K_\varepsilon(x)\,dx - \int_{|x+\frac{\pi\xi}{|\xi|^2}|>|\xi|^{-1}} e^{-ix\xi} K_\varepsilon(x + \tfrac{\pi\xi}{|\xi|^2})\,dx,\end{aligned}$$

where we have written

$$e^{\pi i} = e^{\pi i \xi \cdot \frac{\xi}{|\xi|^2}}$$

and performed a change of variables in the second integral.

Now let us rewrite the integral above as the sum of three pieces, namely,

$$\int_{|x|\geq 2\pi|\xi|^{-1}} e^{-ix\xi}\big[K_\varepsilon(x) - K_\varepsilon(x+\tfrac{\pi\xi}{|\xi|^2})\big]\,dx, \tag{6.7}$$

$$\int_{|\xi|^{-1}\leq|x|\leq 2\pi|\xi|^{-1}} e^{-ix\xi}K_\varepsilon(x)\,dx, \tag{6.8}$$

$$-\int_R e^{-ix\xi}K_\varepsilon(x+\tfrac{\pi\xi}{|\xi|^2})\,dx,. \tag{6.9}$$

where R is the region

$$R = \{|x| \leq 2\pi|\xi|^{-1}, \quad |x+\tfrac{\pi\xi}{|\xi|^2}| > |\xi|^{-1}\}.$$

The term (6.7) is bounded uniformly by property (c). The second term (6.8) is bounded uniformly by property (a). For the third term, we note that in the region R we have

$$|\xi|^{-1} \leq |x+\tfrac{\pi\xi}{|\xi|^2}| \leq 3\pi|\xi|^{-1},$$

and hence term (6.9) is again uniformly bounded by property (a). This completes the proof. $\square$

Our next main result states that Calderón–Zygmund operators are weak type $(1,1)$ and strong type (p,p) for all $1<p<\infty$.

Theorem 6.4.4. *Let K be a Calderón–Zygmund kernel and K_ε be as before. Then the following hold (uniformly in $\varepsilon>0$):*

- $|\{|K_\varepsilon * f| > \alpha\}| \lesssim \alpha^{-1}\|f\|_{L^1}$,
- $\|K_\varepsilon * f\|_{L^p} \lesssim \|f\|_{L^p}$ *for all* $1<p<\infty$.

In particular, $f \mapsto K*f = \lim_{\varepsilon\to 0} K_\varepsilon * f$ extends continuously from Schwartz space to a bounded map on L^p for $1<p<\infty$.

Proof. We show the bounds and leave the extension as an exercise. Given $\alpha>0$ and $f\in L^1$, we perform a Calderon–Zygmund decomposition and write $f=g+b$, where the support of b is a union of nonoverlapping cubes Q_k, with

$$\tfrac{1}{|Q_k|}\int b = 0, \quad \tfrac{1}{|Q_k|}\int |b| \lesssim \alpha,$$

and $|g|\lesssim\alpha$ a.e.

We then have

$$|\{|K_\varepsilon * f| > \alpha\}| \leq |\{|K_\varepsilon * g| > \tfrac{1}{2}\alpha\}| + |\{|K_\varepsilon * b| > \tfrac{1}{2}\alpha\}|.$$

The contribution of g is handled in a straightforward fashion. We have by Tchebychev's inequality,

$$|\{|K_\varepsilon * g| > \tfrac{1}{2}\alpha\}| \lesssim \alpha^{-2}\|K_\varepsilon * g\|_{L^2}^2 \lesssim \alpha^{-2}\|g\|_{L^2}^2 \lesssim \alpha^{-1}\|f\|_{L^1},$$

where we recall that $|g| \lesssim \alpha$ and the definition of g (cf. Remark 6.3.11).

Now consider the contribution of b. We let Q_k^* denote the cube centered at x_k (the center of Q_k) and dilated by $2\sqrt{d}$. Then

$$|\cup Q_k^*| \leq \sum |Q_k^*| = \sum (2\sqrt{d})^d |Q_k| \lesssim \alpha^{-1}\|f\|_{L^1}$$

by construction of the Q_k, and hence we only need to show

$$|\{x \in \left(\cup Q_k^*\right)^c : |K_\varepsilon * b(x)| > \tfrac{1}{2}\alpha\}| \lesssim \alpha^{-1}\|f\|_{L^1}.$$

We begin by with an application of Tchebychev's inequality to bound this measure by

$$\alpha^{-1}\int_{\cap (Q_k^*)^c} |K_\varepsilon * b|\,dx. \tag{6.10}$$

We now use the mean zero condition on b to write

$$K_\varepsilon * b(x) = \sum_k \int_{Q_k} [K_\varepsilon(x-y) - K_\varepsilon(x-x_k)]b(y)\,dy.$$

Thus

$$\begin{aligned}(6.10) &\lesssim \alpha^{-1}\int_{\cap (Q_k^*)^c} \sum_k \int_{Q_k} |K_\varepsilon(x-y) - K_\varepsilon(x-x_k)|\,|b(y)|\,dy\,dx \\ &\lesssim \sum_k \alpha^{-1}\int_{Q_k}\int_{(Q_k^*)^c} |K_\varepsilon(x-y) - K_\varepsilon(x-x_k)|\,dx\,|b(y)|\,dy.\end{aligned}$$

Now write

$$\int_{(Q_k^*)^c} |K_\varepsilon(x-y) - K_\varepsilon(x-x_k)|\,dx = \int_{-\{x_k\}+(Q_k^*)^c} |K_\varepsilon(x+x_k-y) - K_\varepsilon(x)|\,dx.$$

Now we claim that $|x| \geq 2|x_k - y|$ for x, y in the appropriate sets above. Indeed, we first have $|x_k - y| \leq \sqrt{d}\ell(Q_k)$ for $y \in Q_k$, while $|x| \geq 2\sqrt{d}\ell(Q_k)$.

Thus condition (c) applies and this integral is bounded uniformly, leading to

$$(6.10) \lesssim \alpha^{-1} \sum_k \int_{Q_k} |b(y)|\, dy \lesssim \alpha^{-1} \|f\|_{L^1},$$

as desired.

Marcinkiewicz interpolation now yields (p, p) bounds for $1 < p \leq 2$ (uniformly in ϵ). It remains to treat the case $2 < p < \infty$. To this end, we fix $2 < p < \infty$ and write

$$\begin{aligned}
\|K_\varepsilon * f\|_{L^p} &= \sup_{\|g\|_{L^{p'}}=1} \iint K_\varepsilon(x-y) f(y) \bar{g}(x)\, dx\, dy \\
&= \sup \int f(y) \overline{\int K_\varepsilon(x-y) g(y)\, dx}\, dy \\
&= \sup \langle f, \bar{K}_\varepsilon(-\cdot) * g \rangle \\
&\lesssim \|f\|_{L^p} \sup \|\bar{K}_\varepsilon(-\cdot) * g\|_{L^{p'}} \\
&\lesssim \|f\|_{L^p},
\end{aligned}$$

where we have used $L^{p'}$ boundedness for $\bar{K}_\varepsilon(-\cdot)$, which follows from the fact that $1 < p' \leq 2$. This completes the proof. □

Remark 6.4.5. Let us briefly summarize the ideas of the proofs above. Essentially, what we showed is that the three conditions defining a Calderon–Zygmund kernel K guarantee that $\hat{K}$ is bounded, which yields the $L^2 \to L^2$ bounds by Plancherel. Then, using a Calderon–Zygmund decomposition and item (c), we showed that Calderon–Zygmund kernels have weak type $(1, 1)$ bounds. Interpolation yields (p, p) bounds for $1 < p < 2$, and a duality argument yields (p, p) bounds for $2 < p < \infty$.

Remark 6.4.6. Boundedness in L^1 and L^∞ can fail. To see this, consider again the Hilbert transform

$$Hf(x) = \tfrac{1}{\pi} \int \frac{f(x-y)}{y}\, dy.$$

Let $f = \chi_{[a,b]} \in L^\infty \cap L^1$. We claim that

$$H\chi_{[a,b]}(x) = \tfrac{1}{\pi} \log\Big|\frac{x-a}{x-b}\Big|,$$

which belongs to neither L^1 nor L^∞. Indeed, we can write

$$H\chi_{[a,b]}(x) = \tfrac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |y| \leq \varepsilon^{-1}} \frac{\chi_{[a,b]}(x-y)}{y}\, dy.$$

Now note that $\chi_{[a,b]}(x-y) \neq 0$ if and only if $x-b \leq y \leq x-a$. Thus if $x > b$, then for ε sufficiently small we get

$$H\chi_{[a,b]}(x) = \tfrac{1}{\pi}\int_{x-b}^{x-a} \tfrac{1}{y}\,dy = \tfrac{1}{\pi}\log\left|\tfrac{x-a}{x-b}\right|.$$

Similar considerations treat the cases $a < x < b$ and $x < a$ (choosing ε small enough), and so the identity follows. (See the exercises.)

6.5 Exercises

Exercise 6.5.1. Show that that

$$x \mapsto |x|^{-1}\log(|x| + |x|^{-1})^{-\theta}$$

belongs to $L^1(\mathbb{R})$ if and only if $\theta > 1$. Use this fact to fill in the details in Remark 6.1.6.

Exercise 6.5.2. Prove the estimates (6.5) and (6.6). *Hint:* Write the norms in terms of the integral of the distribution function.

Exercise 6.5.3. Use the Hardy–Littlewood maximal inequality (Theorem 6.3.1) to prove the Lebesgue differentiation theorem (Proposition 6.3.4).

Exercise 6.5.4. Fill in the details of the proof of Theorem 6.3.5.

Exercise 6.5.5. Show that if $f_n(x) = \chi_{[2^{n-1},2^n]}(x)$ then $f = \{f_n\} \in L^\infty$ but $|\bar{M}f(x)|^2 \equiv \infty$.

Exercise 6.5.6. Fill in the details in Remark 6.4.6.

Chapter 7

Classical harmonic analysis, part II

In this chapter we primarily focus on the topics of Littlewood–Paley theory and the theory of oscillatory integrals.

We define a partition of unity to be used throughout the chapter as follows. We let $\varphi : \mathbb{R}^d \to [0,1]$ be a smooth C^∞ function such that

$$\varphi(x) = \begin{cases} 1 & |x| \leq 1.4 \\ 0 & |x| > 1.42. \end{cases}$$

We let $\psi : \mathbb{R}^d \to [0,1]$ be given by $\psi(x) = \varphi(x) - \varphi(2x)$. For $N \in 2^{\mathbb{Z}}$, we set $\psi_N(x) = \psi(\frac{x}{N})$. It follows that

$$\sum_{N \in 2^{\mathbb{Z}}} \psi_N(x) = 1 \quad \text{almost everywhere.}$$

Indeed,

$$\sum_{N_1 \leq N \leq N_2} \psi_N(x) = \varphi(\tfrac{x}{N_2}) - \varphi(\tfrac{2x}{N_1}).$$

In what follows, sums over N will be understood to be indexed by $N \in 2^{\mathbb{Z}}$.

7.1 Mihlin multiplier theorem

Recall the notion of a Fourier multiplier operator, i.e. an operator of the form

$$T_m = \mathcal{F}^{-1} m \mathcal{F} \quad \text{for some} \quad m : \mathbb{R}^d \to \mathbb{C}.$$

It is a simple consequence of Plancherel's theorem that T_m maps $L^2 \to L^2$ boundedly whenever $m \in L^\infty$. The following theorem concerns the more general question of L^p boundedness for Fourier multiplier operators.

Theorem 7.1.1 (Mihlin multiplier theorem). *Suppose* $m : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ *satisfies*

$$|\partial_\xi^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} \tag{7.1}$$

uniformly in $\xi \in \mathbb{R}^d \backslash \{0\}$ *and for all multiindices* α *of order* $0 \leq |\alpha| \leq \lceil \frac{d+1}{2} \rceil$. *Then* $T_m = \mathcal{F}^{-1} m \mathcal{F}$ *is bounded on* L^p *for all* $1 < p < \infty$.

Proof. Observe that T_m is given by convolution with $\check{m} = F^{-1}m$. Now, as just mentioned, the $L^2 \to L^2$ bound for T_m follows from the assumption that m is bounded (which is just (7.1) with $|\alpha| = 0$).

Thus, revisiting the proof of L^p boundedness for Calderón–Zygmund operators (cf. Remark 6.4.5 following Theorem 6.4.4), we only need to verify that condition (c) holds for $\check{m}$, i.e.

$$\int_{|x| \geq 2|y|} |\check{m}(x+y) - \check{m}(x)|\, dx \lesssim 1 \quad \text{uniformly in} \quad y.$$

Recall that this condition would be implied by the estimate $|\nabla \check{m}(x)| \lesssim |x|^{-(d+1)}$ uniformly in $x \in \mathbb{R}^d \backslash \{0\}$ (cf. Lemma 6.4.2). Let us first see that this stronger condition holds if we assume (7.1) holds up to $|\alpha| \leq d+2$.

We write

$$m(\xi) = \sum_N m_N(\xi), \quad \text{where} \quad m_N = \psi_N m.$$

By the product rule,

$$\begin{aligned} |\partial_\xi^\alpha [\xi m_N]| &= \Big| \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1,\alpha_2} \partial_\xi^{\alpha_1}(\xi m(\xi)) \partial_\xi^{\alpha_2} \psi_N \Big| \\ &\lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} N^{-|\alpha_2|} |\xi|^{1-|\alpha_1|} \chi_{\{|\xi| \sim N\}}. \end{aligned}$$

Thus

$$\begin{aligned} \|x^\alpha \nabla \check{m}_N\|_{L^\infty} &\lesssim \|\partial_\xi^\alpha [\xi m_N]\|_{L^1} \\ &\lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} \int_{|\xi| \sim N} |\xi|^{1-|\alpha_1|} N^{-|\alpha_2|}\, d\xi \\ &\lesssim N^{d+1-|\alpha|}. \end{aligned}$$

Applying this with $|\alpha| = 0$ and $|\alpha| = d + 2$ yields

$$|\nabla \check{m}_N(x)| \lesssim \min\{N^{d+1}, N^{-1}|x|^{-(d+2)}\}.$$

Thus

$$|\nabla \check{m}(x)| \lesssim \sum_{N \le |x|^{-1}} N^{d+1} + \sum_{N > |x|^{-1}} N^{-1}|x|^{-(d+2)} \lesssim |x|^{-(d+1)},$$

as needed.

Let us now prove condition (c) assuming (7.1) holds only up to $|\alpha| \le \lceil \frac{d+1}{2} \rceil$. By Plancherel and the computation above,

$$\begin{aligned} \int |x^\alpha \check{m}_N(x)|^2 \, dx &\sim \int |\partial_\xi^\alpha m_N(\xi)|^2 \, d\xi \\ &\lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} \int_{|\xi| \sim N} |\xi|^{-2|\alpha_1|} N^{-2|\alpha_2|} \, d\xi \\ &\lesssim N^{d - 2|\alpha|}. \end{aligned}$$

Using Cauchy–Schwarz and applying the above with $|\alpha| = 0$, we find

$$\int_{|x| \le R} |\check{m}_N(x)| \, dx \lesssim (NR)^{\frac{d}{2}}.$$

On the other hand, applying the above with $|\alpha| = \lceil \frac{d+1}{2} \rceil$,

$$\begin{aligned} \int_{|x| > R} |\check{m}_N(x)| \, dx &\le \left(\int |x^\alpha \check{m}_N(x)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{|x| > R} |x|^{-2|\alpha|} \, dx \right)^{\frac{1}{2}} \\ &\lesssim (NR)^{\frac{d}{2} - \lceil \frac{d+1}{2} \rceil}. \end{aligned}$$

Choosing $R \sim \frac{1}{N}$, we find

$$\int |\check{m}_N(x)| \, dx \lesssim 1 \quad \text{uniformly in} \quad N.$$

Arguing in the same way, we have

$$\int |\partial^\beta \check{m}_N(x)| \, dx \lesssim N^{|\beta|}.$$

In particular, this shows

$$\int |\check{m}_N(x + y) - \check{m}_N(x)| \, dx \lesssim N|y|.$$

Thus we have

$$\begin{aligned}\int_{|x|\geq 2|y|} |\check{m}(x+y)-\check{m}(x)|\,dx &\leq \sum_N \int_{|x|\geq 2|y|} |\check{m}_N(x+y)-\check{m}_N(x)|\,dx \\ &\lesssim \sum_{N\leq |y|^{-1}} N|y| + \sum_{N>|y|^{-1}} \int_{|x|\geq |y|} |\check{m}_N(x)|\,dx \\ &\lesssim 1 + \sum_{N>|y|^{-1}} (N|y|)^{\frac{d}{2}-\lceil\frac{d+1}{2}\rceil} \lesssim 1.\end{aligned}$$

This completes the proof. □

Remark 7.1.2. This result is sharp in the sense that L^1 and L^∞ bounds can fail. To see this, let us re-use the Hilbert transform example, which essentially corresponds to taking $\check{m}(x) = \frac{1}{x}$ in $d = 1$. By using contour integration (say), one can verify that $m(\xi)$ is a multiple of the signum function. In particular, it satisfies (7.1). However, as we saw before, the Hilbert transform is not bounded on L^1 or L^∞.

Remark 7.1.3. One application of the Mihlin multiplier theorem is the following 'Schauder' type estimate, which is useful in the setting of elliptic PDE: for any $i, j = 1, \ldots, d$ and $1 < p < \infty$,

$$\|\tfrac{\partial^2 f}{\partial x_i \partial x_j}\|_{L^p} \lesssim \|\Delta f\|_{L^p}$$

(uniformly for $f \in \mathcal{S}$). Indeed, this is equivalent to the L^p boundedness of the Fourier multiplier operator

$$m_{ij}(\xi) := \tfrac{\xi_i \xi_j}{|\xi|^2},$$

which is a consequence of the Mihlin multiplier theorem. (This can also be deduced by using boundedness of Riesz transforms twice.)

7.2 Littlewood–Paley theory

Definition 7.2.1. For $N \in 2^{\mathbb{Z}}$, we define the Littlewood–Paley projection operators P_N via

$$\hat{f}_N(\xi) = \widehat{P_N f}(\xi) = \psi_N(\xi)\hat{f}(\xi).$$

In particular,

$$f_N(x) = \int N^d \check{\psi}(Ny) f(x-y)\,dy.$$

Note that P_N is not an actual projection, in the sense that $P_N^2 \neq P_N$. We further define $P_{\leq N}$ by

$$\widehat{f_{\leq N}}(\xi) = \widehat{P_{\leq N} f}(\xi) = \varphi(\tfrac{\xi}{N})\hat{f}(\xi).$$

Finally, we have $P_{>N} = I - P_{\leq N}$ and $P_{N \leq \cdot \leq M} = \sum_{N \leq K \leq M} P_K$.

Remark 7.2.2. There is an alternate definition of frequency projection that utilizes heat flow. Recall that the solution to the heat equation

$$u_t = \Delta u, \quad u(0,x) = f(x)$$

is given by $u(t,x)$ satisfying

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{f}(\xi).$$

We may alternately write this as $u(t) = e^{t\Delta} f$. Suppose f is a nice function (e.g. $f \in \mathcal{S}$). Then at time $t > 0$, $\hat{u}$ will mostly be concentrated where $|\xi| \leq \frac{1}{\sqrt{t}}$. In particular, we can consider

$$\tilde{P}_{\leq N} f := e^{\Delta/N^2} f$$

to represent a projection of f to frequencies $\leq N$. We could then define

$$\tilde{P}_N f = e^{\Delta/N^2} f - e^{4\Delta/N^2} f.$$

This viewpoint is useful in settings in which one may not have a nice notion of a Fourier transform, but one can still solve the heat equation (e.g. on manifolds).

We next prove some basic properties of the Littlewood–Paley operators.

Proposition 7.2.3. *The following hold:*

(a) P_N and $P_{\leq N}$ are bounded on L^p for $1 \leq p \leq \infty$ (uniformly in N).

(b) We have the pointwise bound

$$|f_N(x)| + |f_{\leq N}(x)| \lesssim [Mf](x)$$

for all N, where M denotes the Hardy–Littlewood maximal function.

(c) For $1 < p < \infty$ and $f \in L^p$, the sum $\sum_N f_N$ converges in L^p to f.

(d) *For $1 \leq p \leq q \leq \infty$, we have*

$$\|f_N\|_{L^q} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|f_N\|_{L^p}$$

and

$$\|f_{\leq N}\|_{L^q} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|f_{\leq N}\|_{L^p}.$$

(e) *For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, we have*

$$\||\nabla|^s f_N\|_{L^p} \sim N^s \|f_N\|_{L^p}.$$

In particular, for $s > 0$,

$$\||\nabla|^s f_{\leq N}\|_{L^p} \lesssim N^s \|f\|_{L^p},$$
$$\|f_{>N} f\|_{L^p} \lesssim N^{-s} \||\nabla|^s f\|_{L^p}.$$

Remark 7.2.4. The estimates in (d) and (e) are known as Bernstein estimates. Item (c) may fail in L^1 and L^∞. To see this, first observe that for each N, we have that $P_{\leq N} f \in C^\infty$ (since $P_{\leq N}$ is convolution with a smooth function), but C^∞ is not dense in L^∞. Alternately, note that $f \equiv 1$ belongs to L^∞, while

$$f_N(x) = \int \check{\psi}(y)\, dy \equiv 0.$$

To see why (c) fails in L^1, note that any individual piece f_N (and hence any finite sum of pieces) has mean zero, while mean zero functions are not dense in L^1. Indeed,

$$\int f_N(x)\, dx = \hat{f}_N(0) = 0.$$

The proof of (c) above, however, will show that $f_{\leq N} \to f$ in L^1 as $N \to \infty$.

Proof. Item (a) follows from the fact that P_N and $P_{\leq N}$ are given by convolution with L^1 functions with uniformly bounded L^1-norms (in N). In particular, by Young's convolution inequality and a change of variables,

$$\|P_N f\|_{L^p} = \|\mathcal{F}^{-1}[\psi_N] * f\|_{L^p} \leq \|\mathcal{F}^{-1}[\psi_N]\|_{L^1} \|f\|_{L^p} \leq \|f\|_{L^p},$$

and similarly for $P_{\leq N}$. This argument also proves (d), since

$$\|\psi_N * f\|_{L^q} \lesssim \|\mathcal{F}^{-1}[\psi_N]\|_{L^r} \|f\|_{L^p}$$

for $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$. By a change of variables, one readily checks that

$$\|\mathcal{F}^{-1}[\psi_N]\|_{L^r} \lesssim N^{\frac{d}{p}-\frac{d}{q}},$$

which yields (d).

Item (b) follows from the general fact that convolution with a spherically symmetric L^1-normalized function is always controlled by the maximal function, which we state as Lemma 7.2.5 below.

Next consider (c). Writing $\sum_{N_1 \leq N \leq N_2} f_N = f_{\leq N_2} - f_{\leq N_1}$, the problem reduces to showing $f_{\leq N} \to f$ as $N \to \infty$ and $f_{\leq N} \to 0$ as $f \to 0$. For the first point, we observe

$$\|f - f_{\leq N}\|_{L^p}^p = \int \left| \int \check{\varphi}(y)[f(x - \tfrac{y}{N}) - f(x)]\, dy \right|^p dx,$$

where we have used $\int \check{\varphi} = 1$. The result now follows from continuity of translations in L^p and the fact that $\int_{|x|>R} |\check{\varphi}|\, dy$ can be made arbitrarily small by choosing R large enough. The details are left as an exercise. Note that this part of the argument works even when $p = 1$.

For the second point, recalling that $|P_{\leq N} f| \lesssim Mf$ pointwise, we see that by dominated convergence (and the maximal function estimate) it would be sufficient to establish $P_{\leq N} f \to 0$ pointwise. This holds, for example, for $f \in \mathcal{S}$, since

$$\|P_{\leq N} f\|_{L^\infty} \lesssim N^d \|\check{\varphi}\|_{L^\infty} \|f\|_{L^1} \to 0 \quad \text{as} \quad N \to 0.$$

Thus we have $P_{\leq N} f \to 0$ in L^p for $f \in \mathcal{S}$, and the result for $f \in L^p$ follows from density and L^p boundedness of $P_{\leq N}$.

Finally, we turn to (e). It suffices to prove the bound for f_N, for the remaining two estimates can then be obtained by summation over $M \leq N$ or $M > N$. We consider the Fourier multiplier operator with multiplier

$$m_N(\xi) = N^{-s} |\xi|^s \tilde{\psi}_N(\xi),$$

where $\tilde{\psi}_N$ is a slight fattening of ψ_N. As $\tilde{\psi}_N$ is supported away from $\xi = 0$, we have $m_N \in \mathcal{S}$. Thus $\mathcal{F}^{-1}(m_N) \in \mathcal{S}$ and a change of variables shows

$$\|\mathcal{F}^{-1}(m_N)\|_{L^1} \lesssim 1$$

uniformly in N. Since $\tilde{P}_N P_N = P_N$, we deduce from Young's inequality that

$$\||\nabla|^s f_N\|_{L^p} \lesssim N^s \|f_N\|_{L^p}.$$

However, since $s \in \mathbb{R}$ was arbitrary, the argument above also shows the reverse inequality. This completes the proof. $\square$

The following lemma was used in the proof above and may be of independent interest.

Lemma 7.2.5. *Suppose $K \in \mathcal{S}$ is nonnegative. For any $N > 0$, we have*

$$N^d \int K(N(x-y))|f(y)|\,dy \lesssim_K Mf[x].$$

Proof. Without loss of generality, assume $f \geq 0$. Then

$$\begin{aligned}
&\int N^d K(N(x-y)) f(y)\,dy \\
&\quad\lesssim N^d \int_{N|x-y|\leq 1} f(y)\,dy + \sum_{R>1} \int_{R<N|x-y|<2R} N^d K(N(x-y)) f(y)\,dy \\
&\quad\lesssim Mf(x) + \sum_{R>1} \tfrac{1}{R}(\tfrac{N}{R})^d \int_{N|x-y|\leq 2R} |N(x-y)|^{d+1} K(N(x-y)) f(y)\,dy \\
&\quad\lesssim Mf(x) + \sum_{R>1} R^{-1} Mf(x) \lesssim Mf(x),
\end{aligned}$$

where we sum over dyadic $R > 1$ and the implicit constants depend only on $\|\langle x\rangle^{d+1} K\|_{L^\infty}$. □

Our next result is the Littlewood–Paley square function estimate.

Theorem 7.2.6 (Littlewood–Paley square function estimate)**.** *Let*

$$(Sf)(x) := \Big(\sum_{N\in 2^{\mathbb{Z}}} |f_N(x)|^2\Big)^{\frac{1}{2}}.$$

Then

$$\|Sf\|_{L^p} \sim \|f\|_{L^p} \quad \textit{for all} \quad 1 < p < \infty.$$

The proof we present will make use of a probabilistic result known as Khinchin's inequality.

Lemma 7.2.7 (Khinchin's inequality)**.** *Let X_n be independent identically distributed random variables on a probability space with $X_n = \pm 1$ with equal probability. For any $0 < p < \infty$,*

$$\Big(\mathbb{E}\big\{\big|\sum c_n X_n\big|^p\big\}\Big)^{\frac{1}{p}} \sim_p \Big(\sum |c_n|^2\Big)^{\frac{1}{2}}$$

for any $\{c_n\} \in \ell^2$.

Here independence of X and Y means that

$$\mathbb{E}\{f(X)g(y)\} = \mathbb{E}\{f(X)\}\,\mathbb{E}\{g(Y)\}$$

for any measurable, bounded f, g. Note that $\mathbb{E}\{X_n\} = 0$ for the random variables defined above.

Proof. Without loss of generality, take $c_n \in \mathbb{R}$. By Tchebychev's inequality, for any t we can write

$$\begin{aligned}\mathbb{P}\{\sum c_n X_n > \lambda\} &\le e^{-\lambda t}\,\mathbb{E}\{e^{t\sum c_n X_n}\} \\ &\le e^{-\lambda t}\prod_n \mathbb{E}\{e^{tc_n X_n}\} \\ &\le e^{-\lambda t}\prod_n \tfrac{1}{2}\{e^{tc_n} + e^{-tc_n}\} \\ &\le e^{-\lambda t}\prod_n e^{t^2 c_n^2/2} \le e^{-\lambda t} e^{t^2\sum c_n^2/2}.\end{aligned}$$

Now choose $t = \frac{\lambda}{\sum c_n^2}$ to get the bound

$$\mathbb{P}\{\big|\sum c_n X_n\big| > \lambda\} \le 2e^{-\frac{\lambda^2}{2\sum c_n^2}}.$$

This implies

$$\begin{aligned}\Big(\mathbb{E}\{|\sum c_n X_n|^p\}\Big)^{\frac{1}{p}} &= \Big(\int_0^\infty p\lambda^{p-1}\,\mathbb{P}\{|\sum c_n X_n| > \lambda\}\,d\lambda\Big)^{\frac{1}{p}} \\ &\le \Big(\int_0^\infty p\lambda^{p-1} 2e^{-\frac{\lambda^2}{2\sum c_n^2}}\,d\lambda\Big)^{\frac{1}{p}} \\ &\lesssim_p \Big(\sum c_n^2\Big)^{\frac{1}{2}},\end{aligned}$$

where in the final inequality we have made the substitution

$$\mu = (\sum c_n^2)^{-\frac{1}{2}}\lambda$$

and used finiteness of the integral $\int e^{-\mu^2/2}\mu^{p-1}\,d\mu$.

It remains to establish the reverse inequality. We claim that

$$\sum c_n^2 = \mathbb{E}\{\big|\sum c_n X_n\big|^2\}.$$

Indeed,

$$\begin{aligned}\mathbb{E}\{\big|\sum c_n X_n\big|^2\} &= \mathbb{E}\{\sum_{n,m} c_n c_m X_n X_m\} \\ &= \sum_n c_n^2\,\mathbb{E}\{X_n^2\} + \sum_{n\neq m} c_n c_m\,\mathbb{E}\{X_n\}\,\mathbb{E}\{X_m\} \\ &= \sum_n c_n^2 + \sum_{n\neq m} c_n c_m\,\mathbb{E}\{X_n\}\,\mathbb{E}\{X_m\} = \sum_n c_n^2,\end{aligned}$$

where we used $\mathbb{E}\{X_n^2\} = 1$ and independence.

Thus, for $1 < p < \infty$ we may use Hölder's inequality and the inequality proved above to get

$$\begin{aligned}\sum c_n^2 &= \mathbb{E}\{|\sum c_n X_n|^2\} \\ &\leq \big(\mathbb{E}\{|\sum c_n X_n|^p\}\big)^{\frac{1}{p}}\big(\mathbb{E}\{|\sum c_n X_n|^{p'}\}\big)^{\frac{1}{p'}} \\ &\leq \big(\mathbb{E}\{|\sum c_n X_n|^p\}\big)^{\frac{1}{p}}\big(\sum c_n^2\big)^{\frac{1}{2}},\end{aligned}$$

which yields the result.

Finally, for $0 < p \leq 1$, we argue similarly to get

$$\begin{aligned}\sum c_n^2 &= \mathbb{E}\{|\sum c_n X_n|^{p/2}|\sum c_n X_n|^{2-p/2}\} \\ &\leq \big(\mathbb{E}\{|\sum c_n X_n|^p\}\big)^{1/2}\big(\mathbb{E}\{|\sum c_n X_n|^{4-p}\big)^{1/2} \\ &\lesssim \big(\mathbb{E}\{|\sum c_n X_n|^p\}\big)^{1/2}\big(\sum c_n^2\big)^{1-p/4}.\end{aligned}$$

Rearranging now yields the desired inequality. □

We turn to the proof of Theorem 7.2.6.

Proof of Theorem 7.2.6. Let X_N be independent identically distributed random variables with $X_N = \pm 1$ with equal probability. By Khinchin's inequality,

$$(Sf)(x) \sim_p \left(\mathbb{E}\{\big|\sum X_N f_N(x)\big|^p\}\right)^{\frac{1}{p}}.$$

Thus

$$\begin{aligned}\|Sf\|_{L^p}^p &\sim \mathbb{E}\{\int \big|\sum X_N f_N(x)\big|^p\,dx\} \\ &= \mathbb{E}\{\|\sum X_N f_n\|_{L^p}^p\} = \mathbb{E}\{\|\check{m}_X * f\|_{L^p}^p\},\end{aligned}$$

where

$$m_X(\xi) := \sum_N X_N \psi_N(\xi).$$

Let us now show that m_X is a Mihlin multiplier (with bounds independent of the value of the X_N), which will imply $\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$. We compute

$$|\partial_\xi^\alpha m_X(\xi)| = \left|\sum_N X_N N^{-|\alpha|}(\partial_\xi^\alpha \psi)(\tfrac{\xi}{N})\right| \lesssim |\xi|^{-|\alpha|},$$

where we use the fact that for $\psi \in C_c^\infty(\mathbb{R}^d\backslash\{0\})$ only finitely many of the terms above will contribute to the sum.

Remark 7.2.8. The proof shows that the bound $\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$ holds for a wider class of 'square functions'. In particular, instead of using the multiplier ψ (that defines P_1), one could use any C_c^∞ function supported in $\mathbb{R}^d\backslash\{0\}$.

It remains to establish $\|f\|_{L^p} \lesssim \|Sf\|_{L^p}$. This will actually depend on the fact that ψ_N is a partition of unity. Define

$$\tilde{P}_N = P_{\frac{N}{2}} + P_N + P_{2N}, \quad \text{so that} \quad \tilde{P}_N P_N = P_N.$$

Then by duality, Proposition 7.2.3(c), and the estimate proved above,

$$\begin{aligned}
\langle f, g\rangle &= \int \sum_N \tilde{P}_N P_N f \bar{g}\, dx \\
&= \int \sum_N \tilde{P}_N f \overline{\tilde{P}_N g}\, dx \\
&\lesssim \int \Big(\sum_N |P_N f(x)|^2\Big)^{\frac{1}{2}} \Big(\sum_N |\tilde{P}_N g(x)|^2\Big)^{\frac{1}{2}} dx \\
&\lesssim \|Sf\|_{L^p} \|\tilde{S}g\|_{L^{p'}} \\
&\lesssim \|Sf\|_{L^p} \|g\|_{L^{p'}}.
\end{aligned}$$

This completes the proof. □

In the following, we will establish some 'fractional calculus' estimates that are useful in applications to PDE. We begin with the following corollary.

Corollary 7.2.9. *The following hold:*

$$\||\nabla|^s f\|_{L^p} \sim_{s,p} \left\|\Big(\sum_N N^{2s}|f_N(x)|^2\Big)^{\frac{1}{2}}\right\|_{L^p} \tag{7.2}$$

for all $s \in \mathbb{R}$ and $1 < p < \infty$, and

$$\| |\nabla|^s f \|_{L^p} \sim_{s,p} \Big\| \Big(\sum_N N^{2s} |f_{>N}(x)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \tag{7.3}$$

for all $s > 0$ and $1 < p < \infty$.

Proof. Using the (proof of) the square function estimate, we first observe

$$\Big\| \Big(\sum_N N^{2s} |P_N |\nabla|^{-s} g|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim \|g\|_{L^p}.$$

Taking $g = |\nabla|^s f$ now yields RHS(7.2)$\lesssim$ LHS(7.2). For the reverse inequality, we use Plancherel, Cauchy–Schwarz, Hölder, and the estimate just established to write

$$\begin{aligned}
\langle g, h \rangle_{L^2} &= \int \sum N^{-s} |\nabla|^{-s} P_N g \cdot N^{-s} |\nabla|^s \tilde{P}_N h \, dx \\
&\lesssim \int \Big(\sum_N N^{2s} |P_N |\nabla|^{-s} g|^2 \Big)^{\frac{1}{2}} \Big(N^{-2s} |\tilde{P}_N |\nabla|^s h|^2 \Big)^{\frac{1}{2}} \, dx \\
&\lesssim \| \big(\sum N^{2s} |P_N |\nabla|^{-s} g|^2 \big)^{\frac{1}{2}} \|_{L^p} \| \big(\sum N^{-2s} |\tilde{P}_N |\nabla|^s h|^2 \big)^{\frac{1}{2}} \|_{L^{p'}} \\
&\lesssim \| \big(\sum N^{2s} |P_N |\nabla|^{-s} g|^2 \big)^{\frac{1}{2}} \|_{L^p} \|h\|_{L^{p'}}.
\end{aligned}$$

Taking the supremum over $h \in L^{p'}$ and applying this to $g = |\nabla|^s f$ yields LHS(7.2)$\lesssim$RHS(7.2).

We turn to (7.3). We will show RHS(7.2)$\sim$RHS(7.3). Using

$$f_N = f_{\geq N} - f_{\geq 2N}$$

and the triangle inequality, we readily deduce RHS(7.2)$\lesssim$RHS(7.3). For the reverse, we estimate as follows:

$$\begin{aligned}
\sum N^{2s} |f_{>N}|^2 &\leq \sum N^{2s} \sum_{N_1, N_2 \geq N} |f_{N_1}| |f_{N_2}| \\
&\leq 2 \sum N^{2s} \sum_{N \leq N_1 \leq N_2} \tfrac{1}{N_1^s N_2^s} N_1^s |f_{N_1}| N_2^s |f_{N_2}| \\
&\lesssim \sum_{N_1 \leq N_2} \sum_{N \leq N_1} \tfrac{N^{2s}}{N_1^s N_2^s} N_1^s |f_{N_1}| N_2^s |f_{N_2}| \\
&\lesssim \sum_{N_1 \leq N_2} \tfrac{N_1^s}{N_2^s} N_1^s |f_{N_1}| N_2^s |f_{N_2}| \lesssim \sum_N N^{2s} |f_N|^2,
\end{aligned}$$

where in the last step we have used Schur's test (Lemma A.3.4). This completes the proof. □

We turn to the following fractional calculus estimates due to Christ and Weinstein [6].

Theorem 7.2.10 (Fractional product rule)**.** *Let* $s > 0$ *and*

$$\tfrac{1}{p} = \tfrac{1}{p_1} + \tfrac{1}{p_2} = \tfrac{1}{q_1} + \tfrac{1}{q_2},$$

with $1 < p, q, p_j, q_j < \infty$. *Then*

$$\||\nabla|^s(fg)\|_{L^p} \lesssim \||\nabla|^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\||\nabla|^s g\|_{L^{q_2}}.$$

Proof. We use

$$\||\nabla|^s(fg)\|_{L^p} \sim \|\big(\sum N^{2s}|P_N(fg)|^2\big)^{\frac{1}{2}}\|_{L^p}.$$

Now write

$$fg = f_{>N/4}g + f_{\leq N/4}g_{>N/4} + f_{\leq N/4}g_{\leq N/4},$$

so that

$$P_N(fg) = P_N(f_{>N/4}g) + P_N(f_{\leq N/4}g_{>N/4}).$$

Thus (using Proposition 7.2.3)

$$\begin{aligned}\sum N^{2s}|P_N(fg)|^2 &\leq \sum N^{2s}|P_N(f_{>N/4}g)|^2 + \sum N^{2s}|P_N(f_{\leq N/4}g_{\leq N/4})|^2\\ &\lesssim \sum N^{2s}|M(f_{>N/4}g)|^2 + \sum N^{2s}|M(Mfg_{>N/4})|^2.\end{aligned}$$

Now, by the vector maximal inequality and the corollary,

$$\begin{aligned}\|\big(\sum N^{2s}|M(f_{>N/4}g)|^2\big)^{\frac{1}{2}}\|_{L^p} &\lesssim \|(\sum |N^s f_{>N/4}|^2)^{1/2}g\|_{L^p}\\ &\lesssim \|g\|_{L^{p_2}}\||\nabla|^s f\|_{L^{p_1}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\big(\sum N^{2s}|M(Mfg_{>N/4})|^2\big)^{\frac{1}{2}}\|_{L^p} &\lesssim \|Mf(\sum |N^s g_{>N/4}|^2)^{1/2}\|_{L^p}\\ &\lesssim \|Mf\|_{L^{q_1}}\||\nabla|^s g\|_{L^{q_2}}\\ &\lesssim \|f\|_{L^{q_1}}\||\nabla|^s g\|_{L^{q_2}}.\end{aligned}$$

This completes the proof. □

Theorem 7.2.11 (Fractional chain rule)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be such that*

$$|F(u) - F(v)| \leq |u - v|[G(u) + G(v)] \quad \textit{for some} \quad G : C \to [0, \infty).$$

Then for any $0 < s < 1$, $1 < p, p_1 < \infty$ and $1 < p_2 \leq \infty$ with

$$\tfrac{1}{p} = \tfrac{1}{p_1} + \tfrac{1}{p_2},$$

we have

$$\| |\nabla|^s (F \circ u) \|_{L^p} \lesssim \|G(u)\|_{L^{p_2}} \| |\nabla|^s u \|_{L^{p_1}}.$$

We will need the following lemma, similar in spirit to Lemma 7.2.5

Lemma 7.2.12. *Suppose $|h(x)| \leq g(x)$, where g is a radial decreasing function with $\lim_{r \to \infty} r^d g(r) = 0$. Then*

$$|(h * f)(x)| \lesssim \|g\|_{L^1} (Mf)(x).$$

Proof. By the fundamental theorem of calculus, we have

$$g(y) = \int_{|y|}^{\infty} \chi_{(0,\rho)}(|y|)(-\tfrac{\partial g}{\partial r})(\rho)\, d\rho.$$

Thus

$$\begin{aligned}
|(h * f)(x)| &\leq \int \int_0^{\infty} \chi_{(0,\rho)}(|y|)(-\tfrac{\partial g}{\partial r})(\rho) |f(x - y)|\, d\rho\, dy \\
&\leq \int_0^{\infty} \int_{|y| < \rho} |f(x - y)|\, dy (-\tfrac{\partial g}{\partial r})(\rho)\, d\rho \\
&\lesssim Mf(x) \int_0^{\infty} \rho^d (-\tfrac{\partial g}{\partial r})(\rho)\, d\rho \\
&\lesssim Mf(x) \int_0^{\infty} g(\rho) \rho^{d-1}\, d\rho \lesssim \|g\|_{L^1} Mf(x).
\end{aligned}$$

Note that to integrate by parts used $r^d g(r) \to 0$ as $r \to \infty$. This completes the proof. □

Proof of the fractional chain rule. We write

$$\| |\nabla|^s F(u) \|_{L^p} \sim \| (\sum N^{2s} |P_N F(u)|^2)^{\frac{1}{2}} \|_{L^p}.$$

Now, using that $\int \check{\psi} = 0$, we write

$$[P_N(F(u))](x) = \int N^d \check{\psi}(Ny) [F \circ u(x - y) - F \circ u(x)]\, dy,$$

so that

$$|[P_N(F(u))](x)| \leq \int N^d|\check{\psi}(Ny)||u(x-y)-u(x)|[G\circ u(x-y)+G\circ u(x)]\,dy. \tag{7.4}$$

We decompose

$$|u(x-y)-u(x)| \leq |u_{>N}(x-y)| + |u_{>N}(x)| + \sum_{K\leq N} |u_K(x-y)-u_K(x)|.$$

We will now prove

$$|u_K(x-y)-u_K(x)| \lesssim K|y|\big(Mu_K(x-y)+Mu_K(x)\big). \tag{7.5}$$

It suffices to treat the case $K|y|<1$. We may apply the fattened projection $\tilde{P}_K$ (abusing notation and writing the corresponding convolution kernel as $\check{\psi}$) and write

$$\begin{aligned} u_K(x-y)-u_K(x) &= \int K^d\check{\psi}(Kz)[u_K(x-y-z)-u_K(x-z)]\,dz \\ &= \int K^d[\check{\psi}(K(z-y))-\check{\psi}(Kz)]u_K(x-z)\,dy \\ &= \int K^d \int_0^1 Ky\cdot\nabla\check{\psi}(Kz-\theta Ky)\,d\theta\, u_K(x-z)\,dz. \end{aligned}$$

Thus, using the lemma above,

$$\begin{aligned} |u_K(x-y)-u_K(x)| &\lesssim K|y| \int \tfrac{K^d}{(1+K|z|)^{100d}} u_K(x-y)\,dy \\ &\lesssim K|y| Mu_K(x) \|K^d(1+K|z|)^{100d}\|_{L^1} \\ &\lesssim K|y|Mu_K(x), \end{aligned}$$

which is acceptable.

Continuing from (7.4), we bound $|P_N(F\circ u)(x)|$ by

$$\begin{aligned} &|P_N(F\circ u)(x)| \\ &\quad \leq M(u_{>N}G\circ u)(x) + M(u_{>N})(x)G\circ u(x) \\ &\qquad + |u_{>N}(x)|M(G\circ u)(x) + |u_{>N}(x)(G\circ u)(x) \\ &\qquad + \sum_{K\leq N}\int N^dK|y||\check{\psi}(Ny)|[Mu_K(x-y)+Mu_K(x)] \\ &\qquad\qquad \times [G\circ u(x-y) - G\circ u(x)]\,dy. \end{aligned}$$

The contribution of the first four terms can be bounded by

$$M(u_{>N}G\circ u)(x) + M(u_{>N})(x)M(G\circ u)(x).$$

The contribution of the sum can be bounded by

$$\sum_{K\leq N} \tfrac{K}{N}\{M(Mu_K\, G\circ u)(x) + M[Mu_K]M[G\circ u](x)\}.$$

Now we can estimate

$$\begin{aligned}
&\|(\textstyle\sum N^{2s}|P_N F(u)|^2)^{\frac12}\|_{L^p} \\
&\quad\lesssim \|(\textstyle\sum N^{2s}|M(u_{>N}G\circ u)|^2)^{\frac12}\|_{L^p} && (7.6)\\
&\qquad + \|(\textstyle\sum N^{2s}|M(u_{>N})M(G\circ u)|^2)^{\frac12}\|_{L^p} && (7.7)\\
&\qquad + \|(\textstyle\sum N^{2s}|\sum_{K\leq N} \tfrac{K}{N} M(Mu_K G\circ u)|^2)^{\frac12}\|_{L^p} && (7.8)\\
&\qquad + \|(\textstyle\sum N^{2s}|\sum_{K\leq N} \tfrac{K}{N} M(Mu_K)M(G\circ u)|^2)^{\frac12}\|_{L^p}. && (7.9)
\end{aligned}$$

It remains to bound these four terms.

First, by the vector maximal inequality and the corollary to the square function estimate,

$$\begin{aligned}
(7.6) &\lesssim \|G\circ u(\textstyle\sum N^{2s}|u_{>N}|^2)^{\frac12}\|_{L^p} \\
&\lesssim \|G\circ u\|_{L^{p_2}}\||\nabla|^s u\|_{L^{p_1}}.
\end{aligned}$$

Arguing similarly,

$$\begin{aligned}
(7.7) &\lesssim \|M(G\circ u)\|_{L^{p_2}}\|(\textstyle\sum |M(N^s u_{>N})|^2)^{\frac12}\|_{L^{p_1}} \\
&\lesssim \|G\circ u\|_{L^{p_2}}\||\nabla|^s u\|_{L^{p_1}}.
\end{aligned}$$

Note that $p_2 = \infty$ is allowed.

For (7.8) and (7.9), we need the following general inequality:

$$\sum N^{2s}\Big|\sum_{K\leq N} \tfrac{K}{N} c_K\Big|^2 \lesssim \sum N^{2s}|c_N|^2 \quad \text{provided} \quad s<1. \tag{7.10}$$

Using this and arguing as above suffices to treat (7.8) and (7.9). The proof of (7.10) (and (7.8) and (7.9)) is left as an exercise. □

7.3 Oscillatory integrals

In this section we discuss the theory of oscillatory integrals. There are two types of integrals one often considers.

Oscillatory integrals of the first kind are written

$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\,dx,$$

where $\lambda > 0$, $\phi : \mathbb{R}^d \to \mathbb{R}$, and $\psi : \mathbb{R}^d \to \mathbb{C}$. In this case we are interested in the asymptotic behavior of $I(\lambda)$ as $\lambda \to \infty$.

Oscillatory integrals of the second kind are written

$$T_\lambda f(x) = \int e^{i\lambda\phi(x,y)}K(x,y)f(y)\,dy,$$

where $\lambda > 0$, $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, and $f : \mathbb{R}^d \to \mathbb{C}$. In this case, we are interested in estimates on the operator norm of T_λ as $\lambda \to \infty$.

We begin by considering oscillatory integrals of the first kind in dimension $d = 1$.

Proposition 7.3.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{C}$ be smooth functions. Suppose ψ has compact support inside an interval (a, b) and $\phi'(x) \neq 0$ for all $x \in [a, b]$. Then*

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx$$

satisfies

$$|I(\lambda)| \lesssim_N \lambda^{-N} \quad \textit{for all} \quad N \geq 0.$$

Note that without the assumption of compact support inside (a, b), the best possible decay is λ^{-1}, which is realized by $\phi(x) = x$ and $\psi(x) = 1$.

Proof. First, the bound $|I(\lambda)| \lesssim 1$ is immediate. Next, we write

$$e^{i\lambda\phi(x)} = \tfrac{1}{i\lambda\phi'(x)}\tfrac{d}{dx}e^{i\lambda\phi(x)}$$

and integrate by parts. This yields

$$I(\lambda) = -\int e^{i\lambda\phi(x)}\tfrac{d}{dx}\big[\tfrac{1}{\lambda\phi'(x)}\psi(x)\big]\,dx,$$

so that

$$|I(\lambda)| \lesssim \lambda^{-1}.$$

To continue, define the operator D via

$$Df(x) = \tfrac{1}{i\lambda\phi'(x)}\tfrac{d}{dx}f(x).$$

The computation above shows that the adjoint D^t is given by

$$D^t f(x) = -\tfrac{d}{dx}\big[\tfrac{1}{i\lambda\phi'(x)}f(x)\big].$$

For any $N \geq 0$, we may write $e^{i\lambda\phi} = D^N e^{i\lambda\phi}$, and so

$$I(\lambda) = \int_a^b e^{i\lambda}(D^t)^N \psi\, dx.$$

This yields

$$\begin{aligned} |I(\lambda)| &\lesssim \int |(D^t)^N\psi|\, dx \\ &\lesssim \lambda^{-N}\sum_{k=0}^N \sum_{|\beta|+|\alpha_1|+\cdots+|\alpha_k|=N}\bigg\| \frac{\partial^\beta\psi(\partial^{\alpha_1}\phi)\cdots(\partial^{\alpha_k}\phi)}{(\phi')^{N+k}}\bigg\|_{L^1([a,b])} \\ &\lesssim_N \lambda^{-N}. \end{aligned}$$

This completes the proof. □

Proposition 7.3.2 (Van der Corput Lemma)**.** *Let ϕ be real-valued and smooth. Let $k \geq 1$ and suppose*

$$|\phi^{(k)}(x)| \geq 1 \quad \textit{for all} \quad x \in [a,b].$$

If $k = 1$, assume additionally that ϕ' is monotone. Then

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\, dx$$

satisfies

$$|I(\lambda)| \lesssim_k \lambda^{-\frac{1}{k}},$$

where the implicit constant is independent of λ, ϕ, a, b.

Note that if $k = 1$ then one needs more than just $|\phi'| \geq 1$. This can be seen first by noting that

$$\bigg|\int_a^b e^{i\phi(x)}\, dx\bigg| \geq \bigg|\int_a^b \cos(\phi(x))\, dx\bigg|.$$

Now, if ϕ is chosen so that ϕ' is very large when $\cos\phi < 0$ and ϕ' is very small when $\cos\phi > 0$, then one can arrange that

$$\left|\int_a^b \cos(\phi(x))\,dx\right| \to \infty \quad \text{as} \quad b \to \infty.$$

Proof of Van der Corput. First consider $k = 1$. Then an integration by parts yields

$$I(\lambda) = \tfrac{1}{i\lambda}\big[\tfrac{e^{i\lambda\phi(b)}}{\phi'(b)} - \tfrac{e^{i\lambda\phi(a)}}{\phi'(a)}\big] + \tfrac{1}{i\lambda}\int_a^b e^{i\lambda\phi}\,\tfrac{d}{dx}\big(\tfrac{1}{\phi'(x)}\big)\,dx.$$

As the integral is bounded by

$$\left|\int_a^b \tfrac{d}{dx}\big(\tfrac{1}{\phi'(x)}\big)\,dx\right| \lesssim 1$$

(since ϕ' is assumed to be monotone), the result follows in this case.

For $k \geq 2$, we proceed by induction. Suppose the result holds at level k. Replacing ϕ by $-\phi$ if necessary, we may assume that

$$\phi^{(k+1)}(x) \geq 1 \quad \text{for all} \quad x \in [a,b].$$

In particular, $\phi^{(k)}$ is increasing, so there is at most one point $c \in [a,b]$ such that $\phi^{(k)}(c) = 0$.

Case 1. Suppose there exists $c \in [a,b]$ so that $\phi^{(k)}(c) = 0$. Then for $\delta > 0$ we have

$$|\phi^{(k)}(x)| \geq \delta \quad \text{for all} \quad x \in [a,b]\backslash(c-\delta, c+\delta).$$

We write

$$I(\lambda) = \int_a^{c-\delta} e^{i\lambda\phi}\,dx + \int_{c-\delta}^{c+\delta} e^{i\lambda\phi}\,dx + \int_{c+\delta}^b e^{i\lambda\phi}\,dx.$$

We estimate by the change of variables $x = \delta^{-\frac{1}{k}}y$:

$$\begin{aligned}\left|\int_a^{c-\delta} e^{i\lambda\phi}\,dx\right| &= \left|\delta^{-\frac{1}{k}}\int_{\delta^{\frac{1}{k}}a}^{\delta^{\frac{1}{k}}(c-\delta)} e^{i\lambda\phi(\delta^{-\frac{1}{k}}y)}\,dy\right| \\ &\lesssim \delta^{-\frac{1}{k}}\lambda^{-\frac{1}{k}},\end{aligned}$$

where we have used the inductive hypothesis and the fact that

$$|\partial^k\phi(\delta^{-\frac{1}{k}}y)| \geq \delta^{-1}\delta \geq 1.$$

The contribution of $(c+\delta, b)$ is treated similarly, while the contribution of $(c-\delta, c+\delta)$ is bounded by the length of the interval, i.e. 2δ. In particular,

$$\left|\int_a^b e^{i\lambda\phi}\,dx\right| \lesssim \delta + (\delta\lambda)^{-\frac{1}{k}} \lesssim \lambda^{-\frac{1}{k+1}},$$

as can be seen by choosing $\delta \sim \lambda^{-\frac{1}{k+1}}$.

Case 2. Suppose $\phi^{(k)} \neq 0$ for $x \in [a,b]$. If $\phi^{(k)}(a) > 0$ then we have $\phi^{(k)}(x) \geq \delta$ for $x \in [a+\delta, b]$. Then we can write

$$\begin{aligned}\left|\int_a^b e^{i\lambda\phi}\,dx\right| &\leq \left|\int_a^{a+\delta} e^{i\lambda\phi}\,dx\right| + \left|\int_{a+\delta}^b e^{i\lambda\phi}\,dx\right| \\ &\leq \delta + (\delta\lambda)^{-\frac{1}{k}} \\ &\lesssim \lambda^{-\frac{1}{k+1}}\end{aligned}$$

choosing δ as above. If $\phi^{(k)}(a) < 0$ then we have $\phi^{(k)}(b) < 0$ and so $\phi^{(k)}(x) < -\delta$ for $x \in (a, b-\delta)$. Then we can argue similarly. This completes the proof. □

Corollary 7.3.3. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth. Let $k \geq 1$ and assume that*

$$|\phi^{(k)}(x)| \geq 1 \quad \textit{for all} \quad x \in [a,b].$$

If $k = 1$, assume additionally that ϕ' is monotone. Then

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx\right| \lesssim_k \lambda^{-\frac{1}{k}}\left[|\psi(b)| + \|\psi'\|_{L^1([a,b])}\right].$$

Proof. We write

$$\begin{aligned}\int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx &= \int_a^b \psi(x)\tfrac{d}{dx}\int_a^x e^{i\lambda\phi(y)}\,dy\,dx \\ &= \psi(b)\int_a^b e^{i\lambda\phi(y)}\,dy - \int_a^b \psi'(x)\int_a^x e^{i\lambda\phi(y)}\,dy\,dx.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_a^b e^{i\lambda\phi}\psi\,dx\right| &\leq |\psi(b)|\left|\int_a^b e^{i\lambda\phi}\,dy\right| \\ &\quad + \sup_{x\in[a,b]}\left|\int_a^x e^{i\lambda\phi}\,dy\right|\|\psi'\|_{L^1} \\ &\lesssim |\psi(b)|\lambda^{-\frac{1}{k}} + \lambda^{-\frac{1}{k}}\|\psi'\|_{L^1},\end{aligned}$$

where in the final step we use the van der Corput lemma. □

Proposition 7.3.4 (Stationary phase). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth. Assume ϕ has a nondegenerate critical point at x_0, that is,*

$$\phi'(x_0) = 0 \quad \text{and} \quad \phi''(x_0) \neq 0.$$

If $\psi : \mathbb{R} \to \mathbb{C}$ is smooth and supported in a sufficiently small neighborhood of x_0, then

$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\,dx$$

satisfies

$$I(\lambda) = (2\pi i)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}[\phi''(x_0)]^{-\frac{1}{2}}e^{i\lambda\phi(x_0)}\psi(x_0) + \mathcal{O}(\lambda^{-\frac{3}{2}}) \quad \text{as} \quad \lambda \to \infty.$$

Proof. Let us first get the decay rate.

Let $a \in C_c^\infty$ satisfy $a(x) = 1$ for $|x| \leq 1$ and $a(x) = 0$ for $|x| > 2$. We write

$$I(\lambda) = I_1(\lambda) + I_2(\lambda),$$

where

$$I_1(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)a(\lambda^{\frac{1}{2}}(x - x_0))\,dx,$$
$$I_2(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)[1 - a(\lambda^{\frac{1}{2}}(x - x_0))]\,dx.$$

Thus by a change of a variables,

$$|I_1(\lambda)| \lesssim \lambda^{-\frac{1}{2}}.$$

On the other hand, using integration by parts (and the fact that $\phi'(x) \neq 0$ on the support of ψ away from $x = x_0$), we can get

$$|I_2(\lambda)| \lesssim_N \lambda^{-N}$$

for any N.

To get the exact coefficients, we argue as follows. By Taylor's theorem and $\phi'(x_0) = 0$

$$\phi(x) - \phi(x_0) = \tfrac{1}{2}\phi''(x_0)(x - x_0)^2\big\{1 + \eta(x)\big\},$$

where η is smooth and $\eta(x) = \mathcal{O}(|x - x_0|)$.

Now let U be a small neighborhood of x_0 so that (i) $|\eta| < 1$ on U and (ii) $\phi' = 0$ on $U\backslash\{x_0\}$. We make the change of variables

$$y = (x - x_0)\{1 + \eta(x)\}^{\frac{1}{2}},$$

which is a diffeomorphism from U to a small neighborhood of $y = 0$. Assume that ψ is supported in U.

Now we write

$$\begin{aligned} I(\lambda) &= e^{i\lambda\phi(x_0)} \int_U e^{i\lambda[\phi(x)-\phi(x_0)]}\psi(x)\, dx \\ &= e^{i\lambda\phi(x_0)} \int e^{i\lambda\phi''(x_0)y^2/2}\psi_1(y)\, dy, \end{aligned}$$

where $\psi_1 \in C_c^\infty$ is supported in a neighborhood of $y = 0$. Set $\lambda_1 = \lambda\phi''(x_0)/2$.

Introducing another C_c^∞ function ψ_2 (equal to one on the support of ψ_1), we can write

$$\int e^{i\lambda_1 y^2}\psi_1(y)\, dy = \int e^{i\lambda_1 y^2} e^{-y^2} e^{y^2}\psi_1(y)\psi_2(y)\, dy.$$

We now use Taylor expansion to write

$$e^{y^2}\psi_1(y) = \sum_{j=0}^N a_j y^j + y^N R_N(y) = P(y) + y^N R_N(y)$$

for some $N \geq 2$. Note $a_0 = \psi(x_0)$.

Thus

$$\int e^{i\lambda_1 y^2} e^{-y^2} e^{y^2}\psi_1\psi_2\, dy = \sum_{j=0}^N a_j \int e^{i\lambda_1 y^2} e^{-y^2} y^j\, dy \tag{7.11}$$

$$+ \int e^{i\lambda_1 y^2} e^{-y^2} P[\psi_2 - 1]\, dy \tag{7.12}$$

$$+ \int e^{i\lambda_1 y^2} e^{-y^2} y^N R_N \psi_2\, dy. \tag{7.13}$$

First consider (7.11). Then, using the change of variables $z = y(1-i\lambda_1)^{\frac{1}{2}}$ and using $(1+z)^{-a} = 1 - az + \mathcal{O}(z^2)$, we have

$$\begin{aligned} \int e^{i\lambda_1 y^2} e^{-y^2} y^j\, dy &= \int e^{-y^2(1-i\lambda_1)} y^j\, dy \\ &= (1-i\lambda_1)^{-\frac{1}{2}-\frac{j}{2}} \int e^{-z^2} z^j\, dz \\ &= (-i\lambda_1)^{-\frac{1}{2}-\frac{j}{2}} (1 + i\lambda_1^{-1})^{-\frac{1}{2}-\frac{j}{2}} \int e^{-z^2} z^j\, dz \\ &= (-i\lambda_1)^{-\frac{1}{2}-\frac{j}{2}} (1 + \mathcal{O}(\lambda_1^{-1})) \int e^{-z^2} z^j\, dz. \end{aligned}$$

Thus the leading term of (7.11) is

$$\psi(x_0)\sqrt{\pi}[\tfrac{-i\phi''(x_0)\lambda}{2}]^{-\frac{1}{2}},$$

as desired. The next terms are all $\mathcal{O}(\lambda_1^{-\frac{3}{2}}) = \mathcal{O}(\lambda^{-\frac{3}{2}})$ (note that the integral in the $j = 1$ case vanishes).

We would like to show that (7.12) and (7.13) are $\mathcal{O}(\lambda^{-\frac{3}{2}})$.

For term (7.12), we note that

$$e^{-y^2}P(y)[\psi_2(y) - 1]$$

is supported away from zero, so that we may integrate by parts to deduce (7.12) is $\lesssim_m \lambda^{-m}$ for any $m \geq 0$.

We turn to (7.13). We write

$$\int e^{i\lambda_1 y^2} y^N e^{-y^2} R_N \psi_2 \, dy = \int e^{i\lambda_1 y^2} y^N e^{-y^2} R_N \psi_2 a(\tfrac{y}{\varepsilon})\, dy \tag{7.14}$$

$$+ \int e^{i\lambda_1 y^2} y^N b(y)[1 - a(\tfrac{y}{\varepsilon})]\, dy, \tag{7.15}$$

where

$$b(y) := e^{-y^2} R_N(y)\psi_2(y).$$

Now,

$$|(7.14)| \lesssim \int |y|^N |a(\tfrac{y}{\varepsilon})|\, dy \lesssim \varepsilon^{N+1}.$$

To deal with (7.15), we write

$$e^{i\lambda_1 y^2} = \tfrac{1}{2i\lambda_1 y}\tfrac{d}{dy} e^{i\lambda_1 y^2}.$$

Then

$$(7.15) = \int e^{i\lambda_1 y^2}\big[-\tfrac{d}{dy}\tfrac{1}{2iy\lambda_1}\big]^m \big[y^N b(y)(1 - a(\tfrac{y}{\varepsilon}))\big]\, dy.$$

Thus, choosing $m > N + 1$,

$$\begin{aligned} |(7.15)| &\lesssim \lambda_1^{-m} \sum_{k=0}^{m} \sum_{\alpha_1+\alpha_2+\alpha_3=m-k} \int \frac{y^{N-\alpha_1}|\partial^{\alpha_2} b|\varepsilon^{-\alpha_3}|\partial^{\alpha_3}[1-a](\frac{y}{\varepsilon})|}{|y|^{m+k}}\, dy \\ &\lesssim \lambda_1^{-m} \sum_{k,\alpha} \int_{|y|\geq\varepsilon} |y|^{N-m-(k+\alpha_1)}\varepsilon^{-\alpha_3}\, dy \\ &\lesssim \lambda_1^{-m} \sum_{k,\alpha} \varepsilon^{N+1-m-(k+\alpha_1+\alpha_3)} \\ &\lesssim \lambda_1^{-m}\varepsilon^{N+1-2m}. \end{aligned}$$

Now choose $\varepsilon \sim \lambda_1^{-\frac{1}{2}}$, so that

$$\varepsilon^{N+1} \sim \lambda_1^{-m} \varepsilon^{N+1-2m}.$$

We have

$$|(7.15)| \lesssim \lambda_1^{-\frac{N+1}{2}} \lesssim \lambda_1^{-\frac{3}{2}}$$

for $N \geq 2$. This completes the proof. □

Example 7.3.1 (Linear Schrödinger equation). Consider the linear Schrödinger equation

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u, \\ u(0,x) = u_0(x), \end{cases}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$. We can solve this using the Fourier transform. With

$$\hat{f}(\xi) = (2\pi)^{-\frac{1}{2}} \int e^{-ix\xi} f(x)\, dx,$$

we have

$$i\partial_t \hat{u}(t,\xi) = \tfrac{1}{2}|\xi|^2 \hat{u}(t,\xi),$$

so that

$$u(t,x) = (2\pi)^{-\frac{d}{2}} \int e^{ix\xi - it\xi^2/2} u_0(x)\, dx.$$

Stationary phase allows us to describe the long-time behavior of solutions. In particular, we write

$$e^{ix\xi - it\xi^2/2} = e^{it\Phi(\xi;t,x)}, \quad \Phi(\xi;t,x) = \tfrac{x}{t}\xi - \tfrac{1}{2}\xi^2.$$

We compute the critical points of Φ:

$$\partial_\xi \Phi = \tfrac{x}{t} - \xi = 0 \quad \text{for} \quad \xi = \xi_0 := \tfrac{x}{t}.$$

As $\partial_\xi^2 \phi \equiv 1$, we have that ξ_0 is a nondegenerate critical point.

Thus, stationary phase yields

$$u(t,x) = (2\pi)^{-\frac{1}{2}} (2\pi i)^{\frac{1}{2}} t^{-\frac{1}{2}} (-1)^{\frac{1}{2}} e^{it(\frac{x}{t}\xi_0 - \frac{1}{2}\xi_0^2)} \hat{u}_0(\xi_0) + \mathcal{O}(t^{-\frac{3}{2}})$$

as $t \to \infty$. Simplifying this we get the **Fraunhofer approximation**

$$u(t,x) \sim (it)^{-\frac{1}{2}} e^{ix^2/2t} \hat{u}_0(\tfrac{x}{t}).$$

Roughly, this states that the long-time spatial distribution is determined by the initial momentum distribution.

Example 7.3.2. Consider the Klein–Gordon equation

$$u_{tt} - u_{xx} + m^2 u = 0.$$

Using the Fourier transform, this becomes

$$\hat{u}_{tt} = -(\xi^2 + m^2)\hat{u},$$

and so

$$\hat{u}(t,\xi) = A(\xi)e^{it\sqrt{\xi^2+m^2}} + B(\xi)e^{-it\sqrt{\xi^2+m^2}}$$

for some A, B defined in terms of the initial data. In particular, to understand the asymptotic behavior we need to understand the asymptotics of

$$\int e^{ix\xi \pm it\sqrt{\xi^2+m^2}}\varphi(\xi)\,d\xi.$$

Consider first

$$\Phi = \tfrac{x}{t}\xi + \sqrt{\xi^2+m^2},$$

so that

$$\Phi' = \tfrac{x}{t} + \xi(\xi^2+m^2)^{-\frac{1}{2}}, \quad \Phi'' = \tfrac{m^2}{(\xi^2+m^2)^{\frac{3}{2}}} \neq 0.$$

The stationary point of Φ occurs when

$$\tfrac{x}{t} = -\xi(\xi^2+m^2)^{-\frac{1}{2}}.$$

Squaring both sides leads to

$$\xi^2 = \frac{m^2(\frac{x}{t})^2}{1-(\frac{x}{t})^2},$$

and thus we get a stationary point if $|\frac{x}{t}| < 1$. Considering separately the cases $x < 0$ and $x > 0$, we find that the stationary point is given by

$$\xi_0 = \frac{-m(\frac{x}{t})}{\sqrt{1-(\frac{x}{t})^2}}$$

In this case, we get that

$$\Phi(\xi_0) = m\sqrt{1-(\tfrac{x}{t})^2} \quad \text{and} \quad \Phi''(\xi_0) = m^{\frac{3}{2}}\left[1-(\tfrac{x}{t})^2\right]^{-\frac{3}{2}},$$

and so

$$\int e^{ix\xi+it\sqrt{\xi^2+m^2}}\varphi(\xi)\,d\xi \sim (2\pi i)^{\frac{1}{2}}t^{-\frac{1}{2}}m^{-\frac{3}{4}}[1-(\tfrac{x}{t})^2]^{\frac{3}{4}}e^{imt\sqrt{1-(\frac{x}{t})^2}}\varphi\Big(\tfrac{-m\frac{x}{t}}{\sqrt{1-(\frac{x}{t})^2}}\Big)$$

provided $|\frac{x}{t}| < 1$.

For the other phase, observe

$$\begin{aligned}
&\int e^{ix\xi - it\sqrt{\xi^2+m^2}}\psi(\xi)\,d\xi \\
&\quad = \overline{\int e^{-ix\xi+it\sqrt{\xi^2+m^2}}\bar{\psi}(\xi)\,d\xi} \\
&\quad = \overline{\int e^{ix\xi+it\sqrt{\xi^2+m^2}}\bar{\psi}(-\xi)\,d\xi} \\
&\quad \sim (-2\pi i)^{\frac12} t^{-\frac12} m^{-\frac34}[1-(\tfrac{x}{t})^2]^{\frac34} e^{-imt\sqrt{1-(\frac{x}{t})^2}}\overline{\psi}\big(\tfrac{m\frac{x}{t}}{\sqrt{1-(\frac{x}{t})^2}}\big).
\end{aligned}$$

Now return to the PDE. Suppose $u|_{t=0} = f$ for some $f : \mathbb{R} \to \mathbb{R}$ and $\partial_t u|_{t=0} = 0$. (We leave the more general case as an exercise). Solving for A and B above then yields $A = B = \frac12 \hat{f}$. Noting that $\overline{\hat{f}}(-\xi) = \hat{f}(\xi)$ (for real-valued f), we deduce

$$u(t,x) \sim t^{-\frac12} m^{-\frac34}[1-(\tfrac{x}{t})^2]^{\frac34}\,\mathrm{Re}\big\{i^{\frac12} e^{imt\sqrt{1-(\frac{x}{t})^2}}\hat{f}\big(\tfrac{-m\frac{x}{t}}{\sqrt{1-(\frac{x}{t})^2}}\big)\big\}$$

for $|x| < t$. Alternately, writing $\rho = (t^2 - |x|^2)^{\frac12}$, we can write

$$u(t,x) \sim_m \rho^{-\frac12}\,\mathrm{Re}[i^{\frac12} e^{im\rho}\hat{f}(m\rho)], \quad |x| < t.$$

We turn to the higher dimensional case. We begin with the following.

Proposition 7.3.5. *Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be smooth and compactly supported. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth, with $\nabla\phi$ nonzero on the support of ψ. Then*

$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\,dx$$

obeys

$$|I(\lambda)| \lesssim_N \lambda^{-N} \quad \textit{for all} \quad N > 0.$$

Proof. The case $N = 0$ is immediate. Let us demonstrate the $N = 1$ case; the extension to $N \geq 2$ then follows from iteration.

We use the fact that

$$\nabla e^{i\lambda\phi} = i\lambda\nabla\phi\, e^{i\lambda\phi},$$

so that

$$e^{i\lambda\phi} = \frac{\nabla\phi\cdot\nabla e^{i\lambda\phi}}{i\lambda|\nabla\phi|^2}.$$

We can then write

$$
\begin{aligned}
I(\lambda) &= \int \frac{\nabla\phi\cdot\nabla e^{i\lambda\phi}}{i\lambda|\nabla\phi|^2}\psi\,dx \\
&= \sum_{j=1}^{d}\int -\partial_j\left[\frac{\partial_j\phi}{i\lambda|\nabla\phi|^2}\psi\right]e^{i\lambda\phi}\,dx,
\end{aligned}
$$

and so

$$
|I(\lambda)| \lesssim \lambda^{-1}\left\|\nabla\cdot\left[\frac{\nabla\phi\,\psi}{|\nabla\phi|^2}\right]\right\|_{L^1} \lesssim \lambda^{-1}.
$$

As mentioned above, the case $N \geq 2$ follows from iteration. $\square$

We skip the analogue of van der Corput's lemma, which would yield a bound of $\lambda^{-\frac{1}{|\alpha|}}$ whenever one has lower bounds on $\partial^\alpha\phi$.

Instead, we will move to stationary phase in higher dimensions. The result is the following.

Proposition 7.3.6. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth. Assume ϕ has a nondegenerate critical point at x_0. If $\psi : \mathbb{R}^d \to \mathbb{C}$ is smooth and supported in a sufficiently smooth neighborhood of x_0, then*

$$
I(\lambda) := \int e^{i\lambda\phi(x)}\psi(x)\,dx
$$

satisfies

$$
I(\lambda) = \frac{(2\pi i)^{\frac{d}{2}}e^{i\lambda\phi(x_0)}\psi(x_0)}{\left(\det[\frac{\partial^2\phi}{\partial x_i\partial x_j}(x_0)]\right)^{\frac{1}{2}}}\lambda^{-\frac{d}{2}} + \mathcal{O}(\lambda^{-\frac{d}{2}-1})
$$

as $\lambda \to \infty$.

We also write $(\frac{\partial^2\phi}{\partial x_i\partial x_j}) = D^2\phi$.

The proof is similar in spirit to the $d = 1$ case. The key step there was to use a change of variables to turn the phase into an exactly quadratic phase. The necessary result in higher dimensions is the following:

Lemma 7.3.7 (Morse lemma)**.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth with a nondegenerate critical point at x_0. Then there exists a smooth local change of variables $y = y(x)$ such that $y(x_0) = 0$, $\frac{\partial y}{\partial x}|_{x_0} = Id$, and*

$$
\phi(x) = \phi(x_0) + \sum_{j=1}^{d}\tfrac{1}{2}\lambda_j y_j^2,
$$

where λ_j are the eigenvalues of $D^2\phi(x_0)$.

With the Morse lemma in hand, the proof of the stationary phase lemma is very similar to the proof in one dimension. So, we will conclude this section just by proving the Morse lemma.

Proof of the Morse lemma. Noting that $D^2\phi(x_0)$ is symmetric, we may (by a change of variables) assume that

$$D^2\phi(x_0) = \operatorname{diag}(\lambda_1, \ldots, \lambda_d).$$

Now we can write

$$\phi(x) = \phi(x_0) + \nabla\phi(x_0)\cdot(x - x_0) + \int_0^1 (1-t)\tfrac{d^2}{dt^2}[\phi(x_0 + t(x - x_0))]\,dt,$$

as one can check by integrating by parts in the final integral. In particular,

$$\begin{aligned}\phi(x) - \phi(x_0) &= \sum_{i,j=1}^d (x - x_0)_i(x - x_0)_j \int_0^1 (1-t)\frac{\partial^2\phi}{\partial x_i \partial x_j}(x_0 + t(x - x_0))\,dt \\ &=: \sum_{i,j=1}^d (x - x_0)_i(x - x_0)_j m_{ij}(x).\end{aligned}$$

Observe that m_{ij} is smooth, with $m_{ij} = m_{ji}$ and $m_{ij}(x_0) = \frac{1}{2}\frac{\partial^2\phi}{\partial x_i\partial x_j}(x_0)$.

We proceed by induction. Suppose we have found a smooth local change of variables $y = y(x)$ so that

$$\phi(x) = \phi(x_0) + \tfrac{1}{2}\lambda_1 y_1^2 + \cdots + \tfrac{1}{2}\lambda_{r-1}y_{r-1}^2 + \sum_{i,j\geq r} y_i y_j \tilde{m}_{ij}(y),$$

where $y(x_0) = 0$, $\frac{\partial y}{\partial x}|_{x=x_0} = Id$, and the $\tilde{m}_{ij}$ are smooth and symmetric.

Now we compute

$$\partial_{x_i}\partial_{x_j}(\tfrac{1}{2}\lambda_k y_k^2) = \lambda_k(\partial_{x_i}y_k\partial_{x_j}y_k) + \lambda_k y_k(\partial_{x_i}\partial_{x_j}y_k).$$

In particular, at $x = x_0$ we have

$$\partial_{x_i}\partial_{x_j}(\tfrac{1}{2}\lambda_k y_k^2)|_{x=x_0} = \lambda_k\delta_{ij}.$$

Thus

$$D^2(\tfrac{1}{2}\lambda_1 y_1^2 + \cdots + \tfrac{1}{2}\lambda_{r-1}y_{r-1}^2)|_{x=x_0} = \operatorname{diag}\{\lambda_1, \ldots, \lambda_{r-1}, 0, \ldots, 0\}.$$

Using the fact that

$$D^2(\phi(x) - \phi(x_0))|_{x=x_0} = \operatorname{diag}\{\lambda_1, \ldots, \lambda_d\},$$

we deduce

$$D^2\bigg(\sum_{i,j\geq r} y_i y_j \tilde{m}_{ij}(y)\bigg) = \operatorname{diag}\{0, \ldots, 0, \lambda_r, \ldots, \lambda_d\}.$$

In particular, this implies

$$[\tilde{m}_{ij}(y(x_0))]_{i,j\geq r} = [\tilde{m}_{ij}(0)]_{i,j\geq r} = \tfrac{1}{2}\operatorname{diag}\{\lambda_r, \ldots, \lambda_d\},$$

because if any y_k survives undifferentiated then the contribution of the term will be zero.

We would now like to define a change of variables y' so that $y'_j = y_j$ for any $j \neq r$, while

$$\sum_{i,j\geq r} y_i y_j \tilde{m}_{ij}(y) = \tfrac{1}{2}\lambda_1 (y'_r)^2 + \sum_{i,j\geq r+1} y_i y_j \tilde{m}'_{ij}(y)$$

for some smooth symmetric $\tilde{m}'_{ij}$. This will imply all of the desired properties for the new variable y'. Writing

$$\sum_{i,j\geq r} y_i y_j \tilde{m}_{ij}(y) = \tilde{m}_{rr}(y) y_r^2 + y_r \sum_{j\geq r+1} \tilde{m}_{jr}(y) y_j + \sum_{i,j\geq r+1} y_i y_j \tilde{m}_{ij}(y),$$

we see that we should take

$$y'_r = \sqrt{\frac{\tilde{m}_{rr}(y)}{\frac{1}{2}\lambda_r}}\bigg(y_r + \sum_{j\geq r+1} \frac{\tilde{m}_{jr}(y)}{\tilde{m}_{rr}(y)} y_j\bigg).$$

With this change of variables, one can now verify all of the desired properties and hence complete the induction. □

We conclude this section with one sample result regarding oscillatory integrals of the second kind. We will merely scratch the surface of a very rich subject.

Define the family of operators T_λ by

$$(T_\lambda f)(\xi) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(x,\xi)} \psi(x,\xi) f(x)\, dx,$$

where $\lambda > 0$, $\psi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, and $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth. Assume that on the support of ψ the Hessian of Φ is nonzero:

$$\det\big(\tfrac{\partial^2 \Phi(x,\xi)}{\partial x_i \partial \xi_j}\big) \neq 0.$$

We will prove:

Proposition 7.3.8. *Under the assumptions above, we have*

$$\|T_\lambda\|_{L^2 \to L^2} \lesssim \lambda^{-\frac{d}{2}}.$$

Remark 7.3.9. Note that for fixed λ, we can easily obtain L^2 boundedness:

$$\|T_\lambda f\|_{L^2_\xi} \leq \int \|\psi(x,\xi)\|_{L^2_\xi} |f(x)|\, dx \leq \|\psi\|_{L^2_{x,\xi}} \|f\|_{L^2}.$$

The interesting point is to obtain decay as $\lambda \to \infty$. Note also that if $\Phi(x,\xi) = -2\pi x\xi$, then after rescaling we get that the family of operators

$$T'_\lambda f(\xi) = \int e^{-2\pi i x\xi} \psi(\lambda^{-\frac{1}{2}} x, \lambda^{-\frac{1}{2}} \xi) f(x)\, dx$$

is uniformly bounded on L^2 as $\lambda \to \infty$. This recovers the Fourier transform if $\psi(0,0) = 1$.

Proof of Proposition 7.3.8. Let T_λ^* denote the adjoint of T_λ. By the method of TT^*, it suffices to prove that the $L^2 \to L^2$ norm of $T_\lambda T_\lambda^*$ is bounded by λ^{-d}. (See Exercise 7.4.5.)

We may write

$$T_\lambda T_\lambda^* f(\xi) = \int K_\lambda(\xi,\eta) f(\eta)\, d\eta,$$

where

$$K_\lambda(\xi,\eta) = \int e^{i\lambda[\Phi(x,\xi) - \Phi(x,\eta)]} \psi(x,\xi) \bar{\psi}(x,\eta)\, dx.$$

We will prove bounds on K_λ.

Now let us denote by $M(x,\xi)$ the matrix $\frac{\partial^2 \Phi}{\partial x_i \partial \xi_j}$. Given $a \in \mathbb{R}^n$, denote by ∇_x^a differentiation in the a direction.

For fixed (ξ,η), denote

$$\Delta = \Delta(x,\xi,\eta) = \nabla_x^{a(x)} [\Phi(x,\xi) - \Phi(x,\eta)]$$

where $a(x) \in \mathbb{R}^d$ is to be determined. We have

$$\Delta = M(x,\xi) a(x) \cdot (\xi - \eta) + \mathcal{O}(|\xi - \eta|^2).$$

As M is invertible, we may take

$$a(x) = a(x,\xi,\eta) = M(x,\xi)^{-1} \tfrac{\xi - \eta}{|\xi - \eta|},$$

so that
$$M(\xi,\eta)a(x)\cdot(\xi-\eta)=|\xi-\eta|.$$

Now, if the support of ψ is sufficiently small, then we can guarantee
$$|\Delta(x,\xi,\eta)|\geq c|\xi-\eta| \quad \text{for} \quad (\xi,\eta)\in \text{supp } K_\lambda.$$

Now set $D_x=[i\lambda\Delta(x,\xi,\eta)]^{-1}\nabla_x^{a(x)}$. We use the identity
$$(D_x)^N(e^{i\lambda[\Phi(x,\xi)-\Phi(x,\eta)]})=e^{i\lambda[\Phi(x,\xi)-\Phi(x,\eta)]}$$

and integrate by parts N times to obtain
$$K_\lambda(\xi,\eta)=\int_{\mathbb{R}^d}e^{i\lambda[\Phi(x,\xi)-\Phi(x,\eta)]}(D_x^T)^N[\psi(x,\xi)\bar{\psi}(x,\eta)]\,dx.$$

This yields
$$|K_\lambda(\xi,\eta)|\lesssim_N (1+\lambda|\xi-\eta|)^{-N}:=A_\lambda(|\xi-\eta|)$$

for any $N\geq 0$. Thus
$$|T_\lambda T_\lambda^* f(\xi)|\lesssim [A_\lambda * |f|](\xi),$$

so that (applying Young's inequality and a change of variables), we get
$$\|T_\lambda T_\lambda^* f\|_{L^2}\lesssim \|A_\lambda\|_{L^1}\|f\|_{L^2}\lesssim \lambda^{-d}.$$

This implies the desired result provided the support of ψ is sufficiently small. To deal with the more general case, we employ a partition of unity to split the support of ψ up into a finite number of sufficiently small pieces. This completes the proof. $\square$

7.4 Exercises

Exercise 7.4.1. Complete the proof of (c) in Proposition 7.2.3.

Exercise 7.4.2. Prove (7.10) and complete the proof of (7.8) and (7.9). *Hint:* Expand the inner sum and change the order of integration. Performing the sum in N leads to a bound of
$$\sum_{K\leq L}(\tfrac{K}{L})^{1-s}K^s c_K L^s c_L,$$

which can then be dealt with by Schur's test.

Exercise 7.4.3. Show that

$$\lim_{N\to\infty} N\int_0^\pi t\sin(\tfrac{1}{t})\cos(Nt)\,dt = 0.$$

[Thanks to D. Grow for suggesting this exercise!]

Exercise 7.4.4. Prove the stationary phase lemma in higher dimensions, using the Morse lemma to find the change of variables that makes the phase exactly quadratic.

Exercise 7.4.5. Let $T : L^2 \to L^2$ and let T^* be its adjoint. Show that

$$\|T\|^2 = \|T^*\|^2 = \|TT^*\|.$$

Exercise 7.4.6. In this exercise, you will prove the stationary phase lemma in the special case of an exactly quadratic phase: Show that for $u \in C_c^\infty(\mathbb{R}^d)$ and $N \geq 1$, we have

$$\int e^{ix\cdot Qx/2h}u(x)\,dx = \sum_{k=0}^{N-1} \frac{(2\pi)^{\frac{d}{2}}h^{k+\frac{d}{2}}e^{i\frac{\pi}{4}\operatorname{sgn}Q}}{(2i)^k|\det Q|^{\frac{1}{2}}k!}((D_x\cdot Q^{-1}D_x)^k u)(0)+S_N(u,h),$$

where $D_x = -i\nabla_x$ and

$$|S_N(u,h)| \lesssim_{d,Q,u,N} h^{N+\frac{d}{2}}.$$

Here $\operatorname{sgn} Q$ denotes the signature of Q, which is a nondegenerate real symmetric $d\times d$ matrix.

Hint: Use Plancherel and the Fourier transform to prove

$$\int e^{ix\cdot Qx/2h}u(x)\,dx = h^{\frac{d}{2}}e^{i\frac{\pi}{4}\operatorname{sgn}Q}|\det Q|^{-\frac{1}{2}}\int e^{-ih\xi\cdot Q^{-1}\xi/2}\hat{u}(\xi)\,d\xi.$$

Then use Taylor's theorem to write

$$e^{it} = \sum_{k=0}^{N-1}\frac{(it)^k}{k!} + \mathcal{O}(\tfrac{|t|^N}{N!}).$$

Exercise 7.4.7. Let σ denote the surface measure of the sphere $S \subset \mathbb{R}^d$ with $d \geq 2$. Show that

$$|\check{\sigma}(x)| \lesssim \langle x\rangle^{-\frac{d-1}{2}},$$

where

$$\check{\sigma}(x) = \int_S e^{ix\xi}\,d\sigma(\xi) \quad \text{and} \quad \langle x\rangle := \sqrt{1+|x|^2}.$$

Hint: As $d\sigma$ is invariant under rotations, we can write

$$\check{\sigma}(x) = \int_S e^{i|x|\xi_d} d\sigma(\xi) \sim \int_0^\pi e^{i|x|\cos\theta} [\sin\theta]^{\frac{d}{2}} d\theta$$

where θ is the angle between x and e_d. To estimate this integral, use stationary phase and the van der Corput lemma.

Chapter 8

Modern harmonic analysis, part I

8.1 Semiclassical analysis

In this section we discuss some of the most basic concepts and results in semiclassical analysis, following [19]. This subject was developed largely in order to give rigorous meaning to the 'Bohr correspondence principle', which informally states that one recovers classical mechanics from quantum mechanics in the limit $h \to 0$ (where h is Planck's constant). One of the key tools in semiclassical analysis is known as pseudodifferential calculus, which in turn has applications in a wide range of related fields (e.g. partial differential equations and other areas of mathematical physics).

We begin with some basic definitions.

1. Let g be a nonnegative smooth function on $\mathbb{R}^n$. If for all multi-indices α we have
$$\partial^\alpha g = \mathcal{O}(g)$$
uniformly over $\mathbb{R}^n$, then we call g a **order function** on $\mathbb{R}^n$. For example,
$$1, \quad \langle z\rangle^m = (1+|z|^2)^{\frac{m}{2}} \quad \text{and} \quad e^{\langle z\rangle}$$
are order functions. If g is an order function, so is $1/g$ (check!).

2. Given an order function g, we define $S_d(g)$ to be the set of all functions $a = a(x;h)$ (defined on $\mathbb{R}^d \times (0,h_0]$ for some $h_0 > 0$) that are smooth in x and satisfy
$$\partial_x^\alpha a(x;h) = \mathcal{O}(g)$$

uniformly over (x, h) for any multi-index α. Frequently, one works with the order function $g = 1$, noting that $a \in S_d(g)$ if and only if $ag^{-1} \in S_d(1)$.

3. Let a and $\{a_j\}$ belong to $S_d(g)$. We write

$$a \sim \sum_{j=0}^{\infty} h^j a_j$$

if for any N, α there exists $h_{N,\alpha} > 0$ so that

$$\left| \partial_x^\alpha \left(a - \sum_{j=0}^{N} h^j a_j \right) \right| \lesssim_{N,\alpha} h^N g$$

uniformly on $\mathbb{R}^d \times (0, h_{N,\alpha}]$. If $a \sim 0$ in $S_d(g)$, then we write $a = \mathcal{O}(h^\infty)$ in $S_d(g)$.

We have the following result.

Proposition 8.1.1. *For any sequence $\{a_j\}$ of symbols in $S_d(g)$, there exists $a \in S_d(g)$ such that $a \sim \sum h^j a_j$ in $S_d(g)$. Furthermore, a is unique up to the addition of an $\mathcal{O}(h^\infty)$ symbol.*

Remark 8.1.2. One calls a a **resummation** of the formal symbol $\sum h^j a_j$.

Proof. We only sketch the proof; the complete details may be found in [19, Lemma 2.3.3]. Without loss of generality, take $g \equiv 1$. One first constructs a sequence $\varepsilon_j \to 0$ such that if $|\alpha| \leq j$,

$$\|(1 - \chi(\tfrac{\varepsilon_j}{h}))\partial^\alpha a_j\|_{L_x^\infty} \leq h^{-1}$$

for h small enough, where χ is a bump function. One then defines

$$a(x; h) = \sum h^j \big(1 - \chi(\tfrac{\varepsilon_j}{h})\big) a_j(x; h),$$

which for any $h > 0$ is actually only a finite sum. One can then verify that $a \sim \sum h^j a_j$. $\square$

Example 8.1.1 (WKB approximation). The resummation of a formal series can be used to construct approximate eigenfunctions for $1d$ Schrödinger operators.

In particular, let V be a smooth function on $\mathbb{R}$ and suppose $V(x_0) < E$. For x near x_0, we define solutions to the 'eikonal equation'

$$(\varphi_\pm')^2 = E - V$$

by setting

$$\varphi_\pm(x) = \pm \int_{x_0}^x \sqrt{E - V(y)}\, dy.$$

Using this, a direct computation shows

$$(-h^2\partial_x^2 + V - E)(ae^{i\phi/h}) = -2ih\sqrt{\varphi'}\big[(a\sqrt{\varphi})' - ih\tfrac{a''}{2\sqrt{\varphi'}}\big]e^{i\varphi/h}$$

for $\varphi = \varphi_\pm$ and a smooth near x_0 (check!). We then recursively define $a_j^\pm$ by solving the following transport equations:

$$(a_0^\pm\sqrt{\varphi_\pm'})' = 0, \quad (a_j^\pm\sqrt{\varphi_\pm'})' - i\tfrac{(a_{j-1}^\pm)''}{2\sqrt{\varphi_\pm'}} = 0$$

for $j \geq 1$. We then let $a_\pm(x;h)$ be a resummation of the formal symbol $\sum h^j a_j^\pm$. By construction one can check that

$$(-h^2\partial_x^2 + V - E)u_\pm(x,h) = \mathcal{O}(h^\infty), \quad \text{where} \quad u_\pm = a_\pm e^{i\varphi_\pm/h}.$$

Indeed, this follows from the fact that the formal series

$$\sum_{j\geq 0} h^j\big[\big(a_j^\pm\sqrt{\varphi_\pm'}\big)' - ih\tfrac{(a_j^\pm)''}{2\sqrt{\varphi_\pm'}}\big]$$

is identically zero.

These approximate solutions are called WKB solutions (after Wentzel, Kramers, and Brillouin).

We only considered the case $V(x_0) < 0$. In the case $V(x_0) = E$ (called a 'turning point)', this technique breaks down. In this case one can instead use a power series expansion for $V(x) - E$. Solving the ODE to first order leads to an equation known as the Airy equation, which is solved with special functions (the Airy functions). One then needs to patch together the approximate solutions away from and near the turning points (which will only be possible for special values of E).

Pseudodifferential operators.

We next define the **semiclassical Fourier transform** of a Schwartz function u on $\mathbb{R}^n$ by

$$\mathcal{F}_h u(\xi) = \hat{u}(\xi) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi/h}u(x)\, dx.$$

This is an L^2-isomorphism with inverse

$$\mathcal{F}_h^{-1}v(x) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi/h} v(\xi)\, d\xi.$$

As with the standard Fourier transform, $\mathcal{F}_h$ extends to an isomorphism on tempered distributions. Writing $D_x = -i\partial_x$, we have

$$\mathcal{F}_h(hD_x u) = \xi \mathcal{F}_h u \quad \text{and} \quad \mathcal{F}_h(xu) = -hD_\xi \mathcal{F}_h u,$$

generalizing the familiar identities obtained for the standard Fourier transform.

Expanding out $\mathcal{F}_h^{-1}\mathcal{F}_h u = u$ leads to the identity

$$u(x) = (2\pi h)^{-n} \iint e^{i(x-y)\xi/h} u(y)\, dy\, d\xi.$$

Our next goal is to make sense of more general operators of the form

$$u(x) \mapsto (2\pi h)^{-n} \iint e^{i(x-y)\xi/h} a(x,y,\xi) u(y)\, dy\, d\xi$$

for some kernel $a(x,y,\xi)$. This requires that we make sense of the integrals

$$I(a) = I(a;x,y) = \int e^{i(x-y)\xi/h} a(x,y,\xi)\, d\xi. \tag{8.1}$$

Suppose $a(x,y,\xi) \in S_{3n}(\langle\xi\rangle^m)$. If $m < -n$, the integral $I(a)$ converges and for $u \in C_c^\infty(\mathbb{R}^n)$ we may define

$$A_a u(x,h) = \int e^{i(x-y)\xi/h} a(x,y,\xi) u(y)\, dy\, d\xi.$$

Now observe that

$$Le^{i(x-y)\xi/h} = e^{i(x-y)\xi/h}, \quad \text{where} \quad L := \tfrac{1}{1+\xi^2}(1 - h\xi D_y).$$

Thus we can write

$$A_a u(x,h) = I_k u(x) = \int e^{i(x-y)\xi/h} [{}^tL(\xi, hD_y)]^k (au)\, dy\, d\xi$$

where ^{t} denotes transpose and we have

$$({}^tL)^k(au) = \left(\tfrac{1+h\xi D_y}{1+\xi^2}\right)^k (au) = \mathcal{O}(\langle\xi\rangle^{m-k})$$

uniformly as $|\xi| \to \infty$. Thus we have

- $I_k u(x)$ converges provided $m < k - n$,
- $I_{k+\ell} u = I_k u$ for all $\ell \geq 0$.

Thus for any $m \in \mathbb{R}$, $a \in S_{3n}(\langle \xi \rangle^m)$, and $u \in C_c^\infty(\mathbb{R}^n)$, we may define

$$A_a u(x,h) = \int e^{i(x-y)\xi/h} [{}^t L(\xi, hD_y)]^k (au)\, dy\, d\xi$$

for any $k > m + n$. One can check that A_a defines a continuous linear operator from C_c^∞ to C^∞; in particular, (by the 'Schwartz kernel theorem') we may find a distribution K on $\mathbb{R}^n \times \mathbb{R}^n$ (called the **distribution kernel** of A_a) such that

$$\langle A_a u, v \rangle = \langle K, v \otimes u \rangle,$$

where $u, v \in C_c^\infty$, $\otimes$ denotes tensor product, and $\langle \cdot, \cdot \rangle$ denotes the pairing of distributions and test functions. We denote the distribution kernel by (8.1).

Example 8.1.2. If $a = 1$, then choosing $k > n$ we may verify that

$$A_a u(x) = (2\pi h)^n u(x),$$

so that

$$\int e^{i(x-y)\xi/h}\, d\xi = (2\pi h)^n \delta(y - x)$$

in the sense of oscillatory integrals.

In light of the above, we make the following definition.

Definition 8.1.3. Given $a \in S_{3n}(\langle \xi \rangle^m)$ and $u \in C_c^\infty(\mathbb{R}^n)$, define

$$\mathrm{Op}_h(a)u(x;h) = (2\pi h)^{-n} \int e^{i(x-y)\xi/h} a(x,y,\xi) u(y)\, dy\, d\xi.$$

Then $\mathrm{Op}_h(a)u \in C^\infty(\mathbb{R}^n)$. For any $\nu \in \mathbb{R}$, the operator

$$h^{-\nu}\mathrm{Op}_h(a) : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is called the **semiclassical pseudodifferential operator of symbol** $h^{-\nu} a$. We say $h^{-\nu}\mathrm{Op}_h(a)$ is of degree m and order ν.

Proposition 8.1.4. *For $a \in S_{3n}(\langle \xi \rangle^m)$, we can extend $Op_h(a)$ to a map from $\mathcal{S} \to \mathcal{S}$ or from $\mathcal{S}' \to \mathcal{S}'$.*

Proof. Let us show that $\mathrm{Op}_h(a) : \mathcal{S} \to \mathcal{S}$. We write

$$x^\beta \partial_x^\alpha I_k u(x) = \int x^\beta \partial_x^\alpha [e^{i(x-y)\xi/h} ({}^tL)^k (au)]\, dy\, d\xi$$

and split the integral into two regions I_1 and I_2, where

$$I_1 = \{|x-y| \le \tfrac{1}{2}|x|\} \quad \text{and} \quad I_2 = \{|x-y| > \tfrac{1}{2}|x|\}.$$

Here we recall $L = (1+\xi^2)^{-1}(1 - h\xi D_y)$. For $k > m + n + |\alpha|$, the integral over I_1 is uniformly bounded since

$$x^\beta \langle \xi \rangle^{m+|\alpha|-k} \langle y \rangle^{-\gamma} = \mathcal{O}(\langle \xi \rangle^{m+|\alpha|-k} \langle y \rangle^{|\beta|-\gamma})$$

for any $\gamma > 0$ on $\{|x-y| \le \frac{1}{2}|x|\}$. Choosing $\gamma > |\beta| + n$, this contribution is integrable with respect to y and ξ.

For the remaining region, we write

$$\tilde{L} = \tfrac{1}{1+|x-y|^2}(1 + h(x-y)D_\xi).$$

Integrating by parts N times with respect to ξ, the integral over I_2 is written as a sum of terms of the form

$$\int_{|x-y| \ge \frac{1}{2}|x|} x^\beta e^{i(x-y)\xi/h} ({}^t\tilde{L})^N [\xi^{\alpha_1} \partial_x^{\alpha_2} ({}^tL)^k (au)]\, dy\, d\xi$$

(with $\alpha_1 + \alpha_2 = \alpha$). Choosing $N \ge |\beta|$, the contribution of I_2 is uniformly bounded. This shows $\mathrm{Op}_h(a)u \in \mathcal{S}$, and in fact these estimates suffice to show that the mapping is continuous. □

Example 8.1.3 (Semiclassical differential operators). If

$$a(x,y,\xi) = \sum_{|\alpha| \le m} b_\alpha(x) \xi^\alpha$$

with $b_\alpha \in S_n(1)$, then

$$\mathrm{Op}_h(a) = \sum_{|\alpha| \le m} b_\alpha(x)(hD_x)^\alpha.$$

If

$$a(x,y,\xi) = \sum_{|\alpha| \le m} b_\alpha(y) \xi^\alpha$$

with $b_\alpha \in S_n(1)$, then

$$\mathrm{Op}_h(a) = \sum_{|\alpha| \le m} (hD_x)^\alpha b_\alpha(x).$$

Remark 8.1.5. If we replace $e^{i(x-y)\xi/h}$ with $e^{i\varphi(x,y,\xi)/h}$ where φ is a phase function, then we are led to 'Fourier integral operators', often abbreviated FIOs.

Remark 8.1.6. Given $a(x, y, \xi)$, we define

$$a^*(x,y,\xi) = \overline{a(y,x,\xi)}.$$

Then the operator

$$[\mathrm{Op}_h(a)]^* := \mathrm{Op}_h(a^*)$$

is the **formal adjoint** of $\mathrm{Op}_h a$, which satisfies

$$\langle [\mathrm{Op}_h(a)]^* u, v\rangle = \langle u, \mathrm{Op}_h(a)v\rangle$$

for all $u, v \in \mathcal{S}$.

Composition of pseudodifferential operators.

Let $a \in S_{3n}(\langle\xi\rangle^k)$ and $b \in S_{3n}(\langle x\rangle^{k'})$, and let $A = \mathrm{Op}_h(a)$ and $B = \mathrm{Op}_h(b)$. The composition of A and B is defined formally by

$$\begin{aligned}(A \circ B)u(x) &= (2\pi h)^{-n} \int e^{i(x-y)\xi/h} a(x,y,\xi) Bu(y)\, dy\, d\xi \\ &= (2\pi h)^{-n} \int e^{i(x-z)\eta/h} c_h(x,z,\eta) u(z)\, dz\, d\eta,\end{aligned}$$

where

$$c_h(x,z,\eta) := (2\pi h)^{-n} \int e^{i(x-y)(\xi-\eta)/h} a(x,y,\xi) b(y,z,\eta)\, dy\, d\xi.$$

To show that $A \circ B$ is again a pseudodifferential operator, we need to verify that $c_h \in S_{3n}(\langle\xi\rangle^m)$ for some m. We will prove the following:

Theorem 8.1.7 (Composition)**.** *Given $a \in S_{3n}(\langle\xi\rangle^m)$ and $b \in S_{3n}(\langle\xi\rangle^{m'})$, there exists $c \in S_{3n}(\langle x\rangle^{m+m'})$ such that*

$$Op_h(a) \circ Op_h(b) = Op_h(c).$$

A choice for c is given by

$$a\#b(x,y,\xi) := (2\pi h)^{-n} \int e^{i(x-z)(\eta-\xi)/h} a(x,z,\eta) b(z,y,\xi)\, dz\, d\eta,$$

which satisfies

$$a\#b \sim \sum_{|\alpha|\geq 0} \tfrac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_z^\alpha \partial_\eta^\alpha \big(a(x,z,\eta) b(z,y,\xi)\big)\Big|_{\eta=\xi,\ z=x}$$

in $S_{3n}(\langle\xi\rangle^{m+m'})$.

To prove this, we rely on the method of stationary phase, specifically in the form of Exercise 7.4.6. In particular, an application of that result with $d = 2n$, and Q the block matrix with $-I$ in the top-right and bottom-left corners, we may deduce

$$(2\pi h)^{-n} \int e^{-ixy/h} u(x,y)\, dx\, dy = \sum_{|\alpha| \le N-1} \tfrac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_x^\alpha \partial_y^\alpha u(0,0) + S_N \tag{8.2}$$

for $u \in C_c^\infty(\mathbb{R}^{2n})$, where

$$S_N \lesssim h^N \sum_{|\alpha+\beta| \le 2n+1} \|\partial_x^\alpha \partial_y^\beta (\partial_x \partial_y)^N u\|_{L^1(\mathbb{R}^{2n})}.$$

Proof of Theorem 8.1.7. Proceeding as above, we may write

$$\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b) u(x) = \lim_{\varepsilon,\delta \to 0^+} (2\pi h)^{-n} \int e^{i(x-y)\xi/h - \varepsilon\langle\xi\rangle} c_\delta(x,y,\xi) u(y)\, dy\, d\xi,$$

where

$$c_\delta(x,y,\xi) := (2\pi h)^{-n} \int e^{i(x-z)(\eta-\xi)/h - \delta\langle z\rangle - \delta\langle\eta\rangle} a(x,z,\eta) b(z,y,\xi)\, dz\, d\eta.$$

We will show that $c_\delta = \mathcal{O}(\langle\xi\rangle^{m+m'})$ uniformly in δ, so that we may pass to a limit c_0 as $\delta \to 0$ (by dominated convergence). We will then show $c_0 \in S_{3n}(\langle\xi\rangle^{m+m'})$, which allows us to send $\varepsilon \to 0$ (and interpret the resulting integral in the sense of oscillatory integrals).

We turn to the details. We define

$$L_1 = \left[1 + \tfrac{|\eta-\xi|^2}{h^2} + \tfrac{|x-z|^2}{h^2}\right]^{-1} [1 - \tfrac{\eta-\xi}{h} D_z + \tfrac{x-z}{h} D_\eta].$$

We next let $\chi_1 \in C_c^\infty(\mathbb{R})$ satisfy

$$\chi_1(s) = \begin{cases} 1 & |s| \le 1 \\ 0 & |s| \ge 2 \end{cases}$$

and let $\chi(x,y) = \chi_1(|x-y|)$. Choosing $k \ge |m| + 2n + 1$, we can write

$$\begin{aligned} &c_\delta(x,y,\xi) \\ &\quad = (2\pi h)^{-n} \int e^{i(x-z)(\eta-\xi)/h} [{}^t L_1]^k \left\{ e^{-\delta\langle z\rangle - \delta\langle\eta\rangle} a(x,z,\eta) b(z,y,\xi) \right\} dz\, d\eta \\ &\quad =: d_\delta(x,y,\xi) + e_\delta(x,y,\xi) + f_\delta(x,y,\xi), \end{aligned}$$

where d_δ includes the cutoff $1-\chi(\xi,\eta)$ before a, e_δ includes $\chi(\xi,\eta)[1-\chi(x,z)]$, and f_δ includes $\chi(\xi,\eta)\chi(x,z)$.

We regard d_δ and e_δ as perturbative terms. In particular, we can write

$$\begin{aligned}(2\pi h)^n d_\delta(x,y,\xi) &= \int_{|\xi-\eta|\geq 1} \mathcal{O}\left\{\frac{\langle\eta\rangle^m\langle\xi\rangle^{m'}}{(1+h^{-1}|\xi-\eta|+h^{-1}|x-z|)^k}\right\} dz\, d\eta \\ &= \int \mathcal{O}\left\{\frac{\langle\eta\rangle^m\langle\xi\rangle^{m'}}{(1+(2h)^{-1}|\xi-\eta|)^{k-n-\frac{1}{2}}}\right\} d\eta.\end{aligned}$$

Thus, for example, if $m\geq 0$, we can deduce (writing $\langle\eta\rangle^m \lesssim \langle\xi\rangle^m + \langle\xi-\eta\rangle^m$) that

$$|(2\pi h)^n d_\delta(x,y,\xi)| \lesssim h^{k-n-\frac{1}{2}}\langle\xi\rangle^{m+m'},$$

which is acceptable. We leave the remaining case $m<0$ as an exercise (*hint:* split into regions where $|\eta|\leq \frac{1}{2}\langle\xi\rangle$ and $|\eta|>\frac{1}{2}\langle\xi\rangle$ and estimate each piece separately). Similarly one can deduce that

$$|(2\pi h)^n e_\delta(x,y,\xi)| \lesssim h^{k-n-\frac{1}{2}}\langle\xi\rangle^{m+m'}$$

for $k\geq |m|+2n+1$. We leave this estimate as an exercise. We also note that one can get the same estimates for any number of derivatives, i.e.

$$|\partial^\alpha d_\delta(x,y,\xi)| + |\partial^\alpha e_\delta(x,y,\xi)| = \mathcal{O}(h^\infty\langle\xi\rangle^{m+m'})$$

uniformly over (x,y,ξ) and $\delta>0$. In fact, taking derivatives just produces powers of $|x-z|$ or $|\eta-\xi|$, which can always be overcome by choosing k larger.

It remains to consider f_δ, which (undoing the integration by parts) has the form

$$\begin{aligned}&f_\delta(x,y,\xi)\\ &= (2\pi h)^{-n}\int e^{i(x-z)(\eta-\xi)/h}\chi(\xi,\eta)\chi(x,z)e^{-\delta\langle z\rangle-\delta\langle\eta\rangle}a(x,z,\eta)b(z,y,\xi)\,dz\,d\eta.\end{aligned}$$

We will understand the behavior of this integral through the stationary phase theorem (in the form (8.2)). We write $z'=z-x$ and $\eta'=\eta-\xi$, so that f_δ has the form

$$f_\delta(x,y,\xi) = (2\pi h)^{-n}\int e^{-iz'\eta'/h}u^\delta_{x,y,\xi}(z',\eta')\,dz'\,d\eta',$$

with appropriate

$$u^\delta_{x,y,\xi}(z',\eta') \in C_c^\infty(\mathbb{R}^n_{z'}\times\mathbb{R}^n_{\eta'}).$$

In particular, by (8.2),

$$f_\delta(x,y,\xi) = \sum_{|\alpha|\le N-1} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_z^\alpha\partial_\eta^\alpha u^\delta_{x,y,\xi}(z,\eta)\Big|_{z=0,\eta=0} + S_N,$$

where

$$\begin{aligned}|S_N| &\lesssim h^N \sum_{|\alpha+\beta|\le 2n+1} \|\partial_z^\alpha\partial_\eta^\beta(\partial_z\partial_\eta)^N u^\delta_{x,y,\xi}\|_{L^1(\mathbb{R}^n_z\times\mathbb{R}^n_\eta)} \\ &\lesssim h^N \int_{|\eta-\xi|\le 2,\ |x-z|\le 2} \langle\eta\rangle^m\langle\xi\rangle^{m'}\,dz\,d\eta \\ &\lesssim h^N\langle\xi\rangle^{m+m'}.\end{aligned}$$

In fact, we can get the same bound for any derivatives of f.

Collecting the estimates, we can deduce that $c_\delta(x,y,\xi)\to c_0(x,y,\xi)$, where

$$c_0(x,y,\xi) = (2\pi h)^{-n}\int e^{i(x-z)(\eta-\xi)/h}({}^tL_1)^k(a(x,z,\eta)b(z,y,\xi))\,dz\,d\eta.$$

Furthermore, since the estimates above were uniform in δ, we can deduce that $c_0\in S_{3n}(\langle\xi\rangle^{m+m'})$. Finally, taking the limit as $\delta\to 0$ in the stationary phase approximation for f, we can deduce the asymptotic expansion for $a\#b$. This completes the proof. $\square$

Note that when a is a polynomial in ξ, so that $\mathrm{Op}_h(a)$ is a differential operator, then the formula giving $a\#b$ is an exact formula.

We now give an application of the composition theorem. We call a symbol $a\in S_d(g)$ **elliptic** if

$$|a|\gtrsim g$$

uniformly on $\mathbb{R}^d\times(0,h_0]$ for some h_0. The following result shows how we may use the composition theorem to invert elliptic symbols up to errors that are $\mathcal{O}(h^\infty)$. (One can compare this to the case of Fourier multiplier operators; in this case, if $|m(\xi)|\gtrsim 1$ then the inverse operator is simply the operator with symbol $\frac{1}{m(\xi)}$.)

Proposition 8.1.8. *Let $a\in S_{3n}(\langle\xi\rangle^m)$ be elliptic. Then there exists $b\in S_{3n}(\langle\xi\rangle^{-m})$ such that*

$$Op_h(a)\circ Op_h(b) = 1 + Op_h(r), \quad \text{where} \quad r=\mathcal{O}(h^\infty) \quad \text{in} \quad S_{3n}(1).$$

Similarly, $Op_h(b)\circ Op_h(a) = 1+Op_h(r')$ for some $\mathcal{O}(h^\infty)$ symbol r'.

Proof. Let us only sketch the first claim. The idea is to construct b in the form $b \sim \sum h^j b_j$ in such a way to guarantee that $a\#b \sim 1$, where $a\#b$ is as in the composition theorem. To do this, we firstly set $b_0 = \frac{1}{a} \in S_{3n}(\langle\xi\rangle^{-m})$ (cf. the chain rule). We can then define b_j for $j \geq 1$ recursively. For example, the linear in h terms in the sum will involve c_α for $|\alpha| = 1$ and b_0, along with c_α for $|\alpha| = 0$ and b_1, which we use to define b_1 (so that the total contribution is zero). Proceeding in this way, we can construct b as desired. $\square$

Quantization and symbolic calculus.

Classical observables are given as functions of the position $x \in \mathbb{R}^n$ and momentum $\xi \in \mathbb{R}^n$. We would therefore like to define pseudodifferential operators with symbols depending only on $2n$ variables (x, ξ). There is an inherent nonuniqueness here—indeed, the symbol $x_j\xi_j$ could be associated with either $x_j \cdot hD_{x_j}$ or $hD_{x_j} \cdot x_j$.

Now, for $a \in S_{2n}(\langle\xi\rangle^m)$ and $t \in [0, 1]$, we have

$$a((1-t)x + ty, \xi) \in S_{3n}(\langle\xi\rangle^m).$$

Thus we may define

$$\mathrm{Op}_h^t(a) := \mathrm{Op}_h(a((1-t)x + ty, \xi)).$$

In particular, we have:

- If $t = 0$, we get the standard or 'left' quantization.
- If $t = \frac{1}{2}$, we get the Weyl quantization (denoted by $\mathrm{Op}_h^W(a)$).
- If $t = 1$, we get the 'right' quantization.

The Weyl quantization yields a symmetric operator whenever a is real-valued; for this reason it is useful in the setting of quantum mechanics.

We will state two results regarding quantization of symbols; for the details see [19, Section 2.7].

Proposition 8.1.9. *Given $b = b(x, y, \xi) \in \mathcal{S}_{3n}(\langle\xi\rangle^m)$ and $t \in [0, 1]$, there exists unique $b_t(x, \xi) \in S_{2n}(\langle\xi\rangle^m)$ such that $Op_h(b) = Op_h^t(b_t)$. In fact,*

$$b_t(x, \xi) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i(\xi' - \xi)\theta/h} b(x + t\theta, x - (1-t)\theta, \xi')\, d\xi'\, d\theta$$

in the sense of oscillatory integrals, and

$$b_t(x,\xi) \sim \sum \frac{(-1)^{|\alpha|}h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha \partial_\theta^\alpha b(x+t\theta, x-(1-t)\theta, \xi)|_{\theta=0}$$

in $S_{2n}(\langle\xi\rangle^m)$.

In this result, b_t is called the symbol of index t of $B = \mathrm{Op}_h(b)$, and we denote

$$b_t = \sigma_t(B).$$

When $t = \frac{1}{2}$, we call $b_{\frac{1}{2}} = b^W$ the Weyl symbol of $\mathrm{Op}(b)$.

This proposition is proven by seeking b_t satisfying

$$\int e^{i(x-y)\xi/h} b(x,y,\xi)\, d\xi = \int e^{i(x-y)\xi/h} b_t((1-t)x+ty,\xi)\, d\xi.$$

This ultimately leads to the oscillatory integral above; the asymptotic expansion is a consequence of the stationary phase theorem (after isolating the relevant part of the integral, as in the proof of the composition theorem).

In the case $t = 0$, one can write

$$\sigma_0(B)(x,\xi;h) = e^{-ix\xi/h} B(e^{i(\cdot)\xi/h}).$$

Example 8.1.4. If $V = V(x) \in S_n(1)$, then for every $t \in [0,1]$ we get

$$\sigma_t(-h^2\Delta + V) = \xi^2 + V(x),$$

which is independent of t.

The next result we state concerns composition:

Theorem 8.1.10 (Symbolic calculus)**.** *Let* $a = a(x,\xi) \in S_{2n}(\langle\xi\rangle^m)$ *and* $b = b(x,\xi) \in S_{2n}(\langle\xi\rangle^{m'})$. *For all* $t \in [0,1]$, *there exists unique* $c_t \in S_{2n}(\langle\xi\rangle^{m+m'})$ *such that*

$$Op_h^t(a) \circ Op_h^t(b) = Op_h^t(c_t).$$

In fact, one can write down a formula and asymptotic expansion for the symbol c_t in the preceding theorem. The proof proceeds by applying the composition theorem to write the composition in the form $\mathrm{Op}_h(c)$ for some symbol c, which then satisfies $\mathrm{Op}_h(c) = \mathrm{Op}_h^t(c_t)$ for suitable c_t (by the previous theorem). The asymptotic expansion is again a consequence of the stationary phase lemma.

If A and B are pseudodifferential operators with symbols in $S_{2n}(\langle\xi\rangle^m)$ and $S_{2n}(\langle\xi\rangle^{m'})$, then for all $t \in [0,1]$ one can show

$$\sigma_t(A \circ B) = \sigma_t(A)\sigma_t(B) + \mathcal{O}(h) \quad \text{in} \quad S_{2n}(\langle\xi\rangle^{m+m'}).$$

In the case of the *commutator* of two operators, namely $[A,B] := AB - BA$, we instead get

$$\sigma_t([A,B]) = \tfrac{h}{i}\{a,b\} + \mathcal{O}(h^2) \quad \text{in} \quad S_{2n}(\langle\xi\rangle^{m+m'}),$$

where $\{a,b\}$ is the *Poisson bracket* defined by

$$\{a,b\} = \tfrac{\partial a}{\partial \xi}\tfrac{\partial b}{\partial x} - \tfrac{\partial a}{\partial x}\tfrac{\partial b}{\partial \xi}.$$

A symbol $a \in S_{2n}(\langle\xi\rangle^m)$ is said to be *classical* if it admits an expansion of the form

$$a(x,\xi;h) \sim \sum_{j\geq 0} h^j a_j(x,\xi),$$

where $a_j \in S_{2n}(\langle\xi\rangle^m)$ do not depend on h, and a_0 is not identically zero. For $\nu \in \mathbb{R}$, we call $h^\nu a_0(x,\xi)$ the *principal symbol* of the classical pseudodifferential operator $A = h^\nu \mathrm{Op}_h^t(a)$. In particular, changing the quantization does not affect the classical character of a pseudodifferential operator, nor does the principal symbol depend on the choice of quantization. We define

$$h^\nu a_0 = \sigma_p(A).$$

Then one has

$$\sigma_p(AB) = \sigma_p(A)\sigma_p(B) \quad \text{and} \quad \sigma_p([A,B]) = \tfrac{h}{i}\{\sigma_p(A), \sigma_p(B)\}.$$

L^2 boundedness.

Our final topic will be to consider the L^2 boundedness of pseudodifferential operators. To this point, we have only considered such operators as acting on $\mathcal{S}$ or $\mathcal{S}'$. We will prove the following result.

Theorem 8.1.11 (Calderón–Vaillancourt)**.** *Let $a \in S_{3n}(1)$. Then there exists $M = M(n)$ such that*

$$\| Op_h(a)\|_{L^2\to L^2} \lesssim_n \sum_{|\alpha|\leq M} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^{3n})}.$$

We will need the following lemma, which may be of independent interest.

Lemma 8.1.12 (Cotlar–Stein Lemma). *Let H be a Hilbert space, $\{A_\mu\}_{\mu\in\mathbb{Z}^d}$ a family of bounded linear operators on H, and $\omega : \mathbb{Z}^d \to [0,\infty)$ satisfying the following:*

- *For all $\mu, \nu \in \mathbb{Z}^d$,*
$$\|A_\mu A_\nu^*\| + \|A_\mu^* A_\nu\| \le \omega(\mu - \nu).$$
- $C_0 := \sum_\mu \sqrt{\omega(\mu)} < \infty$.

Then for any $M \ge 0$, we have
$$\Big\| \sum_{|\mu|\le M} A_\mu \Big\| \le C_0.$$

Proof. First set $S = \sum_{|\mu|\le M} A_\mu$. Using Exercise 7.4.5, we have
$$\|S\|^{2m} = \|S^*S\|^m.$$
Next, observe that
$$\|S^*S\|^m = \|(S^*S)^m\| \quad \text{for} \quad m \ge 1.$$
Indeed the $\ge$ direction is clear. For the reverse, we argue essentially as in (A.3), using the fact that S^*S is a bounded, positive, self-adjoint operator. In particular, we can write $S^*S = [(S^*S)^m]^{\frac{1}{m}}$ and use the general bound $\|T^\theta\| \le \|T\|^\theta$; this yields the desired estimate.

Now observe that
$$S^*S = \Big(\sum_{|\mu|,|\nu|\le M} A_\mu^* A_\nu \Big)^m = \sum_{|\mu_\ell|,|\nu_\ell|\le M} A_{\mu_1}^* A_{\nu_1} \cdots A_{\mu_m}^* A_{\nu_m}.$$
By assumption, each summand obeys the bound
$$\|\cdot\| \le \|A_{\mu_1}^* A_{\nu_1}\| \cdots \|A_{\mu_m}^* A_{\nu_m}\| \le \omega(\mu_1 - \nu_1) \cdots \omega(\mu_m - \nu_m).$$
Using instead the bound
$$\|A_\mu\|^2 = \|A_\mu^* A_\mu\| \le \omega(0),$$
we can also bound each summand by
$$\|\cdot\| \le \sqrt{\omega(0)}\,\omega(\nu_1 - \mu_2) \cdots \omega(\nu_{m-1} - \mu_m)\sqrt{\omega(0)}.$$

Taking the geometric mean of the previous two estimates yields the bound

$$\|\cdot\| \leq [\omega(0)\omega(\mu_1-\nu_1)\omega(\nu_1-\mu_2)\cdots\omega(\mu_m-\nu_m)]^{\frac{1}{2}}.$$

Thus, continuing from above, we perform one sum at a time (starting from the ν_m sum, then μ_m, then ν_{m-1}, ...) to get

$$\begin{aligned}\|(S^*S)^m\| &\leq \sum_{|\mu_\ell|,|\nu_\ell|\leq M} [\omega(0)\omega(\mu_1-\nu_1)\omega(\nu_1-\mu_2)\cdots\omega(\mu_m-\nu_m)]^{\frac{1}{2}} \\ &\leq \sum_{|\mu_1|\leq M} \sqrt{\omega(0)}C_0^{2m-1} \leq (2M+1)^d\sqrt{\omega(0)}C_0^{2m-1}.\end{aligned}$$

Hence

$$\|S\|^{2m} = \|(S^*S)\|^m \leq (2M+1)^d\sqrt{\omega(0)}C_0^{2m-1}.$$

Taking the $\frac{1}{2m}$ root of both sides and sending $m\to\infty$ yields the desired result. □

Proof of Theorem 8.1.11. Using Proposition 8.1.9, we may find $b \in S_{2n}(1)$ so that $A = \mathrm{Op}_h^W(b)$. Moreover, by integrating by parts in the integral expression for b, we can control derivatives of b in terms of derivatives of a. Thus we may take $A = \mathrm{Op}_h^W(a)$ for some $a \in S_{2n}(1)$. Furthermore, by rescaling $\xi \mapsto h\xi$, we can further reduce to proving the theorem for the case $h = 1$. That is, we may take $A = \mathrm{Op}_1^W(a)$.

Now let $\chi_0 \in C_c^\infty(\mathbb{R}^{2n})$ yield a partition of unity through the translations $\chi_\mu(z) = \chi_0(z-\mu)$ for $\mu \in \mathbb{Z}^d$. In particular, $\sum_\mu \chi_\mu \equiv 1$. We set $a_\mu = a\chi_\mu$ and observe that

$$|\partial^\alpha a_\mu| \lesssim \sup_{|\beta|\leq|\alpha|} \|\partial^\beta a\|_{L^\infty} \quad \text{uniformly.}$$

Define $A_\mu = \mathrm{Op}_1^W(a_\mu)$, so that

$$Au = \sum_\mu A_\mu u \quad \text{for all} \quad u \in C_c^\infty(\mathbb{R}^n).$$

This series can be summed in L^2, for example. To proceed, we will prove estimates for the operators $A_\mu A_\nu^*$ and then apply the Cotlar–Stein lemma. To this end, we write

$$A_\mu A_\nu^* u(x) = \int K_{\mu,\nu}(x,y)u(y)\,dy,$$

where

$$K_{\mu,\nu}(x,y) = (2\pi)^{-2n} \int e^{i(x\xi - y\eta - z\xi + z\eta)} a_\mu(\tfrac{x+z}{2}, \xi) \bar{a}_\nu(\tfrac{y+z}{2}, \eta)\, dz\, d\eta\, d\xi.$$

Now, a_μ and a_ν are smooth and compactly supported, so that $K_{\mu,\nu}$ is smooth on $\mathbb{R}^{2n}$. We now use the operator

$$L = [1+|x-z|^2+|y-z|^2+|\xi-\eta|^2]^{-1}(1+(x-z)D_\xi-(y-z)D_\eta-(\xi-\eta)D_z),$$

which obeys

$$L[e^{i(x\xi-y\eta-z\xi+z\eta)}] = e^{i(x\xi-y\eta-z\xi+z\eta)}.$$

Thus for any $N \geq 0$, we may integrate by parts N times to get

$$K_{\mu,\nu}(x,y) = (2\pi)^{-2n} \int e^{i(x\xi-y\eta-z\xi+z\eta)} ({}^tL)^N [a_\mu(\tfrac{x+z}{2}, \xi) \bar{a}_\nu(\tfrac{y+z}{2}, \eta)]\, dz\, d\eta\, d\xi.$$

Now, if $|\mu - \nu|$ is large enough, then in order for $a_\mu(t,\tau)\bar{a}_\nu(s,\sigma)$ to be nonzero we must have

$$|\mu - \nu| \sim |t - s| + |\tau - \sigma|.$$

Thus, if we write $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ and define the following set of (y, z, η, ξ),

$$D_{\mu,\nu} = \{|\mu - \nu| \sim |x - y| + |\xi - \eta|, \quad |\xi - \mu_2| \lesssim 1, \quad |\eta - \nu_2| \lesssim 1\},$$

then we find

$$\int |K_{\mu,\nu}(x,y)|\, dy \lesssim \int_{D_{\mu,\nu}} [1 + |x-z| + |y-z| + |\xi-\eta|]^{-N}\, dy\, dz\, d\eta\, d\xi,$$

where the implicit constant depends on

$$\sup_{|\alpha| \leq N} \|\partial^\alpha a\|_{L^\infty}^2.$$

Now we use the fact that

$$|x - z| + |y - z| \geq |x - y|$$

and the definition of $D_{\mu,\nu}$ to get the bound

$$\int |K_{\mu,\nu}(x,y)|\, dy \lesssim \int \frac{[1 + |\mu-\nu|]^{2n+2-N}}{(1+|x-z|)^{n+1}(1+|x-y|)^{n+1}}\, dy\, dz.$$

In particular,

$$\sup_x \int |K_{\mu,\nu}(x,y)|\, dy \lesssim \sup_{|\alpha|\le N} \|\partial^\alpha a\|_{L^\infty}^2 \cdot [1+|\mu-\nu|]^{2n+2-N}.$$

A similar argument yields

$$\sup_y \int |K_{\mu,\nu}(x,y)|\, dx \lesssim \sup_{|\alpha|\le N} \|\partial^\alpha a\|_{L^\infty}^2 \cdot [1+|\mu-\nu|]^{2n+2-N}.$$

Thus by Schur's test (cf. Remark A.3.5), we deduce

$$\|A_\mu A_\nu^*\|_{L^2\to L^2} \lesssim \sup_{|\alpha|\le N} \|\partial^\alpha a\|_{L^\infty}^2 \cdot [1+|\mu-\nu|]^{2n+2-N}$$

uniformly in μ,ν. The same argument handles $A_\mu^* A_\nu$.

We now choose $N = 4n+3$ and apply the Cotlar–Stein lemma with $d = 2n$ and

$$\omega(\mu) \sim (1+|\mu|)^{-2n-1} \sup_{|\alpha|\le N} \|\partial^\alpha a\|_{L^\infty}^2,$$

where the implicit constants depend only on the dimension. In particular, we can get

$$\|\sum_{|\mu|\le M} A_\mu u\|_{L^2} \le C_0 \|u\|_{L^2} \quad \text{uniformly in} \quad M\ge 0 \quad \text{and} \quad u\in L^2,$$

which then implies $\|Au\|_{L^2} \le C_0\|u\|_{L^2}$ for any $u \in L^2$. This completes the proof. □

We close this section with a few applications.

First, combining Proposition 8.1.8 with the L^2 boundedness result, we have that if $a \in S_{3n}(\langle\xi\rangle^m)$ is elliptic then we may find $b \in S_{3n}(\langle\xi\rangle^{-m})$ such that

$$\mathrm{Op}_h(a)\circ \mathrm{Op}_h(b) = 1 + R_1 \quad \text{and} \quad \mathrm{Op}_h(b)\circ\mathrm{Op}_h(a) = 1+R_2,$$

where

$$\|R_1\|_{L^2\to L^2} + \|R_2\|_{L^2\to L^2} = \mathcal{O}(h^\infty).$$

In particular, if $m=0$ and h is sufficiently small, then $\mathrm{Op}_h(a)$ is invertible on L^2, with inverse satisfying

$$\mathrm{Op}_h(a)^{-1} = \mathrm{Op}_h(b) + \mathcal{O}(h^\infty).$$

Finally, we prove the following estimate:

Proposition 8.1.13 (Garding inequality)**.** *Suppose $a \in S_{2n}(1)$ is real-valued and satisfies $a \geq \frac{1}{C}$. Then for any $C_1 > C$, we have*

$$Op_h^W(a) \geq \tfrac{1}{C_1} \quad \text{on} \quad L^2(\mathbb{R}^n)$$

for h sufficiently small, i.e.

$$\langle Op_h^W(a)u, u\rangle \geq \tfrac{1}{C_1}\|u\|_{L^2}^2 \quad \text{for all} \quad u \in L^2.$$

Proof. Let $C < C_2 < C_1$ so that

$$\sqrt{a - \tfrac{1}{C_2}} \in S_{2n}(1).$$

Write

$$B = \mathrm{Op}_h^W(\sqrt{a - \tfrac{1}{C_2}}),$$

so that B is bounded and self-adjoint on $L^2(\mathbb{R}^n)$. By the symbolic calculus, we may write

$$\mathrm{Op}_h^W(a - \tfrac{1}{C_2}) = B^2 + hR, \quad \text{where} \quad \|R\|_{L^2\to L^2} \lesssim 1.$$

Since $B^2 \geq 0$, we may find C' so that

$$\mathrm{Op}_h^W(a - \tfrac{1}{C_2}) \geq -C'h,$$

and hence

$$\mathrm{Op}_h^W(a) \geq \tfrac{1}{C_2} - C'h \geq \tfrac{1}{C_1}$$

for h small enough. □

8.2 Coifman–Meyer multipliers

In this section we will prove a version of the Coifman–Meyer multiplier theorem. This may be viewed as a generalization of the Mihlin multiplier theorem (cf. Theorem 7.1.1) to the case of bilinear operators. In particular, given $m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ we may define the bilinear operator T_m by prescribing its Fourier transform:

$$\mathcal{F}[T_m(f,g)](\xi) = \int_{\mathbb{R}^d} m(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta.$$

Equivalently, we may write

$$T_m(f,g)(x) = \iint e^{2\pi i x\xi} m(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi.$$

Note that if $m \equiv 1$ then $T_m(f,g) = fg$ (cf. Lemma 2.5.11). Similarly, if $m(\xi_1,\xi_2) = a(\xi_1)b(\xi_2)$ then $T_m(f,g) = [T_a f][T_b g]$, where T_a is the Fourier multiplier operator with symbol a.

We may also understand T_m by observing that (formally)

$$T_m(e^{2\pi i x\xi_1}, e^{2\pi i x\xi_2}) = m(\xi_1,\xi_2)e^{2\pi i x(\xi_1+\xi_2)}.$$

Indeed, this follows from $\mathcal{F}[e^{2\pi i x\xi_j}] = \delta(\xi - \xi_j)$. This shows that T_m multiplies plane waves (adding their frequencies) and modulates their amplitude by $m(\xi_1,\xi_2)$.

Our goal will be to prove $L^p \times L^r \to L^q$ mapping properties for bilinear operators of this type. We will consider multipliers/operators of the following type.

Definition 8.2.1. We call $m : \mathbb{R}^d \to \mathbb{R}^d \to \mathbb{C}$ a **Coifman–Meyer** symbol if it obeys

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1,\xi_2)| \lesssim_{|\alpha_1|,|\alpha_2|,d} (|\xi_1| + |\xi_2|)^{-|\alpha_1|-|\alpha_2|}$$

for all multiindices α_1, α_2. We call T_m a **Coifman–Meyer** multiplier.

Remark 8.2.2. This should be compared with the definition of a Mihlin multiplier (see again Theorem 7.1.1). In practice, only finitely many multiindices (depending on the dimension) are needed, but we will not be concerned with this refinement.

Note that the product of two Coifman–Meyer multipliers is again a Coifman–Meyer multiplier.

In the setting of Mihlin multipliers, we sought to prove $L^p \to L^p$ bounds. For Coifman–Meyer multipliers, it is more natural to seek Hölder-type estimates, i.e.

$$\|T_m(f,g)\|_{L^r} \lesssim \|f\|_{L^p}\|g\|_{L^q}, \quad \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{r}.$$

In fact, we will prove the following.

Theorem 8.2.3. *Let m be a Coifman–Meyer symbol. Then the multiplier T_m maps $L^p \times L^q \to L^r$ boundedly for all $1 < p,q,r < \infty$ satisfying*

$$\tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{r}.$$

Remark 8.2.4. This theory can be extended to handle the endpoints $p,q,r \in \{1,\infty\}$; however, we will not pursue this extension here. See e.g. [22] for a clear presentation.

Example 8.2.1. Define

$$m(\xi_1, \xi_2) = \frac{\xi_1 \cdot \xi_2}{|\xi_1|^2 + |\xi_2|^2},$$

i.e.

$$T_m(f, g) = (-\Delta)^{-1}[\nabla f \cdot \nabla g].$$

Then m is a Coifman–Meyer symbol, and hence

$$\|(-\Delta)^{-1}[\nabla f \cdot \nabla g]\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}$$

for $1 < p, q, r < \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

The proof of Theorem 8.2.3 will firstly rely on a 'paraproduct decomposition' for Coifman–Meyer multipliers. In particular, we need the notion of high-high, high-low, and low-high multipliers:

- We call T_m a **high-high paraproduct** if $|\xi_1| \sim |\xi_2|$ on the support of m.
- We call T_m a **low-high paraproduct** if $|\xi_1 + \xi_2| \sim |\xi_2|$ on the support of m.
- We call T_m a **high-low paraproduct** if $|\xi_1 + \xi_2| \sim |\xi_1|$ on the support of m.

We have the following paraproduct decomposiion:

Lemma 8.2.5. *Given a Coifman–Meyer paraproduct T_m, we may decompose*

$$T_m = \pi_{hh} + \pi_{hl} + \pi_{lh},$$

where $\pi_{hh}, \pi_{hl}, \pi_{lh}$ are high-high, high-low, and low-high Coifman–Meyer paraproducts.

Proof. We recall the Littlewood–Paley multipliers ψ_N, φ_N from Section 7.2. In particular, as the ψ_N form a partition of unity (where we take $N \in 2^{\mathbb{Z}}$), we can write for any (ξ_1, ξ_2),

$$\begin{aligned} 1 &= \sum_{N,M} \psi_N(\xi_1)\psi_M(\xi_2) \\ &= \sum_N \psi_N(\xi_1)\varphi_{\frac{N}{8}}(\xi_2) + \sum_N \sum_{\frac{N}{8} \le M \le 8N} \psi_N(\xi_1)\psi_M(\xi_2) + \sum_N \varphi_{\frac{N}{8}}(\xi_1)\psi_N(\xi_2). \end{aligned}$$

These three expressions are high-low, high-high, and low-high multipliers, respectively. If we multiply by $m(\xi_1, \xi_2)$, then we complete the proof of the lemma. $\square$

Thus, to prove the Coifman–Meyer theorem, it suffices to treat low-high and high-high multipliers; the high-low case follows by symmetry.

We will need the following technical lemma.

Lemma 8.2.6. *Let f be a Schwartz function and $N \in 2^{\mathbb{Z}}$. For any x, y, we have the bound*

$$|P_{\le N} f(y)| \lesssim \langle N(y-x)\rangle^d Mf(x),$$

where $\langle x\rangle = \sqrt{1+x^2}$ and M is the Hardy–Littlewood maximal function.

Proof. We begin with the estimate

$$N^d \int |\check{\varphi}(N(y-z))|\,|f(z)|\,dz \lesssim N^d \int \langle N(y-z)\rangle^{-100d}|f(z)|\,dz,$$

where we use the fact that $\check{\varphi}$ is a Schwartz function.

For the region $\langle z-x\rangle \lesssim \langle y-x\rangle$, we estimate

$$\begin{aligned} N^d \int_{\langle z-x\rangle \lesssim \langle y-x\rangle} \langle N(y-z)\rangle^{-100d}|f(z)|\,dz &\lesssim N^d \int_{\langle z-x\rangle \lesssim \langle y-x\rangle} |f(z)|\,dz \\ &\lesssim N^d \langle y-x\rangle^d Mf(x), \end{aligned}$$

which is acceptable. For the remaining region $\langle z-x\rangle \gg \langle y-x\rangle$, we use Lemma 7.2.5 to estimate

$$\begin{aligned} &N^d \int_{\langle z-x\rangle \gg \langle y-x\rangle} \langle N(y-z)\rangle^{-100d}|f(z)|\,dz \\ &\quad\lesssim N^d \int \langle N(z-x)\rangle^{-100d}|f(z)|\,dz \lesssim Mf(x), \end{aligned}$$

which is also acceptable. ∎

Lemma 8.2.7 (High-high paraproducts)**.** *Let π_{hh} be a high-high paraproduct. Then π_{hh} satisfies the bounds appearing in Theorem 8.2.3.*

Proof. We write

$$\pi_{hh}(f,g) = \sum_{N,M} \pi_{hh}(\tilde{P}_N P_N f, \tilde{P}_M P_M g),$$

where $\tilde{P}_N$ denotes the operator corresponding to the fattened Littlewood–Paley multiplier. Now, the operator

$$\pi_{hh}(\tilde{P}_N \cdot, \tilde{P}_M \cdot)$$

is zero unless $N \sim M$; in this case, it is a bilinear multiplier with a symbol m_{NM}, which is a bump function supported where $|\xi_1| \sim |\xi_2| \sim N$. Writing T_{NM} for $T_{m_{NM}}$, we find

$$|\pi_{hh}(f,g)| \leq \sum_{N \sim M} |T_{NM}(P_N f, P_M g)|.$$

Now consider the symbol m_{NM}. We will decompose $m_{NM}(\xi_1, \xi_2)$ using a Fourier series on a torus in $\mathbb{R}^d \times \mathbb{R}^d$ of sidelength CN for large enough $C > 0$:

$$m_{NM}(\xi_1,\xi_2) = \sum_{n_1,n_2 \in \mathbb{Z}^d} c_{n_1,n_2} e^{2\pi i (n_1 \cdot \xi_1 + n_2 \cdot \xi_2)/CN}$$

on the support of $\psi_N(\xi_1)\psi_M(\xi_2)$. Now, using the definition of the Fourier coefficients and integration by parts, the Coifman–Meyer condition guarantees that

$$c_{n_1,n_2} \lesssim (1 + |n_1| + |n_2|)^{-100d} \tag{8.3}$$

(see Exercise 8.3.2).

The advantage of this decomposition is that it factors m into a sum of terms of the form $a(\xi_1)b(\xi_2)$. In particular, we compute

$$\begin{aligned} &T_{NM}(P_N f, P_M g)(x) \\ &\quad = \sum_{n_1,n_2} c_{n_1,n_2} P_N f(x - \tfrac{n_1}{CN}) P_M g(x - \tfrac{n_2}{CN}), \end{aligned}$$

so that

$$\begin{aligned} &|\pi_{hh}(f,g)(x)| \\ &\lesssim \sum_{N \sim M} \sum_{n_1,n_2} (1 + |n_1| + |n_2|)^{-100d} |P_N f(x - \tfrac{n_1}{CN})| \, |P_M g(x - \tfrac{n_2}{CN})|. \end{aligned}$$

We write $P_N f = \tilde{P}_N P_N f$ and use Lemma 8.2.6 to estimate

$$|P_N f(x - \tfrac{n_1}{CN})| \lesssim \langle n_1 \rangle^d M[P_N f](x).$$

Similarly, recalling $N \sim M$,

$$|P_M g(x - \tfrac{n_2}{CN})| \lesssim \langle n_2 \rangle^d M[P_M g](x).$$

Thus

$$\begin{aligned}
|\pi_{hh}(f,g)(x)| &\lesssim \sum_{N\sim M} \sum_{n_1,n_2} \tfrac{\langle n_1\rangle^d \langle n_2\rangle^d}{\langle |n_1|+|n_2|\rangle^{100d}} |MP_N f(x)|\,|MP_M g(x)| \\
&\lesssim \sum_{N\sim M} |MP_N f(x)|\,|MP_M g(x)| \\
&\lesssim \Big(\sum_N |MP_N f(x)|^2\Big)^{\frac12} \Big(\sum_N |MP_N g(x)|^2\Big)^{\frac12},
\end{aligned}$$

and hence, using Hölder's inequality, the vector maximal inequality (Theorem 6.3.8) and the Littlewood–Paley square function estimate (Theorem 7.2.6)

$$\begin{aligned}
\|\pi_{hh}(f,g)\|_{L^r} &\lesssim \Big\| \Big(\sum_N |MP_N f(x)|^2\Big)^{\frac12} \Big(\sum_N |MP_N g(x)|^2\Big)^{\frac12} \Big\|_{L^r} \\
&\lesssim \Big\| \Big(\sum_N |MP_N f(x)|^2\Big)^{\frac12} \Big\|_{L^p} \Big\| \Big(\sum_N |MP_N g(x)|^2\Big)^{\frac12} \Big\|_{L^q} \\
&\lesssim \Big\| \Big(\sum_N |P_N f(x)|^2\Big)^{\frac12} \Big\|_{L^p} \Big\| \Big(\sum_N |P_N g(x)|^2\Big)^{\frac12} \Big\|_{L^q} \\
&\lesssim \|f\|_{L^p} \|g\|_{L^q},
\end{aligned}$$

provided $1 < p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. □

It remains to treat the case of low-high paraproducts.

Lemma 8.2.8 (Low-high paraproducts)**.** *Let π_{lh} be a low-high paraproduct. Then π_{lh} satisfies the bounds appearing in Theorem 8.2.3. (In fact, we may allow $p = \infty$.)*

Proof. We write

$$\pi_{lh}(f,g) = \sum_N \pi_{lh}(f, \tilde{P}_N P_N g) = \sum_N T_{m_N}(P_{\le \frac{N}{8}} f, P_N g),$$

where

$$m_N(\xi_1,\xi_2) = m(\xi_1,\xi_2)\tilde{\varphi}_{\frac{N}{8}}(\xi_1)\tilde{\psi}_N(\xi_2).$$

Here m denotes the multiplier for π_{lh}. In particular m_N is a bump function supported where $|\xi_1| \lesssim N$ and $|\xi_2| \sim N$. Thus, we may perform a Fourier series decomposition for $m_N(\xi_1,\xi_2)$ as before and write

$$\pi_{lh}(f,g)(x) = \sum_N \sum_{n_1,n_2} c_{n_1,n_2} P_{\le \frac{N}{8}} f(x - \tfrac{n_1}{CN}) P_N g(x - \tfrac{n_2}{CN})$$

with

$$|c_{n_1,n_2}| \lesssim \langle |n_1| + |n_2| \rangle^{-100d}.$$

We estimate the L^r norm by duality. We fix $h \in L^{r'}$. Noting that

$$P_{\leq \frac{N}{8}} f P_N g = \tilde{P}_N [P_{\leq \frac{N}{8}} f P_N g],$$

we use Lemma 8.2.6, Proposition 7.2.3(b), the vector maximal inequality (Theorem 6.3.8), and the Littlewood–Paley square function estimate (Theorem 7.2.6), to estimate

$$\begin{aligned}
&\int \sum_N \sum_{n_1,n_2} |c_{n_1,n_2} P_{\leq \frac{N}{8}} f(x - \tfrac{n_1}{CN}) P_N g(x - \tfrac{n_2}{CN}) \tilde{P}_N h(x)| \, dx \\
&\quad \lesssim \int \sum_N \sum_{n_1,n_2} \tfrac{\langle n_1 \rangle^d \langle n_2 \rangle^d}{\langle |n_1| + |n_2| \rangle^{100d}} [Mf] M[P_N g] |\tilde{P}_N h| \, dx \\
&\quad \lesssim \int |Mf| \Big(\sum_N |M[P_N g]|^2 \Big)^{\frac{1}{2}} \Big(\sum_N |\tilde{P}_N h|^2 \Big)^{\frac{1}{2}} dx \\
&\quad \lesssim \|Mf\|_{L^p} \Big\| \Big(\sum_N |M[P_N g]|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q} \Big\| \Big(\sum_N |\tilde{P}_N h|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{r'}} \\
&\quad \lesssim \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^{r'}}.
\end{aligned}$$

Taking the supremum over unit $h \in L^{r'}$ yields the result. □

Proof of Theorem 8.2.3. Combining the high-high and low-high estimates, we complete the proof of Theorem 8.2.3. □

8.3 Exercises

Exercise 8.3.1. Prove (8.2).

Exercise 8.3.2. Prove (8.3).

Chapter 9

Modern harmonic analysis, part II

9.1 Rearrangements and the sharp Gagliardo–Nirenberg inequality

In this section and the next we will consider the problem of existence of optimizers for some functional inequalities. We will consider some particular cases of the Gagliardo–Nirenberg inequality and the Sobolev embedding inequality, namely

$$\|f\|_{L^4} \leq C_{GN}\|f\|_{L^2}^{\frac{1}{4}}\|\nabla f\|_{L^2}^{\frac{3}{4}} \quad \text{for} \quad f \in H^1(\mathbb{R}^3)$$

and

$$\|f\|_{L^6} \leq C_{Sob}\|\nabla f\|_{L^2} \quad \text{for} \quad f \in \dot{H}^1(\mathbb{R}^3).$$

Here we use C_{GN} and C_{Sob} to denote the best possible constant in these inequalities. Our goal will be to prove that there exist functions that attain the best constant. The basic idea is to take an optimizing sequence and try to prove the existence of a limit, which one then proves is an optimizer. However, one must contend with a lack of compactness due to the presence of symmetries that leave the inequalities invariant, namely, translation and scaling invariance. For example, suppose one already knew that there existed an optimizer f^* to one of these estimates. Then $f_n := f^*(b_n x + x_n)$ would be an optimizing sequence for any choice of parameters $b_n \in (0, \infty)$ and $x_n \in \mathbb{R}^3$. However, one can readily choose these parameters so that f_n converges weakly to zero (i.e. $|x_n| \to \infty$, $b_n \to 0$, or $b_n \to \infty$; see the exercises).

Thus, to prove the existence of a limit, we need to restore the loss of compactness. For the case of Gagliardo–Nirenberg, we first perform a rescaling to suitably normalize the sequence. We will restore the loss of compactness due to translations by taking radial decreasing rearrangements and exploiting the compactness of the embedding $H^1_{rad} \hookrightarrow L^4$. For the case of Sobolev embedding, a different approach is needed, as the embedding $\dot{H}^1_{rad} \hookrightarrow L^6$ is not compact. We will use the technique of concentration compactness and profile decompositions, which allows us to understand precisely the ways in which a bounded sequence in $\dot{H}^1$ could fail to be compact. See [18] for an alternate approach.

We begin by proving the following:

Lemma 9.1.1 (Gagliardo–Nirenberg inequality)**.** *There exists $C > 0$ such that for all $f \in H^1(\mathbb{R}^3)$,*

$$\|f\|_{L^4} \leq C\|f\|_{L^2}^{\frac{1}{4}}\|\nabla f\|_{L^2}^{\frac{3}{4}}. \tag{9.1}$$

Remark 9.1.2. This is a special case of a more general range of inequalities of the form

$$\|f\|_{L^p} \lesssim \|f\|_{L^2}^{\theta}\||\nabla|^s f\|_{L^2}^{1-\theta}. \tag{9.2}$$

We leave the investigation of the general case as an exercise.

Proof of (9.1). Using the triangle inequality and Bernstein estimates, we have for any $N_0 \in 2^{\mathbb{Z}}$ the estimate

$$\begin{aligned}
\|f\|_{L^4} &\leq \sum_N \|f_N\|_{L^4} \\
&\lesssim \sum_{N \leq N_0} N^{\frac{3}{4}}\|f\|_{L^2} + \sum_{N > N_0} N^{-\frac{1}{4}}\|\nabla f\|_{L^2} \\
&\lesssim N_0^{\frac{3}{4}}\|f\|_{L^2} + N_0^{-\frac{1}{4}}\|\nabla f\|_{L^2}.
\end{aligned}$$

Optimizing in N_0 yields the result. □

We define the **optimal constant** C_{GN} by

$$C_{GN}^{-1} = \inf\{\|f\|_{L^2}^{\frac{1}{4}}\|\nabla f\|_{L^2}^{\frac{3}{4}} \div \|f\|_{L^4} : f \in H^1(\mathbb{R}^3)\backslash\{0\}\}.$$

We will prove the following:

Theorem 9.1.3. *There exists $f \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that*

$$\|f\|_{L^4} = C_{GN}\|f\|_{L^2}^{\frac{1}{4}}\|\nabla f\|_{L^2}^{\frac{3}{4}}.$$

Remark 9.1.4. Using the Euler–Lagrange equation associated to the optimization of Gagliardo–Nirenberg, one can deduce the existence of solutions to the nonlinear elliptic partial differential equation

$$-\Delta Q + Q - Q^3 = 0, \quad Q : \mathbb{R}^3 \to \mathbb{R}.$$

To prove Theorem 9.1.3, we will need to develop a few tools, specifically, the notion of a radial decreasing rearrangement, and several compactness tools.

We first introduce radial rearrangements. We give here an abbreviated introduction; for more details and further results, see [18, Chapter 3].

For a measurable set $S \subset \mathbb{R}^d$, we define the **radial rearrangement** of S (denoted S^*) to be the ball centered at the origin such that $|S^*| = |S|$. The **radial rearrangement** of a function f is then defined by

$$f^*(x) = \int_0^\infty \chi_{\{|f|>\lambda\}^*}(x)\, d\lambda,$$

where χ_S denotes the characteristic function of S. This can be compared with the level set (or "layer cake") decomposition

$$|f(x)| = \int_0^{|f(x)|} d\lambda = \int_0^\infty \chi_{\{|f|>\lambda\}}(x)\, d\lambda. \tag{9.3}$$

To make sense of this, we only consider functions such that $|\{|f| > \lambda\}|$ is finite for all $\lambda > 0$.

This definition guarantees that $\chi_S^* = \chi_{S^*}$. Indeed, noting that

$$\{\chi_S > \lambda\} = \begin{cases} S & \lambda \in (0,1) \\ \emptyset & \lambda \geq 1, \end{cases}$$

we find

$$\chi_S^*(x) = \int_0^1 \chi_{S^*}(x)\, d\lambda = \chi_{S^*}(x).$$

By construction, the rearrangement f^* of a function f is a nonnegative, radial (i.e. spherically symmetric), decreasing function. Furthermore, the level sets of f^* are the rearrangements of the level sets of $|f|$, that is,

$$\{f^* > \lambda\} = \{|f| > \lambda\}^*.$$

This implies that rearrangements preserve all L^p norms (cf. (A.1)).

We need the following estimate. We only sketch the proof; complete details may be found in [18].

Proposition 9.1.5 (Riesz rearrangement inequality)**.** *For non-negative f, g, h,*

$$\langle f, g * h\rangle \leq \langle f^*, g^* * h^*\rangle.$$

Proof. We first consider the one-dimensional case. Using the decomposition (9.3), we may first reduce to the case when f, g, h are characteristic functions of sets of finite measure. Using approximation by open sets, we further reduce to the case of open sets. Writing open sets as countable unions of open intervals and using monotone convergence, we further reduce the problem to considering finite disjoint unions of open intervals, say

$$f(x) = \sum_{j=1}^{J_1} f_j(x - a_j), \quad g(x) = \sum_{j=1}^{J_2} g_j(x - b_j), \quad h(x) = \sum_{j=1}^{J_3} h_j(x - c_j),$$

where f_j, g_j, h_j are characteristic functions of an interval centered at the origin. Now for $t \in [0, 1]$ let

$$\begin{aligned} I_{jk\ell}(t) &= \iint f_j(x - ta_j)g_k(x - y - tb_k)h_\ell(y - tc_\ell)\, dx\, dy \\ &= \iint f_j(x)g_k(x - y)h_\ell(y + t(a_j - b_k - c_\ell))\, dx\, dy \\ &=: \int u_{jk}(y)h_\ell(y + t(a_j - b_k - c_\ell))\, dy. \end{aligned}$$

We have that u_{jk} is a symmetric decreasing function of y, and so $I_{jk\ell}$ is a decreasing function of t.

We now start sending $t \downarrow 0$. As soon as two intervals corresponding to one of the functions intersect, we stop the process and redefine the function so that it contains an interval that is the union of these two intervals. We now repeat this process finitely many times until we are left with just three intervals centered at the origin with the same measures the intervals comprising f, g, h (i.e. until we have constructed f^*, g^*, h^*). As this process has only increased $I_{jk\ell}$, this implies the result.

To prove the higher dimensional case, we introduce the **Steiner symmetrization**. Given a direction e, we define the Steiner symmetrization of A, denoted A^{*e}, to be the $1d$ symmetrization of A along lines that are parallel to e. Given a rotation ρ of $\mathbb{R}^d$ with $\rho e = e_1$, we define $\rho f(x) = f(\rho^{-1}x)$ and and let $[\rho f]^{*1}$ be the $1d$ symmetrization of ρf with respect to x_1 (keeping $x_1^\perp$ fixed). We then set

$$f^{*e} = \rho^{-1}[(\rho f)^{*1}].$$

This notion of symmetrization preserves measurability of sets and functions (exercise).

We will focus on the case $d = 2$. Again, we may reduce to considering f, g, h to be characteristic functions of finite measure sets, say A, B, C. By the $1d$ rearrangement inequality, we have

$$I(A^{*e}, B^{*e}, C^{*e}) := \langle f^{*e}, g^{*e} * h^{*e} \rangle \geq \langle f, g * h \rangle = I(A, B, C)$$

for all directions $e \in \mathbb{S}^1$.

Now let $\alpha > 0$ be an irrational multiple of 2π and let R_α denote rotation by angle α. Let X, Y denote Steiner symmetrization along the x, y axes. We set

$$A_k = (Y X R_\alpha A)^k$$

and similarly for B_k, C_k. Note $|A_k| \equiv |A|$ and each A_k has reflection symmetry about both the x and y axes. Furthermore,

$$I(A, B, C) \leq I(A_k, B_k, C_k) \leq I(A_{k+1}, B_{k+1}, C_{k+1})$$

for all k. The key is to prove that $\chi_{A_k} \to \chi_{A^*}$ in L^2 along a subsequence, and similarly for B_k, C_k. This suffices to complete the proof, since we estimate along this subsequence

$$\begin{aligned}
&|I(A^*, B^*, C^*) - I(A_k, B_k, C_k)| \\
&\quad \leq \|\chi_{A^*} - \chi_{A_k}\|_{L^2} \|\chi_{B^*}\|_{L^2} \|\chi_{C^*}\|_{L^1} + \|\chi_{A^*}\|_{L^2} \|\chi_{B^*} - \chi_{B_k}\|_{L^2} \|\chi_{C^*}\|_{L^1} \\
&\qquad + \|\chi_{A_k}\|_{L^2} \|\chi_{B_k}\|_{L^1} \|\chi_{C^*} - \chi_{C_k}\|_{L^2} \\
&\quad \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

We now sketch how to prove the desired convergence. We first note that each A_k is of the form

$$A_k = \{(x, y) : |y| < \omega_k(|x|)\}$$

where ω_k is a symmetric decreasing function. We can further reduce to the case that A, B, C are all contained in a ball. Then we can use uniform boundedness of the ω_k and a diagonal argument to find a subsequence on which ω_k converges at every rational point. Using monotonicity, we can get convergence at all but countably many points (where jumps may occur). We can then get L^2 convergence of the χ_{A_k} (and χ_{B_k}, χ_{C_k}) along a subsequence, say to some $\chi_{\tilde{A}}$. To complete the proof, one needs to show that $\tilde{A}$ is a ball, for then it must necessarily be A^*.

To see this, one can introduce $\gamma = e^{-|x|^2}$ consider the sequence $a_k = \|\gamma - \chi_{A_k}\|_{L^2}$, which converges to $a = \|\gamma - \chi_{\tilde{A}}\|_{L^2}$. Using symmetries of γ and the double reflection symmetry of $\tilde{A}$, we can deduce

$$\int \gamma R_\alpha \chi_{\tilde{A}} \leq \int \gamma X R_\alpha \chi_{\tilde{A}} \leq \int \gamma Y X R_\alpha \chi_{\tilde{A}} = \int \gamma R_\alpha \chi_{\tilde{A}},$$

which implies that $R_\alpha \chi_{\tilde{A}} = XY R_\alpha \chi_{\tilde{A}}$. This relies on Exercise 9.3.4. Using this, we find $\chi_{\tilde{A}} = R_{2\alpha} \chi_{\tilde{A}}$. To complete the argument, one relies on the fact that any θ may be approximated by irrational multiples of 2π. □

We will use the Riesz rearrangement inequality to establish the following estimate.

Lemma 9.1.6. *We have*

$$\|\nabla f^*\|_{L^2} \leq \|\nabla f\|_{L^2}.$$

This estimate is known as the Polya–Szegö inequality, and it actually holds in L^p for all $1 \leq p < \infty$.

Proof. Recall the heat kernel $e^{t\Delta}$ introduced in Section 2.6. We compute

$$\begin{aligned}
\|\nabla f\|_{L^2}^2 &= -\langle f, \Delta f\rangle \\
&= \langle f, -\tfrac{d}{dt}[e^{t\Delta} f]|_{t=0}\rangle \\
&= \lim_{t\to 0} \tfrac{1}{t}\Big\{ \|f\|_{L^2}^2 - (4\pi t)^{-\frac{d}{2}} \iint e^{-|x-y|^2/4t} f(x)\bar{f}(y)\, dx\, dy \Big\}.
\end{aligned}$$

Applying the Riesz rearrangement inequality and reversing the steps above, we find

$$\begin{aligned}
\|\nabla f\|_{L^2}^2 &\geq \lim_{t\to 0} \tfrac{1}{t}\Big\{ \|f^*\|_{L^2}^2 - (4\pi t)^{-\frac{d}{2}} \iint e^{-|x-y|^2/4t} f^*(x)\bar{f}^*(y)\, dx\, dy \Big\} \\
&= \|\nabla f^*\|_{L^2}^2.
\end{aligned}$$

This completes the proof. □

The next result we need is a compactness result due to Riesz.

Theorem 9.1.7 (Compactness in L^p)**.** *Suppose $\{f_n\} \subset L^p$ satisfy the following:*

- *Boundedness: there exists $M > 0$ such that $\|f_n\|_{L^p} \leq M$ for all n.*

- *Tightness: for any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{|x|>R} |f_n(x)|^p \, dx < \varepsilon$$

  for all n.

- *Equicontinuity: for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int |f_n(x+y) - f_n(x)|^p \, dx < \varepsilon$$

  for all n and all $|y| < \delta$.

Then f_n converges in L^p along a subsequence.

Proof. Let ϕ be a smooth bump function supported on the unit ball with $\int \phi = 1$. For $R > 0$, we define

$$f_n^R(x) = \phi(\tfrac{x}{R})[R^d \phi(R\cdot) * f_n](x)$$

for each n. For any R, we have that $\{f_n^R\}$ is a sequence of continuous functions on the compact set $\{|x| \leq R\}$. Note that for any $R > 0$, the $\{f_n^R\}$ are uniformly bounded:

$$\|f_n^R\|_{L^\infty} \leq \|\phi\|_{L^\infty} \|R^d \phi(R\cdot) * f_n\|_{L^\infty} \lesssim \|R^d \phi(R\cdot)\|_{L^{p'}} \|f_n\|_{L^p} \lesssim R^{\frac{d}{p}} M$$

for all n. We next show that $\{f_n^R\}$ are equicontinuous for fixed $R > 0$. As ϕ is smooth, it suffices to show that for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$\|R^d \phi(R\cdot) * f_n(x+y) - R^d \phi(R\cdot) * f_n(x)\|_{L^\infty} < \varepsilon$$

for $|y| < \delta$. As convolution is linear, this follows from the L^p-equicontinuity of the f_n and the estimates above.

By the Arzelá–Ascoli theorem, it follows that for any $R > 0$, every subseuqence of $\{f_n^R\}$ has a subsequence that converges in $L^\infty(\{|x| \leq R\})$ and hence in $L^p(\{|x| \leq R\})$.

Now let $\varepsilon > 0$. We claim that there exists $R > 0$ large enough that

$$\|f_n - f_n^R\|_{L^p} < \varepsilon \quad \text{for all} \quad n. \tag{9.4}$$

To see this, write

$$f_n - f_n^R = [1 - \phi(\tfrac{x}{R})] f_n + \phi(\tfrac{x}{R})[f_n - R^d \phi(R\cdot) * f_n].$$

The first term can be made small in L^p (uniformly in n) by employing tightness. For the second term, we safely ignore $\phi(\frac{x}{R})$ and argue as in the proof of approximation to the identity (Lemma A.3.2). Recalling that proof, we see that we may make this term small uniformly in n due to the fact that we have uniform boundedness and uniform equicontinuity for the functions f_n.

Finally, let us show that $\{f_n\}$ has a convergent sequence in L^p. To see this, we will show that for any $\varepsilon > 0$, there exists J such that

$$\{f_n\} \subset \cup_{j=1}^{J} B_\varepsilon(f_j). \tag{9.5}$$

Let us first see that this does the job. We apply this with the sequence $\varepsilon = 2^{-k}$ to find a sequence $f_{n,k}$ satisfying $f_{n,k+1} \in B_{2^{-k}}(f_{n,k})$. Such a subsequence is necessarily Cauchy and hence convergent in L^p.

In fact, by (9.4), it is enough to establish the total boundedness property (9.5) for any $\{f_n^R\}$. However, this follows from sequential compactness. To see this, suppose towards a contradiction that for some ε_0, we have

$$\{f_n^R\} \not\subset \cup_{j=1}^{J} B_{\varepsilon_0}(f_j^R) \quad \text{for any} \quad J.$$

Then we can inductively build a subsequence $f_{n,k}^R$ such that

$$f_{n,k+1}^R \notin B_{\varepsilon_0}(f_{n,k}^R)$$

By construction, this sequence can have no convergent subsequence, which yields a contradiction. This completes the proof. □

Using this, we can establish the following result.

Lemma 9.1.8 (Compact embedding)**.** *Let $H^1_{rad}(\mathbb{R}^3)$ denote the set of radial functions in H^1. Then $H^1_{rad}(\mathbb{R}^3)$ is compactly embedded in $L^4(\mathbb{R}^3)$.*

Remark 9.1.9. This is a special case of a more general result. In particular, $H^1_{rad}(\mathbb{R}^d)$ is compactly embedded in $L^p(\mathbb{R}^d)$ for $2 < p < \frac{2d}{d-2}$ (where the exponent $\frac{2d}{d-2}$ should be taken to be ∞ in dimensions $d = 1, 2$).

Proof. The fact that $H^1 \subset L^4$ is a consequence of Gagliardo–Nirenberg.

We need to show that any bounded sequence $\{f_n\}$ in H^1_{rad} has a subsequence that converges in L^4. We will use Theorem 9.1.7. In particular we need to check boundedness, equicontinuity, and tightness.

Boundedness in L^4 is a consequence of Gagliardo–Nirenberg.

For equicontinuity, we argue as follows. We first apply Gagliardo–Nirenberg to estimate

$$\|f_n(\cdot) - f_n(\cdot + y)\|_{L^4} \lesssim \|f_n(\cdot) - f_n(\cdot + y)\|_{L^2}^{\frac{1}{4}} \|\nabla f_n\|_{L^2}^{\frac{3}{4}},$$

so that it suffices to establish continuity in L^2. In fact, this also follows from H^1 boundedness. Indeed, using Lemma 2.5.8,

$$\begin{aligned}\|f_n(\cdot + y) - f_n(\cdot)\|_{L^2}^2 &\sim \int |\hat{f}_n(\xi)|^2 |e^{iy\cdot\xi} - 1|^2 \, d\xi \\ &\lesssim |y|^2 \int |\xi \hat{f}_n(\xi)|^2 \, d\xi \lesssim |y|^2 \|\nabla f_n\|_{L^2}^2,\end{aligned}$$

which yields equicontinuity.

Finally, we need to prove tightness. Here we rely on the radial symmetry. We write $f_n = f_n(r)$ where $r = |x|$ and denote the radial derivative by ∂_r. By the fundamental theorem of calculus, we have for any radial function f and any $r > 0$,

$$\begin{aligned} r^2 |f(r)|^2 &= \left| 2r^2 \int_r^\infty f(\rho) \partial_\rho f(\rho) \, d\rho \right| \\ &\lesssim \int_0^\infty \rho |f(\rho)| \, \rho |\partial_\rho f(\rho)| \, d\rho \\ &\lesssim \left(\int \rho^2 |f(\rho)|^2 \, d\rho \right)^{\frac{1}{2}} \left(\int \rho^2 |\partial_\rho f(\rho)|^2 \, d\rho \right)^{\frac{1}{2}} \lesssim \|f\|_{H^1}^2, \end{aligned}$$

where in the last line we compute the integral using spherical coordinates. In particular, we have

$$\begin{aligned}\int_{|x|>R} |f_n(x)|^4 \, dx \lesssim R^{-2} \int_{|x|>R} |x|^2 |f_n(x)|^4 \, dx &\lesssim R^{-2} \|f_n\|_{L^2}^2 \||x| f_n\|_{L^\infty}^2 \\ &\lesssim R^{-2} \|f_n\|_{H^1}^4,\end{aligned}$$

which implies tightness. This completes the proof. □

Remark 9.1.10. The estimate used to establish tightness is often called a radial Sobolev embedding inequality.

With the requisite compactness tools in place, we can now prove existence of optimizers for Gagliardo–Nirenberg.

Proof of Theorem 9.1.3. Let

$$J(f) = \frac{\|f\|_{L^2}\|\nabla f\|_{L^2}^3}{\|f\|_{L^4}^4}$$

for $f \in H^1(\mathbb{R}^3)\backslash\{0\}$, so that $C_{GN}^{-4} = \inf J(f)$. We take a sequence $f_n \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying

$$\lim_{n\to\infty} J(f_n) = C_{GN}^{-4}.$$

We now wish to pass to a limit; however, we need to make some modifications to restore the compactness that is lost to scaling and translation symmetries. We will first define

$$g_n(x) = a_n f_n(b_n x)$$

for suitable a_n, b_n. In particular, if we define

$$b_n = \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} \quad \text{and} \quad a_n = \frac{\|f_n\|_{L^2}^{\frac{1}{2}}}{\|\nabla f_n\|_{L^2}^{\frac{3}{2}}},$$

then

$$\|g_n\|_{L^2} = \|\nabla g_n\|_{L^2} = 1.$$

Note that g_n remains an optimizing sequence, i.e. $J(g_n) \to C_{GN}^{-4}$.

Next, we take radial rearrangements and define $h_n = g_n^*$. By Lemma 9.1.6 and the fact that rearrangements preserve L^p-norms, we have that h_n is a bounded sequence in H_{rad}^1 (in fact $\|h_n\|_{L^2} \equiv 1$ and $\|\nabla h_n\|_{L^2} \leq 1$) satisfying

$$J(h_n) \leq J(g_n).$$

In particular, h_n is also an optimizing sequence.

By boundedness in H_{rad}^1, we have that h_n converges to along a subsequence to a limit h, weakly in H^1 and strongly in L^4 (cf. compact embedding and Lemma A.2.3). Using Lemma A.2.3, we deduce

$$J(h) \leq \lim_{n\to\infty} J(h_n) = \inf J(f),$$

thus giving the existence of an optimizer, as desired. $\square$

9.2 Concentration compactness and sharp Sobolev embedding

In this section we will introduce techniques related to 'concentration compactness', specifically the notion of a profile decomposition. We will apply these techniques to prove the existence of optimizers for a Sobolev embedding inequality proved in Section 6.2. These techniques also play an important role in the setting of current research in nonlinear partial differential equations.

Recall from Section 6.2 that we have the following general inequality: there exists $C > 0$ such that for all $f \in \dot{H}^1(\mathbb{R}^3)$,

$$\|f\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}. \tag{9.6}$$

We define the **optimal constant** C_{Sob} by

$$C_{Sob} = \sup\big\{\|f\|_{L^6(\mathbb{R}^3)} \div \|\nabla f\|_{L^2(\mathbb{R}^3)} : f \in \dot{H}^1(\mathbb{R}^3)\backslash\{0\}\big\}. \tag{9.7}$$

Our goal in this section is the following theorem.

Theorem 9.2.1 (Existence of optimizers for Sobolev embedding)**.** *There exists $f \in \dot{H}^1(\mathbb{R}^3)$ so that*

$$\|f\|_{L^6(\mathbb{R}^3)} = C_{Sob}\|\nabla f\|_{L^2(\mathbb{R}^3)},$$

where C_{Sob} is as in (9.7)*.*

Remark 9.2.2. Similar to the case of Gagliardo–Nirenberg, we can use optimizers of Sobolev embedding to find solutions to the nonlinear elliptic PDE $-\Delta W = W^5$.

The basic idea is to take an optimizing sequence and to extract a suitable limit, which can then be shown to be an optimizer. The key to proving convergence is to use a 'profile decomposition', which decomposes an arbitrary bounded sequence in $\dot{H}^1$ into a sum of bubbles (with well-defined spatial positions and scales), plus a remainder that vanishes in the L^6 norm. We then show that an optimizing sequence necessarily contains only one bubble.

The starting point is a refinement of Sobolev embedding that allows us to identify a scale for concentration.

Lemma 9.2.3 (Refined Sobolev embedding)**.** *For $f \in \dot{H}^1(\mathbb{R}^3)$,*

$$\|f\|_{L^6} \lesssim \big[\sup_N \|f_N\|_{L^6}\big]^{\frac{2}{3}}\|\nabla f\|_{L^2}^{\frac{1}{3}},$$

where $f_N = P_N f$ denotes Littlewood–Paley projection.

Proof. We use the Littlewood–Paley square function estimate, Theorem 7.2.6. Then

$$\begin{aligned}\|f\|_{L^6}^6 &\sim \int \Big(\sum_N |f_N|^2\Big)^3\,dx\\ &\sim \sum_{N_1,N_2,N_3} \int |f_{N_1}|^2|f_{N_2}|^2|f_{N_3}|^2\,dx\\ &\lesssim \sum_{N_1\leq N_2\leq N_3} \int |f_{N_1}|^2|f_{N_2}|^2|f_{N_3}|^2\,dx.\end{aligned}$$

Continuing from above and applying Hölder's inequality and Bernstein's inequality (Propostiion 7.2.3), we find

$$\begin{aligned}\|f\|_{L^6}^6 &\lesssim \sum_{N_1\leq N_2\leq N_3} \|f_{N_1}\|_{L^\infty}\|f_{N_1}\|_{L^6}\|f_{N_2}\|_{L^6}^2\|f_{N_3}\|_{L^6}\|f_{N_3}\|_{L^3}\\ &\lesssim \big[\sup_N \|f_N\|_{L^6}\big]^4 \sum_{N_1\leq N_2\leq N_3} \big(\tfrac{N_1}{N_3}\big)^{\frac12}\|\nabla f_{N_1}\|_{L^2}\|\nabla f_{N_3}\|_{L^2}\\ &\lesssim \big[\sup_N \|f_N\|_{L^6}\big]^4 \sum_{N_1\leq N_3} \big(\tfrac{N_1}{N_3}\big)^{\frac12}\log\big(\tfrac{N_3}{N_1}\big)\|\nabla f_{N_1}\|_{L^2}\|\nabla f_{N_3}\|_{L^2}.\end{aligned}$$

Applying Schur's test, we deduce

$$\|f\|_{L^6}^6 \lesssim [\sup_N \|f_N\|_{L^6}^4]\sum_N \|\nabla f_N\|_{L^2}^2,$$

which implies the desired result. □

Next, we combine this result with Hölder's inequality to demonstrate how to extract a bubble of concentration from a bounded sequence in $\dot H^1$.

Proposition 9.2.4 (Inverse Sobolev)**.** *Suppose $\{f_n\}\subset \dot H^1(\mathbb{R}^3)$ is a sequence satisfying*

$$\limsup_{n\to\infty}\|\nabla f_n\|_{L^2}\leq A \quad \textit{and} \quad \liminf_{n\to\infty}\|f_n\|_{L^6}\geq \varepsilon \tag{9.8}$$

for some $A>0$ and $\varepsilon>0$. Passing to a subsequence, there exists $\phi\in\dot H^1$, $\lambda_n\in 2^{\mathbb{Z}}$, and $x_n\in\mathbb{R}^3$ such that

$$\lambda_n^{\frac12} f_n(\lambda_n x + x_n) \rightharpoonup \phi(x)$$

weakly in $\dot H^1$, with

$$\|\nabla\phi\|_{L^2} \gtrsim \varepsilon(\tfrac{\varepsilon}{A})^{\frac{5}{4}}. \tag{9.9}$$

We have the decouplings

$$\lim_{n\to\infty}\big[\|\nabla f_n\|_{L^2}^2 - \|\nabla[f_n-\phi_n]\|_{L^2}^2 - \|\nabla\phi_n\|_{L^2}^2\big] = 0$$

and

$$\lim_{n\to\infty}\big[\|f_n\|_{L^6}^6 - \|f_n-\phi_n\|_{L^6}^6 - \|\phi_n\|_{L^6}^6\big] = 0,$$

where

$$\phi_n(x) := \lambda_n^{-\frac{1}{2}}\phi(\tfrac{x-x_n}{\lambda_n}).$$

Proof. Passing to a subsequence, we may assume the bounds in (9.8) hold for each n. Then using Lemma 9.2.3, we may find a sequence N_n such that

$$\|P_{N_n}f_n\|_{L^6} \gtrsim \varepsilon(\tfrac{\varepsilon}{A})^{\frac{1}{2}}$$

for all n. By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned}\varepsilon(\tfrac{\varepsilon}{A})^{\frac{1}{2}} \lesssim \|P_{N_n}f_n\|_{L^6} &\lesssim \|P_{N_n}f_n\|_{L^2}^{\frac{1}{3}}\|P_{N_n}f_n\|_{L^\infty}^{\frac{2}{3}}\\ &\lesssim N_n^{-\frac{1}{3}}\|\nabla f_n\|_{L^2}^{\frac{1}{3}}\|P_{N_n}f_n\|_{L^\infty}^{\frac{2}{3}}\\ &\lesssim N_n^{-\frac{1}{3}}A^{\frac{1}{3}}\|P_{N_n}f_n\|_{L^\infty}^{\frac{2}{3}}.\end{aligned}$$

Thus

$$\|P_{N_n}f_n|_{L^\infty} \gtrsim \varepsilon N_n^{\frac{1}{2}}(\tfrac{\varepsilon}{A})^{\frac{5}{4}},$$

and hence there exists $x_n\in\mathbb{R}^3$ such that

$$N_n^{-\frac{1}{2}}|P_{N_n}f_n(x_n)\| \gtrsim \varepsilon(\tfrac{\varepsilon}{A})^{\frac{5}{4}}.$$

We now set $\lambda_n = N_n^{-1}$ and define

$$h_n(x) = \lambda_n^{\frac{1}{2}}f_n(\lambda_n x + x_n).$$

As h_n forms a bounded sequence in $\dot H^1(\mathbb{R}^3)$, Alaoglu's theorem implies that h_n converges weakly along a subsequence to some ϕ (cf. Lemma A.2.3). We now claim that ϕ is nonzero; in particular, we have exhibited concentration for f_n at the physical scale λ_n and spatial position x_n.

To prove the claim, we let K denote the convolution kernel of the Littlewood–Paley projection P_1. Then

$$\begin{aligned}\int K(x)h_n(x)\,dx &= \int K(x)\lambda_n^{\frac{1}{2}} f_n(\lambda_n x + x_n)\,dx \\ &= N_n^{-\frac{1}{2}} \int N_n^3 K(N_n(y-x_n)) f_n(y)\,dy = N_n^{-\frac{1}{2}} P_{N_n} f_n(x_n).\end{aligned}$$

In particular, by construction, we have

$$|\langle K, \phi\rangle| = \lim_{n\to\infty} |\langle K, h_n\rangle| \gtrsim \varepsilon(\tfrac{\varepsilon}{A})^{\frac{5}{4}},$$

which by Hölder's inequality and Sobolev embedding implies

$$\varepsilon(\tfrac{\varepsilon}{A})^{\frac{5}{4}} \lesssim \|\phi\|_{L^6} \lesssim \|\nabla\phi\|_{L^2}.$$

The $\dot{H}^1$ decoupling follows from weak convergence and the fact that

$$h_n \rightharpoonup \phi \implies \|\nabla h_n\|_{L^2}^2 - \|\nabla[\phi - h_n]\|_{L^2}^2 \to \|\nabla\phi\|_{L^2}^2$$

(check!).

For the L^6 decoupling, we will need the following refined version of Fatou's lemma due to Brezis and Lieb [1]: if a_n is a sequence of L^p functions with $\limsup_{n\to\infty} \|a_n\|_{L^p} < \infty$ and $a_n \to a$ almost everywhere, then

$$\|a_n\|_{L^p}^p - \|a_n - a\|_{L^p}^p \to \|a\|_{L^p}^p. \tag{9.10}$$

We prove this below. Assuming (9.10) for the moment (see below), we now claim that

$$\lambda_n^{\frac{1}{2}} f_n(\lambda_n x + x_n) \to \phi$$

almost everywhere along a subsequence. To see this, we first observe that $H^1(K) \hookrightarrow L^2(K)$ is a compact embedding for any compact $K \subset \mathbb{R}^3$ (cf. Theorem 9.1.7 and the proof of Lemma 9.1.8). Thus, the weak convergence implies strong L^2 convergence along a subsequence for any compact $K \subset \mathbb{R}^3$. Using a diagonal argument, we can then deduce almost everywhere convergence along a subsequence. (See Exercise 9.3.7.)

Thus, by (9.10), we have

$$\|\lambda_n^{\frac{1}{2}} f_n(\lambda_n x + x_n)\|_{L^6}^6 - \|\lambda_n^{\frac{1}{2}} f_n(\lambda_n x + x_n) - \phi\|_{L^6} \to \|\phi\|_{L^6}^6,$$

which after a change of variables yields the desired L^6 decoupling. □

It remains to prove the refined Fatou lemma due to Brézis and Lieb [1].

Proof of (9.10). Let $\varepsilon > 0$ be arbitrary and define

$$W_{\varepsilon,n} = \big[\big|\,|a_n|^p - |a|^p - |a_n - a|^p\big| - \varepsilon|a_n - a|^p\big]_+,$$

where $+$ denotes the positive part. By assumption, $W_{\varepsilon,n} \to 0$ as $n \to \infty$. On the other hand,

$$\begin{aligned}\big||a_n|^p - |a|^p - |a_n - a|^p\big| &\le \big||a_n|^p - |a_n - a|^p\big| + |a|^p \\ &\le \varepsilon|a_n - a|^p + C_\varepsilon|a|^p + |a|^p,\end{aligned}$$

where we have used the fact that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\big||x+y|^p - |x|^p\big| \le \varepsilon|x|^p + C_\varepsilon|y|^p.$$

To see that this holds, consider the cases $|y| \ll |x|$ and $|y| \gtrsim |x|$ separately; the first case leads to the first term on the right-hand side, while the second case leads to the second term.

Continuing from above, we deduce

$$0 \le W_{\varepsilon,n} \le (1 + C_\varepsilon)|a|^p \in L^1.$$

Thus the dominated convergence theorem implies that $\int W_{\varepsilon,n} \to 0$ as $n \to \infty$. Now we observe that

$$\big||a_n|^p - |a|^p - |a_n - a|^p\big| \le W_{\varepsilon,n} + \varepsilon|a_n - a|^p,$$

and hence we deduce

$$\limsup_{n\to\infty} \int \big||a_n|^p - |a|^p - |a_n - a|^p\big|\,dx \lesssim \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this completes the proof. □

We can now turn to the main technical tool needed for the proof of Theorem 9.2.1, namely, a profile decomposition adapted to the Sobolev embedding inequality.

Proposition 9.2.5 (Profile decomposition)**.** *Let $\{f_n\}$ be a bounded sequence in $\dot H^1(\mathbb{R}^3)$. There exist $J^* \in \{0,1,2,\dots\} \cup \{\infty\}$, profiles $\{\phi^j\}_{j=1}^{J^*}$, positions $\{x_n^j\}_{j=1}^{J^*}$, and scales $\{\lambda_n^j\}_{j=1}^{J^*}$ such that along a subsequence in n we have*

$$f_n(x) = \sum_{j=1}^{J} [\lambda_n^j]^{-\frac12}\phi^j\big(\tfrac{x - x_n^j}{\lambda_n^j}\big) + r_n^J \quad \textit{for} \quad 0 \le J \le J^*.$$

Furthermore, the following properties hold:

- *We have $\dot{H}^1$ decoupling:*

$$\sup_J \limsup_{n\to\infty} \left| \|\nabla f_n\|_{L^2}^2 - \sum_{j=1}^J \|\nabla \phi^j\|_{L^2}^2 - \|\nabla r_n^J\|_{L^2}^2 \right| = 0.$$

- *The remainder vanishes in L^6:*

$$\limsup_{J\to J^*} \limsup_{n\to\infty} \|r_n^J\|_{L^6}^6 = 0,$$

and we have the L^6 decoupling

$$\limsup_{J\to J^*} \limsup_{n\to\infty} \left| \|f_n\|_{L^6}^6 - \sum_{j=1}^J \|\phi^j\|_{L^6}^6 \right| = 0.$$

Proof. We define $r_n^0 \equiv f_n$. If $f_n \to 0$ in L^6, then we stop. Otherwise, we apply inverse Strichartz to identify the first profile ϕ^1 (and the parameters (x_n^1, λ_n^1)) and define

$$r_n^1 = r_n^0 - [\lambda_n^1]^{-\frac{1}{2}} \phi^1\left(\tfrac{x - x_n^1}{\lambda_n^1}\right).$$

If $r_n^1 \to 0$ in L^6, then we stop. Otherwise, we again apply inverse Strichartz to identify the next profile and parameters, defining r_n^2 analogously to above. Proceeding in this way, we construct the profiles, parameters, and remainder terms. We may need to apply this (countably) many times, passing to a subsequence in n each time. This determines whether J^* is finite or infinite.

We now need to verify the properties claimed in the proposition. The $\dot{H}^1$ decoupling follows by induction and by construction.

Let us verify the vanishing of the remainder in L^6. We define

$$\varepsilon_J = \lim_{n\to\infty} \|r_n^J\|_{L^6} \quad \text{and} \quad A_J = \lim_{n\to\infty} \|\nabla r_n^J\|_{L^2}.$$

By construction $A_J \le A_0$. Thus, by (9.9) and construction, we have

$$\|\nabla \phi^j\|_{L^2}^2 \gtrsim \varepsilon_{j-1}^2 \left(\tfrac{\varepsilon_{j-1}}{A_j}\right)^{\frac{5}{2}} \gtrsim \varepsilon_{j-1}^2 \left(\tfrac{\varepsilon_{j-1}}{A_0}\right)^{\frac{5}{2}}.$$

Hence, by $\dot{H}^1$ decoupling,

$$\sum_{j=1}^J \varepsilon_{j-1}^2 \left(\tfrac{\varepsilon_{j-1}}{A_0}\right)^2 \lesssim \sum_{j=1}^J \|\nabla \phi^j\|_{L^2}^2 \lesssim A_0^2,$$

which implies $\varepsilon_j \to 0$ as $j \to \infty$. Finally, the L^6 decoupling follows from induction and the vanishing of the remainder in L^6. □

With the profile decomposition in place, we turn to the proof of Theorem 9.2.1.

Proof of Theorem 9.2.1. We let $\{f_n\}$ be a sequence in $\dot{H}^1\backslash\{0\}$ such that

$$\lim_{n\to\infty} J(f_n) = C_{Sob}^6,$$

where

$$J(f) := \|f\|_{L^6}^6 \div \|\nabla f\|_{L^2}^6$$

Let us first normalize the sequence by replacing

$$f_n \quad \text{with} \quad \tfrac{f_n}{\|\nabla f_n\|_{L^2}},$$

so that $\|\nabla f_n\|_{L^2} \equiv 1$. We now apply the profile decomposition to f_n to write

$$f_n = \sum_{j=1}^{J} [\lambda_n^j]^{-\frac{1}{2}} \phi^j(\tfrac{x-x_n^j}{\lambda_n^j}) + r_n^J$$

along a suitable subsequence. Now observe by the L^6 decoupling and Sobolev embedding,

$$C_{Sob}^6 = \lim_{n\to\infty} J(f_n) = \sum_{j=1}^{J^*} \|\phi^j\|_{L^6}^6 \leq C_{Sob}^6 \sum_{j=1}^{J^*} \|\nabla\phi^j\|_{L^2}^6.$$

On the other hand, by the $\dot{H}^1$ decoupling,

$$\sum_{j=1}^{J^*} \|\nabla\phi^j\|_{L^2}^2 \leq 1.$$

Using the nesting $\ell^2 \subset \ell^6$, this implies that all of the inequalities above are equalities. In particular,

$$\sum_{j=1}^{J^*} \|\nabla\phi^j\|_{L^2}^6 = \sum_{j=1}^{J*} \|\nabla\phi^j\|_{L^2}^2 = 1.$$

Since the ϕ^j are all non-trivial, this implies that there must be only one profile, say ϕ, which satisfies $\|\nabla\phi\|_{L^2} = 1$. We therefore have

$$f_n = \lambda_n^{-\frac{1}{2}} \phi(\tfrac{x-x_n}{\lambda_n}) + r_n,$$

with $\lambda_n^{\frac{1}{2}} f_n(\lambda_n x + x_n) \rightharpoonup \phi$ and $\|\nabla f_n\|_{L^2} = \|\nabla\phi\|_{L^2} = 1$. In particular, $\lambda_n^{\frac{1}{2}} f_n(\lambda_n x + x_n)$ converges strongly to ϕ in H^1 and hence in L^6. In particular, $J(\phi) = C_{Sob}^6$, i.e. ϕ optimizes Sobolev embedding. $\square$

9.3 Exercises

Exercise 9.3.1. Let $f \in L^2(\mathbb{R}^d)$. Show that $g_n := f(x+x_n)$ converges weakly to zero if $|x_n| \to \infty$. Show that $h_n = f(\lambda_n x)$ converges weakly to zero if $\lambda_n \to 0$ or $\lambda_n \to \infty$.

Exercise 9.3.2. Prove that the embedding $\dot{H}^1_{rad}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is not compact. Here the subscript 'rad' denotes the restriction to radial functions, i.e. f satisfying $f(x) = f(|x|)$.

Exercise 9.3.3. Investigate the allowed range of exponents in (9.2).

Exercise 9.3.4. Let f, g be nonnegative functions such that $|\{f > \lambda\}|$ and $|\{g > \lambda\}|$ are finite for all $\lambda > 0$. Then

$$\int_{\mathbb{R}^d} f(x)g(x)\,dx \leq \int_{\mathbb{R}^d} f^*(x)g^*(x)\,dx.$$

If f is strictly radially decreasing, then equality holds if and only if $g = g^*$ a.e.

Exercise 9.3.5. Show that Steiner symmetrization preserves measurability of sets and functions.

Exercise 9.3.6. Prove the existence of optimizers for Gagliardo–Nirenberg by developing an appropriate profile decomposition.

Exercise 9.3.7. Let $K \subset \mathbb{R}^d$ be a compact set. Show that $H^1(K) \hookrightarrow L^2(K)$ is a compact embedding. As a result, show that if $g_n \rightharpoonup g$ weakly in $H^1(K)$, then $g_n \to g$ strongly in $L^2(K)$ along a subsequence.

Chapter 10

Modern harmonic analysis, part III

Our goal in this chapter is to provide an introduction to restriction theory and some related topics, including 'Strichartz estimates' in the setting of Schrödinger equations.

10.1 Restriction theory and Strichartz estimates

In this section, we give a brief introduction to restriction theory, focusing on an early result due to Strichartz [30].

Restriction theory concerns the basic question of when it makes sense to restrict a function's Fourier transform. In particular, given $S \subset \mathbb{R}^n$ (with $n \geq 2$) and a positive measure $d\mu$ supported on S, we can consider the following two problems:

A. For which $p \in [1, 2)$ do we have the following restriction estimate:

$$\|\hat{f}\|_{L^2(d\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad ?$$

B. For which $q \in (2, \infty]$ do we have the estimate

$$\|(F\,d\mu)\hat{}\|_{L^q(\mathbb{R}^n)} \lesssim \|F\|_{L^2(d\mu)} \quad ?$$

By duality, these problems are equivalent provided $q = p'$ (i.e. $\frac{1}{p}+\frac{1}{q} = 1$).

Note that if $f \in L^1$, then the Riemann–Lebesgue lemma tells us that $\hat{f}$ is continuous with $\hat{f} \to 0$ at infinity. In particular, it makes sense to restrict $\hat{f}$ to sets of measure zero. On the other hand, if f is merely in L^2 then $\hat{f} \in L^2$ and it does not necessarily make sense to restrict $\hat{f}$ in this way.

Example 10.1.1. There exists a function belonging to L^p for all $p > 1$ whose Fourier transform is infinite on an entire hyperplane. Let $\psi : \mathbb{R}^{d-1} \to \mathbb{C}$ be a bump function and set

$$f(x) = (1 + |x_1|)^{-1}\psi(x_2, \dots, x_d).$$

Then $f \in L^p$ for all $p > 1$. However, if we now set $S = \{\xi \in \mathbb{R}^d : \xi_1 = 0\}$, then we observe that

$$\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}}\hat{\psi}(\xi_2, \dots, \xi_d) \int \tfrac{e^{-ix_1\xi_1}}{1+|x_1|}\, dx_1 \equiv \infty$$

for $\xi \in S$.

The issue in the previous example is that the hyperplane S has no curvature.

Following [30], we will first focus on the case that S is a quadratic surface; we return to some more additional cases in Section 10.3. For simplicity, we will restrict our attention to the case of the paraboloid

$$S = \{x : x_n = -[x_1^2 + \cdots + x_{n-1}^2]\},$$

although more general quadratic surfaces may be treated by similar methods (see [30]). In our case, the relevant measure $d\mu$ is simple to describe. In particular, for a function F on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} F(x)\, d\mu(x) = \int_{\mathbb{R}^{n-1}} F(y, -|y|^2)\, dy. \tag{10.1}$$

We begin by observing that by Plancherel and Hölder, we may write

$$\|\hat{f}\|_{L^2(d\mu)}^2 = \langle \hat{f}, \hat{f} d\mu\rangle = \langle f, \check{d\mu} * f\rangle \lesssim \|f\|_{L^p} \|\check{d\mu} * f\|_{L^{p'}}.$$

Thus, the restriction estimate in **A.** would hold for a choice of $p \in [1, 2)$ provided we could prove the convolution estimate

$$\|\check{d\mu} * f\|_{L^{p'}} \lesssim \|f\|_{L^p} \tag{10.2}$$

for the same choice of p. We will look for p such that (10.2) holds. We will utilize the Stein interpolation theorem, Theorem 6.1.11.

In what follows, we will use the notation

$$x = (y, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

We introduce the the following analytic family of distributions:

$$K_z(x) = \gamma(z)\big[x_n - |y|^2\big]_+^z, \tag{10.3}$$

where $+$ denotes the positive part. Here $z \in \mathbb{C}$ and $\gamma(z)$ is an analytic function to be determined below with a simple zero at $z = -1$. To connect these distributions to the problem above, let us prove the following:

Lemma 10.1.1. *Define K_z as above and suppose γ has a simple zero at $z = -1$. Then there exists a constant c such that*

$$\lim_{z \to -1} K_z \to c\, d\mu$$

in the sense of distributions.

Proof. Let F be a test function on $\mathbb{R}^n$. Then by a change of variables we have

$$\begin{aligned} \int K_z(x)F(x)\,dx &= \gamma(z)\int\left[\int F(y, x_n + |y|^2)\big[x_n\big]_+^z\,dx_n\right]dy \\ &= \gamma(z)\int \big[F(y,\cdot) * [\,\cdot\,]_+^z\big](-|y|^2)\,dy. \end{aligned}$$

Recalling (10.1), the problem therefore reduces to establishing

$$\gamma(z)[\,\cdot\,]_+^z \to c\delta_0 \quad \text{as} \quad z \to -1, \tag{10.4}$$

where δ_0 is the Dirac delta distribution. Indeed, with (10.4) in place we get

$$\int K_z(x)F(x)\,dx \to c\int F(-|y|^2, y)\,dy = c\int F(x)\,d\mu(x),$$

as desired.

As γ is assumed to have a simple zero at $z = -1$, to prove (10.4) it is enough to establish

$$(z+1)[\,\cdot\,]_+^z \to \delta_0 \quad \text{as} \quad z \to -1,$$

which we now prove. We let H denote the Heaviside distribution

$$H(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0, \end{cases}$$

which satisfies $\partial_\tau H = \delta_0$. We now observe that

$$(z+1)\tau_+^z = \partial_\tau\big[\tau_+\big]^{z+1}.$$

Thus, for any test function φ, we get

$$\langle (z+1)\tau_+^z, \varphi\rangle = -\langle \tau_+^{z+1}, \varphi'\rangle = -\langle \tau_+^{z+1}H, \varphi'\rangle \to -\langle H, \varphi'\rangle = \langle \delta_0, \varphi\rangle$$

as $z \to -1$. This completes the proof. $\square$

In light of the lemma above and (10.2), we have now reduced the restriction estimate **A.** to establishing the $L^p \to L^{p'}$ boundedness for the operator

$$f \mapsto T_{-1}f := \check{K_{-1}} * f,$$

with K_z as in (10.3). Writing $T_z f = K_z * f$, applying Plancherel and Hölder yields the estimates

$$\begin{aligned} \|T_z f\|_{L^2} &\le \|K_z\|_{L^\infty}\|f\|_{L^2}, \quad \text{i.e.} \quad \|T_z\|_{L^2\to L^2} \le \|K_z\|_{L^\infty} \\ \|T_z f\|_{L^\infty} &\le \|\check{K}_z\|_{L^\infty}\|f\|_{L^1} \quad \text{i.e.} \quad \|T_z\|_{L^1\to L^\infty} \le \|\check{K}_z\|_{L^\infty}. \end{aligned}$$

Then, to apply Stein's interpolation result (Theorem 6.1.11) we will look for $x_0 > 1$ and an analytic function γ with a simple zero at $z = 1$ satisfying the following on the strip $-x_0 \le \operatorname{Re} z \le 0$:

(i) $|\gamma(x+iy)|$ has at most exponential growth as $|y| \to \infty$,

(ii) $K_z(x)$ is bounded when $\operatorname{Re} z = 0$.

(iii) $\check{K}_z(x)$ is bounded when $\operatorname{Re} z = -x_0$.

Then, by Theorem 6.1.11, we can interpolate between the estimates at $\operatorname{Re} z = 0$ and $\operatorname{Re} z = -x_0$ to deduce

$$\|T_{-1}\|_{L^p\to L^{p'}} \lesssim 1, \quad \text{where} \quad p = \tfrac{2x_0}{x_0+1}.$$

Lemma 10.1.2. *Let Γ denote the Gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt.$$

Define

$$\gamma(z) = \Gamma(z+1)^{-1}$$

and let K_z be as in (10.3)*. Then γ has a simple zero at $z = -1$ and satisfies items (i) and (ii) above. Furthermore, writing $x = (y, x_n)$ as above, we have*

$$\check{K}_z(x) = c\,\exp\{-\tfrac{i|y|^2}{4x_n}\} i^z x_n^{-[z+\frac{n+1}{2}]} \tag{10.5}$$

for some constant c. In particular, item (iii) holds provided

$$\operatorname{Re} z = -x_0 := -\tfrac{n+1}{2}.$$

This is the key lemma. With Lemma 10.1.2 in place, we conclude:

Theorem 10.1.3 (Restriction estimate for the paraboloid)**.** *The restriction estimates **A.** and **B.** on the paraboloid $S \subset \mathbb{R}^n$ are valid for*

$$p = \tfrac{2(n+1)}{n+3} \quad \textit{and} \quad q = p' = \tfrac{2(n+1)}{n-1},$$

respectively.

We turn to the proof of Lemma 10.1.2.

Proof of Lemma 10.1.2. For basic properties of the Gamma function, we refer the reader to [29] (say). In particular, since Γ is nowhere zero, we get that $\gamma(z)$ is analytic. As Γ has a simple pole at $z = 0$, γ has a simple zero at $z = -1$. One can also check (i) and (ii); for example, for (ii) we observe that for $z = i\sigma$,

$$|K_{i\sigma}(x)| = |\gamma(i\sigma)| \cdot \big||x_n - |y|^2|^{i\sigma}\big| \lesssim |\gamma(i\sigma)|.$$

One can then check (using Stirling's formula, say) that $|\Gamma(1 + i\sigma)| \to \infty$ as $|\sigma| \to \infty$, which yields boundedness of $|\gamma(i\sigma)|$.

Thus, the proof boils down to establishing (10.5). This computation will also explain the origin of the mysterious choice of γ.

We begin with a change of variables and contour integration. We write $x = (y, x_n)$ and the dual variable as $\xi = (\eta, \xi_n)$. Then

$$\begin{aligned} \int e^{i[y\eta + x_n\xi_n]}\big[x_n - |y|^2\big]_+^z \, dx &= \int e^{i[y\eta+\xi_n|y|^2]}\left[\int_0^\infty e^{ix_n\xi_n} x_n^z \, dx_n\right] dy \\ &= i^{z+1}\xi_n^{-(z+1)}\Gamma(z+1)\int e^{i[y\eta+\xi_n|y|^2]}\, dy. \end{aligned}$$

Next, a Gaussian integral computation (i.e. completing the square) implies

$$\int_{\mathbb{R}^{n-1}} e^{i[y\eta+\xi_n|y|^2]}\, dy = c\,\xi_n^{-\frac{n-1}{2}} \exp\big\{-i\tfrac{|\eta|^2}{4\xi_n}\big\}. \tag{10.6}$$

In particular, continuing from above leads to

$$\int e^{i[y\eta+x_n\xi_n]}\big[x_n - |y|^2\big]_+^z \, dx = c\Gamma(z+1)i^z\xi_n^{-[z+\frac{n+1}{2}]} \exp\big\{-\tfrac{i|\eta|^2}{4\xi_n}\big\}.$$

Multiplying both sides by $\gamma(z) = [\Gamma(z+1)]^{-1}$ yields (10.5), as desired. $\square$

Restriction theory remains an active area of research within harmonic analysis. We have truly just scratched the surface by demonstrating a single result. Instead of delving deeper into restriction theory at this point, we will now return to one of the themes appearing frequently in these notes, namely, applications of harmonic analysis to partial differential equations.

We will first show that the restriction estimate for the paraboloid is equivalent for a space-time decay estimate for solutions to the Schrödinger equation. In fact, today these estimates go by the name of Strichartz estimates, in honor of Strichartz and his seminal work [30].

Recall that in Example 7.3.1, we used the Fourier transform to solve the initial-value problem for the linear Schrödinger equation on $\mathbb{R}_t \times \mathbb{R}^d_y$. In particular, the solution to

$$\begin{cases} (i\partial_t + \Delta_y)u = 0 \\ u|_{t=0} = \phi \in L^2(\mathbb{R}^d), \end{cases} \tag{10.7}$$

is given by

$$u(y,t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(y\eta - t|\eta|^2)} \tilde{\phi}(\eta)\, d\eta, \tag{10.8}$$

where we temporarily use $\tilde{\phi}$ to denote the Fourier transform on $\mathbb{R}^d$.

Proposition 10.1.4 (Strichartz estimate)**.** *Let $u : \mathbb{R}^{d+1} \to \mathbb{C}$ be the solution to* (10.7)*. Then*

$$\|u\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{d+1})} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}.$$

Proof. Let $n = d+1$. As above, we set

$$q = \tfrac{2(d+2)}{d} = \tfrac{2(n+1)}{n-1} \quad \text{and} \quad p = q'.$$

We estimate the $L^q(\mathbb{R}^n)$ norm of u by duality. Employing (10.8) and denoting elements of $\mathbb{R}^n$ by $(y,t) \in \mathbb{R}^{d+1}$, we are led to estimate

$$\iint e^{i(y\eta - t|\eta|^2)} \tilde{\phi}(\eta) f(y,t)\, d\eta\, dy\, dt = \int \tilde{\phi}(\eta) \hat{f}(\eta, -|\eta|^2)\, d\eta$$

for $f \in L^p(\mathbb{R}^n)$, where $\hat{f}$ denotes the Fourier transform on $\mathbb{R}^n$. Thus, by Plancherel, (10.1), and Theorem 10.1.3,

$$\begin{aligned} |\langle u, f\rangle| &\lesssim \|\phi\|_{L^2} \|\hat{f}(\eta, -|\eta|^2)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\phi\|_{L^2(\mathbb{R}^d)} \|\hat{f}\|_{L^2(d\mu)} \\ &\lesssim \|\phi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

The result follows. $\square$

10.2 Strichartz estimates

In the previous section we established a space-time estimate for solutions to the linear Schrödinger equation by means of a restriction estimate. In this section we will extend the range of estimates by different methods.

Let us shift notation slightly and denote the solution $u = u(t,x)$ to

$$(i\partial_t + \Delta)u = 0, \quad u(0) = \phi$$

by

$$u(t) = e^{it\Delta}\phi.$$

Here $e^{it\Delta}$ is the Fourier multiplier operator with symbol $e^{-it|\xi|^2}$, i.e.

$$e^{it\Delta} := \mathcal{F}^{-1}e^{-it|\xi|^2}\mathcal{F}.$$

Our goal is to prove space-time estimates for $e^{it\Delta}\phi$. We begin by establishing estimates for fixed time t. By Plancherel, the representation above immediately yields the L^2 bound

$$\|e^{it\Delta}\phi\|_{L^2(\mathbb{R}^d)} \equiv \|\phi\|_{L^2(\mathbb{R}^d)}.$$

On the other hand, using Lemma 2.5.11, we can also write

$$e^{it\Delta}\phi = \mathcal{F}^{-1}[e^{-it|\xi|^2}\mathcal{F}\phi] = (2\pi)^{\frac{d}{2}}\mathcal{F}^{-1}[e^{-it|\xi|^2}] * \phi.$$

However, we have essentially already computed $\mathcal{F}^{-1}[e^{-it|\xi|^2}]$ in (10.6). In particular, we find

$$e^{it\Delta}\phi(x) = (4\pi it)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i|x-y|^2/4t}\phi(y)\,dy,$$

which readily implies the 'dispersive estimate'

$$\|e^{it\Delta}\phi\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}}\|\phi\|_{L^1(\mathbb{R}^d)}.$$

Then, by interpolation we deduce

$$\|e^{it\Delta}\phi\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{-(\frac{d}{2}-\frac{d}{p})}\|\phi\|_{L^{p'}(\mathbb{R}^d)} \quad \text{for} \quad 2 \le p \le \infty, \tag{10.9}$$

where as usual p' denotes the Hölder dual to p, i.e. the solution to $\frac{1}{p}+\frac{1}{p'} = 1$. This will be an essential ingredient in proving the following Strichartz estimates.

Proposition 10.2.1 (Strichartz estimates)**.** *Let $d \geq 1$ and suppose $2 < q, r \leq \infty$ satisfy the scaling condition*

$$\tfrac{2}{q} + \tfrac{d}{r} = \tfrac{d}{2}. \tag{10.10}$$

Then

$$\|e^{it\Delta}\phi\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}. \tag{10.11}$$

Proof. We seek to prove

$$T := e^{it\Delta} \quad \text{maps} \quad L^2 \to L_t^q L_x^r.$$

We will employ the method of TT^*. In particular, we need to compute the adjoint T^* and prove that

$$TT^* : L_t^{q'} L_x^{r'} \to L_t^q L_x^r,$$

where q' and r' denote the dual exponents to q and r.

To compute the adjoint, we rely on the fact that $e^{it\Delta}$ is unitary on L^2 for each t. Thus,

$$\begin{aligned}\langle Tf, G\rangle &= \iint e^{it\Delta} f(x)\overline{G(t,x)}\,dx\,dt \\ &= \int f(x) \int \overline{e^{-it\Delta}G(t,x)}\,dt\,dx,\end{aligned}$$

which shows

$$T^*G(x) = \int_{\mathbb{R}} e^{-is\Delta} G(s,x)\,ds.$$

In particular,

$$TT^*F(t,x) = \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s,x)\,ds. \tag{10.12}$$

We now use the dispersive estimate (10.9), the Hardy–Littlewood–Sobolev inequality (Theorem 6.2.2), and the scaling relation (10.10) to estimate

$$\begin{aligned}\|TT^*F\|_{L_t^q L_x^r} &\lesssim \Big\| \int |t-s|^{-(\frac{d}{2}-\frac{d}{r})} \|F(s)\|_{L_x^{r'}}\,ds \Big\|_{L_t^q} \\ &\lesssim \Big\| |t|^{-\frac{2}{q}} * \|F(t)\|_{L_x^{r'}} \Big\|_{L_t^q} \\ &\lesssim \||t|^{-\frac{2}{q}}\|_{L_t^{\frac{q}{2},\infty}} \|F\|_{L_t^{q'} L_x^{r'}} \\ &\lesssim \|F\|_{L_t^{q'} L_x^{r'}},\end{aligned}$$

yielding the desired bounds for TT^*. Note that the application of Hardy–Littlewood–Sobolev requires $q > 2$.

By the method of TT^* (see e.g. Exercise 3.5.6), we deduce the desired $L^2 \to L^q_t L^r_x$ boundedness of $e^{it\Delta}$. $\square$

Remark 10.2.2. Note that this result covers the special case $q = r = \frac{2(d+2)}{d}$ covered in the previous section. The endpoint case $q = 2$ (which is compatible with (10.10) only for $d \geq 2$) is also allowed provided $d \geq 3$; it is valid in $d = 2$ in the radial setting. However, proving endpoint Strichartz estimates is a much more challenging problem (see [16]).

Apart from the missing endpoint $q = 2$, one may ask about the optimality of the condition (10.10); i.e. are there other exponent pairs for which we may expect an estimate of the form (10.11)? If we insist on putting the function ϕ in L^2, the answer is no. One can check that the scaling relation is necessary by considering $\phi^\lambda(x) := \phi(\lambda x)$. Then we firstly have

$$\|\phi^\lambda\|_{L^2(\mathbb{R}^d)} = \lambda^{-\frac{d}{2}} \|\phi\|_{L^2}.$$

On the other hand, the solution to the linear Schrödinger equation with data ϕ^λ is

$$e^{it\Delta}[\phi^\lambda(\cdot)](x) = [e^{it\lambda^2\Delta}\phi](\lambda x),$$

and so

$$\|e^{it\Delta}[\phi^\lambda(\cdot)]\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} = \lambda^{-\frac{2}{q}-\frac{d}{r}} \|e^{it\Delta}\phi\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)}.$$

In particular, the estimate is only possible if the powers of λ match, which is exactly (10.10).

One may also ask whether (10.11) could hold with (q, r) satisfying (10.10) but $1 \leq q < 2$. The answer is also no. To see this, let us give the following heuristic argument. Fix a nice function ϕ and consider the linear solution $u(t) = e^{it\Delta}\phi$. Supposing the Strichartz estimate holds in $L^q_t L^r_x$, we can find a sufficiently long time interval around the origin (of length T, say) so that 'most' of the norm is contained this interval. We now consider time-translates of u by $\{t_j\}_{j=1}^N$, which are separated by a length $\gg T$. Then, using time-translation symmetry of the equation, we should have

$$\Big\| \sum_{j=1}^N u(t - t_j) \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \gtrsim N^{\frac{1}{q}}$$

(by splitting into the disjoint time intervals where each $u(\cdot - t_j)$ is nontrivial). On the other hand, by linearity, we recognize $\sum u(t - t_j)$ as the solution to the Schrödinger equation with data $\sum e^{-it_j\Delta}\phi$. Thus, applying Strichartz, the fact that $e^{-it_j\Delta}\phi$ and $e^{-it_k\Delta}$ are almost orthogonal (provided $|t_j - t_k|$ is large enough), and Cauchy–Schwarz, we deduce

$$\Big\|\sum_{j=1}^{N} u(t-t_j)\Big\|_{L_t^q L_x^r} \lesssim \Big\|\sum_{j=1}^{N} e^{-it_j\Delta}\phi\Big\|_{L^2} \lesssim N^{\frac{1}{2}}.$$

In particular, if $1 \leq q < 2$, then we can reach a contradiction by choosing N large enough.

One can also prove that the endpoint estimate $(d,q,r) = (2,2,\infty)$ fails in general, but is recovered in the case of spherically-symmetric solutions (see the papers [20, 23]).

Strichartz estimates play an important role in the settling of nonlinear Schrödinger equations. For example, Strichartz estimates are the essential ingredient in the well-posedness theory for equations of the form

$$(i\partial_t + \Delta)u = F(u), \tag{10.13}$$

where F is a nonlinear function of u; common examples include power-type or Hartree nonlinearities, i.e.

$$F(u) = \lambda|u|^p u \quad \text{or} \quad F(u) = \lambda(|x|^{-\gamma} * |u|^2)u.$$

In particular, to connect Strichartz estimates to the well-posedness theory for (10.13), observe that a variation of parameters argument (i.e. looking for a solution in the form $u(t) = e^{it\Delta}v(t)$) leads to an equivalent integral formulation, known as the **Duhamel formula**:

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(u(s))\,ds,$$

where u_0 denotes the initial condition $u|_{t=0}$. In particular, solving the PDE is equivalent to finding a fixed point of the operator

$$\Phi u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(u(s))\,ds.$$

The usual strategy is to utilize the Banach fixed point theorem, which requires proving that Φ is a contraction on a suitable complete metric space. In particular, this requires mapping properties (i.e. estimates) for the operators appearing above.

We may apply Strichartz estimates directly to $e^{it\Delta}$. The remaining operator

$$F(t,x) \mapsto \int_0^t e^{i(t-s)\Delta} F(s,x)\, ds$$

is similar to the operator TT^* appearing in (10.12) (for which we proved estimates); however, it is not identical due to the truncation of the integral.

In the remainder of this section, we will discuss a result due to Christ and Kiselev [5] that allows us to deduce bounds for the truncated operator appearing in the Duhamel formula. The general result is the following:

Theorem 10.2.3 (Christ–Kiselev lemma, [5])**.** *Let X and Y be Banach spaces, and let*

$$T : L^p(\mathbb{R}; X) \to L^q(\mathbb{R}; Y), \quad 1 \le p < q < \infty,$$

be given by an integral transform

$$Tf(t) = \int_{\mathbb{R}} K(t,s) f(s)\, ds, \quad K : \mathbb{R} \times \mathbb{R} \to \mathcal{L}(X, Y).$$

Defining

$$\tilde{T} g(t) = \int_{-\infty}^t K(t,s) g(s)\, ds,$$

we have that $\tilde{T}$ is bounded from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; Y)$.

Remark 10.2.4. In the generality stated above, the assumption $p < q$ is necessary (e.g. one can truncate the Hilbert transform to find a counterexample with $p = q = 2$). We may allow $p = 1$ or $q = \infty$, in which cases $p = q$ is allowed; we leave the investigation of these details and extensions to the reader.

Remark 10.2.5. To apply this in the setting of Strichartz estimates for the Schrödinger equation, we use

$$Tf(t) = \int_{\mathbb{R}} e^{i(t-s)\Delta} f(s)\, ds, \quad p = q', \quad X = L^{r'}, \quad Y = L^r.$$

Thus we have $p < 2 < q$, and $e^{i(t-s)\Delta} \in \mathcal{L}(X, Y)$ by the dispersive estimate.

Sketch of the proof of Theorem 10.2.3. Fixing f satisfying

$$\|f\|_{L^p(\mathbb{R};X)}^p = 1,$$

we aim to prove

$$\|\tilde{T}f\|_{L^q(\mathbb{R};Y)} \lesssim 1.$$

We will use a 'Whitney decomposition' of the region $S = \{x < y\} \subset \mathbb{R}^2$ in order to decompose $K(t,s)f(s)$. In particular, we need the following:

Whitney decomposition (see [27]). We may decompose S into a countable disjoint union of dyadic squares $\{Q\}_{Q\in D}$, i.e. squares of the form

$$Q = I_{jk} \times I_{jl}, \quad I_{jk} = (k2^{-j}, (k+1)2^{-j}], \quad j,k \in \mathbb{Z},$$

such that $l(Q) \sim d(Q, \partial S)$, where $l(Q)$ denotes side-length of Q.

To prove this, one takes Q_x to be the largest dyadic square containing x with $l(Q) < d(Q, \partial S)$. Then verify that this leads to the desired decomposition.

Now, with the nondecreasing function $F : \mathbb{R} \to [0,1]$ defined by

$$F(t) = \int_{-\infty}^{t} \|f(s)\|_X^p \, ds,$$

we claim that for any $t \in \mathbb{R}$, we may decompose

$$K(t,s)f(s) = \sum_{Q\in D} \chi_{\pi_2 Q}(F(t))K(t,s)[\chi_{\pi_1 Q}(F(s))f(s)] \tag{10.14}$$

for almost every $s < t$, where $\pi_1(A\times B) = A$ and $\pi_2(A \times B) = B$. Indeed, if $F(s) < F(t)$, then by properties of the Whitney decomposition there exists a unique $Q(s,t) \in D$ such that $(F(s), F(t)) \in Q(s,t)$. Then by linearity,

$$\begin{aligned} K(t,s)f(s) &= \chi_{Q(s,t)}(F(s),F(t)) \cdot K(t,s)f(s) \\ &= \chi_{\pi_2 Q(s,t)} F(t) K(t,s)\big[\chi_{\pi_1 Q(s,t)}(F(s))f(s)\big] \\ &= \mathrm{RHS}(10.14) \end{aligned}$$

On the other hand, for any t we have by construction

$$\int_{F^{-1}(F(t))} \|f(\tau)\|_X^p \, d\tau = 0,$$

which implies that $f(s) = 0$ for almost every $s < t$ with $F(s) = F(t)$. Thus, in this case we have that both sides of (10.14) are zero (almost surely in s).

Continuing from (10.14), we write

$$\begin{aligned}\tilde{T}f(t) &= \sum_{Q\in D} \chi_{\pi_2 Q}(F(t)) \int_{\mathbb{R}} K(t,s)[\chi_{\pi_1 Q}(F(s))f(s)]\,ds \\ &= \sum_{Q\in D} \chi_{\pi_2 Q}(F(t)) T[(\chi_{\pi_1 Q}\circ F)f](t) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\sum_{Q\in D:\pi_2 Q = I_{jk}} [\chi_{I_{jk}}\circ F] T[(\chi_{\pi_1 Q}\circ F)f],\end{aligned}$$

where we have omitted the terms in which $\pi_2 Q = I_{j,-1}$ or $I_{j,2^j}$ (cf. $F \in [0,1]$). Now observe that for fixed j, we have (using disjointness of supports and the linearity and the boundedness of T)

$$\begin{aligned}\|\tilde{T}f\|^q_{L^q(\mathbb{R};Y)} &\le \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\Big\| \sum_{Q\in D:\pi_2 Q = I_{jk}} (\chi_{I_{jk}}\circ F)\cdot T[(\chi_{\pi_1 Q}\circ F)f] \Big\|^q_{L^q(\mathbb{R};Y)} \\ &\le \sum_{j,k} \Big\| (\chi_{I_{jk}}\circ F) T\Big[\sum_{Q\in D:\pi_2 Q = I_{jk}} (\chi_{\pi_1 Q}\circ F)f\Big]\Big\|^q_{L^q(\mathbb{R};Y)} \\ &\le \sum_{j,k} \Big\| \sum_{Q\in D:\pi_2 Q = I_{jk}} (\chi_{\pi_1 Q}\circ F)f \Big\|^q_{L^p(\mathbb{R};X)} \\ &\le \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\sum_{Q\in D:\pi_2 Q = I_{jk}} \|(\chi_{\pi_1 Q}\circ F)f\|^q_{L^p(\mathbb{R};X)}.\end{aligned}$$

Now, by construction, if $l(Q) = 2^{-j}$ then we have

$$\|(\chi_{\pi_1 Q}\circ F)f\|^q_{L^p(\mathbb{R};X)} \sim 2^{-\frac{qj}{p}}.$$

Furthermore, using properties of the Whitney decomposition, for fixed j,k there are a bounded number of Q such that $\pi_2(Q) = I_{jk}$. Therefore

$$\|\tilde{T}f\|^q_{L^q(\mathbb{R};Y)} \lesssim \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} 2^{-\frac{qj}{p}} \lesssim \sum_{j=0}^{\infty} 2^{-j[\frac{q}{p}-1]} \lesssim 1$$

provided $1 \le p < q < \infty$. This completes the proof. $\square$

We have succeeded in proving a range of Strichartz estimates for the linear Schrödinger equation. Of course, this is not the end of the story;

e.g. we have left out the important endpoint estimate of [16], along with many other extensions and possible refinements. Nor have we discussed in detail how one may apply these estimates to prove local well-posedness for nonlinear Schrödinger equations. For the interested reader, we recommend the reference [3] for a thorough textbook treatment, as well as [17] for an expository introduction into some more advanced techniques.

10.3 Return to restriction theory; Tomas–Stein lemma

In this section, we return to the restriction problem introduced in Section 10.1. Along with the paraboloid (discussed in Section 10.1, this problem has been widely studied in the setting of the sphere and the cone. Just as the restriction theory for the paraboloid is connected to the Schrödinger equation, the cone problem is connected to the linear wave equation.

We will focus on the case of the sphere. Similar to the paraboloid case, we will be interested in estimates of the form

$$\|\hat{f}\big|_S\|_{L^q(S,d\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \tag{10.15}$$

where $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ and $d\sigma$ denotes surface measure on the sphere.

We begin with a few remarks. First, if $p = 1$ then (10.15) holds for all $q \in [1,\infty]$ (cf. Hölder's inequality and Hausdorff–Young). Next, if $p = 2$ then (10.15) fails for all $q \in [1,\infty]$, as $\hat{f}$ may be an arbitrary L^2 function (and hence could be $\equiv \infty$ on S).

Finally, note that if (10.15) holds with some pair (p,q), then it also holds with any $(\tilde{p},\tilde{q})$ with $\tilde{p} \leq p$ and $\tilde{q} \leq q$. To see this, let φ be a bump function with $\hat{\varphi} \equiv 1$ supported on $B(0,2)$. Then

$$\hat{f}|_S = \widehat{f * \varphi}|_S,$$

so, by Hölder, Hausdorff–Young, and Young's inequality, we have

$$\|\hat{f}|_S\|_{L^{\tilde{q}}(S,d\sigma)} \lesssim \|\widehat{f * \varphi}\|_{L^q} \lesssim \|f * \varphi\|_{L^p} \lesssim_\varphi \|f\|_{L^{\tilde{p}}}.$$

Thus the goal is to take p,q as large as possible in (10.15).

As in Section 10.1, we can formulate a dual version of (10.15). Define $\tilde{R} : L^p(\mathbb{R}^d) \to L^q(S,d\sigma)$ by

$$\tilde{R}f = \hat{f}|_S.$$

The adjoint $\tilde{R}^* : L^{q'}(S, d\sigma) \to L^{p'}(\mathbb{R}^d)$ (where primes denote Hölder duals, as usual) is computed by

$$\begin{aligned}\langle \tilde{R}f, g\rangle_{S,d\sigma} &= \int_S \tilde{R}f(\xi)\bar{g}(\xi)d\sigma(\xi)\\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)\overline{\int_S e^{ix\xi} g(\xi)\, d\sigma(\xi)}\, dx = \langle f, (g\, d\sigma)\check{}\,\rangle_{\mathbb{R}^d}.\end{aligned}$$

In particular, $\tilde{R}^* g = (g\, d\sigma)\check{}$, which is bounded if and only if

$$\|(g\, d\sigma)\check{}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}(S,d\sigma)}. \tag{10.16}$$

We begin by finding necessary conditions for $\tilde{R}, \tilde{R}^*$ to be bounded.

Example 10.3.1. Let $g \equiv 1$. Then (10.16) becomes

$$\|\check{\sigma}\|_{L^{p'}(\mathbb{R}^d)} \lesssim 1.$$

As $|\check{\sigma}(x)| = \mathcal{O}(\langle x\rangle^{-\frac{d-1}{2}})$ (cf. Exercise 7.4.7), we find that we must have

$$\tfrac{d-1}{2}p' > d, \quad \text{i.e.} \quad p < \tfrac{2d}{d+1}. \tag{10.17}$$

Example 10.3.2 (The Knapp example). Let $R \gg 1$ and $g = \chi_K$, where K is a spherical cap centered at the south pole ξ_0 of radius R^{-1}. Then $K \subset D$, where D is a disk of radius R^{-1} and thickness R^{-2}; indeed, with $\xi = (\bar{\xi}, \xi_d)$,

$$\xi_d = -\sqrt{1 - |\bar{\xi}|^2} = -\big[1 - \tfrac{1}{2}|\bar{\xi}|^2 + \mathcal{O}(|\xi|^4)\big] = -1 + \mathcal{O}(R^{-2}).$$

Then

$$(g\, d\sigma)\check{}(x) = e^{ix\xi_0}\int_K e^{i(\xi-\xi_0)x}\, d\sigma(\xi).$$

If $|(\xi - \xi_0)x| \lesssim 1$, then $(g\, d\sigma)\check{}(x)$ is of size $\sigma(K) \sim R^{-(d-1)}$.

Now, observe that for $\xi \in K$, we have

$$|x_d(\xi - \xi_0)_d| \lesssim 1 \quad \text{if} \quad |x_d| \lesssim R^2,$$

while

$$|\bar{x}(\overline{\xi - \xi_0})| \lesssim 1 \quad \text{if} \quad |\bar{x}| \lesssim R.$$

In particular, $(g\, d\sigma)\check{}(x) \sim R^{-(d-1)}$ throughout the 'dual tube' T to D, centered at the origin, with height R^2 and radius R. So

$$\|(g\, d\sigma)\check{}\|_{L^{p'}} \gtrsim R^{-(d-1)}|T|^{\frac{1}{p'}} \gtrsim R^{-(d-1)}R^{\frac{d+1}{p'}}.$$

On the other hand,

$$\|g\|_{L^{q'}(S,d\sigma)} \lesssim \sigma(K)^{\frac{1}{q'}} \lesssim R^{-\frac{d-1}{q'}}.$$

Thus for (10.16) to hold, we must have

$$R^{-(d-1)}R^{\frac{d+1}{p'}} \lesssim R^{-\frac{d-1}{q'}}, \quad \text{i.e.} \quad \tfrac{d+1}{p'} \leq \tfrac{d-1}{q}. \tag{10.18}$$

The **restriction conjecture** for the sphere states that the necessary conditions (10.17) and (10.18) are in fact sufficient.

Conjecture 10.3.1 (Restriction conjecture for the sphere)**.** *We have*

$$\|\hat{f}|_S\|_{L^q(d\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \quad \textit{if and only if} \quad p < \tfrac{2d}{d+1}, \quad \tfrac{d+1}{p'} \leq \tfrac{d-1}{q}.$$

This has been resolved completely in $d = 2$ [31], while the full result remains open in higher dimensions.

In the rest of this section, we will discuss a positive result due to Tomas and Stein.

Theorem 10.3.2 (Tomas–Stein)**.** *We have*

$$\|\hat{f}|_S\|_{L^2(S,d\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \quad \textit{whenever} \quad 1 \leq p \leq \tfrac{2(d+1)}{d+3}.$$

Note that when $q = 2$, the necessary conditions in Conjecture 10.3.1 reduce to

$$p \leq \tfrac{2(d+1)}{d+3},$$

so this result is sharp for the choice $q = 2$.

Proof of Theorem 10.3.2. We will prove the result up to (but not including) the endpoint $p = \frac{2(d+1)}{d+3}$. We begin by obtaining the smaller range

$$1 < p \leq \tfrac{4d}{3d+1}$$

by simply relying on the decay of $\check{d\sigma}$; recall the case $p = 1$ always holds.

Denoting

$$Rf = \hat{f}|_S \quad \text{and} \quad R^*g = (g\, d\sigma)\check{},$$

we will employ the method of TT^* and endeavor to prove $R^*R : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$. Note that

$$R^*Rf = (\hat{f}|_S\, d\sigma)\check{} = f * \check{\sigma}.$$

By Hardy–Littlewood–Sobolev (Theorem 6.2.2), we have

$$\|f * \check{\sigma}\|_{L^{p'}} \lesssim \|f\|_{L^p} \|\check{\sigma}\|_{L^{\frac{p}{2(p-1)},\infty}}$$

Now observe that since $|\check{\sigma}| \lesssim |x|^{-\gamma}$ for $\gamma \in [0, \frac{d-1}{2}]$ (cf. Exercise 7.4.7), we have

$$\check{\sigma} \in L^{\frac{p}{2(p-1)},\infty} \quad \text{provided} \quad \tfrac{2d(p-1)}{p} \in (0, \tfrac{d-1}{2}], \quad \text{i.e.} \quad 1 < p \leq \tfrac{4d}{3d+1}.$$

To go beyond this range, we employ the Littlewood–Paley partition of unity and write

$$f * \check{\sigma} = f * (\varphi\check{\sigma}) + \sum_{N>1} f * (\psi_N \check{\sigma}).$$

Because $\varphi \in \mathcal{S}$ and $\sigma \check{\in} L^\infty$, we may use Young's inequality to write

$$\|f * (\varphi\check{\sigma})\|_{L^{p'}} \lesssim \|f\|_{L^p} \|\varphi\check{\sigma}\|_{L^{\frac{p'}{2}}} \lesssim \|f\|_{L^p}.$$

To estimate the sum, we wish to prove

$$\|f * (\psi_N \check{\sigma})\|_{L^{p'}} \lesssim N^{-\varepsilon} \|f\|_{L^p} \quad \text{for some} \quad \varepsilon = \varepsilon(p) > 0, \tag{10.19}$$

for then we may sum to deduce the desired estimate.

To prove (10.19), we will interpolate between an estimate for $p = 1$ and an estimate for $p = 2$.

First, for $p = 1$, we have by Hölder's inequality and the decay of $\check{\sigma}$

$$\|f * (\psi_N \check{\sigma})\|_{L^\infty} \lesssim \|f\|_{L^1} \|\check{\sigma}\|_{L^\infty(|x|\sim N)} \lesssim N^{-(\frac{d-1}{2})} \|f\|_{L^1}.$$

On the other hand, we have by Plancherel and Lemma 2.5.11

$$\|f * (\psi_n \check{\sigma})\|_{L^2} \lesssim \|f\|_{L^2} \|\hat{\psi}_N * d\sigma\|_{L^\infty}.$$

Now, fixing $x \in \mathbb{R}^d$, we may write

$$|\hat{\psi}_N * d\sigma(x)| \leq N^d \int_S |\hat{\psi}(N(x-y))| d\sigma(y) \lesssim \int_S \tfrac{N^d}{\langle N(x-y)\rangle^m} d\sigma$$

for any $m \geq 0$.

To estimate this integral, we split into two regimes: (i) $y \in S$ with $|x-y| \lesssim \frac{1}{N}$, which has a volume of $\sim N^{-(d-1)}$. In this regime the integrand is bounded by N^d. (ii) $y \in S$ with $|x-y| \sim \frac{1}{M}$ for some $M \ll N$, which has a volume bound of $M^{-(d-1)}$ (which could of course be replaced by $\lesssim 1$

if $M \leq 1$). In this case $\langle N(x-y)\rangle \sim \frac{N}{M}$, and if we choose $m = d$ in the estimate above we get that the integrand is bounded by M^d.

Thus

$$|\hat{\psi}_N * d\sigma(x)| \lesssim N^d \cdot N^{-(d-1)} + \sum_{M \leq N} M^d M^{-(d-1)} \lesssim N.$$

We now interpolate between the $(1,\infty)$ and $(2,2)$ estimates and check the range of p for which we end up with a negative power of N. As $\frac{1}{p} = \theta + \frac{1-\theta}{2}$ when $\theta = \frac{2-p}{p}$, we get

$$\|f * (\psi_N \check{\sigma})\|_{L^{p'}} \lesssim N^{-\frac{d-1}{2}\frac{2-p}{p}} N^{2[\frac{p-1}{p}]} \|f\|_{L^p}$$

One can check that this power is negative provided $p < \frac{2(d+1)}{d+3}$. This completes the proof. $\square$

10.4 Exercises

Exercise 10.4.1. Prove (10.6).

Exercise 10.4.2. (*Challenge problem.*) Develop a profile decomposition adapted to the Strichartz estimate in order to prove existence of optimizers to this inequality.

Appendix A

Prerequisite material

The purpose of this chapter is to collect prerequisite material that is used throughout the main body of these notes.

Let us first recall some standard notation to be used throughout these notes.

We use the following multi-index notation. Given $d \geq 1$, a multi-index α is an element of $\mathbb{N}^d$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. We let

$$|\alpha| = \sum_{i=1}^{d} \alpha_i, \quad x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad \partial^\alpha f = \tfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

We write $A \lesssim B$ to denote $A \leq CB$ for some $C > 0$. We write $A \ll B$ to indicate $A \leq cB$ for some suitably small $c > 0$.

We write χ_E for the characteristic function of the set E, that is,

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

A.1 Lebesgue spaces

Given a Lebesgue measurable subset $E \subset \mathbb{R}^d$ of positive measure and $1 \leq p < \infty$, we define $L^p(E)$ to be the space of measurable functions f such that

$$\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The functions f may be either real-valued or complex-valued. The quantity $\| \cdot \|_{L^p(E)}$ defines a norm. When $p = \infty$, we define

$$\|f\|_{L^\infty(E)} = \inf\{M : |\{x \in E : |f(x)| > M\}| = 0\},$$

where $|S|$ denotes the Lebesgue measure of S. The quantity $\|\cdot\|_{L^p(E)}$ also defines a norm. We will often drop the underlying set E and simply write L^p.

To be precise, elements of L^p should be regarded as equivalence classes of functions that are equal almost everywhere; however, we will typically ignore this distinction.

The spaces L^p are vector spaces. Furthermore they are complete with respect to the metric defined by the L^p-norm (namely, $d(f,g) = \|f-g\|_{L^p}$). In particular, they are **Banach spaces**. Furthermore, for $1 \leq p < \infty$, we have that the space L^p is separable (i.e. admits a countable dense subset). On the other hand, L^∞ is not separable.

For spaces of sequences $c = \{c_k\}$ we use

$$\|c\|_{\ell^p} = \left(\sum |c_k|^p\right)^{\frac{1}{p}}, \quad \|c\|_{\ell^\infty} = \sup |c_k|.$$

The ℓ^p spaces are nested; that is, for $p_1 \leq p_2$ we have

$$\ell^{p_1} \subset \ell^{p_2}, \quad \text{with} \quad \|c\|_{\ell^{p_2}} \leq \|c\|_{\ell^{p_1}}.$$

The space L^2 admits an inner product, denoted by

$$\langle f, g\rangle = \int f(x)\bar{g}(x)\,dx,$$

where $\bar{\cdot}$ denotes the complex conjugate. Finiteness of $\langle f, g\rangle$ follows from the Cauchy–Schwarz inequality (or Hölder's inequality):

$$|\langle f, g\rangle| \leq \|f\|_{L^2}\|g\|_{L^2}.$$

The space L^2 is therefore an example of a (separable) **Hilbert space**, i.e. an inner product space that is complete with respect to the metric induced by the inner product.

Given $1 \leq p \leq \infty$, we define $1 \leq p' \leq \infty$ via

$$\tfrac{1}{p} + \tfrac{1}{p'} = 1.$$

We call p' the **dual exponent** to p. It is often useful to compute L^p norms 'by duality', i.e. using the fact that

$$\|f\|_{L^p} = \sup |\langle f, g\rangle|,$$

where the supremum is taken over all $g \in L^{p'}$ with $\|g\|_{L^{p'}} = 1$.

For a measurable function f, we define the **distribution function** of f by

$$\alpha \mapsto |\{x : |f(x)| > \alpha\}|.$$

We can compute the L^p-norm of a function in terms of its distribution function as follows:

$$\int |f|^p \, dx = p \int_0^\infty \alpha^{p-1} |\{|f| > \alpha\}| \, d\alpha. \tag{A.1}$$

For $1 \leq p < \infty$, we define the **weak L^p space** by

$$L^{p,\infty}(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} : \|f\|_{L^{p,\infty}} < \infty\},$$

where the $L^{p,\infty}$ quasi-norm is defined by

$$\|f\|_{L^{p,\infty}} = \sup_{\alpha > 0} \alpha \big|\{|f| > \alpha\}\big|^{\frac{1}{p}}.$$

A quasi-norm refers to a quantity that satisfies all of the hypotheses of a norm except the triangle inequality, which must be replaced by $\|x + y\| \leq C(\|x\| + \|y\|)$ for some $C > 0$. See the exercises.

Note that $L^p \subset L^{p,\infty}$; indeed, by Tchebyshev's inequality we have

$$\alpha^p |\{|f| > \alpha\}| \lesssim \|f\|_{L^p}^p$$

uniformly in α.

We have Minkowski's integral inequality:

$$\|f(x,y)\|_{L_x^p L_y^1} \leq \|f(x,y)\|_{L_y^1 L_x^p}$$

for $1 \leq p \leq \infty$.

Let us briefly discuss one other theory of integration, namely, the Riemann–Stieltjes integral. First recall the definition of functions of bounded variation.

Definition A.1.1. Let $f : [a,b] \to \mathbb{R}$, and let

$$\Gamma = \{x_0, \ldots, x_m\}$$

be a partition of $[a,b]$. Define

$$S_\Gamma = S_\Gamma[f; a, b] = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|.$$

The **variation of** f **over** $[a,b]$ is defined by

$$\|f\|_{BV([a,b])} = \sup_{\Gamma} S_\Gamma[f;a,b].$$

As $0 \le S_\Gamma < \infty$, we have $\|f\|_{BV} \in [0,\infty]$. If $\|f\|_{BV} < \infty$, we say f is of **bounded variation**. We may write $f \in BV([a,b])$. Otherwise, we say f is of **unbounded variation**.

Examples of bounded variation functions are those that are continuously differentiable on an interval. For the reader unfamiliar with the notion of bounded variation, feel free to replace 'function of bounded variation' with 'continuously differentiable function' and 'BV norm' with 'L^∞ norm of the derivative' throughout these notes.

Let us also recall the definition of Riemann–Stieltjes integration.

Definition A.1.2. Let $f,\phi : [a,b] \to \mathbb{R}$. Let $\Gamma = \{x_i\}_{i=0}^m$ be a partition of $[a,b]$ and let $\{\xi_i\}_{i=1}^m$ satisfy

$$x_{i-1} \le \xi_i \le x_i \quad \text{for each} \quad i.$$

The quantity

$$R_\Gamma := \sum_{i=1}^m f(\xi_i)[\phi(x_i) - \phi(x_{i-1})]$$

is called a **Riemann–Stieltjes sum** for Γ.

If

$$I = \lim_{|\Gamma|\to 0} R_\Gamma$$

exists and is finite, then I is called the **Riemann–Stieltjes integral of** f **with respect to** ϕ **on** $[a,b]$, denoted

$$I = \int_a^b f(x)\,d\phi(x) = \int_a^b f\,d\phi.$$

We recall the following results concerning Riemann–Stieltjes integrals.

Proposition A.1.3. *Suppose $f \in C([a,b])$ and $\phi \in BV([a,b])$. Then $\int_a^b f\,d\phi$ exists, and*

$$\left|\int_a^b f\,d\phi\right| \le \|f\|_{L^\infty}\,\|\phi\|_{BV}.$$

Proposition A.1.4 (Integration by parts formula)**.** *If $\int_a^b f\,d\phi$ exists, then so does $\int_a^b \phi\,df$, and*

$$\int_a^b f\,d\phi = [f(b)\phi(b) - f(a)\phi(a)] - \int_a^b \phi\,df.$$

A.2 Hilbert spaces

We record here a few basic results concerning Hilbert spaces. Recall that a Hilbert space is a vector space equipped with an inner product that is complete with respect to the induced norm. In these notes, we restrict our attention to the setting of separable Hilbert spaces (i.e. spaces that admit a countable dense subset).

Lemma A.2.1. *Let H_1 and H_2 be Hilbert spaces. A linear operator $T : H_1 \to H_2$ is bounded if and only if it is continuous.*

Proof. As T is linear, it suffices to verify continuity at 0. This follows directly from boundedness, cf.

$$\|T(f)\|_{H_2} \leq M\|f\|_{H_1}.$$

Conversely, boundedness and linearity readily imply continuity; cf.

$$\|T(f) - T(g)\|_{H_2} \leq M\|f_2 - f_1\|_{H_1}.$$

This completes the proof. □

Lemma A.2.2 (Riesz representation theorem)**.** *Suppose $\ell : H \to \mathbb{R}$ is a continuous linear functional on a real Hilbert space. Then there exists a unique $g \in H$ such that*

$$\ell(f) = \langle f, g\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the Hilbert space inner product.

Proof. Let $\{\varphi_n\}$ be an orthonormal basis for H. To obtain such a basis, start with a countable dense subset of H and apply the Gram–Schmidt algorithm.

Given a function $f \in H$, write $f_n = \langle f, \varphi_n\rangle$ denote the 'Fourier coefficients' of f relative to $\{\varphi_n\}$. (See Section 2.2 for more details.) In particular, we can uniquely specify a function by prescribing its Fourier coefficients.

We define g by prescribing $g_n = \ell(\varphi_n)$. Then by Plancherel's theorem

$$\ell(f) = \sum_n f_n g_n = \langle f, g\rangle.$$

It remains to verify that this procedures defines $g \in H$ and that the result is unique. To this end, define $P_N g = \sum_{n=1}^N \ell(\varphi_n)\varphi_n$ and let us prove that $\|P_N g\|_H$ is uniformly bounded in N. Then, on the one hand, by linearity we have

$$\ell(P_N g) = \sum_{n=1}^N [\ell(\varphi_n)]^2.$$

On the other hand, by boundedness of ℓ, Cauchy–Schwarz, and orthonormality, we have

$$|\ell(P_N g)| \leq M \Big\| \sum_{n=1}^{N} \ell(\varphi_n)\varphi_n \Big\|_H \leq M \Big(\sum_{n=1}^{N} [\ell(\varphi_n)]^2 \Big)^{\frac{1}{2}}.$$

Combining the last two displays yields

$$\|P_N g\|_H = \Big(\sum_{n=1}^{N} [\ell(\varphi_n)]^2 \Big)^{\frac{1}{2}} \leq M,$$

which yields the desired bound.

As for uniqueness, if $\langle f, g\rangle = \langle f, h\rangle$ for all $f \in H$, then applying this with $f = g - h$ immediately yields $g = h$. □

Using the Riesz representation theorem, we can identify H with its dual space via the pairing $\langle f, g\rangle$. We say that a sequence f_n **converges weakly** to f if

$$\langle f_n, g\rangle \to \langle f, g\rangle \quad \text{for all} \quad g \in H.$$

We write $f_n \rightharpoonup f$.

Lemma A.2.3. *The following properties hold concerning weak convergence:*

(a) If $f_n \rightharpoonup f$, then $\|f\| \leq \limsup_{n\to\infty} \|f_n\|$.

(b) If $\{f_n\}$ is bounded, then f_n converges weakly along a subsequence.

Proof. For (a), let $\varepsilon > 0$ and choose a unit vector $g \in H$ so that

$$\langle f, g\rangle > \|f\| - \varepsilon.$$

Thus by weak convergence and Cauchy–Schwarz,

$$\|f\| < \langle f, g\rangle + \varepsilon = \lim_{n\to\infty} \langle f_n, g\rangle + \varepsilon \leq \limsup_{n\to\infty} \|f_n\| + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this implies the result.

We turn to (b). We let $S = \{g_k\}$ be a countable dense subset of H. For each k, the sequence

$$\{\langle f_n, g_k\rangle\}$$

is bounded and hence converges along a subsequence. In particular, by a diagonal argument we may find a subsequence such that $\langle f_n, g_k\rangle$ converges to a limit (say c_k) for all k. We claim that f_n converges weakly along this subsequence.

To see this, we need to define the limit f. By duality, it will suffice to specify the values $\langle f, g\rangle$ for all $g \in H$. To this end, we fix $g \in H$ and take a sequence of $g_k \in S$ converging to g. We will show that $\{c_k\}$ is Cauchy, so that it has a limit c; we will then define $\langle f, g\rangle = c$ and check that this defines an L^2 function to which f_n converge weakly. Let $\varepsilon > 0$ and choose K large enough that $\|g - g_k\| < \varepsilon$ for $k > K$. Now fix $k, \ell > K$. By choosing n large enough, we may guarantee that

$$\big|\langle f_n, g_k\rangle - \langle f_n, g_\ell\rangle - (c_k - c_\ell)\big| < \varepsilon.$$

On the other hand, by Cauchy–Schwarz,

$$|\langle f_n, g_k\rangle - \langle f_n, g_\ell\rangle| = |\langle f_n, g_k - g\rangle - \langle f_n, g_\ell - g\rangle| \leq 2M\varepsilon,$$

where M is the uniform bound for the $\{f_n\}$. Thus $\{c_k\}$ is Cauchy and hence converges to c. An intertwining argument shows that we may uniquely define f as a (linear) functional on H via $\langle f, g\rangle = c$. To see that $f \in H$, we observe that

$$|\langle f, g\rangle| = |\lim_{k\to\infty}\lim_{n\to\infty}\langle f_n, g_k\rangle| \leq M\|g\|,$$

where $g_k \in S$ converges to g. In particular $\|f\| \leq M$. Finally, for weak convergence we again let $g \in H$ and choose a sequence $S \ni g_k \to g$. We then write

$$\langle f - f_n, g\rangle = \langle f - f_n, g - g_k\rangle + \langle f - f_n, g_k\rangle.$$

The first term is $M \cdot o(1)$ as $k \to \infty$, while the second term converges to zero by construction. This completes the proof. $\square$

Remark A.2.4. Item (b) in the previous lemma is a special case of Alaoglu's theorem.

Let $T : H_1 \to H_2$ be a (bounded) linear operator. The **adjoint** of T, denoted T^*, is the linear operator from H_2 to H_1 defined via

$$\langle Tf, g\rangle_{H_2} = \langle f, T^* g\rangle_{H_1}.$$

We call $T : H \to H$ **self-adjoint** or **symmetric** if $T = T^*$.

An operator $T : H_1 \to H_2$ is called **compact** if it maps bounded sets to pre-compact sets. Equivalently, T is compact if whenever $\{f_n\}$ is a bounded sequence in H_1, $\{Tf_n\}$ has a convergent subsequence in H_2. We also have the following:

Lemma A.2.5. *Suppose there exists a sequence of finite-rank operators T_n so that $T_n \to T$. Then T is compact.*

Here finite rank means the range of T_n is finite-dimensional, while the convergence refers to the operator norm

$$\|T\| = \sup_{\|f\| \le 1} \|Tf\|.$$

Proof. Suppose T has finite-rank approximations T_n and $\{f_m\}$ is a bounded sequence. Then f_m has a subsequence f_m^1 so that $T_1 f_m^1$ converges. Similarly, we can take a subsequence f_m^2 of f_m^1 so that $T_2 f_m^2$ converges. For $n > m > K$, we write

$$Tf_n^n - Tf_m^m = (T - T_K)f_n^n + T_K(f_n^n - f_m^m) + (T_K - T)f_m^m.$$

Choosing K large enough, the first and third terms may be made arbitrarily small small due to the fact that $\{f_n^n\}, \{f_m^m\}$ are bounded sequences. For fixed K, the middle term tends to zero as $n, m \to \infty$ because $\{f_n^n\}_{n \ge K}$ is a subsequence of f_K^K. Thus Tf_n^n is Cauchy and hence convergent. This implies T is compact. □

If $X \subset Y$ are two Banach spaces, then we say X is **continuously embedded** in Y if the inclusion map is continuous (i.e. bounded). We say X is **compactly embedded** in Y if the inclusion map is compact.

Lemma A.2.6. *Suppose X and Y are Hilbert spaces and X is compactly embedded in Y. If x_n converges weakly in X, then x_n converges strongly in Y (to the same limit)*

Proof. It suffices to show that weakly convergent sequences are bounded. Indeed, if x_n is bounded, then the compactness of the embedding implies that any subsequence of x_n converges strongly in Y along a further subsequence (to the weak limit of x_n, by uniqueness of limits). This implies that x_n converges strongly in Y.

Let us now show that x_n is bounded. In fact, this will follow from the uniform boundedness principle (Lemma A.3.3). We view each x_n as a bounded linear map from $X \to \mathbb{R}$ via $x_n(y) = \langle x_n, y \rangle$. Then weak convergence implies

$$\sup_n |\langle x_n, y \rangle| < \infty \quad \text{for all} \quad y \in X.$$

By the principle of uniform boundedness, this implies

$$\sup_n \|x_n\|_{X \to \mathbb{R}} = \sup_n \|x_n\|_X < \infty,$$

as desired. In the final step we have used the dual formulation to compute the norm of x_n, i.e. $\|x_n\| = \sup\{|\langle x_n, y\rangle|\}$ where the supremum is taken over all $y \in X$. □

Let us next sketch a proof of the following fundamental fact (a basic version of the spectral theorem).

Theorem A.2.7. *If $T : H \to H$ is compact and symmetric, then it is* ***diagonalizable****. That is, there exists an orthonormal set $\{u_n\}$ if eigenvectors for H and a sequence $\{\lambda_n\} \subset \mathbb{R}$ so that $Tu_n = \lambda_n u_n$ for all n, with $\lambda_n \to 0$. Furthermore,*

$$T(x) = \sum_n \lambda_n \langle x, u_n\rangle u_n$$

for $x \in H$.

The fact that the λ_n are real is a consequence of the symmetry of T (see Exercise A.4.2). Similarly, one can verify that eigenspaces corresponding to distinct eigenvalues are orthogonal. Without loss of generality, we take H to be infinite dimensional.

Sketch of proof. We first show that A has an eigenvalue α_0 with $|\alpha_0| = \|A\|$. We let $\alpha = \|A\| \geq 0$. Using symmetry, one observes

$$\|A\|^2 = \sup_{\|f\|=1} \langle f, A^2 f\rangle,$$

so that we may take a sequence u_n with $\|u_n\| \equiv 1$ and $\langle u_n, A^2 u_n\rangle \to \alpha^2$. By compactness of A (and hence of A^2—exercise), we have $A^2 u_n \to \alpha^2 u$ for some u (along a subsequence). But then

$$\begin{aligned}\|(A^2 - \alpha^2)u_n\|^2 &= \|A^2 u_n\|^2 - 2\alpha^2\langle u_n, A^2 u_n\rangle + \alpha^4 \\ &\leq 2\alpha^2(\alpha^2 - \langle u_n, A^2 u_n\rangle) \to 0.\end{aligned}$$

In particular, $A^2 u_n - \alpha^2 u_n \to 0$ and $A^2 u_n \to \alpha^2 u$, so that $u_n \to u$. Thus $\|u\| = 1$ and $A^2 u = \alpha^2 u$, so that

$$(A + \alpha)(A - \alpha)u = (A - \alpha)(A + \alpha)u = 0.$$

This shows that either α or $-\alpha$ is an eigenvalue of A. In fact, $\|A\|$ must be the largest eigenvalue of A (exercise).

Now let u_0 be a normalized eigenvector corresponding to the largest eigenvalue α_0. Then set

$$H^{(1)} = \{f \in H : \langle u_0, f\rangle = 0\}.$$

This is a Hilbert space with $A : H^{(1)} \to H^{(1)}$; indeed

$$\langle Af, u_0\rangle = \langle f, Au_0\rangle = \alpha\langle f, u_0\rangle = 0 \quad \text{for all} \quad f \in H^{(1)}.$$

Let $A_1 = A|_{H^{(1)}}$. Then A_1 is symmetric and compact, and hence we can find a largest eigenvalue α_1 with normalized eigenvector u_1... we now continue in this way to construct a sequence of normalized eigenvectors $\{u_j\}$ that are mutually orthogonal by construction with eigenvalues α_j.

We claim that $\alpha_j \to 0$. If not, then (passing to a subsequence) we get $\alpha_j \neq 0$ for all j. Now consider the bounded sequence $\frac{1}{\alpha_j}u_j$. Then $\{A\frac{1}{\alpha_j}u_j\}$ has no convergent subsequence, since

$$\|A\tfrac{1}{\alpha_j}u_j - A\tfrac{1}{\alpha_\ell}u_\ell\|^2 = \|u_j - u_\ell\|^2 = 2.$$

This contradicts compactness. The representation of T in terms of eigenvectors is left as an exercise. *Hint:* Show that

$$\|T(x) - \sum_{n=1}^{N} \lambda_n\langle x, u_n\rangle u_n\| \leq |\lambda_n|\|x\|.$$

This completes the proof. □

An operator T is called **positive** (written $T \geq 0$) if $\langle Tf, f\rangle \geq 0$ for all $f \in H$. A bounded operator $T : H \to H$ is called **trace class** if there exists an orthonormal basis $\{\varphi_n\}$ such that

$$\operatorname{tr}(T) := \sum_n \langle T\varphi_n, \varphi_n\rangle < \infty.$$

In fact, in this case the trace $\operatorname{tr}(T)$ is independent of the basis chosen. Trace class operators are compact. We can prove this provided we take a few Fourier analysis type results for granted (see Section 2.2 for details).

We first claim that if T is trace class, then in fact

$$\sum_n \|T\varphi_n\|^2 < \infty. \tag{A.2}$$

Using this, we can complete the proof by exhibiting finite-rank approximations to T. In particular, we set

$$T_N f = \sum_{n\leq N} \langle T_N f, \varphi_n\rangle\varphi_n.$$

Then, using orthogonality and Cauchy–Schwarz, we can write

$$\begin{aligned}
\|(T - T_N)f\|^2 &= \sum_m \left| \sum_{n>N} \langle f, \varphi_n \rangle \langle T\varphi_n, \varphi_m \rangle \right|^2 \\
&\leq \sum_m \left(\sum_{n>N} |\langle f, \varphi_n \rangle|^2 \right) \left(\sum_{n>N} |\langle T\varphi_n, \varphi_m \rangle|^2 \right) \\
&\leq \|f\|^2 \sum_{n>N} \sum_m |\langle T\varphi_n, \varphi_m \rangle|^2 \\
&\leq \|f\|^2 \sum_{n>N} \|T\varphi_n\|^2 = \|f\|^2 \cdot o(1)
\end{aligned}$$

as $N \to \infty$, which implies the result.

It remains to show that trace class operators satisfy (A.2) (which is actually the definition of 'Hilbert–Schmidt' operators). In fact, (A.2) will follow from the more general bound

$$\|Tf\|^2 = \langle T^*Tf, f \rangle \leq \|T\| \langle Tf, f \rangle$$

for positive operators T, as we now prove. As positive operators are automatically self-adjoint, we may replace T^*T with T^2. Multiplying both sides by an arbitrary $\alpha > 0$, we can reduce to the case that $\|T\|$ is as small as we wish, say $\|T\| < \frac{1}{2}$. In this case, we have by Cauchy–Schwarz

$$\langle f, f \rangle \geq \|Tf\| \, \|f\| \geq \langle Tf, f \rangle \geq 0,$$

and so

$$0 \leq \langle f - Tf, f \rangle \leq \langle f, f \rangle.$$

This shows $I - T$ is positive and has norm bounded by 1. Then writing

$$T = I - (I - T)$$

and using the power series expansion for $1 - x$ (which converges whenever $\|x\| \leq 1$), we can define a (unique, positive) square root of T. Then the desired bound follows from

$$\|Tf\|^2 \leq \|T^{\frac{1}{2}}\|^2 \|T^{\frac{1}{2}} f\|^2 \leq \|T\| \langle Tf, f \rangle. \tag{A.3}$$

A.3 Analysis tools

The **convolution** of f and g is defined by

$$f * g(x) = \int f(x - y) g(y) \, dy.$$

We use this both when the functions are defined on all of $\mathbb{R}^d$, as well as when the functions are periodic on some torus. In this section we will focus on the case of $\mathbb{R}^d$.

Definition A.3.1. We call a family of functions $\{K_n\}$ on $\mathbb{R}^d$ a family of **good kernels** if the following three conditions hold:

(i) $\int_{\mathbb{R}^d} K_n(y)\,dy = 1$ for all n,

(ii) there exists M such that for all n we have $\int_{\mathbb{R}^d} |K_n(y)|\,dy \le M$,

(iii) for each $\delta > 0$, we have $\int_{|y|>\delta} |K_n(y)|\,dy \to 0$ as $n \to \infty$.

Example A.3.1. Let $K : \mathbb{R}^d \to \mathbb{R}$ satisfy

$$\int |K(x)|\,dx \lesssim 1 \quad \text{and} \quad \int K(x)\,dx = 1.$$

Then the functions $K_n(x) := n^d K(nx)$ form a family of good kernels.

Lemma A.3.2 (Approximation to the identity)**.** *Let $f \in L^p(\mathbb{R}^d)$ for some $1 \le p < \infty$. Suppose $\{K_n\}$ is a family of good kernels. Then*

$$\lim_{n\to\infty} \|f - f * K_n\|_{L^p} = 0.$$

If f is bounded and continuous, the same result holds in pointwise. If f is bounded and uniformly *continuous (or if f is continuous and tends to zero as $|x| \to \infty$), then the convergence is uniform.*

Proof. Let us prove the L^p result and leave the remaining part as an exercise. Using $\int K_n(y)\,dx \equiv 1$, we see that our task is to prove

$$\lim_{n\to\infty} \int \left| \int K_n(y)[f(x) - f(x-y)]\,dy \right|^p dx = 0.$$

We let $\varepsilon > 0$ and observe that because translations are continuous in L^p, there exists $\delta > 0$ such that

$$\sup_{|y|<\delta} \|f(x) - f(x-y)\|_{L^p_x} < \varepsilon.$$

Thus (writing M as the uniform upper bound for $\|K_n\|_{L^1}$ and applying Minkowski's integral inequality)

$$\begin{aligned}
&\int \left| \int_{|y|<\delta} K_n(y)[f(x) - f(x-y)]\,dy \right|^p dx \\
&\lesssim \sup_{|y|<\delta} \int |f(x) - f(x-y)|^p\,dx \cdot \left| \int K_n(y)\,dy \right|^p dx \\
&\lesssim \varepsilon^p M^p
\end{aligned}$$

uniformly in n. On the other hand, applying Minkowski's integral inequality once again,

$$\begin{aligned}
&\int \left| \int_{|y|>\delta} K_n(y)[f(x) - f(x-y)]\, dy \right|^p dx \\
&\quad \leq \left(\int_{|y|>\delta} |K_n(y)| \left(\int |f(x) - f(x-y)|^p \, dx \right)^{\frac{1}{p}} dy \right)^p \\
&\quad \lesssim \left| \int_{|y|>\delta} K_n(y)\, dy \right|^p \sup_y \int |f(x)|^p + |f(x-y)|^p \, dx \\
&\quad \lesssim o(1) \cdot \int |f(x)|^p \, dx
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

We next record a result known as the principle of uniform boundedness.

Lemma A.3.3 (Uniform boundedness principle). *Let X and Y be Banach spaces and let F be a collection of continuous linear functions from X to Y. If*

$$\sup_{T \in F} \|T(x)\|_Y < \infty \quad \textit{for all} \quad x \in X, \tag{A.4}$$

then

$$\sup_{T \in F} \|T\|_{X \to Y} < \infty. \tag{A.5}$$

Proof. The standard proof relies on the Baire category theorem. Here we present a completely elementary proof appearing in [25]. Let us omit the subscripts from the norms below; the meaning should be clear from context.

First, for any linear operator T and any x, y,

$$\begin{aligned}
\|Ty\| &\leq \|T(\tfrac{1}{2}(x+y) - \tfrac{1}{2}(x-y))\| \\
&\leq \tfrac{1}{2}\big[\|T(x+y)\| + \|T(x-y)\|\big] \\
&\leq \max \|T(x \pm y)\|.
\end{aligned}$$

Thus (using linearity once again), we have for any $r > 0$

$$\|T\| = \tfrac{1}{r} \sup_{\|y\|=r} \|Ty\| \leq \tfrac{1}{r} \sup_{\|y\| \leq r} \max \|T(x \pm y)\| \leq \tfrac{1}{r} \sup_{z \in B(x,r)} \|Tz\|,$$

i.e. for any linear operator we have

$$\sup_{z \in B(x,r)} \|Tz\| \geq r\|T\| \quad \text{for all} \quad x \in X, r > 0.$$

Now suppose (A.5) fails; we will show (A.4) fails as well. We choose a sequence $T_n \in F$ such that $\|T_n\| \geq 4^n$. Define $x_0 = 0$. Proceeding inductively, we may find a sequence x_n such that $x_n \in B(x_{n-1}, 3^{-n})$ and

$$\|T_n x_n\| \geq \tfrac{9}{10} 3^{-n} \|T_n\| \geq \tfrac{9}{10} (\tfrac{4}{3})^n.$$

The sequence x_n is Cauchy and hence converges to some x. In fact, noting that

$$\|x_m - x_n\| \leq \sum_{j=n+1}^{m} \|x_j - x_{j-1}\| \leq \sum_{j=n+1}^{m} 3^{-j} < \tfrac{1}{2} 3^{-n},$$

one finds $\|x - x_n\| \leq \frac{1}{2} \cdot 3^{-n}$. Now

$$\|T_n(x - x_n)\| \leq \|T_n\| \tfrac{1}{2} \cdot 3^{-n}, \quad \text{while} \quad \|T_n x_n\| \geq \tfrac{9}{10} 3^{-n} \|T_n\|,$$

so that

$$\|T_n x\| \geq \tfrac{2}{5} 3^{-n} \|T_n\| \geq \tfrac{2}{5} (\tfrac{4}{3})^n \to \infty,$$

showing the failure of (A.4). □

Lemma A.3.4 (Schur's test). *Let $\{T_{jk}\}$ be a matrix satisfying*

$$\sup_j \sum_k |T_{jk}| \leq C < \infty \quad \textit{and} \quad \sup_k \sum_j |T_{jk}| \leq C < \infty.$$

Then $\|T\|_{\ell^p \to \ell^p} \lesssim C$ for all $1 \leq p \leq \infty$.

Proof. Let us show the proof in the simplest case $p = 2$ and leave the remaining cases as an exercise. Using Cauchy–Schwarz and exchanging the order of summation,

$$\begin{aligned} \|Tf\|_{\ell^2}^2 &= \sum_j \Big| \sum_k T_{jk} f_k \Big|^2 \\ &\lesssim \sum_j \sum_k |T_{jk}| \, |f_k|^2 \, \sum_\ell |T_{j\ell}| \\ &\lesssim \sup_j \sum_\ell |T_{j\ell}| \cdot \sum_k \sum_j |T_{jk}| |f_k|^2 \\ &\lesssim C \cdot \sum_k |f_k|^2 \cdot \sup_k \sum_j |T_{jk}| \lesssim C^2 \|f\|_{\ell^2}^2, \end{aligned}$$

which yields the result. □

Remark A.3.5. There is a completely analogous result concerning operators of the form $Tf(x) = \int K(x,y)f(y)\,dy$ for some integral kernal $K(x,y)$.

The following is a result from complex analysis. It relies on the **maximum principle** (see e.g. [29]): an analytic (also called holomorphic or complex differentiable) function on a bounded domain in the complex plane attains its maximum on the boundary.

Lemma A.3.6 (The three lines lemma)**.** *Let f be analytic on $\{0 \leq \operatorname{Re} z \leq 1\}$. Suppose f satisfies*

$$|f(z)| \leq e^{C|z|^{2-\delta}}$$

for some $C > 0$ and $\delta > 0$. Suppose that $|f(z)| \leq M_0$ when $\operatorname{Re} z = 0$ and $|f(z)| \leq M_1$ when $\operatorname{Re} z = 1$. Then we have

$$|f(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$$

for all z in the strip.

Proof. First suppose that $M_0 \leq 1$ and $M_1 \leq 1$; we will show that $|f(z)| \leq 1$ on the strip.

To this end, we let $\varepsilon > 0$ and set

$$g(z) = e^{\varepsilon z^2} f(z).$$

This is an analytic function, and because of the hypotheses on f we have that $|g(x+iy)| \to 0$ as $y \to \pm\infty$ for any $0 \leq x \leq 1$.

Therefore, applying the maximum principle on a sufficiently large rectangle $[0,1] \times [-R,R]$, we deduce that $|g(z)| \leq 1$ for all z in the strip. As this holds for arbitrary $\varepsilon > 0$, we may send $\varepsilon \to 0$ to deduce $|f(z)| \leq 1$ on the strip.

For the general case we let $h(z) = f(z)M_0^{z-1}M_1^{-z}$. Then h has exponential bounds similar to f and is bounded by 1 on the boundary of the strip. Therefore the previous analysis shows that $|h(z)| \leq 1$ everywhere, which implies the result. $\square$

Remark A.3.7. One must impose some restrictions on the growth of the function f above. Indeed, consider the analytic function $f(z) = \exp\{-ie^{\pi i z}\}$. Then $|f(x+iy)| = \exp\{e^{-\pi y}\sin \pi x\}$. In particular, $|f(x+iy)| = 1$ for $x \in \{0,1\}$ but f is unbounded for $x \in (0,1)$.

Finally, we recall the Arzelá–Ascoli theorem.

Theorem A.3.8 (Arzelá–Ascoli)**.** *Let $K \subset \mathbb{R}^d$ be compact and let $\{f_n\}$ be a bounded, equicontinuous sequence of functions on $C(K)$. Then $\{f_n\}$ has a uniformly convergent subsequence.*

A.4 Exercises

Exercise A.4.1. Let $1 \le p_1 \le p_2 \le \infty$. Show that

$$\|a\|_{\ell^{p_2}} \le \|a\|_{\ell^{p_1}}$$

for all $a \in \ell^{p_1}$.

Exercise A.4.2. Show that if T is a symmetric linear operator on a Hilbert space, then its eigenvalues are necessarily real.

Exercise A.4.3. Show that

$$\|f\|_{L^\infty} = \inf_{|E|=0} \sup_{x \in E^c} |f(x)|.$$

Exercise A.4.4. Prove Schur's test (Lemma A.3.4) for general $1 \le p \le \infty$.

Exercise A.4.5. Show that the functions in Example A.3.1 form a family of good kernels.

Exercise A.4.6. Let $1 \le p < \infty$. Find the best possible C such that

$$\|f+g\|_{L^{p,\infty}} \le C\big\{\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}}\big\}$$

for all f, g.

Exercise A.4.7. Show that if $f_n \rightharpoonup f$ weakly in a Hilbert space H and $\|f_n\| \to \|f\|$, then $f_n \to f$ strongly in H.

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