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A harmonic M-factorial function and applications

Reginald Alfred Brigham II

A HARMONIC M-FACTORIAL FUNCTION AND APPLICATIONS

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REGINALD ALFRED BRIGHAM II

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Ilene H. Morgan
Stephen Clark
David Grow
Robert P. Roe
Michael Hilgers

ABSTRACT

We offer analogs to the falling factorial and rising factorial functions for the set of harmonic numbers, as well as a mixed factorial function called the M-factorial. From these concepts, we develop a harmonic analog of the binomial coefficient and an alternate expression of the harmonic exponential function and establish several identities. We generalize from the harmonic numbers to a general time scale and demonstrate how solutions to some second order eigenvalue problems and partial dynamic equations can be constructed using power series built from the M-factorial function.

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If it were not for the love, prayers, and support of these people and many others, I would not be here today.

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$\sqcup$ disjoint union, page 28

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f^{Δ} delta derivative of f over the relevant time scale, page 4

μ graininess function, page 7

ρ backward jump operator, page 8

σ forward jump operator, page 7

L left shift operator for sequences, page 24

H_k^n generalized harmonic number, page 15

$H_k{}^{\{n,M\}}$ harmonic M-factorial function, page 29

$H_k{}^{\{\overline{n}\}}$ harmonic rising factorial function, page 22

$H_k{}^{\{\underline{n}\}}$ harmonic falling factorial function, page 18

$g_M^n(t, s)$ M-factorial function over $\mathbb{T}$ with $t, s \in \mathbb{T}$, page 72

$x^{\underline{n}}$ falling factorial $x(x-1)\cdots(x-n+1)$, page 13

$\mathcal{H}(k, n)$ an alternative generalized harmonic number, page 16

H_k kth harmonic number, page 1

$\mathcal{R}_k^{(r)}$ hyperharmonic number, page 16

$[n]$ q-analog of n, page 15

$x^{\overline{n}}$ rising factorial $x(x+1)\cdots(x+n-1)$, page 14

T_k kth triangular number, page 80

$\cos_{\mathbb{H}}$ harmonic cosine function, page 60

$\cos_{\mathbb{T}}$ generalized cosine function on the time scale $\mathbb{T}$, page 87

e_H harmonic exponential function, page 56

$e_p(t,s)$ generalized exponential function, page 11

$\sin_{\mathbb{H}}$ harmonic sine function, page 59

$\sin_{\mathbb{T}}$ generalized sine function on the time scale $\mathbb{T}$, page 87

$(x \ominus y)^{\{n,M,s\}}$ M-factorial of a difference over $\mathbb{T}_1$ and $\mathbb{T}_2$, page 93

$(x \oplus y)^{\{n,M,0\}}$ M-factorial of a sum over $\mathbb{T}_1$ and $\mathbb{T}_2$, page 95

M shifting sequence, page 24

M^{*_n} n-adjoint of M, page 26

M^{T_n} n-transpose of M, page 25

$J(M,n,i,k)$ set of all M-increasing n-tuples bounded below by i and above by $k-1$, page 27

$\binom{H_k}{n}_{\mathbb{H}}$ harmonic binomial coefficient, page 38

$\binom{H_k}{n}_{\mathbb{H},M}$ M-shifted harmonic binomial coefficient, page 47

$\begin{bmatrix} k \\ n \end{bmatrix}_r$ r-Stirling number, page 16

$\begin{bmatrix} k \\ n \end{bmatrix}$ Stirling cycle number, page 6

$\begin{Bmatrix} k \\ n \end{Bmatrix}$ Stirling subset number, page 81

$\binom{t}{n}_{\mathbb{T},M}$ M-shifted binomial coefficient over $\mathbb{T}$, page 78

$\mathbb{H}$ the set of harmonic numbers, page 1

$h\mathbb{Z}$ the set of all integer multiples of a constant h, page 3

$\mathbb{N}$ the set of natural numbers $\{1, 2, 3, \ldots\}$, page 15

$\mathbb{N}_0$ the set of whole numbers $\{0, 1, 2, \ldots\}$, page 7

$\mathbb{Q}$ the set of quantum numbers, page 7

$\mathbb{R}$ the set of real numbers, page 3

$\mathbb{T}$ an arbitrary time scale, page 6

$\mathbb{Z}$ the set of integers, page 1

1. INTRODUCTION

The harmonic numbers have been studied extensively, cf. [5], [9]. The harmonic numbers are formed by taking partial sums of the harmonic series. For $k \in \mathbb{Z}$, $k \geq 0$, the kth harmonic number is given by

$$H_k = \sum_{j=1}^{k} \frac{1}{j}.$$

Note that if $k = 0$, this produces the empty sum, which we take to be equal to zero.

We will denote by $\mathbb{H}$ the set of all harmonic numbers. In Section 1.1, we will treat $\mathbb{H}$ as the domain of a class of functions and define a harmonic derivative on these functions. This is similar to, and motivated by, the definition of the quantum numbers and the q-derivative, cf. [12]. Throughout, we will use notation from the study of time scales, so we summarize the critical definitions in Section 1.3. In Section 1.4 we provide an overview of power rules for other finite derivatives.

Harmonic numbers have been generalized in several ways resembling powers. The classic example of this type of generalization, of course, is to take partial sums of the zeta function evaluated at positive integers. In Section 1.5 we present several examples and illustrate how each of these generalizations behaves under harmonic differentiation.

In Section 2 we define harmonic falling and rising factorial functions which obey appropriate power rules, and we show that the two have a relationship which is analogous to the relationship between the classic falling and rising factorial functions. Not content with factorials that fall and rise uniformly, we turn our attention to harmonic mixed factorial functions, defining them so that they behave sometimes like a falling and sometimes a rising factorial function. We call these mixed factorial functions M-factorials.

In Section 3, we use the harmonic M-factorial functions to define some harmonic binomial coefficients.

Zave [22] first gave an expression for what we call the harmonic binomial coefficient $\binom{H_k}{n}_{\mathbb{H}}$ in terms of the Bell polynomials. His notation was unwieldy; it considered his generalization to be a function of n generalized harmonic numbers. Spieß [17] simplified the notation and clarified Zave's concept. He explicitly stated an iterated sum form similar to the one we give in Definition 2.1 and noted the connection to the Stirling cycle numbers that we provide in Lemma 3.2. Both works treat this concept as a generalized harmonic number, but we have also drawn the connection to the harmonic derivative and clarified the distinction between its uses as binomial coefficient and falling factorial analog. In addition, we prove analogs to several classic identities on the binomial coefficients, which we believe to be new and relevant to the role of $\binom{H_k}{n}_{\mathbb{H}}$ as a binomial function analog. These identities include the upper summation identity and the parallel summation identity, cf. [9]. We also provide some analogs to the binomial formula and prove an analog to the Chu-Vandermonde identity.

Similarly, Chu and Yan's generalized harmonic number [4], which we discuss in Sections 1.4 and 2.2, is very much an analog of the shifted binomial coefficient $\binom{k+n-1}{n}$. Through the introduction of the M-factorial and M-shifted binomial coefficient, we unify these two kinds of generalized harmonic numbers as extremes on a spectrum of possibilities.

In Section 4, we construct a series representation of the harmonic exponential function and show that it agrees with the time scales definition of an exponential function on $\mathbb{H}$. We present harmonic polynomials and power series, and illustrate how harmonic analogs of the sine and cosine functions can be constructed using harmonic power series. For the second order dynamic eigenvalue problem with Dirichlet and Neumann boundary conditions, respectively:

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f(\alpha) = f(\beta) = 0, \text{ and} \tag{1.1}$$

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f^{\Delta}(\alpha) = \beta^{\Delta}(L) = 0, \tag{1.2}$$

we provide solutions in the harmonic case. Then, in Section 5, we generalize to all time scales. Alternate definitions of time scales special functions including sine and cosine have been made in [2] and [6]; however, those definitions do not produce functions which are solutions to the above boundary value problems. Solutions to these boundary value problems for the time scales $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ are well known, but we are unable to find solutions in the literature for a general time scale.

In Section 5, we also use the sine and cosine function analogs which are solutions to those two boundary value problems to give Fourier series type solutions to the second order partial dynamic equations for waves and diffusion:

(i) Wave Equation: $u^{\Delta_t \Delta_t \sigma_x} = c^2 u^{\Delta_x \Delta_x \sigma_t}$, and

(ii) Diffusion Equation: $u^{\Delta_t \sigma_x} = k u^{\Delta_x \Delta_x}$

with appropriate initial and boundary conditions. Other partial dynamic equations by the same name are considered in [11]. For example, Jackson refers to the Wave Equation as $u^{\Delta_t \Delta_t} = c^2 u^{\Delta_x \Delta_x}$ without the σ_x or σ_t. Finally, we construct M-factorial functions and M-shifted binomial coefficients in which arguments may come from two different time scales and extend the Vandermonde convolution to this case.

1.1. HARMONIC NUMBERS

Considering functions defined on $\mathbb{H}$, we shall develop a harmonic calculus. We begin by constructing one of the most basic operators: the harmonic differential.

Definition 1.1. Consider an arbitrary function $f : \mathbb{H} \to \mathbb{R}$. The *harmonic differential* of f is

$$d_H f(H_k) = f(H_{k+1}) - f(H_k).$$

Example 1.2. As an exercise, we calculate $d_H H_k$:

$$d_H H_k = H_{k+1} - H_k = \frac{1}{k+1},$$

a result that will be useful momentarily

With the notion of a harmonic differential, it is now possible to define a harmonic derivative.

Definition 1.3. Let $f : \mathbb{H} \to \mathbb{R}$. Then the *harmonic derivative* of f, f^{Δ}, is defined by

$$f^{\Delta}(H_k) = \frac{d_H f(H_k)}{d_H H_k}.$$

Applying Definition 1.1 to the above, we see that

$$f^{\Delta}(H_k) = \frac{f(H_{k+1}) - f(H_k)}{H_{k+1} - H_k}$$

which, thanks to Example 1.2, reduces to

$$f^{\Delta}(H_k) = (k+1)\left(f(H_{k+1}) - f(H_k)\right). \tag{1.3}$$

Some of the properties of the harmonic derivative follow from straightforward calculations. Here we demonstrate the harmonic derivative of the product and quotient of two functions f and g. We begin with the definition,

$$[f(H_k)g(H_k)]^{\Delta} = (k+1)[f(H_{k+1})g(H_{k+1}) - f(H_k)g(H_k)].$$

Adding and subtracting the appropriate terms, we obtain

$$[f(H_k)g(H_k)]^{\Delta} = (k+1)[f(H_{k+1})d_H g(H_k) + d_H f(H_k)g(H_k)].$$

Finally, by distributing the $k+1$, a product rule emerges:

$$[f(H_k)g(H_k)]^\Delta = f(H_{k+1})g^\Delta(H_k) + f^\Delta(H_k)g(H_k). \tag{1.4}$$

Alternatively, by interchanging f and g, we obtain

$$[f(H_k)g(H_k)]^\Delta = f(H_k)g^\Delta(H_k) + f^\Delta(H_k)g(H_{k+1}). \tag{1.5}$$

In order to take the harmonic derivative of the quotient $\dfrac{f(H_k)}{g(H_k)}$ we will use (1.4) to differentiate

$$g(H_k) \cdot \frac{f(H_k)}{g(H_k)} = f(H_k). \tag{1.6}$$

This yields

$$g(H_{k+1})\left[\frac{f(H_k)}{g(H_k)}\right]^\Delta + g^\Delta(H_k) \cdot \frac{f(H_k)}{g(H_k)} = f^\Delta(H_k),$$

which we can rearrange to see

$$\left[\frac{f(H_k)}{g(H_k)}\right]^\Delta = \frac{f^\Delta(H_k)g(H_k) - g^\Delta(H_k)f(H_k)}{g(H_{k+1})g(H_k)}. \tag{1.7}$$

If instead we had used the product rule in (1.5) to differentiate (1.6), then we would have obtained an alternate form of the quotient rule:

$$\left[\frac{f(H_k)}{g(H_k)}\right]^\Delta = \frac{f^\Delta(H_k)g(H_{k+1}) - g^\Delta(H_k)f(H_{k+1})}{g(H_{k+1})g(H_k)}. \tag{1.8}$$

1.2. PERMUTATIONS AND STIRLING NUMBERS

A permutation of k items which maps the ith element to p_i can be represented $(1, 2, \ldots, k) \to (p_1, p_2, \ldots, p_k)$. The same permutation is often represented as $p_1 p_2 \ldots p_k$. For example, 4132 is the permutation that maps $1 \to 4$, $2 \to 1$, $3 \to 3$, and $4 \to 2$.

All permutations can be written as the composition of disjoint cycles. In our example, we can write $4132 = (142)(3)$, since the permutation maps items 1 to 4, 4 to 2, 2 to 1, and leaves 3 alone. The number $\left[{k \atop n} \right]$ of permutations of k items with exactly n cycles is called the *Stirling cycle number* or the unsigned Stirling number of the first kind.

Stirling cycle numbers have a triangle recurrence analogous to the one for the binomial coefficients and a connection to the harmonic numbers (cf. [3], [9], and [18]).

Lemma 1.4. *For $k > 0$,*

$$\left[{k+1 \atop n+1} \right] = k \left[{k \atop n+1} \right] + \left[{k \atop n} \right].$$

Lemma 1.5. *A permutation of k items will have H_k cycles on average. That is,*

$$\frac{1}{k!} \sum_{j=0}^{k} j \left[{k \atop j} \right] = H_k.$$

1.3. TIME SCALES

The standard calculus of functions on $\mathbb{R}$ and the difference calculus of functions defined on $\mathbb{Z}$ have been extensively studied, cf. [13], [14], and [20]. The study of time scales has attempted to bridge the real and finite calculi by considering functions defined on an arbitrary subset $\mathbb{T}$ of the real numbers that is both nonempty and closed. Such a $\mathbb{T}$ is called a *time scale*. A derivative is then taken by use of a modified difference quotient, where the differences are based on the position of points in the time scale rather than at a set distance. Limits are taken as necessary. Time scales calculus was first developed in 1988 by Stephan Hilgers [10] and has been greatly expanded in recent years.

The set of quantum numbers $\mathbb{Q} = \{q^n | n \in \mathbb{N}_0\} \cup 0$ for a fixed $q > 0$ $(q \neq 1)$ is one example of a time scale that has been studied extensively, cf. [12]. The set of harmonic numbers presents another interesting, and often overlooked, time scale. In this time scale, there are many results that are analogous results to familiar ones from differential and difference calculus; some of them are previously known, while some will be derived beginning in Section 2.

Several of the rules in Section 1.1, including the product rule and quotient rule, can be shown for a general time scale. With a product rule and a quotient rule it is natural to wonder about a power rule and a chain rule for the harmonic derivative. A chain rule can be constructed for certain time scales such as $\mathbb{H}$. It will require that an intermediate time scale $\mathbb{T}$ be considered so that $h : \mathbb{H} \to \mathbb{R}$ decomposes as $h = f \circ g$ where $g : \mathbb{H} \to \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. We will derive a chain rule for all time scales, such as $\mathbb{H}$, having a particular property. In order to consider an arbitrary time scale, we need some notation. First we define the forward jump operator.

Definition 1.6. Let $\mathbb{T}$ be a time scale. If t is an element of $\mathbb{T}$, then we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

If $\sigma(t) > t$, then t is said to be *right-scattered*. We say that a time scale $\mathbb{T}$ is right-scattered if, for all $t \in \mathbb{T}$, t is right-scattered. Note that $\mathbb{H}$, as well as many classic time scales such as $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$, are right-scattered. In the set of harmonic numbers, the forward jump operator σ is related to the right shift operator E, commonly used in the study of difference equations, in the following manner: if $t = H_k$, then

$$\sigma(t) = E(H_k) = H_{k+1}.$$

The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

In the time scales literature it is common to see the the function $f^\sigma = f \circ \sigma$, that is

$$f^\sigma(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T}.$$

Definition 1.7. Let $\mathbb{T}$ be a time scale. If t is an element of $\mathbb{T}$, then we define the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

Points in $\mathbb{T}$ may be classified in a variety of ways other than right-scattered. These classifications are summarized in Table 1.1.

Table 1.1. Classification of Points

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$t > \rho(t)$
t left-dense	$t = \rho(t)$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

To discuss the delta derivative we need to consider the set $\mathbb{T}^\kappa$ derived from the time scale $\mathbb{T}$ as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

The delta derivative on a time scale at a right-scattered point t is defined in the same manner as we defined the harmonic derivative; at right-dense points, the definition is in the same manner as the ordinary derivative.

Definition 1.8. If $f : \mathbb{T} \to \mathbb{R}$ is continuous at t, where t is a right-scattered element of $\mathbb{T}^\kappa$, then the *delta derivative* of f exists and is

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

Definition 1.9. If $f : \mathbb{T} \to \mathbb{R}$ and t is a right-dense element of $\mathbb{T}^\kappa$, then the *delta derivative* of f exists and is

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

Ordinary power functions exhibit the following bound under delta differentiation (found in [7]).

Lemma 1.10. *For $t, s \in \mathbb{T}$ with $t \geq s$ and $n \in \mathbb{N}_0$,*

$$((t - s)^{n+1})^\Delta \geq (n + 1)(t - s)^n.$$

Lemma 1.11 (Chain Rule)**.** *Let $\mathbb{T}$ be a right-scattered time scale and $g : \mathbb{T} \to \mathbb{R}$ be an increasing function. If $\mathbb{T}^* = g(\mathbb{T})$, $f : \mathbb{T}^* \to \mathbb{R}$, and $h = f \circ g$, then $h : \mathbb{T} \to \mathbb{R}$. If Δ and Δ^* denote the delta derivatives on $\mathbb{T}$ and $\mathbb{T}^*$, respectively, then*

$$h^\Delta(t) = f^{\Delta^*}(g(t))g^\Delta(t).$$

Proof. Clearly, $\mathbb{T}^* = g(\mathbb{T})$ is also a right-scattered time scale. For any $s \in \mathbb{T}^*$, there exists $t \in \mathbb{T}$ such that $s = g(t)$. Let σ and σ^* denote the forward jump operators on $\mathbb{T}$ and $\mathbb{T}^*$, respectively. For $h : \mathbb{T} \to \mathbb{R}$ as defined above, we have

$$h^{\Delta}(t) = \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} = \frac{h(\sigma(t)) - h(t)}{g(\sigma(t)) - g(t)} \cdot \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t}.$$

Since g is increasing, $g(\sigma(t)) = \sigma^*(s)$. Therefore,

$$h^{\Delta}(t) = \frac{f(\sigma^*(s)) - f(s)}{\sigma^*(s) - s} \cdot \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} = f^{\Delta^*}(s)g^{\Delta}(t) = f^{\Delta^*}(g(t))g^{\Delta}(t).$$

□

Example 1.12. Let $\mathbb{T}$ be a right-scattered time scale, and define $\mathbb{T}^* = 2\mathbb{T}$. If we let $f : \mathbb{T}^* \to \mathbb{R}$ and $h(t) = f(2t)$, then $g(t) = 2t$ and

$$h^{\Delta}(t) = \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} = \frac{h(\sigma(t)) - h(t)}{2\sigma(t) - 2t} \cdot \frac{2\sigma(t) - 2t}{\sigma(t) - t}.$$

Since $2\sigma(t) = \sigma^*(2t)$,

$$h^{\Delta}(t) = \frac{f(\sigma^*(2t)) - f(2t)}{\sigma^*(2t) - 2t} \cdot \frac{2\sigma(t) - 2t}{\sigma(t) - t} = 2f^{\Delta^*}(2t).$$

We now provide some more definitions from the standard literature on time scales, cf. [2], that will be needed to discuss antiderivatives and integrals in a general time scale.

Definition 1.13. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided the right hand limit of f exists at all right-dense points of $\mathbb{T}$ and the left hand limit of f exists at all left-dense points of $\mathbb{T}$.

Definition 1.14. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at all right-dense points of $\mathbb{T}$ and the left hand limit of f exists at all left-dense points of $\mathbb{T}$.

Definition 1.15. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^{\kappa}.$$

In the time scales literature, the so-called cylinder transformation (cf. [2]) is used to define a generalized exponential function. However, Theorems 2.33 and 2.35 from [2] allow us to define the exponential function $e_p : \mathbb{T}^2 \to \mathbb{R}$ as the unique solution of a first order linear dynamic initial value problem.

Definition 1.16. Let $p : \mathbb{T} \to \mathbb{R}$ be regressive and rd-continuous. For each $s \in \mathbb{T}$, let $y_s : \mathbb{T} \to \mathbb{R}$ be the unique solution of the initial value problem

$$y^{\Delta}(t) = p(t)y, \quad y(s) = 1.$$

Then the *generalized exponential function* $e_p : \mathbb{T}^2 \to \mathbb{R}$ is given by

$$e_p(t, s) = y_s(t).$$

Example 1.17. If $\mathbb{T} = \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ is continuous, then the generalized exponential function is given by

$$e_p(t, s) = e^{\int_s^t p(\tau)d\tau}$$

where the exponential on the right side of the equation is the usual exponential function. Therefore, when $p(t) = 1$ and $s = 0$ we have

$$e_1(t, 0) = e^t.$$

Example 1.18. Let $\mathbb{T} = \mathbb{H}$ and $\alpha \in \mathbb{R}$. As shown in [2], the generalized exponential function e_α can be expressed as a binomial coefficient in the following way:

$$e_\alpha(H_k, 0) = \binom{k+\alpha}{k}. \tag{1.9}$$

Specifically, if $\alpha = 1$,

$$e_1(H_k, 0) = \binom{k+1}{k} = k + 1. \tag{1.10}$$

This function is analogous to the ordinary exponential function e^t, in that they solve the same initial value problem with respect to the delta derivative on their respective time scales.

Definition 1.19. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is said to be *pre-differentiable* with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each point $t \in D$.

Theorem 1.20 (Mean Value Theorem)**.** *[2, Corollary 1.68] Suppose f and g are pre-differentiable with D. If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then*

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U^\kappa \cap D} |f^\Delta(t)| \right\} |s - r|.$$

Theorem 1.21. *[2, Theorem 1.70] Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad \textit{for all} \quad t \in D.$$

Any such function F is called a pre-antiderivative *of f. We call F an* antiderivative *of f if $D = \mathbb{T}^\kappa$.*

Definition 1.22. Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function and let F be a pre-antiderivative of f. Then the *indefinite integral* of f is

$$\int f(t)\,\Delta t = F(t) + C,$$

where C is an arbitrary constant. We define the *Cauchy integral* of f,

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \qquad \text{for all } a, b \in \mathbb{T}.$$

Example 1.23. Consider $f : \mathbb{H} \to \mathbb{R}$. Since all points of $\mathbb{H}$ are isolated, f is rd-continuous, and therefore regulated. The Cauchy integral of f from $a = H_l$ to $b = H_m$ is

$$\int_a^b f(t)\,\Delta t = \sum_{j=l}^{m-1} \frac{f(H_j)}{j+1}.$$

We continue now with a discussion of power rules in finite calculus.

1.4. POWER RULES FOR FINITE CALCULUS

When considering derivatives in finite calculus, it is beneficial to have a power rule analogous to that in differential calculus $D[x^n] = nx^{n-1}$. For the ordinary difference operator, this can be achieved by considering the falling factorial function. For a thorough discussion of the topic, see [9] and [13].

Definition 1.24. If $x \in \mathbb{R}$ and n is a nonnegative integer, then the *falling factorial function* $x^{\underline{n}}$ is defined as

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1),$$

when $n \geq 1$, and $x^{\underline{0}} = 1$ for all x.

More generally, the falling factorial function can be defined as

$$x^{\underline{n}} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)},$$

but we will not require such generality here.

A quick calculation (with either definition) shows that, indeed,

$$\Delta[x^{\underline{n}}] = nx^{\underline{n-1}}. \tag{1.11}$$

We can define a rising factorial function $x^{\overline{n}}$ in a manner similar to Definition 1.24.

Definition 1.25. If $x \in \mathbb{R}$ and n is a nonnegative integer, then the *rising factorial function* $x^{\overline{n}}$ is defined as

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1),$$

when $n \geq 0$, and $x^{\overline{0}} = 0$ for all x.

Again, a quick calculation yields a variation of the classic power rule,

$$\Delta[x^{\overline{n}}] = n(x+1)^{\overline{n-1}}.$$

Alternatively, one could define the rising factorial power in terms of the falling factorial power, either by shifting terms or negating them.

Lemma 1.26. *If $x \in \mathbb{R}$ and n is a nonnegative integer, then the rising factorial function $x^{\overline{n}}$ and falling factorial function $(x+n-1)^{\underline{n}}$ satisfy:*

(i) $x^{\overline{n}} = (x+n-1)^{\underline{n}}$;

(ii) $x^{\overline{n}} = (-1)^n(-x)^{\underline{n}}$.

Proof. From their respective definitions,

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1) = (x+n-1)^{\underline{n}}, \quad \text{and}$$

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1) = (-1)^n(-x)(-x-1)\cdots(-x-n+1) = (-1)^n(-x)^{\underline{n}}.$$

□

The issue of notation for these functions has not been settled in the literature. We choose the under/overbar notation because it resembles a power, and has an intuitive distinction as to which is rising and which is falling.

Other finite derivatives have similar power rules. For instance, in the quantum calculus (q-calculus), using the standard notations, we can make the following definitions.

Definition 1.27. For $q \neq 1$ and $n \in \mathbb{N}$, the *q analog of n* is

$$[n] = \frac{q^n - 1}{q - 1}.$$

Notice that since $[n] = q^{n-1} + q^{n-2} + \cdots + 1$, the $\lim_{q\to 1}[n] = n$.

The q-derivative is defined as the delta derivative for the time scale $\mathbb{T} = q^{\mathbb{N}_0}$,

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{1.12}$$

Applying (1.12) to $f(x) = x^n$, we obtain

$$D_q[x^n] = \frac{(qx)^n - x^n}{(q-1)x} = \frac{q^n - 1}{q - 1}x^{n-1} = [n]x^{n-1}.$$

For a full treatment of the q-calculus, see [12].

1.5. GENERALIZED HARMONIC NUMBERS

Several varieties of generalized harmonic numbers have been introduced. The standard generalization, cf. [9], is derived by raising the terms of the sum to an appropriate power. This generalization is closely related to the Riemann zeta function.

Definition 1.28. Let $H_k \in \mathbb{H}$ with $k \geq 1$ and let $n \in \mathbb{N}_0$. Then

$$H_k^n = \sum_{j=1}^{k} \frac{1}{j^n}.$$

Summations have also been used to generalize harmonic numbers. Conway and Guy [5] refer to the following as hyperharmonic numbers.

Definition 1.29. Let $H_k \in \mathbb{H}$ with $k \geq 1$. Set $\mathcal{R}_k^{(1)} = H_k$ and, for $r \in \mathbb{N}$, define

$$\mathcal{R}_k^{(r)} = \sum_{j=1}^{k} \mathcal{R}_j^{(r-1)}.$$

Benjamin et al. [1] give the following expression for the hyperharmonic numbers in terms of r-Stirling numbers:

$$\mathcal{R}_k^{(r)} = \frac{\left[{k+r \atop r+1}\right]_r}{k!}.$$

Other generalizations emphasize the relationship between harmonic numbers and binomial coefficient identities. The following generalization is utilized by Chu and Yan [4] to prove a variety of binomial identities.

Definition 1.30. Let $H_k \in \mathbb{H}$ with $k \geq 1$ and let $n \in \mathbb{N}$. Then

$$\mathcal{H}(k,n) = \sum_{1 \leq j_1 \leq \ldots \leq j_n \leq k} \prod_{l=1}^{n} \frac{1}{j_l}.$$

Example 1.31. We will compute f^Δ for various $f : \mathbb{H} \to \mathbb{R}$ defined as follows.

(i) Let $f(H_k) = H_k^n$; then

$$f^\Delta = (k+1)\left(H_{k+1}^n - H_k^n\right) = \frac{1}{(k+1)^{n-1}}.$$

(ii) Let $f(H_k) = \mathcal{R}_k^{(r)}$; then

$$\begin{aligned} f^{\Delta} &= (k+1)\left(\mathcal{R}_{k+1}^{(r)} - \mathcal{R}_k^{(r)}\right) \\ &= (k+1)\left(\sum_{j=1}^{k+1} \mathcal{R}_j^{(r-1)} - \sum_{j=1}^{k} \mathcal{R}_j^{(r-1)}\right) \\ &= (k+1)\mathcal{R}_{k+1}^{(r-1)}. \end{aligned}$$

(iii) Let $f(H_k) = \mathcal{H}(k,n)$; then

$$\begin{aligned} f^{\Delta}(H_k) &= (k+1)\left(\mathcal{H}(k+1,n) - \mathcal{H}(k,n)\right) \\ &= (k+1)\left[\sum_{1 \le j_1 \le \ldots \le j_n \le k+1} \prod_{l=1}^{n} \frac{1}{j_l} - \sum_{1 \le j_1 \le \ldots \le j_n \le k} \prod_{l=1}^{n} \frac{1}{j_l}\right]. \end{aligned}$$

The second sum cancels all terms in the first sum except those where $j_n = k+1$, therefore

$$f^{\Delta}(H_k) = \frac{k+1}{k+1} \sum_{1 \le j_1 \le \ldots \le j_{n-1} \le k+1} \prod_{l=1}^{n-1} \frac{1}{j_l} = \mathcal{H}(k+1, n-1).$$

The preceding harmonic derivatives suggest a power rule to match that of $\frac{d}{dx}[x^n] = nx^{n-1}$ for $x \in \mathbb{R}$ and $\Delta[x^{\underline{n}}] = nx^{\underline{n-1}}$. (In fact, they are closer to the power rule for the rising factorial function $\Delta[x^{\overline{n}}] = n(x+1)^{\overline{n-1}}$ which we will revisit in Section 2.2.) In the next section we will develop a function that observes such a power rule under the harmonic derivative.

2. HARMONIC NUMBERS FALLING AND RISING

2.1. A HARMONIC FALLING FACTORIAL FUNCTION

In the quest for a harmonic falling factorial function, we first make the definition in terms of an iterated sum. This sum can be obtained in the classic manner (cf. [12]) by iteratively considering harmonic antiderivatives of H_k as defined in Theorem 1.21. For example, to ensure that the harmonic falling factorial satisfies the desired power rule, we define the *second power* of H_k to be an antiderivative of $2H_k$. We naturally pick the antiderivative that evaluates to zero when $k = 0$. So in this manner, the *nth power* of H_k is defined to be the nth antiderivative of $n!$ that is zero when $k = 0$.

Definition 2.1. Suppose $H_k \in \mathbb{H}$. A harmonic analog of the falling factorial function ${H_k}^{\{\underline{n}\}}$ is defined as follows for $n, k \in \mathbb{N}_0$. If $1 \le n \le k$,

$$ {H_k}^{\{\underline{n}\}} = n! \sum_{0 \le j_1 < \ldots < j_n < k} \prod_{l=1}^{n} \frac{1}{j_l + 1}. $$

If $n \ge k+1$, ${H_k}^{\{\underline{n}\}} = 0$. Also, ${H_k}^{\{\underline{0}\}} = 1$. We define ${H_0}^{\{\underline{n}\}} = 0$ for $n \ge 1$ and ${H_0}^{\{\underline{0}\}} = 1$.

These generalized harmonic numbers have been related to the Bell polynomials by Zave [22] and Spieß [17] and have been shown to satisfy many identities [21]. The focus of this discussion will be to demonstrate a number of ways that ${H_k}^{\{\underline{n}\}}$ is analogous to $x^{\underline{n}}$. In Section 2.2 we will give an expression of a rising factorial analog, then generalize in Section 2.4 to a mixed factorial function which rises and falls as needed.

Manipulating the sum in Definition 2.1 leads to a few results which aid in calculations. Of particular note, we expect that raising H_k to the first power should preserve the identity, and indeed the next result demonstrates that it does.

Lemma 2.2. *The following are true of the harmonic falling factorial function defined in Definition 2.1:*

(i) $H_k{}^{\{\underline{k}\}} = 1$;

(ii) $H_k{}^{\{\underline{1}\}} = H_k$;

(iii) For $k \geq 1$, $H_k{}^{\{\underline{k-1}\}} = \dfrac{k+1}{2}$.

Proof. In part *(i)*, if $n = k$, then notice that each j_l has only one possible value, namely $j_l = l - 1$. Consequently,

$$H_k{}^{\{\underline{k}\}} = k! \prod_{l=1}^{k} \frac{1}{l-1+1} = k! \prod_{l=1}^{k} \frac{1}{l} = 1.$$

In part *(ii)*, if $n = 1$, then j_1 ranges from 0 to $k - 1$ and

$$H_k{}^{\{\underline{1}\}} = \sum_{j_1=0}^{k-1} \frac{1}{j_1+1} = \sum_{j=1}^{k} \frac{1}{j} = H_k.$$

In part *(iii)*, we begin by substituting $n = k - 1$ into Definition 2.1 to obtain

$$H_k{}^{\{\underline{k-1}\}} = (k-1)! \sum_{0 \leq j_1 < \ldots < j_{k-1} < k} \prod_{l=1}^{k-1} \frac{1}{j_l+1}.$$

Since $n = k - 1$, then there are exactly k choices for the $(k-1)$-tuple $j_1, j_2, \ldots, j_{k-1}$. So the expression reduces to

$$H_k{}^{\{\underline{k-1}\}} = (k-1)! \sum_{j=1}^{k} \frac{j}{k!}.$$

Canceling out the $(k-1)!$'s yields

$$H_k{}^{\{\underline{k-1}\}} = \frac{1}{k} \sum_{j=1}^{k} j.$$

Recognizing the classic sum identity, we obtain the desired result:

$$H_k^{\{\underline{k-1}\}} = \frac{k+1}{2}.$$

□

Calculations with the iterated sum can be tedious. Fortunately, the proof of Lemma 2.2 *(iii)* suggests a connection between this harmonic falling factorial and the Stirling numbers. This connection provides an alternate expression for $H_k^{\{\underline{n}\}}$.

Theorem 2.3. *Let* $\begin{bmatrix} k \\ n \end{bmatrix}$ *represent the unsigned Stirling number of the first kind. Then*

$$H_k^{\{\underline{n}\}} = \frac{n!}{k!}\begin{bmatrix} k+1 \\ n+1 \end{bmatrix}$$

Proof. If $n > k$, then $H_k^{\{\underline{n}\}}$ and $\begin{bmatrix} k+1 \\ n+1 \end{bmatrix}$ are both equal to 0. When $n = 0$,

$$\frac{0!}{k!}\begin{bmatrix} k+1 \\ 1 \end{bmatrix} = \frac{k!}{k!} = 1 = H_k^{\{\underline{0}\}}.$$

When $1 \leq n \leq k$,

$$H_k^{\{\underline{n}\}} = n! \sum_{0 \leq j_1 < \ldots < j_n < k} \prod_{l=1}^{n} \frac{1}{j_l + 1}.$$

Multiplying the numerator and denominator of the right hand side by $k!$ yields

$$H_k^{\{\underline{n}\}} = \frac{n!}{k!} \sum_{0 \leq j_1 < \cdots < j_n < k} k! \prod_{l=1}^{n} \frac{1}{j_l + 1}.$$

The n $j_l + 1$'s in the product cancel with n terms in the $k!$ leaving $k - n$ distinct $m_l + 1$'s in the numerator, each between 1 and k:

$$H_k^{\{\underline{n}\}} = \frac{n!}{k!} \sum_{0 \leq m_1 < \ldots < m_{k-n} < k} \prod_{l=1}^{k-n} (m_l + 1). \tag{2.1}$$

Recognizing the sum on the right as an unsigned Stirling number of the first kind, cf. [3],

$$H_k^{\{\underline{n}\}} = \frac{n!}{k!}\begin{bmatrix} k+1 \\ n+1 \end{bmatrix}.$$

□

Example 2.4. We now provide some examples.

(i) $H_3^{\{\underline{2}\}} = 2!\left[\frac{1}{1}\frac{1}{2} + \frac{1}{1}\frac{1}{3} + \frac{1}{2}\frac{1}{3}\right] = 2.$

Likewise, from Lemma 2.2, part *(iii)*, $H_3^{\{\underline{2}\}} = \frac{3+1}{2} = 2.$

(ii) $H_4^{\{\underline{2}\}} = 2!\left[\frac{1}{1}\frac{1}{2} + \frac{1}{1}\frac{1}{3} + \frac{1}{1}\frac{1}{4} + \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{1}{4}\right] = \frac{35}{12}.$

(iii) $H_4^{\{\underline{3}\}} = \frac{4+1}{2} = \frac{5}{2}.$

Because of the availability of Stirling number tables, it is relatively simple to calculate $H_k^{\{\underline{n}\}}$ using Theorem 2.3. We provide values of $H_k^{\{\underline{n}\}}$ for small k and n in Table 2.1.

Theorem 2.5. *Whenever $H_k \in \mathbb{H}$ and $n \geq 1$, we have*

$$[H_k^{\{\underline{n}\}}]^{\Delta} = nH_k^{\{\underline{n-1}\}}.$$

Proof. Beginning with the expression for $H_k^{\{\underline{n}\}}$ from Theorem 2.3, we take the harmonic derivative on both sides,

$$[H_k^{\{\underline{n}\}}]^{\Delta} = (k+1)\left[\frac{n!}{(k+1)!}\begin{bmatrix} k+2 \\ n+1 \end{bmatrix} - \frac{n!}{k!}\begin{bmatrix} k+1 \\ n+1 \end{bmatrix}\right].$$

Multiplying through by the $k+1$ and factoring out $\frac{n!}{k!}$,

$$\frac{n!}{k!}\left[\begin{bmatrix} k+2 \\ n+1 \end{bmatrix} - (k+1)\begin{bmatrix} k+1 \\ n+1 \end{bmatrix}\right].$$

Table 2.1. Values of the Harmonic Falling Factorial

k	$H_k^{\underline{\{0\}}}$	$H_k^{\underline{\{1\}}}$	$H_k^{\underline{\{2\}}}$	$H_k^{\underline{\{3\}}}$	$H_k^{\underline{\{4\}}}$	$H_k^{\underline{\{5\}}}$	$H_k^{\underline{\{6\}}}$
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	$\frac{3}{2}$	1	0	0	0	0
3	1	$\frac{11}{6}$	2	1	0	0	0
4	1	$\frac{25}{12}$	$\frac{35}{12}$	$\frac{5}{2}$	1	0	0
5	1	$\frac{137}{60}$	$\frac{15}{4}$	$\frac{17}{4}$	3	1	0
6	1	$\frac{49}{20}$	$\frac{203}{95}$	$\frac{49}{8}$	$\frac{35}{6}$	$\frac{7}{2}$	1

Finally, recognizing the triangle identity (Lemma 1.4) for Stirling numbers of the first kind, we obtain

$$n\frac{(n-1)!}{k!}\begin{bmatrix} k+1 \\ n \end{bmatrix} = nH_k^{\underline{\{n-1\}}}.$$

□

2.2. A HARMONIC RISING FACTORIAL FUNCTION

In the usual difference calculus, it is natural to consider the rising factorial function in terms of the falling factorial function, and we will do the same in Corollary 2.39. We begin, however, by defining the rising factorial in terms of the iterated sum.

Definition 2.6. Let $H_k \in \mathbb{H}$ and $n \in \mathbb{N}_0$. We define a *harmonic rising factorial function* ${H_k}^{\{\overline{n}\}}$ using an iterated sum analogous to Definition 2.1:

$$
{H_k}^{\{\overline{n}\}} = n! \sum_{0 \le j_1 \le \ldots \le j_n \le k-1} \prod_{l=1}^{n} \frac{1}{j_l + 1}.
$$

This definition is related to the alternate generalized harmonic number of Definition 1.30 in the following way.

Lemma 2.7. *For $H_k \in \mathbb{H}$ and $n \in \mathbb{N}$,*

$$
{H_k}^{\{\overline{n}\}} = n! \mathcal{H}(k, n).
$$

Proof. This is accomplished by reindexing from $1 \le j_1 \le j_2 \le \ldots \le j_n \le k$ to $1 \le j_1 + 1 \le j_2 + 1 \le \ldots \le j_n + 1 \le k$ in Definition 1.30. □

We choose to shift the j_l in order to clarify the dependence on the graininess μ, which will help when we generalize these factorial functions to other time scales.

The rising factorial function has a power rule similar to the falling factorial: $[x^{\overline{n}}]^{\Delta} = n(x+1)^{\overline{n-1}}$. The same should hold true for the harmonic rising factorial.

Lemma 2.8. *The harmonic rising factorial function defined above obeys the expected power rule for a rising factorial function,*

$$
{{H_k}^{\{\overline{n}\}}}^{\Delta} = n {H_{k+1}}^{\{\overline{n-1}\}} = n \left[{H_k}^{\{\overline{n-1}\}} \right]^{\sigma}.
$$

Proof. Taking the harmonic derivative on both sides of the equation in Lemma 2.7 using the result of Example 1.31 part *(iii)* yields

$$
{{H_k}^{\{\overline{n}\}}}^{\Delta} = n! \mathcal{H}(k+1, n-1) = n {H_{k+1}}^{\{\overline{n-1}\}} = n \left[{H_k}^{\{\overline{n-1}\}} \right]^{\sigma}.
$$

□

2.3. SHIFTING SEQUENCES

In Section 2.4 we will present the notion of a harmonic mixed factorial which shares some traits with the harmonic rising and falling factorial functions. This mixed factorial will have a power rule which sometimes has a shift (as in the rising factorial) and sometimes does not (as in the falling factorial). In order to accomplish this, we require a way to track the shift. To this end we introduce a shifting sequence M.

Definition 2.9. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a sequence such that $m_i \in \{0, 1\}$ for all nonnegative integers i. We say such an M is a *shifting sequence*.

The exact value of m_0 will not matter, because m_0 will correspond with the constant function, which is shift invariant. For this reason, except as it is needed (for indexing purposes) we may omit m_0 from any list of M. If it is ever not specified, assume $m_0 = 0$. (This is merely a convenience; it should also work with $m_0 = 1$.) Likewise, only terms with index less than or equal to n will be important in most applications. Since there are infinitely many shifting sequences which agree on the terms from 1 to n (which we call the first n terms hereafter), and any one of them will work for such situations, the following equivalence relation will be useful.

Definition 2.10. For the set of shifting sequences, define the equivalence relation $\equiv$ mod n such that $A \equiv B$ mod n if and only if $a_i = b_i$ for all $1 \leq i \leq n$. We say A is n-equivalent to B.

We may refer to a shifting sequence and its equivalence class interchangeably. Note that it is vacuously true that any two shifting sequences will be 0-equivalent.

Three operators on the space of shifting sequences will prove useful in later discussions. Of these, the left shift operator is well known.

Definition 2.11. Denote by L the *left shift operator* which maps a sequence $\{a_0, a_1, a_2, \ldots\} \to \{a_1, a_2, a_3, \ldots\}$.

It is clear from the definition that any shifting sequence will remain a shifting sequence after applying L. This will also be true for the remaining two operators, the transpose and the adjoint.

Definition 2.12. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence. We say a sequence $S = \{s_i\}_{i\in\mathbb{N}_0}$ is an *n-transpose* of M if $s_i = m_{n-i+1}$ for all $1 \leq i \leq n$ or if $S = M$ for $n = 0$. Denote by M^{T_n} the n-transpose of M such that $s_0 = m_0$ and $s_i = m_i$ for all $i > n$.

Theorem 2.13. *Any n-transpose of a shifting sequence is a shifting sequence. Additionally, for a given $n \geq 1$ the set of n-transposes of a shifting sequence M forms an n-equivalence class.*

Proof. The first statement is obviously true for $n = 0$.

Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ and $n \in \mathbb{N}$. Clearly any rearrangement of M will preserve the condition $m_i \in \{0, 1\}$ for all $i \in \mathbb{N}$, so any n-transpose of M remains a shifting sequence.

If A, B are any two n-transposes of M, then Definition 2.12 gives $a_i = m_{n-i+1} = b_i$ for all $1 \leq i \leq n$. Therefore, A is n-equivalent to B. □

Theorem 2.14. *If M is a shifting sequence, S is an n-transpose of M, and $0 \leq j \leq n$, then*

$$L^j(M) \equiv S^{T_{n-j}} \ mod\ (n - j)$$

Proof. If $n = j$, then $n - j = 0$, and any two sequences are 0-equivalent. So let $n \in \mathbb{N}$, $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence, $S = \{s_i\}_{i\in\mathbb{N}_0}$ be an n-transpose of M, and $0 \leq j \leq n-1$. Denote the ith element of $L^j(M)$ by l_i and the ith element of $S^{T_{n-j}}$ by t_i. From Definition 2.11, $l_i = m_{i+j}$ for all $i \in \mathbb{N}$.

Then applying Definition 2.12 twice, we obtain for all $1 \leq i \leq n - j$

$$\begin{aligned} t_i &= s_{n-j-i+1} \\ &= m_{n-(n-j-i+1)+1} \\ &= m_{i+j} \\ &= l_i. \end{aligned}$$

Therefore, $L^j(M)$ is $(n - j)$-equivalent to $S^{T_{n-j}}$. □

A consequence of Theorem 2.14 is that M is an n-transpose of any n-transpose of M. The following shifting sequences are actually n-transposes of themselves.

Lemma 2.15. *The shifting sequences* $M_F = \{0, 0, 0, \ldots\}$ *and* $M_R = \{1, 1, 1, \ldots\}$ *satisfy* $M \equiv M^{T_n}$ *mod n for all positive integers n.*

The remaining operator we define on the space of shifting sequences calls to mind the idea of a conjugate transpose, or adjoint.

Definition 2.16. Let $M = \{m_i\}_{i \in \mathbb{N}_0}$ be a shifting sequence. We say a sequence $S = \{s_i\}_{i \in \mathbb{N}_0}$ is an *n-adjoint* of M if $s_i = 1 - m_{n-i+1}$ for all $1 \leq i \leq n$. Denote by M^{*_n} the n-adjoint of M such that $s_0 = 1 - m_0$ and $s_i = 1 - m_i$ for all $i > n$.

The proof of the next result is exactly like the one for Theorem 2.13.

Theorem 2.17. *Any n-adjoint of a shifting sequence is a shifting sequence. Additionally, for a given* $n \geq 1$ *the set of n-adjoints of a shifting sequence M forms an n-equivalence class.*

Theorem 2.18. *If M is a shifting sequence, S is an n-adjoint of M, and* $0 \leq j \leq n$*, then*

$$L^j(M) \equiv S^{*_{n-j}} \; mod \; (n - j)$$

Proof. If $n = j$, then $n - j = 0$, and any two sequences are 0-equivalent. So let $M = \{m_i\}_{i \in \mathbb{N}_0}$ be a shifting sequence, $S = \{s_i\}_{i \in \mathbb{N}_0}$ be an n-adjoint of M, and $0 \leq j \leq n-1$. Denote the ith element of $L^j(M)$ as l_i, and the ith element of $S^{*_{n-j}}$ as t_i. From Definition 2.11, $l_i = m_{i+j}$ for all $i \in \mathbb{N}_0$.

Then applying Definition 2.16 twice, we obtain for all $1 \leq i \leq n - j$

$$\begin{aligned} t_i &= 1 - s_{n-j-i+1} \\ &= 1 - (1 - m_{n-(n-j-i+1)+1}) \\ &= m_{i+j} \\ &= l_i. \end{aligned}$$

Therefore, $L^j(M)$ is $(n - j)$-equivalent to $S^{*_{n-j}}$. □

As with the n-transpose, a consequence of the Theorem 2.18 is that $[M^{*_n}]^{*_n}$ is n-equivalent to M for any positive integer n. The following sequences are n-adjoints of each other for every positive integer n.

Lemma 2.19. *The shifting sequences M_F and M_R of Lemma 2.15 satisfy $M_F \equiv M_R^{*_n}$ mod n and $M_R \equiv M_F^{*_n}$ mod n for all nonnegative integers n.*

Definition 2.20. Let $M = \{m_i\}_{i \in \mathbb{N}_0}$ be a shifting sequence. We say an integer sequence $\{a_i\}_{i \in \mathbb{N}}$ or n-tuple $(a_n) = (a_1, a_2, \ldots, a_n)$ is *M-increasing* if $a_i < a_{i+1} + m_i$ for all $i \in \mathbb{N}$ or all $1 \leq i \leq n - 1$, respectively.

Definition 2.21. Let $M = \{m_i\}_{i \in \mathbb{N}}$ be a shifting sequence. For nonnegative integers l, k, denote by $J(M, n, l, k)$ the set of all M-increasing integer n-tuples bounded below by l $(j_1 \geq l)$ and above by $k - 1$ $(j_n \leq k - 1)$.

Example 2.22. Let $M_E = \{m_i\}_{i\in\mathbb{N}_0}$ where $m_i = 1$ if i is even and $m_i = 0$ otherwise. The set of all M_E-increasing 4-tuples bounded below by 0 and above by 4 is

$$\begin{aligned} J(M_E, 4, 0, 5) = \{&(0, 1, 1, 2), (0, 1, 1, 3), (0, 1, 1, 4), (0, 1, 2, 3), (0, 1, 2, 4), \\ &(0, 1, 3, 4), (0, 2, 2, 3), (0, 2, 2, 4), (0, 2, 3, 4), (0, 3, 3, 4), \\ &(1, 2, 2, 3), (1, 2, 2, 4), (1, 2, 3, 4), (1, 3, 3, 4), (2, 3, 3, 4)\}. \end{aligned}$$

In several later discussions we will sum over all $(j_n) \in J(M, n, l, k)$. The next four lemmas will prove useful in those instances.

Lemma 2.23. *If M and S are $(n-1)$-equivalent shifting sequences and n, l, k are nonnegative integers with $l < k$, then $J(M, n, l, k) = J(S, n, l, k)$.*

Lemma 2.24. *If M is a shifting sequence and n, l, k are nonnegative integers with $l < k$, then $J(M, n, l, k + 1)$ can be partitioned as*

$$J(M, n, l, k + 1) = \{((j_{n-1}), k) : (j_{n-1}) \in J(M, n - 1, l, k + m_{n-1})\} \sqcup J(M, n, l, k).$$

Proof. The set $\{((j_{n-1}), k) : (j_{n-1}) \in J(M, n-1, l, k+m_{n-1})\}$ consists of the M-increasing n-tuples with $j_n = k$, while $J(M, n, l, k)$ is the set of M-increasing n-tuples with $j_n \leq k-1$. These sets are disjoint and cover $J(M, n, l, k + 1)$. □

Lemma 2.25. *If M is a shifting sequence and n, l, k are nonnegative integers with $l < k$, then $J(M, n, l, k)$ can be partitioned as*

$$J(M, n, l, k) = \{(l, (j_{n-1})) : (j_{n-1}) \in J(L(M), n - 1, l + 1 - m_1, k)\} \sqcup J(M, n, l + 1, k).$$

Proof. The set $\{(l, (j_{n-1})) : (j_{n-1}) \in J(L(M), n-1, l+1-m_1, k)\}$ consists of the M-increasing n-tuples whose first element is l (with the shift L accounting for the fact that m_1 has already been used), and $J(M, n, l+1, k)$ is the set of M-increasing n-tuples with $j_1 \geq l+1$. □

Lemma 2.26. *If M, S are $(n-1)$-equivalent shifting sequences and n, l, k are nonnegative integers with $l < k$, then*

$$\sum_{(j_n)\in J(M,n,l,k)} f(j_1, j_2, \ldots, j_n) = \sum_{(j_n)\in J(S,n,l,k)} f(j_1, j_2, \ldots, j_n).$$

2.4. A HARMONIC M-FACTORIAL FUNCTION

While the formulas $[H_k{}^{\{\underline{n}\}}]^{\Delta} = nH_k{}^{\{\underline{n-1}\}}$ and $[H_k{}^{\{\overline{n}\}}]^{\Delta} = n[H_k{}^{\{\overline{n-1}\}}]^{\sigma}$ are useful, we seek a function f which satisfies the more general power rule $f^{\Delta}(H_k, n) = nf^{\sigma^{m_{n-1}}}(H_k, n-1)$ where m_i is the ith element of a shifting sequence M.

Definition 2.27. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence, n be a nonnegative integer, and $H_k \in \mathbb{H}$. We define a *harmonic M-factorial function $H_k{}^{\{n,M\}}$* as follows:

$$H_k{}^{\{n,M\}} = n! \sum_{(j_n)\in J(M,n,0,k)} \prod_{l=1}^{n} \frac{1}{j_l + 1}.$$

Example 2.28. If M is $(n-1)$-equivalent to $M_F = \{0, 0, 0, \ldots\}$ from Lemma 2.15, then $H_k{}^{\{n,M\}} = H_k{}^{\{\underline{n}\}}$ is the harmonic falling factorial function.

Example 2.29. If M is $(n-1)$-equivalent to $M_R = \{1, 1, 1, \ldots\}$ from Lemma 2.15, then $H_k{}^{\{n,M\}} = H_k{}^{\{\overline{n}\}}$ is the harmonic rising factorial function.

Lemma 2.30. *Whenever $H_k \in \mathbb{H}$, $n \geq 1$, and M and S are $(n-1)$-equivalent shifting sequences,*

$$H_k{}^{\{n,M\}} = H_k{}^{\{n,S\}}.$$

Proof. This is a consequence of Lemma 2.23. □

Lemma 2.31. *Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence. Whenever $H_k \in \mathbb{H}$ and $n \geq 1$, we have*

$$\left[{H_k}^{\{n,M\}}\right]^{\Delta} = n\left[{H_k}^{\{n-1,M\}}\right]^{\sigma^{m_{n-1}}}.$$

Proof. Taking the harmonic derivative on both sides of Definition 2.27 yields

$$\left[{H_k}^{\{n,M\}}\right]^{\Delta} = (k+1)\left[n!\sum_{(j_n)\in J(M,n,0,k+1)}\prod_{l=1}^{n}\frac{1}{j_l+1} - n!\sum_{(j_n)\in J(M,n,0,k)}\prod_{l=1}^{n}\frac{1}{j_l+1}\right].$$

Using Lemma 2.24, the first sum can be divided into the cases where $j_n \leq k-1$ and $j_n = k$. When $j_n \leq k-1$, $(j_n) \in J(M,n,0,k)$, and so those terms of the first sum will cancel with the second sum leaving only the case where $j_n = k$ in the first sum. When $j_n = k$, then $j_{n-1} \leq j_n + m_{n-1} = k + m_{n-1}$ and the remaining sum can be written as

$$\sum_{\substack{(j_n)\in J(M,n,0,k+1)\\ j_n=k}}\prod_{l=1}^{n}\frac{1}{j_l+1} = \frac{1}{k+1}\sum_{(j_{n-1})\in J(M,n-1,0,k+m_{n-1})}\prod_{l=1}^{n-1}\frac{1}{j_l+1}.$$

Factoring an n out of the factorial and canceling the $k+1$ with the $\frac{1}{k+1}$ yields

$$(k+1)\left[\frac{n}{k+1}(n-1)!\sum_{(j_{n-1})\in J(M,n-1,0,k+m_{n-1})}\prod_{l=1}^{n-1}\frac{1}{j_l+1}\right] = n\left[{H_k}^{\{n-1,M\}}\right]^{\sigma^{m_{n-1}}}.$$

□

2.5. HARMONIC M-FACTORIAL OF A DIFFERENCE

In order to ensure that a given harmonic polynomial has a specific zero, we will now extend the definition of the harmonic M-factorial to allow for a difference of harmonic numbers. To define a formula for an expression of the form $(x-y)^{\{n,M\}}$ for $x, y \in \mathbb{H}$, we first need to consider the possibilities for $x - y$.

Theorem 2.32 (Erdős and Niven, 1946)**.** *No two partial sums of the harmonic series can be equal; that is, it is not possible that*

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n} = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{b},$$

unless $a = m$ and $b = n$.

Proof. Cf. [8]. □

A consequence of Theorem 2.32 is, for any $x - y \neq 0$ where $x, y \in \mathbb{H}$, x and y are unique. We now define the M-factorial of a difference using a recursive integral definition.

Definition 2.33. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence, n be a nonnegative integer, and $x, y \in \mathbb{H}$. We define $(x-y)^{\{n,M\}}$ as follows:

$$(x-y)^{\{0,M\}} \equiv 1 \quad \text{and}$$

$$(x-y)^{\{n,M\}} = \int_y^x \left(\sigma^{m_{n-1}}(\tau) - y\right)^{\{n-1,M\}} \Delta\tau.$$

With an integral definition, the harmonic derivative of $(x-y)^{\{n,M\}}$ can be calculated using Theorem 1.21 and Definition 1.22.

Lemma 2.34. *If $M = \{m_i\}_{i\in\mathbb{N}_0}$ is a shifting sequence, n is a nonnegative integer, and $x, y \in \mathbb{H}$, then*

$$\left[(x-y)^{\{n,M\}}\right]^{\Delta_x} = n\left(\sigma^{m_{n-1}}(x) - y\right)^{\{n-1,M\}}.$$

Theorem 2.35. *If M is a shifting sequence, n is a nonnegative integer, and $x, y \in \mathbb{H}$, where $x = H_k$ and $y = H_i$ with $k \geq i$, then:*

$$(x-y)^{\{n,M\}} = n! \sum_{(j_n)\in J(M,n,i,k)} \prod_{l=1}^{n} \frac{1}{j_l+1}.$$

Proof. Let $x = H_k$. For fixed $y = H_i \in \mathbb{H}$, let $f(x,n) = n! \sum_{(j_n)\in J(M,n,i,k)} \prod_{l=1}^{n} \frac{1}{j_l+1}$. Then using the same argument as Lemma 2.31, we obtain

$$f^{\Delta}(x,n) = nf(\sigma^{m_{n-1}}(x), n-1).$$

So, f and $(x-y)^{\{n,M\}}$ satisfy the same first order linear dynamic equation. Also, $f(y,n) = (y-y)^{\{n,M\}}$ for all $n \in \mathbb{N}$. □

Corollary 2.36. *If M is a shifting sequence and n is a nonnegative integer, then $(x-0)^{\{n,M\}} = x^{\{n,M\}}$.*

Lemma 2.37. *If $M = \{m_i\}_{i\in\mathbb{N}}$ is a shifting sequence, n is a nonnegative integer, and $x, y \in \mathbb{H}$ with $x \geq y$, then*

$$\left[(x-y)^{\{n,M\}}\right]^{\Delta_y} = -n\left(x - \sigma^{1-m_1}(y)\right)^{\{n-1,L(M)\}}.$$

Proof. Let $M = \{m_i\}_{i\in\mathbb{N}}$ be a shifting sequence, n be a nonnegative integer, and $x, y \in \mathbb{H}$ with $x \geq y$. Then $x = H_k$ and $y = H_i$ with $k \geq i$, and we can take the harmonic derivative with respect to y on both sides of the equation from Theorem 2.35:

$$\begin{aligned}\left[(x-y)^{\{n,M\}}\right]^{\Delta_y} &= \left[n! \sum_{(j_n)\in J(M,n,i,k)} \prod_{l=1}^{n} \frac{1}{j_l+1}\right]^{\Delta_y} \\ &= (i+1)\left[n! \sum_{(j_n)\in J(M,n,i+1,k)} \prod_{l=1}^{n} \frac{1}{j_l+1} - n! \sum_{(j_n)\in J(M,n,i,k)} \prod_{l=1}^{n} \frac{1}{j_l+1}\right].\end{aligned}$$

From Lemma 2.25, we know that the second sum will cancel with all the terms of the first sum leaving only those with $j_1 = i$ which may be factored out to obtain

$$\left[(x-y)^{\{n,M\}}\right]^{\Delta_y} = -n\frac{i+1}{i+1}\left[(n-1)! \sum_{(j_{n-1})\in J(L(M),n-1,i+1-m_1,k)} \prod_{l=1}^{n-1} \frac{1}{j_l+1}\right].$$

Applying Theorem 2.35 one more time yields

$$\left[(x-y)^{\{n,M\}}\right]^{\Delta_y} = -n\left(x-\sigma^{1-m_1}(y)\right)^{\{n-1,L(M)\}}.$$

□

Theorem 2.38. *If M is a shifting sequence, n is a positive integer, and $x, y \in \mathbb{H}$, then*

$$(x-y)^{\{n,M\}} = (-1)^n (y-x)^{\{n,M^{*(n-1)}\}}$$

where $M^{*(n-1)}$ is the $(n-1)$-adjoint of M from Definition 2.16.

Proof. The proof is by induction on n. Let M be a shifting sequence and $x, y \in \mathbb{H}$ with x fixed. When $n = 1$, $M^{*(1-1)} = M$ and

$$(x-y)^{\{1,M\}} = x-y = -(y-x) = (-1)^1 (y-x)^{\{1,M\}}.$$

Now suppose $(x-y)^{\{n-1,M\}} = (-1)^{n-1}(y-x)^{\{n-1,M^{*(n-2)}\}}$ for all shifting sequences M. Without loss of generality we may assume $x \geq y$ since the fact that M is $(n-2)$-equivalent to $[M^{*(n-2)}]^{*(n-2)}$ implies that the statement holds with x and y interchanged. Then, since the $(n-1)$the element of $M^{*(n-1)}$ is $1-m_1$ (Definition 2.16), the harmonic derivative of the right hand side with respect to y is

$$\left[(-1)^n (y-x)^{\{n,M^{*(n-1)}\}}\right]^{\Delta_y} = n(-1)^n\left(\sigma^{1-m_1}(y)-x\right)^{\{n-1,M^{*(n-1)}\}}$$

by Lemma 2.34. Utilizing the inductive hypothesis yields

$$\Big[(-1)^n(y-x)^{\{n,M^{*(n-1)}\}}\Big]^{\Delta_y} = -n\Big(x-\sigma^{1-m_1}(y)\Big)^{\{n-1,[M^{*(n-1)}]^{*(n-2)}\}}.$$

By Theorem 2.18, $[M^{*(n-1)}]^{*(n-2)}$ is $(n-2)$-equivalent to $L(M)$, and therefore

$$\Big[(-1)^n(y-x)^{\{n,M^{*(n-1)}\}}\Big]^{\Delta_y} = -n\Big(x-\sigma^{1-m_1}(y)\Big)^{\{n-1,L(M)\}}.$$

Since $x \geq y$, we obtain from Lemma 2.37

$$\Big[(-1)^n(y-x)^{\{n,M^{*(n-1)}\}}\Big]^{\Delta_y} = \Big[(x-y)^{\{n,M\}}\Big]^{\Delta_y}.$$

Hence, $(x-y)^{\{n,M\}}$ and $(-1)^n(y-x)^{\{n,M^{*(n-1)}\}}$ have the same harmonic derivative and when $y = x$ they are both zero. Equality is therefore shown. □

As a result of Theorem 2.38, we obtain an analog to the classic relationship between rising and falling factorials.

Corollary 2.39. *For $x \in \mathbb{H}$ and n a nonnegative integer,*

$$x^{\{\underline{n}\}} = (-1)^n(0-x)^{\{\overline{n}\}} \quad \textit{and}$$

$$x^{\{\overline{n}\}} = (-1)^n(0-x)^{\{\underline{n}\}}.$$

Proof. For the first equation, let $M = M_F = \{0,0,0,\ldots\}$ and $y = 0$ in Theorem 2.38. For the second, let $M = M_R = \{1,1,1,\ldots\}$ and $y = 0$. □

Theorem 2.40. *If $M = \{m_i\}_{i\in\mathbb{N}_0}$ is a shifting sequence, n is a nonnegative integer, and $x, y \in \mathbb{H}$, then*

$$\Big[(x-y)^{\{n,M\}}\Big]^{\Delta_y} = -n\Big(x-\sigma^{1-m_1}(y)\Big)^{\{n-1,L(M)\}}.$$

Proof. The case where $x \geq y$ is established by Lemma 2.37. Let M be a shifting sequence, n be a nonnegative integer, and $x, y \in \mathbb{H}$ with $x < y$. Then from Theorem 2.38 and Lemma 2.34 we obtain

$$\begin{aligned}\left[(x-y)^{\{n,M\}}\right]^{\Delta_y} &= \left[(-1)^n (y-x)^{\{n,M^{*(n-1)}\}}\right]^{\Delta_y} \\ &= n(-1)^n \left(\sigma^{1-m_1}(y) - x\right)^{\{n-1,M^{*(n-1)}\}} \\ &= -n\left(x - \sigma^{1-m_1}(y)\right)^{\{n-1,L(M)\}}.\end{aligned}$$

□

Theorem 2.41. *If* $x, y \in \mathbb{H}$ *and* $n \in \mathbb{N}_0$*, then*

$$(x-y)^{\{n,M\}} = \sum_{j=0}^{n} \binom{n}{j} x^{\{j,L^{n-j}(M)\}} (0-y)^{\{n-j,M\}}.$$

Proof. For a shifting sequence M and any fixed $y \in \mathbb{H}$, let $f(x,n,M) = \sum_{j=0}^{n} \binom{n}{j} x^{\{j,L^{n-j}(M)\}} (0-y)^{\{n-j,M\}}$. Then $f(0,n,M) = (0-y)^{\{n,M\}}$, and

$$\begin{aligned}f^{\Delta}(x,n,M) &= \left[\sum_{j=0}^{n} \binom{n}{j} x^{\{j,L^{n-j}(M)\}} (0-y)^{\{n-j,M\}}\right]^{\Delta_x} \\ &= \sum_{j=0}^{n} \binom{n}{j} \left[x^{\{j,L^{n-j}(M)\}}\right]^{\Delta_x} (0-y)^{\{n-j,M\}}.\end{aligned}$$

Applying Lemma 2.34 we obtain

$$f^{\Delta}(x,n,M) = \sum_{j=1}^{n} \binom{n}{j} j \sigma^{m_{n-1}}(x)^{\{j-1,L^{n-j}(M)\}} (0-y)^{\{n-j,M\}}.$$

Absorbing the j into the binomial coefficient and reindexing yields

$$f^{\Delta}(x,n,M) = n\sum_{j=0}^{n-1}\binom{n-1}{j}\sigma^{m_{n-1}}(x)^{\{j,L^{n-1-j}(M)\}}(0-y)^{\{n-1-j,M\}}$$
$$= nf(\sigma^{m_{n-1}}(x),n-1,M).$$

Therefore, $f(x,n,M)$ and $(x-y)^{\{n,M\}}$ solve the same first order linear initial value problem with respect to x.

For fixed $x \in \mathbb{H}$, let $g(y,n,M) = \sum_{j=0}^{n}\binom{n}{j}x^{\{j,L^{n-j}(M)\}}(0-y)^{\{n-j,M\}}$. Then $g(0,n,M) = (x-0)^{\{n,M\}} = x^{\{n,M\}}$, and

$$g^{\Delta}(y,n,M) = \left[\sum_{j=0}^{n}\binom{n}{j}x^{\{j,L^{n-j}(M)\}}(0-y)^{\{n-j,M\}}\right]^{\Delta_y}$$
$$= \sum_{j=0}^{n}\binom{n}{j}x^{\{j,L^{n-j}(M)\}}\left[(0-y)^{\{n-j,M\}}\right]^{\Delta_y}.$$

This time, applying Theorem 2.40 we obtain

$$g^{\Delta}(y,n,M) = \sum_{j=0}^{n-1}\binom{n}{j}jx^{\{j,L^{n-j}(M)\}}(-1)(n-j)\left(0-\sigma^{1-m_1}(y)\right)^{\{n-1-j,L(M)\}}.$$

Absorbing the $n-j$ into the binomial coefficient yields

$$g^{\Delta}(y,n,M) = -n\sum_{j=0}^{n-1}\binom{n-1}{j}x^{\{j,L^{n-1-j}(L(M))\}}\left(0-\sigma^{1-m_1}(y)\right)^{\{n-1-j,L(M)\}}$$
$$= -ng(\sigma^{1-m_1}(y),n-1,L(M)).$$

Therefore, $g(y,n,M)$ and $(x-y)^{\{n,M\}}$ solve the same first order linear initial value problem with respect to y. Equality is therefore shown. □

Note the similarity of Theorem 2.41 to the familiar Binomial Theorem $(a + b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$ with $a = x$ and $b = -y$. We will see an analogous expression for $(x + y)^{\{n,M\}}$ $(x, y \in \mathbb{H})$ in Section 3.4 and further generalizations in Section 5.4.

3. SOME HARMONIC BINOMIAL COEFFICIENTS

The ordinary binomial coefficient can be defined in terms of the falling factorial function by the equation $\binom{k}{n} = \frac{k^{\underline{n}}}{n!}$. The binomial coefficient $\binom{k+n-1}{n} = \frac{k^{\overline{n}}}{n!}$ is sometimes called the shifted binomial coefficient (cf. [15]). In this section, we define analogs to these coefficients using the harmonic falling factorial and M-factorial we have developed. We also explore several results analogous to familiar binomial coefficient identities.

3.1. A HARMONIC BINOMIAL COEFFICIENT

Definition 3.1. If ${H_k}^{\{\underline{n}\}}$ is defined, then we can define a *harmonic binomial coefficient*

$$\binom{H_k}{n}_{\mathbb{H}} = \frac{{H_k}^{\{\underline{n}\}}}{n!}.$$

Lemma 3.2. *The harmonic binomial coefficient defined by Definition 3.1 is related to the unsigned Stirling numbers of the first kind in the following way:*

$$\binom{H_k}{n}_{\mathbb{H}} = \frac{1}{k!}\begin{bmatrix} k+1 \\ n+1 \end{bmatrix}.$$

Proof. The result is obtained by dividing both sides of Theorem 2.3 by $n!$. □

Corollary 3.3. *The special case of Lemma 3.2 with* $n = 1$ *gives a classic identity relating the harmonic numbers and the Stirling numbers:*

$$\begin{bmatrix} k+1 \\ 2 \end{bmatrix} = k!H_k.$$

Using the data from Table 2.1, we provide values of $\binom{H_k}{n}_{\mathbb{H}}$ for small k and n in Table 3.1.

Lemma 3.4. *Whenever* $H_k \in \mathbb{H}$ *and* $n \geq 1$, $\binom{H_k}{n}_{\mathbb{H}}^{\Delta} = \binom{H_k}{n-1}_{\mathbb{H}}$.

Proof. Let $H_k \in \mathbb{H}$ and $n \geq 1$. Then, using Theorem 2.5, we can compute

$$\binom{H_k}{n}_{\mathbb{H}}^{\Delta} = \left[\frac{H_k^{\{\underline{n}\}}}{n!}\right]^{\Delta} = \frac{\left[H_k^{\{\underline{n}\}}\right]^{\Delta}}{n!} = \frac{nH_k^{\{\underline{n-1}\}}}{n!} = \frac{H_k^{\{\underline{n-1}\}}}{(n-1)!} = \binom{H_k}{n-1}_{\mathbb{H}}.$$

□

Note that, if $n = 0$, then $\binom{H_k}{n}_{\mathbb{H}}^{\Delta} = 0$.

Table 3.1. Values of the Harmonic Binomial Coefficient

k	$\binom{H_k}{0}_{\mathbb{H}}$	$\binom{H_k}{1}_{\mathbb{H}}$	$\binom{H_k}{2}_{\mathbb{H}}$	$\binom{H_k}{3}_{\mathbb{H}}$	$\binom{H_k}{4}_{\mathbb{H}}$	$\binom{H_k}{5}_{\mathbb{H}}$	$\binom{H_k}{6}_{\mathbb{H}}$
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	$\frac{3}{2}$	$\frac{1}{2}$	0	0	0	0
3	1	$\frac{11}{6}$	1	$\frac{1}{6}$	0	0	0
4	1	$\frac{25}{12}$	$\frac{35}{24}$	$\frac{5}{12}$	$\frac{1}{24}$	0	0
5	1	$\frac{137}{60}$	$\frac{15}{8}$	$\frac{17}{24}$	$\frac{3}{24}$	$\frac{1}{120}$	0
6	1	$\frac{49}{20}$	$\frac{203}{190}$	$\frac{49}{48}$	$\frac{35}{144}$	$\frac{7}{240}$	$\frac{1}{720}$

Theorem 3.5. *Whenever* $H_k \in \mathbb{H}$ *and* $n \geq 1$, *the harmonic binomial coefficient defined in Definition 3.1 observes the following Pascal rule:*

$$\binom{H_{k+1}}{n}_{\mathbb{H}} = \binom{H_k}{n}_{\mathbb{H}} + \frac{1}{k+1}\binom{H_k}{n-1}_{\mathbb{H}}.$$

Proof. Suppose $H_k \in \mathbb{H}$ and $n \geq 1$. Then, from Lemma 3.4,

$$\binom{H_k}{n-1}_{\mathbb{H}} = \binom{H_k}{n}_{\mathbb{H}}^{\Delta} = (k+1)\left[\binom{H_{k+1}}{n}_{\mathbb{H}} - \binom{H_k}{n}_{\mathbb{H}}\right],$$

and the result follows immediately. □

Partly because of its relationship to the Stirling cycle numbers, the harmonic binomial coefficient can be shown to satisfy many identities analogous to those for the binomial coefficient. More will be shown in Section 3.2, but the following result is useful for providing a combinatorial interpretation for the harmonic binomial coefficient.

Lemma 3.6. *For $H_k \in \mathbb{H}$ and $n \geq 0$,*

$$\binom{H_k}{n}_{\mathbb{H}} = \frac{1}{k!}\sum_{j=0}^{k}\binom{j}{n}\left[{k \atop j}\right].$$

Proof. Divide through by $k!$ in the row sum identity for Stirling cycle numbers (cf. (6.16) from [9]):

$$\left[{k+1 \atop n+1}\right] = \sum_{j=0}^{k}\binom{j}{n}\left[{k \atop j}\right].$$

□

Theorem 3.7. *The harmonic binomial coefficient $\binom{H_k}{n}_{\mathbb{H}}$ is the average number of ways to choose n cycles from the cycle decomposition of a permutation of k items.*

Proof. For each j in the summation in Lemma 3.6, the term of the summation is the number of k-permutations with j cycles multiplied by the number of ways to pick n cycles from those j. There are $k!$ total k-permutations, so, by summing over all the valid j's and dividing by $k!$, we obtain the result. □

This leads to a combinatorial interpretation of $H_k{}^{\{\underline{n}\}}$, as well. Since $H_k{}^{\{\underline{n}\}} = n!\binom{H_k}{n}_{\mathbb{H}}$, we have the following result.

Corollary 3.8. *$H_k{}^{\{\underline{n}\}}$ is the average number of ways to permute n cycles chosen from the cycle decomposition of a permutation of k items.*

Letting $n = 1$ in either identity leads to a classic relationship between the Stirling cycle numbers and the harmonic numbers stated earlier in Lemma 1.5.

Corollary 3.9. *The average number of cycles in a permutation of k items is the kth harmonic number, i.e.,*

$$\frac{1}{k!}\sum_{j=0}^{k} j \begin{bmatrix} k \\ j \end{bmatrix} = H_k.$$

3.2. SOME IDENTITIES

There exist harmonic analogs to many familiar binomial coefficient identities. The following are consequences of the Pascal rule.

Lemma 3.10. *For $H_k \in \mathbb{H}$ and $1 \leq n \leq k-1$,*

$$\frac{1}{n+1} H_k{}^{\{\underline{n+1}\}} = \sum_{j=n}^{k-1} \frac{1}{j+1} H_j{}^{\{\underline{n}\}}.$$

Proof. Looking at the expression

$$H_k{}^{\{\underline{n+1}\}} = (n+1)! \sum_{0 \leq j_1 < \ldots < j_{n+1} < k} \prod_{l=1}^{n+1} \frac{1}{j_l + 1},$$

it is easy to see that j_{n+1} attains all values ranging from n to $k-1$. Factoring out those terms gives

$$H_k{}^{\{\underline{n+1}\}} = (n+1) \sum_{j=n}^{k-1} \frac{1}{j+1} \left[n! \sum_{0 \leq j_1 < \ldots < j_n < j} \prod_{l=1}^{n} \frac{1}{j_l + 1} \right].$$

Dividing through by $n+1$, we obtain

$$\frac{1}{n+1}H_k{}^{\{\underline{n+1}\}} = \sum_{j=n}^{k-1}\frac{1}{j+1}\left[n!\sum_{0\le j_1<\ldots<j_n<j}\prod_{l=1}^{n}\frac{1}{j_l+1}\right]$$
$$= \sum_{j=n}^{k-1}\frac{1}{j+1}H_j{}^{\{\underline{n}\}}.$$

□

The sum in Lemma 3.10 may begin at $j=0$ if we wish since $H_j{}^{\{\underline{n}\}}=0$ for $0\le j<n$. Dividing through by $n!$, we obtain a harmonic analog to the upper summation identity for the binomial coefficient, cf. [9].

Theorem 3.11 (Upper Summation)**.** *For* $H_k \in \mathbb{H}$ *and* $1 \le n \le k-1$,

$$\binom{H_k}{n+1}_{\mathbb{H}} = \sum_{j=0}^{k-1}\frac{1}{j+1}\binom{H_j}{n}_{\mathbb{H}}.$$

Completing the summation in Lemma 3.10 yields an alternate form, and is suggestive of the Pascal rule.

Corollary 3.12. *For* $H_k \in \mathbb{H}$ *and* $1 \le n \le k-1$,

$$\frac{1}{n+1}H_k{}^{\{\underline{n+1}\}} + \frac{1}{k+1}H_k{}^{\{\underline{n}\}} = \sum_{j=n}^{k}\frac{1}{j+1}H_j{}^{\{\underline{n}\}}.$$

Proof. The result may be obtained by adding $\dfrac{1}{k+1}H_k{}^{\{\underline{n}\}}$ to both sides of the equation in Lemma 3.10. □

We now use these upper summation identities to provide an alternate proof of the Pascal rule in Theorem 3.5.

Theorem 3.13. *Whenever $H_k \in \mathbb{H}$ and $n \geq 1$, the harmonic binomial coefficient defined in Definition 3.1 observes the following Pascal rule:*

$$\binom{H_{k+1}}{n}_{\mathbb{H}} = \binom{H_k}{n}_{\mathbb{H}} + \frac{1}{k+1}\binom{H_k}{n-1}_{\mathbb{H}}.$$

Proof. Dividing through by $n!$ in Corollary 3.12, we obtain

$$\frac{H_k{}^{\{\underline{n+1}\}}}{(n+1)!} + \frac{1}{k+1}\frac{H_k{}^{\{\underline{n}\}}}{n!} = \frac{1}{n!}\sum_{j=n}^{k}\frac{1}{j+1}H_j{}^{\{\underline{n}\}}.$$

Then using Lemma 3.10 on the right hand side gives

$$\binom{H_k}{n+1}_{\mathbb{H}} + \frac{1}{k+1}\binom{H_k}{n}_{\mathbb{H}} = \binom{H_{k+1}}{n+1}_{\mathbb{H}}.$$

Reindexing and rearranging yields the result. □

The Pascal rule also leads us to an analog for the Parallel Summation Identity, cf. [9].

Lemma 3.14. *For $H_k \in \mathbb{H}$ and $1 \leq n \leq k$,*

$$\binom{H_{k+1}}{n}_{\mathbb{H}} = \sum_{j=0}^{n}\frac{1}{(k+1)^{\underline{j}}}\binom{H_{k-j}}{n-j}_{\mathbb{H}}.$$

Proof. We begin with the sum

$$\sum_{j=0}^{n-1}\frac{1}{(k+1)^{\underline{j}}}\binom{H_{k-j}}{n-j}_{\mathbb{H}}.$$

Using the Pascal rule to expand $\binom{H_{k-j}}{n-j}_{\mathbb{H}}$ as a difference, we obtain

$$\sum_{j=0}^{n-1}\frac{1}{(k+1)^{\underline{j}}}\left[\binom{H_{k-j+1}}{n-j}_{\mathbb{H}} - \frac{1}{k-j+1}\binom{H_{k-j}}{n-j-1}_{\mathbb{H}}\right].$$

This sum telescopes to

$$\binom{H_{k+1}}{n}_{\mathbb{H}} - \frac{1}{(k+1)^{\underline{n}}}\binom{H_{k-n+1}}{0}_{\mathbb{H}}.$$

Therefore,

$$\sum_{j=0}^{n-1} \frac{1}{(k+1)^{\underline{j}}}\binom{H_{k-j}}{n-j}_{\mathbb{H}} = \binom{H_{k+1}}{n}_{\mathbb{H}} - \frac{1}{(k+1)^{\underline{n}}}\binom{H_{k-n+1}}{0}_{\mathbb{H}}.$$

Since $\binom{H_{k-n+1}}{0}_{\mathbb{H}} = 1 = \binom{H_{k-n}}{0}_{\mathbb{H}}$ we can rearrange the terms to achieve the desired result:

$$\sum_{j=0}^{n} \frac{1}{(k+1)^{\underline{j}}}\binom{H_{k-j}}{n-j}_{\mathbb{H}} = \binom{H_{k+1}}{n}_{\mathbb{H}}.$$

□

Theorem 3.15 (Parallel Summation)**.** *For $H_k \in \mathbb{H}$ and $1 \le n \le k$,*

$$\binom{H_{k+n+1}}{n}_{\mathbb{H}} = \sum_{j=0}^{n} \frac{1}{(k+1)^{\underline{n-j}}}\binom{H_{k+j}}{j}_{\mathbb{H}}.$$

Proof. The result is obtained by replacing k with $k+n$ in Lemma 3.14 and reindexing. □

The alternating partial sum of the kth row of Pascal's triangle is given by the following sum (cf. [9]):

$$\sum_{j\le n} \binom{k+1}{j}(-1)^j = (-1)^n\binom{k}{n}.$$

We give a harmonic analog of this sum, again based on the harmonic Pascal rule.

Lemma 3.16. *For $H_k \in \mathbb{H}$ and $1 \le n \le k-1$,*

$$\sum_{0\le j\le n} \frac{(-1)^j}{(k+1)^{n-j}}\binom{H_{k+1}}{j}_{\mathbb{H}} = (-1)^n\binom{H_k}{n}_{\mathbb{H}}.$$

Proof. We begin with the sum

$$\sum_{j=1}^{n} \frac{(-1)^j}{(k+1)^{n-j}}\binom{H_{k+1}}{j}_{\mathbb{H}}.$$

Now we expand each of the $\binom{H_{k+1}}{j}_{\mathbb{H}}$ with the Pascal rule from Theorem 3.5, yielding

$$\sum_{j=1}^{n} \frac{(-1)^j}{(k+1)^{n-j}} \left[\binom{H_k}{j}_{\mathbb{H}} + \frac{1}{k+1} \binom{H_k}{j-1}_{\mathbb{H}} \right].$$

This alternating sum telescopes to

$$\frac{(-1)^n}{(k+1)^0} \binom{H_k}{n}_{\mathbb{H}} + \frac{(-1)^1}{(k+1)^n} \binom{H_k}{0}_{\mathbb{H}}.$$

Therefore,

$$\sum_{1 \le j \le n} \frac{(-1)^j}{(k+1)^{n-j}} \binom{H_{k+1}}{j}_{\mathbb{H}} = (-1)^n \binom{H_k}{n}_{\mathbb{H}} - \frac{1}{(k+1)^n} \binom{H_k}{0}_{\mathbb{H}}.$$

Since $\binom{H_k}{0}_{\mathbb{H}} = 1 = \binom{H_{k+1}}{0}_{\mathbb{H}}$, we rearrange the terms to obtain

$$\sum_{0 \le j \le n} \frac{(-1)^j}{(k+1)^{n-j}} \binom{H_{k+1}}{j}_{\mathbb{H}} = (-1)^n \binom{H_k}{n}_{\mathbb{H}}.$$

□

Alternatively, by replacing j with $n - j$, we could have written Lemma 3.16 as

$$\sum_{0 \le j \le n} \left(\frac{-1}{k+1} \right)^j \binom{H_{k+1}}{n-j}_{\mathbb{H}} = \binom{H_k}{n}_{\mathbb{H}}. \tag{3.1}$$

We will explore some other binomial formulas in Section 3.4, but we can show this binomial formula identity now.

Theorem 3.17. *For $H_k \in \mathbb{H}$ and $1 \le n \le k$,*

$$\sum_{n=0}^{k} k! \binom{H_k}{k-n}_{\mathbb{H}} x^n = \prod_{j=1}^{k} (1 + jx).$$

Proof. The coefficient of x^m in $\prod_{j=1}^{k}(1+jx)$ is the sum

$$\sum_{0<j_1<j_2<\cdots<j_m\le k} j_1 \cdot j_2 \cdot \cdots \cdot j_m.$$

This expression looks similar to the definition of the harmonic falling factorial. In fact, if we replace n with $k-m$ in (2.1), we can see that it is equivalent to

$$k^{\underline{m}} H_k{}^{\underline{\{k-m\}}}.$$

Multiplying and dividing by $(k-m)!$ yields

$$k!\binom{H_k}{k-m}_{\mathbb{H}}.$$

Therefore, the coefficient of x^m in the product is $k!\binom{H_k}{k-m}_{\mathbb{H}}$. Consequently, $\prod_{j=1}^{k}(1+jx)$ is the generating function for the sequence $\{k!\binom{H_k}{k-m}_{\mathbb{H}}\}_{m\in\mathbb{N}_0}$. Thus,

$$\sum_{n=0}^{\infty} k!\binom{H_k}{k-n}_{\mathbb{H}} x^n = \prod_{j=1}^{k}(1+jx).$$

However, summing past k only produces superfluous zeros. We may, therefore, eliminate those terms, yielding the desired result:

$$\sum_{n=0}^{k} k!\binom{H_k}{k-n}_{\mathbb{H}} x^n = \prod_{j=1}^{k}(1+jx).$$

□

Corollary 3.18. *For $H_k \in \mathbb{H}$ and $1 \le n \le k$,*

$$\sum_{n=0}^{k} \binom{H_k}{k-n}_{\mathbb{H}} x^n = \prod_{j=1}^{k}\left(\frac{1}{j}+x\right).$$

Proof. Dividing j out of the jth factor of $\prod_{j=1}^{k}(1+jx)$ gives

$$\sum_{n=0}^{k} k!\binom{H_k}{k-n}_{\mathbb{H}} x^n = \prod_{j=1}^{k} j\left(\frac{1}{j}+x\right).$$

Splitting the right hand side into two products, we obtain

$$\sum_{n=0}^{k} k!\binom{H_k}{k-n}_{\mathbb{H}} x^n = \left(\prod_{j=1}^{k} j\right)\left(\prod_{j=1}^{k}\left(\frac{1}{j}+x\right)\right).$$

Recognizing $\prod_{j=1}^{k} j$ as $k!$, we divide that out of both sides. Hence,

$$\sum_{n=0}^{k} \binom{H_k}{k-n}_{\mathbb{H}} x^n = \prod_{j=1}^{k}\left(\frac{1}{j}+x\right).$$

□

3.3. THE M-SHIFTED BINOMIAL COEFFICIENT

In elementary combinatorics, the shifted binomial coefficient $\binom{k+n-1}{n} = \frac{k^{\overline{n}}}{n!}$ counts the number of ways to choose n items from a set of k with replacement (cf. [15]). Here we will develop a shifted harmonic binomial coefficient. Rather than consider the shifted binomial coefficient based only on the harmonic rising factorial, we define the M-shifted binomial coefficient based on the harmonic M-factorial function of Definition 2.27.

Definition 3.19. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence and n be a nonnegative integer. We define the *M-shifted harmonic binomial coefficient* $\binom{H_k}{n}_{\mathbb{H},M}$ as follows:

$$\binom{H_k}{n}_{\mathbb{H},M} = \frac{{H_k}^{\{n,M\}}}{n!}.$$

Example 3.20. If $M = \{0, 0, 0, \ldots\}$, then

$$\binom{H_k}{n}_{\mathbb{H},M} = \binom{H_k}{n}_{\mathbb{H}}.$$

Example 3.21. If $M = \{1, 1, 1, \ldots\}$, then

$$\binom{H_k}{n}_{\mathbb{H},M} = \frac{H_k{}^{\{\overline{n}\}}}{n!}.$$

Lemma 3.22. *Whenever $H_k \in \mathbb{H}$, $n \geq 1$, and M and S are $(n-1)$-equivalent shifting sequences,*

$$\binom{H_k}{n}_{\mathbb{H},M} = \binom{H_k}{n}_{\mathbb{H},S}.$$

Proof. This is a consequence of Lemma 2.30. □

Lemma 3.23. *Whenever $H_k \in \mathbb{H}$, $M = \{m_i\}_{i\in\mathbb{N}_0}$ is a shifting sequence, and $n \geq 1$,*

$$\binom{H_k}{n}_{\mathbb{H},M}^{\Delta} = \binom{H_k}{n-1}_{\mathbb{H},M}^{\sigma^{m_{n-1}}}.$$

Proof. Let $H_k \in \mathbb{H}$, $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence, and $n \geq 1$. Beginning with Definition 3.19, we take the harmonic derivative using Lemma 2.31:

$$\binom{H_k}{n}_{\mathbb{H},M}^{\Delta} = \left[\frac{H_k{}^{\{n,M\}}}{n!}\right] = \frac{n\left[H_k{}^{\{n-1,M\}}\right]^{\sigma^{m_{n-1}}}}{n!} = \frac{\left[H_k{}^{\{n-1,M\}}\right]^{\sigma^{m_{n-1}}}}{(n-1)!} = \binom{H_k}{n-1}_{\mathbb{H},M}^{\sigma^{m_{n-1}}}.$$

□

Theorem 3.24. *Whenever $H_k \in \mathbb{H}$, $M = \{m_i\}_{i\in\mathbb{N}_0}$ is a shifting sequence, and $n > 1$, the M-shifted binomial coefficient observes the following Pascal rule:*

$$\binom{H_{k+1}}{n}_{\mathbb{H},M} = \binom{H_k}{n}_{\mathbb{H},M} + \frac{1}{k+1}\binom{H_k}{n-1}_{\mathbb{H},M}^{\sigma^{m_{n-1}}}.$$

Proof. We begin with Lemma 3.23 and use (1.3) to expand the harmonic derivative to obtain

$$\begin{aligned}\binom{H_k}{n-1}_{\mathbb{H},M}^{\sigma^{m_{n-1}}} &= \binom{H_k}{n}_{\mathbb{H},M}^{\Delta} \\ &= (k+1)\left[\binom{H_{k+1}}{n}_{\mathbb{H},M} - \binom{H_k}{n}_{\mathbb{H},M}\right].\end{aligned}$$

Dividing through by $k+1$ and rearranging, we obtain the result

$$\binom{H_{k+1}}{n}_{\mathbb{H},M} = \binom{H_k}{n}_{\mathbb{H},M} + \frac{1}{k+1}\binom{H_k}{n-1}_{\mathbb{H},M}^{\sigma^{m_{n-1}}}.$$

□

Theorem 3.25 (Upper Summation)**.** *Whenever* $H_k \in \mathbb{H}$*,* $M = \{m_i\}_{i\in\mathbb{N}_0}$ *is a shifting sequence, and* $0 \le n \le k-1$*,*

$$\binom{H_k}{n+1}_{\mathbb{H},M} = \sum_{j\le k-1} \frac{1}{j+1}\binom{H_j}{n}_{\mathbb{H},M}^{\sigma^{m_n}}.$$

Proof. The proof is by induction on k. When $k=1$, then $n=0$ and the sum

$$\sum_{j\le 0} \frac{1}{j+1}\binom{H_j}{0}_{\mathbb{H},M}^{\sigma^{m_0}} = \binom{H_0}{0}_{\mathbb{H},M} = 1 = \binom{H_1}{1}_{\mathbb{H},M},$$

since $\binom{H_j}{0}_{\mathbb{H},M} \equiv 1$ for all nonnegative integers j. Note that the σ can be ignored. Also note that, if $n = k-1$, the summands vanish for all $j \le k-1$ and the statement becomes

$$\binom{H_k}{k}_{\mathbb{H},M} = \frac{1}{k}\binom{H_{k-1}}{k-1}_{\mathbb{H},M}^{\sigma^{m_{k-1}}},$$

which is true by Theorem 3.24 since $\binom{H_{k-1}}{k}_{\mathbb{H},M} = 0$. Now, suppose

$$\binom{H_{k-1}}{n+1}_{\mathbb{H},M} = \sum_{j\le k-2} \frac{1}{j+1}\binom{H_j}{n}_{\mathbb{H},M}^{\sigma^{m_n}}$$

is true for some $k-1$ and all $0 \leq n \leq k-2$. Using Theorem 3.24 on $\binom{H_k}{n+1}_{\mathbb{H},M}$, we obtain

$$\binom{H_k}{n+1}_{\mathbb{H},M} = \binom{H_{k-1}}{n+1}_{\mathbb{H},M} + \frac{1}{k}\binom{H_{k-1}}{n}_{\mathbb{H},M}^{\sigma^{m_n}}.$$

Applying the inductive hypothesis to $\binom{H_{k-1}}{n+1}_{\mathbb{H},M}$ yields

$$\binom{H_k}{n+1}_{\mathbb{H},M} = \sum_{j \leq k-1} \frac{1}{j+1}\binom{H_j}{n}_{\mathbb{H},M}^{\sigma^{m_n}}.$$

□

Corollary 3.26. *Whenever* $H_k \in \mathbb{H}$*,* $M = \{m_i\}_{i \in \mathbb{N}_0}$ *is a shifting sequence, and* $0 \leq n \leq k-1$*,*

$$\frac{{H_k}^{\{n+1,M\}}}{n+1} = \sum_{j \leq k-1} \frac{1}{j+1}\left[{H_j}^{\{n,M\}}\right]^{\sigma^{m_n}}.$$

Proof. The result is obtained by multiplying through by $n!$ in Theorem 3.25. □

3.4. HARMONIC M-BINOMIAL FORMULAS

In this section we extend the harmonic M-factorial to a sum of harmonic numbers using a definition analogous to the Binomial Formula. We also prove an analog of the Vandermonde's Convolution and the Chu-Vandermonde identity, cf. [9].

Recall that, when we defined $(x-y)^{\{n,M\}}$ in Section 2.5,we first needed to consider the possibilities for the quantity $x-y$. We now need to do something similar with $x+y$ as the first step to define a formula for an expression of the form $(x+y)^{\{n,M\}}$ for $x, y \in \mathbb{H}$. As a consequence of Theorem 2.32 we know, if $x, y \in \mathbb{H}$, then the sum $x+y$ cannot be written as the sum of any other pair of harmonic numbers.

Definition 3.27. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence and $x, y \in \mathbb{H}$. We may extend the harmonic M-factorial function as follows:

$$(x+y)^{\{n,M\}} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} x^{\{j,L^{n-j}(M)\}} (0-y)^{\{n-j,M\}}. \tag{3.2}$$

(Recall that L is the left shift operator on sequences from Definition 2.11. Note the similarity to the expression for $(x-y)^{\{n,M\}}$ from Theorem 2.41.)

Likewise, we extend the harmonic M-shifted binomial coefficients

$$\binom{x+y}{n}_{\mathbb{H},M} = \frac{(x+y)^{\{n,M\}}}{n!} \text{ and} \tag{3.3}$$

$$\binom{x-y}{n}_{\mathbb{H},M} = \frac{(x-y)^{\{n,M\}}}{n!}. \tag{3.4}$$

This next lemma gives an alternate expression for the harmonic M-factorial of a sum.

Lemma 3.28. *If M is a shifting sequence, $x, y \in \mathbb{H}$, and $n \in \mathbb{N}_0$, then*

$$(x+y)^{\{n,M\}} = \sum_{j=0}^{n} \binom{n}{j} x^{\{j,L^{n-j}(M)\}} y^{\{n-j,M^{*n-j-1}\}}.$$

Proof. By Theorem 2.38, $(-1)^{n-j}(0-y)^{\{n-j,M\}} = y^{\{n-j,M^{*n-j-1}\}}$. □

These extensions obey the analogous power rules to $(x-y)^{\{n,M\}}$ and $\binom{H_k}{n}_{\mathbb{H},M}$ respectively under harmonic differentiation with respect to x and y.

Theorem 3.29. *If M is a shifting sequence, $x, y \in \mathbb{H}$, and $n \in \mathbb{N}_0$, then*

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_x} = n(\sigma^{m_{n-1}}(x)+y)^{\{n-1,M\}}, \textit{ and}$$

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_y} = n\left(x+\sigma^{1-m_1}(y)\right)^{\{n-1,L(M)\}}.$$

Proof. We begin with (3.2). Taking the harmonic derivative with respect to x on both sides, we obtain

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_x} = \left[\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}x^{\{j,L^{n-j}(M)\}}(0-y)^{\{n-j,M\}}\right]^{\Delta_x}.$$

The derivative passes through the sum, yielding

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_x} = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\left[x^{\{j,L^{n-j}(M)\}}\right]^{\Delta_x}(0-y)^{\{n-j,M\}}.$$

Once again we utilize Lemma 2.31:

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_x} = \sum_{j=1}^{n}\binom{n}{j}(-1)^{n-j}j\sigma^{m_{n-1}}(x)^{\{j-1,L^{n-j}(M)\}}(0-y)^{\{n-j,M\}}.$$

Factoring an n out of each binomial coefficient yields

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_x} = n\sum_{j=1}^{n}\binom{n-1}{j-1}(-1)^{n-j}\sigma^{m_{n-1}}(x)^{\{j-1,L^{n-j}(M)\}}(0-y)^{\{n-j,M\}}.$$

Finally, reindexing the sum to begin at zero we obtain the desired result:

$$\begin{aligned}\left[(x+y)^{\{n,M\}}\right]^{\Delta_x} &= n\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{n-1-j}\sigma^{m_{n-1}}(x)^{\{j,L^{n-1-j}(M)\}}(0-y)^{\{n-1-j,M\}}\\ &= n(\sigma^{m_{n-1}}(x)+y)^{\{n-1,M\}}.\end{aligned}$$

To show the identity for the delta derivative with respect to y, we again start with (3.2). Taking the harmonic derivative with respect to y on both sides, we obtain

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_y} = \left[\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}x^{\{j,L^{n-j}(M)\}}(0-y)^{\{n-j,M\}}\right]^{\Delta_y}.$$

The derivative passes through the sum, yielding

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_y} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} x^{\{j,L^{n-j}(M)\}} \left[(0-y)^{\{n-j,M\}}\right]^{\Delta_y}.$$

This time we will utilize Theorem 2.40 to obtain

$$\left[(x+y)^{\{n,M\}}\right]^{\Delta_y} = \sum_{j=0}^{n-1} \binom{n}{j} (-1)^{n-j} x^{\{j,L^{n-j}(M)\}} (-1)(n-j)\left(0-\sigma^{1-m_1}(y)\right)^{\{n-j-1,L(M)\}}.$$

Absorbing the $n-j$ into the binomial coefficient allows us to factor an n out of each summand yielding

$$\begin{aligned}\left[(x+y)^{\{n,M\}}\right]^{\Delta_y} &= n\sum_{j=1}^{n} \binom{n-1}{j} (-1)^{n-1-j} x^{\{j,L^{n-1-j}(L(M))\}} \left(0-\sigma^{1-m_1}(y)\right)^{\{n-1-j,L(M)\}} \\ &= n\left(x+\sigma^{1-m_1}(y)\right)^{\{n-1,L(M)\}}.\end{aligned}$$

□

As a result of this, the extended binomial coefficients also behave as expected under harmonic differentiation with respect to both x and y.

Corollary 3.30. *If M is a shifting sequence, $x, y \in \mathbb{H}$, and $n \geq 0$, then*

(i) $\binom{x+y}{n}_{\mathbb{H},M}^{\Delta_x} = \binom{\sigma^{m_{n-1}}(x)+y}{n-1}_{\mathbb{H},M}$;

(ii) $\binom{x+y}{n}_{\mathbb{H},M}^{\Delta_y} = \binom{x+\sigma^{1-m_1}(y)}{n-1}_{\mathbb{H},L(M)}$;

(iii) $\binom{x-y}{n}_{\mathbb{H},M}^{\Delta_x} = \binom{\sigma^{m_{n-1}}(x)-y}{n-1}_{\mathbb{H},M}$; *and*

(iv) $\binom{x-y}{n}_{\mathbb{H},M}^{\Delta_y} = -\binom{x-\sigma^{1-m_1}(y)}{n-1}_{\mathbb{H},L(M)}$.

Theorem 3.31. *If $x, y \in \mathbb{H}$ and $n \geq 0$, then the following hold:*

(i) $(x+y)^{\{\underline{n}\}} = \sum_{j=0}^{n} \binom{n}{j} x^{\{\underline{j}\}} y^{\{\overline{n-j}\}}$;

(ii) $(x+y)^{\{\overline{n}\}} = \sum_{j=0}^{n} \binom{n}{j} x^{\{\overline{j}\}} y^{\{\underline{n-j}\}}$;

(iii) $\binom{x+y}{n}_{\mathbb{H}} = \dfrac{(x+y)^{\{\underline{n}\}}}{n!}$; *and*

(iv) $\binom{x-y}{n}_{\mathbb{H}} = \dfrac{(x-y)^{\{\underline{n}\}}}{n!}$.

Proof. These are a result of $M_F = \{0,0,0,\ldots\}$ and $M_R = \{1,1,1,\ldots\}$ being constant sequences. □

Lemma 3.32. *If M is a shifting sequence and $x, y \in \mathbb{H}$, then*

$$(x+y)^{\{n,M\}} = \sum_{j=0}^{n} n! \binom{x}{j}_{\mathbb{H},L^{n-j}(M)} \binom{y}{n-j}_{\mathbb{H},M^{*n-j-1}}.$$

Proof. We begin with the the expression from Lemma 3.28,

$$\begin{aligned}
(x+y)^{\{n,M\}} &= \sum_{j=0}^{n} \binom{n}{j} x^{\{j,L^{n-j}(M)\}} y^{\{n-j,M^{*n-j-1}\}} \\
&= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{\{j,L^{n-j}(M)\}} y^{\{n-j,M^{*n-j-1}\}} \\
&= \sum_{j=0}^{n} n! \frac{x^{\{j,L^{n-j}(M)\}}}{j!} \frac{y^{\{n-j,M^{*n-j-1}\}}}{(n-j)!} \\
&= \sum_{j=0}^{n} n! \binom{x}{j}_{\mathbb{H},L^{n-j}(M)} \binom{y}{n-j}_{\mathbb{H},M^{*n-j-1}}.
\end{aligned}$$

□

Theorem 3.33 (Vandermonde's Convolution)**.** *If M is a shifting sequence and $x, y \in \mathbb{H}$, then*

$$\binom{x+y}{n}_{\mathbb{H},M} = \sum_{j=0}^{n} \binom{x}{j}_{\mathbb{H},L^{n-j}(M)} \binom{y}{n-j}_{\mathbb{H},M^{*n-j-1}}.$$

Proof. The proof follows directly from Lemma 3.32 by dividing through by $n!$. □

Letting $M = M_F = \{0,0,0,\ldots\}$ yields the equivalent convolution for the harmonic binomial coeffiecient.

Corollary 3.34. *If* $x, y \in \mathbb{H}$*, then*

$$\binom{x+y}{n}_{\mathbb{H}} = \sum_{j=0}^{n} \binom{x}{j}_{\mathbb{H}} \binom{y}{n-j}_{\mathbb{H},M_R}.$$

A consequence of Corollary 3.34 is an analog to the well known Chu-Vandermonde identity for the binomial coefficient.

Corollary 3.35. *For* $H_k \in \mathbb{H}$*,*

$$\binom{H_k + H_k}{k}_{\mathbb{H}} = \sum_{j=0}^{k} \binom{H_k}{j}_{\mathbb{H}} \binom{H_k}{k-j}_{\mathbb{H},M_R}.$$

Proof. Since $H_k \in \mathbb{H}$, we have $k \in \mathbb{N}_0$. Let $x = y = H_k$ and $n = k$ in Corollary 3.34. □

4. HARMONIC POWER SERIES AND APPLICATIONS

In 4.1 we define a harmonic exponential as the infinite sum of terms of the form $\frac{H_k^{\{\underline{j}\}}}{j!}$ and show that our series definition is equivalent to the standard harmonic exponential function (cf. [2]). We then develop a class of harmonic polynomials that allow us to consider more of this kind of power series built of M-factorial functions. Using these harmonic power series, we construct harmonic analogs to the sine and cosine function and demonstrate that these functions are solutions of a classic boundary value problem.

4.1. A HARMONIC EXPONENTIAL FUNCTION

Definition 4.1. A harmonic analog of the classic exponential function e^x is

$$e_H(H_k) = \sum_{n=0}^{\infty} \binom{H_k}{n}_{\mathbb{H}} = \sum_{n=0}^{\infty} \frac{H_k^{\{\underline{n}\}}}{n!}.$$

Since $H_k^{\{\underline{n}\}} = 0$ for $n \geq k+1$, convergence is assured. In fact, we could use the equivalent expression

$$e_H(H_k) = \sum_{n=0}^{k} \binom{H_k}{n}_{\mathbb{H}} = \sum_{n=0}^{k} \frac{H_k^{\{\underline{n}\}}}{n!},$$

but the infinite sum actually simplifies the task of proving the following result.

Theorem 4.2. $e_H^{\Delta} = e_H$.

Proof. From Lemma 3.4,

$$\begin{aligned} e_H^{\Delta}(H_k) &= \left[\sum_{n=0}^{\infty} \binom{H_k}{n}_{\mathbb{H}}\right]^{\Delta} = \sum_{n=0}^{\infty} \binom{H_k}{n}_{\mathbb{H}}^{\Delta} = \sum_{n=1}^{\infty} \binom{H_k}{n-1}_{\mathbb{H}} = \sum_{n=0}^{\infty} \binom{H_k}{n}_{\mathbb{H}} \\ &= e_H(H_k). \end{aligned}$$

□

Theorem 4.3. *If $H_k \in \mathbb{H}$, then $e_H(H_k) = k+1$.*

Proof. If we let $f(H_k) = k+1$, then

$$f^{\Delta}(H_k) = (k+1)[f(H_{k+1}) - f(H_k)] = k+1 = f(H_k),$$

and $f(H_1) = 2 = e_H(H_1)$. Therefore, by uniqueness of the first order linear dynamic equation [2], the identity is established. □

4.2. HARMONIC POWER SERIES

In the previous section, we formally defined a series expression for a harmonic exponential function. In order to consider power series definitions of other functions, we shall develop some harmonic polynomials.

Definition 4.4. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence. For $H_k, h \in \mathbb{H}$, we define a *harmonic polynomial* in H_k, $P(H_k)$, to be a linear combination of ${H_k}^{\{n,M\}}$. If

$$P(H_k) = \sum_{n=0}^{N} a_n (H_k - h)^{\{n,M\}},$$

where $a_n \in \mathbb{R}$ for $0 \le n \le N$ and $a_N(H_k - h)^{\{N,M\}} \neq 0$, we say $P(H_k)$ is an Nth degree harmonic polynomial.

From the definition, it should be clear that the set of harmonic polynomials forms a linear space over $\mathbb{R}$.

Definition 4.5 (Harmonic Power Series)**.** Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence. Let h be an element of $\mathbb{H}$, $\{a_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers, and $P_0(H_k), P_1(H_k), P_2(H_k), \ldots$ be a sequence of harmonic polynomials satisfying the following four conditions:

(i) $P_0(H_k) = 1$;

(ii) Either $P_n = 0$ or the degree of P_n is n;

(iii) $P_n(h) = 0$ for $n \in \mathbb{N}$;

(iv) $P_n{}^{\Delta} = [P_{n-1}]^{\sigma^{m_{n-1}}}$ for $n \geq 1$.

Then $\sum_{n=0}^{\infty} a_n P_n(H_k)$ is a *harmonic power series*.

The special cases of Definition 4.5 where $M = \{0, 0, 0, \ldots\}$ or $M = \{1, 1, 1, \ldots\}$ have been explored thoroughly in the time scales calculus (cf. [2]). In the next two remarks we confirm this relationship.

Remark 4.6. If we choose $P_n(H_k) = \dfrac{1}{n!}(H_k - h)^{\{\underline{n}\}}$ then the harmonic power series in Definition 4.5 match the generalized polynomials used in time scales calculus Taylor's Formula for the case $\mathbb{T} = \mathbb{H}$ (cf. [2] Theorem 1.113).

Remark 4.7. Likewise, if we choose $P_n(H_k) = \dfrac{1}{n!}(H_k - h)^{\{\overline{n}\}}$ then Definition 4.5 gives the set of generalized polynomials from the other Taylor's Formula used in time scales calculus for the case $\mathbb{T} = \mathbb{H}$ (cf. [2] Theorem 1.111).

We will not restrict our considerations to either of these cases. The following examples illustrate two sequences of harmonic polynomials not usually considered in the time scales power series.

Example 4.8. Let $M_O = \{m_i\}_{i \in \mathbb{N}_0}$ where $m_i = 1$ if i is odd and $m_i = 0$ otherwise. For $h \in \mathbb{H}$, let $P_n(H_k)$ be defined by

$$P_n(H_k) = \frac{(H_k - h)^{\{n, M_O\}}}{n!}.$$

From the definition, it is clear that any nonzero P_n is an nth degree generalized polynomial, $P_0 = 1$, and $P_n(h) = 0$ for $n \geq 1$.

By Lemma 2.34, the delta derivative of P_n is

$$P_n^{\Delta}(H_k) = \begin{cases} 0 & \text{if } n = 0 \\ P_{n-1}\left(\sigma^{m_{n-1}}(H_k)\right) & \text{if } n \geq 1. \end{cases}$$

Therefore, $\{P_n\}$ satisfy the conditions of Definition 4.5.

Example 4.9. Let $M_E = \{m_i\}_{i \in \mathbb{N}_0}$ where $m_i = 1$ if i is even and $m_i = 0$ otherwise. For $h \in \mathbb{H}$, let $Q_n(H_k)$ be defined by

$$Q_n(H_k) = \frac{(H_k - h)^{\{n, M_E\}}}{n!}.$$

Likewise, $\{Q_n\}$ also satisfies the conditions of Definition 4.5.

4.3. HARMONIC SINE AND COSINE AND THE SECOND ORDER EIGENVALUE PROBLEM

We now consider power series definitions of classic functions such as the sine and cosine functions. It might be tempting to define the harmonic sine as the alternating series composed of the odd terms in the series expansion of the harmonic exponential function from Definition 4.1, and likewise to define the harmonic cosine as the alternating series composed of the even terms. However, functions defined in this manner do not satisfy the second order self-adjoint dynamic equation

$$f^{\Delta\Delta} + f^{\sigma} = 0. \tag{4.1}$$

In order to solve the eigenvalue problems in (1.1) and (1.2), we will need harmonic sine and cosine functions that are solutions to (4.1). With this in mind, we make the following definitions.

Definition 4.10. Let $h \in \mathbb{H}$. We define the *harmonic sine function* $\sin_{\mathbb{H}}$ with phase shift h in terms of harmonic power series, using the polynomials P_n from Example 4.8 as follows:

$$\sin_{\mathbb{H}}(H_k - h) = \sum_{j=0}^{\infty} (-1)^j P_{2j+1}(H_k) = \sum_{j=0}^{\infty} \frac{(-1)^j (H_k - h)^{\{2j+1, M_O\}}}{(2j+1)!}.$$

Definition 4.11. Let $h \in \mathbb{H}$. We likewise define the *harmonic cosine function* $\cos_{\mathbb{H}}$ with phase shift h in terms of harmonic power series using the polynomials Q_n from Example 4.9 as follows:

$$\cos_{\mathbb{H}}(H_k - h) = \sum_{j=0}^{\infty} (-1)^j Q_{2j}(H_k) = \sum_{j=0}^{\infty} \frac{(-1)^j (H_k - h)^{\{2j, M_E\}}}{(2j)!}.$$

Convergence of both of these power series is assured since, for fixed k, $(H_k - h)^{\{n, M_O\}} = 0$ and $(H_k - h)^{\{n, M_E\}} = 0$ whenever n is sufficiently large.

Example 4.12. Let $h \in \mathbb{H}$. We will verify that $\sin_{\mathbb{H}}(0) = 0$ and $\cos_{\mathbb{H}}(0) = 1$. We begin with the definition of $\sin_{\mathbb{H}}$,

$$\sin_{\mathbb{H}}(0) = \sin_{\mathbb{H}}(h - h) = \sum_{j=0}^{\infty} (-1)^j P_{2j+1}(h).$$

From Example 4.8 we know $P_n(h) = 0$ for all $n \geq 1$. Therefore,

$$\sin_{\mathbb{H}}(0) = 0.$$

Next we consider $\cos_{\mathbb{H}}$, whose definition is

$$\cos_{\mathbb{H}}(0) = \cos_{\mathbb{H}}(h - h) = \sum_{j=0}^{\infty} (-1)^j Q_{2j}(h).$$

From Example 4.9, we know $Q_n(h) = 0$ for all $n \geq 1$ and $Q_0(h) = 1$. Thus, we obtain

$$\cos_{\mathbb{H}}(0) = 1.$$

Notice that the polynomials in the power series for $\sin_{\mathbb{H}}$ are the subsequence of $\{P_n\}$ from Example 4.8 with odd n, and those for $\cos_{\mathbb{H}}$ are the even subscripted $\{Q_n\}$.

It is natural to consider what relationship the harmonic derivatives ${\sin_{\mathbb{H}}}^{\Delta}$ and ${\cos_{\mathbb{H}}}^{\Delta}$ have with the original functions. Referring once again to their definitions, we explore these relationships in the following lemmas.

Lemma 4.13. *Let $H_k, h \in \mathbb{H}$ and P_n be defined as in Example 4.8. Then* $\sin_{\mathbb{H}}$ *is twice differentiable and the first two harmonic derivatives of* $\sin_{\mathbb{H}}$ *are*

$${\sin_{\mathbb{H}}}^{\Delta}(H_k - h) = \sum_{j=0}^{\infty} (-1)^j P_{2j}(H_k)$$

and

$$\sin_{\mathbb{H}}^{\Delta\Delta}(H_k - h) = -\sin_{\mathbb{H}}^{\sigma}(H_k - h).$$

Proof. Taking the harmonic derivative on both sides of the definition of $\sin_{\mathbb{H}}$, we obtain

$${\sin_{\mathbb{H}}}^{\Delta}(H_k - h) = \left[\sum_{j=0}^{\infty} (-1)^j P_{2j+1}(H_k)\right]^{\Delta} = \sum_{j=0}^{\infty} (-1)^j {P_{2j+1}}^{\Delta}(H_k).$$

The calculation of the harmonic derivative of P_{2j+1} from Example 4.8 yields the first result

$$\begin{aligned} {\sin_{\mathbb{H}}}^{\Delta}(H_k - h) &= \sum_{j=0}^{\infty} (-1)^j P_{2j}\left(\sigma^{m_{2j}}(H_k)\right) \\ &= \sum_{j=0}^{\infty} (-1)^j P_{2j}(H_k) \end{aligned}$$

since $m_{2j} = 0$. Taking the harmonic derivative on both sides again,

$$\sin_{\mathbb{H}}^{\Delta\Delta}(H_k - h) = \sum_{j=1}^{\infty} (-1)^j {P_{2j}}^{\Delta}(H_k)$$

which from Example 4.8 (recalling $m_{2j-1} = 1$) evaluates to

$$\sin_{\mathbb{H}}^{\Delta\Delta}(H_k - h) = \sum_{j=1}^{\infty} (-1)^j P_{2j-1}\left(\sigma(H_k)\right).$$

Reindexing the sum, we obtain the second result:

$$\sin_{\mathbb{H}}^{\Delta\Delta}(H_k - h) = -\sum_{j=0}^{\infty} (-1)^j P_{2j+1}^{\sigma}(H_k) = -\sin_{\mathbb{H}}^{\sigma}(H_k - h).$$

□

A similar argument proves the corresponding lemma for $\cos_{\mathbb{H}}$.

Lemma 4.14. *Let $H_k, h \in \mathbb{H}$ and Q_n be defined as in Example 4.9. Then* $\cos_{\mathbb{H}}$ *is twice differentiable and the first two harmonic derivatives of* $\cos_{\mathbb{H}}$ *are*

$${\cos_{\mathbb{H}}}^{\Delta}(H_k - h) = -\sum_{j=0}^{\infty} (-1)^j Q_{2j+1}(H_k)$$

and

$$\cos_{\mathbb{H}}^{\Delta\Delta}(H_k - h) = -\cos_{\mathbb{H}}^{\sigma}(H_k - h).$$

Proof. Taking the harmonic derivative on both sides of the definition of $\cos_{\mathbb{H}}$, we obtain

$${\cos_{\mathbb{H}}}^{\Delta}(H_k - h) = \left[\sum_{j=0}^{\infty} (-1)^j Q_{2j}(H_k)\right]^{\Delta} = \sum_{j=0}^{\infty} (-1)^j {Q_{2j}}^{\Delta}(H_k).$$

Using the calculation for the harmonic derivative of Q_{2j} from Example 4.9 (this time $m_{2j-1} = 0$ and $m_{2j} = 1$) we obtain

$$\cos_{\mathbb{H}}{}^{\Delta}(H_k - h) = \sum_{j=1}^{\infty} (-1)^j Q_{2j-1}(H_k).$$

Reindexing the sum yields the first result:

$$\cos_{\mathbb{H}}{}^{\Delta}(H_k - h) = -\sum_{j=0}^{\infty} (-1)^j Q_{2j+1}(H_k).$$

Taking the harmonic derivative on both sides again,

$$\cos_{\mathbb{H}}^{\Delta\Delta}(H_k - h) = -\sum_{j=0}^{\infty} (-1)^j Q_{2j+1}{}^{\Delta}(H_k),$$

which from Example 4.9 we know to be

$$\cos_{\mathbb{H}}^{\Delta\Delta}(H_k - h) = -\sum_{j=0}^{\infty} (-1)^j Q_{2j}\left(\sigma(H_k)\right) = -\cos_{\mathbb{H}}{}^{\sigma}(H_k - h).$$

□

Thus, taking the second harmonic derivative of $\sin_{\mathbb{H}}$ and $\cos_{\mathbb{H}}$ is analogous to taking the classic second derivatives of sin and cos, since $\sigma(t) = t$ for $t \in \mathbb{R}$. The next result demonstrates that we can use the functions $\sin_{\mathbb{H}}$ and $\cos_{\mathbb{H}}$ to construct a general solution to the second order harmonic dynamic equation $f^{\Delta\Delta} + f^{\sigma} = 0$. This is the dynamic equation from the harmonic boundary value problem stated in (1.1) with $\lambda = 1$.

Corollary 4.15. *The harmonic sine and cosine functions* $\sin_{\mathbb{H}}$ *and* $\cos_{\mathbb{H}}$ *are linearly independent solutions of the self-adjoint harmonic dynamic equation* $f^{\Delta\Delta} + f^{\sigma} = 0$.

Proof. The fact that $\sin_{\mathbb{H}}$ and $\cos_{\mathbb{H}}$ are solutions follows directly from Lemmas 4.13 and 4.14. The fact that the are linearly independent can be verified by using the Wronskian. A corollary to Abel's Theorem from [2] states that the Wronskian of any two solutions of (1.1) is independent of H_k. Therefore, for any $H_k \in \mathbb{H}$,

$$\begin{aligned} W(\sin_{\mathbb{H}}, \cos_{\mathbb{H}})(H_k) &= \cos_{\mathbb{H}}(H_k - h)\sin_{\mathbb{H}}{}^{\Delta}(H_k - h) - \sin_{\mathbb{H}}(H_k - h)\cos_{\mathbb{H}}{}^{\Delta}(H_k - h) \\ &= \cos_{\mathbb{H}}(h - h)\sin_{\mathbb{H}}{}^{\Delta}(h - h) - \sin_{\mathbb{H}}(h - h)\cos_{\mathbb{H}}{}^{\Delta}(h - h) \\ &= \cos_{\mathbb{H}}(0)\sin_{\mathbb{H}}{}^{\Delta}(0) - \sin_{\mathbb{H}}(0)\cos_{\mathbb{H}}{}^{\Delta}(0) \\ &= 1 \cdot 1 - 0 \cdot 0 \\ &= 1. \end{aligned}$$

□

Remark 4.16. If $H_k, h \in \mathbb{H}$, then the equation

$$\cos_{\mathbb{H}}(H_k - h)\sin_{\mathbb{H}}{}^{\Delta}(H_k - h) - \sin_{\mathbb{H}}(H_k - h)\cos_{\mathbb{H}}{}^{\Delta}(H_k - h) = 1$$

can be viewed as an analog of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$.

We now turn our attention to developing a solution to the the second order harmonic eigenvalue problem (1.1) for λ other than 1, first with Dirichlet boundary conditions:

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f(\alpha) = f(\beta) = 0. \tag{4.2}$$

Since $\cos_{\mathbb{H}}$ does not satisfy the Dirichlet boundary conditions, we now focus our attention on the harmonic sine function $\sin_{\mathbb{H}}$. Instead of the previous definition, we will construct a more general definition that incorporates a frequency.

Definition 4.17. For $H_k, h \in \mathbb{H}$ and $\omega \in \mathbb{R}$, we define the *harmonic sine function* with frequency ω and phase shift h by

$$\sin_{\mathbb{H}}(H_k - h; \omega) = \sum_{j=0}^{\infty} (-1)^j \omega^{2j+1} P_{2j+1}(H_k).$$

When $\omega = 1$ or $h = 0$, they will be suppressed. We write, for example,

$$\sin_{\mathbb{H}}(H_k) = \sin_{\mathbb{H}}(H_k - 0; 1).$$

From Example 4.8 we can calculate the first and second harmonic derivatives of this harmonic sine as follows:

$$\sin_{\mathbb{H}}{}^{\Delta}(H_k - h; \omega) = \omega \sum_{j=0}^{\infty} (-1)^j \omega^{2j} P_{2j}(H_k).$$

We can take the harmonic derivative of both sides of the equation above to find the second harmonic derivative:

$$\sin_{\mathbb{H}}{}^{\Delta\Delta}(H_k - h; \omega) = \left[\omega \sum_{j=0}^{\infty} (-1)^j \omega^{2j} P_{2j}(H_k) \right]^{\Delta}.$$

Passing the Δ through the sum yields

$$\sin_{\mathbb{H}}{}^{\Delta\Delta}(H_k - h; \omega) = \omega \sum_{j=0}^{\infty} (-1)^j \omega^{2j} P_{2j}{}^{\Delta}(H_k).$$

We factor out an ω and imitate the proof of Lemma 4.13 to obtain

$$\sin_{\mathbb{H}}{}^{\Delta\Delta}(H_k - h; \omega) = \omega^2 \sum_{j=1}^{\infty} (-1)^j \omega^{2j-1} P_{2j-1}{}^{\sigma}(H_k).$$

Finally, reindexing gives the identity

$$\sin_{\mathbb{H}}{}^{\Delta\Delta}(H_k - h; \omega) = -\omega^2 \sum_{j=0}^{\infty} (-1)^j \omega^{2j+1} P_{2j+1}{}^{\sigma}(H_k) = -\omega^2 \sin_{\mathbb{H}}{}^{\sigma}(H_k - h; \omega). \quad (4.3)$$

Theorem 4.18. *If* $\lambda > 0$ *and* α, β *are elements of* $\mathbb{H}$ *such that* $\sin_{\mathbb{H}}(\beta - \alpha; \sqrt{\lambda}) = 0$*, then* $\sin_{\mathbb{H}}(H_k - \alpha; \sqrt{\lambda})$ *is a solution of the harmonic boundary value problem*

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f(\alpha) = f(\beta) = 0.$$

Example 4.19. For $\alpha = 0$ and $\beta = H_2 = \frac{3}{2}$, we have

$$\sin_{\mathbb{H}}(H_2, \sqrt{\lambda}) = \sum_{j=0}^{\infty} (-1)^j (\sqrt{\lambda})^{2j+1} \frac{H_2{}^{\{2j+1, M_O\}}}{(2j+1)!}.$$

Since, when $j \geq 2$, $H_2{}^{\{2j+1, M_O\}} = 0$, this reduces to

$$\sin_{\mathbb{H}}(H_2, \sqrt{\lambda}) = (\lambda)^{\frac{1}{2}} \frac{H_2{}^{\{1, M_O\}}}{1!} - (\lambda)^{\frac{3}{2}} \frac{H_2{}^{\{3, M_O\}}}{3!}.$$

Upon simplifying and equating to zero, we obtain

$$\frac{3}{2}\lambda^{\frac{1}{2}} - \frac{1}{2}\lambda^{\frac{3}{2}} = 0.$$

Multiplying both sides by $\frac{2}{\sqrt{\lambda}}$ yields the equation

$$3 - \lambda = 0.$$

Thus when $\beta = H_2$, $\lambda = 3$. From (4.3), $\sin_{\mathbb{H}}{}^{\Delta\Delta}(H_k; \sqrt{3}) = -3\sin_{\mathbb{H}}{}^{\sigma}(H_k; \sqrt{3})$. Using the same argument as in Example 4.12 we see that $\sin_{\mathbb{H}}(0; \sqrt{3}) = 0$. Therefore, $\sin_{\mathbb{H}}(H_k; \sqrt{3})$ is the eigenfunction corresponding to $\lambda = 3$. That is to say, $\sin_{\mathbb{H}}(H_k; \sqrt{3})$ is a solution of the harmonic boundary value problem

$$f^{\Delta\Delta} + 3f^{\sigma} = 0; f(0) = f(H_2) = 0.$$

Example 4.20. Let $\alpha = 0$ and $\beta = H_3$, then

$$\sin_{\mathbb{H}}(H_3, \sqrt{\lambda}) = \sum_{j=0}^{\infty} (-1)^j (\sqrt{\lambda})^{2j+1} \frac{{H_3}^{\{2j+1, M_O\}}}{(2j+1)!}.$$

Since, when $j \geq 3$, ${H_3}^{\{2j+1, M_O\}} = 0$, this reduces to

$$\sin_{\mathbb{H}}(H_3, \sqrt{\lambda}) = \frac{\lambda^{\frac{1}{2}} {H_3}^{\{1, M_O\}}}{1!} - \frac{\lambda^{\frac{3}{2}} {H_3}^{\{3, M_O\}}}{3!} + \frac{\lambda^{\frac{5}{2}} {H_3}^{\{5, M_O\}}}{5!}.$$

Upon simplifying and equating to zero, we obtain

$$\frac{11}{6}\lambda^{\frac{1}{2}} - \frac{13}{12}\lambda^{\frac{3}{2}} + \frac{1}{12}\lambda^{\frac{5}{2}} = 0.$$

Multiplying both sides by $\frac{12}{\sqrt{\lambda}}$ yields the quadratic equation

$$\lambda^2 - 13\lambda + 22 = 0.$$

Thus when $\alpha = 0$ and $\beta = H_3$, there are two eigenvalues of (4.2): $\lambda_1 = 2$ and $\lambda_2 = 11$. From (4.3),

$$\sin_{\mathbb{H}}{}^{\Delta\Delta}(H_k; \sqrt{2}) = -2\sin_{\mathbb{H}}{}^{\sigma}(H_k; \sqrt{2})$$

and

$$\sin_{\mathbb{H}}{}^{\Delta\Delta}(H_k; \sqrt{11}) = -11\sin_{\mathbb{H}}{}^{\sigma}(H_k; \sqrt{11}).$$

Using the same argument as in Example 4.12 we see that $\sin_{\mathbb{H}}(0; \sqrt{2}) = 0$ and $\sin_{\mathbb{H}}(0; \sqrt{11}) = 0$. Therefore, $f_1 = \sin_{\mathbb{H}}(H_k; \sqrt{2})$ and $f_2 = \sin_{\mathbb{H}}(H_k; \sqrt{11})$ are the eigenfunctions corresponding to the eigenvalues λ_1 and λ_2, respectively. We may conclude that f_1 and f_2, respectively, are solutions of the harmonic boundary value problems

$$f^{\Delta\Delta} + 2f^{\sigma} = 0; f(0) = f(H_3) = 0$$

and

$$f^{\Delta\Delta} + 11f^{\sigma} = 0; f(0) = f(H_3) = 0.$$

In general, if $\alpha = 0$ the $N-1$ solutions of the $(N-1)$st degree polynomial equation $\lambda^{-\frac{1}{2}} \sin_{\mathbb{H}}(H_N, \sqrt{\lambda}) = 0$ will be eigenvalues of (4.2) with $\beta = H_N$.

With explicit solutions to the Dirichlet problem in (4.2), it is natural to consider whether $\cos_{\mathbb{H}}$ can be used to solve the corresponding Neumann problem

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f^{\Delta}(\alpha) = f^{\Delta}(\beta) = 0. \tag{4.4}$$

To begin, just as we did for $\sin_{\mathbb{H}}$ we will extend the definition of $\cos_{\mathbb{H}}$ to allow for a frequency.

Definition 4.21. For $H_k, h \in \mathbb{H}$ and $\omega \in \mathbb{R}$, we define the *harmonic cosine function* with frequency ω and phase shift h by

$$\cos_{\mathbb{H}}(H_k - h; \omega) = \sum_{j=0}^{\infty} (-1)^j \omega^{2j} Q_{2j}(H_k).$$

When $\omega = 1$ or $h = 0$, they will be suppressed. We write, for example,

$$\cos_{\mathbb{H}}(H_k) = \cos_{\mathbb{H}}(H_k - h; 1).$$

Just as with $\sin_{\mathbb{H}}$, we'll calculate ${\cos_{\mathbb{H}}}^{\Delta\Delta}$ using Example 4.9. First,

$${\cos_{\mathbb{H}}}^{\Delta}(H_k - h; \omega) = \sum_{j=1}^{\infty} (-1)^j \omega^{2j} Q_{2j-1}(H_k).$$

Reindexing yields

$$\cos_{\mathbb{H}}{}^{\Delta}(H_k - h; \omega) = -\omega \sum_{j=0}^{\infty} (-1)^j \omega^{2j+1} Q_{2j+1}(H_k).$$

We can take the harmonic derivative of both sides of the equation above to find the second harmonic derivative:

$$\cos_{\mathbb{H}}{}^{\Delta\Delta}(H_k - h; \omega) = -\omega \sum_{j=0}^{\infty} (-1)^j \omega^{2j+1} Q_{2j}{}^{\sigma}(H_k).$$

Factoring out an ω yields the identity

$$\cos_{\mathbb{H}}{}^{\Delta\Delta}(H_k - h; \omega) = -\omega^2 \sum_{j=0}^{\infty} (-1)^j \omega^{2j} Q_{2j}{}^{\sigma}(H_k) = -\omega^2 \cos_{\mathbb{H}}{}^{\sigma}(H_k - h; \omega). \tag{4.5}$$

Theorem 4.22. *If $\lambda > 0$ and α and β are elements of $\mathbb{H}$ such that $\cos_{\mathbb{H}}{}^{\Delta}(\beta - \alpha; \sqrt{\lambda}) = 0$, then $\cos_{\mathbb{H}}(H_k - \alpha; \sqrt{\lambda})$ is a solution of the harmonic boundary value problem with Neumann conditions*

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f^{\Delta}(\alpha) = f^{\Delta}(\beta) = 0.$$

Example 4.23. For $\alpha = 0$ and $\beta = H_2$, we have

$$\cos_{\mathbb{H}}{}^{\Delta}(H_2, \sqrt{\lambda}) = -\sqrt{\lambda} \left(\frac{\lambda^{\frac{1}{2}} H_2{}^{\{1,M_E\}}}{1!} - \frac{\lambda^{\frac{3}{2}} H_2{}^{\{3,M_E\}}}{3!} \right).$$

Upon simplifying and equating to zero, we obtain

$$-\frac{1}{2}\lambda + \frac{3}{2}\lambda^2 = 0.$$

Dividing both sides by $\dfrac{\lambda}{2}$, which must not be zero, yields the equation

$$3\lambda - 1 = 0.$$

Thus when $\alpha = 0$ and $\beta = H_2$, $\lambda = \frac{1}{3}$. From (4.5), $\cos_{\mathbb{H}}{}^{\Delta\Delta}(H_k; \sqrt{\frac{1}{3}}) = -\frac{1}{3}\cos_{\mathbb{H}}{}^{\sigma}(H_k; 3)$. Using the same argument as in Example 4.12 we see that $\cos_{\mathbb{H}}{}^{\Delta}(0; \sqrt{\frac{1}{3}}) = 0$. Therefore, $\cos_{\mathbb{H}}(H_k; \sqrt{\frac{1}{3}})$ is the eigenfunction corresponding to $\lambda = \frac{1}{3}$. That is to say, $\cos_{\mathbb{H}}(H_k; \sqrt{\frac{1}{3}})$ is a solution of the harmonic boundary value problem

$$f^{\Delta\Delta} + \frac{1}{3}f^{\sigma} = 0; f^{\Delta}(0) = f^{\Delta}(H_2) = 0.$$

Example 4.24. Let $\alpha = 0$ and $\beta = H_3$, then

$$\cos_{\mathbb{H}}{}^{\Delta}(H_3, \sqrt{\lambda}) = -\sqrt{\lambda}\left(\frac{\lambda^{\frac{1}{2}} H_3{}^{\{1,M_E\}}}{1!} - \frac{\lambda^{\frac{3}{2}} H_3{}^{\{3,M_E\}}}{3!} + \frac{\lambda^{\frac{5}{2}} H_3{}^{\{5,M_E\}}}{5!}\right).$$

Upon simplifying and equating to zero, we obtain

$$\frac{11}{6}\lambda - \frac{7}{12}\lambda^2 + \frac{1}{36}\lambda^3 = 0.$$

From this, we see the roots will be $\frac{21 \pm \sqrt{177}}{2}$. Thus when $\alpha = 0$ and $\beta = H_3$, there are two eigenvalues of (4.4): $\lambda_1 = \frac{21 + \sqrt{177}}{2}$ and $\lambda_2 = \frac{21 - \sqrt{177}}{2}$. From (4.5),

$$\cos_{\mathbb{H}}{}^{\Delta\Delta}\left(H_k; \sqrt{\frac{21 + \sqrt{177}}{2}}\right) = -\left(\frac{21 + \sqrt{177}}{2}\right)\cos_{\mathbb{H}}{}^{\sigma}\left(H_k; \sqrt{\frac{21 + \sqrt{177}}{2}}\right)$$

and

$$\cos_{\mathbb{H}}{}^{\Delta\Delta}\left(H_k; \sqrt{\frac{21 - \sqrt{177}}{2}}\right) = -\left(\frac{21 - \sqrt{177}}{2}\right)\cos_{\mathbb{H}}{}^{\sigma}\left(H_k; \sqrt{\frac{21 - \sqrt{177}}{2}}\right).$$

Using the same argument as in Example 4.12 we see that $\cos_{\mathbb{H}}{}^{\Delta}\left(0, \sqrt{\frac{21 \pm \sqrt{177}}{2}}\right) = 0$. Therefore, $f_1 = \cos_{\mathbb{H}}\left(H_k; \sqrt{\frac{21 + \sqrt{177}}{2}}\right)$ and $f_2 = \cos_{\mathbb{H}}\left(H_k; \sqrt{\frac{21 - \sqrt{177}}{2}}\right)$ are the eigenfunctions corresponding to the eigenvalues λ_1 and λ_2, respectively. We may conclude that f_1 and f_2, respectively, are solutions of the harmonic boundary value problems

$$f^{\Delta\Delta} + \left(\frac{21 + \sqrt{177}}{2}\right) f^{\sigma} = 0; f^{\Delta}(0) = f^{\Delta}(H_3) = 0$$

and

$$f^{\Delta\Delta} + \left(\frac{21 - \sqrt{177}}{2}\right) f^{\sigma} = 0; f^{\Delta}(0) = f^{\Delta}(H_3) = 0.$$

In general, the $N - 1$ solutions of the $(N - 1)$st degree polynomial equation $\lambda^{-1}\cos_{\mathbb{H}}{}^{\Delta}(H_N, \sqrt{\lambda}) = 0$ will be eigenvalues of (4.4) with $\alpha = 0$ and $\beta = H_N$.

5. GENERALIZATIONS TO OTHER TIME SCALES AND POSSIBILITIES FOR FUTURE RESEARCH

The harmonic M-factorials of Definition 2.27 and Definition 2.33 were constructed using antiderivatives. This method can be extended to a general time scale $\mathbb{T}$. In addition, if $\mathbb{T} = \{a_k\}_{k\in\mathbb{Z}}$ is the image of an increasing function $a : \mathbb{Z} \to \mathbb{R}$, then $\mathbb{T}$ is right-scattered, and therefore the graininess $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$. These time scales will possess an iterated sum form of the M-factorial function that will very much resemble the harmonic M-factorial. For time scales which contain 0, we will show that several of the results proved earlier for the harmonic numbers apply.

5.1. GENERALIZATION FROM $\mathbb{H}$ TO OTHER $\mathbb{T}$

In this section we define the M-factorial function over $\mathbb{T}$. As with the harmonic M-factorial of Definition 2.33, we will define it recursively in terms of antiderivatives. Because $\mathbb{T}$ may not be right-scattered, we first make the definition in terms of the iterated integral. Also, because we do not necessarily have an analog to Theorem 2.32 for a general $\mathbb{T}$, we will not use the minus sign.

Definition 5.1. Let $M = \{m_i\}_{i\in\mathbb{N}_0}$ be a shifting sequence. For $s, t \in \mathbb{T}$ and $n \geq 0$, define the *M-factorial function* over $\mathbb{T}$, $g_M^n : \mathbb{T}^2 \to \mathbb{R}$, recursively as follows:

$$g_M^0(t, s) \equiv 1 \quad \text{and}$$

$$g_M^n(t, s) = n \int_s^t g_M^{n-1}\left(\sigma^{m_{n-1}}(\tau), s\right) \Delta\tau.$$

As with Lemma 2.34, the derivative is obtained using Theorem 1.21 and Definition 1.22.

Lemma 5.2. *Let* $M = \{m_i\}_{i\in\mathbb{N}_0}$ *be a shifting sequence. If* $s, t \in \mathbb{T}$ *and* $n \geq 0$*, then*

$$\left[g_M^n(t,s)\right]^{\Delta_t} = n g_M^{n-1}\left(\sigma^{m_{n-1}}(t), s\right).$$

Lemma 5.3. *If* $a, b \in \mathbb{R}$ *and* $t, s \in \mathbb{T}\cap[a,b]$*, where* $\mathbb{T}\cap[a,b]$ *is right-dense, then for any shifting sequence* M *and all nonnegative integers* n*,*

$$g_M^n(t,s) = (t-s)^n$$

Proof. Since $\mathbb{T}\cap[a,b]$ is right dense, the delta derivative is the ordinary derivative and $\sigma(t) = t$. Therefore, both sides of the equation are solutions to the same ordinary initial value problem. □

Theorem 5.4. *If* $t, s \in \mathbb{T}$ *with* $t > s$ *and* $\mathbb{T}\cap[s,t]$ *is right-scattered, then there is an increasing sequence* $\{a_k\}_{k\in\mathbb{N}}$ *such that* $t = a_k$ *and* $s = a_i$ *with* $k > i$ *and, for every shifting sequence* M*,*

$$g_M^n(t,s) = n! \sum_{(j_n)\in J(M,n,i,k)} \prod_{l=1}^{n} \mu(a_{j_l}).$$

Proof. For any fixed s, a comparison of the delta derivatives along with $g_M^n(s,s) = 0$ for all $n \geq 1$ yields the result. □

As a result, time scales with such a right-scattered interval will share the following identities with the harmonic M-factorial (Theorem 2.38 and Theorem 2.40).

Corollary 5.5. *If* $a, b \in \mathbb{R}$ *and* $t, s \in \mathbb{T}\cap[a,b]$*, where* $\mathbb{T}\cap[a,b]$ *is right-scattered or right-dense, then*

$$g_M^n(t,s) = (-1)^n g_{M^{*(n-1)}}^n(s,t), \text{ and}$$

$$\left[g_M^n(t,s)\right]^{\Delta_s} = -n g_{L(M)}^{n-1}\left(t, \sigma^{1-m_1}(s)\right).$$

Proof. If $\mathbb{T} \cap [a, b]$ is right-dense, then repeated application of Lemma 5.3 yields

$$g^n_M(t, s) = (t - s)^n = (-1)^n (s - t)^n = (-1)^n g^n_{M^{*(n-1)}}(s, t)$$

and

$$\left[g^n_M(t, s)\right]^{\Delta_s} = \left[(t - s)^n\right]^{\Delta_s} = -n(t - s)^{n-1} = -n g^{n-1}_{L(M)}\left(t, \sigma^{1-m_1}(s)\right).$$

If $\mathbb{T} \cap [a, b]$ is right-scattered, the results are obtained by imitating the proofs of Lemma 2.37, Theorem 2.38, and Theorem 2.40 using Theorem 5.4 in place of Theorem 2.35. □

As with the harmonic numbers, two special cases are worth mentioning. For $M_F = \{0, 0, 0, \ldots\}$ and $M_R = \{1, 1, 1, \ldots\}$ we call $g^n_{M_F}(t, s)$ and $g^n_{M_R}(t, s)$ the falling and rising factorial functions over $\mathbb{T}$, respectively. The functions $h_n(t, s) = \dfrac{g^n_{M_F}(t, s)}{n!}$ and $g_n(t, s) = \dfrac{g^n_{M_R}(t, s)}{n!}$ are commonly referred to as the Taylor monomials in time scales literature (cf. [2] and [7]). In [2] it is shown that the Taylor monomials have the negation property of Corollary 5.5 regardless of the time scale, summarized here as follows.

Lemma 5.6. *For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$,*

$$g_n(t, s) = (-1)^n h_n(s, t),$$

which is equivalent to

$$g^n_{M_R}(t, s) = (-1)^n g^n_{M_F}(s, t).$$

The next result can be found in [7].

Lemma 5.7. *For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$ with $t \geq s$,*

$$h_n(t, s) \leq \frac{(t - s)^n}{n!}.$$

We now present an analogous result when $t < s$.

Lemma 5.8. *For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$ with $t < s$,*

$$|g^n_{M_F}(t,s)| \leq (s-t)^n.$$

Proof. Let $t, s \in \mathbb{T}$ with $t < s$. Suppose the statement is true for some $n-1 \in \mathbb{N}_0$. Then

$$\begin{aligned} |g^n_{M_F}(t,s)| &= |n \int_s^t g^{n-1}_{M_F}(\tau,s)\ \Delta\tau| \\ &\leq n \int_t^s |g^{n-1}_{M_F}(\tau,s)|\,\Delta\tau. \end{aligned}$$

Applying the inductive hypothesis yields

$$|g^n_{M_F}(t,s)| \leq n \int_t^s (s-\tau)^{n-1}\,\Delta\tau.$$

Applying Lemma 1.10, we obtain

$$\begin{aligned} |g^n_{M_F}(t,s)| &\leq n \int_t^s \frac{[(s-\tau)^n]^\Delta}{n}\,\Delta\tau \\ &= (s-t)^n. \end{aligned}$$

□

The next few results, which will be useful when we discuss power series, provide some properties of any M-factorial function.

Lemma 5.9. *Let M be a shifting sequence. For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$ with $t \geq s$,*

$$0 \leq g^n_M(t,s) \leq g^n_{M_R}(t,s).$$

Proof. The left hand inequality is obvious as $g^{n-1}_M(t,s) \geq 0$ implies that $g^n_M(t,s)$ is nondecreasing on $(s,\infty)_{\mathbb{T}}$, and we have $g^n_M(s,s) = 0$.

We show the right hand inequality by induction. Assume the statement holds for some $n-1 \in \mathbb{N}_0$. Then

$$\begin{aligned} g_M^n(t,s) &= n\int_s^t g_M^{n-1}\left(\sigma^{m_{n-1}}(\tau), s\right)\Delta\tau \\ &\le n\int_s^t g_M^{n-1}\left(\sigma(\tau), s\right)\Delta\tau \\ &\le n\int_s^t g_{M_R}^{n-1}\left(\sigma(\tau), s\right)\Delta\tau \\ &= g_{M_R}^n(t,s). \end{aligned}$$

□

Lemma 5.10. *Let M be a shifting sequence. For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$ with $t < s$,*

$$g_M^n(t,s) \le 0 \text{ if } n \text{ is odd, and}$$

$$g_M^n(t,s) \ge 0 \text{ if } n \text{ is even.}$$

Proof. By induction. $g_M^0(t,s) \equiv 1$. Assume the statement holds for all whole numbers up to $n-1$ and note $g_M^n(s,s) = 0$ for $n \ge 1$. Then if n is odd,

$$\left[g_M^n(t,s)\right]^\Delta = n g_M^{n-1}\left(\sigma^{m_{n-1}}(t), s\right) \ge 0$$

implies that $g_M^n(t,s)$ is nondecreasing for $t \in (-\infty, s)_\mathbb{T}$. Therefore, $g_M^n(t,s) \le g_M^n(s,s) = 0$.

The proof is almost identical for even n. □

Corollary 5.11. *Let M be a shifting sequence. For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$ with $t < s$,*

$$\left|g_M^n(\sigma(t), s)\right| \le \left|g_M^n(t,s)\right|.$$

Proof. This is clear when $n = 0$, since both sides of the inequality are 1.

We now use Lemma 5.10 to prove the result when n is positive. If n is odd, then

$$g_M^n(t,s) \leq g_M^n(\sigma(t),s) \leq g_M^n(s,s) = 0,$$

and if $n > 0$ is even,

$$0 = g_M^n(s,s) \leq g_M^n(\sigma(t),s) \leq g_M^n(t,s).$$

□

Lemma 5.12. *Let M be a shifting sequence. For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$ with $t < s$,*

$$|g_M^n(t,s)| \leq |g_{M_F}^n(t,s)|.$$

Proof. By induction. It is clear that

$$g_M^0(t,s) \equiv g_{M_F}^0(t,s) \equiv 1.$$

Suppose the statement holds for some $n-1 \in \mathbb{N}_0$. Then,

$$\begin{aligned} |g_M^n(t,s)| &= |n \int_s^t g_M^{n-1}(\sigma^{m_{n-1}}(\tau),s)\ \Delta\tau| \\ &\leq n \int_t^s |g_M^{n-1}(\sigma^{m_{n-1}}(\tau),s)|\,\Delta\tau. \end{aligned}$$

Applying Corollary 5.11 and then the inductive hypothesis, we obtain

$$\begin{aligned} |g_M^n(t,s)| &\leq n \int_t^s |g_M^{n-1}(\tau,s)|\,\Delta\tau \\ &\leq n \int_t^s |g_{M_F}^{n-1}(\tau,s)|\,\Delta\tau. \end{aligned}$$

From Lemma 5.10 we have $|g^{n-1}_{M_F}(\tau,s)| = (-1)^{n-1} g^{n-1}_{M_F}(\tau,s)$, therefore

$$\begin{aligned} |g^{n}_{M}(t,s)| &\le (-1)^n n \int_s^t g^{n-1}_{M_F}(\tau,s)\,\Delta\tau \\ &= (-1)^n g^{n}_{M_F}(t,s) = |g^{n}_{M_F}(t,s)|. \end{aligned}$$

□

We now provide a bound for any M-factorial function.

Theorem 5.13. *Let M be a shifting sequence. For $n \in \mathbb{N}_0$ and $t, s \in \mathbb{T}$,*

$$|g^{n}_{M}(t,s)| \le |(t-s)^n|.$$

Proof. If $t \ge s$, then

$$\begin{aligned} |g^{n}_{M}(t,s)| &\le |g^{n}_{M_R}(t,s)| \\ &= |(-1)^n g^{n}_{M_F}(s,t)| \\ &= |g^{n}_{M_F}(s,t)| \\ &\le (t-s)^n \\ &= |(t-s)^n|. \end{aligned}$$

If $t < s$, then

$$|g^{n}_{M}(t,s)| \le |g^{n}_{M_F}(t,s)| \le (s-t)^n = |(t-s)^n|.$$

□

Definition 5.14. Let M be a shifting sequence and $\mathbb{T}$ be a time scale containing 0. For $t \in \mathbb{T}$ we define the *M-shifted binomial coefficient* over $\mathbb{T}$ as

$$\binom{t}{n}_{\mathbb{T},M} = \frac{g_M^n(t,0)}{n!}.$$

Lemma 5.15. *If M is a shifting sequence, $\mathbb{T}$ is a time scale containing 0, and $t \in \mathbb{T}$, then*

$$\binom{t}{n}_{\mathbb{T},M}^{\Delta} = \binom{\sigma^{m_{n-1}}(t)}{n-1}_{\mathbb{T},M}.$$

Theorem 5.16. *If $M = \{m_i\}_{i\in\mathbb{N}}$ is a shifting sequence and $t \geq 0$ is an element of a time scale $\mathbb{T}$ containing 0, then the M-shifted binomial coefficient over $\mathbb{T}$ satisfies the following Pascal rule for all $n \geq 1$:*

$$\binom{\sigma(t)}{n}_{\mathbb{T},M} = \binom{t}{n}_{\mathbb{T},M} + \mu(t)\binom{\sigma^{m_{n-1}}(t)}{n-1}_{\mathbb{T},M}.$$

Proof. If t is a right-dense point in $\mathbb{T}$, then $\mu(t) = 0$ and $\sigma(t) = t$.

If t is a right-scattered point in $\mathbb{T}$, then the result may be obtained by equating the expressions for $\binom{t}{n}_{T,M}^{\Delta}$ to obtain

$$\frac{1}{\mu(t)}\binom{\sigma(t)}{n}_{\mathbb{T},M} - \binom{t}{n}_{\mathbb{T},M} = \binom{t}{n}_{\mathbb{T},M}^{\Delta} = \binom{\sigma^{m_{n-1}}(t)}{n-1}_{\mathbb{T},M},$$

and rearranging as in the proof of Theorem 3.24. □

Theorem 5.17 (Upper Summation)**.** *If $M = \{m_i\}_{i\in\mathbb{N}_0}$ is a shifting sequence, $t, 0 \in \mathbb{T}$ with $t > 0$, and $\mathbb{T} \cap [0,t]$ is right-scattered, then for all $n \geq 0$*

$$\binom{t}{n+1}_{\mathbb{T},M} = \sum_{\substack{0\leq s<t \\ s\in\mathbb{T}}} \mu(s)\binom{\sigma^{m_n}(s)}{n}_{\mathbb{T},M}.$$

We will return to the consideration of general time scales, but we will first derive some results for one specific time scale that shares some commonalities with the set of harmonic numbers.

Example 5.18 (Triangular Numbers)**.** If $\mathbb{T} = \{T_k\}_{k\in\mathbb{N}_0}$ is the set of triangular numbers defined by the sums

$$T_k = \sum_{j=0}^{k} j = \frac{k(k+1)}{2},$$

then the graininess is

$$\mu(T_k) = T_{k+1} - T_k = k + 1.$$

Therefore, the triangular falling factorial, $T_k^{\underline{n}}$, is

$$\begin{aligned} T_k^{\underline{n}} &= n! \sum_{0\le j_1<\ldots<j_n<k} \prod_{l=1}^{n} \mu(T_{j_l}) \\ &= n! \sum_{0\le j_1<\ldots<j_n<k} \prod_{l=1}^{n} (j_l + 1) \\ &= n! \begin{bmatrix} k+1 \\ k-n+1 \end{bmatrix}. \end{aligned}$$

Continuing as before with the harmonic numbers, the triangular binomial coefficient is obtained by dividing the triangular falling factorial by $n!$:

$$\binom{T_k}{n}_{\mathbb{T}} = \begin{bmatrix} k+1 \\ k-n+1 \end{bmatrix}.$$

The triangular rising factorial, $T_k^{\overline{n}}$, is

$$\begin{aligned} T_k^{\overline{n}} &= n! \sum_{0\le j_1\le\ldots\le j_n<k} \prod_{l=1}^{n} \mu(T_{j_l}) \\ &= n! \sum_{0\le j_1\le\ldots\le j_n<k} \prod_{l=1}^{n} (j_l + 1) \\ &= n! \begin{Bmatrix} n+k \\ k \end{Bmatrix}, \end{aligned}$$

where $\left\{ k \atop n \right\}$ is the Stirling subset number or the Stirling number of the second kind (cf. [3], [9], and [18]).

Theorem 5.19. *The triangular binomial coefficient defined in the previous example possesses analogs to the Pascal Rule, Upper Summation identity, and Parallel Summation identity:*

(i) (Pascal Rule): $\binom{T_{k+1}}{n}_{\mathbb{T}} = \binom{T_k}{n}_{\mathbb{T}} + (k+1)\binom{T_k}{n-1}_{\mathbb{T}}$*;*

(ii) (Upper Summation): $\binom{T_k}{n+1}_{\mathbb{T}} = \sum_{j \le k-1} (j+1)\binom{T_j}{j}_{\mathbb{T}}$*; and*

(iii) (Parallel Summation): $\binom{T_{k+n+1}}{n}_{\mathbb{T}} = \sum_{j \le n} (k+n+1)^{\underline{n-j}}\binom{T_{k+j}}{j}_{\mathbb{T}}$.

Proof. Expanding $\binom{T_k}{n}_{\mathbb{T}}^{\Delta} = \binom{T_k}{n-1}_{\mathbb{T}}$ we obtain

$$\binom{T_k}{n-1}_{\mathbb{T}} = \frac{\binom{T_{k+1}}{n}_{\mathbb{T}} - \binom{T_k}{n}_{\mathbb{T}}}{k+1}.$$

Rearranging yields (i):

$$\binom{T_{k+1}}{n}_{\mathbb{T}} = \binom{T_k}{n}_{\mathbb{T}} + (k+1)\binom{T_k}{n-1}_{\mathbb{T}}.$$

To show the Upper Summation identity, we begin with the definition and apply a Stirling number identity (cf. (6.23) from [9]):

$$\begin{aligned}
\binom{T_k}{n+1}_{\mathbb{T}} &= \frac{T_k^{\underline{n+1}}}{(n+1)!} \\
&= \begin{bmatrix} k+1 \\ k-(n+1)+1 \end{bmatrix} \\
&= \begin{bmatrix} k+1 \\ k-n \end{bmatrix} \\
&= \sum_{j=0}^{k-n} (n+j) \begin{bmatrix} n+j \\ j \end{bmatrix} \\
&= \sum_{j=n-1}^{k-1} (j+1) \begin{bmatrix} j+1 \\ j-n+1 \end{bmatrix} \\
&= \sum_{j \le k-1} (j+1) \binom{T_j}{n}_{\mathbb{T}}.
\end{aligned}$$

As with the Upper Summation identity, the Parallel Summation identity relies on a Stirling number identity (cf. (6.21) from [9]):

$$\begin{aligned}
\binom{T_{k+n+1}}{n}_{\mathbb{T}} &= \begin{bmatrix} k+n+2 \\ k+2 \end{bmatrix} \\
&= (k+n+1)! \sum_{j=0}^{k+n+1} \frac{1}{j!} \begin{bmatrix} j \\ k+1 \end{bmatrix} \\
&= (k+n+1)! \sum_{j=k+1}^{k+n+1} \frac{1}{j!} \begin{bmatrix} j \\ k+1 \end{bmatrix} \\
&= (k+n+1)! \sum_{j=0}^{n} \frac{1}{(k+j+1)!} \begin{bmatrix} k+j+1 \\ k+1 \end{bmatrix} \\
&= \sum_{j=0}^{k+n+1} (k+n+1)^{\underline{n-j}} \binom{T_{k+j}}{j}_{\mathbb{T}}.
\end{aligned}$$

□

In Section 4.2, we developed power series for the case $\mathbb{T} = \mathbb{H}$. We can do the same for a more general $\mathbb{T}$. As before, we begin by defining the space of polynomials over $\mathbb{T}$.

Definition 5.20. Let $s, t \in \mathbb{T}$. For a fixed s we define a *generalized polynomial* in t, $P(t, s)$, to be a linear combination of $g_M^n(t, s)$ for $0 \leq n \leq N$. If

$$P(t, s) = \sum_{n=0}^{N} a_n g_M^n(t, s),$$

where $a_n \in \mathbb{R}$ for $0 \leq n \leq N$ and $a_N g_M^N(t, s) \not\equiv 0$, we say $P(t, s)$ is an Nth degree generalized polynomial.

Definition 5.21 (Time Scales Power Series)**.** Let $M = \{m_i\}_{i \in \mathbb{N}}$ be a shifting sequence. Let s, t be elements of $\mathbb{T}$, $\{a_n\}_{n \in \mathbb{N}_0}$ be a sequence of real numbers, and $P_0(t, s), P_1(t, s), P_2(t, s), \ldots$ be a sequence of generalized polynomials in t satisfying the following four conditions:

(i) $P_0(t, s) = 1$;

(ii) The degree of P_n is n or $P_n \equiv 0$;

(iii) $P_n(s, s) = 0$ for $n \in \mathbb{N}$;

(iv) $P_n{}^{\Delta} = P_{n-1}^{\sigma^{m_{n-1}}}$ for $n \geq 1$.

Then $\sum_{n=0}^{\infty} a_n P_n(t, s)$ is a *power series* over $\mathbb{T}$.

Theorem 5.22. *If M is a shifting sequence and $t, s \in \mathbb{T}$, then $\sum_{n=0}^{\infty} a_n g_M^n(t, s)$ is a* power series *over $\mathbb{T}$. Furthermore, if the power series $\sum_{n=0}^{\infty} a_n (t - s)^n$ converges uniformly on some interval D, then $\sum_{n=0}^{\infty} a_n g_M^n(t, s)$ converges uniformly on $D \cap \mathbb{T}$.*

Proof. Clearly the monomials $g_M^n(t, s)$ satisfy the conditions of Definition 5.21. From Theorem 5.13, we have

$$\sum_{n=0}^{\infty} |a_n g_M^n(t, s)| \leq \sum_{n=0}^{\infty} |a_n (t - s)^n|,$$

therefore uniform convergence of the time scales power series follows immediately from uniform convergence of the standard power series. □

In order to consider power series solutions of dynamic equations, it will be necessary to take delta derivatives term by term. The next theorem is a time scales analog to the classic real analysis result showing that pointwise convergence of a function sequence and uniform convergence of the sequence of derivatives imply two things: the original sequence converges uniformly to some function and the sequence of derivatives converges to the derivative of that same function (cf. [16, Theorem 7.17]). The proof we give largely imitates that presentation. Note that the scope of this result is not limited to power series.

Theorem 5.23. *Suppose $\{f_n\}$ is a sequence of functions where, for all $n \in \mathbb{N}_0$, $f_n : \mathbb{T} \to \mathbb{R}$ is delta-differentiable on some interval $[a, b] \cap \mathbb{T}$. Suppose also that $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b] \cap \mathbb{T}$. If $\{f_n^\Delta\}$ converges uniformly on $[a, b] \cap \mathbb{T}$, then $\{f_n\}$ converges uniformly on $[a, b] \cap \mathbb{T}$ to a function f and, for all $x \in [a, b] \cap \mathbb{T}$,*

$$f^\Delta(x) = \lim_{n\to\infty} f_n^\Delta(x).$$

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that whenever $m, n \geq N$, we have

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$$

and

$$|f_n^\Delta(t) - f_m^\Delta(t)| < \frac{\epsilon}{2(b-a)}$$

for all $t \in [a, b] \cap \mathbb{T}$.

Applying Theorem 1.20 to the function $f_n - f_m$ shows that, for $x, t \in [a, b] \cap \mathbb{T}$,

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{\epsilon|x-t|}{2(b-a)} \leq \frac{\epsilon}{2} \tag{5.1}$$

whenever $m, n \geq N$.

Using (5.1), for $x \in [a, b] \cap \mathbb{T}$ and $m, n \geq N$ we obtain

$$\begin{aligned}|f_n(x) - f_m(x)| &= |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0) + f_n(x_0) - f_m(x_0)| \\ &\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore $\{f_n\}$ converges uniformly on $[a, b] \cap \mathbb{T}$. Let $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [a, b] \cap \mathbb{T}$.

If $x \in [a, b] \cap \mathbb{T}^\kappa$ is right-scattered, then the uniform convergence of $\{f_n\}$ on $[a, b] \cap \mathbb{T}$ and the definition

$$f_n^\Delta(x) = \frac{f_n(\sigma(x)) - f_n(x)}{\sigma(x) - x}$$

imply that

$$\begin{aligned}\lim_{n\to\infty} f_n^\Delta(x) &= \lim_{n\to\infty} \frac{f_n(\sigma(x)) - f_n(x)}{\sigma(x) - x} \\ &= \frac{f(\sigma(x)) - f(x)}{\sigma(x) - x} \\ &= f^\Delta(x).\end{aligned}$$

Now fix a right-dense element of $x \in [a, b] \cap \mathbb{T}$. For $t \in [a, b] \cap \mathbb{T}$ with $t \neq x$, define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \quad \text{and} \quad \phi(t) = \frac{f(t) - f(x)}{t - x}.$$

Then for any $n \in \mathbb{N}_0$,

$$\lim_{t\to x} \phi_n(t) = f_n^\Delta(x).$$

Another consequence of (5.1) is that, whenever $m, n \geq N$,

$$|\phi_n(t) - \phi_m(t)| = \frac{|f_n(t) - f_m(t) - f_n(x) + f_m(x)|}{|t - x|} \leq \frac{\epsilon}{2(b-a)}.$$

Therefore $\{\phi_n\}$ converges uniformly to $\phi(t)$ for $t \in [a, b] \cap \mathbb{T}$ with $t \neq x$.

Uniform convergence allows us to interchange the limits (cf. [16, Theroem 7.11]) in the following equality to obtain:

$$\begin{aligned}\lim_{n\to\infty} f_n^{\Delta}(x) &= \lim_{n\to\infty}\lim_{t\to x}\phi_n(t) \\ &= \lim_{t\to x}\lim_{n\to\infty}\phi_n(t) \\ &= \lim_{t\to x}\phi(t) \\ &= f^{\Delta}(x).\end{aligned}$$

□

Definition 5.24. Let $M_O = \{m_i\}_{i\in\mathbb{N}_0} = \{0, 1, 0, 1, 0, \ldots\}$ and $M_E = \{m_i\}_{i\in\mathbb{N}_0} = \{1, 0, 1, 0, 1, \ldots\}$. For $s, t \in \mathbb{T}$, we define the sequences $\{P_n\}_{n\in\mathbb{N}_0}$ and $\{Q_n\}_{n\in\mathbb{N}_0}$ as follows:

$$P_n(t, s) = \frac{g^n_{M_O}(t, s)}{n!}$$

and

$$Q_n(t, s) = \frac{g^n_{M_E}(t, s)}{n!}.$$

From the definition, it is clear that any nonzero P_n and Q_n are nth degree generalized polynomials, $P_0 = Q_0 = 1$, and $P_n(s, s) = Q_n(s, s) = 0$ for $n \geq 1$.

The following delta derivatives of P_n and Q_n are arrived at by calculations analogous to those in Examples 4.8 and 4.9:

$$P_n^{\Delta}(t,s) = \begin{cases} 0 & \text{if } n = 0 \\ P_{n-1}\left(\sigma^{m_{n-1}}(t), s\right) & \text{if } n \geq 1 \end{cases}$$

and

$$Q_n^{\Delta}(t,s) = \begin{cases} 0 & \text{if } n = 0 \\ Q_{n-1}\left(\sigma^{m_{n-1}}(t), s\right) & \text{if } n \geq 1. \end{cases}$$

Therefore $\{P_n\}_{n\in\mathbb{N}_0}$ and $\{Q_n\}_{n\in\mathbb{N}_0}$, together with the delta derivative, satisfy the conditions of Definition 5.21.

With the goal in mind of generalizing the results in Theorems 4.18 and 4.22, we define analogs to the sine and cosine functions in much the same way as we defined the harmonic sine and cosine functions of Definitions 4.17 and 4.21. Using the polynomials we have just defined, we provide power series definitions which exhibit the desired behaviors.

Definition 5.25. Let $s, t \in \mathbb{T}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be the polynomial sequences from Definition 5.24. For $\omega \in \mathbb{R}$, we define the *generalized sine function* $\sin_{\mathbb{T}}$ on $\mathbb{T}$ with frequency ω and phase shift s by

$$\sin_{\mathbb{T}}(t,s;\omega) = \sum_{j=0}^{\infty} (-1)^j \omega^{2j+1} P_{2j+1}(t,s).$$

Definition 5.26. Let, $s, t \in \mathbb{T}$ and $\{Q_n\}_{n\in\mathbb{N}_0}$ be the polynomial sequences from Definition 5.24. For $\omega \in \mathbb{R}$, we define the *generalized cosine function* $\cos_{\mathbb{T}}$ on $\mathbb{T}$ with frequency ω and phase shift s by

$$\cos_{\mathbb{T}}(t,s;\omega) = \sum_{j=0}^{\infty} (-1)^j \omega^{2j} Q_{2j}(t,s).$$

As before, $\sin_{\mathbb{T}}(t) = \sin_{\mathbb{T}}(t,0;1)$ and $\cos_{\mathbb{T}}(t) = \cos_{\mathbb{T}}(t,0;1)$.

Lemma 5.27. *If $t \in \mathbb{T}$, $\omega \in \mathbb{R}$, and $\{P_n\}_{n\in\mathbb{N}_0}$ and $\{Q_n\}_{n\in\mathbb{N}_0}$ are the polynomial sequences from Definition 5.24, then* $\sin_{\mathbb{T}}$ *and* $\cos_{\mathbb{T}}$ *are twice delta differentiable and their first two derivatives are as follows:*

(i) ${\sin_{\mathbb{T}}}^{\Delta}(t,s;\omega) = \omega \sum_{j=0}^{\infty} (-1)^j \omega^{2j} P_{2j}(t,s)$*;*

(ii) ${\sin_{\mathbb{T}}}^{\Delta\Delta}(t,s;\omega) = -\omega^2 {\sin_{\mathbb{T}}}^{\sigma}(t,s;\omega)$*;*

(iii) ${\cos_{\mathbb{T}}}^{\Delta}(t,s;\omega) = -\omega \sum_{j=0}^{\infty} (-1)^j \omega^{2j+1} Q_{2j+1}(t,s)$*; and*

(iv) ${\cos_{\mathbb{T}}}^{\Delta\Delta}(t,s;\omega) = -\omega^2 {\cos_{\mathbb{T}}}^{\sigma}(t,s;\omega)$.

Proof. For a fixed s, let $\{f_n\}$ be the sequence of partial sums of $\sum_{j=0}^{\infty} (-1)^j \omega^{2j+1} P_{2j+1}(t,s)$, such that

$$f_n(t) = \sum_{j=0}^{n} (-1)^j \omega^{2j+1} P_{2j+1}(t,s) = \sum_{j=0}^{n} \frac{(-1)^j \omega^{2j+1} g_{M_O}^{2j+1}(t,s)}{(2j+1)!}.$$

Then $\{f_n\}, \{f_n^{\Delta}\}$, and $\{f_n^{\Delta\Delta}\}$ are uniformly convergent on $[a,b] \cap \mathbb{T}$ by Theorem 5.22. Therefore, by Theorem 5.23, $f_n^{\Delta} \to {\sin_{\mathbb{T}}}^{\Delta}$ and $f_n^{\Delta\Delta} \to {\sin_{\mathbb{T}}}^{\Delta\Delta}$. The proof for $\cos_{\mathbb{T}}$ proceeds similarly. □

From their definitions and Lemma 5.27 it should be clear that

$$\sin_{\mathbb{T}}(s,s;\omega) = {\cos_{\mathbb{T}}}^{\Delta}(s,s;\omega) = 0 \quad \text{and} \tag{5.2}$$

$$\cos_{\mathbb{T}}(s,s;\omega) = {\sin_{\mathbb{T}}}^{\Delta}(s,s;\omega) = 1. \tag{5.3}$$

Lemma 5.28. *If $s, t \in \mathbb{T}$ and $\lambda > 0$, the time scale sine and cosine functions* $\sin_{\mathbb{T}}(t,s;\sqrt{\lambda})$ *and* $\cos_{\mathbb{T}}(t,s;\sqrt{\lambda})$ *are linearly independent solutions of the self-adjoint dynamic equation* $f^{\Delta\Delta} + \lambda f^{\sigma} = 0$.

Again, we have an analog to the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$.

Corollary 5.29. *If $t \in \mathbb{T}$, then*

$$\cos_{\mathbb{T}}(t,s)\sin_{\mathbb{T}}{}^{\Delta}(t,s) - \sin_{\mathbb{T}}(t,s)\cos_{\mathbb{T}}{}^{\Delta}(t,s) = 1.$$

Theorem 5.30. *If $\lambda > 0$ and β is an element of $\mathbb{T}$ such that $\sin_{\mathbb{T}}(\beta,\alpha;\sqrt{\lambda}) = 0$, then $\sin_{\mathbb{T}}(t,\alpha;\sqrt{\lambda})$ is a solution of the boundary value problem with Dirichlet conditions*

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f(\alpha) = f(\beta) = 0.$$

Theorem 5.31. *If $\lambda > 0$ and β is an element of $\mathbb{T}$ such that $\cos_{\mathbb{T}}{}^{\Delta}(\beta,\alpha;\sqrt{\lambda}) = 0$, then $\cos_{\mathbb{T}}(t,\alpha;\sqrt{\lambda})$ is a solution of the boundary value problem with Neumann conditions*

$$f^{\Delta\Delta} + \lambda f^{\sigma} = 0; f^{\Delta}(\alpha) = f^{\Delta}(\beta) = 0.$$

Solutions to the self-adjoint ordinary differential equation with the Dirichlet and Neumann boundary conditions are useful in solving some separable second order differential equations (cf. [19]). In the following two sections, we will consider two partial dynamic equations, the Wave Equation and the Diffusion Equation, with appropriate boundary conditions.

5.2. SOLUTIONS TO THE DYNAMIC WAVE EQUATION

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be time scales. In this section we exhibit solutions to the Wave Equation

$$u^{\Delta_t\Delta_t\sigma_x} = c^2 u^{\Delta_x\Delta_x\sigma_t} \tag{5.4}$$

for $x \in \mathbb{T}_1 \cap (\alpha, \beta)$ and $t \in \mathbb{T}_2 \cap (s, r)$ (with $\alpha, \beta \in \mathbb{T}_1$ and $s, r \in \mathbb{T}_2$) subject to the homogeneous Dirichlet or Neumann boundary conditions

$$u(\alpha, t) = u(\beta, t) = 0 \quad \text{or} \tag{5.5}$$

$$u^{\Delta_x}(\alpha, t) = u^{\Delta_x}(\beta, t) = 0, \tag{5.6}$$

respectively, for $t \in \mathbb{T}_2 \cap (s, r)$ and certain initial conditions of the type

$$u(x, s) = \phi(x) \tag{5.7}$$

$$u^{\Delta_t}(x, s) = \psi(x) \tag{5.8}$$

for $x \in \mathbb{T}_1 \cap (\alpha, \beta)$.

We can find solutions to the Wave Equation (5.4) by the method of separation of variables. If we suppose that $u(x, t)$ is the product of a function of only x by a function of only t, (i.e., $u(x, t) = X(x)T(t)$) and plug this u into the Wave Equation, we obtain

$$X^{\sigma_x}(x)T^{\Delta_t\Delta_t}(t) = c^2 X^{\Delta_x\Delta_x}(x)T^{\sigma_t}(t).$$

Dividing through by $c^2 X^{\sigma_x}(x)T^{\sigma_t}(t)$ yields

$$\frac{T^{\Delta_t\Delta_t}(t)}{c^2 T^{\sigma_t}(t)} = \frac{X^{\Delta_x\Delta_x}(x)}{X^{\sigma_x}(x)} = -\lambda.$$

From this equation it is clear that λ must be a constant, and so X and T must satisfy the following self-adjoint ordinary dynamic equations which we have already solved:

$$T^{\Delta_t\Delta_t} = -c^2\lambda T^{\sigma_t} \text{ and } X^{\Delta_x\Delta_x} = -\lambda X^{\sigma_x}. \tag{5.9}$$

So, the following two theorems are natural consequences of Lemma 5.28, Theorem 5.30, and Theorem 5.31.

Theorem 5.32. *Let $N \in \mathbb{N}$, $s \in \mathbb{T}_2$, and $\alpha, \beta \in \mathbb{T}_1$. If $\lambda_n > 0$ and $\sin_{\mathbb{T}_1}(\beta, \alpha; \sqrt{\lambda_n}) = 0$ for all $n \in \mathbb{N}$ with $n \leq N$, then*

$$u(x,t) = \sum_{n \leq N} \left(A_n \cos_{\mathbb{T}_2}(t, s; c\sqrt{\lambda_n}) + B_n \sin_{\mathbb{T}_2}(t, s; c\sqrt{\lambda_n}) \right) \sin_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n})$$

is a solution of the Wave Equation (5.4) on $\mathbb{T}_1 \cap [\alpha, \beta] \times \mathbb{T}_2 \cap (s, r)$ with homogeneous Dirichlet conditions (5.5) and initial conditions (5.7) and (5.8) where

$$\phi(x) = \sum_{n \leq N} A_n \sin_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n})$$

and

$$\psi(x) = \sum_{n \leq N} c\sqrt{\lambda_n} B_n \sin_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n}).$$

Theorem 5.33. *Let $N \in \mathbb{N}$, $s \in \mathbb{T}_2$, and $\alpha, \beta \in \mathbb{T}_1$. If $\lambda_n > 0$ and ${\cos_{\mathbb{T}_1}}^{\Delta}(\beta, \alpha; \sqrt{\lambda_n}) = 0$ for all $n \in \mathbb{N}$ with $n \leq N$, then*

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n \leq N} \left(A_n \cos_{\mathbb{T}_2}(t; c\sqrt{\lambda_n}) + B_n \sin_{\mathbb{T}_2}(t; c\sqrt{\lambda_n}) \right) \cos_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n})$$

is a solution of the Wave Equation (5.4) on $\mathbb{T}_1 \cap [\alpha, \beta] \times \mathbb{T}_2 \cap (s, r)$ with homogeneous Neumann conditions (5.6) and initial conditions (5.7) and (5.8) where

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n \leq N} A_n \cos_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n})$$

and

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n \leq N} c\sqrt{\lambda_n} B_n \cos_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n}).$$

These sums resemble partial sums of the Fourier series, but we will leave the consideration of their convergence as $N \to \infty$ for another discussion.

5.3. SOLUTIONS TO THE DYNAMIC DIFFUSION EQUATION

As in Section 5.2 let $\mathbb{T}_1$ and $\mathbb{T}_2$ be time scales. In this section we provide solutions to the Diffusion Equation

$$u^{\Delta_t \sigma_x} = k u^{\Delta_x \Delta_x} \tag{5.10}$$

for $x \in \mathbb{T}_1 \cap (\alpha, \beta)$ and $t \in \mathbb{T}_2 \cap (s, r)$ (with $\alpha, \beta \in \mathbb{T}_1$ and $s, r \in \mathbb{T}_2$) subject to the homogeneous Dirichlet (5.5) or Neumann boundary conditions (5.6) and certain initial condition of the form (5.7).

Separation of variables will work for the Diffusion Equation as well. The process is almost exactly the same as for the Wave Equation, except that X and T satisfy the ordinary dynamic equations

$$X^{\Delta_x \Delta_x} = -\lambda X^{\sigma_x} \text{ and } T^{\Delta_t} = -k\lambda T. \tag{5.11}$$

So, the next theorems follow naturally from Theorem 5.30 and Theorem 5.31. Recall that $e_p(t, s)$ was given in Definition 1.16. As with the Wave Equation, we'll first consider the homogeneous Dirichlet boundary conditions (5.5) for the Diffusion Equation (5.10).

Theorem 5.34. *Let $N \in \mathbb{N}$, $s \in \mathbb{T}_2$, and $\alpha, \beta \in \mathbb{T}_1$. If $\lambda_n > 0$ and $\sin_{\mathbb{T}_1}(\beta, \alpha; \sqrt{\lambda_n}) = 0$ for all $n \in \mathbb{N}$ with $n \leq N$, then*

$$u(x, t) = \sum_{n \leq N} A_n e_{-k\lambda_n}(t, s) \sin_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n})$$

is a solution of the Diffusion Equation (5.10) on $\mathbb{T}_1 \cap [\alpha, \beta] \times \mathbb{T}_2 \cap (s, r)$ with homogeneous Dirichlet conditions (5.5) and initial condition (5.7) where

$$\phi(x) = \sum_{n \leq N} A_n \sin_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n}).$$

Theorem 5.35. *Let $N \in \mathbb{N}$, $s \in \mathbb{T}_2$, and $\alpha, \beta \in \mathbb{T}_1$. If $\lambda_n > 0$ and ${\cos_{\mathbb{T}_1}}^{\Delta}(\beta; \sqrt{\lambda_n}) = 0$ for all $n \in \mathbb{N}$ with $n \le N$, then*

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n \le N} A_n e_{-k\lambda_n}(t,s) \cos_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n})$$

is a solution of the Diffusion Equation (5.10) on $\mathbb{T}_1 \cap [\alpha, \beta] \times \mathbb{T}_2 \cap (s, r)$ with Neumann conditions (5.6) and initial condition (5.7) where

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n \le N} A_n \cos_{\mathbb{T}_1}(x, \alpha; \sqrt{\lambda_n}).$$

5.4. MIXED TIME SCALES

In Section 2.5 and Section 3.4 we defined some expressions in terms of a binomial convolution. In a similar manner, we define an analog of the binomial theorem where the arguments are in two (potentially different) time scales. In this section we denote by $\mathbb{T}_1$ and $\mathbb{T}_2$ time scales with a non-empty intersection. Additionally, we denote by g_M and h_M the M-factorial functions over $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. For convenience, for $a, b \in \mathbb{R}$ with $b \ge a$ we will refer to the interval $[a, b] \cap \mathbb{T}$ interchangeably as $[a, b]$ or $[b, a]$.

Definition 5.36. Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{R}$. We say the interval $[a, b]$ is *uniform* if either $[a, b] \cap \mathbb{T}$ is right-scattered or $[a, b] \cap \mathbb{T}$ is right-dense.

Definition 5.37. Let M be a shifting sequence. For $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $s \in \mathbb{T}_1 \cap \mathbb{T}_2$, and $n \ge 0$, with $[s, x]$ and $[y, s]$ uniform, we define the *mixed binomial difference* $(x \ominus y)^{\{n,M,s\}}$ in the following way:

$$(x \ominus y)^{\{n,M,s\}} = \sum_{j=0}^{n} \binom{n}{j} g^{j}_{L^{n-j}(M)}(x, s)\, h^{n-j}_{M}(s, y).$$

Lemma 5.38. *Let $M = \{m_0, m_1, m_2, \ldots\}$ be a shifting sequence. For $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $s \in \mathbb{T}_1 \cap \mathbb{T}_2$, and $n \geq 0$, with $[s, x]$ and $[y, s]$ uniform, we have the following delta derivative identities:*

$$\left[(x \ominus y)^{\{n,M,s\}}\right]^{\Delta_x} = n\left(\sigma^{m_{n-1}(x)} \ominus y\right)^{\{n-1,M,s\}};$$

$$\left[(x \ominus y)^{\{n,M,s\}}\right]^{\Delta_y} = -n\left(x \ominus \sigma^{1-m_1}(y)\right)^{\{n-1,L(M),s\}}.$$

Theorem 5.39. *Let $M = \{m_0, m_1, m_2, \ldots\}$ be a shifting sequence. If $x, s \in \mathbb{T}$, $[s, x]$ is uniform, and $n \in \mathbb{N}_0$, then*

$$(x \ominus s)^{\{n,M,s\}} = g_M^n(x, s).$$

Proof. Let M be a shifting sequence. Let $x, s \in \mathbb{T}$ such that $[s, x]$ is uniform. Choose $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{T}$. Then for all $n \in \mathbb{N}_0$,

$$\begin{aligned}(x \ominus s)^{\{n,M,s\}} &= \sum_{j=0}^{n} \binom{n}{j} g^j_{L^{n-j}(M)}(x, s)\, h_M^{n-j}(s, s) \\ &= \binom{n}{n} g^n_{L^0(M)}(x, s) \\ &= g_M^n(x, s).\end{aligned}$$

□

Theorem 5.40. *Let $M = \{m_0, m_1, m_2, \ldots\}$ be a shifting sequence. If $x, y, 0 \in \mathbb{T}$ with $[0, x]$ and $[y, 0]$ uniform, then for all $n \in \mathbb{N}_0$,*

$$(x \ominus y)^{\{n,M,0\}} = g_M^n(x, y).$$

Proof. Let M be a shifting sequence. Let $\mathbb{T}$ be a time scale containing 0 and $x, y \in \mathbb{T}$ such that $[0, x]$ and $[y, 0]$ are uniform. Choose $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{T}$. Then for all $n \in \mathbb{N}_0$,

$$(x \ominus y)^{\{n,M,0\}} = \sum_{j=0}^{n} \binom{n}{j} g^j_{L^{n-j}(M)}(x, 0)\, h_M^{n-j}(0, y).$$

This is just the convolution form of $g_M^n(x,y)$, which can be shown by imitating the proof of Theorem 2.41 using Lemma 5.38 for the derivative identities. □

Definition 5.41. Let M be a shifting sequence. For $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $0 \in \mathbb{T}_1 \cap \mathbb{T}_2$, and $n \geq 0$, with $[0,x]$ and $[y,0]$ uniform, we define the *mixed binomial sum* $(x \oplus y)^{\{n,M,0\}}$ in the following way:

$$(x \oplus y)^{\{n,M,0\}} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} g^{j}_{L^{n-j}(M)}(x,0)\, h^{n-j}_{M}(0,y).$$

The binomial sum presented here is related to the sum from Definition 3.27. In part due to Corollary 5.5 we see the following relationship.

Lemma 5.42. *Let M be a shifting sequence. For $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $0 \in \mathbb{T}_1 \cap \mathbb{T}_2$, and $n \geq 0$, with $[0,x]$ and $[y,0]$ uniform, then*

$$(x \oplus y)^{\{n,M,0\}} = \sum_{j=0}^{n} \binom{n}{j} g^{j}_{L^{n-j}(M)}(x,0)\, h^{n-j}_{M^{*n-j-1}}(y,0).$$

Lemma 5.43. *Let $M = \{m_0, m_1, m_2, \ldots\}$ be a shifting sequence. For $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $0 \in \mathbb{T}_1 \cap \mathbb{T}_2$, and $n \geq 0$, with $[0,x]$ and $[y,0]$ uniform, we have the following delta derivative identities:*

$$\left[(x \oplus y)^{\{n,M,0\}}\right]^{\Delta_x} = n\left(\sigma^{m_{n-1}(x)} \oplus y\right)^{\{n-1,M,0\}};$$

$$\left[(x \oplus y)^{\{n,M,0\}}\right]^{\Delta_y} = n\left(x \oplus \sigma^{1-m_1}(y)\right)^{\{n-1,L(M),0\}}.$$

With a binomial sum well defined, we proceed as in Section 3.4 and extend the definition of a generalized binomial coefficient in terms of the M-factorial function.

Definition 5.44. Let M be a shifting sequence. For $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $0 \in \mathbb{T}_1 \cap \mathbb{T}_2$, and $n \geq 0$, with $[0,x]$ and $[y,0]$ uniform, we extend the M-shifted binomial coefficient in the following way:

$$\binom{x \oplus y}{n}_{\mathbb{T}_1,\mathbb{T}_2,M} = \frac{(x \oplus y)^{\{n,M,0\}}}{n!}.$$

As a consequence of Lemma 5.43, we see that this extension behaves as expected under delta differentiation.

Theorem 5.45. *Let* $M = \{m_0, m_1, m_2, \ldots\}$ *be a shifting sequence. For* $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $0 \in \mathbb{T}_1 \cap \mathbb{T}_2$, *and* $n \geq 0$, *with* $[0, x]$ *and* $[y, 0]$ *uniform, we have the following delta derivative identities:*

$$\binom{x \oplus y}{n}_{\mathbb{T}_1,\mathbb{T}_2,M}^{\Delta_x} = \binom{\sigma^{m_{n-1}(x)} \oplus y}{n-1}_{\mathbb{T}_1,\mathbb{T}_2,M};$$

$$\binom{x \oplus y}{n}_{\mathbb{T}_1,\mathbb{T}_2,M}^{\Delta_y} = \binom{x \oplus \sigma^{1-m_1}(y)}{n-1}_{\mathbb{T}_1,\mathbb{T}_2,L(M)}.$$

Combining Definition 5.44 with Lemma 5.42 yields the following analog to Vandermonde's Convolution.

Theorem 5.46 (Vandermonde's Convolution)**.** *Let* M *be a shifting sequence. For* $x \in \mathbb{T}_1$, $y \in \mathbb{T}_2$, $0 \in \mathbb{T}_1 \cap \mathbb{T}_2$, *and* $n \geq 0$, *with* $[0, x]$ *and* $[y, 0]$ *uniform, then*

$$\binom{x \oplus y}{n}_{\mathbb{T}_1,\mathbb{T}_2,M} = \sum_{j=0}^{n} \binom{x}{j}_{\mathbb{T}_1,L^{n-j}(M)} \binom{y}{n-j}_{\mathbb{T}_2,M^{*n-j-1}}.$$

Example 5.47. Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$. For $x, y, s \in \mathbb{R}$, $n \in \mathbb{N}_0$, and any shifting sequence M, $(x \ominus y)^{\{n,M,s\}} = (x - y)^n$ and $(x \oplus y)^{\{n,M,0\}} = (x + y)^n$.

Example 5.48. Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{H}$. For $x, y, s \in \mathbb{H}$, $n \in \mathbb{N}_0$, and any shifting sequence M, $(x \ominus y)^{\{n,M,s\}} = (x - y)^{\{n,M\}}$ and $(x \oplus y)^{\{n,M,0\}} = (x + y)^{\{n,M\}}$.

5.5. POSSIBILITIES FOR FUTURE RESEARCH

We have already shown analogs for several classic binomial coefficient identities, but there remain many more which we have not demonstrated. An absorption identity (analogous to $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$) would be particularly useful, as many of the classic identities which we have omitted rely upon this foundation. Of similar usefulness would be a symmetry identity (analogous to $\binom{n}{k} = \binom{n}{n-k}$).

With both the set of harmonic numbers and the set of triangular numbers, we saw how the M-factorial was related to the Stirling cycle numbers and how Stirling number identities corresponded to $\mathbb{T}$-binomial coefficient identities. It seems likely that other umbral sequences may be brought out of the shadows through a correspondence to a specific $\mathbb{T}$ and M.

The definition of harmonic power series in Definition 5.21 suggests the possibility of a unified Taylor's Formula for $\mathbb{H}$ and perhaps $\mathbb{T}$, especially since an appropriate choice of P_n coincides with either class of generalized polynomial in the two existing time scales Taylor's Formulae, cf. [2].

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VITA

Reginald A. Brigham II was born in St. Louis, Missouri. He graduated from Gateway High School in 1995. After a few years of working and attending St. Louis Community College part-time, he entered University of Missouri – Rolla in 2002. He graduated in May 2004 with a Bachelor of Science in Applied Mathematics. In December 2006, he graduated from Missouri University of Science and Technology with a Master of Science in Mathematics. In May 2017, he received his PhD in Mathematics from Missouri University of Science and Technology.