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A SIMPLIFIED MATHEMATICAL METHOD
FOR PETROLEUM PROPERTY EVALUATION

BY

CHARLES G. EDWARDS

A

THESIS

submitted to the faculty of the

SCHOOL OF MINES AND METALLURGY OF THE UNIVERSITY OF MISSOURI

in partial fulfillment of the work required for the

Degree of

MASTER OF SCIENCE

MINING ENGINEERING-PETROLEUM ENGINEERING OPTION

Rolla, Missouri

1957

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ACKNOWLEDGEMENTS

The author takes this opportunity to express sincere appreciation to Professor J. Paul Roston, Petroleum Engineering, who first suggested the need for a simplified evaluation method. His guidance has been invaluable in designing the method to fulfill practical requirements.

The author also expresses appreciation to Professor R. E. Lee, Mathematics Dept., who suggested application of the bounded error theorem, and to Professor R. F. Bruzewski, Mining Engineering, who advised the photographic reproduction of the illustrations.

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I. INTRODUCTION

The economic evaluation of a petroleum reservoir involves two fundamental operations; 1) predicting future production behaviour, 2) determining the present worth of this future production. This thesis purposes to eliminate the necessity of repeated calculation which is required in performing the second operation under existing methods.

Because the first operation, that of predicting future behaviour, is necessarily an estimate, there is an apparent tendency to employ approximate methods in the determination of present worth. The evaluation method developed in this paper confines estimation to the first operation, where it rightly belongs, and treats all subsequent work with mathematical exactness.

The estimation of petroleum reserves and future production rates has received considerable attention in the past. Various techniques for predicting behaviour have been devised and successfully applied to petroleum reservoirs. Where production is obtained by gas cap and solution gas drives, the application of decline curves can often supply a dependable prediction of future rates. It has been found, in these cases, that the instantaneous rate of uncurtailed production declines in a regular manner and is a function of the conditions within the reservoir at the instant in question. Since these conditions are in turn a function of past production, it is justifiable to extrapolate the decline curve to predict future rates. Such an extrapolation is of course purely empirical and is based on the assumption that the future behaviour of a reservoir will be governed by whatever trend or mathematical relationship is apparent in its past performance.

II. REVIEW OF THE LITERATURE

The mathematical analysis and development of various production rate-time and cumulative production-time curves has been presented in detail by Arps, Cutler, Larkey, and others. However, these analyses have been concerned with the primary operation of curve fitting and prediction. To the best knowledge of the author, no work comparable to the development of the evaluation method presented herein has been published.

III. DISCUSSION

General Decline Curve Types

Three types of production decline curves are commonly recognized: constant percentage decline, hyperbolic decline, and harmonic decline. The several curve types are shown in Figs. A, C, and E, which have been plotted from sample production data contained in Appendix A. The curve equations have been derived in the literature by the application of numerical analysis. However, considering that the method described herein is intended for practical application, the general curve equations will be derived from graphical inspection wherever possible.

All three curves have a common characteristic; they can be made rectilinear through the selection of an appropriate coordinate system. If a curve can be straightened in this manner, then its extrapolation is greatly simplified, and merely consists in extending the line. The equation of the curve can be easily written in this form by applying the general rectilinear equation $y = mx + b$.

Constant Percentage Decline

Any constant percentage decline curve, such as the one shown in Fig. A, can be straightened by graphing the logarithm of the production rate versus actual time. This can be conveniently accomplished by graphing actual rate and time values on a semi-logarithmic coordinate system as shown in

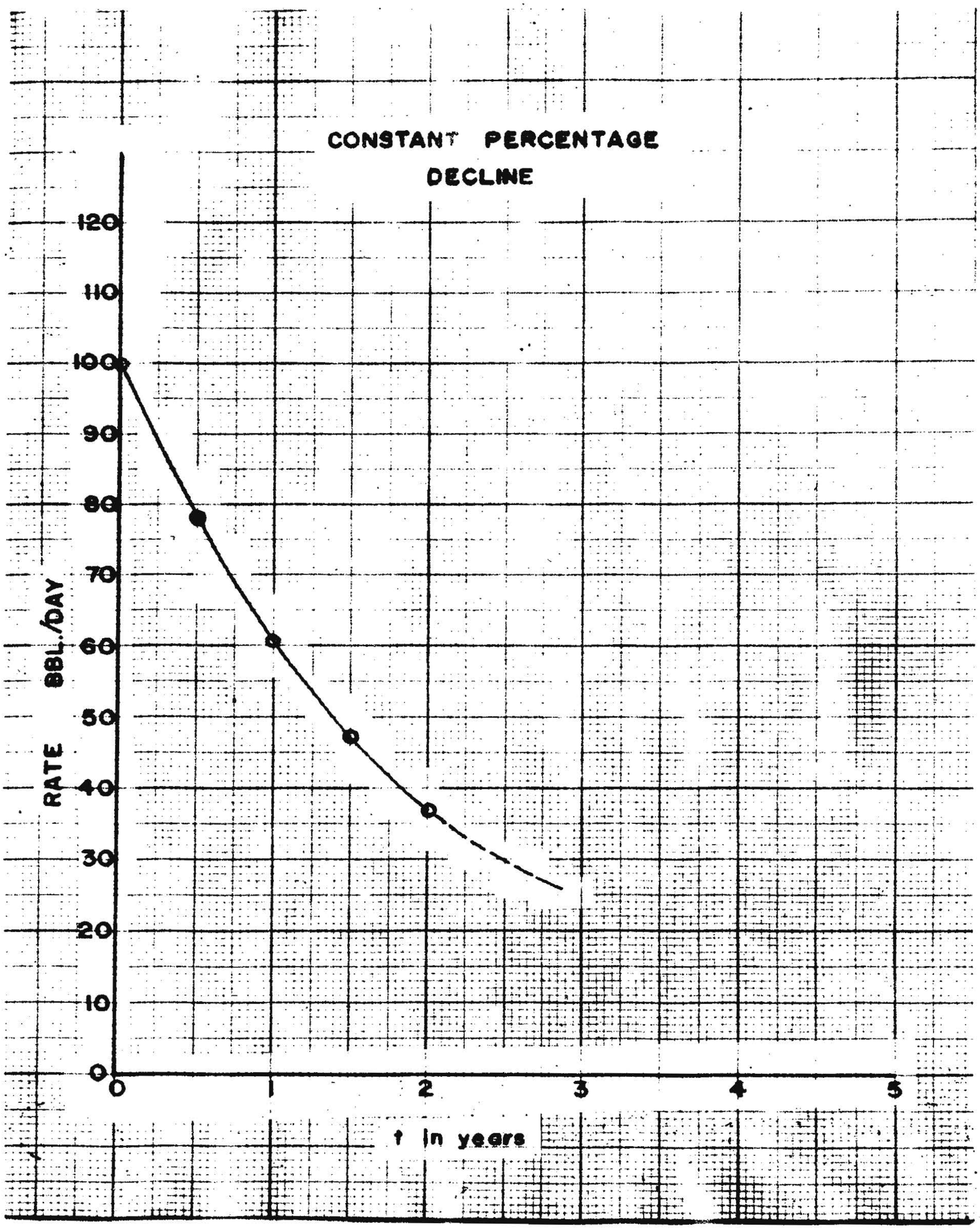
CONSTANT PERCENTAGE DECLINE
RATE BBL./DAY
120
110
100
90
80
70
60
50
40
30
20
10
0
0
1
2
3
4
5
t in years

FIG. A

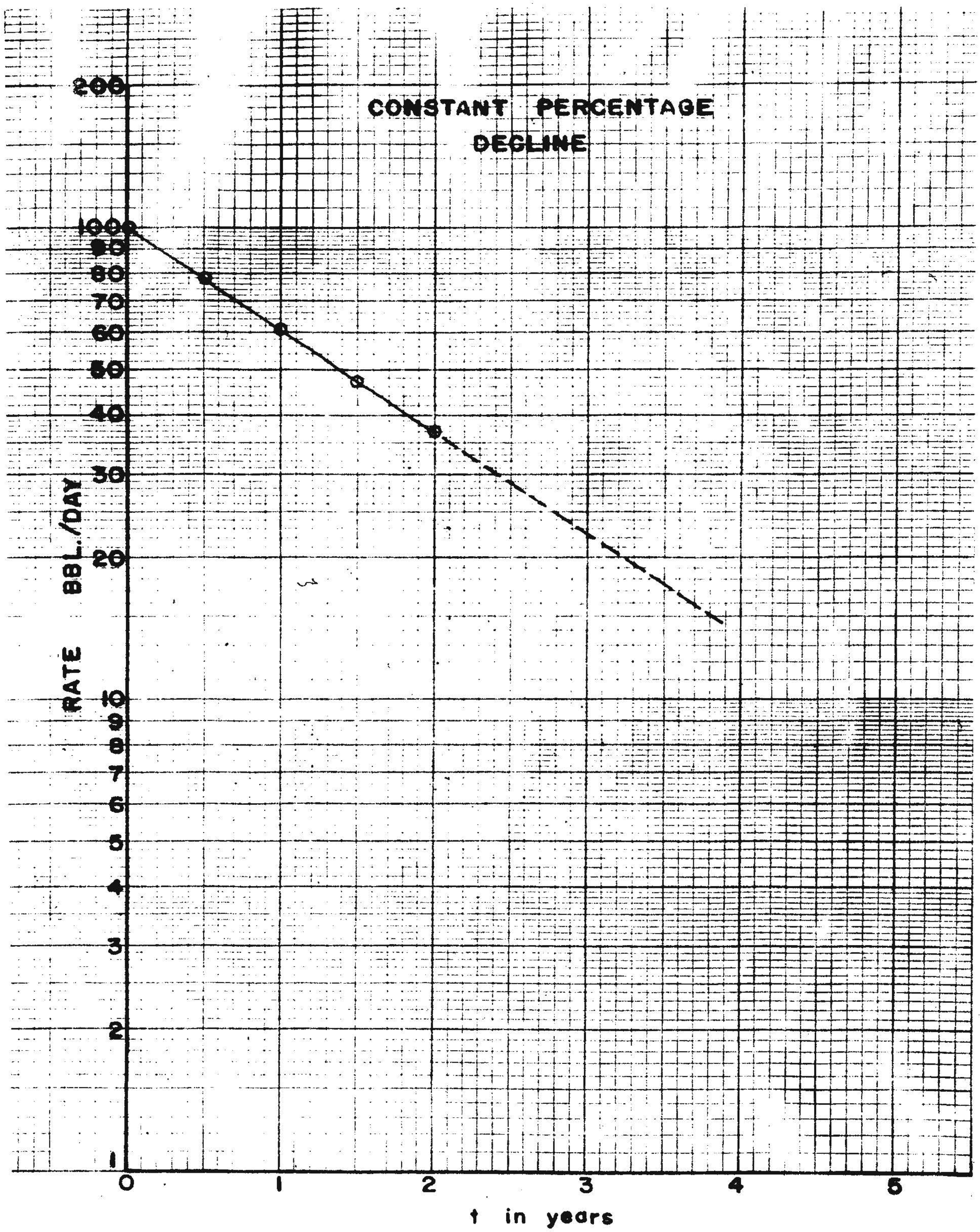
CONSTANT PERCENTAGE
DECLINE
RATE BBL./DAY
200
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
t in years

FIG. B

Fig. B.

Remembering that the ordinate represents the logarithm of the rate, an equation can now be written by applying the general equation for a straight line

$$y = mx + b$$

The curve then becomes

$$\ln q = kt + \ln q_o$$

or more familiarly

$$q = q_o e^{kt} \qquad (1)$$

where q = rate at any time t (bbl. per day)

$$k = \frac{d(\ln q)}{dt} = \frac{\ln \frac{q_1}{q_o}}{t_1 - t_o}$$

t = time (yrs.)

q_o = initial rate at time t_o

The specific equation for the curve in Fig. B can now be written as shown on page 31, Appendix A, the necessary constants being calculated or supplied directly from the graph.

Hyperbolic Decline

Hyperbolic curves, such as the one shown in Fig. C, can be straightened in the following manner. The logarithm of the rate and of the time are plotted rather than the actual values themselves. This can be conveniently accomplished by graphing the actual values on a fully logarithmic coordinate system. The resulting curve is shifted[1] to achieve linearity as indicated in Fig. D.

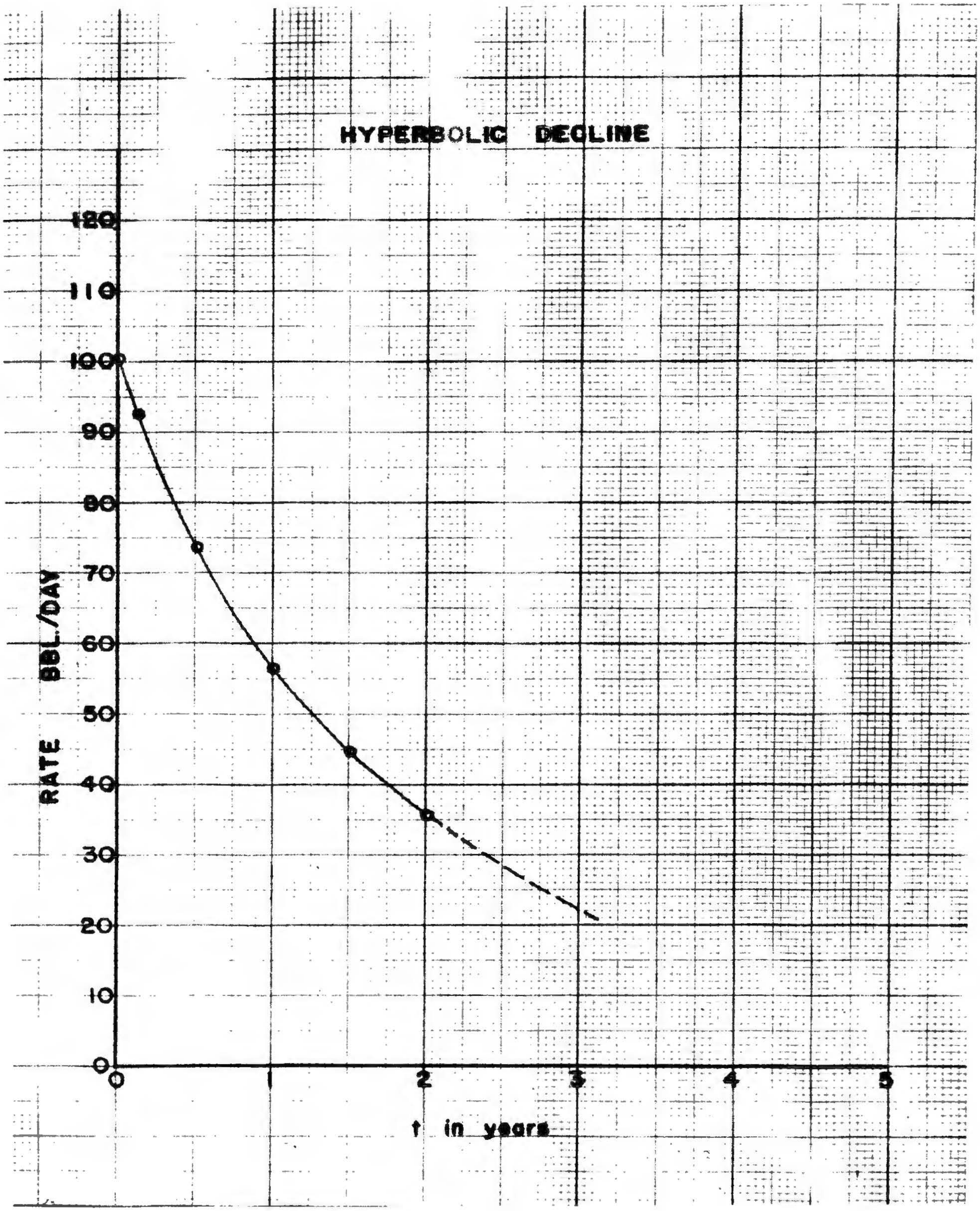
HYPERBOLIC DECLINE
RATE BBL./DAY
120
110
100
90
80
70
60
50
40
30
20
10
0
0
1
2
3
4
5
t in years

FIG. C

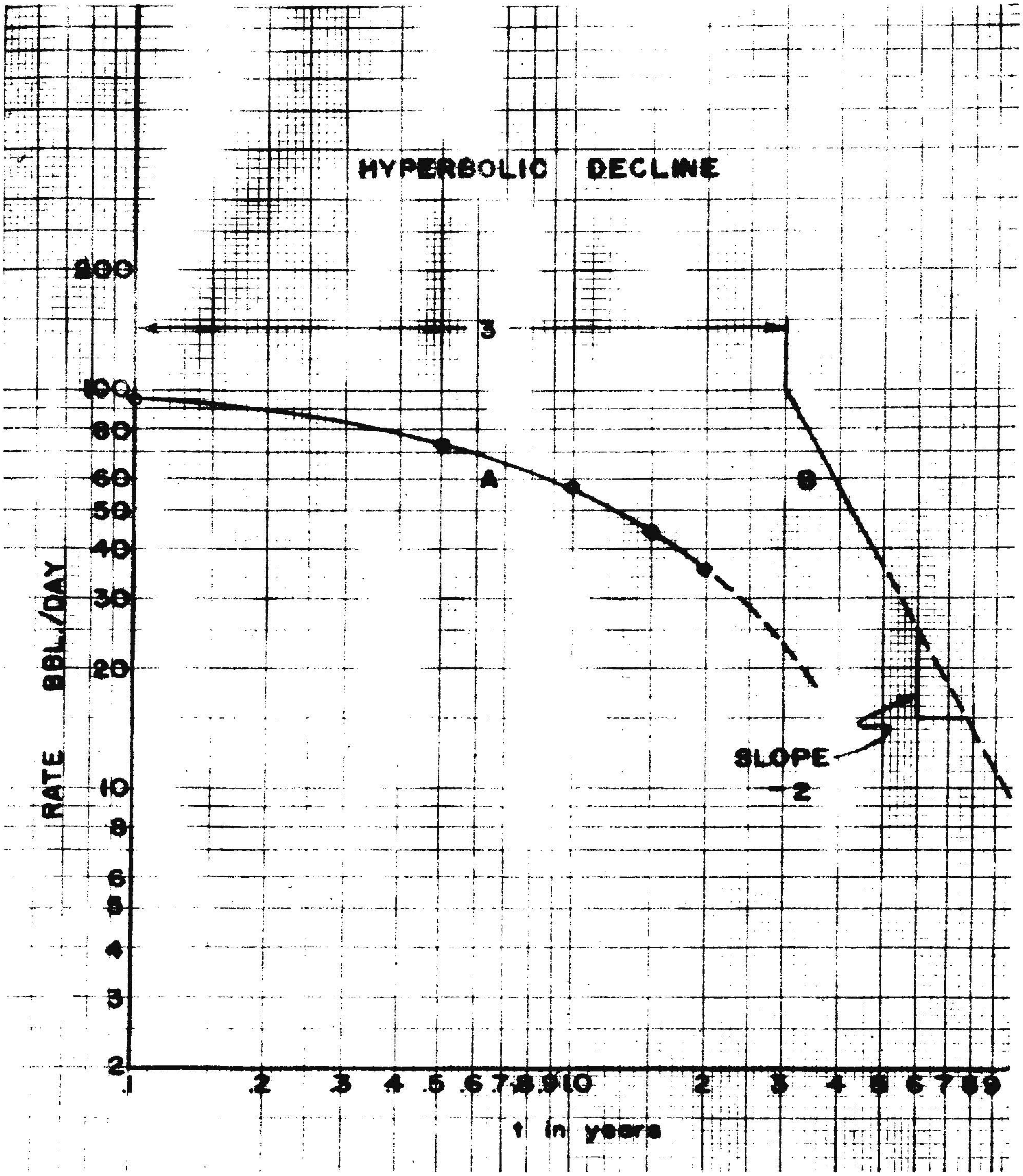

FIG. D

Note: The slope shown above is not characteristic of all hyperbolic decline curves. It has been determined by the sample production data in Appendix A, p. 32.

[1] Lipka, J., Graphical and Mechanical Computation, Part II, N.Y., John Wiley & Sons, 1921, p. 149 et. seq..

Remembering as before that the ordinate and abscissa represent the logarithms of rate and time, the curve equation can again be written by applying the general straight line form $y = mx + b$

The curve then becomes

$$\ln q = m \cdot \ln (1+t/c) + \ln q_o$$

or more familiarly

$$q = q_o(1 + t/c)^m \quad (2)$$

where c is the shift constant (yrs.) required to produce linearity.

This is equivalent to the analytical equation of Arps;[2]

$$q = q_o(tb/a_o + 1)^{-1/b}$$

See p. 35a

[2] Arps, J. J., Analysis of Decline Curves, Transactions of the A.I.M.E.-Petroleum Technology & Development, Vol. 160 p. 239.

Harmonic Decline

Harmonic decline involves the special case of the hyperbolic form where the exponent **m** is equal to -1. However, because of the mathematically indeterminate qualities of the special case it will be treated separately in all subsequent development.

Substituting m = -1 in eq. 2 for hyperbolic decline yields the following expression for harmonic decline:

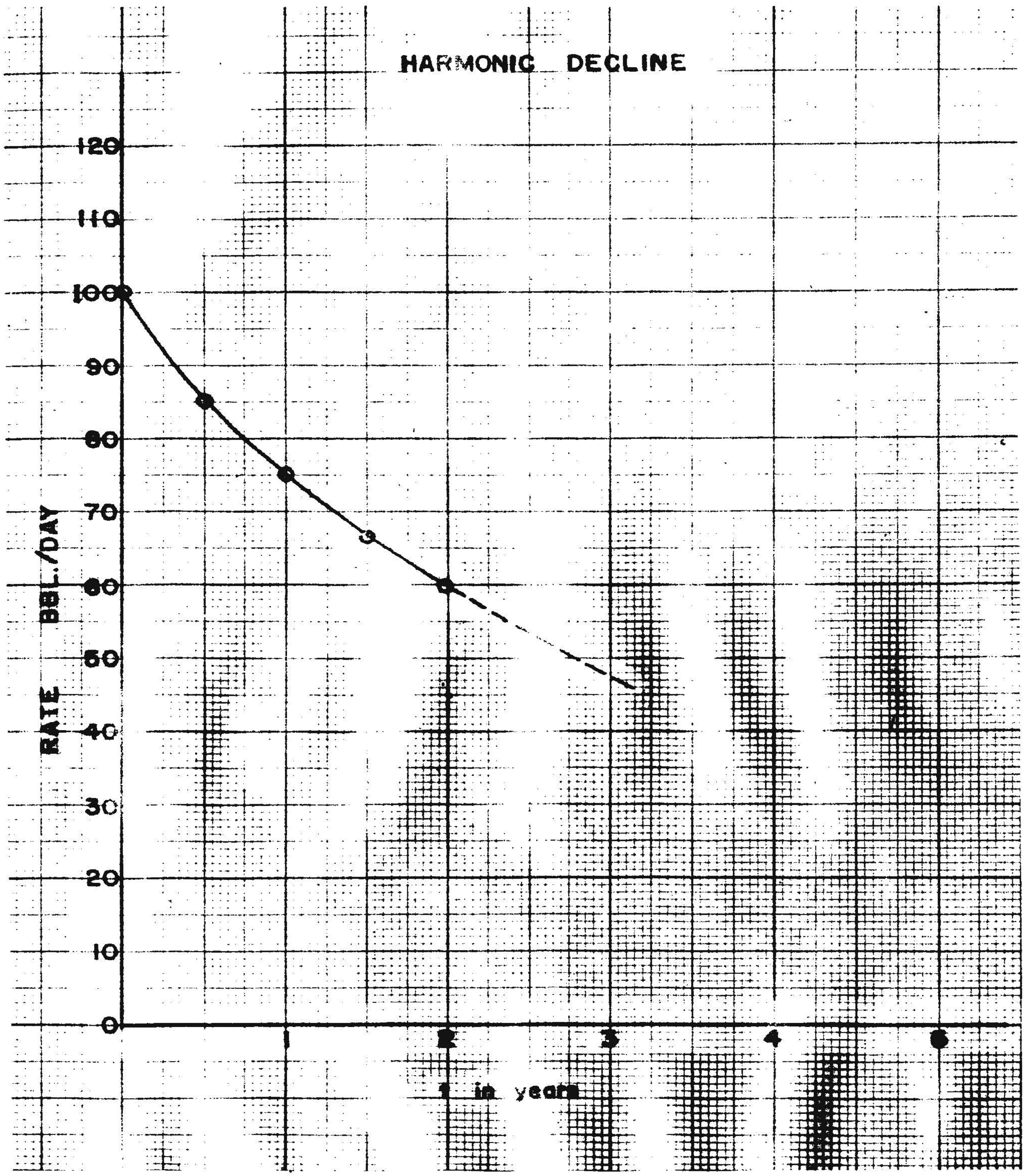
HARMONIC DECLINE
RATE BBL./DAY
120
110
100
90
80
70
60
50
40
30
20
10
0
1
2
3
4
t in years

FIG. E

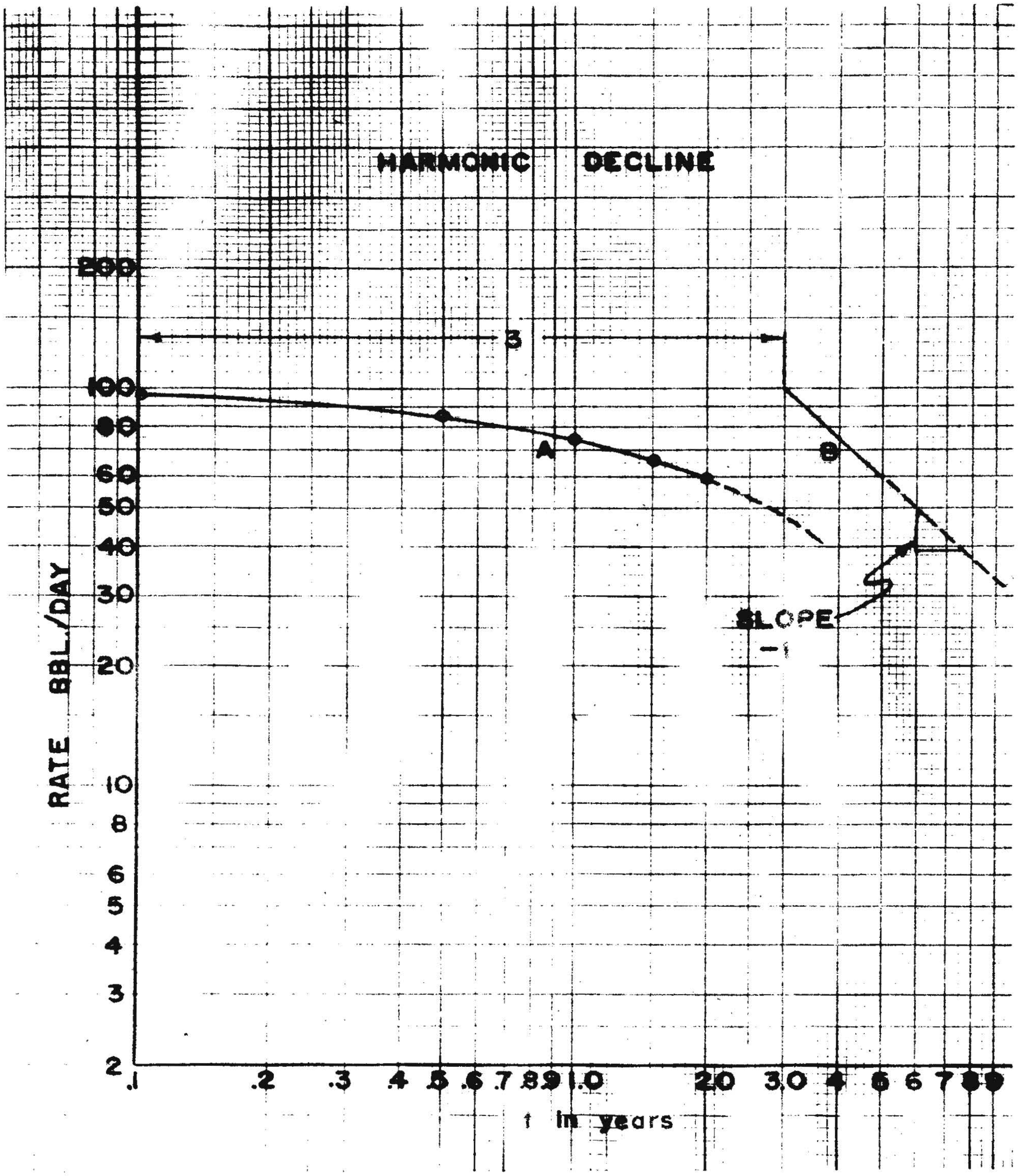
HARMONIC DECLINE
3
A
B
SLOPE
-1
RATE BBL./DAY
200
100
80
60
50
40
30
20
10
8
6
5
4
3
2
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
2.0
3.0
4
5
6
7
8
9
t in years

FIG. F

$$q = \frac{q_o}{1 + t/c} \tag{3}$$

The shifting process is the same as for hyperbolic decline and is indicated in Fig. F. For a thorough discussion of curve shifting the reader is directed to J. J. Lipka's "Graphical and Mechanical Computation" cited in the first footnote.

IV.. ECONOMIC CONSIDERATIONS

Operating Expenses and Net Price

Operating expenses can be divided into two general categories; fixed operating expenses (F) and variable operating expenses (V), Variable operating expenses consist of lifting costs, etc., and can be most conveniently treated by calculating the expense per barrel of produced oil and subtracting this amount from the market price. Thus the Net Price (M) is equal to the market price minus the variable operating expenses.

Fixed operating expenses are relatively independent of production and consist of salaries, overhead, etc.. They must be treated in a different manner and will be discussed later. Fixed and variable operating expenses will vary from field to field and must be determined from experience.

Economic Limit

It will be assumed that production will continue until an Economic Limit is reached, i.e. until the value of the production is equal to the daily fixed costs. To discuss this limit it will be necessary to modify the production equations to cash equations. The cash rate equations corresponding to the three production equations are shown in Table I.

TABLE I
Cash Rate Equations

Constant Percentage Decline	Hyperbolic Decline	Harmonic Decline
(4)	(5)	(6)
$q_c = Mq_o e^{kt}$	$q_c = Mq_o(1 + t/c)^m$	$q_c = \frac{Mq_o}{1 + t/c}$

where q_c is the daily cash production and M is the net price per barrel.

The economic limit will be reached when daily cash production is equal to the constant fixed costs. Setting q_c equal to the fixed cost (F) and solving for t will give the time of abandonment (t_e). The expressions for t_e appear in Table II.

TABLE II.
Economic Limit Equations

Constant Percentage Decline	Hyperbolic Decline	Harmonic Decline
(7)	(8)	(9)
$t_e = \frac{\ln \frac{F}{Mq_o}}{k}$	$t_e = c\left[\frac{F}{Mq_o}\right]^{1/m} - c$	$t_e = \frac{cMq_o}{F} - c$

V. ECONOMIC EVALUATION

General Procedure

Having determined the cash rate equations for various decline curves, it is now possible to apply the techniques of economic evaluation. The problem of obtaining a valuation involves three operations; 1) determining the cumulative cash income in each discount interval, 2) discounting the cash income by selecting a suitable interest rate, 3) summing the individually discounted incomes to obtain a total present worth. The following development will adopt a yearly discount interval. This, however, in no way limits the ordinates of the production rate graph. Each decline type requires a different method of solution and will therefore be treated separately.

Constant Percentage Decline

Step I. Determining the cumulative cash income in each discount interval.

Let Q equal the cumulative production in barrels. Then the production rate, q, of eq. 1 is the derivative of Q taken with respect to time.

$$q = \frac{dQ}{dt} = q_o e^{kt}$$

or

$$dQ = q_o e^{kt} dt$$

If the units of q and q_o are bbl./day then the right side can be multiplied by 365 and the equation can be integrated over any year to determine annual cumulative production.

Using the generalized limits, t and t-1, the annual cumulative production takes the form

$$Q_t = \int_{t-1}^{t} 365 q_o e^{kt} dt \qquad (10)$$

where Q_t is the cumulative production over the t^{th} year. Or in terms of the cash income

$$Q_{ct} = \int_{t-1}^{t} 365\ M q_o e^{kt} dt \qquad (11)$$

where Q_{ct} is the cumulative cash income over the t^{th} year.

Step II. Discounting the cash income.

The present worth of this cash income is equal to the product of the annual cumulative income and the appropriate discount factor, D_t.[3] Assuming the use of simple discount

---

[3] Paine, P., Oil Property Valuation, N.Y., John Wiley and Sons, 1942, p. 128.

---

gives the following expression for D_t.

$$D_t = \frac{1}{(1+i)^t} \qquad \text{where } i = \text{interest rate}$$

Since $\frac{1}{1+i}$ is a constant for any specific interest rate, this term may be represented by b. Then $D_t = b^t$, and the present worth of the t^{th} year's income, P_t, is equal to $Q_{ct} \cdot D_t$, or from eq. 11

$$P_t = b^t \int_{t-1}^{t} 365 M q_o e^{kt} dt \qquad (12)$$

Step III. Summing the discounted incomes.

The total present worth of future income, P_T, is the sum of these annual present worths.

$$P_T = \sum_{t=1}^{t_e} b^t \int_{t-1}^{t} 365 M q_o e^{kt} dt \qquad (13)$$

where t_e is determined by the economic limit equations in Table II.

Performing the integration and extracting constants from the summation results in the expression

$$P_T = \frac{365Mq_o}{k} \sum_{t=1}^{t_e} b^t \left[e^{kt}\right]_{t-1}^{t}$$

$$P_T = \frac{365Mq_o}{k} \sum_{t=1}^{t_e} b^t (e^{kt} - e^{kt-k})$$

$$P_T = \frac{365Mq_o}{k} \sum_{t=1}^{t_e} b^t e^{kt} (1 - e^{-k})$$

$$P_T = \frac{365Mq_o(1 - e^{-k})}{k} \sum_{t=1}^{t_e} b^t e^{kt} \qquad (14)$$

Now in the geometric series $\sum_{t=0}^{\infty} b^t e^{kt}$ if $|be^k| < 1$ then the series converges to $\frac{1}{1 - be^k}$.[4] See p. 35.

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[4] Scott, E. J., Transform Calculus with an Introduction to Complex Variables, N.Y., Harper & Bros., 1955, p. 167.

The geometric series in the present worth equation begins with the term t_1, while the series above begins with the term t_o. To adjust for this difference the t_o term will be subtracted from the convergence value.

Then

$$\sum_{t=1}^{\infty} b^t e^{kt} = \frac{1}{1 - be^k} - 1 = \frac{be^k}{1 - be^k} \qquad (15)$$

This expression can be substituted in eq. 14 for the summation to give

$$P_T = \frac{365Mq_o(1 - e^{-k})}{k} \left[\frac{be^k}{1 - be^k}\right]$$

This is the present worth of all production from $t = o$ to $t = \infty$.

The economic limit will be reached when $t = t_e$, as given in Table II, at which time production will be discontinued. It is therefore required to find the total present worth of production prior to t_e. To determine this value the partial sum of the geometric series must be found. The summation has been rearranged below.

$$\sum_{t=1}^{n} b^t e^{kt} = \sum_{t=1}^{n} (be^k)^t$$

This is similar to the general series $\sum_{p=1}^{n} r^{p-1}$, the partial sum of which has the form

$$S_n = \frac{(1-r^n)}{1-r} \qquad (16)$$

where S_n is the sum of the series up to and including the n^{th} term.[5]

---

[5] Sherwood, G. E. F., and Taylor, A. E., Calculus, N.Y., Prentice-Hall, 1949, p. 372.

The total present worth series involves the exponent t rather than t-1 and the partial sum must be adjusted accordingly. Comparing the first few terms of both series will demonstrate the necessary adjustment.

$$\sum_{t=1}^{n} (be^k)^t = be^k + (be^k)^2 + (be^k)^3 + (be^k)^4 + \ldots$$

$$\sum_{t=1}^{n} (be^k)^{t-1} = 1 + be^k + (be^k)^2 + (be^k)^3 + \ldots$$

But from eq. 16

$$\sum_{t=1}^{n} (be^k)^{t-1} = \frac{1 - (be^k)^n}{1 - be^k}$$

Therefore $$\sum_{t=1}^{t_e} (be^k)^t = \frac{1 - (be^k)^{t_e}}{1 - be^k} - 1 + (be^k)^{t_e}$$

or $$\sum_{t=1}^{t_e} (be^k)^t = be^k \frac{\left[1 - (be^k)^{t_e}\right]}{1 - be^k} \qquad (17)$$

Substituting this for the series in eq. 14 gives

$$P_T = \frac{365Mq_o(1 - e^{-k})}{k} \frac{be^k\left[1 - (be^k)^{t_e}\right]}{1 - be^k} \qquad (18)$$

Hyperbolic Decline

Step I. Determining the cumulative production over the discount interval.

The differential production for any time increment can be derived from the production rate equation, eq. 2, as follows.

$$q = \frac{dQ}{dt} = q_o(1 + t/c)^m$$

$$dQ = q_o(1 + t/c)^m dt$$

As in constant percentage decline this expression can be integrated over any year to determine annual production. Using the general limits t and t-1 the annual production takes the form

$$Q_t = \int_{t-1}^{t} 365 q_o (1 + t/c)^m dt \qquad (19)$$

or in terms of cash income

$$Q_{ct} = \int_{t-1}^{t} 365 M q_o (1 + t/c)^m dt \qquad (20)$$

Step II. Discounting the cash income.

The present worth of this cash income is equal to the product, $Q_{ct} \cdot D_t$, as discussed before.

$$P_t = b^t \int_{t-1}^{t} 365 M q_o (1 + t/c)^m dt \qquad (21)$$

Step III. Summing the discounted production.

The total present worth of future income is the sum of these present worths.

$$P_T = \sum_{t=1}^{t_e} b^t \int_{t-1}^{t} 365 M q_o (1 + t/c)^m dt \qquad (22)$$

where t_e is determined by the economic limit conditions in Table II.

Performing the integration and extracting constants from the summation results in the expression

$$P_T = 365Mq_o \sum_{t=1}^{t_e} b^t \left[\frac{(1 + t/c)^{m+1}}{m+1} \right]_{t-1}^{t}$$

$$P_T = 365Mq_o c \sum_{t=1}^{t_e} \left\{ \frac{b^t (1 + t/c)^{m+1} - b^t \left[1 - \frac{(t-1)}{c} \right]^{m+1}}{m+1} \right\}$$

$$P_T = \frac{365Mq_o}{c^m(m+1)} \left[\sum_{t=1}^{t_e} b^t (c+t)^{m+1} - \sum_{t=1}^{t_e} b^t (c - 1 + t)^{m+1} \right] \quad (23)$$

<u>Harmonic Decline</u>

Harmonic decline is a special case of hyperbolic decline, i.e. $m = -1$. However, the present worth expression for hyperbolic decline, eq. 23, becomes mathematically indeterminate when this particular value is substituted for m. It will therefore be necessary to begin development of the present worth equation from the annual cash income equation which is determinate for the special case.

Step I. Determining the cumulative production over the discount interval.

Substituting $m = -1$ in eq. 20 results in the following integral:

$$Q_{ct} = \int_{t-1}^{t} \frac{365Mq_o\,dt}{1+t/c} \qquad (24)$$

Step II. Discounting the cash income. As before, the present worth of this income is equal to the product $Q_{ct} \cdot D_t$, or

$$P_t = b^t \int_{t-1}^{t} \frac{365Mq_o\,dt}{1+t/c} \qquad (25)$$

Step III. Summing the discounted incomes.

The total present worth is equal to the sum of these annual present worths.

$$P_T = \sum_{t=1}^{t_e} b^t \int_{t-1}^{t} \frac{365Mq_o\,dt}{1+t/c} \qquad (26)$$

Performing the integration and extracting constants from the summation yields

$$P_T = 365Mq_oc \sum_{t=1}^{t_e} b^t \Big[\ln\ (1+t/c)\Big]_{t-1}^{t}$$

$$P_T = 365Mq_oc \sum_{t=1}^{t_e} b^t \ln \frac{c+t}{c-1+t} \qquad (27)$$

Fixed Operating Expenses

The fixed operating expenses as discussed on p. 14 are independent of production and are assumed constant. This daily expense must be subtracted from the daily income and thus constitutes a new series of terms which must be discounted and summed in the same manner as production. The method of solution will be demonstrated for harmonic decline. Considering eq. 24,

if $\frac{Mq_o}{1+t/c}$ = daily cash income

and F = daily fixed operating expense

then the daily net income is equal to $\frac{Mq_o}{1+t/c}$ minus F. This is the amount requiring discount and summation. Subtracting F in eq. 24 yields

$$Q_{ct} = 365\int_{t-1}^{t}\left[\frac{Mq_o}{1+t/c} - F\right]dt \qquad (28)$$

Discounting the net annual cumulative income:

$$P_t = b^t\int_{t-1}^{t}\frac{365Mq_o dt}{1+t/c} - b^t\int_{t-1}^{t}365F dt \qquad (29)$$

Summing the discounted net income:

$$P_T = \sum_{t=1}^{t_e} b^t\int_{t-1}^{t}\frac{365Mq_o dt}{1+t/c} - \sum_{t=1}^{t_e} b^t\int_{t-1}^{t}365F dt \qquad (30)$$

The first term on the right has been solved in eq. 27. The second term after integration becomes $\sum_{t=1}^{t_e} 365Fb^t$, which is similar to the general series $\sum_{t=1}^{t_e} r^{p-1}$ treated on p. 19. The partial sum is found in the same manner as before.

$$\sum_{t=1}^{te} Fb^t = S_{te} = \frac{Fb(1 - b^{te})}{1 - b}$$

Substituting this term for the summation in eq. 30, and rewriting the solution of the first term as given by eq. 27, yields

$$P_T = 365\ Mq_oc \sum_{t=1}^{te}\left[b^t \ln \frac{c+t}{c-1+t}\right] - \frac{365Fb(1 - b^{te})}{1 - b} \qquad (31)$$

This represents the total present worth of future net income for harmonic production. The second term on the right is identical for all three decline types and can be attached to the expression for present worth, eq. 18 and eq. 23, without alteration.

Royalties

Royalties, when payable, are generally expressed as a fraction of the daily production rate. From the operator's standpoint this has the effect of multiplying daily production by one minus the royalty fraction, i.e. net production = q(1-R), where R is the royalty fraction.

The introduction of a royalty payment necessitates a generalization of the economic limit conditions in Table II. These are easily adjusted, however, by multiplication of the Mq_o terms by (1-R).

VI. CONCLUSIONS

Generalized Economic Limit and Present Worth Equations

The present worth equations have been derived and the effects of fixed operating expenses and royalty payments have been discussed. To include these factors the present worth equations were modified. Appearing below are the generalized present worth equations together with the corresponding economic limit conditions.

Constant Percentage Decline

$$P_{T_o} = \frac{365M(1-R)q_o be^k(1-e^{-k})\ 1-(be^k)^{t_e}}{k(1-be^k)} - \frac{365Fb(1-b^{t_e})}{1-b} \qquad (32)$$

where

$$t_e = \frac{\ln \frac{F}{(1-R)Mq_o}}{k} \qquad (33)$$

Hyperbolic Decline

$$P_{T_o} = \frac{365Mq_o(1-R)}{(m+1)c^m}\left[\sum_{t=1}^{t_e} b^t(c+t)^{m+1} - \sum_{t=1}^{t_e} b^t(c-1+t)^{m+1}\right] - 365Fb\frac{(1-b^{t_e})}{1-b} \qquad (34)$$

where

$$t_e = c\left[\frac{F}{(1-R)Mq_o}\right]^{1/m} - c \qquad (35)$$

Harmonic Decline

$$P_{T_o} = 365Mq_o(1-R)c\sum_{t=1}^{t_e} b^t \ln \frac{c+t}{c-1+t} - 365Fb\frac{(1-b^{t_e})}{1-b} \qquad (36)$$

where

$$t_e = \frac{c(1-R)Mq_o}{F} - c \qquad (37)$$

In the preceding equations the term P_{T_o} represents the total present worth of future production from the operator's standpoint. A list of all other symbols follows.

<u>List of Symbols</u>

$b = \frac{1}{1+i}$

c = shift constant (yrs.) required to produce linearity of the production equation

$D_t = \frac{1}{(1+i)^t}$ = discount factor

e = base of natural logarithms

F = fixed operating expenses

$k = \frac{\ln (q_1/q_o)}{t_1 - t_o}$ = exponent for constant percentage decline

i = interest rate expressed as a decimal fraction

ln = logarithm to the base e

m = slope of the straightened production curve

M = net price per bbl.

P_T = total present worth

P_{T_o} = total operator's present worth

P_t = present worth of t^{th} year's production

Q = cumulative production (bbl.)

Q_t = cumulative production for t^{th} year

Qct = cumulative cash income for t^{th} year

q = production rate (bbl. per day)

q_c = cash value of daily production

q_o = initial daily production

q_1 = daily production at any specific time t_1

S_n = sum of a series through the n^{th} term

t = time (yrs.) since initial production

t_e = abandonment time, economic limit

List of Symbols (contd.)

t_1 = any specific time with corresponding production q_1.

V = variable operating expense

Royalty Holder's Present Worth

The preceding equations can be used to calculate present worth from the royalty holder's standpoint with the following changes. Substitute R for (1-R), market price per bbl. for M, and drop the term, $\frac{365Fb(1 - b^{te})}{1 - b}$, from each present worth equation.

Taxes

Because of the variation in federal income tax rates, no consideration has been given to their effect on the total present worth. If the pertinent taxes are of a predictable nature, it may be possible to modify the present worth equations to accomodate a tax factor.

Numerical Evaluation of the Generalized Present Worth Equations

To facilitate the numerical evaluation of the foregoing equations several graphical and mechanical aids have been prepared. These appear in the appendices accompanied by an example solution. Appendix A contains sample production data for the curves shown in Figs. A - F and indicates the general procedure for obtaining specific equations from the graphed production curves.

Constant percentage evaluations can be most easily made since there are no series involved. The economic limit, t_e, is calculated from a study of operating expense. This value, together with the other specific conditions of the problem are then substituted in eq.32 to determine the total present worth.

Hyperbolic evaluations are more difficult since they involve solving the two series in eq. 34. The term b^t is the familiar discount factor obtainable from any fundamental economics text.[6]

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6 Bullinger, C. E., Engineering Economic Analysis, N.Y., McGraw-Hill Book Co., 1950, p. 379.

A nomogram has been prepared for determining the $(c+t)^{m\ 1}$ and $(c-1+t)$ terms. The construction and use of the nomogram are discussed in Appendix B.

Harmonic Decline also necessitates a series solution. The values of the series have been calculated for two common interest rates and for eight shift constants in the graphs of Appendix C. An example solution is included.

A slide rule has been devised to facilitate the rapid calculation of additional series which have not been graphed. The rule employs three interest rates and twenty shift constants. All four digit numbers appearing on the rule are decimal fractions. When the slide rule is adjusted to the proper shift constant, the value of the individual terms in the series is equal to the product of the production factor and the discount factor appearing opposite. The individual products are then summed to yield the series value.

VII. SUMMARY

Several generalized equations for the economic evaluation of petroleum production have been derived for normal production patterns. To facilitate the numerical solution of these equations, several graphical and mechanical aids have been prepared. Much of the tedious work of repeated calculation required under existing methods has been eliminated. The generalized equations contain those economic factors generally considered in a practical evaluation and at no time have the problems been purposely oversimplified for the sake of a simple theoretical solution. Every effort has been made to develop and present a practical evaluation method applicable to a wide range of production trends.

VIII. APPENDICES: APPENDIX A

Constant Percentage Decline

Characteristic production data

Initial production: 100 bbl./day, January, 1950.

Production for succeeding two years:

Month	Year	Production (bbl./day)
January	1950	100.0
July	1950	77.9
January	1951	60.6
July	1951	47.2
January	1952	36.8

The production data has been plotted in Fig. A and Fig. B. The decline curve is a straight line on the semi-log coordinate system of Fig. B. Therefore it has the general equation $q = q_o e^{kt}$,

where q = production at time t

q_o = initial production

t = time

$$k = \frac{\ln \frac{q_1}{q_o}}{t_1 - t_o}$$

The constants, q_o and k, can be evaluated from the known production history.

q_o = 100 bbl./day

$$k = \frac{\ln \frac{36.8}{100}}{2} = -.5$$

By substituting in the general equation, the curve equation becomes $q = 100e^{-.5t}$

Hyperbolic Decline

Characteristic production data

Initial production: 100 bbl. /day, January, 1950.

Production for the succeeding two years:

Month	Year	Production (bbl./day)
January	1950	100.0
July	1950	73.5
January	1951	56.3
July	1951	44.5
January	1952	36.0

The production has been plotted in Fig. C and shifted for linearity in Fig. D. The curve can be straightened by a three year shift. Therefore it has the general equation $q = q_o(1 + t/c)^m$

where c = shift constant in years

m = slope of shifted curve

The constants in the equation can be evaluated by inspection from Fig. D.

q_o = 100 bbl./day

c = 3 yrs.

m = -2

By substitution in the general equation, the curve equation becomes $q = 100(1 + t/3)^{-2}$

Harmonic Decline

Characteristic production data

Initial production: 100 bbl./day, January, 1950

Production for the succeeding two years:

Month	Year	Production (bbl./day)
January	1950	100.0
July	1950	85.6
January	1951	75.0
July	1951	66.7
January	1952	60.0

The production has been plotted in Fig. E and shifted for linearity in Fig. F. The curve can be straightened by a three year shift. Therefore it has the general equation $q = q_o\ (1 + t/c)^m$. The constants in the equation can be evaluated by inspection from Fig. F.

q_o = 100 bbl./day

$c = 3$ yrs.

$m = -1$

By substitution in the general equation the curve equation becomes

$$q = \frac{100}{1 + t/3}$$

APPENDIX B

Convergence Test for Series

It will sometimes be useful in computing series values to be able to estimate the remainder after any arbitrary number of terms have been calculated. This is especially true when the field has a long production life requiring the calculation of a considerable number of terms. The following convergence test and dependent theorem[7] may be used to determine the maximum error bound.

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[7] Middlemiss, R. R., Differential and Integral Calculus, Second Edition, N.Y., McGraw-Hill Book Co., 1946, p. 407.

Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = p < 1$$

Let $S_k = \sum_{n=1}^{k} u_n$ be the sum of the first k terms, k satisfying the conditions that $\frac{u_{k+1}}{u_k} = r < 1$, and for $n > k$ the value of this ratio is less than r.

Then $R_k < \frac{ru_k}{1-r}$, i.e. the error or remainder after k terms is less than $\frac{r}{1-r}$ times the last term retained.

Example

The series in the hyperbolic present worth equation, eq. 34, has the form $\sum_{t=1}^{n} b^t(c+t)^{m+1}$. Forming the prescribed ratio yields $\frac{u_{k+1}}{u_k} = \frac{b^{t\ 1}(c-1+t)^{m+1}}{b^t(c+t)^{m+1}}$

$$\lim_{k \to \infty} \frac{u_{k+1}}{u_k} = \lim \frac{b^{t+1}(c-1+t)^{m+1}}{b^t(c+t)^{m+1}} = b$$

Since $b = \dfrac{1}{1+i}$, and i assumes only positive values, b must always be less than one. For hyperbolic decline m is always less than -1. Therfore m+1 will be negative and the necessary conditions are satisfied.

$$\frac{u_{k+1}}{u_k} = r < 1$$

Also for all $n > k$ the value of the ratio is less than r.

Therefore the remainder after k terms is given by the the expression $R_k < \dfrac{bu_k}{1-b}$.

Condition for Convergence

As discussed on p. 18, the geometric series, $\sum_{t=0}^{\infty} b^t e^{kt}$, will converge to $\dfrac{1}{1 - be^k}$ if $|be^k| < 1$. Now $b = \dfrac{1}{1+i}$, where i takes only positive values. Therefore b must always be less than one. Also, $k = \dfrac{\ln (q_1/q_0)}{t_1 - t_0}$, and since the production decreases consistently, k must be negative. Then e^k will always be less than one. Therefore the product, be^k, and its absolute value, $|be^k|$, must always be less than one for the case of declining production.

Discussion of Arps' Hyperbolic Decline Equation

Definition of terms (Arps):

P = production rate (bbl. per unit t)

P = loss in production rate during unit t

$a = \frac{P}{\Delta P}$ = loss ratio per unit t

$b = \Delta\left[\frac{P}{\Delta P}\right]$ = change of loss ratio per unit t

or in equivalent derivatives,

$$P = dP/dt$$

$$a = -\frac{P}{dP/dt}$$

$$b = -\frac{d\left[\frac{P}{dP/dt}\right]}{dt}$$

The minus signs have been inserted to make a and b positive constants.

Derivation:

Integration of the preceding equation yields

$$\frac{P}{dP/dt} = -bt + C$$

To evaluate the constant of integration, C, substitute the initial conditions $a = a_o$ when $t = 0$.

Then $C = -a_o$

and $\frac{P}{dP/dt} = -bt - a_o$

or $\frac{dP}{P} = \frac{-\,dt}{a_o + bt}$

A second integration yields

$$\ln P = -1/b \ln (bt + a_o) + \ln C_1$$

To evaluate this constant of integration, C_1, substitute the initial conditions $P = P_o$ when $t = 0$.

Then $C_1 = P_o(a_o)^{1/b}$

Substitution of this value for C_1 yields

$$\ln P = -1/b \ln (bt + a_o) + \ln P_o(a_o)^{1/b}$$

or $$P = P_o(tb/a_o + 1)^{-1/b}$$

"This expression ... shows that horizontal shifting to the right over a distance a_o/b is necessary ... for straightening the curve on log-log paper."

Therefore a_o/b is equal to the shift constant, c, used throughout this paper, and the equation above is equivalent to the equation derived graphically on p. 10, i.e. $q = q_o(1 + t/c)^m$.

Nomogram for Series Evaluation

The total present worth equation for hyperbolic decline contains the series $\sum_{t=1}^{n}(c+t)^{m+1}$ and $\sum_{t=1}^{n}(c-1+t)^{m+1}$. A nomogram has been prepared to determine the terms of the series more quickly.

Construction of the nomogram

The generalized form of the preceding series may be written $\sum_{t=1}^{n} T_t = \sum_{t=1}^{n} V_t^p$

where $p = m+1$
$V = (c+t)$ or $(c-1+t)$
$T = V^p$

The desired range of the variables must be determined. From practical considerations these have been assumed as follows: T to vary from .001 to 1.00

V to vary from 1.0 to 100

m to vary from -1 to $-\infty$

Using logarithms, the equation $T = V^p$ may be written in the form $\frac{x}{y} = \frac{z}{w}$ or $\frac{\ln T}{\ln V} = \frac{p}{1}$.

From the range of T the scale modulus for x (ln T) is found to be $M_x = M_{\ln T} = \frac{10}{3} = 3.33$ (using a 10 in. scale)

Similarly the modulus for y (ln V) is found to be

$M_y = M_{\ln V} = \frac{10}{2} = 5$ (using a 10 in. scale)

The distance, L, from the point T = 1 along the diagonal to V = 1 is given, for various values of p by the equation

$$L = \frac{pKM_x}{M_y - pM_x}$$

where K is the length of the diagonal.

A table of p values appears below. These values were computed from the equation and laid off on the diagonal.[8]

-m	-p	L
1.1	0.1	0.71
1.2	0.2	1.33
1.3	0.3	1.88
1.4	0.4	2.39
1.5	0.5	2.85
1.6	0.6	3.24
1.7	0.7	3.60
1.8	0.8	3.94
1.9	0.9	4.25
2.0	1.0	4.47
2.5	1.5	5.66
3.0	2.0	6.48
4.0	3.0	7.55
5.0	4.0	8.25
6.0	5.0	8.75
7.0	6.0	9.10
8.0	7.0	9.34

Use of the nomogram

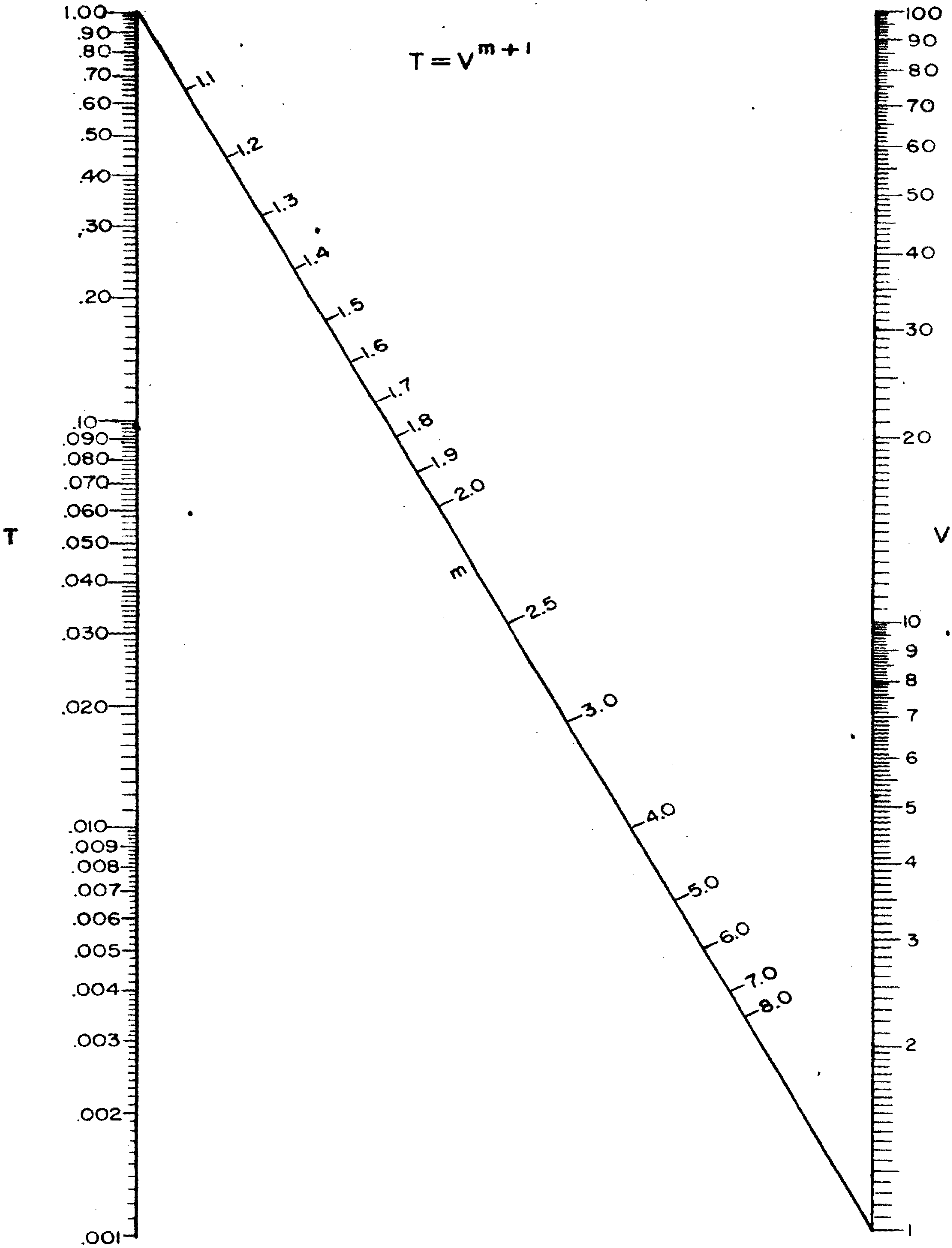
To obtain the numerical value of any term in the series $(c+t)^{m+1}$ or $(c-1+t)^{m+1}$:

1. Let V = (c+t) or (c-1+t)
2. Connect the known V and m values with a straight edge and extend the line to its intersection with the T scale.
3. Read the value of the term at the point of intersection.

[8] Swett, G. W., Construction of Alignment Charts, N.Y., John Wiley & Sons, 1928, p. 59.

SERIES EVALUATION NOMOGRAM

HYPERBOLIC DECLINE



APPENDIX C

Example Evaluation for Harmonic Decline

The production data on p. 33 and the following economic values will be used to calculate the total present worth of production up to the time of abandonment.

Interest rate = 8%

Market price: $2.60 per bbl.

Variable operating expenses: $0.30 per bbl.

Fixed operating expense: $25.00 per day

Royalty: one eighth

Step I. Calculation of the "break even" or abandonment time.

From eq. 37
$$t_e = \frac{(1-R)cMq_o}{F} - c$$

$$t_e = \frac{(1-1/8)(3)(2.30)(100)}{25.00} - 3$$

$$t_e = 21 \text{ yrs.}$$

Step II. Determination of the Series Value.

From the graph on p. 41, when t = 21 and c = 3, the Series Value = 1.26

Step III. Substitution of the preceding values in the total present worth equation, eq. 36.

$$P_{T_o} = (365)(2.60 - 0.30)(1-1/8)(100)(3)(1.26) - \frac{(365)(25)(1/1.08)\left[1-(1/1.08)^{21}\right]}{(1-1/1.08)}$$

$$P_{T_o} = 278{,}000 - 90{,}900$$

$$P_{T_o} = \$187{,}100$$

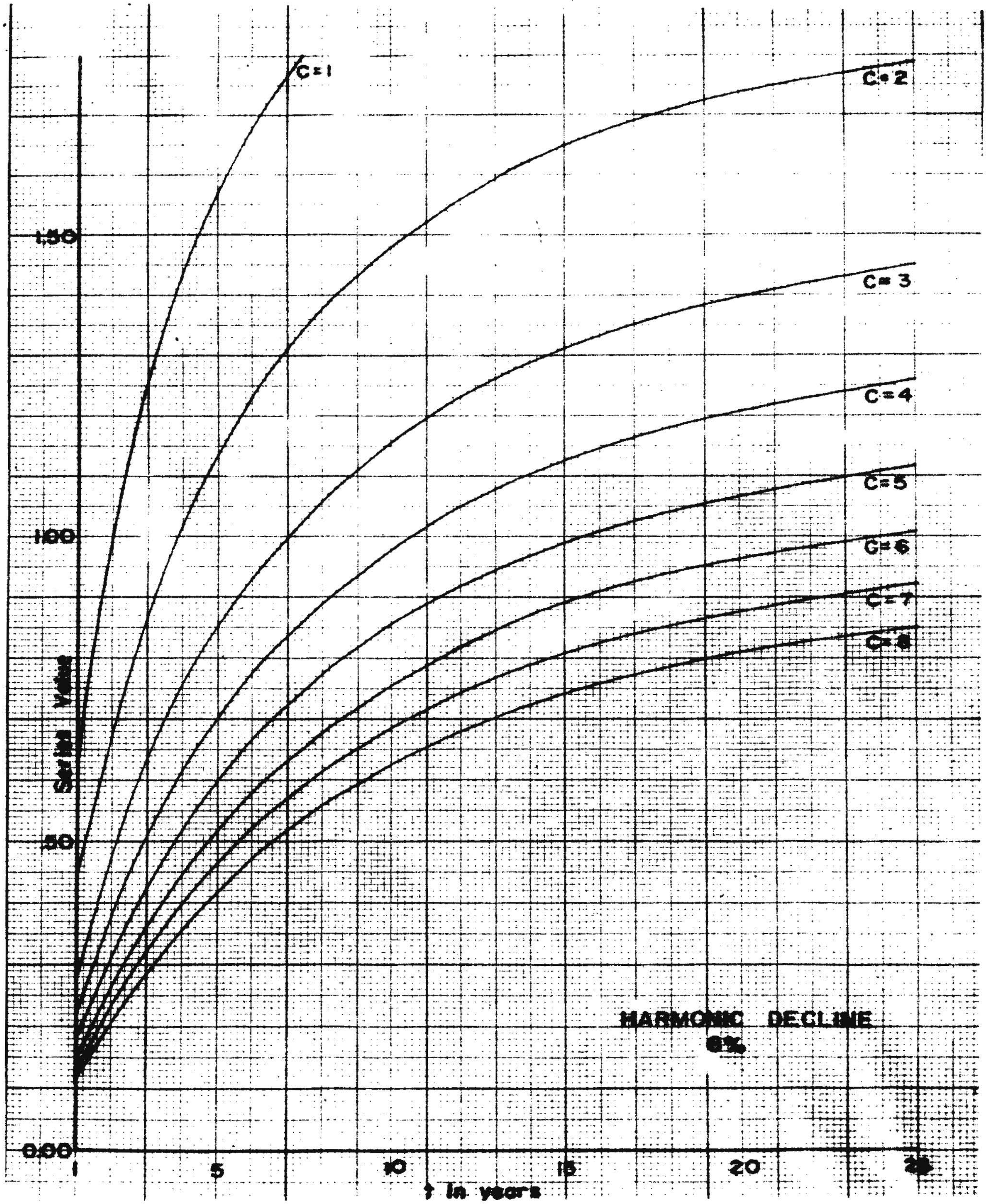

C=1
C=2
C=3
C=4
C=5
C=6
C=7
C=8
1.50
1.00
.50
0.00
Series Value
1
5
10
15
20
25
t in years
HARMONIC DECLINE
6%

FIG. M

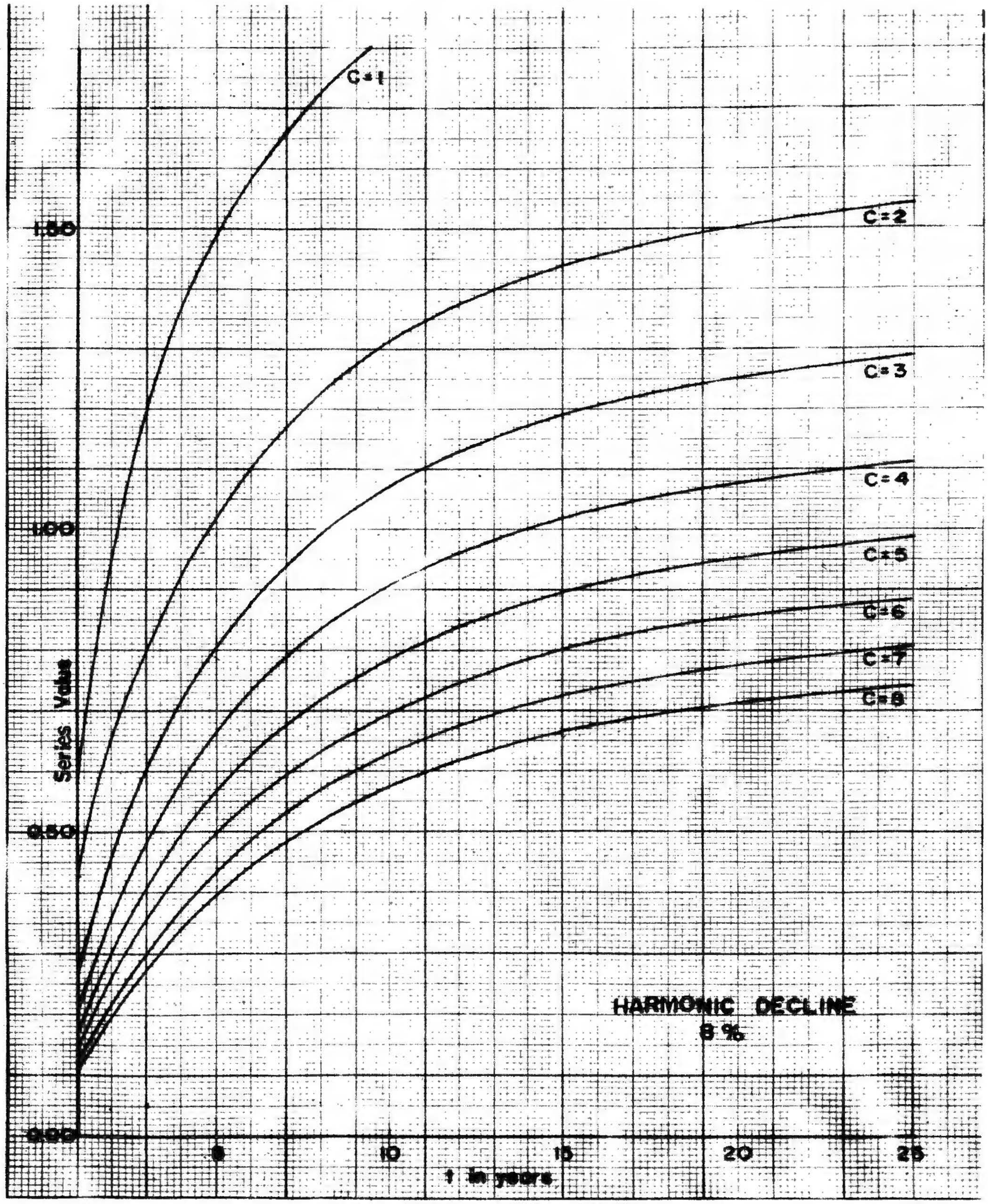

C=1
C=2
C=3
C=4
C=5
C=6
C=7
C=8
1.50
1.00
0.50
0.00
Series Value
5
10
15
20
25
t in years
HARMONIC DECLINE
8 %

FIG. I

SERIES CALCULATIONS

6%

Harmonic Decline

t	C = 1		C = 2		C = 3		C = 4	
1	.6055	.6055	.4309	.4309	.2714	.2714	.2105	.2105
2	.4065	1.0120	.2560	.6869	.1986	.4700	.1623	.3728
3	.2415	1.2535	.1874	.8743	.1531	.6231	.1294	.5022
4	.1768	1.4303	.1444	1.0187	.1221	.7452	.1058	.6080
5	.1362	1.5665	.1152	1.1339	.0998	.8450	.0880	.6960
6	.1087	1.6752	.0941	1.2280	.0830	.9280	.0743	.7703
7	.0888	1.7640	.0783	1.3063	.0701	.9981	.0634	.8337
8	.0739	1.8379	.0661	1.3724	.0598	1.0579	.0546	.8883
9	.0624	1.9003	.0564	1.4288	.0515	1.1094	.0474	.9357
10	.0532	1.9535	.0406	1.4774	.0447	1.1541	.0414	.9771
11	.0458	1.9993	.0422	1.5196	.0390	1.1931	.0363	1.0134
12	.0398	2.0391	.0368	1.5564	.0343	1.2274	.0321	1.0455
13	.0347	2.0738	.0323	1.5877	.0303	1.2577	.0284	1.0739
14	.0305	2.1043	.0285	1.6172	.0268	1.2845	.0253	1.0992
15	.0269	2.1312	.0253	1.6425	.0238	1.3083	.0226	1.1218
16	.0239	2.1551	.0225	1.6650	.0213	1.3296	.0203	1.1421
17	.0212	2.1763	.0201	1.6851	.0190	1.3486	.0181	1.1602
18	.0189	2.1952	.0180	1.7031	.0171	1.3657	.0163	1.1765
19	.0170	2.2122	.0161	1.7192	.0154	1.3811	.0147	1.1912
20			.0145	1.7337	.0139	1.3950	.0133	1.2045
21			.0131	1.7468	.0125	1.4075	.0120	1.2165
22			.0118	1.7586	.0113	1.4188	.0109	1.2274
23			.0107	1.7693	.0103	1.4291	.0099	1.2373
24			.0097	1.7790	.0093	1.4384	.0090	1.2463
25			.0088	1.7878	.0085	1.4469	.0082	1.2545

SERIES CALCULATIONS 6%
Harmonic Decline

t	c = 5		c = 6		c = 7		c = 8	
1	.1720	.1720	.1454	.1454	.1259	.1259	.1111	.1111
2	.1372	.3092	.1188	.2642	.1048	.2307	.0938	.2049
3	.1121	.4213	.0989	.3631	.0885	.3192	.0800	.2849
4	.0933	.5146	.0835	.4466	.0755	.3947	.0689	.3528
5	.0787	.5933	.0712	.5178	.0650	.4597	.0598	.4136
6	.0674	.6607	.0613	.5791	.0564	.5167	.0522	.4658
7	.0579	.7186	.0532	.6323	.0493	.5654	.0459	.5117
8	.0502	.7688	.0465	.6788	.0433	.6087	.0405	.5522
9	.0439	.8127	.0408	.7196	.0382	.6469	.0359	.5881
10	.0385	.8512	.0360	.7556	.0338	.6807	.0319	.6200
11	.0340	.8852	.0319	.7875	.0301	.7108	.0285	.6485
12	.0301	.9153	.0284	.8159	.0269	.7377	.0255	.6740
13	.0268	.9421	.0253	.8412	.0240	.7617	.0229	.6969
14	.0239	.9630	.0227	.8639	.0216	.7833	.0206	.7175
15	.0214	.9844	.0204	.8843	.0194	.8027	.0186	.7361
16	.0192	1.0036	.0183	.9026	.0175	.8202	.0168	.7529
17	.0173	1.0208	.0165	.9191	.0158	.8360	.0152	.7681
18	.0156	1.0365	.0149	.9340	.0143	.8503	.0137	.7818
19	.0141	1.0506	.0135	.9475	.0130	.8633	.0125	.7943
20	.0127	1.0633	.0122	.9600	.0116	.8749	.0113	.8056
21	.0115	1.0778	.0111	.9711	.0107	.8856	.0103	.8159
22	.0105	1.0883	.0101	.9812	.0097	.8953	.0094	.8253
23	.0095	1.0978	.0092	.9904	.0089	.9042	.0086	.8339
24	.0087	1.1065	.0084	.9988	.0081	.9123	.0079	.8418
25	.0079	1.1144	.0076	1.0064	.0074	.9197	.0072	.8490

SERIES CALCULATIONS
Harmonic Decline

8%

t	C = 1		C = 2		C = 3		C = 4	
1	.5943	.5943	.4229	.4229	.2664	.2664	.2066	.2066
2	.3916	.9859	.2466	.6695	.1913	.4577	.1563	.3629
3	.2284	1.2142	.1771	.8143	.1447	.6024	.1224	.4853
4	.1640	1.3783	.1340	.9276	.1130	.7157	.0981	.5834
5	.1241	1.5023	.1049	1.0185	.0909	.8066	.0802	.6636
6	.0972	1.5955	.0842	1.1026	.0742	.8808	.0664	.7300
7	.0779	1.6774	.0687	1.1713	.0615	.9423	.0556	.7856
8	.0636	1.7410	.0569	1.2283	.0515	.9938	.0470	.8326
9	.0527	1.7938	.0477	1.2759	.0435	1.0373	.0400	.8726
10	.0442	1.8379	.0403	1.3162	.0371	1.0744	.0343	.9069
11	.0373	1.8752	.0343	1.3506	.0318	1.1062	.0296	.9365
12	.0318	1.9070	.0294	1.3800	.0274	1.1336	.0256	.9621
13	.0273	1.9343	.0257	1.4057	.0237	1.1573	.0223	.9844
14	.0235	1.9577	.0220	1.4277	.0206	1.1779	.0195	1.0039
15	.0203	1.9781	.0191	1.4468	.0180	1.1960	.0170	1.0209
16	.0177	1.9958	.0167	1.4635	.0158	1.2177	.0150	1.0359
17	.0155	2.0112	.0146	1.4781	.0139	1.2256	.0132	1.0491
18	.0135	2.0248	.0128	1.4910	.0122	1.2378	.0116	1.0607
19	.0119	2.0266	.0113	1.5023	.0108	1.2486	.0103	1.0710
20			.0100	1.5122	.0095	1.2581	.0091	1.0801
21			.0088	1.5211	.0085	1.2666	.0081	1.0882
22			.0078	1.5289	.0075	1.2741	.0072	1.0954
23			.0070	1.5358	.0067	1.2808	.0064	1.1018
24			.0062	1.5420	.0060	1.2867	.0057	1.1075
25			.0055	1.5475	.0053	1.2920	.0051	1.1126

SERIES CALCULATIONS
Harmonic Decline

8%

t	c = 5		c = 6		c = 7		c = 8	
1	.1688	.1688	.1427	.1427	.1236	.1236	.1091	.1091
2	.1322	.3010	.1145	.2572	.1010	.2246	.0904	.1995
3	.1060	.4070	.0935	.3507	.0837	.3083	.0756	.2751
4	.0866	.4936	.0774	.4281	.0700	.3783	.0639	.3390
5	.0717	.5653	.0649	.4930	.0592	.4375	.0545	.3935
6	.0601	.6254	.0548	.5478	.0504	.4879	.0467	.4402
7	.0508	.6762	.0467	.5945	.0432	.5311	.0403	.4805
8	.0432	.7194	.0400	.6345	.0373	.5684	.0348	.5153
9	.0371	.7565	.0345	.6690	.0323	.6007	.0303	.5456
10	.0320	.7885	.0299	.6989	.0281	.6288	.0265	.5721
11	.0277	.8162	.0260	.7249	.0245	.6533	.0232	.5953
12	.0241	.8403	.0227	.7476	.0215	.6748	.0204	.6157
13	.0210	.8613	.0199	.7675	.0189	.6937	.0179	.6336
14	.0184	.8797	.0175	.7850	.0166	.7103	.0158	.6494
15	.0162	.8959	.0153	.8003	.0147	.7250	.0140	.6634
16	.0142	.9101	.0136	.8139	.0130	.7380	.0124	.6758
17	.0126	.9227	.0120	.8259	.0115	.7495	.0110	.6868
18	.0112	.9339	.0106	.8367	.0102	.7597	.0098	.6966
19	.0099	.9438	.0095	.8462	.0091	.7688	.0087	.7053
20	.0088	.9526	.0084	.8546	.0081	.7769	.0078	.7131
21	.0078	.9604	.0075	.8621	.0072	.7841	.0070	.7201
22	.0069	.9673	.0067	.8688	.0065	.7906	.0062	.7263
23	.0062	.9735	.0060	.8748	.0058	.7964	.0056	.7319
24	.0055	.9790	.0053	.8801	.0052	.8016	.0049	.7368
25	.0049	.9839	.0048	.8849	.0046	.8062	.0045	.7413

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