



## Scheduling Jobs with Linear Model of Simultaneous Ageing and Learning Effects

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*Abstract.* In the paper, we introduce some new scheduling model in which learning and aging effects are both considered simultaneously. In this model the actual processing time of the jobs depends only on its position in a schedule and can be described by the piecewise linear function. For single-processor problem with introduced model, we show that the problem of minimizing the makespan criterion for independent jobs with release dates is strongly NP-hard, but some special cases of this problem are polynomially solvable. Based on those special cases, we propose 4 heuristic algorithms and we experimentally examine their usefulness for solving the general problem.

*Keywords:* sequencing, single machine, learning effect, ageing effect, computational complexity

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### 1. INTRODUCTION

Classical scheduling models assume that the processing time of a job is a given, fixed value. However, in models of many real life systems we cannot make this assumption. Thus the processing time of a job (or other job parameters) is treated as a variable dependent on the schedule itself. The scheduling models that assume variable jobs processing times include among others: resource dependent processing times (Shabtay and Steiner, 2007; Janiak *et al.*, 2007), deteriorating environments (Cheng *et al.*, 2004; Bachman *et al.*, 2002), etc. Recently, in the scientific literature many papers regarding the learning effect (Biskup, 2008; Janiak and Śnieżyk, 2004), and the ageing effect (Yang and Yang, 2010; Janiak and Rudek, 2010) were published, which also assume that the value of processing time of a job is a variable dependent on the schedule.

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In this paper we introduce some new scheduling model which takes into account both learning, and ageing effect simultaneously. In this model, the actual processing time of a job depends on its position in the schedule and is described by piecewise linear function. This function is dependent on the position of a job in the sequence of jobs and describes three phases of processor efficiency: the learning phase in which processing time of a job decreases, maturity phase in which processing time of a job is constant, and the ageing phase, in which job processing time increases.

On the basis of the introduced model we formulate the single processor scheduling problem with given job release dates and the makespan (the schedule length) minimization objective. We show that considered problem is in general strongly NP-hard, but we identify several special cases of this problem which can be polynomially solved. Based on these special cases we construct 4 heuristic algorithms for solving general case of the problem and verify their efficiency by experimental analysis.

The remainder of the paper is organized as follows. In the next section we shortly present state of research in the domain of scheduling problems with both learning, and ageing effect. For the complete and up-to-date presentation of the results in the area of scheduling with learning effect and ageing effect we refer the reader to Janiak *et al.* (2011). In section 3 we formulate precisely mentioned above model of job processing time and the considered makespan minimization problems. In Section 4 we present some properties of the problem and describe its special, polynomially solvable cases. Section 5 is devoted to the proposed heuristic algorithms together with experiments testing their efficiency. Finally, in Section 6 we conclude the paper pointing out some directions of future research in this subject.

## 2. LITERATURE REVIEW

The learning and ageing phenomena were introduced into the scheduling area in the beginning of the XXI century and since then attracted many researchers all over the world. Both learning effect and ageing effect were initially considered separately and recently there are many attempts to combine both this effect into a single model. In what follows, we present some basic and most important results available in the scheduling literature separately for the learning effect models, ageing effect models, and the models that combine learning and ageing effects. The reader interested in the deeper look into the results presented in the literature in this matter, is referred to the paper by Janiak *et al.* (2011).

The learning effect itself (not in the scheduling area) was firstly observed by Wright (1936) while performing research on the workers efficiency in airplanes factory. He proposed the exponential shape of the learning curve (the curve that describes the increase of worker efficiency over the number of tasks he performed). Another research introduced many other shapes of the learning curve such as the S-shaped curve by Jordan (1965) and Carlson and Rowe (1976). The detailed description and interpretation of the learning curves can be found in (Jaber and Bonney, 1999; Badiru, 1992; Dutton and Thomas, 1984).

The first results regarding the learning effect in scheduling area are due to Biskup (1999) and Cheng and Wang (2000). In (Biskup, 1999) author used the model in-

troduced by Wright (1936), i.e. the exponential, non-increasing job processing time function dependent on its position in the schedule. The properties developed by Biskup were then used by Mosheiov in (Mosheiov, 2001b; Mosheiov and Sidney, 2003; Mosheiov, 2001a). Independently, Cheng and Wang (2000) the linear model of the learning curve was introduced. Future work with linear model of learning curve was conducted in (Bachman and Janiak, 2004). Beside model in which processing time of a job is dependent on its position in the sequence, the model with the processing time dependent on the time that passed since start of schedule to the beginning of job processing were introduced in (Kuo and Yang, 2006). Future work in this area include (Biskup, 2008; Yin *et al.*, 2009). Unfortunately, in the mentioned papers there are not convincing, reasonable real-life examples of application of such scheduling problems with learning effects.

The ageing effect is a very similar phenomenon to the aging effect, however, the function that describes the job processing time is non-decreasing, i.e. the processing of a task takes longer if it is processed later in the sequence. This phenomenon models the degeneration of the processor with the number of jobs it processed or over time. The exponential ageing effect in the scheduling area was introduced by Mosheiov (2001a). Then Janiak and Śnieżyk (2004) and Bachman and Janiak (2004) the linear model of ageing was introduced. More complex models were considered by Wang *et al.* (2009), Cheng *et al.* (2008), and Janiak and Śnieżyk (2005a). Similarly, to the learning effects the time dependent models were considered in (Janiak and Śnieżyk, 2005a; Gawiejnowicz and Kononov, 2010). Again, in all those cited papers the convincing practical application of considered scheduling problems is not presented which puts in question the motivation of the research on the ageing effect in the scheduling area.

The model that includes both learning and ageing effects simultaneously in a single model was introduced in by Lee (2004). He considered a model in which processing time of a job is dependent on its position in the sequence (the learning effect), however, deteriorates with its starting time (ageing effect). The research on similar models (combining the learning effect and deteriorating jobs) were continued by Wang (2007), Yang and Kuo (2009), and Toksari *et al.* (2010). Another kind of model that combines the learning and ageing effects was introduced by Sun (2009). In this model the actual processing time of a job increases with its position in the sequence (the ageing effect), but decreases with the total processing time of jobs already processed (the learning effect). The research on this type of models was continued in (Cheng *et al.*, 2010; Wang and Liu, 2009).

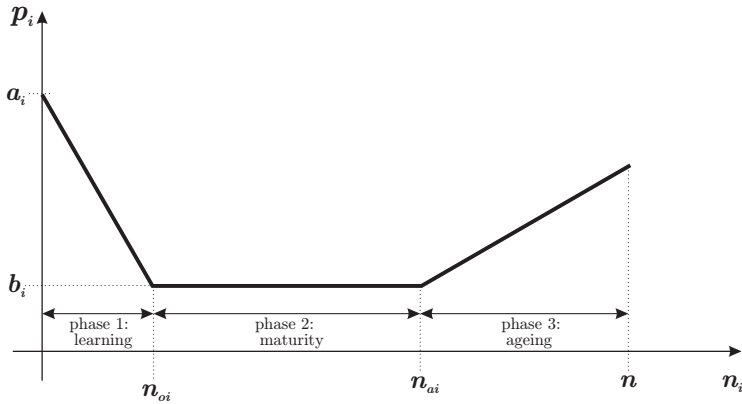
In this section we briefly presented some basic results available in the scheduling literature that deals with the learning and aging phenomena. The reader interested in more detailed survey is again referred to the paper by Janiak *et al.* (2011) which is complete and up-to-date.

The models available in the literature that combine both learning and ageing effects in a single model are very non-realistic (even bizarre) and again lack the convincing real-life system of their application. Moreover, the interpretation of these models is somehow difficult since they are extensions of complex models of learning and ageing effects. Thus, in this paper we introduce a new model of simultaneous learning and ageing that is easy to interpret which can be viewed as a basic linear

model and we hope to find for it use in scheduling problems some convincing real-life applications in computer engineering, or technical, or economical environment.

### 3. PROBLEM FORMULATION

Before we formulate the problem that is considered in the paper, first we introduce the learning-aging model described by the piecewise linear curve, which is depicted in Figure 1.



**Fig. 1.** *The learning-aging curve.*

In the considered model, the processing time of a job (say  $i$ ) which is executed by the machine on the position  $n_i$  in the sequence of  $n$  jobs is given by the following function:

$$p_i(n_i) = a_i - v_i \min\{n_i, n_{oi}\} + w_i \max\{0, n_i - n_{ai}\}, \quad (1)$$

where:

- $a_i > 0$  – the initial processing time of job  $i$ ,
- $v_i \geq 0$  – the learning ratio of job  $i$ ,
- $w_i \geq 0$  – the ageing ratio of job  $i$ ,
- $n_{oi} \geq 1$  – the learning threshold – the number of the last position at which the learning effect can be observed; the end of the learning phase and the beginning of the maturity phase,
- $n_{ai} \leq n$  – the ageing threshold – the number of the first position at which the aging effect can be observed; the end of the maturity phase and the beginning of the aging phase.

In the sequence of  $n$  jobs every job can be scheduled at one of  $n$  positions, so the processing time can have  $n$  (potentially) different values. On the other hand, the processing time of a job can be viewed as a function of the number of jobs processed before this job, which is actually equal to the job position decremented by 1. Looking at the Figure 1, we have to remember that the domain of the function is discrete, not continuous, as the figure could suggest.

On the basis of introduced above model (1) we formulate the following scheduling problem. There is a given set  $J = \{1, \dots, n\}$  of  $n$  non-preemptive jobs to be processed on a single processor. Every job  $i \in J$  is defined by its release date  $r_i \geq 0$ , i.e. the time at which the job is available for processing, and the parameters of its processing time  $p_i(n_i)$  given in (1), i.e. values  $a_i, v_i, w_i, n_{oi}, n_{ai}$ . All the parameters have to take values which ensure that for every position  $n_i$  the processing time has a non-negative value. The objective of the problem is to find a sequence of jobs (i.e. the permutation of the set  $J$ ) such that the schedule length (makespan)  $C_{max} = \max_{i \in J} \{C_i\}$  is minimized, where  $C_i$  is the completion time of job  $i$ . For convenience, by  $S_i$  we denote the starting time of job  $i$ . It is clear that  $C_i = S_i + p_i(n_i)$ .

As mentioned, the sequence of job processing can be defined by a permutation  $\pi = (\pi(1), \dots, \pi(n))$  of the set  $J$ , where  $\pi(k)$  denotes the job scheduled as  $k$ th in this sequence. For a given permutation  $\pi$  the completion time of job scheduled at position  $k$  can be calculated using the following formulae:

$$C_{\pi(k)} = \max\{C_{\pi(k-1)}, r_{\pi(k)}\} + p_{\pi(k)}(k) \quad (2)$$

for  $k = 1, \dots, n$ , where  $\pi(0) = 0$  and  $C_0 = 0$ .

In the following sections, the defined above problem will be denoted as *PLA*.

#### 4. PROBLEM PROPERTIES

Bachman and Janiak (2004) showed that the similar problem with job processing time given as the linear function of job position  $p_i(k) = a_i - b_i k$ , which models the learning effect only, is NP-hard in the strong sense. Similarly Janiak and Śnieżyk (2005b) showed that the problem with processing times given as  $p_i(k) = a_i + b_i k$  is also NP-hard in strong sense. From this facts, since both these problem are the special cases of *PLA*, it follow that *PLA* is also NP-hard in the strong sense. Thus, we can formulate the following corollary.

**Corollary 1** *The problem PLA is NP-hard in strong sense.*

On the other hand, it can be showed that another special case of *PLA* denoted here as *PLA0*, in which  $r_i = 0$  for all  $i \in J$  can be solved in polynomial time.

**Proposition 1** *The problem PLA0 can be solved in polynomial time  $O(n^3)$ .*

**Proof.** Let  $x_{ij}$  are binary variables that indicate that if  $x_{ij} = 1$  then job  $i$  is scheduled at  $j$ th position and  $x_{ij} = 0$  otherwise. In this situation we can formulate the problem *PLA0* as follows.

Minimize:

$$\sum_{i=1}^n \sum_{j=1}^n (a_i - v_i \min\{j, n_{oi}\} + w_i \max\{0, j - n_{ai}\}) x_{ij},$$

subject to:

$$\sum_{i=1}^n x_{ij} = 1, j = 1, \dots, n,$$

$$\sum_{j=1}^n x_{ij} = 1, i = 1, \dots, n,$$

$$x_{ij} \in \{0, 1\}.$$

As it can be seen the above formulation is an assignment problem instance which can be solved in  $O(n^3)$  time (e.g. Papadimitrou and Steiglitz, 1982; Ji *et al.*, 1997). ■

Let  $I$  denote some instance of the problem  $PLA$  and  $I_0$  be the modification of  $I$  such that  $r_i = 0$  for all  $i \in J$ . It is clear that  $I_0$  is an instance of  $PLA0$ . Let  $\pi_0$  denote the optimal solution of  $I_0$ , obtained according to Property 1. If we apply the solution  $\pi_0$  to the original instance  $I$  we will have some feasible but not necessarily optimal solution. However, solution  $\pi_0$  is optimal for  $PLA$  if the following property holds.

**Proposition 2** *If  $S_i \geq r_i$  for all  $i \in J$  in solution  $\pi_0$ , then  $\pi_0$  is the optimal solution to  $PLA$ .*

**Proof.** It is easy to see, that in such case, the release dates of jobs don't have impact on the makespan value. So the obtained solution is the optimal one. ■

Finally, the following two properties, can be proved by simple job interchange argument.

**Proposition 3** *For the problem  $PLA$ , if  $v_i = v$ ,  $n_{oi} = n_o$ ,  $w_i = w$ , and  $n_{ai} = n_a$  for all  $i \in J$  then the optimal solution to the problem can be obtained by ERD rule, i.e., by sorting jobs according the non-decreasing values of their release dates ( $r_i \nearrow$ ).*

**Proof.** Assume that the permutation  $\pi$  is optimal and doesn't meet the statement of the property. Thus, there exists at least one pair of jobs  $\pi(k)$  and  $\pi(k+1)$  in  $\pi$  such that

$$r_{\pi(k)} > r_{\pi(k+1)}. \quad (3)$$

Let  $\pi'$  be a permutation constructed from  $\pi$  such that jobs  $\pi(k)$  and  $\pi(k+1)$  are interchanged. We will show that the completion time of the  $k+1$ th job in  $\pi'$  is less (or at most equal) to the  $k+1$ th job in  $\pi$ , i.e.  $C_{\pi'(k+1)} \leq C_{\pi(k+1)}$ . This observation leads to the conclusion that  $C_{max}(\pi) \geq C_{max}(\pi')$  and  $\pi$  cannot be the optimal permutation.

Taking into account the formulae (2) and the fact that  $C_{\pi(i)} = C_{\pi'(i)}$ , for  $i = 1, \dots, k-1$  we can write the following equations:

$$\begin{aligned} C_{\pi(k)} &= \max\{C_{\pi(k-1)}, r_{\pi(k)}\} + a_{\pi(k)} - v \min\{k, n_0\} + w \max\{0, k - n_a\}, \\ C_{\pi(k+1)} &= \max\{C_{\pi(k)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - v \min\{k+1, n_0\} + w \max\{0, k+1 - n_a\}, \\ C_{\pi'(k)} &= \max\{C_{\pi(k-1)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - v \min\{k, n_0\} + w \max\{0, k - n_a\}, \\ C_{\pi'(k+1)} &= \max\{C_{\pi'(k)}, r_{\pi(k)}\} + a_{\pi(k)} - v \min\{k+1, n_0\} + w \max\{0, k+1 - n_a\}. \end{aligned} \quad (4)$$

We will show that the value  $\Delta = C_{\pi(k+1)} - C_{\pi'(k+1)}$  is always non-negative.

Following (4) we have:

$$\begin{aligned} \Delta &= C_{\pi(k+1)} - C_{\pi'(k+1)} = \\ &= \max\{C_{\pi(k)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - v \min\{k+1, n_0\} + w \max\{0, k+1 - n_a\} - \\ &= \max\{C_{\pi'(k)}, r_{\pi(k)}\} - a_{\pi(k)} + v \min\{k+1, n_0\} - w \max\{0, k+1 - n_a\} = \\ &= \max\{C_{\pi(k)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - \max\{C_{\pi'(k)}, r_{\pi(k)}\} - a_{\pi(k)}. \end{aligned}$$

Noticing that  $C_{\pi(k-1)} < C_{\pi(k)}$  and  $C_{\pi'(k-1)} < C_{\pi'(k)}$ , and taking into account (3) we have to consider the following 4 exhaustive cases:

1)  $r_{\pi(k+1)} < r_{\pi(k)} \leq C_{\pi(k+1)}$

We have

$$\begin{aligned} \Delta &= \max\{C_{\pi(k)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - \max\{C_{\pi'(k)}, r_{\pi(k)}\} - a_{\pi(k)} = \\ &C_{\pi(k)} + a_{\pi(k+1)} - C_{\pi'(k)} - a_{\pi(k)} = \\ &C_{\pi(k-1)} + a_{\pi(k)} - v \min\{k, n_0\} + w \max\{0, k - n_a\} + a_{\pi(k+1)} - \\ &C_{\pi(k-1)} - a_{\pi(k+1)} + v \min\{k, n_0\} - w \max\{0, k - n_a\} - a_{\pi(k)} = 0. \end{aligned}$$

2)  $r_{\pi(k+1)} \leq C_{\pi(k-1)} < r_{\pi(k)} \leq C_{\pi'(k)}$

We have

$$\begin{aligned} \Delta &= \max\{C_{\pi(k)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - \max\{C_{\pi'(k)}, r_{\pi(k)}\} - a_{\pi(k)} = \\ &C_{\pi(k)} + a_{\pi(k+1)} - C_{\pi'(k)} - a_{\pi(k)} = \\ &r_{\pi(k)} + a_{\pi(k)} - v \min\{k, n_0\} + w \max\{0, k - n_a\} + a_{\pi(k+1)} - \\ &C_{\pi(k-1)} - a_{\pi(k+1)} + v \min\{k, n_0\} - w \max\{0, k - n_a\} - a_{\pi(k)} = \\ &r_{\pi(k)} - C_{\pi(k-1)} > 0. \end{aligned}$$

3)  $r_{\pi(k+1)} \leq C_{\pi(k-1)} < C_{\pi'(k)} \leq r_{\pi(k)}$

We have

$$\begin{aligned} \Delta &= \max\{C_{\pi(k)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - \max\{C_{\pi'(k)}, r_{\pi(k)}\} - a_{\pi(k)} = \\ &C_{\pi(k)} + a_{\pi(k+1)} - r_{\pi(k)} - a_{\pi(k)} = \\ &r_{\pi(k)} + a_{\pi(k)} - v \min\{k, n_0\} + w \max\{0, k - n_a\} + a_{\pi(k+1)} - r_{\pi(k)} - a_{\pi(k)} = \\ &a_{\pi(k+1)} - v \min\{k, n_0\} + w \max\{0, k - n_a\} = p_{\pi(k+1)} > 0. \end{aligned}$$

4)  $C_{\pi(k-1)} < r_{\pi(k+1)} < r_{\pi(k)} \leq C_{\pi'(k)}$

We have

$$\begin{aligned} \Delta &= \max\{C_{\pi(k)}, r_{\pi(k+1)}\} + a_{\pi(k+1)} - \max\{C_{\pi'(k)}, r_{\pi(k)}\} - a_{\pi(k)} = \\ &C_{\pi(k)} + a_{\pi(k+1)} - C_{\pi'(k)} - a_{\pi(k)} = \\ &r_{\pi(k)} + a_{\pi(k)} - v \min\{k, n_0\} + w \max\{0, k - n_a\} + a_{\pi(k+1)} - \\ &r_{\pi(k+1)} - a_{\pi(k+1)} + v \min\{k, n_0\} - w \max\{0, k - n_a\} + a_{\pi(k)} = \\ &r_{\pi(k)} - r_{\pi(k+1)} > 0. \end{aligned}$$

We have shown, that in every case the value of  $\Delta$  is non-negative, which completes the proof.  $\blacksquare$

**Proposition 4** For the problem PLA, if  $v_i = v$ ,  $n_{oi} = n_o$ ,  $w_i = w$ , and  $r_i = 0$  for all  $i \in J$  then the optimal solution to the problem can be obtained by sorting jobs according the non-decreasing values of their parameters  $n_{ai}$  ( $n_{ai} \nearrow$ ).

**Proof.** The proof is very similar to the proof of Property 3 and therefore will be omitted.  $\blacksquare$

## 5. HEURISTIC ALGORITHMS

The considered problem is strongly NP-hard so it is highly unlikely to find an algorithm which can solve this problem in polynomial time. Therefore we propose a few efficient heuristics to find an optimal solution. To solve the problem we constructed four algorithms based on Properties 1–4. These algorithms are denoted respectively as  $r_i \nearrow$ ,  $n_{ai} \nearrow$ , *ASSIGNMENT*, and *NEH*.

In the  $r_i \nearrow$  algorithm ties are broken according to the non-decreasing values of the parameter  $v_i n_{oi} - w_i n_{ai}$ . In the  $n_{ai} \nearrow$  algorithm ties are broken according to the non-increasing values of the product  $w_i n_{ai}$ . The *NEH* algorithm is a direct adaptation of the insertion procedure proposed by Nawaz *et al.* (1983).

All tests were made on the computer with Intel Core<sup>TM</sup> 2 Duo 3.00GHz, 2GB RAM and Windows 7. We constructed four sets of instances with different parameters of jobs. For every set, we described the interval of numbers for every parameter of job in the instance. Parameters for every particular job in the instance are uniformly distributed from those interval.

**Test set 1** (small  $r_i$  values compared with  $a_i$ , small  $v_i$  and  $w_i$  values compared with  $a_i$ )

$$r_i \in [10, 50n], a_i \in [50, 100], v_i \in [0.1, 1], w_i \in [0.1, 1], n_{oi} \in [0, n/2-1], n_{ai} \in [n/2, n-1]$$

where  $n$  is a number of jobs.

**Test set 2** (large  $r_i$  values compared with  $a_i$ , small  $v_i$  and  $w_i$  values compared with  $a_i$ )

$$r_i \in [10, 100n], a_i \in [50, 100], v_i \in [0.1, 1], w_i \in [0.1, 1], n_{oi} \in [0, n/2-1], n_{ai} \in [n/2, n-1]$$

where  $n$  is a number of jobs.

**Test set 3** (small  $r_i$  values compared with  $a_i$ , large  $v_i$  and  $w_i$  values compared with  $a_i$ )

$$r_i \in [10, 8n], a_i \in [5, 10], v_i \in [0.1, 1], w_i \in [0.1, 1], n_{oi} \in [0, n/2-1], n_{ai} \in [n/2, n-1]$$

where  $n$  is a number of jobs.

**Test set 4** (large  $r_i$  values compared with  $a_i$ , large  $v_i$  and  $w_i$  values compared with  $a_i$ )

$$r_i \in [10, 40n], a_i \in [5, 10], v_i \in [0.1, 1], w_i \in [0.1, 1], n_{oi} \in [0, n/2-1], n_{ai} \in [n/2, n-1]$$

where  $n$  is a number of jobs.

To compare the efficiency of the algorithms, we define the relative gap to the best (optimal) objective value as:

$$\frac{C_{max}^A - C_{max}^{OPT}}{C_{max}^{OPT}} \times [100\%],$$

where  $C_{max}^A$  is a criterion value found by an algorithm

$A \in \{r_i \nearrow, n_{ai} \nearrow, \textit{ASSIGNMENT}, \textit{NEH}\}$ , for a given instance, and  $C_{max}^{OPT}$  is an optimal value of the criterion for an instance with  $n \leq 9$ , which was generated by an exhaustive search method, and the best found value found by all the algorithms for  $n > 9$ .



**Table 1.** The efficiency of the algorithms for the instances of the Test set 1.

n	$r_i \nearrow$	$n_{oi} \nearrow$	ASSIGN	NEH
9	0.2	30.6	1.6	1.2
50	0.2	51.8	1.0	6.8
100	0.3	59.9	0.5	9.9
average	0.2	47.4	1.0	6.0

**Table 2.** The efficiency of the algorithms for the instances of the Test set 2.

n	$r_i \nearrow$	$n_{oi} \nearrow$	ASSIGN	NEH
9	0.0	35.9	5.9	2.2
50	0.0	49.3	10.8	8.5
100	0.0	48.6	11.9	8.6
average	0.0	44.6	9.5	6.4

**Table 3.** The efficiency of the algorithms for the instances of the Test set 3.

n	$r_i \nearrow$	$n_{oi} \nearrow$	ASSIGN	NEH
9	0.4	37.5	5.0	2.7
50	0.0	44.5	9.8	20.4
100	0.1	48.1	11.0	31.5
average	0.2	43.4	8.6	18.2

**Table 4.** The efficiency of the algorithms for the instances of the Test set 4.

n	$r_i \nearrow$	$n_{oi} \nearrow$	ASSIGN	NEH
9	0.0	8.4	1.5	0.1
50	0.0	8.0	2.5	1.9
100	0.0	10.1	3.2	4.5
average	0.0	8.8	2.4	2.2

**Table 5.** The average efficiency of the algorithms for the instances of all the Test sets.

n	$r_i \nearrow$	$n_{oi} \nearrow$	ASSIGN	NEH
average	0.1	36.1	5.4	8.2

The results of tests for every set are presented in Tables 1 to 4. For every set, algorithm and values of  $n \in \{9, 50, 100\}$ , we calculated a performance ratio. Last row of the table is an average performance ratio over the test set for every algorithm. In Table 5, we can find the values of total performance ratio, which is an average value of average performance ratios over all test sets.

According to our expectations, the efficiency of algorithms which are based on the release date values is the best. The rest of the algorithms are based on different parameters of jobs and the release dates do not influence on the order of jobs in a solution. If there is a job with large value of release date and the algorithm schedule this job on the first position, the solution generated by this algorithm can be far from an optimal one. The efficiency of those algorithms depends on the relation between parameters of the job, but it increases if  $r_i$  values are small in comparison with  $a_i$ .

## 6. CONCLUSIONS

In this paper we introduced a new model of scheduling problems, which takes into account the learning and ageing effects simultaneously. In this model the processing time of a job is described by piecewise linear function of its position in the jobs sequence and consists of three phases of the processor efficiency: the learning phase, the maturity phase, and the ageing phase. On the basis of this model we formulated the single machine makespan minimization problem and showed its computational complexity (strong NP-hardness) as well as its special cases that can be solved in polynomial time.

On the basis on showed problem properties we proposed 4 heuristic algorithms to solve general case of the problem. The efficiency of proposed algorithms is verified by experimental analysis and shows their practical applicability in the environments that accepts relative error about 10 percent to the optimal solution.

The model introduced in this paper can be extended to other machine environments (such as parallel machines, shop problems), as well as introduce more complex (nonlinear) shapes of the learning-ageing functions. However, it would be sense to consider (from practical point of view) such kind of problems if some reasonable real-life examples of applications of the considered models are found.

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