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A theoretical analysis of the free vibrations of a clamped circular plate with damping

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A THEORETICAL ANALYSIS OF THE FREE VIBRATIONS OF A CLAMPED CIRCULAR PLATE WITH DAMPING

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ABSTRACT

A theoretical analysis of the free vibrations of a clamped circular plate was undertaken. The effects of transverse shear were considered for application to relatively thick plates in which there exist sharp transients and frequencies corresponding to higher modes. The method adopted by R. D. Mindlin was used to develope the differential equation of motion for the plate. The damping effect due to the resistance of the medium in which the plate vibrates was also considered. This damping was assumed to be due to the generation of pressure waves caused by the displacement of the plate.

A product solution of the form $R(r)e^{P_1 t}$ was assumed. The equation governing the frequencies of vibration was obtained by the application of the boundary conditions. Of the several sets of roots of this equation, one set of roots has been discussed in this investigation; namely, the roots of the Bessel function of the first kind and zero order. Accordingly, it was found that two values of P_1 exist for every root of the frequency equation. As a result, the general form of the time dependent part of the solution was the sum of two different forms of exponential functions. The initial conditions for the displacement and velocity were successfully applied to the solution to evaluate the arbitrary constants. The solution involved the combination of two different roots of Bessel functions of the first kind and zero order.

Deflection curves for a specific example were calculated and plotted by combining the roots of the Bessel functions in several ways. Three different classes of deflection curves were found to exist in the nineteen solutions computed. It was observed that the maximum deflection did not always occur at the center of the plate. The solutions in which one root of the Bessel function of the first kind and zero order does not associate with two frequencies is believed to give the best results. Since the solution requires that the initial displacement function be known, the example presented is at best an approximation because the initial displacement function assumed did not include shear deformation. It should be emphasized that only one possibility of the roots of the frequency equation has been discussed in this investigation. Therefore, the solution presented is a particular solution. However, it is hoped that this is a significant first step in the study of the stated problem.

ACKNOWLEDGEMENT

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r, θ, z	:	Cylindrical coordinates
ε & γ	:	Strain components
σ & τ	:	Stress components
M_r, M_θ	:	Unit Moments
$M_{r\theta}$	:	Unit Twist
Q_r, Q_θ	:	Unit Shear Forces
q_1, q_2	:	Stresses on the faces of the plate
ψ_r, ψ_θ	:	Negative of the slope in respective direction neglecting shear deformation
E, G, ρ, μ	:	Young's Modulus of Elasticity, Modulus of Rigidity, Density, and Poisson's Ratio for the material of the plate
R, h	:	Radius and Thickness of the plate
u, v, ω	:	Radial, Tangential and Transverse components of displacement of a point
ρ_3, C_3	:	The density and the velocity of wave propogation in medium I
$q(r,\theta,t)$	:	The net forcing function
D, B, P, B_1 A, Q	:	The constants defined by equations 23-a.
C_1, C_2	:	The constants defined by equations 27 and 28.
J_0, J_1	:	Bessel functions of first kind
Y_0, Y_1	:	Bessel functions of second kind
a_1, a_2, a_3, a_4	:	Arbitrary constants
α_n, α_m	:	Roots of J_o
ρ_4, C_4	:	The density and the velocity of wave propogation in medium II

z_1, z_2	:	Arbitrary constants
b_1, b_2, b_3, b_4, b	:	Arbitrary constants
U_1, U_2, U_3, U_4	:	The constants defined by equations 49.
$A_1, A_2, \ldots A_{19}$	:	Solutions
I	:	Defined by equation G-1.
P, V	:	Pressure and volume of a quantity of a gas.
γ_1	:	The ratio of specific heat at constant pressure and specific heat at constant volume for a gas.
∇^2	:	Laplace operator
v_1	:	Auxiliary function of (r,θ,t)

Subscripts:

r	:	Refers to the radial direction.
θ	:	Refers to the tangential direction.
z	:	Refers to the transverse direction.
$r\theta$	:	Refers to a plane perpendicular to the r axis and a direction parallel to the θ axis.
rz	:	Refers to a plane perpendicular to the r axis and a direction parallel to the z axis.
θz	:	Refers to a plane perpendicular to the θ axis and a direction parallel to the z axis.

I. INTRODUCTION

The use of a plate as a component in machinery and equipment is very common. Therefore, the analysis of the behavior of plates under various loading conditions and environments has received considerable attention. The attempts which have been made to analyze the behavior of a plate during vibrations are numerous and diverse.

In general, the method used is to solve the mathematical equations derived from the mathematical simulation of conditions of vibration. In the classical theory proposed by Lagrange, it is assumed that normals to the middle surface of the plate remain straight, normal, and unstrained as the plate deforms and thus does not include any effects of transverse shear on the deformations. In the theory given by E. Reissner [1]*, it is assumed that the components of stress at a point in the plate depend on the z coordinate at that point, and does include the effects of transverse shear on the deformation of the plate. Most current authors agree that a consistent theory should require that three conditions, known as Poisson's boundary conditions, should be satisfied by the solution where only two boundary conditions were used with the classical theory. In the theory given by Reissner, it is required that the solution satisfy three conditions at the

*Numbers in the rectangular brackets refer to the references at the end.

boundaries of the plate. It is generally agreed in most of the literature that the consideration of the effects of transverse shear in the theory improves the analytical results significantly. The theory given by Reissner is applicable to small deflections of thin plates. It can also be applied to the case of small deflections of moderately thick plates. The effects of transverse shear play an important role as the thickness of the plate increases. The classical theory shows that the velocity of straight crested waves is inversely proportional to the wave length, but this result is valid in the case of linear, three dimensional theory of elasticity only for waves which are long compared to the thickness of the plate. Therefore, the effects of transverse shear are important when sharp transients and higher modes of vibrations are present. The effects of transverse shear on the flexural motion of a plate can be taken into account, as has been done by R. D. Mindlin [2], by inclusion of the terms for rotary inertia and shear deformation in the equations of motion.

There is a resistance to the flexural motion of a plate from the medium in which the plate is vibrating. As the plate vibrates, it generates waves in the medium on either side of it and the pressure due to these waves resists the motion of the plate. This resistance depends upon the density of the medium and the velocity of wave propogation in the medium. The case of forced vibrations of a clamped, rectangular plate in a fluid media has been investigated by Yeh and Martinek [3], though without consideration of the effects of transverse shear i.e. they did not include the terms of rotary inertia and shear deformation in the differential equation of motion.

It seems from the literature survey that Reissner's theory is the most accurate of the available theories. The difficulty in the application of Reissner's theory is the solution of the differential equations one obtains. This investigation was undertaken in order to provide an analytical solution for the case of free vibrations of a clamped circular plate which includes transverse shear and damping.

II. REVIEW OF LITERATURE

The free vibrations of a thin, isotropic, elastic, circular disk of a constant thickness were first studied by Poisson (1829) [4]. G. Kirchoff has also investigated the case of free vibrations of an undamped, circular plate with free edges. D. Á. Goldhammer (1910) [5] considered the damping of the vibrations of plates for the first time. Timoshenko has given a solution for the case of rectangular plates with free edges. Voigt has also given a solution for two opposite edges of a rectangular plate simply supported and the other two edges free. Robertson [6] has considered circular plates with concentrated mass with a free circumference and a clamped circumference. Many more like Rayleigh, Meyer Zu Capellen, Debye, Franke, Ritz, Iguchi, Young, etc. have made valuable contributions to the subject. The work of these investigators was primarily based on the classical theory of plates proposed by Lagrange.

The theory proposed by E. Reissner [1] is one of the most recent and most successful attempts to improve upon the classical theory. Following the lines of development of Reissner, R. D. Mindlin [2] has proposed a theory of vibrations of plates which accounts for the effects of transverse shear. This theory removes the discrepancies of classical theory as it assumes linear variation of stress in the direction perpendicular to the plane of the plate throughout the thickness of the plate. The theory given by Mindlin yields three differential equations of motion of a plate which can be reduced to one differential equation of motion in terms of the transverse displacement of the plate. The case of free vibrations of a circular plate with free edges for elastic material has been considered by Mindlin [7, 8, 9]. F. F. Ehrich [10] has given a matrix solution for the vibration modes of nonuniform disks. A complete analytical solution for forced vibrations of a clamped, circular plate with viscous damping has been given by Jaroslav Pachner [11]. However, he neglected the terms for rotary inertia and shear deformation in the differential equation of motion. The damping due to the resistance to the motion of the plate from the medium in which the plate vibrates has been considered by Morse [12] and Vogel [13].

III. DEVELOPMENT OF THE DIFFERENTIAL EQUATION

The equations of motion for the vibrating plate can be developed in a very simple and straightforward manner by use of various equations in the theory of elasticity in three dimensions. The equations of motion and expressions for unit moments, shear forces and twist are developed for small deflections of plates. The equations of motion and the expressions for unit moments, shear forces and twist are developed in cylindrical coordinates for the case of axially-unsymmetric deflections of circular plates. The theory is developed on the basis of Reissner's theory and in the same manner used by R. D. Mindlin [2]. The theory is developed as follows:

For an elemental volume in cylindrical coordinates as shown in Figure 1, the strain-displacement relations in three dimensional theory of elasticity are given by [14],

$$\varepsilon_r = [\frac{\partial u}{\partial r} + \frac{1}{2}(\frac{\partial u}{\partial r})^2 + \frac{1}{2}(\frac{\partial v}{\partial r})^2 + \frac{1}{2}(\frac{\partial \omega}{\partial r})^2]$$

$$\varepsilon_\theta = \frac{1}{r}\ [\frac{\partial u}{\partial \theta} + u + \frac{1}{2r}(\frac{\partial v}{\partial \theta} + u)^2 + \frac{1}{2r}(\frac{\partial \omega}{\partial \theta})^2 + \frac{1}{2r}(\frac{\partial u}{\partial \theta} - v)^2]$$

$$\varepsilon_z = [\frac{\partial \omega}{\partial z} + \frac{1}{2}(\frac{\partial \omega}{\partial z})^2 + \frac{1}{2}(\frac{\partial u}{\partial z})^2 + \frac{1}{2}(\frac{\partial u}{\partial z})^2] \tag{1}$$

$$\gamma_{r\theta} = \frac{\partial u}{r\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}(\frac{\partial u}{\partial r})(\frac{\partial u}{\partial \theta} - v) + \frac{1}{r}(\frac{\partial v}{\partial r})(\frac{\partial v}{\partial \theta} + u) + \frac{1}{r}(\frac{\partial \omega}{\partial r})(\frac{\partial \omega}{\partial \theta})$$

$$\gamma_{rz} = \frac{\partial \omega}{\partial r} + \frac{\partial u}{\partial z} + (\frac{\partial \omega}{\partial z})(\frac{\partial \omega}{\partial r}) + (\frac{\partial u}{\partial z})(\frac{\partial u}{\partial r}) + (\frac{\partial v}{\partial z})(\frac{\partial v}{\partial r})$$

$$\gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{\partial \omega}{r\partial \theta} + \frac{1}{r}(\frac{\partial v}{\partial \theta} + u)(\frac{\partial v}{\partial z}) + \frac{1}{r}(\frac{\partial \omega}{\partial \theta})(\frac{\partial \omega}{\partial z}) + \frac{1}{r}(\frac{\partial u}{\partial \theta} - v)(\frac{\partial u}{\partial z})$$

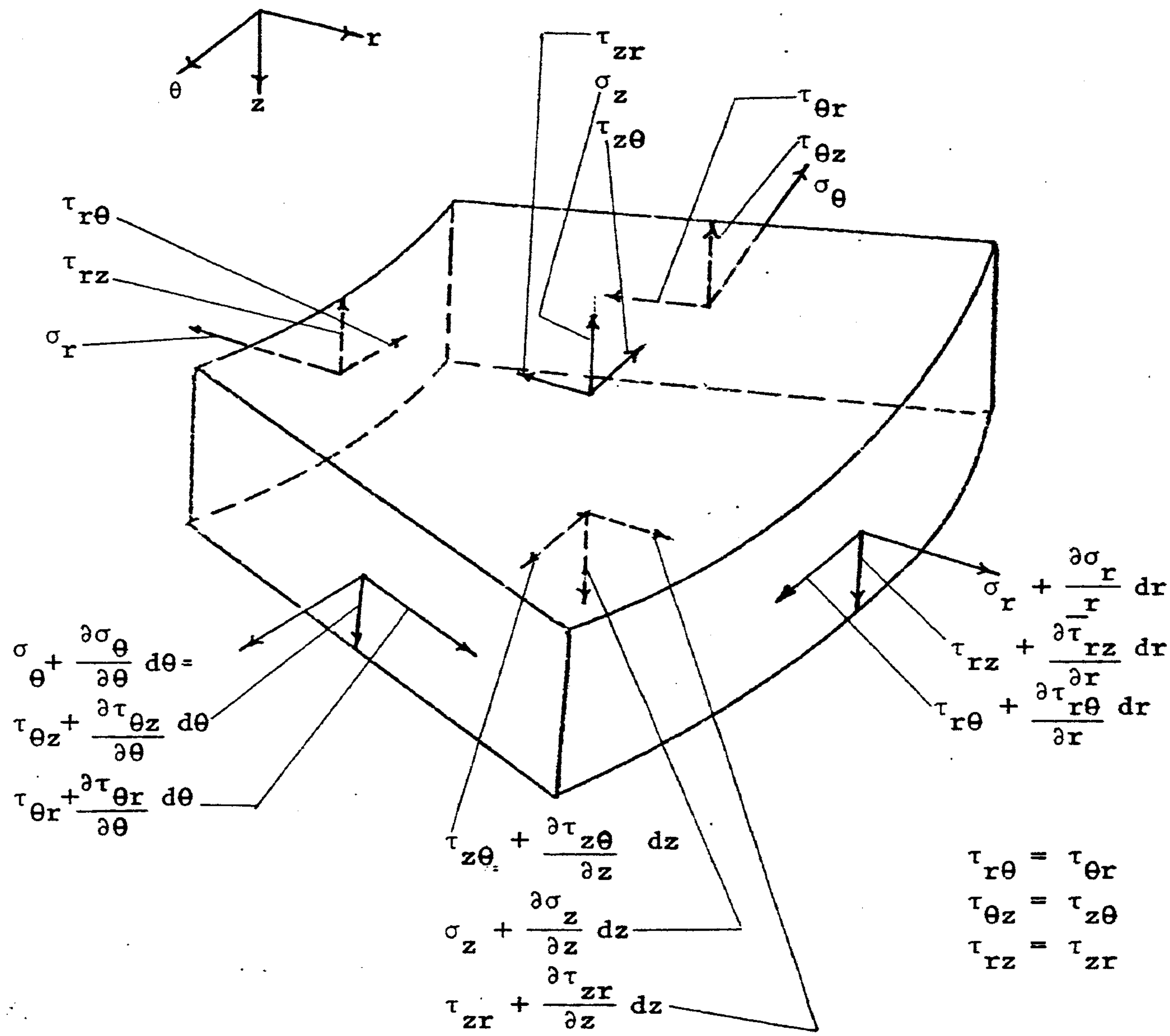

Figure 1: Components of Stress on an Element in Cylindrical Coordinates

For small displacements, higher powers of small quantities are neglected. Neglecting powers higher than one, equations one reduce to [14],

$$\begin{aligned}
\varepsilon_r &= \frac{\partial u}{\partial r} \\
\varepsilon_\theta &= \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\varepsilon_z &= \frac{\partial \omega}{\partial z} \\
\gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\
\gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial r} \\
\gamma_{\theta z} &= \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial \omega}{\partial \theta}
\end{aligned} \tag{2}$$

In the three dimensional theory of elasticity, stress-strain relations are given by [15],

$$\begin{aligned}
\varepsilon_r &= \frac{\sigma_r}{E} - \mu\frac{\sigma_\theta}{E} - \mu\frac{\sigma_z}{E} \\
\varepsilon_\theta &= -\mu\frac{\sigma_r}{E} + \frac{\sigma_\theta}{E} - \mu\frac{\sigma_z}{E} \\
\varepsilon_z &= -\mu\frac{\sigma_r}{E} - \mu\frac{\sigma_\theta}{E} + \frac{\sigma_z}{E} \\
\gamma_{r\theta} &= \frac{\tau_{r\theta}}{G} \\
\gamma_{\theta z} &= \frac{\tau_{\theta z}}{G} \\
\gamma_{rz} &= \frac{\tau_{rz}}{G}
\end{aligned} \tag{3}$$

While considering the equations of motion, it is assumed that there are no body forces such as gravity. The equations of motion in the three dimensional theory of elasticity are [16],

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = \rho \frac{\partial^2 v}{\partial t^2} \qquad (4)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho \frac{\partial^2 \omega}{\partial t^2}$$

The derivation of the two of the above three equations, i.e., equations 1 and 4 can be found in any book on the theory of elasticity. The derivations of equations 1 and 4 are given in Appendix- A.

The unit moments, shear forces and twist for a circular plate are defined as [2]

$$M_r = \int_{-h/2}^{h/2} \sigma_r z \, dz$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z \, dz$$

$$M_{r\theta} = \int_{-h/2}^{h/2} \tau_{r\theta} z \, dz \qquad (5)$$

$$Q_r = \int_{-h/2}^{h/2} \tau_{rz} \, dz$$

$$Q_\theta = \int_{-h/2}^{h/2} \tau_{\theta z} \, dz$$

The directions of positive moments, shear forces and twists on a typical plate element in cylindrical coordinates are shown in

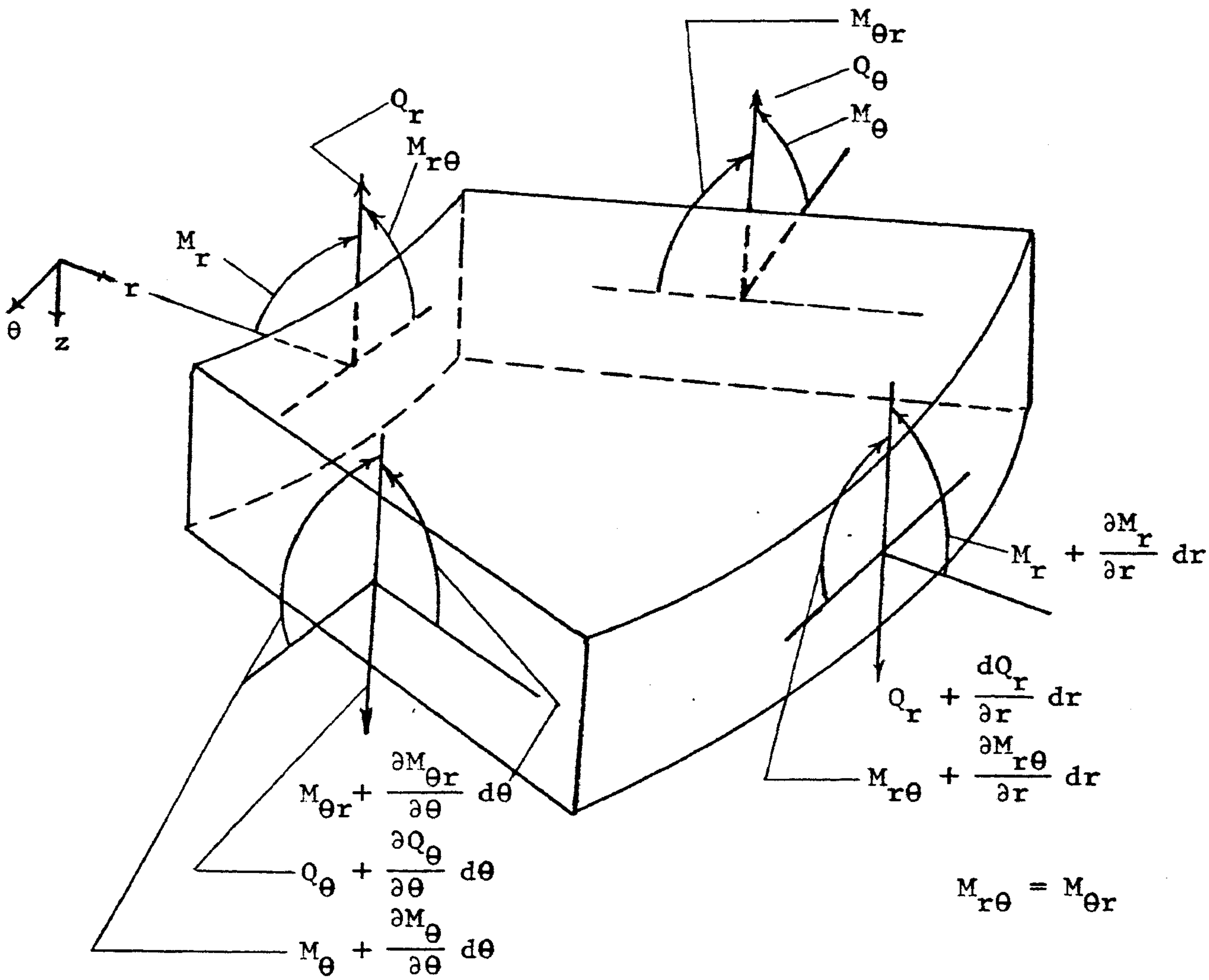

Figure 2: Directions of Positive Moments, Shear Forces and Twist on a Plate Element

Figure 2. The unit moments, shear forces and twists shown in Figure 2 are known as Plate Stress Components. To obtain relations for the plate stress components in terms of plate displacements, the procedure is as follows.

In the three dimensional theory of elasticity, there are six components of stress and six components of strain. The six components of strain are related to three components of displacement and the six components of stress are related to the six components of strain by means of Hooke's law. Of the six components of strain, the relation for the strain in the z direction in terms of the six components of stress is ignored, because for small deflections, the elongation per unit length in the z direction is negligible. Now five equations for strain in terms of the 6 components of stress are solved for σ_r, σ_θ, $\tau_{r\theta}$, τ_{rz}, $\tau_{\theta z}$ in terms of ε_r, ε_θ, $\gamma_{r\theta}$, γ_{rz}, $\gamma_{\theta z}$ and σ_z. The relations for the stress components are then integrated over the plate thickness to convert the ordinary stress components to the plate stress components . For the case of a thin plate, the boundary conditions for stress components are

$$\sigma_z\big|_{+h/2} = -q_1(r,\theta,t) \quad , \quad \sigma_z\big|_{-h/2} = -q_2(r,\theta,t) \ ,$$
$$\tau_{rz}\big|_{\pm h/2} = 0 \quad \text{and} \quad \tau_{\theta z}\big|_{\pm h/2} = 0 \ . \tag{6}$$

The expressions for the plate stress components , which have integrations of strain components and integrations of σ_z , are altered in two respects.

a) Integrations containing σ_z are dropped, because the displacements are small and when vibrations are free, the value of σ_z is very small. Further, the thickness of the plate is very small compared to the other dimensions. Therefore, the contributions of the integrations containing σ_z are negligible.

b) The coefficients of the integrals containing γ_{rz} and $\gamma_{\theta z}$ are multiplied by a constant. In this manner, the modulus of rigidity G is replaced by G' where $G' = k^2 x G$. The value of constant k^2 is chosen in order that the relations for the plate stress components could be used even when the wave lengths during vibration are short compared to the thickness of the plate and so as to give better analytical results for sharp transients and for the modes of vibrations of higher order. The value of the constant k^2 has been found by Mindlin [2] to be $\pi^2/12$. Therefore, the expressions for the plate stress components become,

$$M_r = \int_{-h/2}^{h/2} \sigma_r z\,dz = D\left[\frac{12}{h^3}\int_{-h/2}^{h/2} \varepsilon_r z\,dz + \frac{12\mu}{h^3}\int_{-h/2}^{h/2} \varepsilon_\theta z\,dz\right]$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z\,dz = D\left[\frac{12\mu}{h^3}\int_{-h/2}^{h/2} \varepsilon_r z\,dz + \frac{12}{h^3}\int_{-h/2}^{h/2} \varepsilon_\theta z\,dz\right]$$

$$M_{r\theta} = (1-\mu)\frac{D}{2}\left[\frac{12}{h^3}\int_{-h/2}^{h/2} \gamma_{r\theta}\, z\,dz\right] \tag{7}$$

$$Q_r = G'h\left[\frac{1}{h}\int_{-h/2}^{h/2} \gamma_{rz}\,dz\right]$$

$$Q_\theta = G'h\left[\frac{1}{h}\int_{-h/2}^{h/2} \gamma_{\theta z}\,dz\right]$$

where $G' = K^2 x G$ and

$$D = Eh^3/12(1-\mu^2)$$

The deduction of the first of the equations 7 is given in Appendix-B.

Now, by using the strain-displacement relations, the relations for plate stress components can be obtained. Some approximations to the radial and tangential components of displacement are now made. It is assumed that the radial and tangential components of the displacement of a point, u and v respectively, vary linearly with its distance from the r-θ plane, i.e., its z coordinate. The assumption is that

$$u(r,\theta,t) = Z\,\psi_r(r,\theta,t)$$

and (8)

$$v(r,\theta,t) = Z\,\psi_\theta(r,\theta,t)$$

where ψ_r is the negative of the slope at a point in the radial direction neglecting shear deformation and ψ_θ is the negative of the slope at a point in the tangential direction [17]. In figure 3, β_1 is the slope of the neutral plane at a point neglecting shear deformation and β_2 is the angle of shear at the neutral plane. The classical theory does not assume the existance of β_2; therefore, the total slope at a point is given by $\beta_1 + \beta_2$. The ψ's in equation 8 are the negative of the β_1's in their respective directions.

Now, using the strain-displacement relations in equations 2, the following expressions for the plate stress components in terms of the plate displacement components ω, ψ_r and ψ_θ can be obtained.

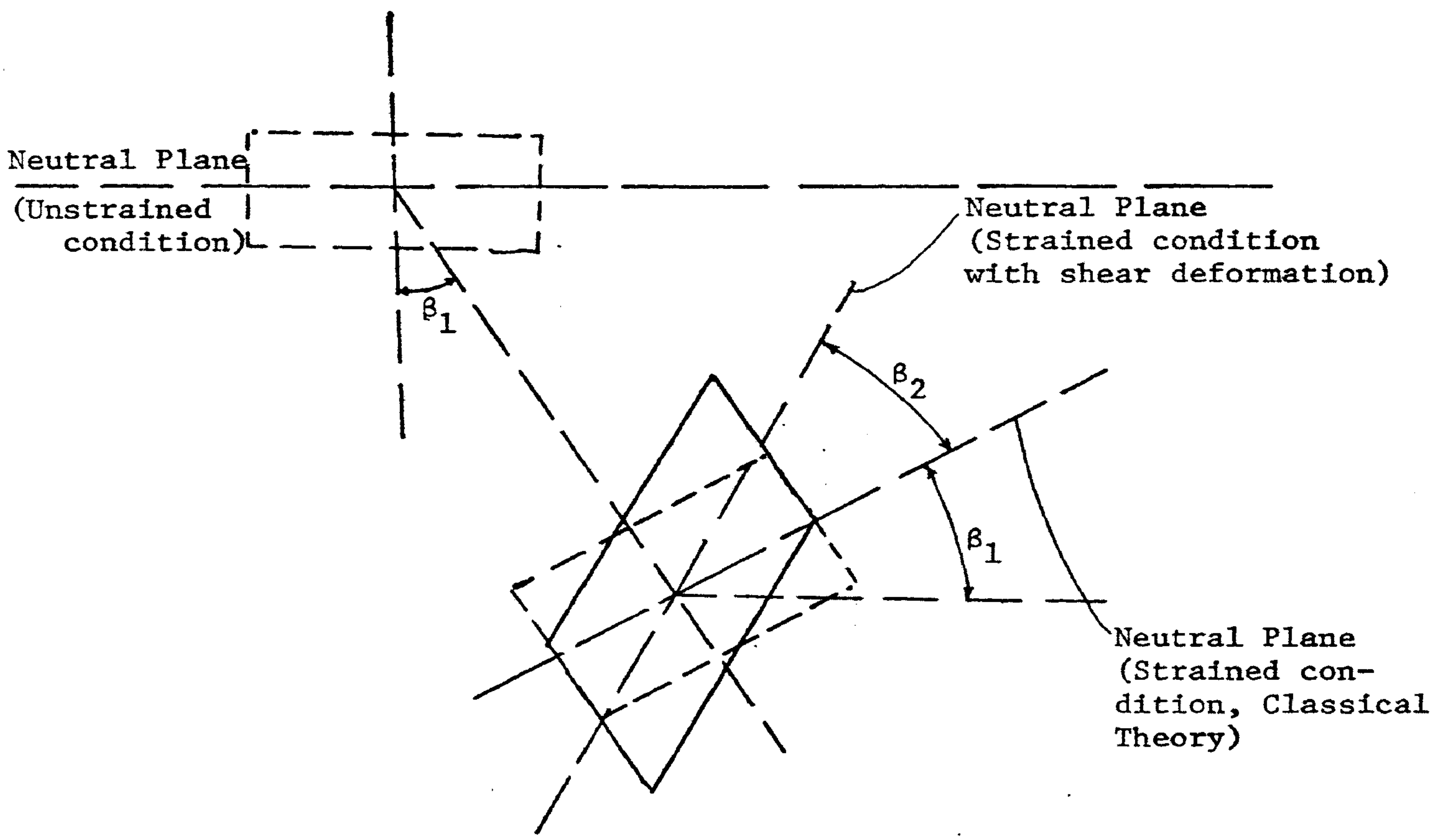

Figure 3: Slope of a differential element in a strained condition

$$M_r = D[\frac{\partial\psi_r}{\partial r} + \frac{\mu}{r}(\psi_r + \frac{\partial\psi_\theta}{\partial\theta})]$$

$$M_\theta = D\ [\mu \frac{\partial\psi_r}{\partial r} + \frac{1}{r}(\psi_r + \frac{\partial\psi_\theta}{\partial\theta})]$$

$$M_{r\theta} = (1 - \mu)\frac{D}{2}[\frac{1}{r}(\frac{\partial\psi_r}{\partial\theta} - \psi_\theta) + \frac{\partial\psi_\theta}{\partial r}] \qquad (9)$$

$$Q_r = G'h\ (\frac{\partial\omega}{\partial r} + \psi_r)$$

$$Q_\theta = G'h\ \frac{1}{r}(\frac{\partial\omega}{\partial\theta} + \psi_\theta)$$

The next step in the development of the theory is to convert the equations of motion which are in terms of ordinary stress components to the equations of motion in terms of the plate stress components. To do this we multiply the first two of the equations 4 by zdz and integrate over the plate thickness. The third equation of equations 4 is simply integrated over the thickness of the plate, by using the definitions of the plate stress components. Following are the equations of motion in terms of the plate stress components.

$$\frac{\partial M_r}{\partial r} + \frac{\partial M_r}{r\partial\theta} + \frac{M_r - M_\theta}{r} - Q_r = \frac{\rho h^3}{12}\frac{\partial^2\psi_r}{\partial t^2}$$

$$\frac{\partial M_{r\theta}}{\partial r} + \frac{\partial M_\theta}{r\partial\theta} + \frac{2M_{r\theta}}{r} - Q_\theta = \frac{\rho h^3}{12}\frac{\partial^2\psi_\theta}{\partial t^2} \qquad (10)$$

$$\frac{\partial Q_r}{\partial r} + \frac{\partial Q_\theta}{r\partial\theta} + \frac{Q_r}{r} + q(r,\theta,t) = \rho h\frac{\partial^2\omega}{\partial t^2}$$

where $q(r,\theta,t) = q_2(r,\theta,t) - q_1(r,\theta,t)$

The deduction of the first of equations 10 is shown in Appendix C.

It is evident that when the expressions for the quantities M_r, $M_{r\theta}$, M_θ, Q_r, Q_θ from equations 9 are substituted in equations 10, three equations involving the three dependent variables ψ_r, ψ_θ, ω will be obtained. These three equations could be uncoupled in the manner similar to that used by R. D. Mindlin [2] in rectangular coordinates. The procedure of uncoupling the equations in cylindrical coordinates for a spherical shell has been given by Kalnins [18]. Appendix D gives the procedure to uncouple the equations for a plate. The single equation is then,

$$(\nabla^2 - \frac{\rho}{G'} \frac{\partial^2}{\partial t^2}) (D\nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2})\omega + \rho h \frac{\partial^2 \omega}{\partial t^2} = (1 - \frac{D}{G'h} \nabla^2 + \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2}) q(r,\theta,t) \qquad (11)$$

To account for the resistance of the medium to the motion of the plate, it is evident that as the plate vibrates, it generates waves in the medium on either side of it. It is assumed that the waves generated by the motion of the plate are simple plane waves and there are no losses due to dispersion and scattering. Figure 4 shows two waves generated by the motion of a plate. At the instant shown in Figure 4, the backward pressure P_4 acting on the plate due to the wave V_4 is given from Appendix-E by [19],

$$P_4 = -\rho_4 C_4 \frac{\partial \omega}{\partial t} \qquad (12)$$

where ρ_4 is the density of medium II and C_4 is the velocity of wave propagation in medium II. The forward pressure P_3 acting

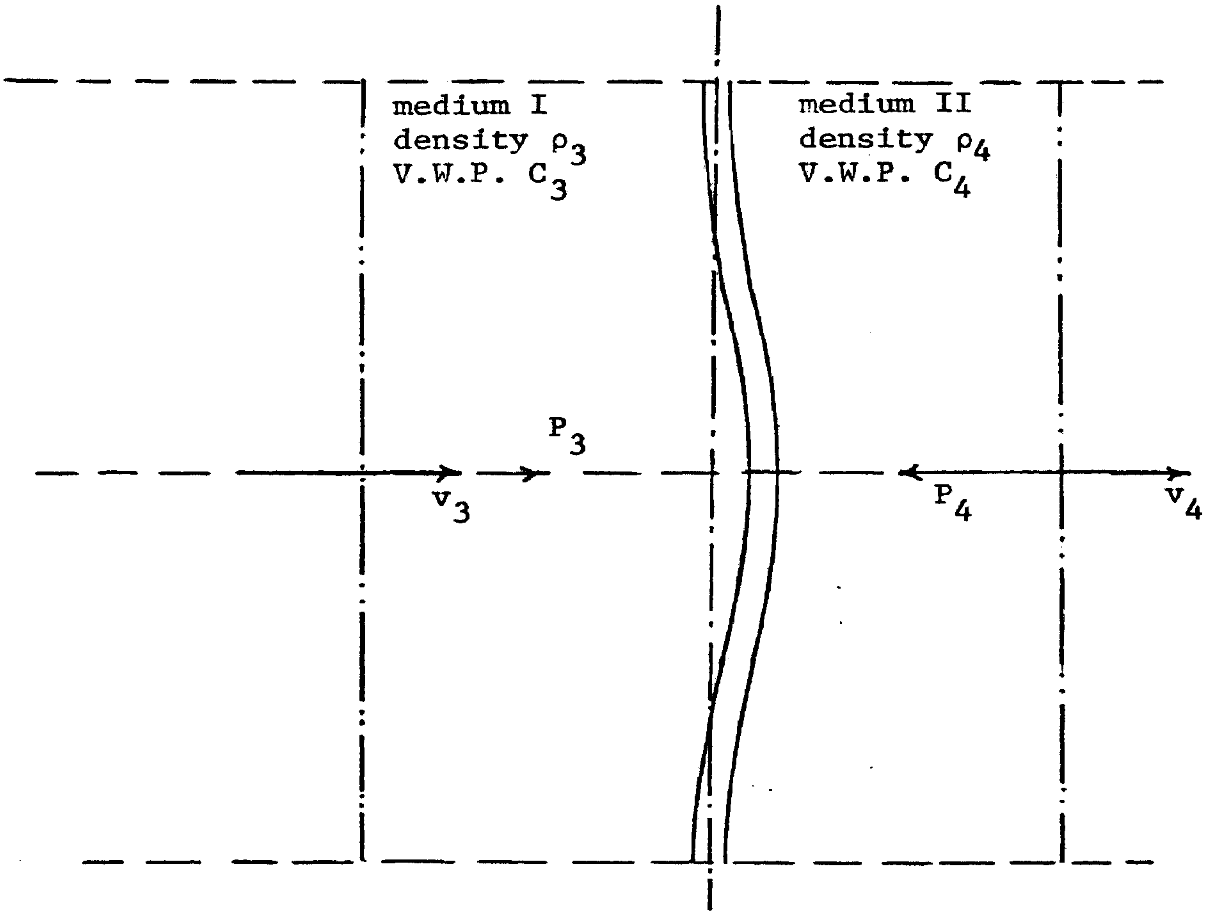

Figure 4: Waves Generated by Flexural Motion of a Plate

on the plate due to the wave V_3 is given by [19],

$$P_3 = \rho_3 C_3 \frac{\partial \omega}{\partial t} \tag{13}$$

where ρ_3 is the density of medium I and C_3 is the velocity of wave propagation in medium I. Therefore, the net resistance to the motion of the plate is given by,

$$P_4 - P_3 = -(\rho_3 C_3 + \rho_4 C_4) \frac{\partial \omega}{\partial t} \tag{14}$$

If an arbitrary forcing function $f(r,\theta,t)$ were applied to the plate, the resultant forcing function would be written as the algebraic sum of the forcing function and the resisting force. Since the problem under study is that of free vibrations, no external forcing function exists. Therefore, $q(r,\theta,t)$, which was initially assumed to be

$$q(r,\theta,t) = \sigma_z\big|_{z=h/2} - \sigma_z\big|_{z=-h/2} = q_2(r,\theta,t) - q_1(r,\theta,t) \tag{15}$$

is now given by,

$$q(r,\theta,t) = P_4 - P_3 = -(\rho_3 C_3 + \rho_4 C_4) \frac{\partial \omega}{\partial t} \tag{16}$$

After substitution of equation 16 in equation 11, the differential equation of motion becomes,

$$\left(\nabla^2 - \frac{\rho}{G'} \frac{\partial^2}{\partial t^2}\right) \left(D\nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2}\right) \omega + \rho h \frac{\partial^2 \omega}{\partial t^2}$$

$$+ (\rho_3 C_3 + \rho_4 C_4) \left(\frac{\partial}{\partial t} - \frac{D}{G'h} \nabla^2 \frac{\partial}{\partial t} + \frac{\rho h^2}{12G'} \frac{\partial^3}{\partial t^3}\right) \omega = 0. \tag{17}$$

The expressions for moments, shear forces, twist and the equations of motion stated in the above discussion are for axially-unsymmetric deflection of circular plates. For axially-symmetric deflections of a plate, there will not be any tangential displacement and the transverse displacement ω will not be a function of θ. Therefore,

$v = z\ \psi_\theta\ (r,\theta,t) \ = 0$ which implies that

$\psi_\theta(r,\theta,t) \ = 0,$

$$\frac{\partial\omega}{\partial\theta} = 0 \quad \text{and} \quad \frac{\partial^2\omega}{\partial\theta^2} = 0. \tag{18}$$

The Laplace operator ∇^2 which was

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \qquad \text{reduces to}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) . \tag{19}$$

The expressions for moments, shear forces and twist for axially-symmetric deflections of a plate are,

$$M_r = D\ \left[\frac{\partial\psi_r}{\partial r} + \frac{\mu}{r}\ \psi_r\right]$$

$$M_\theta = D\ \left[\mu\ \frac{\partial\psi_r}{\partial r} + \frac{1}{r}\ \psi_r\right] \tag{20}$$

$$M_{r\theta} = 0$$

$$Q_r = G'h\ \left(\frac{\partial\omega}{\partial r} + \psi_r\right)$$

$$Q_\theta = 0$$

The equations of motion for axially-symmetric deflections of a plate in terms of plate stress components become,

$$\frac{\partial M_r}{\partial r} + \frac{M_r - M_\theta}{r} - Q_r = \frac{\rho h^3}{12} \frac{\partial^2 \psi_r}{\partial t^2}$$

$$\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + q(r,t) = \rho h \frac{\partial^2 \omega}{\partial t^2} \qquad (21)$$

The equation of motion in one variable ω, transverse displacement, for axially-symmetric deflections of a plate is,

$$(\nabla^2 - \frac{\rho}{G'} \frac{\partial^2}{\partial t^2}) (D \nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2})$$

$$+ \rho h \frac{\partial^2 \omega}{\partial t^2} + (\rho_3 C_3 + \rho_4 C_4) (\frac{\partial}{\partial t} - \frac{D}{G'h} \nabla^2 \frac{\partial}{\partial t} + \frac{\rho h^2}{12G'} \frac{\partial^3}{\partial t^3}) \omega$$

$$= 0 \qquad (22)$$

where $\nabla^2 = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})$.

In this derivation, the equations of motion for the plate have been developed in a manner similar to that of Mindlin. The effect of resistance of the medium was considered as a forcing function, but since this forcing function depends upon the transverse displacement, because of the assumption that the waves generated by the vibrating plate are simple plane waves, the equation of motion remained in terms of one variable, i.e., transverse displacement of the plate.

IV. THE SOLUTION OF THE DIFFERENTIAL EQUATION

The differential equation to be solved for the vibrations of a circular plate with axially symmetrical deflections, rotary inertia, shear deformation and resistance from a fluid media along with boundary conditions and initial conditions can be written as,

$$[(\nabla^2 - \frac{\rho}{G'}\frac{\partial^2}{\partial t^2})(D\nabla^2 - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}) + \rho h\frac{\partial^2}{\partial t^2} + (1 - \frac{D\nabla^2}{G'h} + \frac{\rho h^2}{12G'}\frac{\partial^2}{\partial t^2})$$

$$(\rho_3 C_3 + \rho_4 C_4)\frac{\partial}{\partial t}]\ \omega = 0 \tag{22}$$

Boundary Conditions:
1) $\omega(0,t)$ must be finite
2) $\omega(R,t) = 0$
3) $\psi_r(R,t) = 0$ (22-a)

since $\psi_r(r,t)$ is the slope without shear deformation.

Initial Conditions:
4) $\omega(r,0) = f(r)$
5) $\frac{\partial\omega}{\partial t}(r,0) = g(r)$
6) as $t \to \infty$, $\omega(r,t) \to 0$ (22-b)

When simplified, the equation becomes

$$[D\nabla^4 - B\nabla^2\frac{\partial^2}{\partial t^2} - P\nabla^2\frac{\partial}{\partial t} + B_1\frac{\partial^4}{\partial t^4} + Q\frac{\partial^3}{\partial t^3} + \rho h\frac{\partial^2}{\partial t^2} + A\frac{\partial}{\partial t}]\ \omega = 0 \tag{23}$$

where the constants D, B, P, B_1, Q, and A involving material constants are given by the relations,

$$D = Eh^3/12(1 - \mu^2)$$

$$B = (\frac{\rho^2 h^3}{12} + \frac{\rho D}{G'}) \tag{23-a}$$

$$Q = (\rho_3 C_3 + \rho_4 C_4) \frac{\rho h^2}{12G'}$$

$$B_1 = \frac{\rho^2 h^3}{12G'} \tag{23-a}$$

$$P = (\rho_3 C_3 + \rho_4 C_4) \frac{D}{G'h}$$

$$A = (\rho_3 C_3 + \rho_4 C_4).$$

It is assumed that the solution of equation (23) is of the form,

$$\omega(r,t) = R(r)\, e^{P_1 t}$$

Substitution of the above solution in the differential equation yields,

$$[D\nabla^4 - BP_1^2\nabla^2 - PP_1\nabla^2 + B_1P_1^4 + QP_1^3 + \rho h P_1^2 + AP_1]\, R(r)e^{P_1 t} = 0 \tag{24}$$

Therefore,

$$[D\nabla^4 + (-BP_1^2 - PP_1)\nabla^2 + (B_1P_1^4 + QP_1^3 + \rho h P_1^2 + AP_1)]\, R(r) = 0 \tag{25}$$

The equation can be expressed as

$$(\nabla^2 + c_1^2)\,(\nabla^2 + c_2^2)\, R(r) = 0 \tag{26}$$

where

$$c_1^2 = \frac{1}{2D}\left[(-BP_1^2 - PP_1) - \sqrt{(-BP_1^2-PP_1)^2-4D(B_1P_1^4+QP_1^3+\rho h P_1^2+AP_1)}\,\right] \tag{27}$$

$$c_2^2 = \frac{1}{2D}\left[(-BP_1^2 - PP_1) + \sqrt{(-BP_1^2-PP_1)^2-4D(B_1P_1^4+QP_1^3+\rho h P_1^2+AP_1)}\,\right] \tag{28}$$

Due to the fact that the equation is linear, the solution for R(r) can be written as

$$R(r) = R_1(r) + R_2(r) \tag{29}$$

where $R_1(r)$ and $R_2(r)$ are the solutions of,

$$(\nabla^2 + C_1^{\;2})\ R_1(r) = 0$$
$$(\nabla^2 + C_2^{\;2})\ R_2(r) = 0 \qquad (30)$$

respectively.

So,

$$R_1(r) = a_1 J_0(rC_1) + a_3 Y_0(rC_1)$$
$$R_2(r) = a_2 J_0(rC_2) + a_4 Y_0(rC_2) \qquad (31)$$

and

$$R(r) = a_1 J_0(rC_1) + a_2 J_0(rC_2) + a_3 Y_0(rC_1) + a_4 Y_0(rC_2) \qquad (32)$$

It is known from the theory of differential equations that the form of the solution to the differential equation changes if the roots of the characteristic function are equal or one of them is zero. Therefore, it is noted that the solution given by equation 32 to the differential equation 25 is valid only when C_1 and C_2 are not equal and neither is zero. In this discussion we will investigate only those cases where C_1 and C_2 satisfy the above-mentioned conditions.

The boundary conditions are

1) $\omega(0,t)$ must be finite
2) $\omega(R,t) = 0$ (22-a)
3) $\psi_r(R,t) = 0$

The first condition gives $a_3 = a_4 = 0$. The $\psi_r(r,t)$ is obtained in the following manner:

Substitution of equations 20 into equations 21 gives

$$[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{h^3}{12D}\frac{\partial^2}{\partial t^2}]\ \psi_r - \frac{G'h}{D}\ [\psi_r + \frac{\partial\omega}{\partial r}] = 0$$

$$[\frac{\partial}{\partial r} + \frac{1}{r}]\ \psi_r + [\frac{\partial^2\omega}{dr^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{\rho}{G'}\ \frac{\partial^2\omega}{\partial t^2} - \frac{q(r,t)}{G'h}] = 0 \tag{33}$$

Assuming that

$$\omega(r,t) = R(r)e^{P_1 t} \qquad \text{and} \qquad \psi_r(r,t) = \psi_r(r)e^{P_1 t}$$

and substituting for q(r,t) from equation 16 into equations 33 gives

$$[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{\rho h^3}{12D} P_1^{\ 2}]\ \psi_r - \frac{G'h}{D}\ [\psi_r + \frac{\partial R(r)}{\partial r}] = 0$$

$$[\frac{\partial}{\partial r} + \frac{1}{r}]\ \psi_r + [\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\rho}{G'} P_1^{\ 2} - \frac{P_1}{G'h} A]\ R(r) = 0 \tag{33-a}$$

Differentiating the second of equations 33-a with respect to r and substracting the result from the first of equations 33-a gives

$$\psi_r(r,t) = -\ [\frac{\rho h^3}{12D} P_1^{\ 2} + \frac{G'h}{D}]^{-1}\ \frac{\partial}{\partial r}\ [\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \delta_1]\ \omega(r,t)$$

where (34)

$$\delta_1 = [\frac{\rho}{G'} P_1^{\ 2} + \frac{P_1}{G'h} A - \frac{G'h}{D}]$$

Substitution of equation 32 in the boundary conditions 22-a with the help of equations 34 yields,

$$a_1 J_0(RC_1) + a_2 J_0(RC_2) = 0 \tag{35-a}$$

$$a_1 d_1 C_1 J_1(RC_1) + a_2 d_2 C_2 J_1(RC_2) = 0 \tag{35-b}$$

where

$$d_{1,2} = [C_{1,2}^{\ 2} + \delta_1]$$

It should be noted that since

$$C_1 \neq C_2 \qquad , \qquad d_1 \neq d_2$$

The constant a_2 from equation 35-b is given by

$$a_2 = -a_1 \frac{d_1 C_1 J_1(RC_1)}{d_2 C_2 J_1(RC_2)} \tag{35-c}$$

Substitution of this value of a_2 in equation 35-a yields,

$$a_1 J_0(RC_1) - a_1 \frac{d_1 C_1 J_1(RC_1)}{d_2 C_2 J_1(RC_2)} J_0(RC_2) = 0 \tag{35-d}$$

Therefore,

$$\frac{a_1}{d_2 C_2 J_1(RC_2)} [d_2 C_2 J_1(RC_2) J_0(RC_1) - d_1 C_1 J_1(RC_1) J_0(RC_2)] = 0 \tag{35-e}$$

It is noted if a_1 in equation 35-e is zero, a_2 in equation 35-c is also zero and that gives a trivial solution. Therefore,

$$d_2 C_2 J_1(RC_2) J_0(RC_1) - d_1 C_1 J_1(RC_1) J_0(RC_2) = 0 \tag{36}$$

Equation 36 is the equation governing the frequencies of vibration because C_1, C_2, d_1, d_2 are functions of P_1. It is noted that the roots of equation 36 would satisfy the differential equation and the boundary conditions. Some of the roots of equation 36 are

$$\begin{aligned}
d_1 &= 0 \quad , & J_0(RC_1) &= 0 \\
d_1 &= 0 \quad , & J_1(RC_2) &= 0 \\
d_2 &= 0 \quad , & J_1(RC_1) &= 0 \\
d_2 &= 0 \quad , & J_0(RC_2) &= 0 \\
J_1(RC_1) &= 0, & J_1(RC_2) &= 0 \\
J_0(RC_1) &= 0, & J_0(RC_2) &= 0
\end{aligned} \tag{36-a}$$

In this investigation, only the last set of roots in sets 36-a are considered. The other possibilities for satisfaction of equation 36 are not herein investigated. So 'RC_1' and 'RC_2' are two unequal and nonzero roots of $J_0(\alpha)$. Now, knowing the values of 'RC_1' and 'RC_2', the values of 'P_1' can be obtained from equations 27 and 28 as follows:

$$C_1^{\,2} = \frac{(-BB_1^{\,2}-PP_1)}{2D} - \sqrt{\frac{(-BB_1^{\,2}-PP_1)^2}{4D^2} - \frac{(B_1P_1^{\,4}+QP_1^{\,3}+PhP_1^{\,2}+AP_1)}{D}} \tag{27}$$

$$C_2^{\,2} = \frac{(-BP_1^{\,2}-PP_1)}{2D} + \sqrt{\frac{(-BP_1^{\,2}-PP_1)^2}{4D^2} - \frac{(B_1P_1^{\,4}+QP_1^{\,3}+PhP_1^{\,2}+AP_1)}{D}} \tag{28}$$

Now, add the two equations to obtain,

$$(C_1^{\,2} + C_2^{\,2}) = \frac{(-BP_1^{\,2} - PP_1)}{D} \tag{37}$$

and multiply the two equations to obtain,

$$C_1^{\,2}C_2^{\,2} = \frac{(B_1P_1^{\,4}+QP_1^{\,3}+PhP_1^{\,2}+AP_1)}{D} \tag{38}$$

It is noted that if P_1 satisfies equations 37 and 38, it also satisfied equations 27 and 28. Further, C_1 and C_2 have the values α_n/R and α_m/R, where α_n and α_m are the roots of $J_0(\alpha)$.

It is necessary to find the values of P_1 which satisfy both the equations 37 and 38. Consider various combinations of α_n and α_m and find numerically for particular values of B, P, Q, A, D, B_1, h, R, and ρ: the greatest common divisor of the two polynomials

$$BP_1^2 + PP_1 + D(C_1^2 + C_2^2) = 0 \tag{39}$$

$$B_1P_1^4 + QP_1^3 + \rho hP_1^2 + AP_1 - DC_1^2C_2^2 = 0 \tag{40}$$

After obtaining the greatest common divisor of the two polynomials, equate the greatest common divisor to zero and the roots of the resulting equation will be the values of P_1 which will satisfy both of the equations 39 and 40.

A package subroutine to find the greatest common divisor of two polynomials is available in the U.M.R. computer center under the title PGCD. This subroutine was used to obtain the greatest common divisor of the equations 39 and 40. When the first ten roots of $J_o(\alpha)$ were considered for the material 347 Stainless Steel , it was found that the polynomial given by equation 39 is the greatest common divisor of equation 39 and 40 for all combinations of α_n and α_m except when n = 1 and m = 2. Therefore, the values of P_1 which satisfy both of the equations 39 and 40 are given by

$$P_1 = \frac{-P}{2B} \pm \sqrt{\frac{P^2}{4B^2} - \frac{D}{B}(C_1^2 + C_2^2)} \tag{41}$$

where C_1 and C_2 take the values corresponding to α_n and α_m.

The computer program which gives the greatest common divisor of equations 39 and 40 and the results of the computation are presented in Appendix-F.

From the results in Appendix-F, the values of P_1 which satisfy both the equation 27 and 28 are given by the roots of

$$BP_1^{\,2} + PP_1 + \frac{D}{R^2}\left(\alpha_n^{\,2} + \alpha_m^{\,2}\right) = 0 \tag{39}$$

where α_n and α_m are two unequal and non-zero roots of $J_o(\alpha)$. It is evident from the results given in Appendix-F that for n = 1 and m = 2, equation 39 does not give the values of P_1 which satisfy both the equations 27 and 28 for all other combinations of the first ten roots of $J_o(\alpha)$, equation 39 gives the value of P_1 which satisfy both the equations 27 and 28. So except for the n = 1 and m = 2 combination, the values of P_1 are given by

$$P_1 = \frac{-P}{2B} \pm \quad \frac{P^2}{4B^2} - \frac{D}{Br^2}\left(\alpha_n^{\,2} + \alpha_m^{\,2}\right) \tag{41}$$

Since there are two values of P_1 for every combination of α_n and α_m, let S_1 and S_2 be the two values of P_1. Then, the solution for the deflection curve is given by

$$\omega(r,t) = [a_1 J_o(rC_1) + a_2 J_o(rC_2)][Z_1 e^{s_1 t} + Z_2 e^{s_2 t}]_\ell \tag{42}$$

where Z_1 and Z_2 are arbitrary constants. The above solution can also be written as,

$$\omega(r,t) = \sum_{\ell=1}^{\infty} [b_2 e^{s_1 t} J_o(rC_1) + b_3 e^{s_2 t} J_o(rC_1) + b_4 e^{s_1 t} J_o(rC_2) + b_5 e^{s_2 t} J_o(rC_2)]_\ell \tag{43}$$

where ℓ stands for a particular combination of α_n and α_m, e.g., for n = 1 and m = 3, ℓ = 1. Then, for another combination n = 2 and m = 4, ℓ = 2, etc. The arbitrary constants b_2, b_3, b_4, and b_5 are evaluated by using the initial conditions that

5) $\omega(r,0) = f(r)$

6) $\frac{\partial\omega}{\partial t}(r,0) = g(r)$ (22-b)

7) as $t \to \infty$, $\omega(r,t) \to 0$

Applying the first two of the above initial conditions, the result is

$$\sum_{\ell=1}^{\infty} [(b_2 + b_3)\ J_o(rC_1) + (b_4 + b_5)\ J_o(rC_2)]_\ell = f(r)$$
$$\sum_{\ell=1}^{\infty} [(s_1b_2 + s_2b_3)\ J_o(rC_1) + (s_1b_4 + s_2b_5)\ J_o(rC_2)]_\ell = g(r) \tag{44}$$

From the orthogonality condition for Bessel functions and Fourier-Bessel series

$$(b_2 + b_3) = U_1 \tag{45}$$

$$(b_4 + b_5) = U_2 \tag{46}$$

$$(s_1b_2 + s_2b_3) = U_3 \tag{47}$$

$$(s_1b_4 + s_2b_5) = U_4 \tag{48}$$

where U_1, U_2, U_3, and U_4 are given by

$$U_1 = \frac{{}_o\int^R rf(r)\ J_o(rC_1)dr}{{}_o\int^R r\ [J_o(rC_1)\ ]^2 dr}$$

$$U_2 = \frac{{}_o\int^R rf(r)\ J_o(rC_2)dr}{{}_o\int^R r\ [J_o(rC_2)]^2 dr} \tag{49}$$

$$U_3 = \frac{{}_o\int^R rg(r)\ J_o(rC_1)dr}{{}_o\int^R r\ [J_o(rC_1)]^2 dr}$$

$$U_4 = \frac{{}_o\int^R rg(r)\ J_o(rC_2)dr}{{}_o\int^R r\ [J_o(rC_2)]^2 dr}$$

It is evident that the values of the arbitrary constants b_2 and b_3 can be obtained from equations 45 and 47. Similarly, the values of the arbitrary constants b_4 and b_5 can be obtained from equations 46 and 48. In order to satisfy the third initial condition, S_1 and S_2 must be negative. We observe from the expressions for the terms B, P, and D given in equations 23-a that the terms B, P, and $\frac{D}{R^2}(\alpha_n{}^2 + \alpha_m{}^2)$ in equation 41 are all positive. Therefore, the values of S_1 and S_2 given by

$$S_{1,2} = \left[\frac{-P}{2B} \pm \sqrt{\frac{P^2}{4B^2} - \frac{D}{BR^2}(\alpha_n{}^2 + \alpha_m{}^2)} \right] \qquad (41)$$

can either be negative, positive, or complex. Neither S_1 and S_2 can be zero because $\frac{D}{BR^2}(\alpha_n{}^2 + \alpha_m{}^2)$ cannot be zero for a finite plate. The roots S_1 or S_2 could be positive if

$$\frac{-P}{2B} \text{ is less than } \left[\sqrt{\frac{P^2}{4B^2} - \frac{D}{BR^2}(\alpha_n{}^2 + \alpha_m{}^2)} \right].$$

However, if this inequality were true,

$$-\frac{D}{BR^2}(\alpha_n{}^2 + \alpha_m{}^2) \text{ is greater than zero,}$$

which is impossible. Therefore, S_1 and S_2 can be either negative or complex. When S_1 and S_2 are negative, the arbitrary constants b_2, b_3, b_4, and b_5 are obtained from equations 45, 46, 47, and 48.

When S_1 and S_2 are complex, they are given by

$$S_{1,2} = \left[\frac{-P}{2B} \pm \omega_\ell i \right] \qquad (50)$$

where $\omega_\ell{}^2 = [\frac{D}{BR^2}(\alpha_n{}^2 + \alpha_m{}^2) - \frac{P^2}{4B^2}]$

The solution for the deflection curves is then expressed as

$$\omega(r,t) = \sum_{\ell=1}^{\infty} e^{-\frac{P}{2B}t} J_o(rC_1) \quad [b_2 \cos \omega_\ell t + b_4 \sin \omega_\ell t]$$
$$+ \sum_{\ell=1}^{\infty} e^{-\frac{P}{2B}t} J_o(rC_2) \quad [b_3 \cos \omega_\ell t + b_5 \sin \omega_\ell t)] \qquad (51)$$

Since there are infinite roots of $J_o(\alpha)$, there are infinite combinations of α_n, and α_m. As C_1 and C_2 are given by

$$C_1 = \frac{\alpha_n}{R} \quad \text{and} \quad C_2 = \frac{\alpha_m}{R} \quad ,$$

the solution can also be written as

$$\omega(r,t) = \sum_{\ell=1}^{\infty} e^{-\frac{P}{2B}t} \quad [b_2 \cos \omega_\ell t + b_4 \sin \omega_\ell t] \quad J_o(r\frac{\alpha_n}{R})$$
$$+ \sum_{\ell=1}^{\infty} e^{-\frac{P}{2B}t} \quad [b_3 \cos \omega_\ell t + b_5 \sin \omega_\ell t] \quad J_o(r\frac{\alpha_m}{R})$$

where

$$\omega_\ell{}^2 = [\frac{D(\alpha_n{}^2 + \alpha_m{}^2)}{BR^2} - \frac{P^2}{4B^2}] . \qquad (52)$$

Of the infinite combinations of α_n and α_m, possible for the evaluation of ω_ℓ, it is observed that if both m and n are allowed to vary from zero to infinity, there will be duplication of the combinations of α_n and α_m and also ω_ℓ. Therefore, either of the values of m or n should be restricted in such a way that this repetition does not occur.

Let m be such that it is always greater than n. Table (IV-1) gives the values of the frequencies ω_ℓ in ascending order for

TABLE: (IV-1)

The Frequencies in the Ascending Order and the Values of α_n and α_m

n	m	Roots of $J_0(\alpha)$		The Value of Frequency ω_ℓ
		α_n	α_m	
1	3	2.4048	8.6537	14856.54
2	3	5.5200	8.6537	16978.35
1	4	2.4048	11.7915	19905.85
2	4	5.5200	11.7915	21535.80
3	4	8.6537	11.7915	24193.25
1	5	2.4048	14.9309	25015.50
2	5	5.5200	14.9509	26331.02
3	5	8.6537	14.9309	28545.50
4	5	11.7915	14.9309	31470.17

corresponding values of α_n and α_m along with the values of the indices n and m. It can be observed that the frequencies ω_ℓ follow a particular pattern as far as the combinations of n and m are concerned. If n is equal to (m-1), the next combination of n and m which gives the frequency ω_ℓ immediately higher than the previous one, is given by increasing m to m+1 and n equal to one.

It is noted that the third of the initial conditions in equation 22-b is satisfied by the solution expressed in the form of equation 52, because as t tends to infinity, $e^{\frac{-P}{2B}t}$ tends to zero. Use of the first initial condition in the solution yields

$$f(r) = \sum_{\ell=1}^{\infty} [b_2 \; J_0(r \frac{\alpha_n}{R}) + b_3 \; J_0(r \frac{\alpha_m}{R})] \tag{53}$$

as an initial displacement.

Use of the property of orthogonality for the Bessel functions and Fourier-Bessel series, allows evaluation of the arbitrary constants b_2 and b_3 as follows,

$$b_2 \int_0^R r \; [J_0(r \frac{\alpha_n}{R})]^2 dr = \int_0^R rf(r) \; J_0(r \frac{\alpha_n}{R}) dr \tag{54}$$

$$b_3 \int_0^R r \; [J_0(r \frac{\alpha_m}{R})]^2 dr = \int_0^R rf(r) \; J_0(r \frac{\alpha_m}{R}) dr$$

From Wayland [20],

$$b_2 = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R rf(r) \; J_0(r \frac{\alpha_n}{R}) \; dr \tag{55}$$

and

$$b_3 = \frac{2}{R^2 J_1{}^2(\alpha_n)} \int_0^R rf(r) \; J_0(r \frac{\alpha_m}{R}) \; dr \tag{56}$$

Use of the second initial condition for the initial velocity allows the arbitrary constants b_4 and b_5 to be evaluated as follows,

$$\frac{\partial\omega(r,t)}{\partial t} = \frac{-P}{2B}\,\omega(r,t) + \sum_{\ell=1}^{\infty} e^{\frac{-P}{2B}t}\,\omega_\ell\, J_0(r\,\frac{\alpha_n}{R})\,[-b_2\sin\omega_\ell t + b_4\cos\omega_\ell t]$$
$$+ \sum_{\ell=1}^{\infty} e^{\frac{-P}{2B}t}\,\omega_\ell\, J_0\,(r\,\frac{\alpha_m}{R})[-b_3\sin\omega_\ell t + b_5\cos\omega_\ell t] \qquad (57\text{-a})$$

$$\frac{\partial\omega(r,0)}{\partial t} = \frac{-P}{2B}\,\omega(r,0) + \sum_{\ell=1}^{\infty}\omega_\ell\,[b_4\,J_0(r\,\frac{\alpha_n}{R}) + b_5\,J_0(r\,\frac{\alpha_m}{R})] \qquad (57\text{-b})$$

so

$$[g(r) + \frac{P}{2B}\,f(r)] = \sum_{\ell=1}^{\infty}\omega_\ell\,[J_0(r\,\frac{\alpha_n}{R})b_4 + b_5\,J_0(r\,\frac{\alpha_m}{R})] \qquad (57\text{-c})$$

Now, b_4 and b_5 are obtained in the same manner used to obtain b_2 and b_3.

$$b_4 = \frac{2}{R^2 J_1^{\,2}(\alpha_n)\omega_\ell}\int_o^R r[g(r) + \frac{P}{2B}\,f(r)]\,J_0(r\,\frac{\alpha_n}{R})dr \qquad (58)$$

$$b_5 = \frac{2}{R^2 J_1^{\,2}(\alpha_m)\omega_\ell}\int_o^R r[g(r) + \frac{P}{2B}\,f(r)]\,J_0(r\,\frac{\alpha_m}{R})dr \qquad (59)$$

For the case under consideration the oscillatory solution for the differential equation of motion for a plate is given by equations 52, 55, 56, 58, and 59.

V. THE CHARACTERISTICS OF THE SOLUTION

The characteristics of the solution are best illustrated by an example.

SPECIFICATIONS:

MATERIAL: 347 Stainless Steel.

DENSITY: $\rho = 0.2845$ lbm./in^3.

YOUNG'S MODULUS OF ELASTICITY: $E = 28.43 \times 10^6$ lbs./in^2.

MODULUS OF RIGIDITY: $G = 10.98 \times 10^6$ lbs./in^2.

POISSON'S RATIO: $\mu = 0.3$

We assume the following dimensions for the plate.

RADIUS: $R = 3.0$ in.

THICKNESS: $h = 3/16$ in.

The medium in which the plate vibrates is assumed to be air. For air, the product of the density [22] and the velocity of wave propogation [23] is

$$\rho_3 C_3 = \rho_4 C_4 = 0.565 \text{ lbm/in}^2\text{-sec.} \tag{60}$$

Further, it is assumed that the plate has an initial displacement given by,

$$f(r) = \omega_{max} [1 - (\frac{r}{R})^2]^2 \tag{61}$$

This equation is the static deflection curve for a flat, uniformly loaded, circular plate which is clamped around the circumference [24]. The initial velocity of the plate is assumed to be zero. Therefore,

$$g(r) = 0 \tag{62}$$

ω_{max} is the initial displacement at the center of the plate. For $g(r) = 0$ and the initial displacement function $f(r)$, the arbitrary constants b_2, b_3, b_4, b_5 can be evaluated by means of equations 55, 56, 58, and 59. The integration for the integrals in equations 55, 56, 58, and 59, is carried out in Appendix-G with the result that

$$
{}_{o}\int^{R} f(r)\ J_0(r\theta_1)dr = \omega_{max}\ J_1(R\theta_1)\ [\frac{-8}{R\theta_1^{\ 3}} + \frac{64}{R^3\theta_1^{\ 5}}] + \omega_{max}\ J_0(R\theta_1)\ [\frac{32}{R^2\theta_1^{\ 4}}] \tag{63}
$$

where θ_1 is a constant.

To evaluate the arbitrary constants b_2 and b_4, substitute $\theta_1 = \frac{\alpha_n}{R}$ in equation 63 and to evaluate the arbitrary b_3 and b_5 substitute $\theta_1 = \frac{\alpha_m}{R}$ in equation 63. These substitutions yield,

$$
b_2 = \frac{2\omega_{max}}{J_1(\alpha_n)}\ [\frac{-8}{\alpha_n^{\ 3}} + \frac{64}{\alpha_n^{\ 5}}] \tag{64}
$$

$$
b_3 = \frac{2\omega_{max}}{J_1(\alpha_m)}\ [\frac{-8}{\alpha_m^{\ 3}} + \frac{64}{\alpha_m^{\ 5}}] \tag{65}
$$

$$
b_4 = \frac{P\omega_{max}}{\omega_\ell B J_1(\alpha_n)}\ [\frac{-8}{\alpha_n^{\ 3}} + \frac{64}{\alpha_n^{\ 5}}] \tag{66}
$$

$$
b_5 = \frac{P\omega_{max}}{\omega_\ell B J_1(\alpha_m)}\ [\frac{-8}{\alpha_m^{\ 3}} + \frac{64}{\alpha_m^{\ 5}}] \tag{67}
$$

It can be noted from the form of the expressions for the arbitrary constants b_2, b_3, b_4 and b_5, that the deflection at any point in the plate can be expressed as a product of maximum initial deflection at the center, ω_{max}, and some function of radius and time.

TABLE: (V-1)

The Solutions of group 1

A_1		A_2		A_3		A_4	
n	m	n	m	n	m	n	m
1	3	1	3	1	3	1	3
2	4	2	4	2	4	2	4
–	–	5	6	5	6	5	6
–	–	–	–	7	8	7	8
–	–	–	–	–	–	9	10

TABLE: (V-2)

The Solutions of group 2

A_5		A_6		A_7	
n	m	n	m	n	m
1	3	1	3	1	3
2	4	2	4	2	4
3	5	3	5	3	5
–	–	4	6	4	6
–	–	–	–	5	7

TABLE: (V-3)

The Solutions of group 3

A_8		A_9		A_{10}		A_{11}	
n	m	n	m	n	m	n	m
1	3	1	3	1	3	1	3
2	3	2	3	2	3	2	3
–	–	3	4	3	4	3	4
–	–	–	–	4	5	4	5
–	–	–	–	–	–	5	6

TABLE: (V-4)

The Solutions of group 4

A_{12}		A_{13}	
n	m	n	m
1	3	1	3
2	3	2	3
3	4	3	4
3	5	3	5
		3	6

TABLE: (V-5)

The Solutions of group 5

A_{14}		A_{15}		A_{16}		A_{17}		A_{18}		A_{19}	
n	m	n	m	n	m	n	m	n	m	n	m
1	4	1	5	1	6	1	7	1	8	1	10
2	4	2	5	2	6	2	7	2	8	2	10
3	4	3	5	3	6	3	7	3	8	3	10
		4	5	4	6	4	7	4	8	4	10
				5	6	5	7	5	8	5	10
						6	7	6	8	6	10
								7	8	7	10
										8	10
										9	10

It is noted that there are an infinite number of solutions depending upon the combinations of α_n and α_m. Further, there are infinite ways to express the solution as the sum of several solutions. The values of $\omega(r,t)/\omega_{max}$ for this case were computed for nineteen different solutions, each one as the sum of several solutions. In tables (V-1) through (V-5), these nineteen solutions have been tabulated.

For the solutions in table (V-1), it was assumed that one root of the Bessel function of the first kind and zero order does not associate with two frequencies in table (IV-1). For the solutions in table (V-2), it was assumed that the values of n and m do not repeat in their respective columns. It was assumed for the solution in table (V-3), that the values of n do not repeat in that column but the value of m has no restrictions. The solutions in table (V-4) incorporate the assumption that all possible combinations with m = 3 should be considered. Those in table (V-5) require that for a particular value of m, n takes the value one to m-1.

The computer program to evaluate $\omega(r,t)/\omega_{max}$ is given in Appendix H along with the results of the computations for three values of time (t = 0.0, t = 0.05, and t = 0.1) and for the nineteen different solutions. The graphical results are shown in figures 5 through 23. Figure 24 shows the graph of the assumed initial deflection curve f(r) with which various solutions which contain a finite number of terms were compared.

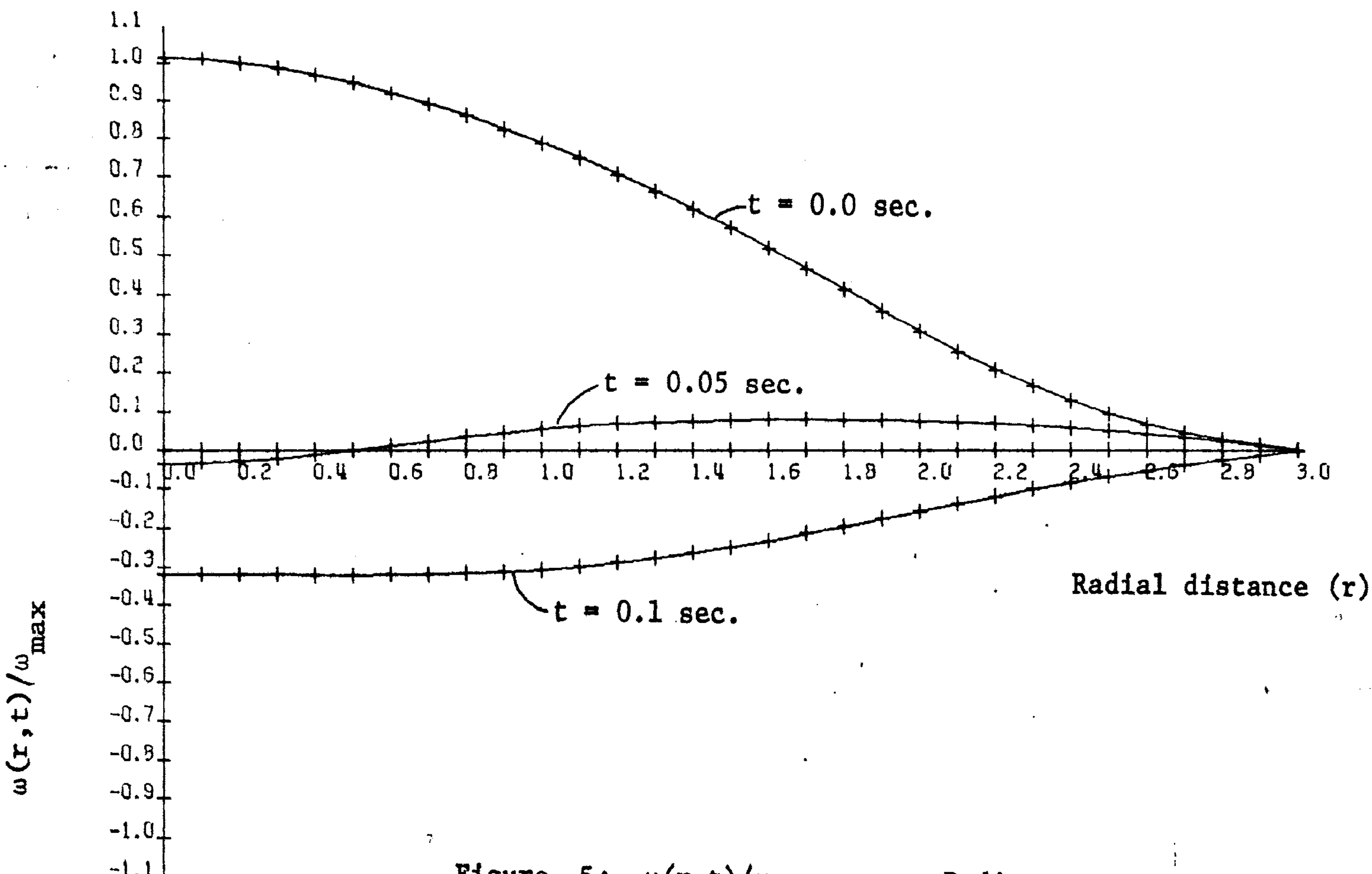

Figure -5: $\omega(r,t)/\omega_{max}$ versus Radius for Solution A_1.

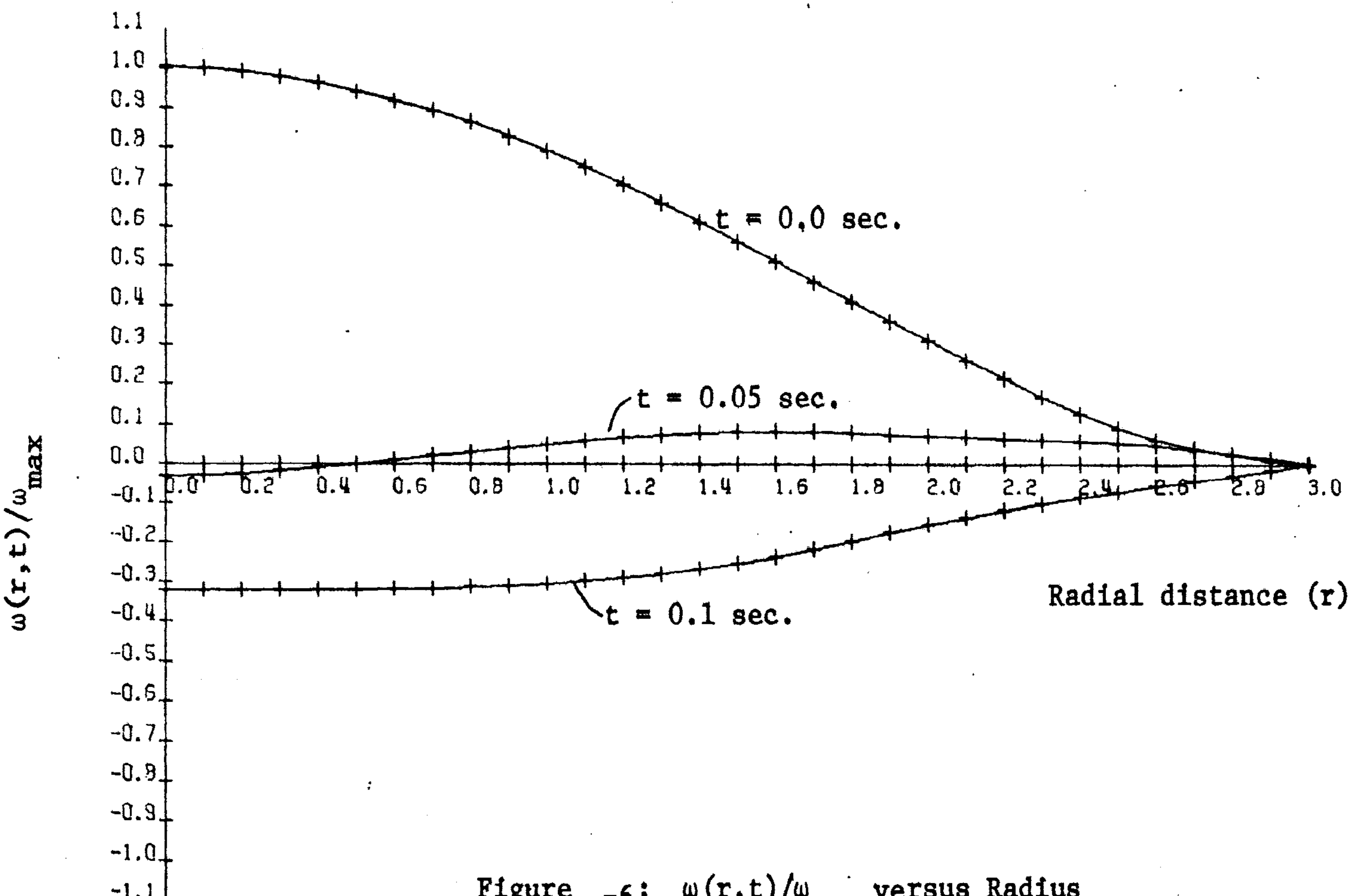

Figure 6: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_2.

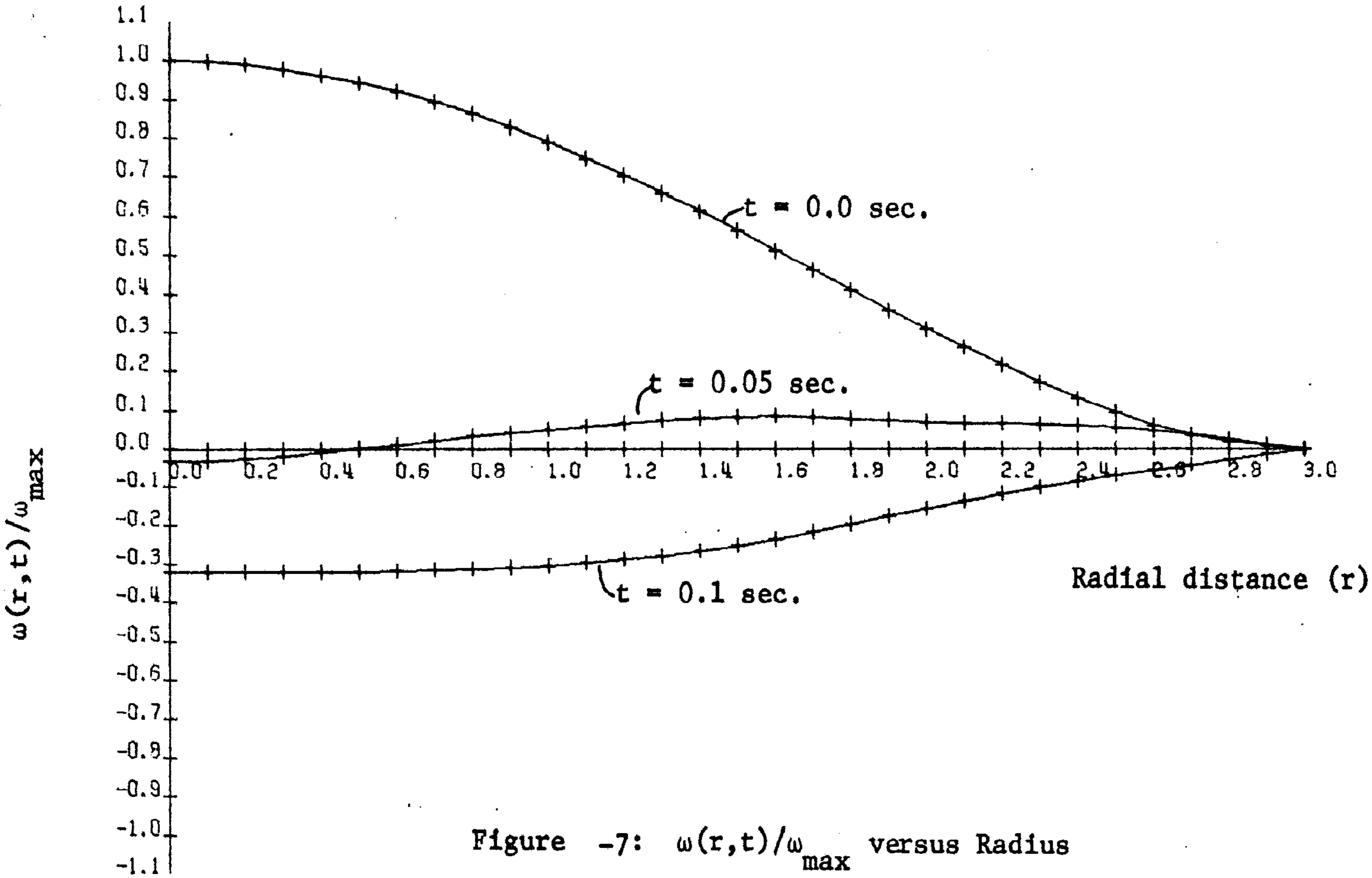

Figure -7: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_3.

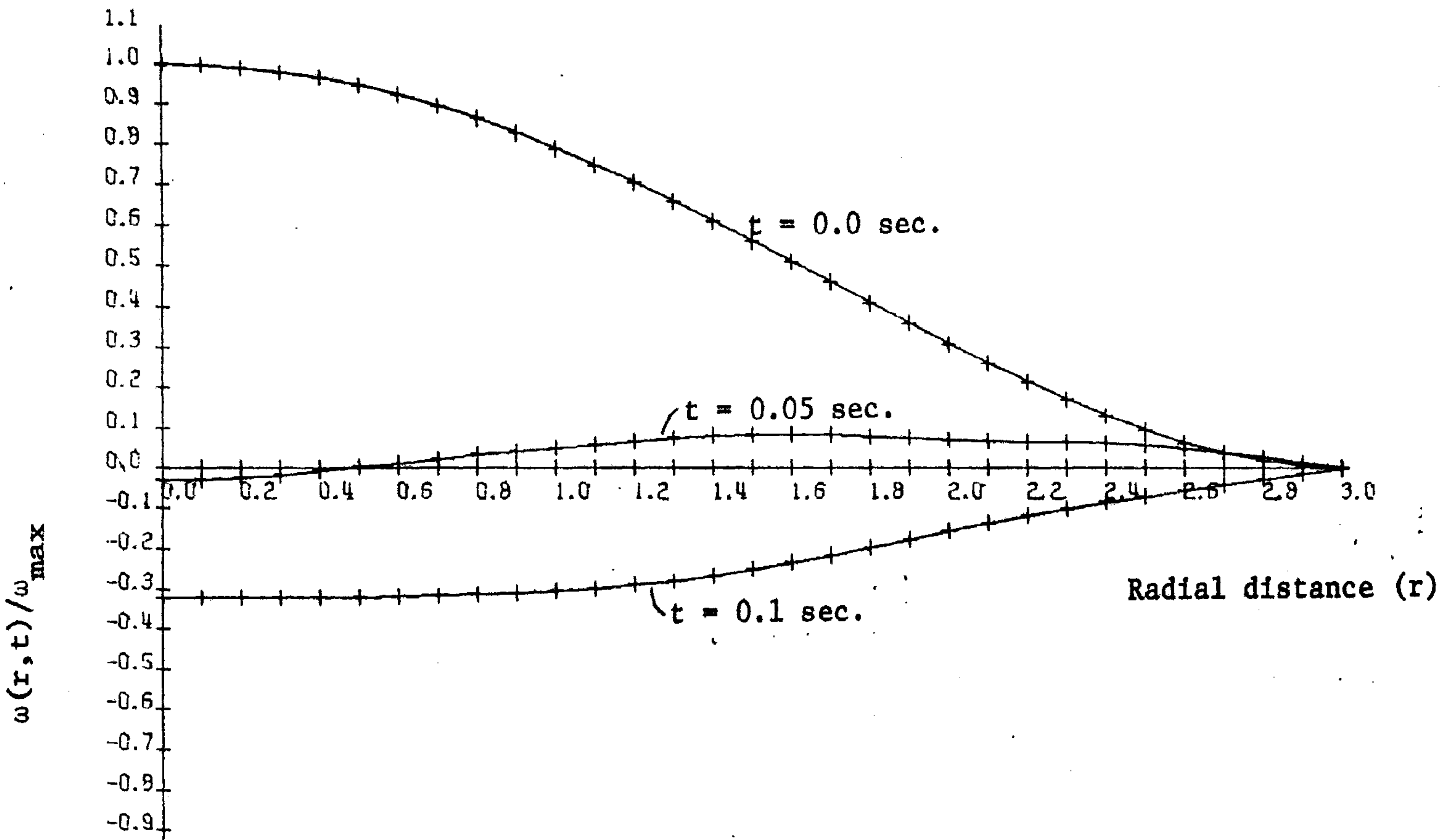

Figure -8 : $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_4.

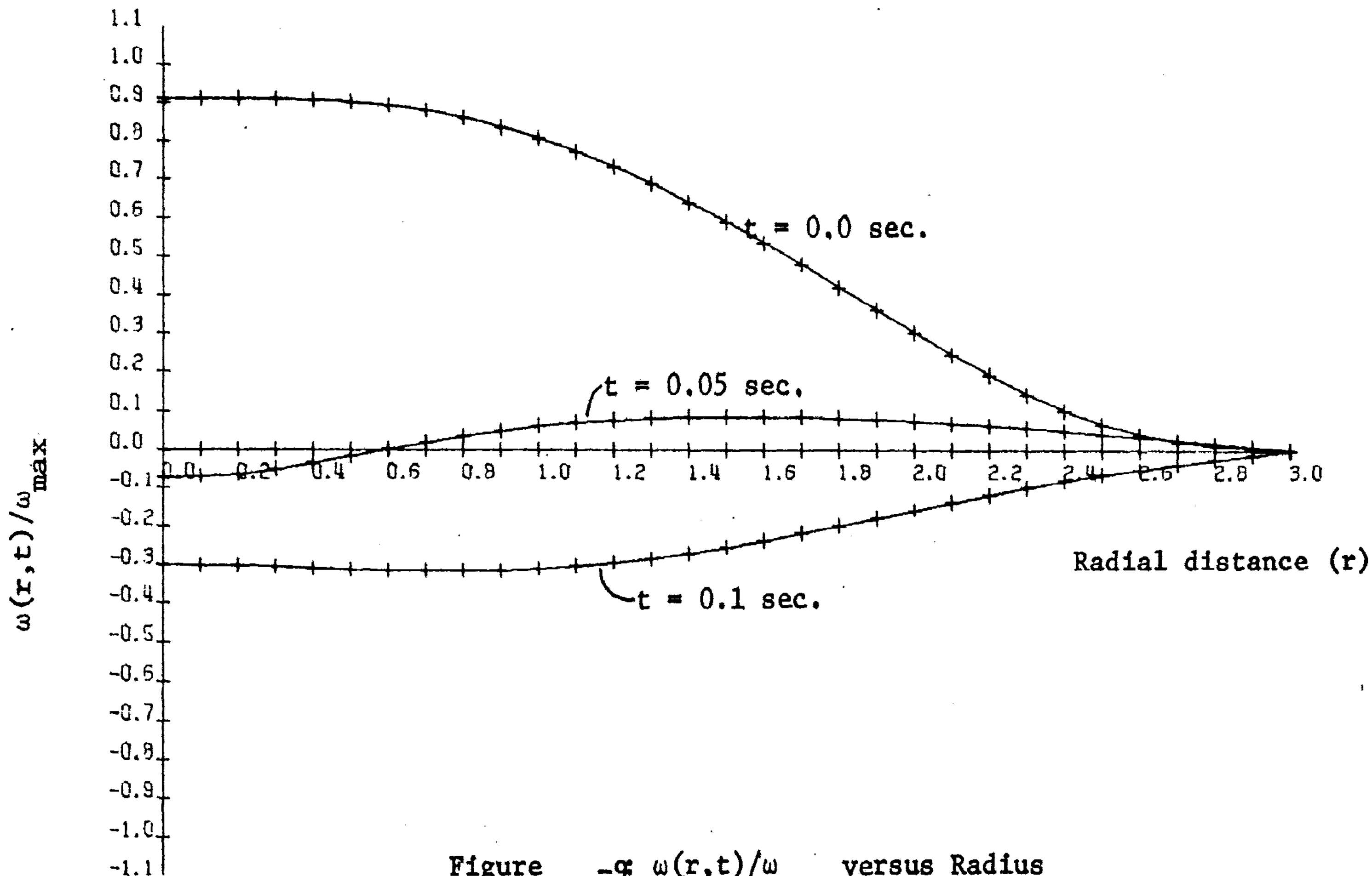

Figure 9: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_5.

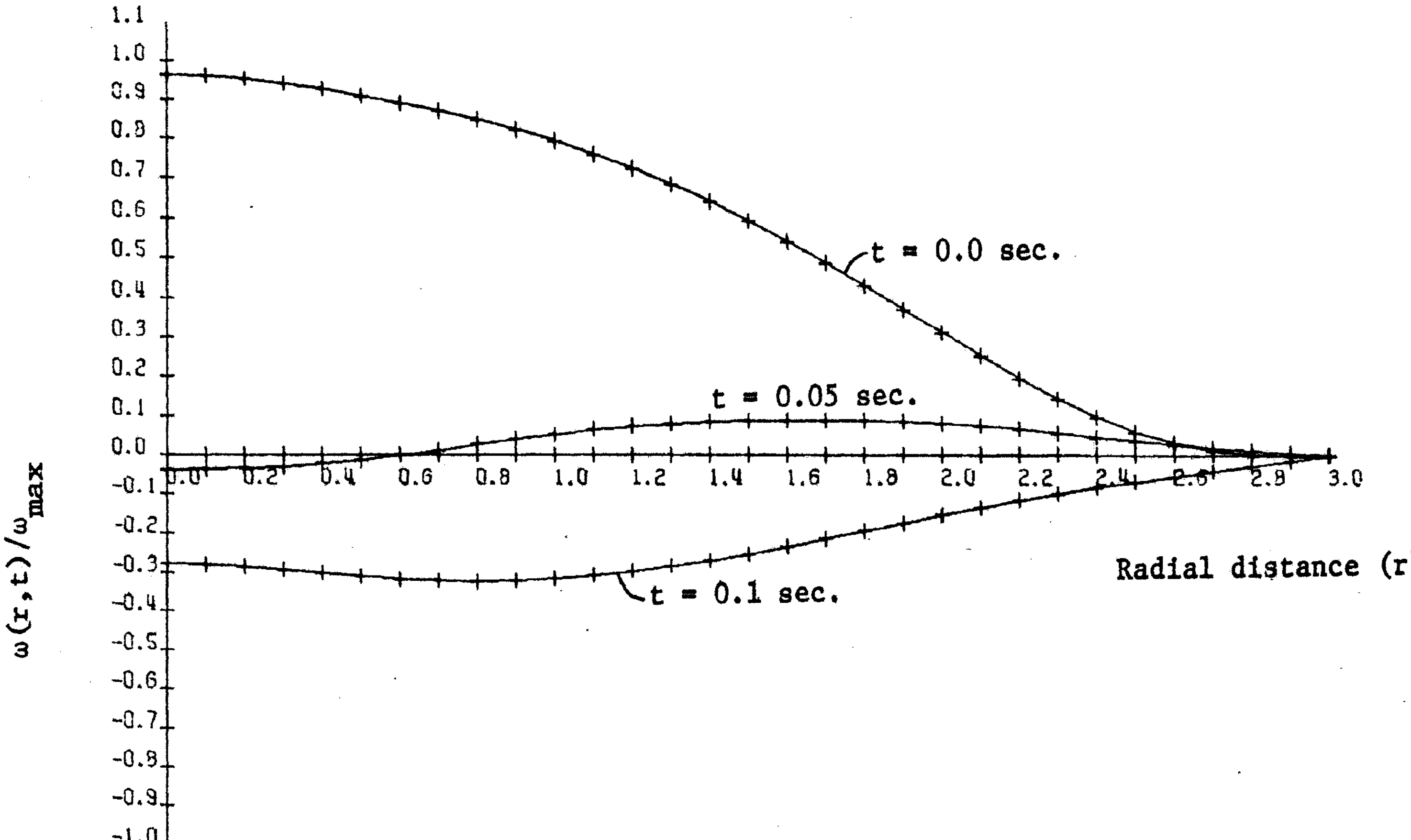

Figure -10: $\omega(r,t)/\omega_{max}$ versus Radius for Solution A_6.

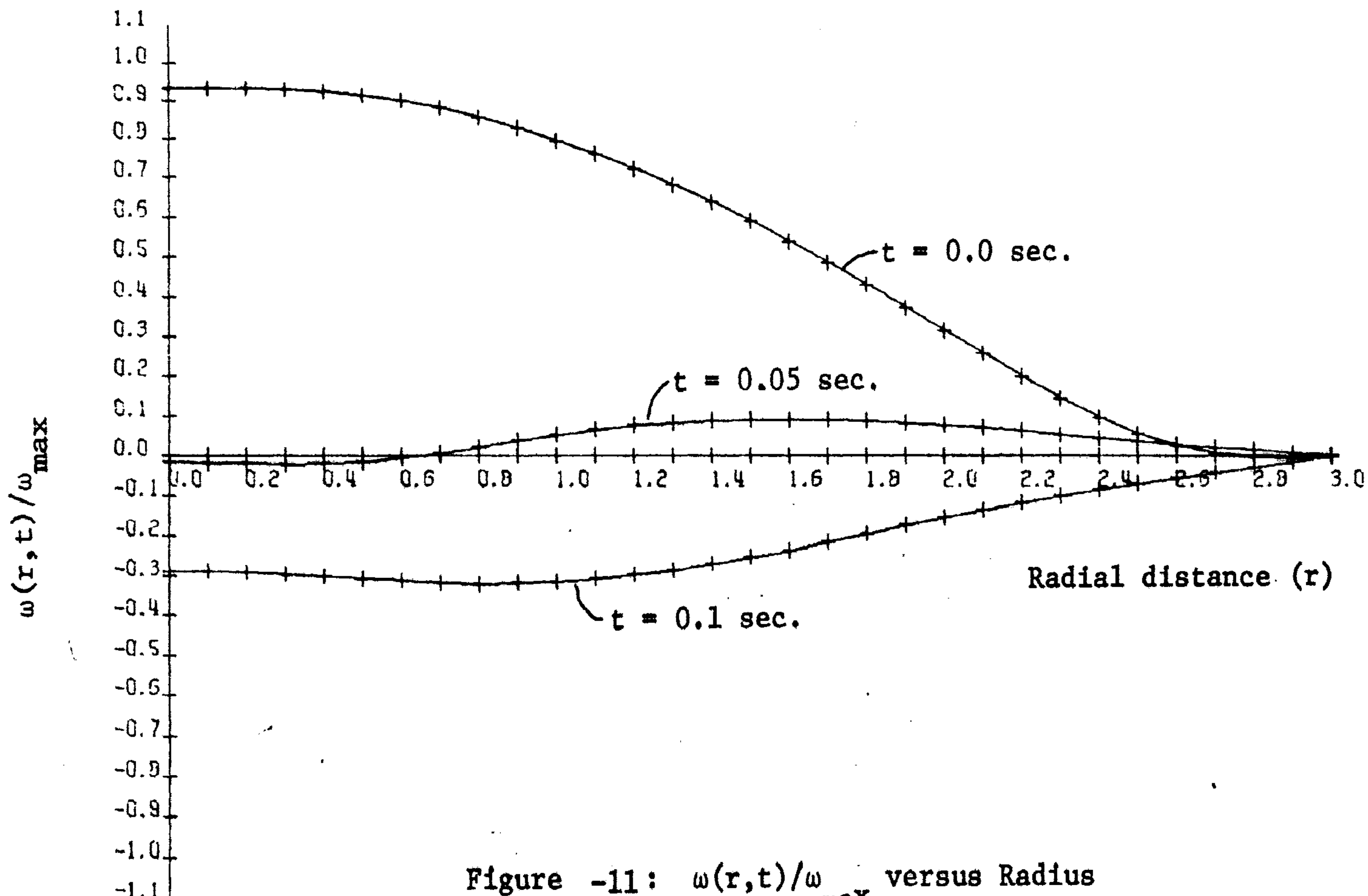

Figure -11: $\omega(r,t)/\omega_{max}$ versus Radius for Solution A_7.

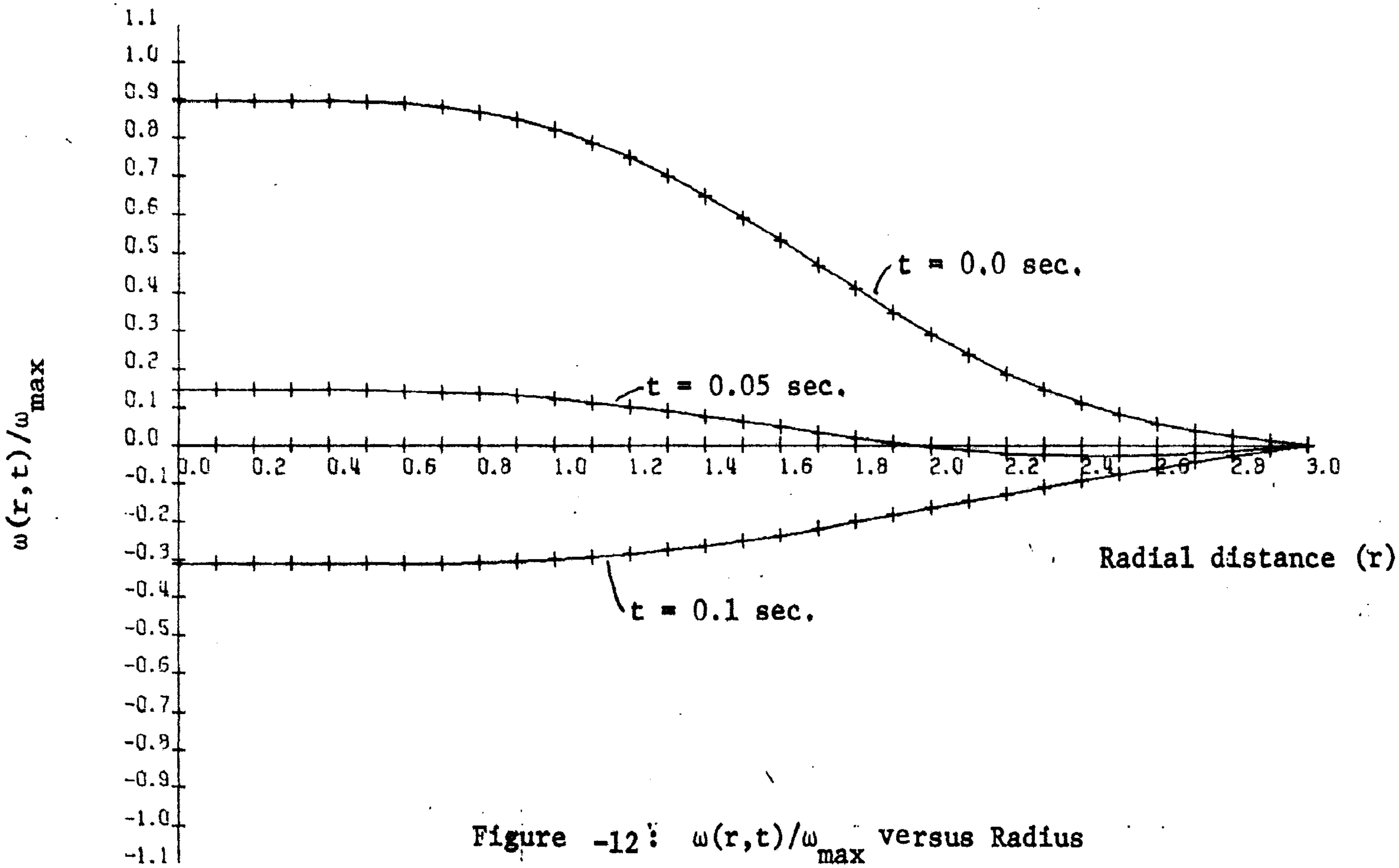

Figure -12: $\omega(r,t)/\omega_{max}$ versus Radius for Solution A_8.

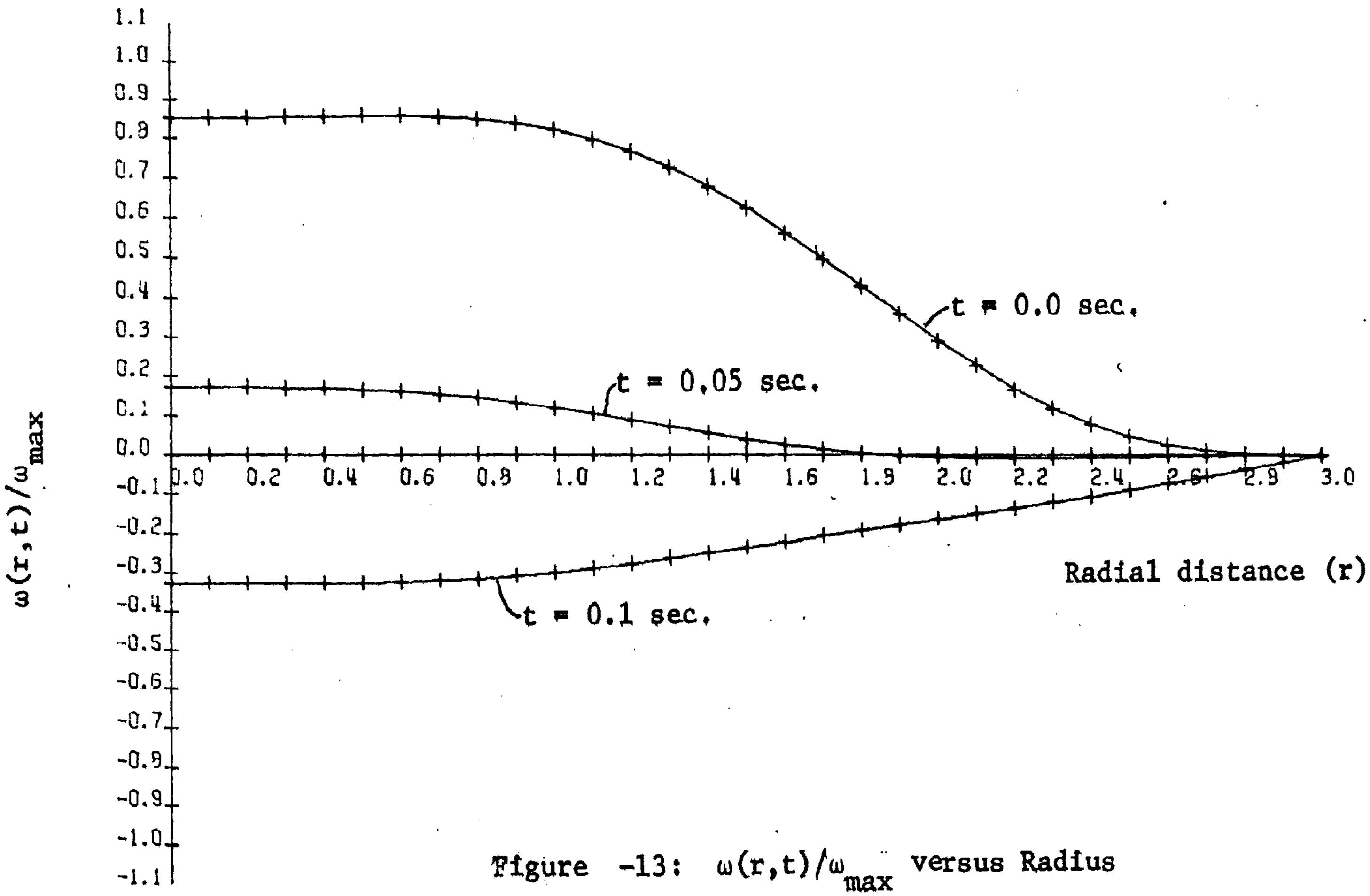

Figure -13: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_9.

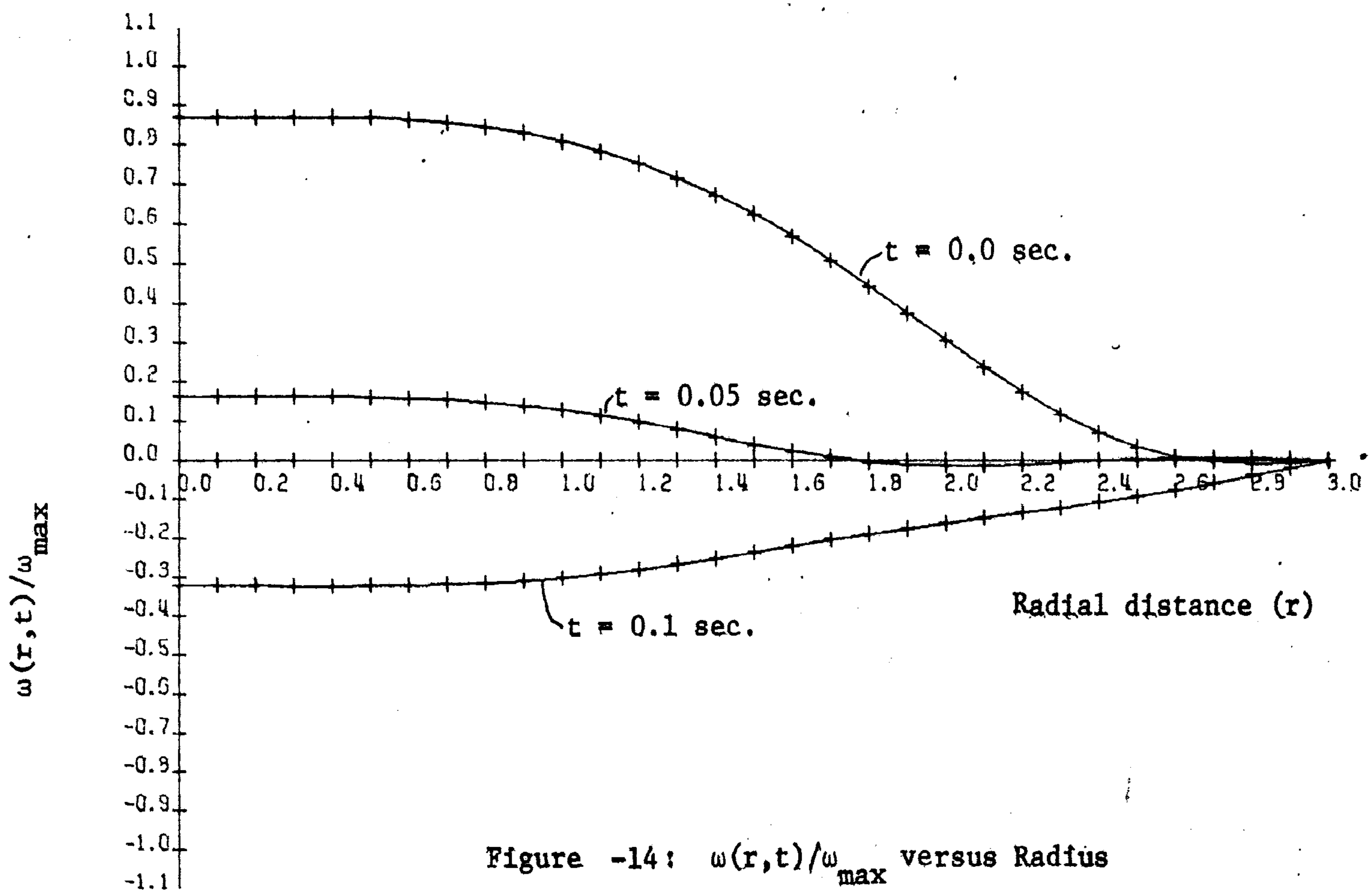

Figure -14: $\omega(r,t)/\omega_{max}$ versus Radius for Solution A_{10}.

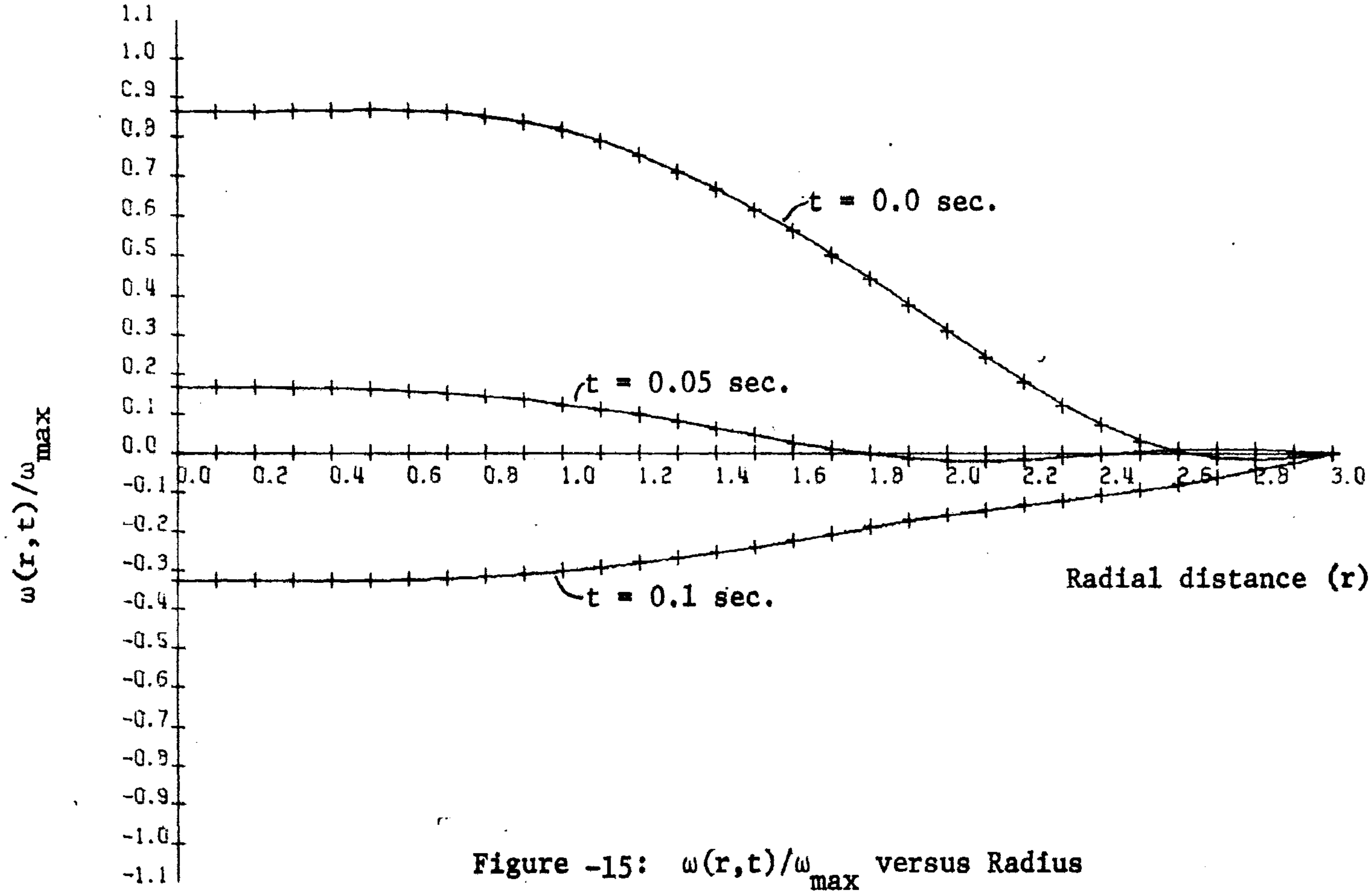

Figure -15: $\omega(r,t)/\omega_{max}$ **versus Radius**

for Solution A_{11}.

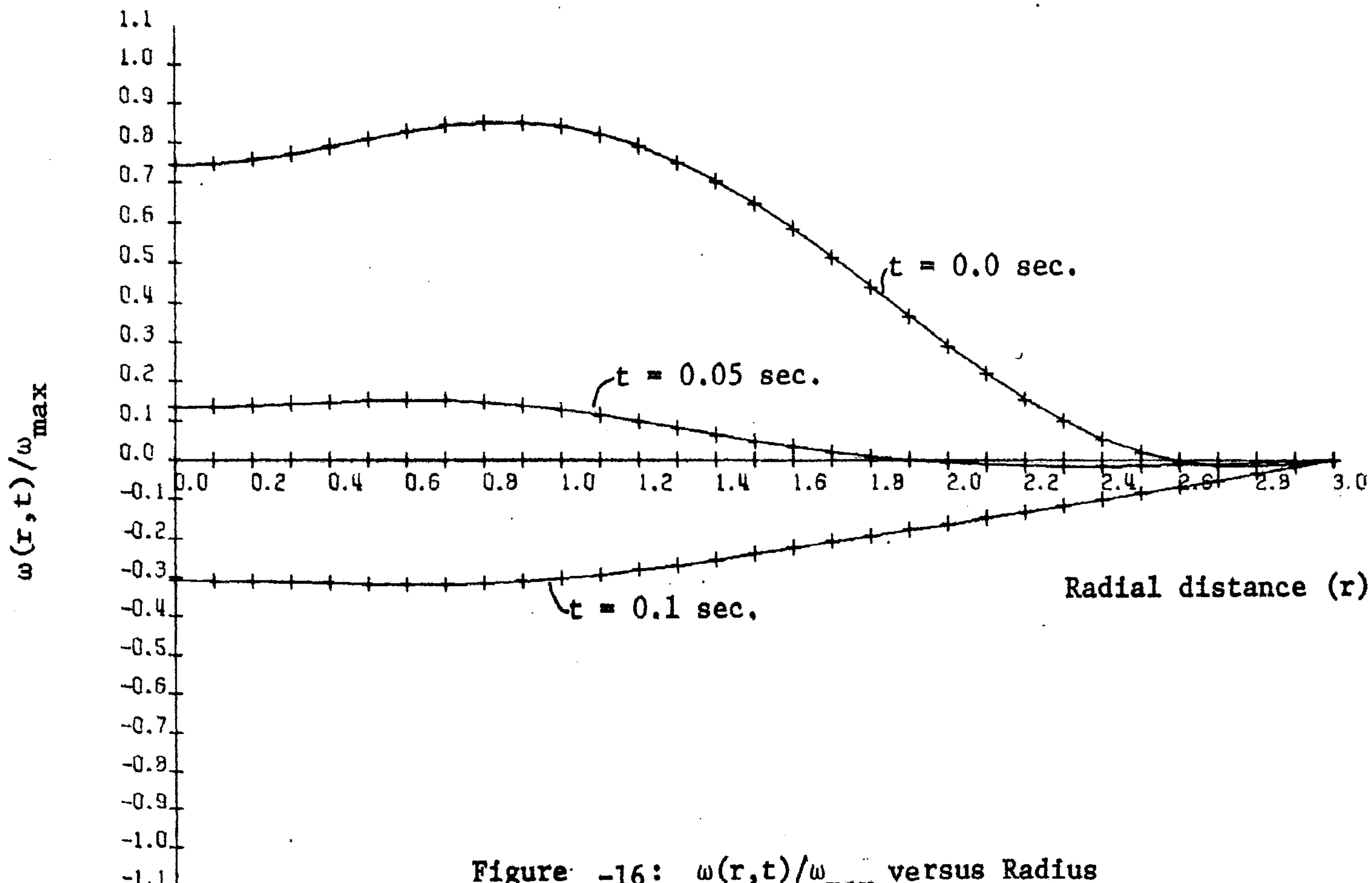

Figure -16: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_{12}.

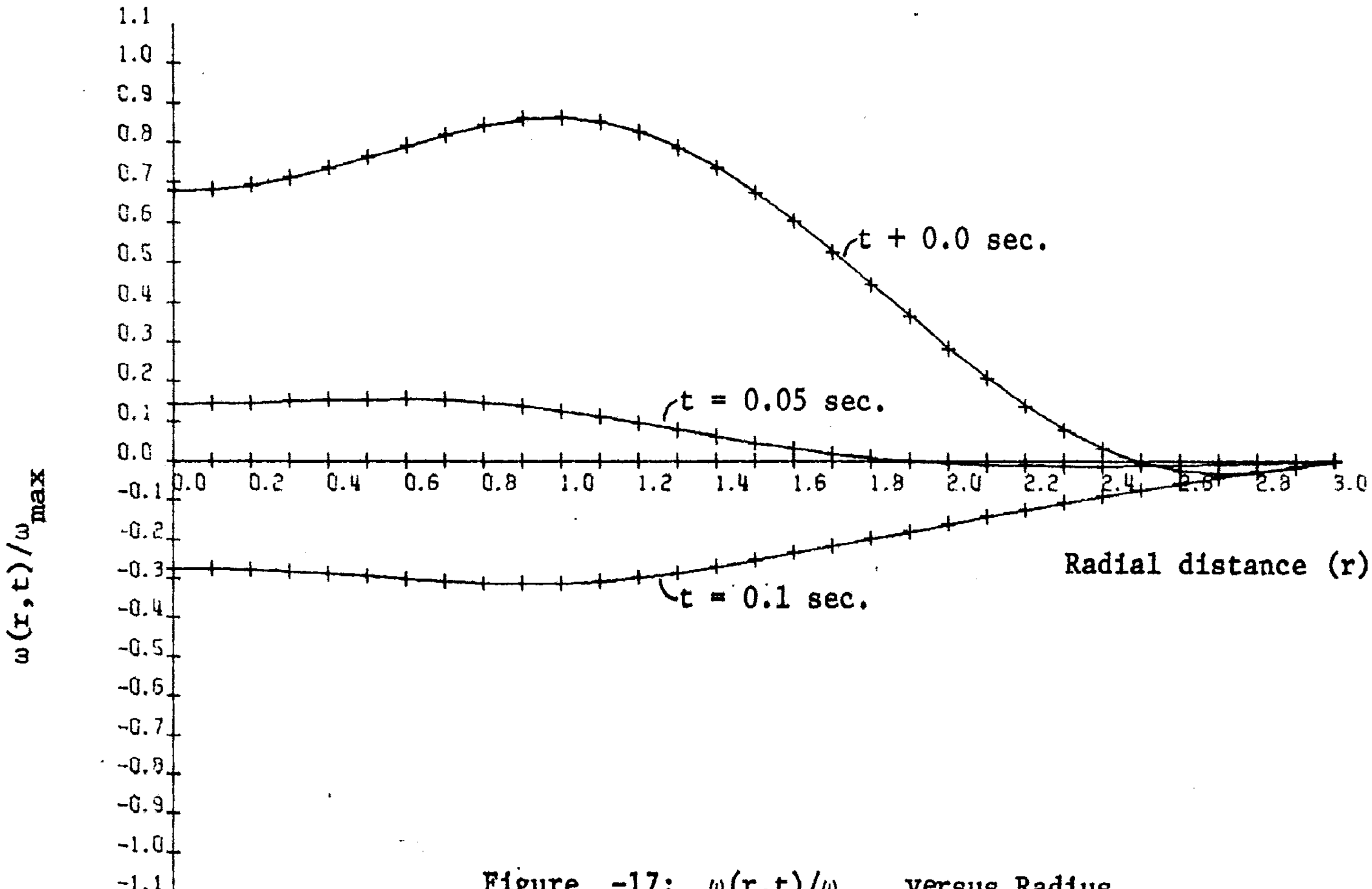

Figure -17: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_{13}.

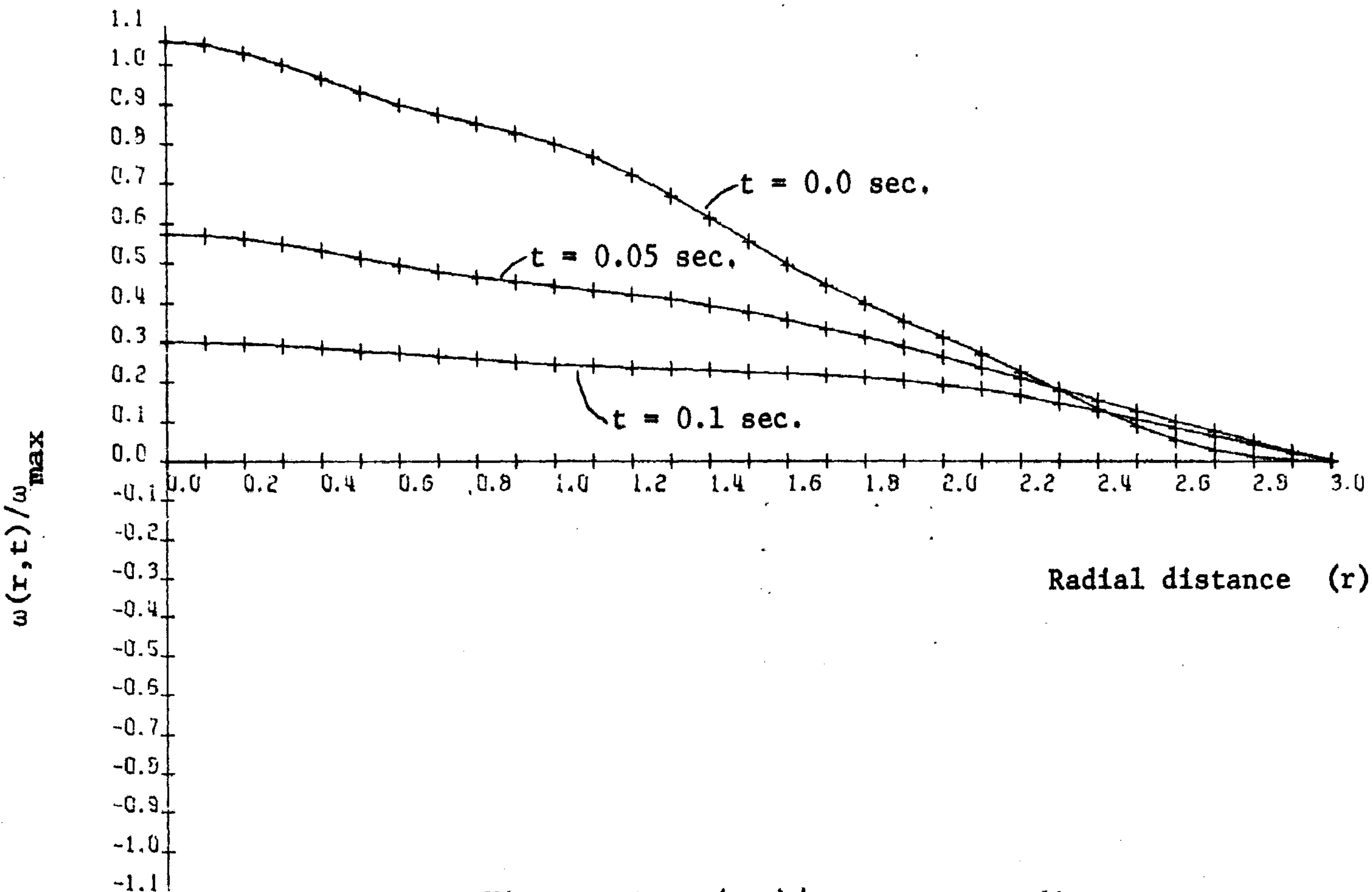

Figure -18: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_{14}.

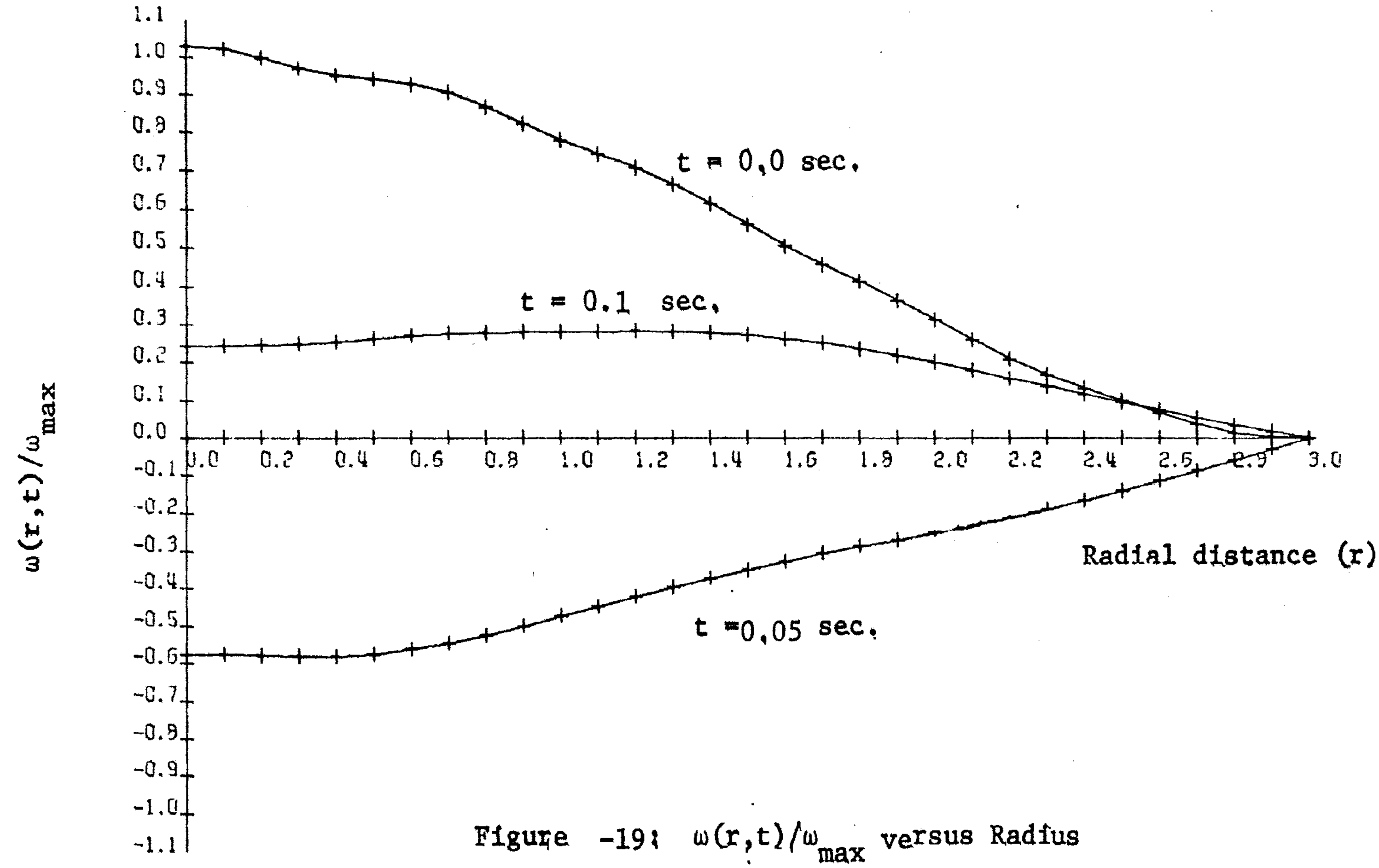

Figure -19: $\omega(r,t)/\omega_{max}$ versus Radius for Solution A_{15}.

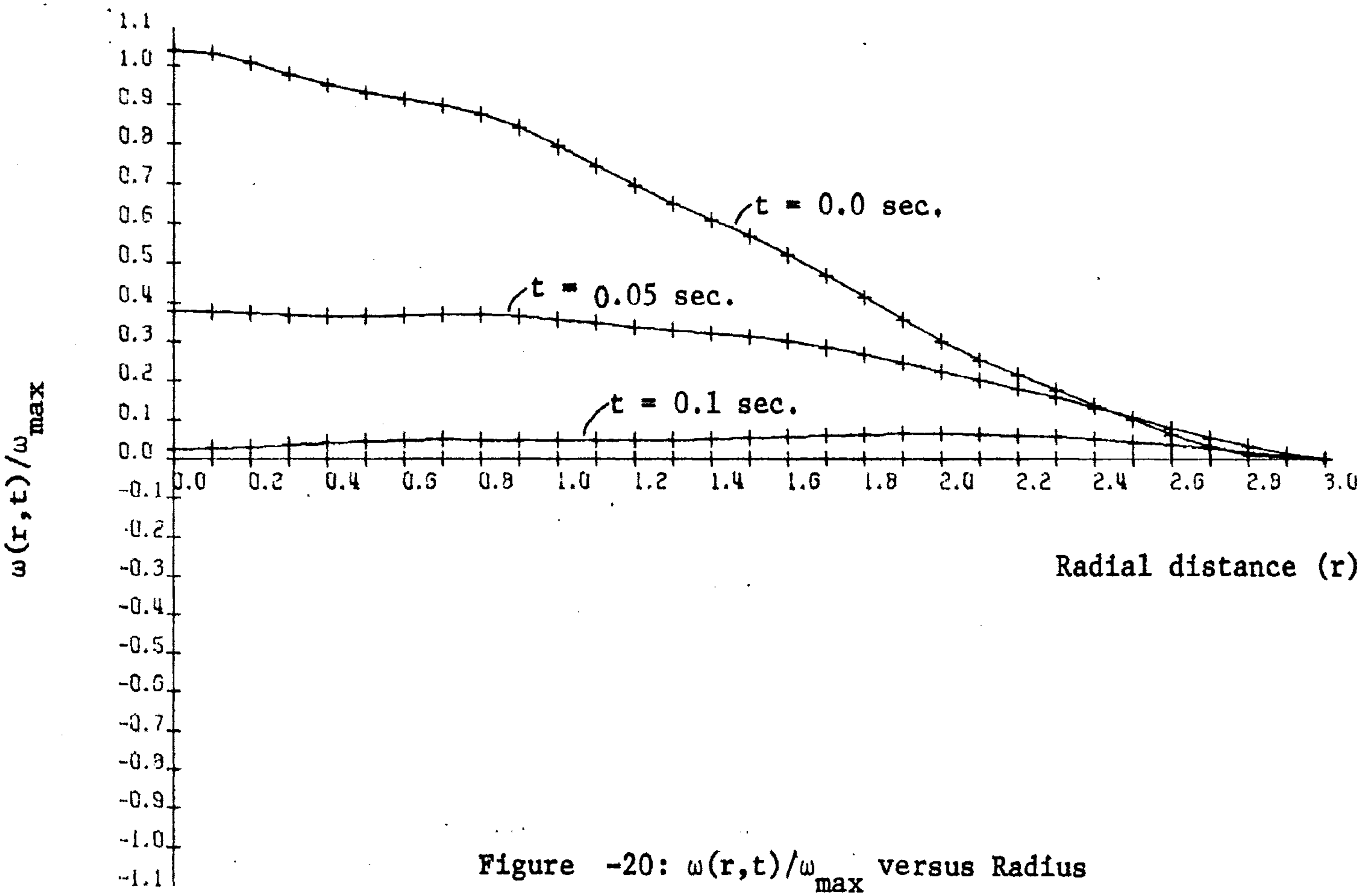

Figure -20: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_{16}.

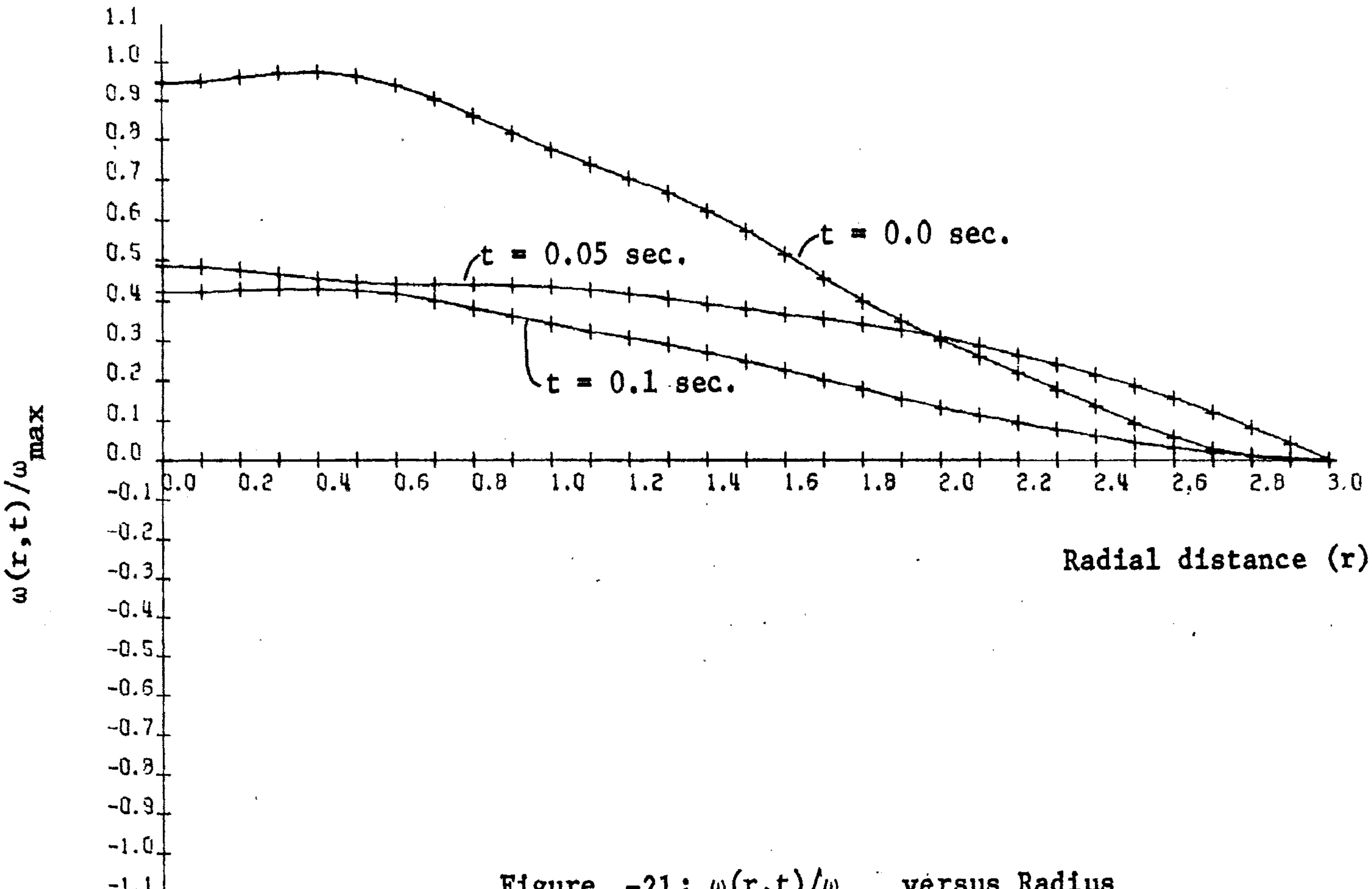

Figure -21: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_{17}.

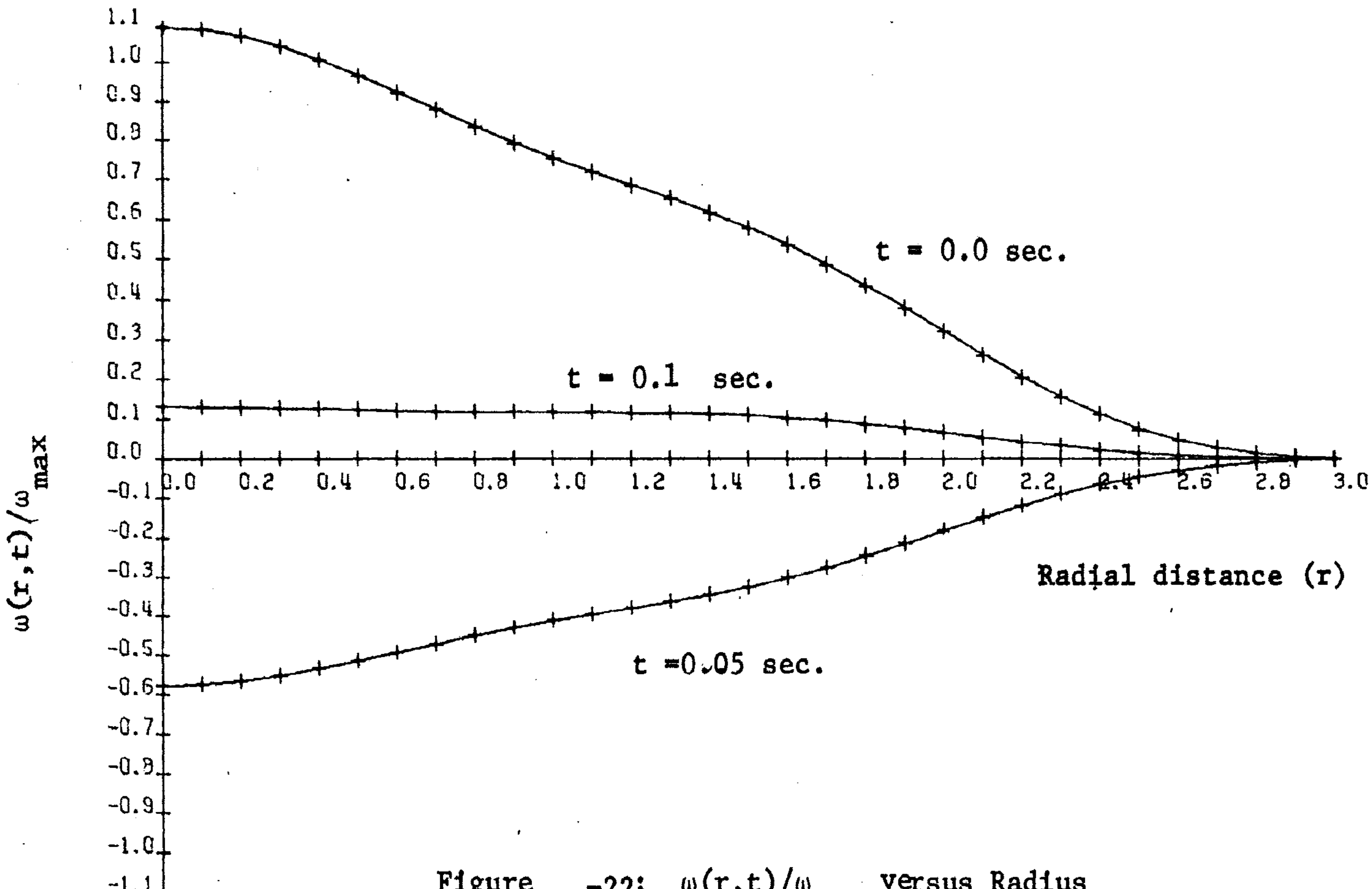

Figure 22: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_{18}.

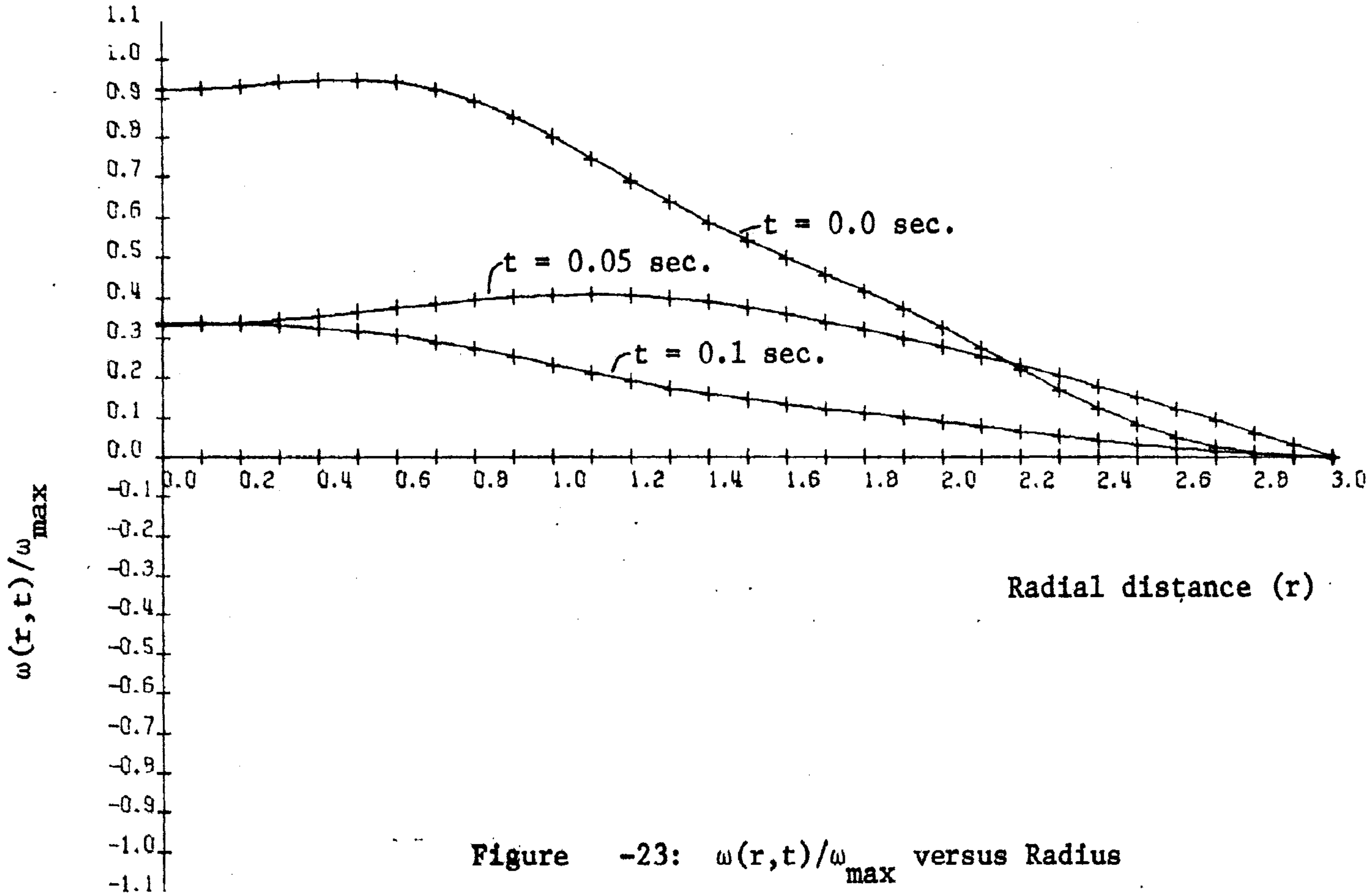

Figure -23: $\omega(r,t)/\omega_{max}$ versus Radius

for Solution A_{19}.

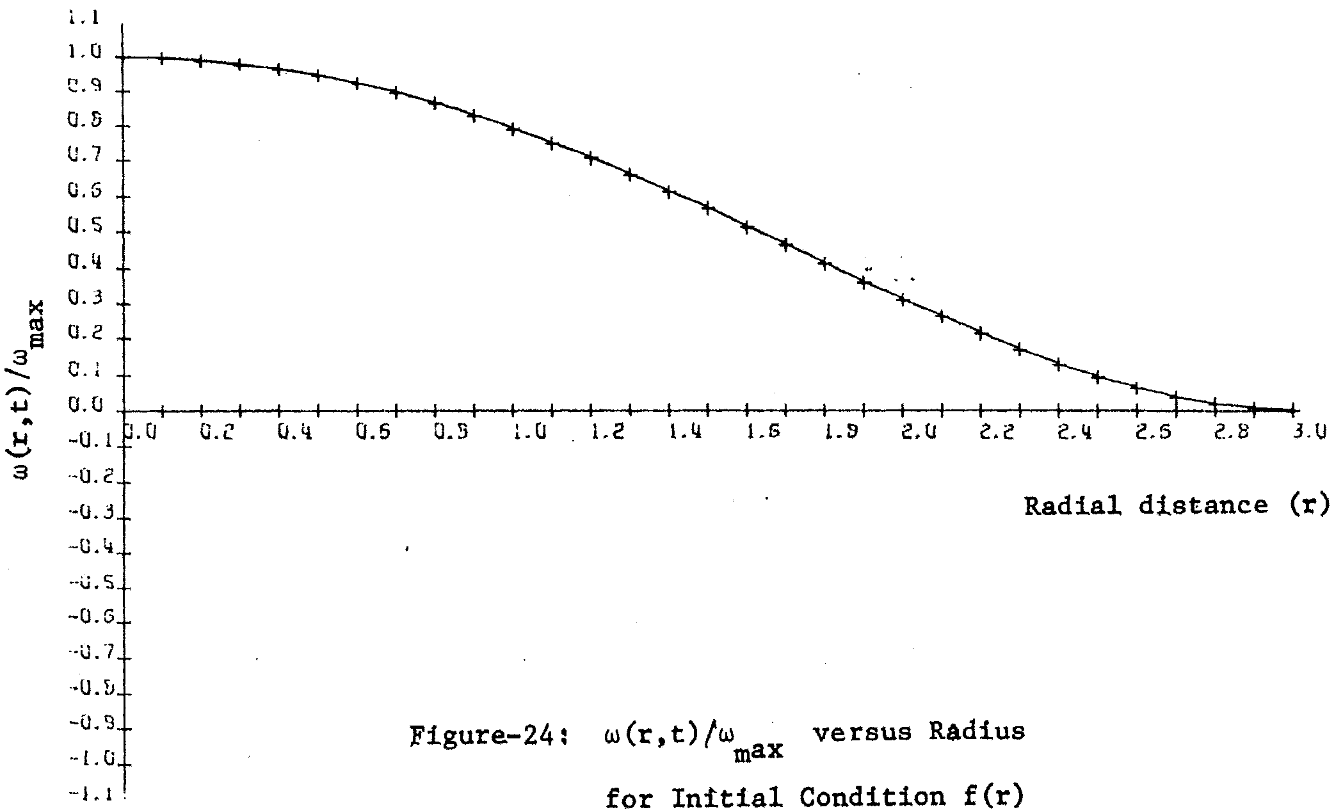

Figure-24: $\omega(r,t)/\omega_{max}$ versus Radius

for Initial Condition f(r)

VI. SUMMARY AND CONCLUSIONS

The case of free vibrations of a circular plate has been discussed in this investigation. One single differential equation of motion has been deduced in Chapter III by the method suggested by E. Reissner and in the similar way as has been done by R. D. Mindlin. In this development, it was assumed that displacements of the plate were axially symmetric and small compared to the dimensions of the plate. Further it was assumed that the values of the stress component σ_z were small and so the integrations containing σ_z over the thickness of the plate were neglected. It was also assumed that the radial and the tangential displacements were linear functions of the z coordinate and independent of shear. While considering the effects of the resistance of a fluid medium to the motion of the plate, the assumption of simple plane wave generation was made and so a simple relation for the pressure at a point in the wave in terms of the density of the medium and the velocity of wave propogation in the medium was used.

In Chapter IV, the solution to the differential equation of motion of the plate has been presented for the case when the plate is clamped around its circumference. A product solution was assumed in which the function of time was exponential in form. The equation governing the frequencies of vibration was obtained by application of the boundary conditions. Of the several sets of roots of this equation, one set of roots has been discussed in this investigation; namely, the roots of the Bessel function of first kind and zero order. The solution presented requires that the roots of the Bessel function be unequal and non-zero.

For the illustration, the material of the plate was assumed to be stainless steel 347. The plate was assumed to be vibrating in the medium of air. The radius and the thickness of the plate were assumed to be 3.0 inches and 3/16 inch respectively. The plate had an initial displacement described by the classical theory corresponding to a uniformly distributed load but with no initial velocity. When the results of the computations of $\omega(r,t)/\omega_{max}$ by nineteen different solutions, each one as the sum of several solutions given in tables (V-1) through (V-5) were plotted, it was found that three different types of curves are given by those nineteen solutions.

The first type was given by solutions A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19}. The curves are shown in figures (18, 19, 20, 21, 22, 23). It was noted that the approximation to the initial deflection curves was very crude. The approximation for the initial deflection at the center of the plate given by these solutions was with minimum error of 3.0 percent. Further it was found that this approximation was better when the value of m was increased. In the other two curves for $t = 0.05$ and $t = 0.1$, either deflections at all points were positive or they were all negative. It was observed that the deflections went on increasing up to a certain distance from the center of the plate and afterwards they decrease to zero at the boundary of the plate.

Figures 12, 13, 14, 15, 16, 17 show the plots of the results of the solutions A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}. The approximation to the initial deflection curve by these solutions is not even as good as that of solutions A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19}. In this case also it was noted that the deflections near the center of the plate are less than those at some greater values of r. This is prominent in solutions A_{13} and A_{12}.

Solutions A_1, A_2, A_3, A_4, A_5, A_6, and A_7 whose plots are shown in figures (5, 6, 7, 8, 9, 10, 11), give the third type of curves for $\omega(r,t)/\omega_{max}$. For solution A_4, the approximation to the initial deflection curve is very good with an error less than 0.2 percent for the deflection at the center. The approximation becomes better by increasing the number of terms in the solution. This increase in accuracy can be seen from the plots of solutions A_1, A_2, A_3 and A_4. This increase in accuracy is also true for solutions A_5, A_6 and A_7. In this case, the deflection also goes on increasing for some distance from the center of the plate and then decreases to zero at the boundary of the plate. The increase in the deflection in this case was not more than the order of 10^{-1}. From the approximation to the initial deflection curve, the solutions A_1, A_2, A_3, A_4 were better than the solutions A_5, A_6, A_7.

Though all the solutions show the characteristic of a hump at the center of the plate, very few solutions like A_{10}, A_{11}, A_{12}, A_{13} show a very pronounced hump in the approximation to the initial deflection curve. The probable reason might lie in the fact that solutions of the type used generally employ one Bessel function associated with one frequency but in this case every solution of the nineteen solutions is the sum of several solutions and each one of these several solutions is the sum of two Bessel functions with which one frequency is associated. From the solutions computed, though no specific conclusion as to which solution would give best results could be drawn, it could be believed that the soltuions A_1, A_2, A_3, and A_4 would give best results, as they satisfy the initial conditions better than the other solutions. However, a comparison with practical results would show which of these nineteen solutions give results close to the actual data. Since the solution requires that the initial displacement function be known, the example presented is at best an approximation because the initial displacement function assumed did not include shear deformation. It should be emphasized that only one possibility of the roots of the frequency equation has been discussed in this investigation. Therefore, the solution presented is a particular solution. However, it is hoped that this is a significant first step in the study of the stated problem.

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VITA

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APPENDIX-A

Derivation of Equations 1 and 4

In Figure A-1, PQ is a line segment located in a plate before deformation and P*Q* is the same line segment after deformation of the plate has occurred. The length of the line segment PQ is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{A-1}$$

Let the point P displace to P* and let the coordinates of the point P* be x*, y*, z*.

Then,

$$\begin{aligned} x^* &= x + u' \\ y^* &= y + v' \\ z^* &= z + \omega' \end{aligned} \tag{A-2}$$

where u', v' and ω' are the components of displacement of point P in the x, y and z directions, respectively.

Let the point Q(x + dx, y + dy, z + dz) displace to Q*(x* + dx*, y* + dy*, z* + dz*), where dx*, dy* and dz* are given by

$$\begin{aligned} dx^* &= \frac{\partial x^*}{\partial x} dx + \frac{\partial x^*}{\partial y} dy + \frac{\partial x^*}{\partial z} dz \\ dy^* &= \frac{\partial y^*}{\partial x} dx + \frac{\partial y^*}{\partial y} dy + \frac{\partial y^*}{\partial z} dz \\ dz^* &= \frac{\partial z^*}{\partial x} dx + \frac{\partial z^*}{\partial y} dy + \frac{\partial z^*}{\partial z} dz \end{aligned} \tag{A-3}$$

The length of the line element P*Q* is given by

$$ds^{*2} = dx^{*2} + dy^{*2} + dz^{*2} \tag{A-4}$$

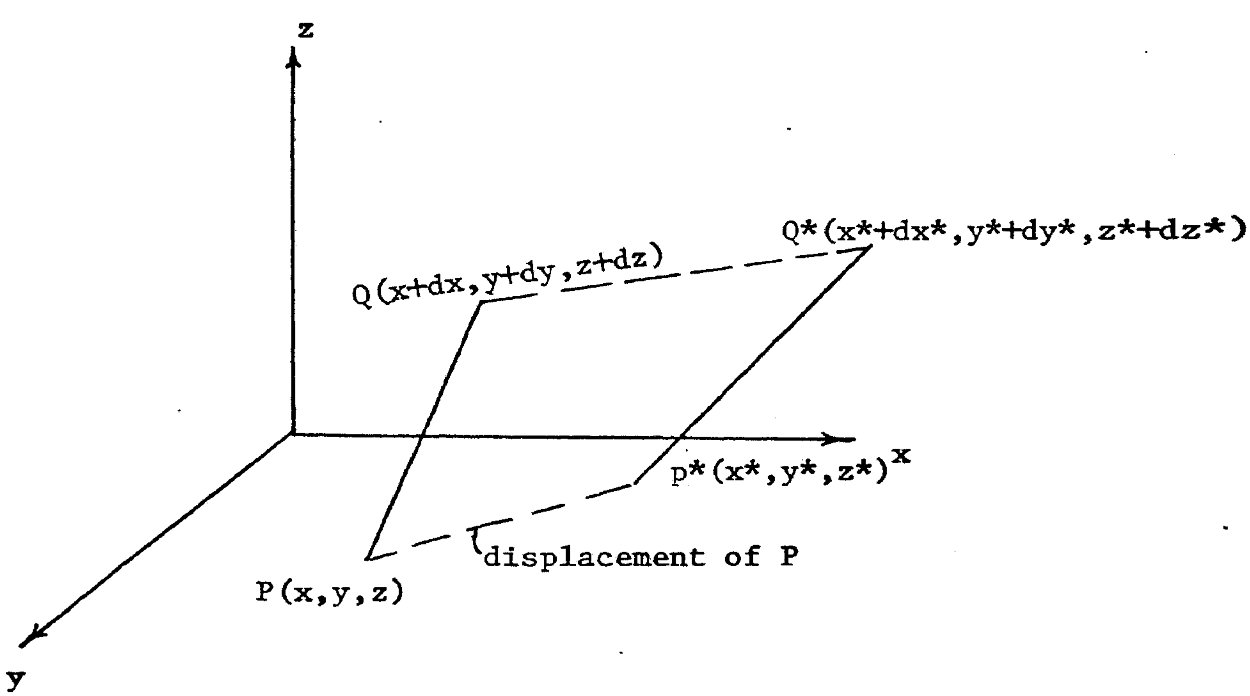

Figure A-1: Displacement of a Line Segment due to Plate Deformation

From equations A-2,

$$\frac{\partial x^*}{\partial x} = \frac{\partial u'}{\partial x} + 1 \; ; \; \frac{\partial x^*}{\partial y} = \frac{\partial u'}{\partial y} \; ; \; \frac{\partial x^*}{\partial z} = \frac{\partial u'}{\partial z}$$

$$\frac{\partial y^*}{\partial x} = \frac{\partial v'}{\partial y} \; ; \; \frac{\partial y^*}{\partial y} = \frac{\partial v'}{\partial y} + 1 \; ; \; \frac{\partial y^*}{\partial z} = \frac{\partial v'}{\partial z} \qquad \text{(A-5)}$$

$$\frac{\partial z^*}{\partial x} = \frac{\partial \omega'}{\partial x} \; ; \; \frac{\partial z^*}{\partial y} = \frac{\partial \omega'}{\partial y} \; ; \; \frac{\partial z^*}{\partial z} = \frac{\partial \omega'}{\partial z} + 1.$$

Therefore from equations A-1, A-3, A-4, and A-5

$$\begin{aligned}(ds^{*2} - ds^2) = {} & [(\frac{\partial u'}{\partial x})^2 + (\frac{\partial v'}{\partial x})^2 + (\frac{\partial \omega'}{\partial x})^2 + 2\,\frac{\partial u'}{\partial x}]\,dx^2 \\ & + [(\frac{\partial u'}{\partial y})^2 + (\frac{\partial v'}{\partial y})^2 + (\frac{\partial \omega'}{\partial y})^2 + 2\,\frac{\partial v'}{\partial y}]\,dy^2 \\ & + [(\frac{\partial u'}{\partial z})^2 + (\frac{\partial v'}{\partial z})^2 + (\frac{\partial \omega'}{\partial z})^2 + 2\,\frac{\partial \omega'}{\partial z}]\,dz^2 \qquad \text{(A-6)} \\ & + [2\frac{\partial u'}{\partial x}\,\frac{\partial u'}{\partial y} + 2\frac{\partial v'}{\partial x}\,\frac{\partial v'}{\partial y} + 2\frac{\partial \omega'}{\partial x}\,\frac{\partial \omega'}{\partial y} + 2\frac{\partial v'}{\partial x} + 2\frac{\partial u'}{\partial y}]dxdy \\ & + [2\frac{\partial u'}{\partial y}\,\frac{\partial u'}{\partial z} + 2\frac{\partial v'}{\partial y}\,\frac{\partial v'}{\partial z} + 2\frac{\partial \omega'}{\partial y}\,\frac{\partial \omega'}{\partial z} + 2\frac{\partial v'}{\partial z} + 2\frac{\partial \omega'}{\partial y}]dydz \\ & + [2\frac{\partial u'}{\partial x}\,\frac{\partial u'}{\partial z} + 2\frac{\partial v'}{\partial x}\,\frac{\partial v'}{\partial z} + 2\frac{\partial \omega'}{\partial x}\,\frac{\partial \omega'}{\partial z} + 2\frac{\partial \omega'}{\partial x} + 2\frac{\partial u'}{\partial z}]dxdz\end{aligned}$$

The transformation of the displacement components from rectangular to cylindrical coordinates is accomplished with the help of Fig. A-2. It can be seen from this figure that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad \text{(A-7)}$$

$$\begin{aligned} u' &= u\cos\theta - v\sin\theta \\ v' &= u\sin\theta + v\cos\theta \\ \omega' &= \omega \end{aligned} \qquad \text{(A-8)}$$

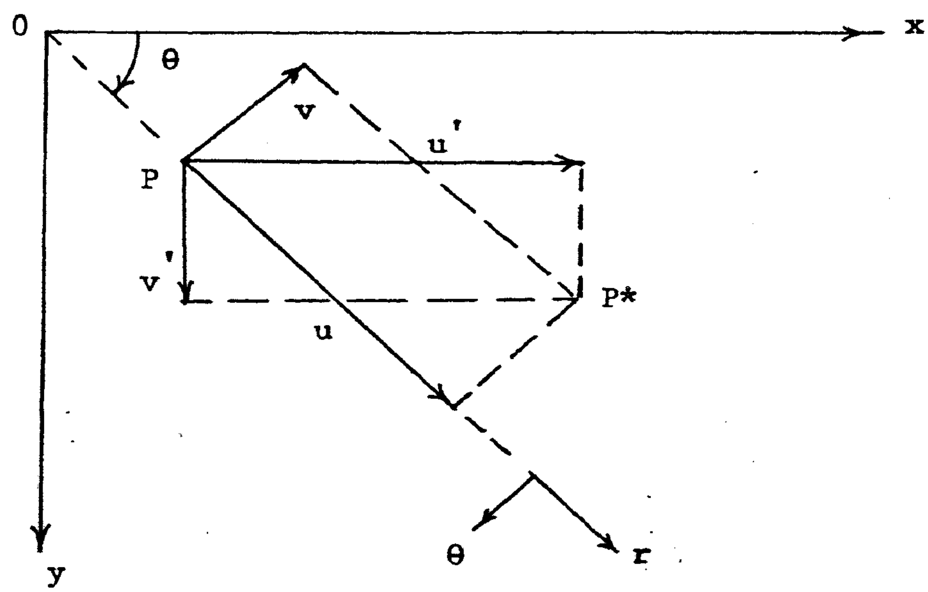

Figure A-2: Transformation of the Displacement Components from Rectangular to Cylindrical Coordinates

These equations are used to transform the terms in Equation A-6 as follows:

$$\frac{\partial u'}{\partial x} = \frac{\partial u'}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u'}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\sin\theta}{r}\left[-\frac{\partial u}{\partial \theta}\cos\theta + u\sin\theta + \frac{\partial v}{\partial \theta}\sin\theta + v\cos\theta\right]$$

$$+ \cos\theta\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial v}{\partial r}\sin\theta\right]$$

$$\frac{\partial u'}{\partial y} = \frac{\partial u'}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u'}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\cos\theta}{r}\left[\frac{\partial u}{\partial \theta}\cos\theta - u\sin\theta - \frac{\partial v}{\partial \theta}\sin\theta - v\cos\theta\right]$$

$$+ \sin\theta\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial v}{\partial r}\sin\theta\right] \qquad \text{(A-9)}$$

similarly,

$$\frac{\partial v'}{\partial x} = \frac{\sin\theta}{r}\left[-\frac{\partial u}{\partial \theta}\sin\theta - u\cos\theta - \frac{\partial v}{\partial \theta}\cos\theta - v\sin\theta\right]$$

$$+ \cos\theta\left[\frac{\partial u}{\partial r}\sin\theta + \frac{\partial v}{\partial r}\cos\theta\right]$$

$$\frac{\partial v'}{\partial y} = \frac{\cos\theta}{r}\left[\frac{\partial u}{\partial \theta}\sin\theta + u\cos\theta + \frac{\partial v}{\partial \theta}\cos\theta - v\sin\theta\right]$$

$$+ \sin\theta\left[\frac{\partial u}{\partial r}\sin\theta + \frac{\partial v}{\partial r}\cos\theta\right] \qquad \text{(A-10)}$$

It is noted that as θ approaches zero, $\sin\theta$ approaches zero and $\cos\theta$ approaches 1.

Therefore, in the limit

$$dx = dr$$

$$dy = r d\theta$$

$$dz = dz. \qquad \text{(A-11)}$$

For this case, Equations A-9 and A-10 reduce to,

$$\frac{\partial u'}{\partial x}\Big|_{\theta\to 0} = \frac{\partial u}{\partial r},$$

$$\frac{\partial u'}{\partial y}\Big|_{\theta\to 0} = \frac{\partial u}{r\partial\theta} - \frac{v}{r},$$

$$\frac{\partial v'}{\partial x}\Big|_{\theta\to 0} = \frac{\partial v}{\partial r}, \quad \frac{\partial \omega}{\partial y}\Big|_{\theta\to 0} = \frac{\partial \omega}{r\partial\theta} \qquad \text{(A-12)}$$

$$\frac{\partial v'}{\partial y}\Big|_{\theta\to 0} = \frac{\partial v}{r\partial\theta} + \frac{u}{r}, \quad \frac{\partial \omega'}{\partial x}\Big|_{\theta\to 0} = \frac{\partial \omega}{\partial r},$$

and equation A-6 becomes

$$(ds^{*2} - ds^2) = 2\,\varepsilon_r\, dr^2 + 2\,\varepsilon_\theta\, r^2 d\theta^2 + 2\,\varepsilon_z\, dz^2 + 2\gamma_{r\theta}\, r dr d\theta + 2\gamma_{rz}\, dr dz + 2\gamma_{\theta z}\, r d\theta dz$$

where ε_r, ε_θ, ε_z, $\gamma_{r\theta}$, γ_{rz} and $\gamma_{\theta z}$ are given by

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial \omega}{\partial r}\right)^2\right.$$

$$\varepsilon_\theta = \frac{\partial v}{r\partial\theta} + \frac{u}{r} + \frac{1}{2}\left[\left(\frac{\partial u}{r\partial\theta} - \frac{v}{r}\right)^2 + \left(\frac{\partial v}{r\partial\theta} + \frac{u}{r}\right)^2 + \left(\frac{\partial \omega}{r\partial\theta}\right)^2\right]$$

$$\varepsilon_z = \frac{\partial \omega}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial \omega}{\partial z}\right)^2\right]$$

$$\gamma_{r\theta} = \left[\frac{\partial u}{r\partial\theta} - \frac{v}{r} + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial r}\left(\frac{\partial u}{r\partial\theta} - \frac{v}{r}\right) + \frac{\partial v}{\partial r}\left(\frac{\partial v}{r\partial\theta} + \frac{u}{r}\right) + \frac{\partial \omega}{\partial r}\frac{\partial \omega}{r\partial\theta}\right]$$

$$\gamma_{rz} = \left[\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial r} + \frac{\partial u}{\partial r}\;\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\;\frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial r}\;\frac{\partial \omega}{\partial z}\right]$$

$$\gamma_{\theta z} = \left[\frac{\partial v}{\partial z} + \frac{\partial \omega}{r\partial\theta} + \frac{\partial u}{\partial z}\left(\frac{\partial u}{r\partial\theta} - \frac{v}{r}\right) + \frac{\partial v}{\partial z}\left(\frac{\partial v}{r\partial\theta} + \frac{u}{r}\right) + \frac{\partial \omega}{\partial z}\frac{\partial \omega}{r\partial\theta}\right] \qquad \text{(A-13)}$$

In large deflection theory, the strain ε_L of a line element is defined as

$$\varepsilon_L = \frac{1}{2}\left[\left(\frac{ds^*}{ds}\right)^2 - 1\right]$$

which, from Equation A-13 is given by

$$\varepsilon_L = \varepsilon_r \ell^2 + \varepsilon_\theta m^2 + \varepsilon_z n^2 + \gamma_{r\theta} \ell m + \gamma_{\theta z} mn + \gamma_{rz} \ell n$$

where ε_r, ε_θ, ε_z, $\varepsilon_{r\theta}$, ε_{rz} and $\varepsilon_{\theta z}$ are components of strain and ℓ, m, n are direction cosines of the line element PQ before deformation. In cylindrical coordinates ℓ, m, n are given by

$$\ell = \frac{dr}{ds} \quad , \quad m = \frac{rd\theta}{ds} \quad \text{and} \quad n = \frac{dz}{ds} \quad .$$

Therefore, the strain components are given by,

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial \omega}{\partial r}\right)^2 \right]$$

$$\varepsilon_\theta = \left(\frac{\partial v}{r\partial\theta} + \frac{u}{r}\right) + \frac{1}{2} \left[\left(\frac{\partial v}{r\partial\theta} + \frac{u}{r}\right)^2 + \left(\frac{\partial u}{r\partial\theta} - \frac{v}{r}\right)^2 + \left(\frac{\partial \omega}{r\partial\theta}\right)^2 \right]$$

$$\varepsilon_z = \frac{\partial \omega}{\partial z} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial \omega}{\partial z}\right)^2 \right] \qquad \text{(A-14)}$$

$$\gamma_{r\theta} = \frac{\partial u}{r\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{r\partial\theta} - \frac{v}{r}\right) + \left(\frac{\partial v}{\partial r}\right)\left(\frac{\partial u}{r\partial\theta} + \frac{u}{r}\right) + \left(\frac{\partial \omega}{\partial r}\right)\left(\frac{\partial \omega}{r\partial\theta}\right)$$

$$\gamma_{rz} = \frac{\partial \omega}{\partial r} + \frac{\partial u}{\partial z} + \left(\frac{\partial \omega}{\partial z}\right)\left(\frac{\partial \omega}{\partial r}\right) + \left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial u}{\partial r}\right) + \left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial v}{\partial r}\right)$$

$$\gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{\partial \omega}{r\partial\theta} + \left(\frac{\partial v}{r\partial\theta} + \frac{u}{r}\right)\frac{\partial v}{\partial z} + \left(\frac{\partial u}{r\partial\theta} - \frac{v}{r}\right)\frac{\partial u}{\partial z} + \frac{\partial \omega}{r\partial\theta}\frac{\partial \omega}{\partial z}$$

which are the same as given by Equations 1.

The equations of motion in cylindrical coordinates are derived as follows:

Figure A-3 shows the stresses acting on the six side faces of a differential element in cylindrical coordinates. The thickness of the element in the radial direction is dr, the thickness in the tangential direction is $(r + \frac{dr}{2})d\theta$ and the thickness in the transverse

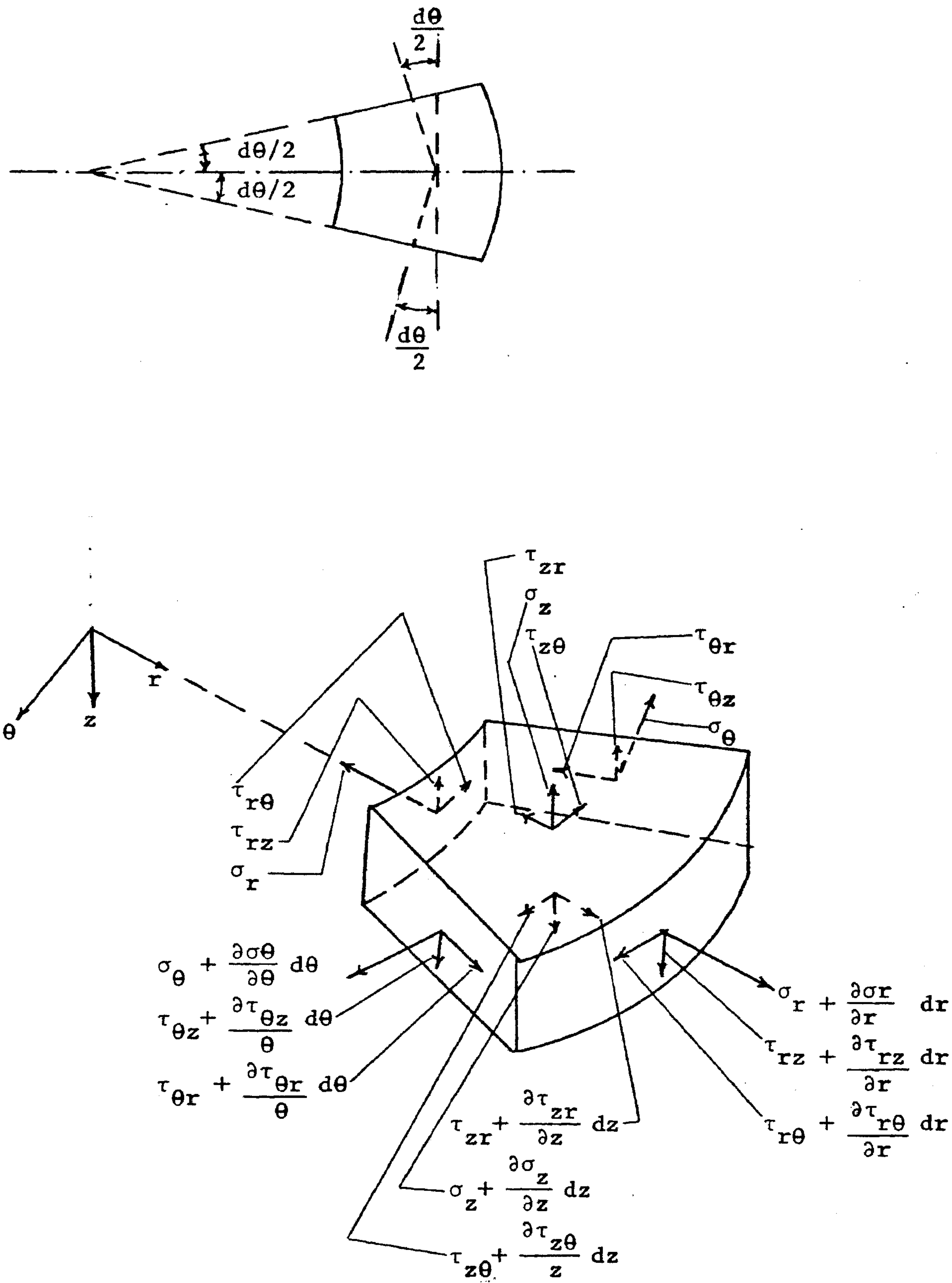

Figure A-3: Stresses on Six Faces of an Element of Plate direction is dz. Equating the sum of the forces in the radial direction to the inertia force in the radial direction yields, assuming the force in the direction of increasing r as positive,

$$(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr)(r + dr)\, d\theta dz + (\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d\theta)\, drdz \cos \frac{d\theta}{2}$$

$$+ (\tau_{rz} + \frac{\partial \tau rz}{\partial z} dz)(r + \frac{dr}{2})\, d\theta dr - \sigma_r\, rd\theta dz$$

$$- \tau_{r\theta}\, drdz \cos \frac{d\theta}{2} - \tau_{rz}\, (r + \frac{dr}{2})\, d\theta dr - \sigma_\theta \sin \frac{d\theta}{2} drdz$$

$$- (\sigma_\theta + \frac{\partial \sigma\theta}{\partial \theta} d\theta) \sin \frac{d\theta}{2} drdz$$

$$= \rho\; rd\theta\; dr\; dz\; \frac{\partial^2 u}{\partial t^2} \qquad \text{(A-15)}$$

where u is the component of displacement in the radial direction and ρ is the density of the material which is assumed to be constant. Since $d\theta$ is an infinitesimal angle,

$$\sin \frac{d\theta}{2} = \frac{d\theta}{2} \quad \text{and} \quad \cos \frac{d\theta}{2} = 1.$$

So on simplification the first equation of motion is

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho\, \frac{\partial^2 u}{\partial t^2} \qquad \text{(A-16)}$$

Similarly, summing the forces in the tangential and transverse directions and assuming v and ω as the displacement components in the tangential and transverse directions, respectively, the other two equations of motion can be obtained as

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \sigma_\theta}{r \partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\, \frac{\tau_{r\theta}}{r} = \rho\, \frac{\partial^2 v}{\partial t^2} \qquad \text{(A-17)}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{\theta z}}{r \partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho\, \frac{\partial^2 \omega}{\partial t^2} \qquad \text{(A-18)}$$

Equations A-16, A-17, and A-18 are the same as Equations 4.

APPENDIX-B

The Derivation of the First of Equations 7

From the stress strain relations given in equations 3, it can be seen that the expressions for $\tau_{r\theta}$, $\tau_{\theta z}$, τ_{rz} will not involve ε_r , ε_θ , ε_z , and σ_z . To obtain the expression for σ_r , two simultaneous equations, namely the first and the second of equations 3, are available. For σ_r , the result is

$$\sigma_r = \frac{E}{(1-\mu^2)} [\varepsilon_r + \mu\varepsilon_\theta] + \frac{\mu}{(1-\mu)} \sigma_z \qquad \text{(B-1)}$$

Multiplying both sides of equation B-1 by zdz and integrating the resultant equation over the thickness of the plate yields

$$\int_{-h/2}^{+h/2} \sigma_r zdz = \frac{E}{(1-\mu^2)} [\int_{-h/2}^{h/2} \varepsilon_r zdz + \mu \int_{-h/2}^{h/2} \varepsilon_\theta zdz] + \frac{\mu}{(1-\mu)} \int_{-h/2}^{h/2} \sigma_z zdz \qquad \text{(B-2)}$$

But by definition

$$\int_{-h/2}^{h/2} \sigma_r zdz = M_r \quad ;$$

Therefore,

$$M_r = \frac{E}{(1-\mu^2)} [\int_{-h/2}^{h/2} \varepsilon_r zdz + \mu \int_{-h/2}^{h/2} \varepsilon_\theta zdz] + \frac{\mu}{(1-\mu)} \int_{-h/2}^{h/2} \sigma_z zdz \qquad \text{(B-3)}$$

Now, neglecting the contribution of the term containing the integral of σ_z for the reasons that the value of σ_z is small and the thickness of the plate is small compared to the other dimensions of the plate, the result is

$$M_r = \frac{Eh^3}{12(1-\mu^2)} \left[\frac{12}{h^3} \int \varepsilon_r z dz + \mu \frac{12}{h^3} \int \varepsilon_\theta z dz\right] \qquad \text{(B-4)}$$

which is the same as the first of equations 7.

APPENDIX-C

Deduction of the Simplified Equations of Motion

The first of the equations of motion is

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{(\sigma_r - \sigma_\theta)}{r} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \text{(C-1)}$$

Multiplying both the sides of equation C-1 by 'zdz' and then integrating over the thickness of the plate yields

$$\int_{-h/2}^{h/2} \frac{\partial \sigma_r}{\partial r}\, zdz + \int_{-h/2}^{h/2} \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}\, zdz + \int_{-h/2}^{h/2} \frac{\partial \tau_{rz}}{\partial z}\, zdz + \int_{-h/2}^{h/2} \frac{(\sigma_r - \sigma_\theta)}{r}\, zdz$$

$$= \rho \int_{-h/2}^{h/2} \frac{\partial^2 u}{\partial t^2}\, zdz \qquad \text{(C-2)}$$

Using the definitions for 'M_r', 'M_θ', and '$M_{r\theta}$' in equation C-2 gives

$$\frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + \int_{-h/2}^{h/2} \frac{\partial \tau_{rz}}{\partial z}\, zdz + \frac{(M_r - M_\theta)}{r} = \rho \int_{-h/2}^{h/2} \frac{\partial^2 u}{\partial t^2}\, zdz \qquad \text{(C-3)}$$

Now, when the assumptions for the radial and tangential components of displacement, namely

$u = z\,\psi_r\,(r,\theta,t)$ and $v = z\,\psi_\theta\,(r,\theta,t)$, are substituted

in the fifth of equations 2, the result is

$$\tau_{rz} = \frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial r} = \psi_r + \frac{\partial \omega}{\partial r} \qquad \text{(C-4)}$$

It can be seen from equations C-4 that τ_{rz} is not a function of z. To evaluate the third term in equation C-3, use the identity

$$\int u dv = uv - \int v\, du \qquad \text{(C-5)}$$

Let

$$I = \int_{-h/2}^{h/2} \frac{\partial \tau_{rz}}{\partial z}\, zdz.$$

Further, let

$$u = z \quad \text{and} \quad dv = \frac{\partial \tau_{rz}}{\partial z}\, dz$$

then,

$$du = dz \quad \text{and} \quad v = \tau_{rz}$$

Therefore,

$$I = \int_{-h/2}^{h/2} \frac{\partial \tau_{rz}}{\partial z}\, z dz = [z\, \tau_{rz}]_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \tau_{rz}\, dz$$

Use of the definition of Q_r (equations 5) and the boundary conditions (equations 6) for τ_{rz} in the preceeding equation gives

$$I = \int_{-h/2}^{h/2} \frac{\partial \tau_{rz}}{\partial z}\, z dz = - Q_r \tag{C-6}$$

Hence, equation C-3 becomes,

$$\frac{\partial M_r}{\partial r} + \frac{\partial M_{r\theta}}{r\partial\theta} + \frac{M_r - M_\theta}{r} - Q_r = \rho \frac{\partial^2}{\partial t^2} \int_{-h/2}^{h/2} z^2 \psi_r dz \tag{C-7}$$

Since ψ_r is not a function of z, equation B-1 can be written in the form

$$\frac{\partial M_r}{\partial r} + \frac{\partial M_{r\theta}}{r\partial\theta} + \frac{M_r - M_\theta}{r} - Q_r = \frac{\rho h^3}{12} \frac{\partial^2 \psi_r}{\partial t^2} \tag{C-8}$$

which is the same as the first of equations 10.

APPENDIX-D

The Procedure to Uncouple the Equations of Motion

To uncouple the differential equations of motion in terms of unit moments, shear forces and twist, the procedure used here is similar to that used by Kalnins [18] for spherical shells. The procedure is as follows:

Let

$$Q_r = \frac{\partial v_1}{\partial r} \qquad \text{and}$$

$$Q_\theta = \frac{\partial v_1}{r\partial\theta} \tag{D-1}$$

where v_1 is an auxiliary function of r, θ and t. Since,

$$\psi_r = -\frac{\partial\omega}{\partial r} + \frac{Q_r}{G'h} \quad ,$$

$$\psi_r = -\frac{\partial\omega}{\partial r} + \frac{1}{G'h}\frac{\partial v_1}{\partial r} \tag{D-2}$$

and since,

$$\psi_\theta = -\frac{\partial\omega}{r\partial\theta} + \frac{Q_\theta}{G'h} \quad ,$$

$$\psi_\theta = -\frac{\partial\omega}{r\partial\theta} + \frac{1}{G'h}\frac{\partial v_1}{r\partial\theta} \tag{D-3}$$

With these substitutions, the expressions for unit moments, shear forces and twist become,

$$M_r = D\,[\frac{\partial\psi_r}{\partial r} + \frac{\mu}{r}\,(\psi_r + \frac{\partial\psi_\theta}{\partial\theta})]$$

$$M_r = -D\,[\frac{\partial^2\omega}{\partial r^2} + \frac{\mu}{r}\,(\frac{\partial\omega}{\partial r} + \frac{\partial^2\omega}{r\partial\theta^2})]$$

$$+ \frac{D}{G'h}\,[\frac{\partial^2 v_1}{\partial r_2} + \frac{\mu}{r}\,(\frac{\partial v_1}{\partial r} + \frac{\partial^2 v_1}{r\partial\theta^2})] \tag{D-4}$$

$$M_\theta = D[\mu \frac{\partial^2 \omega}{\partial r_2} + \frac{1}{r} (\psi_r + \frac{\partial \psi_\theta}{\partial \theta})]$$

$$M_\theta = -D[\mu \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} (\frac{\partial \omega}{\partial r} + \frac{\partial^2 \omega}{r \partial \theta^2})]$$

$$+ \frac{D}{G'h} [\mu \frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r} (\frac{\partial v_1}{\partial r} + \frac{\partial^2 v_1}{r \partial \theta^2})] \quad \text{(D-5)}$$

$$M_{r\theta} = (1-\mu) \frac{D}{2} [\frac{\partial \psi_\theta}{\partial r} - \frac{\partial_\theta}{r} + \frac{\partial \psi_r}{r \partial \theta}]$$

$$M_{r\theta} = (1-\mu) D[\frac{\partial \omega}{r^2 \partial \theta} - \frac{\partial^2 \omega}{r \partial r \partial \theta} - \frac{1}{G'h} \frac{\partial v_1}{r^2 \partial \theta} + \frac{1}{G'h} \frac{\partial^2 v_1}{r \partial r \partial \theta}] \quad \text{(D-6)}$$

The substitution of M_r, M_θ, and $M_{r\theta}$ in the first two of the equations of motion gives,

$$-D \frac{\partial}{\partial r} (\nabla^2 \omega) + \frac{D}{G'h} \frac{\partial}{\partial r} (\nabla^2 v_1) - \frac{\partial v_1}{\partial r} + \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} (\frac{\partial \omega}{\partial r})$$

$$- \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2} (\frac{\partial v_1}{\partial r}) = 0 \quad \text{(D-7)}$$

$$-D \frac{\partial}{\partial \theta} (\nabla^2 \omega) + \frac{D}{G'h} \frac{\partial}{\partial \theta} (\nabla^2 v_1) - \frac{\partial v_1}{\partial \theta}$$

$$+ \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} (\frac{\partial \omega}{\partial \theta}) - \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2} (\frac{\partial v_1}{\partial \theta}) = 0 \quad \text{(D-8)}$$

Differentiating the equation D-7 with respect to r gives,

$$-D \frac{\partial^2}{\partial r^2} (\nabla^2 \omega) + \frac{D}{G'h} \frac{\partial^2}{\partial r^2} (\nabla^2 v_1) - \frac{\partial^2 v_1}{\partial r^2}$$

$$+ \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} (\frac{\partial^2 \omega}{\partial r^2}) - \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2} (\frac{\partial^2 v_1}{\partial r^2}) = 0 \quad \text{(D-9)}$$

Dividing the equation D-7 by r, gives,

$$-D \frac{1}{r} \frac{\partial}{\partial r} (\nabla^2 \omega) + \frac{D}{G'h} \frac{1}{r} \frac{\partial}{\partial r} (\nabla^2 v_1) - \frac{1}{r} \frac{\partial v_1}{\partial r}$$

$$+ \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} (\frac{1}{r} \frac{\partial \omega}{\partial r}) - \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2} (\frac{1}{r} \frac{\partial v_1}{\partial r}) = 0 \quad \text{(D-10)}$$

Differentiating the equation D-8 with respect to θ and dividing the same by r^2 yields,

$$-D \frac{\partial^2}{r^2 \partial\theta^2} (\nabla^2 \omega) + \frac{D}{G'h} \frac{\partial^2}{r^2 \partial\theta^2} (\nabla^2 v_1) - \frac{\partial^2 v_1}{r^2 \partial\theta^2}$$

$$+ \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 \omega}{r^2 \partial\theta^2}\right) - \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 v_1}{r^2 \partial\theta^2}\right) = 0 \qquad \text{(D-11)}$$

Now, addition of equations D-9, D-10, and D-11 gives

$$-D \nabla^4 \omega + \frac{D}{G'h} \nabla^4 v_1 - \nabla^2 v_1 + \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} (\nabla^2 \omega)$$

$$- \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2} (\nabla^2 v_1) = 0 \qquad \text{(D-12)}$$

For the third of the equations of motion, substitution of Q_r and Q_θ give

$$\nabla^2 v_1 + q(r,\theta,t) = \rho h \frac{\partial^2}{\partial t^2} \qquad \text{(D-13)}$$

Now, substitution of $\nabla^2 v_1$ from equation D-13 into the equation D-12 gives after simplification,

$$[(D\nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2})(\nabla^2 - \frac{\rho}{G'} \frac{\partial^2}{\partial t^2}) + \rho h \frac{\partial^2}{\partial t^2}] \omega$$

$$- [1 - \frac{D}{G'h} \nabla^2 + \frac{\rho h^2}{12G'} \frac{\partial^2}{\partial t^2}] q(r,\theta,t) = 0 \qquad \text{(D-14)}$$

which is the same as equation 11.

APPENDIX-E

The Derivation of Equations 12 and 13

Equation (12) is the solution of two wave equations in terms of effective pressure and effective particle velocity. They are

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} \qquad \text{(E-1)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{(E-2)}$$

The assumption of simple plane wave generation by a plate during vibrations implies that

(1) Media on both the sides of the plate are such that the simplified small amplitude wave equation is valid, which means that

The frequencies of vibration are high and

$$\frac{\omega}{C_1} >> \frac{1}{R} \qquad \text{and} \qquad \frac{\omega}{C_2} >> \frac{1}{R}$$

(2) The pressure at any point in the wave at any time is a function of one variable only (i.e. the distance along the direction of propogation of the wave).

(3) There are no losses due to turbulence and scattering; therefore, the wave does not lose energy as it propogates.

Equations E-1 and E-2 are derived as follows: Referring to figure E-1, let the pressure 'p' increase from the left hand surface to the right hand surface by the amount $\frac{\partial p}{\partial x}\Delta x$.

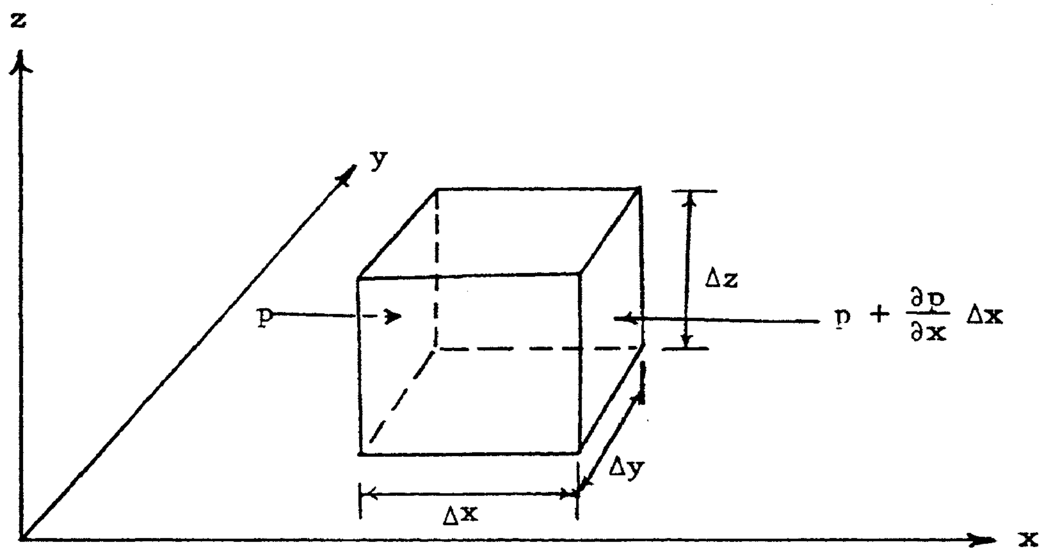

Figure (E-1): Small Volume of Medium in which Pressure Increases from Left Face to Right Face by the Amount $\frac{\partial p}{\partial x}$.

Summing the forces in the direction of propogation the result is

$$p\Delta y\Delta z \; - \; p\Delta y\Delta z - \frac{\partial p}{\partial x}\Delta x\Delta y\Delta z \; = \; \Sigma F_x \tag{E-3}$$

Equating the sum of the forces to the inertial force in the 'X' direction, yields

$$-\frac{\partial p}{\partial x}\Delta x\Delta y\Delta z = \rho' \frac{\partial u}{\partial t}\Delta x\Delta y\Delta z \tag{E-4}$$

where ρ' is the space average of the instantaneous density of the medium in the box $(\Delta x, \Delta y, \Delta z)$ and u is the average velocity of the medium in the box. If the change in the density of the medium is small enough, the instantaneous density is equal to the average density of the medium. In the limit as $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$, the equation becomes

$$-\frac{\partial p}{\partial x} = \rho_o \frac{\partial u}{\partial t} \tag{E-5}$$

The next step is to consider the change in the pressure, temperature and volume of the medium in the box. Assume that the change satisfies the equation

$$PV^{\gamma_1} = \text{constant} \tag{E-6}$$

and that it is consistent with assumption 3. For an ideal gas,

$$PV = RT \tag{E-7}$$

In the differential form, this equation is

$$\frac{dp}{P} = -\frac{\gamma_1 dv}{V} \; . \tag{E-8}$$

Let

$$P = P_o + p \; ; \quad V = V_o + v_1$$

where P_o and V_o represent the undisturbed pressure and volume respectively and p and v_1 represent the incremental pressure and volume respectively. Therefore,

$$\frac{p}{P_o} = -\frac{\gamma_1 v_1}{V_o} \quad \text{and } V_o = \Delta x \Delta y \Delta z \tag{E-9}$$

Differentiating both the sides of the previous equation with respect to time yields

$$\frac{1}{P_o}\frac{\partial p}{\partial t} = \frac{-\gamma_1}{V_o}\frac{\partial v_1}{\partial t} \tag{E-10}$$

Now apply the equation of continuity. Referring to figure (E-2), if in the interval of time, the particles on the left hand side of the box are displaced through x_1, the particles on the right hand side of the box move through $(x_1 + \frac{\partial x_1}{\partial x}\Delta x)$ and the change or increment in volume will be

$$v_1 = \frac{\partial x_1}{\partial x}\Delta x \Delta y \Delta z \qquad \text{or}$$

$$v_1 = V_o \frac{\partial x_1}{\partial x} \tag{E-11}$$

Differentiating the above equation with respect to time,

$$\frac{\partial v_1}{\partial t} = V_o \frac{\partial u}{\partial x} \tag{E-12}$$

where $\frac{\partial x_1}{\partial t}$ is the instantaneous velocity of the particles in the box. The combination of equations (E-10) and (E-12) gives

$$\frac{\partial p}{\partial t} = -\gamma_1 P_o \frac{\partial u}{\partial x} \tag{E-13}$$

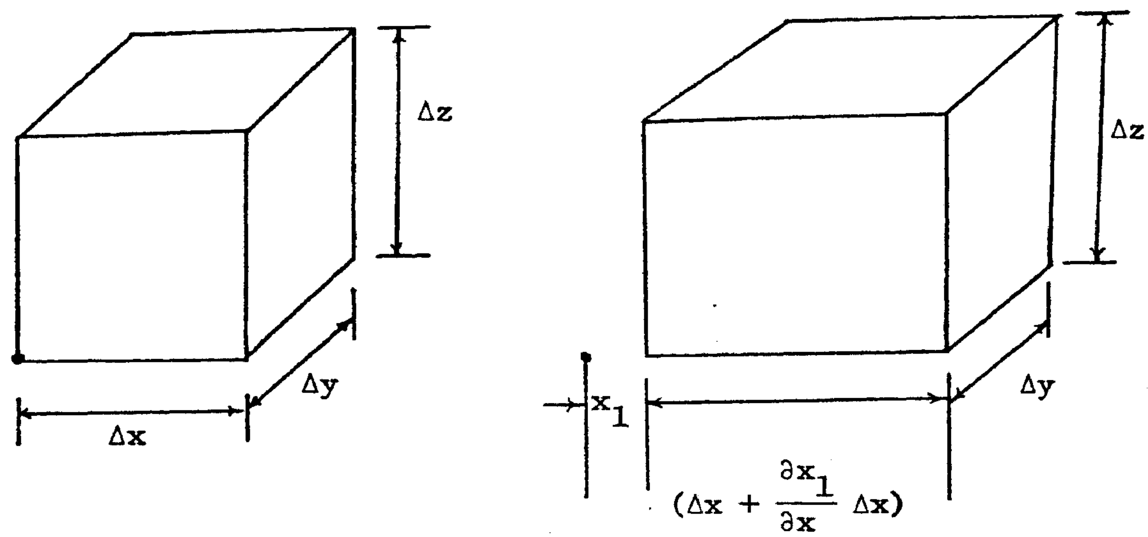

Figure E-2: Change in the Volume of the Fluid Element with Change in Position

Differentiating the above equation with respect to time yields

$$\frac{\partial^2 p}{\partial t^2} = - \gamma_1 P_o \frac{\partial^2 u}{\partial t \partial x} \quad \text{(E-14)}$$

Differentiating equation (E-4) with respect to x , the result is

$$- \frac{\partial^2 p}{\partial x^2} = \rho_o \frac{\partial^2 u}{\partial x \partial t} \quad \text{(E-15)}$$

Elimination of u from equations (E-14) and (E-15) produces,

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 p}{\partial t^2} \quad \text{(E-16)}$$

where

$$C^2 = \frac{\gamma_1 P_o}{\rho_o}$$

Similarly, eliminate p from equations E-4 and E-12 to obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 u}{\partial t^2} \quad \text{(E-17)}$$

The general solution of equation (E-16) is written as

$$p(x,t) = f_1(t - \frac{x}{c}) + f_2(t + \frac{x}{c}) \quad \text{(E-18)}$$

where f_1 and f_2 are two arbitrary functions with continuous first and second derivatives. Two important facts are observed from the general solution (E-18).

(1) The pressure at any point x in space can be separated into two components: an outgoing or forward traveling wave and a backward traveling wave, $f_1(t - \frac{x}{c})$ and $f_2(t + \frac{x}{c})$ respectively.

(2) Regardless of the shape of forward traveling wave, it is propogated without change of shape. To show this, suppose that at

$t = t_1$, the pressure at $x = 0$ is given by $p = f_1(t_1)$ and at the time $t = t_1 + t_2$, the wave has traveled through the distance t_2c. At this time, the pressure is $p = f_1(t_1 + t_2 - t_2 \frac{c}{c}) = f_1(t_1)$. This means that the pressure has propogated without change. The same argument can be made for a backward traveling wave, which travels in the -X direction.

For simplicity, it is assumed that the local particle velocity in the medium in which the plate vibrates is much slower than the plate velocity. Then, for the reaction of the particles in the medium to the motion of the plate, the steady state solutions of equations E-16 and E-17 can be used.

For equation E-15, the steady state solution is written in the form

$$p(x,t) = \sum_{\nu} P_{\nu}(x,t) = \sum_{\nu} \sqrt{2}\, R_e \quad e^{j\omega_\nu t}[P_{(+)}{}^{\nu} e^{-j\omega_\nu \frac{x}{c}} + P_{(-)}{}^{\nu} e^{j\omega_\nu \frac{x}{c}}] \quad \text{(E-19)}$$

where the (+) and (-) subscripts represent forward and backward traveling waves. 'R_e' means that only the real part of the complex functions should be considered. The square root of two is introduced so that P_+^{ν} and P_-^{ν} can represent the root mean square values of the complex functions in the time dimension. Substituting $k = \frac{\omega}{c}$, any one term of the above series solution is given by

$$p(x,t) = \sqrt{2}\ [P_{(+)} e^{jk(ct - x)} + P_{(-)} e^{jk(ct + x)}] . \quad \text{(E-20)}$$

Similarly,

$$u(x,t) = \sqrt{2}\ [U_{(+)} e^{jk(ct-x)} + U_{(-)} e^{jk(ct+x)}] \quad \text{(E-21)}$$

For calculation of pressure in the positive direction of propogation of a plane wave, only the first term in equations E-20 and E-21 is considered and for calculating the pressure in the direction opposite to the direction of propogation, only the second term in equations E-20 and E-21 is considered.

For forward traveling wave, differentiation of equations E-20 and E-21 yields

$$\frac{\partial u}{\partial t} = j\omega \;\; u(x,t)$$

$$\frac{\partial p}{\partial x} = -j\frac{\omega}{c} \;\; p(x,t) \qquad \text{(E-22)}$$

Substitution of these derivatives in equation E-5 yields

$$p(x,t) = \rho_o \; C \; u(x,t) \qquad \text{(E-23)}$$

For a backward traveling wave, the result is

$$p(x,t) = - \rho_o C \; u(x,t) \qquad \text{(E-24)}$$

Equation E-23 and E-24 are equivalent to equations 12 and 13 respectively.

APPENDIX-F

The Computer Program to Evaluate the Greatest Common Divisor of Two Polynomials

NOMENCLATURE:

AL(I)	–	α_i ($i = 1, \ldots 10$), the roots of $J_o(\alpha)$
RHOVWP	–	$\rho_3 C_3 = \rho_4 C_4$, the product of the density and the velocity of wave propogation in air
UM	–	μ, the Poisson's ratio
G	–	Modulus of Rigidity
E	–	Modulus of Elasticity
H	–	h, the thickness of the plate
R	–	the radius of the plate
GP	–	$G' = K^2 G$
D	–	$Eh^3/(2(1-\mu^2)$, a constant
B	–	$(\frac{\rho h^3}{12} + \frac{\rho D}{G'})$, a constant
P	–	$(\frac{2D}{G'h} \rho_3 C_3)$, a constant
B_1	–	$(\frac{\rho^2 h^3}{12G'})$, a constant
Q	–	$(2 \frac{\rho h^2}{12G'} \rho_3 C_3)$, a constant
A	–	$2 \rho_3 C_3$, a constant
C_1	=	$\frac{\alpha_n}{R}$
C_2	=	$\frac{\alpha_m}{R}$

(X(I),I = 1,5)	–	The coefficients of the equation 40.
(Y(I),I = 1,3)	–	The coefficients of the equation 39. to be replaced by the coefficients of the greatest common divisor.
IY	–	The number of coefficients in the polynomial representing Y(I)
IER	–	If IER = 0, no error

```
      DIMENSION X(6), Y(6), WO(6), AL(20)
      READ(1,14)(AL(I),I=1,10)
   14 FORMAT(5E14.7)
      RHOVWP = 0.565
      RHO=0.2845
      UM=0.3
      G=10.98E6
      E=28.43E6
      H=3/16.
      R=3.0
      GP=3.14159**2/12*G
      D=E*H**3/ 12.0/(1-UM*UM)
      B=(RHO*H**3/12.0)+(RH)*D/GP)
      P=(D/GP/H)*(2.0*RHOVWP)
      B1=RHO*RHO*H**3/12./GP
      Q-RHO*H*H*(2.*RHOVWP)/12./GP
      A=(2.*RHOVWP)
      DO 2 K=1,10
      DO 2 J=1,10
      IF(K-J)1,2,2
    1 C1=Al(K) /R
      C2=AL(J)/R
      X(1)=-D*C1**2*C2**2
      X(2)=A
      X(3)=RHO*H
      X(4)=Q
      X(5)=B1
      Y(1)=D*(D1**2+C2**2)
      Y(2)=P
      Y(3)=B
      IY=3
      WRITE(3,14)(Y(I),I=1,3)
      CALL PGCD (X,5,Y,IY,WO,0.00005,IER)
      WRITE(3,101)(Y(I),I=1,3),IY,IER,AL(K),AL(J)
  101 FORMAT(3E14.7,10X,2I6,4X,2F14.4,/)
    2 CONTINUE
      STOP
      END
```

IY	IER	Input Coefficients of Polynomial Y(I)			Computed Coefficients of Greatest Common Divisor Y(I)			α_n	α_m
		Y(1)	Y(2)	Y(3)	Y(1)	Y(2)	Y(3)		
1	0	0.6913E-05	0.1145E-01	0.6969E-03	0.2894E-12	0.1420E-05	0.5285E-01	2.4048252	5.5200777
3	0	0.1538E-06	0.1145E-01	0.6969E-03	0.1538E-06	0.1145E-01	0.6969E-03	2.4048252	8.6537275
3	0	0.2761E-06	0.1145E-01	0.6969E-03	0.2761E-06	0.1145E-01	0.6969E-03	2.4048252	11.7915335
3	0	0.4361E-06	0.1145E-01	0.6969E-03	0.4361E-06	0.1145E-01	0.6969E-03	2.4048252	14.9309168
3	0	0.6337E-06	0.1145E-01	0.6969E-03	0.6337E-06	0.1145E-01	0.6969E-03	2.4048252	18.0710602
3	0	0.8689E-06	0.1145E-01	0.6969E-03	0.8689E-06	0.1145E-01	0.6969E-03	2.4048252	21.2116241
3	0	0.1141E-07	0.1145E-01	0.6969E-03	0.1141E-07	0.1145E-01	0.6969E-03	2.4048252	24.3524628
3	0	0.1452E-07	0.1145E-01	0.6969E-03	0.1452E-07	0.1145E-01	0.6969E-03	2.4048252	27.4934692
3	0	0.1800E-07	0.1145E-01	0.6969E-03	0.1800E-07	0.1145E-01	0.6969E-03	2.4048252	30.6345978
3	0	0.2009E-06	0.1145E-01	0.6969E-03	0.2009E-06	0.1145E-01	0.6969E-03	5.5200777	8.6537275
3	0	0.3232E-06	0.1145E-01	0.6969E-03	0.3232E-06	0.1145E-01	0.6969E-03	5.5200777	11.7915335
3	0	0.4832E-06	0.1145E-01	0.6969E-03	0.4832E-06	0.1145E-01	0.6969E-03	5.5200777	14.9309168
3	0	0.6808E-06	0.1145E-01	0.6969E-03	0.6808E-06	0.1145E-01	0.6969E-03	5.5200777	18.0710602
3	0	0.9160E-06	0.1145E-01	0.6969E-03	0.9106E-06	0.1145E-01	0.6969E-03	5.5200777	21.2116241
3	0	0.1188E-07	0.1145E-01	0.6969E-03	0.1188E-07	0.1145E-01	0.6969E-03	5.5200777	24.3524628

IY	IER	Input Coefficients of Polynomial Y(I)			Computed Coefficients of Greatest Common Divisor Y(I)			α_n	α_m
		Y(1)	Y(2)	Y(3)	Y(1)	Y(2)	Y(3)		
3	0	0.1499E-07	0.1145E-01	0.6969E-03	0.1499E-07	0.1145E-01	0.6969E-03	5.5200777	27.4934692
3	0	0.1847E-07	0.1145E-01	0.6969E-03	0.1847E-07	0.1145E-01	0.6969E-03	5.5200777	30.6345978
3	0	0.4079E-06	0.1145E-01	0.6969E-03	0.4079E-06	0.1145E-01	0.6969E-03	8.6537275	11.7915335
3	0	0.5678E-06	0.1145E-01	0.6969E-03	0.5678E-06	0.1145E-01	0.6969E-03	8.6537275	14.9309168
3	0	0.7655E-06	0.1145E-01	0.6969E-03	0.7655E-06	0.1145E-01	0.6969E-03	8.6537275	18.0710602
3	0	0.1000E-07	0.1145E-01	0.6969E-03	0.1000E-07	0.1145E-01	0.6969E-03	8.6537275	21.2116241
3	0	0.1273E-07	0.1145E-01	0.6969E-03	0.1273E-07	0.1145E-01	0.6969E-03	8.6537275	24.3524628
3	0	0.1584E-07	0.1145E-01	0.6969E-03	0.1584E-07	0.1145E-01	0.6969E-03	8.6537275	27.4934692
3	0	0.1932E-07	0.1145E-01	0.6969E-03	0.1932E-07	0.1145E-01	0.6969E-03	8.6537275	30.6345978
3	0	0.6902E-06	0.1145E-01	0.6969E-03	0.6902E-06	0.1145E-01	0.6969E-03	11.7915335	14.9309168
3	0	0.8878E-06	0.1145E-01	0.6969E-03	0.8878E-06	0.1145E-01	0.6969E-03	11.7915335	18.0710602
3	0	0.1123E-07	0.1145E-01	0.6969E-03	0.1123E-07	0.1145E-01	0.6969E-03	11.7915335	21.2116241
3	0	0.1395E-07	0.1145E-01	0.6969E-03	0.1395E-07	0.1145E-01	0.6969E-03	11.7915335	24.3524628
3	0	0.1706E-07	0.1145E-01	0.6969E-03	0.1706E-07	0.1145E-01	0.6969E-03	11.7915335	27.4934692
3	0	0.2054E-07	0.1145E-01	0.6969E-03	0.2054E-07	0.1145E-01	0.6969E-03	11.7915335	30.6345978

IY	IER	Input Coefficients of Polynomial Y(I)			Computed Coefficients of Greatest Common Divisor Y(I)			α_n	α_m
		Y(1)	Y(2)	Y(3)	Y(1)	Y(2)	Y(3)		
3	0	0.1047E-07	0.1145E-01	0.6969E-03	0.1047E-07	0.1145E-01	0.6969E-01	14.9309168	18.0710602
3	0	0.1283E-07	0.1145E-01	0.6969E-03	0.1283E-07	0.1145E-01	0.6969E-01	14.930916	21.2116241
3	0	0.1555E-07	0.1145E-01	0.6969E-03	0.1555E-07	0.1145E-01	0.6969E-01	14.930916	24.3524628
3	0	0.1866E-07	0.1145E-01	0.6969E-03	0.1866E-07	0.1145E-01	0.6969E-03	14.930916	27.4934692
3	0	0.2214E-07	0.1145E-01	0.6969E-03	0.2214E-07	0.1145E-01	0.6969E-03	14.9309168	30.6345978
3	0	0.1480E-07	0.1145E-01	0.6969E-03	0.1480E-07	0.1145E-01	0.6969E-03	18.0710602	21.2116241
3	0	0.1753E-07	0.1145E-01	0.6969E-03	0.1753E-07	0.1145E-01	0.6969E-03	18.0710602	24.3524628
3	0	0.2064E-07	0.1145E-01	0.6969E-03	0.2064E-07	0.1145E-01	0.6969E-03	18.0710602	27.4934692
3	0	0.2412E-07	0.1145E-01	0.6969E-03	0.2412E-07	0.1145E-01	0.6969E-03	18.0710602	30.6345978
3	0	0.1988E-07	0.1145E-01	0.6969E-03	0.1988E-07	0.1145E-01	0.6969E-03	21.2116241	24.3524628
3	0	0.2299E-07	0.1145E-01	0.6969E-03	0.2299E-07	0.1145E-01	0.6969E-03	21.2116241	27.4934692
3	0	0.2647E-07	0.1145E-01	0.6969E-03	0.2647E-07	0.1145E-01	0.6969E-03	21.2116241	30.6345978
3	0	0.2572E-07	0.1145E-01	0.6969E-03	0.2572E-07	0.1145E-01	0.6969E-03	24.3524628	27.4934692
3	0	0.2920E-07	0.1145E-01	0.6969E-03	0.2920E-07	0.1145E-01	0.6969E-03	24.3524628	30.6345978
3	0	0.3230E-07	0.1145E-01	0.6969E-03	0.3230E-07	0.1145E-01	0.6969E-03	27.4934692	30.6345978

APPENDIX-G

The Evaluation of Integrals in Equations 55, 56, 58 and 59

The integral to be evaluated is

$$I = \int_o^R rf(r)\ J_o(r\theta_1)dr \tag{G-1}$$

where

$$f(r) = \omega_{max}\ [1 - (\frac{r}{R})^2]^2 \tag{G-1a}$$

and θ_1 is any constant. With the substitution of f(r) in the integral, it becomes

$$I = \omega_{max} \int_o^R r\ (1 - \frac{2\ r^2}{R^2} + \frac{r^4}{R^4})\ J_o(r\theta_1)dr$$

or

$$I = \omega_{max} \int_o^R r\ J_o(r\theta_1)dr + \omega_{max} \int_o^R (\frac{-2}{R^2})\ r^3 J_0(r\theta_1)dr$$

$$+ \omega_{max} \int_o^R (\frac{1}{R^4})\ r^5\ J_o(r\theta_1)dr \tag{G-2}$$

Let

$$I_1 = \int_o^R r\ J_o(r\theta_1)dr$$

$$I_2 = \int_o^R r^3\ J_o(r\theta_1)dr$$

$$I_3 = \int_o^R r^5\ J_o(r\theta_1)dr$$

Then,

$$I = \omega_{max}\ [I_1 - \frac{2\ I_2}{R^2} + \frac{I_3}{R^4}] \tag{G-3}$$

I_1 is obtained by use of the identity

$$\frac{d}{dx}\,[x\ J_1(x)] = x\ J_o(x)$$

or

$$\int x\ J_o(x)dx = x\ J_1(x)$$

To transform the previous equation to cylindrical coordinates, let

$$x = r\theta_1 \qquad \text{So} \quad dx = dr\ \theta_1 .$$

The equation then becomes

$$\int r\ J_o(r\theta_1)dr = \frac{r}{\theta_1}\ J_1(r\theta_1). \tag{G-4}$$

Therefore,

$$I_1 = \frac{r}{\theta_1}\ J_1(r\theta_1)$$

I_2 and I_3 are obtained from integration by parts with the use of the identity

$$\int udv = uv - \int vdu \ .$$

For I_2 , let $u = r^2$ and $dv = rJ_o(r\theta_1)dr$. Then, $du = 2rdr$ and $v = \frac{r}{\theta_1}\ J_1(r\theta_1)dr$. With these substitutions,

$$I_2 = \frac{r^3}{\theta_1}\ J_1(r\theta_1) - \frac{2}{\theta_1} \int r^2\ J_1(r\theta_1)dr$$

The last integral in this equation is also integrated by parts.

Let $u = r^2$ and $dv = J_1(r\theta_1)dr$

$$du = 2rdr \text{ and } v = \frac{-J_o(r\theta_1)}{\theta_1}$$

With these substitutions, the integral reads

$$I_2 = \frac{r^3}{\theta_1} J_1(r\theta_1) + \frac{2r^2}{\theta_1^2} J_o(r\theta_1) - \frac{4}{\theta_1^2} \int r J_o(r\theta_1)dr$$

or

$$I_2 = \frac{r^3}{\theta_1} J_1(r\theta_1) + \frac{2r^2}{\theta_1^2} J_o(r\theta_1) - \frac{4r}{\theta_1^3} J_1(r\theta_1) \qquad \text{(G-5)}$$

For I_3 , let $u = r^4$ and $dv = r J_o(r\theta_1)dr$. Then, $du = 4r^3 dr$ and $v = \frac{r}{\theta_1} J_1(r\theta_1)$.

In this case,

$$I_3 = \frac{r^5}{\theta_1} J_1(r\theta_1) - \frac{4}{\theta_1} \int r^4 J_1(r\theta_1)dr.$$

To evaluate the last integral in the previous equation, let $u = r^4$ and $dv = J_1(r\theta_1)dr$, so that $du = 4r^3 dr$ and $v = -\frac{J_o(r\theta_1)}{\theta_1}$

Then,

$$I_3 = \frac{r^5}{\theta_1} J_1(r\theta_1) + \frac{4r^4}{\theta_1^2} J_o(r\theta_1) - \frac{16}{\theta_1^2} \int r^3 J_o(r\theta_1)dr$$

or

$$I_3 = \frac{r^5}{\theta_1} J_1(r\theta_1) - \frac{4r^4}{\theta_1^2} J_o(r\theta_1) - \frac{16 I_2}{\theta_1^2} . \qquad \text{(G-6)}$$

It is recalled that

$$I = \omega_{max} [I_1 - \frac{2I_2}{R^2} + \frac{I_3}{R^4}].$$

Substitution of I_3 yields

$$I = \omega_{max} [I_1 - I_2 (\frac{2}{R^2} + \frac{16}{\theta_1^2 R^4}) + \frac{r^5}{\theta_1 R^4} J_1(r\theta_1) + \frac{4r^4}{\theta_1^2 R^4} J_o(r\theta_1)] \qquad \text{(G-7)}$$

Then, substitution of the relations for I_1 and I_2 , gives

$$I = [\frac{r}{\theta_1} \; J_1(r\theta_1) + \frac{r^5}{\theta_1 R^4} J_1(r\theta_1) + \frac{4r^4}{\theta_1{}^2 R^4} \; J_o(r\theta_1)$$

$$- (\frac{2}{R^2} + \frac{16}{\theta_1{}^2 R^4}) \; [\frac{r^3}{\theta_1} J_1(r\theta_1) - \frac{4r}{\theta_1{}^3} J_1(r\theta_1) + \frac{2r^2}{\theta_1{}^2} J_o(r\theta_1)]] \; \omega_{max}$$

When evaluated from zero to R , the result is

$$I = [J_1(r\theta_1) \; [\frac{2\not{R}}{\not{\theta}_1} - \frac{2\not{R}}{\not{\theta}_1} - \frac{8}{R\theta_1{}^3} + \frac{64}{R^3\theta_1{}^5}]$$

$$+ J_o(R\theta_1) \; [\frac{32}{R^2\theta_1{}^4}] \;] \; \omega_{max} \qquad \text{(G-8)}$$

APPENDIX-H

The Computer Program to Evaluate the Values of $\omega(r,t)/\omega_{max}$

NOMENCLATURE:

ALN(I)	–	α_n
ALM(I)	–	α_m
JX	–	A number to indicate the solution
N	–	The number of terms in the solution
RHOVMP	–	$\rho_3 C_3 = \rho_4 C_4$, the product of the density and the velocity of wave propogation in air
UM	–	μ, the Poisson's ratio
G	–	Modulus of Rigidity
E	–	Modulus of Elasticity
H	–	h, the thickness of the plate
R	–	the radius of the plate
GP	–	$G' = k^2 G$
D	–	$Eh^3/12(1-\mu^2)$, a constant
B	–	$(\frac{\rho h^3}{12} + \frac{\rho D}{G'})$, a constant
P	–	$(\frac{2\ D}{G'h}\ \rho_3 C_3)$, a constant
B_1	–	$(\frac{\rho^2 h^3}{12G'})$, a constant
Q	–	$(2\ \frac{\rho h^2}{12G'}\ \rho_3 C_3)$, a constant
A	–	$2\ \rho_3 C_3$, a constant
R_1	–	$-P/2B$; a constant
AR	–	The radial distance (r)
T	–	An instant in time.

$\omega(NN)$ - The value of $\omega(r,t)/\omega_{max}$

BESJ - The package subroutine to evaluate the Bessel function of the first kind when the argument is non-zero and positive.

z_1 - (rC_1)

z_2 - (rC_2)

B_2, B_3, B_4, B_5 - The arbitrary constants b_2, b_3, b_4, and b_5 in the solution.

OM - ω_ℓ; the circular frequency

```
C     TO COMPUTE DEFLECTION CURVE.
      DIMENSION W(50),ALN(50),ALM(50)
      READ(1,20)JX,N
   20 FORMAT(2I3)
      READ(1,14)(ALN(I),ALM(I),I=1,N)
   14 FORMAT(6F12.7)
      RHOVWP=0.565
      RHO=0.2845
      UM=0.3
      G=10.98E6
      E=28.43E6
      H=3/16.
      R=3.0
      GP=3.14159**2/12*G
      D=E*H**3/ 12.0/(1-UM*UM)
      B =(RHO*H**3/12.)+(RHO*D/GP)
      P=(D.GP/H)*(2.0*RHOVWP)
      B1=RHO*RHO*H**3/12./GP
      Q=RHO*H*H*(2.*RHOVWP)/12./GP
      A =(2.*RHOVWP)
      R1=-P/(2*B)
    3 T=0.0
      DO 11 MM=1,3
      WRITE(3,15)
   15 FORMAT(/,1X,'TIME', 8X,'RAD.DISTANCE',5X,'DEFLECTION')
      AR=0.0
      DO 12 NN=1,31
      W(NN)=0.0
      DO 41 J=1,N
   40 BETAJ=ALN(J)
      BETAK=ALM(J)
      RADD=D*(BETAJ**2+BETAK**2)/B/R/R-R1**2
      OM=SQRT(RADD)
      C1=BETAJ/R
      C2=BETAK/R
      Z3=BETAJ
      Z4=BETAK
      CALL BESJ (Z3,1,BJ1Z3,0.00005,IER)
      CALL BESJ (Z4,1,BJ1Z4,0.00005,IER)
```

```
      B2=2./BJ1Z3*(-8./Z3**3+64./Z3**5)
      B3=2./BJ1Z4*(-8./Z4**3+64./Z4**5)
      B4=P/B/OM/BJ1Z3*(-8./Z3**3+64./Z3**5)
      B5=P/B/OM/BJ1Z4*(-8./Z4**3+64./Z4**5)
      Z1=AR*C1
      Z2=AR*C2
      IF(AR)51,52,51
   51 CALL BESJ (Z1,0,BJ0Z1,0.00005,IER)
      CALL BESJ (Z2,0,BJ0Z2,0.00005,IER)
      GO TO 41
   52 BJ0Z1=1.0
      BJ0Z2=1.0
   41 W(NN)=W(NN)+EXP(R1*T)*COS(OM*T)*(B2*BJ0Z1+B3*BJ0Z2)+EXP(R1*T)*SIN(M*T)*B
     1OM*T)*(B4*BJ0Z1+B5*BJ0Z2)
   42 WRITE(3,17)T,AR,W(NN)
   17 FORMAT(F5.2,10X,F5.2,8X,E14.7)
   12 AR=AR+0.1
   11 T=T+0.05
      STOP
      END
```

Solution A_1

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.1013946E 01
0.0	0.10	0.1011008E 01
0.0	0.20	0.1002315E 01
0.0	0.30	0.9882092E 00
0.0	0.40	0.9692109E 00
0.0	0.50	0.9459502E 00
0.0	0.60	0.9190823E 00
0.0	0.70	0.8892119E 00
0.0	0.80	0.8568207E 00
0.0	0.90	0.8222250E 00
0.0	1.00	0.7855611E 00
0.0	1.10	0.7468035E 00
0.0	1.20	0.7058167E 00
0.0	1.30	0.6624276E 00
0.0	1.40	0.6165123E 00
0.0	1.50	0.5680817E 00
0.0	1.60	0.5173540E 00
0.0	1.70	0.4647977E 00
0.0	1.80	0.4111453E 00
0.0	1.90	0.3573619E 00
0.0	2.00	0.3045814E 00
0.0	2.10	0.2540081E 00
0.0	2.20	0.2068061E 00
0.0	2.30	0.1639743E 00
0.0	2.40	0.1262383E 00
0.0	2.50	0.9396768E-01
0.0	2.60	0.6712538E-01
0.0	2.70	0.4526157E-01
0.0	2.80	0.2755675E-01
0.0	2.90	0.1289926E-01
0.0	3.00	0.1328239E-05

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	-0.3485333E-01
0.05	0.10	-0.3328106E-01
0.05	0.20	-0.2867746E-01
0.05	0.30	-0.2137007E-01
0.05	0.40	-0.1186707E-01
0.05	0.50	-0.8057354E-03
0.05	0.60	0.1111097E-01
0.05	0.70	0.2318124E-01
0.05	0.80	0.3476746E-01
0.05	0.90	0.4534714E-01
0.05	1.00	0.5454805E-01
0.05	1.10	0.6216126E-01
0.05	1.20	0.6813538E-01
0.05	1.30	0.7255149E-01
0.05	1.40	0.7558352E-01
0.05	1.50	0.7745230E-01
0.05	1.60	0.7837814E-01
0.05	1.70	0.7853913E-01
0.05	1.80	0.7804257E-01
0.05	1.90	0.7691020E-01
0.05	2.00	0.7508063E-01
0.05	2.10	0.7242799E-01
0.05	2.20	0.6879175E-01
0.05	2.30	0.6401378E-01
0.05	2.40	0.5797743E-01
0.05	2.50	0.5064061E-01
0.05	2.60	0.4205962E-01
0.05	2.70	0.3239890E-01
0.05	2.80	0.2192598E-01
0.05	2.90	0.1099226E-01
0.05	3.00	0.1258163E-05

Solution A_1

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3188004E 00
0.10	0.10	-0.3189566E 00
0.10	0.20	-0.3193830E 00
0.10	0.30	-0.3199555E 00
0.10	0.40	-0.3204779E 00
0.10	0.50	-0.3206963E 00
0.10	0.60	-0.3203175E 00
0.10	0.70	-0.3190316E 00
0.10	0.80	-0.3165342E 00
0.10	0.90	-0.3125493E 00
0.10	1.00	-0.3068499E 00
0.10	1.10	-0.2992740E 00
0.10	1.20	-0.2897380E 00
0.10	1.30	-0.2782425E 00
0.10	1.40	-0.2648730E 00
0.10	1.50	-0.2497954E 00
0.10	1.60	-0.2332455E 00
0.10	1.70	-0.2155132E 00
0.10	1.80	-0.1969259E 00
0.10	1.90	-0.1778274E 00
0.10	2.00	-0.1585579E 00
0.10	2.10	-0.1394339E 00
0.10	2.20	-0.1207329E 00
0.10	2.30	-0.1026784E 00
0.10	2.40	-0.8543086E-01
0.10	2.50	-0.6908518E-01
0.10	2.60	-0.5367005E-01
0.10	2.70	-0.3915325E-01
0.10	2.80	-0.2545290E-01
0.10	2.90	-0.1244934E-01
0.10	3.00	-0.1374605E-05

Solution A_2

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.1005596E 01
0.0	0.10	0.1002778E 01
0.0	0.20	0.9945222E 00
0.0	0.30	0.9813704E 00
0.0	0.40	0.9640342E 00
0.0	0.50	0.9431870E 00
0.0	0.60	0.9192733E 00
0.0	0.70	0.8924179E 00
0.0	0.80	0.8624504E 00
0.0	0.90	0.8290463E 00
0.0	1.00	0.7919284E 00
0.0	1.10	0.7510568E 00
0.0	1.20	0.7067437E 00
0.0	1.30	0.6596442E 00
0.0	1.40	0.6106338E 00
0.0	1.50	0.5606099E 00
0.0	1.60	0.5102943E 00
0.0	1.70	0.4601067E 00
0.0	1.80	0.4101666E 00
0.0	1.90	0.3604211E 00
0.0	2.00	0.3108684E 00
0.0	2.10	0.2617894E 00
0.0	2.20	0.2139121E 00
0.0	2.30	0.1684207E 00
0.0	2.40	0.1267977E 00
0.0	2.50	0.9052116E-01
0.0	2.60	0.6068970E-01
0.0	2.70	0.3768521E-01
0.0	2.80	0.2097713E-01
0.0	2.90	0.9124607E-02
0.0	3.00	0.8796477E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	-0.2965109E-01
0.05	0.10	-0.2815283E-01
0.05	0.20	-0.2382190E-01
0.05	0.30	-0.1710873E-01
0.05	0.40	-0.8641366E-02
0.05	0.50	0.9160286E-03
0.05	0.60	0.1099196E-01
0.05	0.70	0.2118350E-01
0.05	0.80	0.3125950E-01
0.05	0.90	0.4109669E-01
0.05	1.00	0.5058048E-01
0.05	1.10	0.5951093E-01
0.05	1.20	0.6755775E-01
0.05	1.30	0.7428581E-01
0.05	1.40	0.7924640E-01
0.05	1.50	0.8210808E-01
0.05	1.60	0.8277702E-01
0.05	1.70	0.8146209E-01
0.05	1.80	0.7865232E-01
0.05	1.90	0.7500386E-01
0.05	2.00	0.7116300E-01
0.05	2.10	0.6757927E-01
0.05	2.20	0.6436384E-01
0.05	2.30	0.6124309E-01
0.05	2.40	0.5762882E-01
0.05	2.50	0.5278819E-01
0.05	2.60	0.4606980E-01
0.05	2.70	0.3711985E-01
0.05	2.80	0.2602585E-01
0.05	2.90	0.1334430E-01
0.05	3.00	0.1537686E-05

Solution A_2

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3216150E 00
0.10	0.10	-0.3217312E 00
0.10	0.20	-0.3220100E 00
0.10	0.30	-0.3222610E 00
0.10	0.40	-0.3222231E 00
0.10	0.50	-0.3216278E 00
0.10	0.60	-0.3202531E 00
0.10	0.70	-0.3179507E 00
0.10	0.80	-0.3146362E 00
0.10	0.90	-0.3102496E 00
0.10	1.00	-0.3047032E 00
0.10	1.10	-0.2978400E 00
0.10	1.20	-0.2894255E 00
0.10	1.30	-0.2791808E 00
0.10	1.40	-0.2668547E 00
0.10	1.50	-0.2523143E 00
0.10	1.60	-0.2356255E 00
0.10	1.70	-0.2170946E 00
0.10	1.80	-0.1972557E 00
0.10	1.90	-0.1767959E 00
0.10	2.00	-0.1564382E 00
0.10	2.10	-0.1368105E 00
0.10	2.20	-0.1183372E 00
0.10	2.30	-0.1011792E 00
0.10	2.40	-0.8524221E-01
0.10	2.50	-0.7024705E-01
0.10	2.60	-0.5583973E-01
0.10	2.70	-0.4170749E-01
0.10	2.80	-0.2767111E-01
0.10	2.90	-0.1372190E-01
0.10	3.00	-0.1525838E-05

Solution A_3

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.1002851E 01
0.0	0.10	0.1000115E 01
0.0	0.20	0.9921951E 00
0.0	0.30	0.9797950E 00
0.0	0.40	0.9636572E 00
0.0	0.50	0.9441895E 00
0.0	0.60	0.9213133E 00
0.0	0.70	0.8946278E 00
0.0	0.80	0.8637677E 00
0.0	0.90	0.8287435E 00
0.0	1.00	0.7900481E 00
0.0	1.10	0.7484759E 00
0.0	1.20	0.7047752E 00
0.0	1.30	0.6593573E 00
0.0	1.40	0.6122542E 00
0.0	1.50	0.5633478E 00
0.0	1.60	0.5127259E 00
0.0	1.70	0.4609250E 00
0.0	1.80	0.4088987E 00
0.0	1.90	0.3577093E 00
0.0	2.00	0.3081450E 00
0.0	2.10	0.2605105E 00
0.0	2.20	0.2147666E 00
0.0	2.30	0.1709511E 00
0.0	2.40	0.1296459E 00
0.0	2.50	0.9216595E-01
0.0	2.60	0.6027077E-01
0.0	2.70	0.3545991E-01
0.0	2.80	0.1816446E-01
0.0	2.90	0.7230077E-02
0.0	3.00	0.5974738E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	-0.3114090E-01
0.05	0.10	-0.2959770E-01
0.05	0.20	-0.2508497E-01
0.05	0.30	-0.1796381E-01
0.05	0.40	-0.8845985E-02
0.05	0.50	0.1460170E-02
0.05	0.60	0.1209928E-01
0.05	0.70	0.2238302E-01
0.05	0.80	0.3197458E-01
0.05	0.90	0.4093233E-01
0.05	1.00	0.4955992E-01
0.05	1.10	0.5811005E-01
0.05	1.20	0.6648928E-01
0.05	1.30	0.7413006E-01
0.05	1.40	0.8012587E-01
0.05	1.50	0.8359414E-01
0.05	1.60	0.8409679E-01
0.05	1.70	0.8190620E-01
0.05	1.80	0.7796407E-01
0.05	1.90	0.7353193E-01
0.05	2.00	0.6968474E-01
0.05	2.10	0.6688505E-01
0.05	2.20	0.6482762E-01
0.05	2.30	0.6261653E-01
0.05	2.40	0.5917474E-01
0.05	2.50	0.5368095E-01
0.05	2.60	0.4584241E-01
0.05	2.70	0.3591200E-01
0.05	2.80	0.2449919E-01
0.05	2.90	0.1231598E-01
0.05	3.00	0.1384527E-05

Solution A_3

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3220273E 00
0.10	0.10	-0.3221311E 00
0.10	0.20	-0.3223596E 00
0.10	0.30	-0.3224976E 00
0.10	0.40	-0.3222798E 00
0.10	0.50	-0.3214771E 00
0.10	0.60	-0.3199465E 00
0.10	0.70	-0.3176185E 00
0.10	0.80	-0.3144382E 00
0.10	0.90	-0.3102950E 00
0.10	1.00	-0.3049858E 00
0.10	1.10	-0.2982277E 00
0.10	1.20	-0.2897213E 00
0.10	1.30	-0.2792238E 00
0.10	1.40	-0.2666112E 00
0.10	1.50	-0.2519029E 00
0.10	1.60	-0.2352601E 00
0.10	1.70	-0.2169716E 00
0.10	1.80	-0.1974462E 00
0.10	1.90	-0.1772033E 00
0.10	2.00	-0.1568474E 00
0.10	2.10	-0.1370026E 00
0.10	2.20	-0.1182087E 00
0.10	2.30	-0.1007990E 00
0.10	2.40	-0.8481419E-01
0.10	2.50	-0.6999987E-01
0.10	2.60	-0.5590268E-01
0.10	2.70	-0.4204187E-01
0.10	2.80	-0.2809375E-01
0.10	2.90	-0.1400658E-01
0.10	3.00	-0.1568238E-05

Solution A_4

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.1001672E 01
0.0	0.10	0.9989998E 00
0.0	0.20	0.9913529E 00
0.0	0.30	0.9795404E 00
0.0	0.40	0.9641516E 00
0.0	0.50	0.9451494E 00
0.0	0.60	0.9220544E 00
0.0	0.70	0.8945349E 00
0.0	0.80	0.8628368E 00
0.0	0.90	0.8276789E 00
0.0	1.00	0.7897384E 00
0.0	1.10	0.7492371E 00
0.0	1.20	0.7060236E 00
0.0	1.30	0.6600508E 00
0.0	1.40	0.6117722E 00
0.0	1.50	0.5620561E 00
0.0	1.60	0.5117045E 00
0.0	1.70	0.4610578E 00
0.0	1.80	0.4100989E 00
0.0	1.90	0.3589685E 00
0.0	2.00	0.3083864E 00
0.0	2.10	0.2595187E 00
0.0	2.20	0.2133830E 00
0.0	2.30	0.1703554E 00
0.0	2.40	0.1303419E 00
0.0	2.50	0.9355122E-01
0.0	2.60	0.6116068E-01
0.0	2.70	0.3510841E-01
0.0	2.80	0.1689449E-01
0.0	2.90	0.6136768E-02
0.0	3.00	0.4504476E-06
TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	-0.3147603E-01
0.05	0.10	-0.2991505E-01
0.05	0.20	-0.2532455E-01
0.05	0.30	-0.1803621E-01
0.05	0.40	-0.8705337E-02
0.05	0.50	0.1733237E-02
0.05	0.60	0.1231009E-01
0.05	0.70	0.2235660E-01
0.05	0.80	0.3170976E-01
0.05	0.90	0.4062950E-01
0.05	1.00	0.4947183E-01
0.05	1.10	0.5832660E-01
0.05	1.20	0.6684434E-01
0.05	1.30	0.7432729E-01
0.05	1.40	0.7998872E-01
0.05	1.50	0.8322662E-01
0.05	1.60	0.8380622E-01
0.05	1.70	0.8194387E-01
0.05	1.80	0.7830542E-01
0.05	1.90	0.7389009E-01
0.05	2.00	0.6975335E-01
0.05	2.10	0.6660289E-01
0.05	2.20	0.6443399E-01
0.05	2.30	0.6244706E-01
0.05	2.40	0.5937276E-01
0.05	2.50	0.5407501E-01
0.05	2.60	0.4609555E-01
0.05	2.70	0.3581201E-01
0.05	2.80	0.2413792E-01
0.05	2.90	0.1200498E-01
0.05	3.00	0.1342702E-05

Solution A_4

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.10	0.0	-0.3216986E 00
0.10	0.10	-0.3218198E 00
0.10	0.20	-0.3221246E 00
0.10	0.30	-0.3224265E 00
0.10	0.40	-0.3224176E 00
0.10	0.50	-0.3217448E 00
0.10	0.60	-0.3201532E 00
0.10	0.70	-0.3175926E 00
0.10	0.80	-0.3141785E 00
0.10	0.90	-0.3099980E 00
0.10	1.00	-0.3048993E 00
0.10	1.10	-0.2984400E 00
0.10	1.20	-0.2900695E 00
0.10	1.30	-0.2794172E 00
0.10	1.40	-0.2664766E 00
0.10	1.50	-0.2515424E 00
0.10	1.60	-0.2349751E 00
0.10	1.70	-0.2170086E 00
0.10	1.80	-0.1977809E 00
0.10	1.90	-0.1775545E 00
0.10	2.00	-0.1569148E 00
0.10	2.10	-0.1367258E 00
0.10	2.20	-0.1178226E 00
0.10	2.30	-0.1006327E 00
0.10	2.40	-0.8500832E-01
0.10	2.50	-0.7038629E-01
0.10	2.60	-0.5615095E-01
0.10	2.70	-0.4194380E-01
0.10	2.80	-0.2773944E-01
0.10	2.90	-0.1370156E-01
0.10	3.00	-0.1527219E-05

Solution A_5

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.9102716E 00
0.0	0.10	0.9103835E 00
0.0	0.20	0.9104837E 00
0.0	0.30	0.9099138E 00
0.0	0.40	0.9077013E 00
0.0	0.50	0.9027318E 00
0.0	0.60	0.8939253E 00
0.0	0.70	0.8804005E 00
0.0	0.80	0.8615811E 00
0.0	0.90	0.8372387E 00
0.0	1.00	0.8074630E 00
0.0	1.10	0.7725781E 00
0.0	1.20	0.7330352E 00
0.0	1.30	0.6893067E 00
0.0	1.40	0.6418207E 00
0.0	1.50	0.5909472E 00
0.0	1.60	0.5370466E 00
0.0	1.70	0.4805590E 00
0.0	1.80	0.4221179E 00
0.0	1.90	0.3626429E 00
0.0	2.00	0.3033843E 00
0.0	2.10	0.2458920E 00
0.0	2.20	0.1919058E 00
0.0	2.30	0.1431637E 00
0.0	2.40	0.1011721E 00
0.0	2.50	0.6697303E-01
0.0	2.60	0.4094760E-01
0.0	2.70	0.2270506E-01
0.0	2.80	0.1108467E-01
0.0	2.90	0.4268076E-02
0.0	3.00	0.3427194E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	-0.7248443E-01
0.05	0.10	-0.6980538E-01
0.05	0.20	-0.6200989E-01
0.05	0.30	-0.4978942E-01
0.05	0.40	-0.3419362E-01
0.05	0.50	-0.1649294E-01
0.05	0.60	0.1979597E-02
0.05	0.70	0.1998292E-01
0.05	0.80	0.3649538E-01
0.05	0.90	0.5079675E-01
0.05	1.00	0.6249791E-01
0.05	1.10	0.7151681E-01
0.05	1.20	0.7801497E-01
0.05	1.30	0.8230793E-01
0.05	1.40	0.8476979E-01
0.05	1.50	0.8575189E-01
0.05	1.60	0.8552605E-01
0.05	1.70	0.8426005E-01
0.05	1.80	0.8202535E-01
0.05	1.90	0.7882702E-01
0.05	2.00	0.7464606E-01
0.05	2.10	0.6948203E-01
0.05	2.20	0.6338328E-01
0.05	2.30	0.5646003E-01
0.05	2.40	0.4887904E-01
0.05	2.50	0.4084220E-01
0.05	2.60	0.3255771E-01
0.05	2.70	0.2421142E-01
0.05	2.80	0.1594701E-01
0.05	2.90	0.7859342E-02
0.05	3.00	0.9004430E-06

Solution A_5

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3005620E 00
0.10	0.10	-0.3012547E 00
0.10	0.20	-0.3032280E 00
0.10	0.30	-0.3061817E 00
0.10	0.40	-0.3096571E 00
0.10	0.50	-0.3130933E 00
0.10	0.60	-0.3158918E 00
0.10	0.70	-0.3174815E 00
0.10	0.80	-0.3173716E 00
0.10	0.90	-0.3151904E 00
0.10	1.00	-0.3107028E 00
0.10	1.10	-0.3038082E 00
0.10	1.20	-0.2945262E 00
0.10	1.30	-0.2829710E 00
0.10	1.40	-0.2693251E 00
0.10	1.50	-0.2538178E 00
0.10	1.60	-0.2367097E 00
0.10	1.70	-0.2182859E 00
0.10	1.80	-0.1988561E 00
0.10	1.90	-0.1787563E 00
0.10	2.00	-0.1583472E 00
0.10	2.10	-0.1380060E 00
0.10	2.20	-0.1181116E 00
0.10	2.30	-0.9901732E-01
0.10	2.40	-0.8102119E-01
0.10	2.50	-0.6433624E-01
0.10	2.60	-0.4906486E-01
0.10	2.70	-0.3518512E-01
0.10	2.80	-0.2255514E-01
0.10	2.90	-0.1093094E-01
0.10	3.00	-0.1201231E-05

Solution A_6

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.0	0.0	0.9639264E 00
0.0	0.10	0.9612756E 00
0.0	0.20	0.9535915E 00
0.0	0.30	0.9416085E 00
0.0	0.40	0.9263110E 00
0.0	0.50	0.9086647E 00
0.0	0.60	0.8893519E 00
0.0	0.70	0.8686110E 00
0.0	0.80	0.8462046E 00
0.0	0.90	0.8215294E 00
0.0	1.00	0.7938182E 00
0.0	1.10	0.7623633E 00
0.0	1.20	0.7266970E 00
0.0	1.30	0.6866711E 00
0.0	1.40	0.6424340E 00
0.0	1.50	0.5943316E 00
0.0	1.60	0.5427927E 00
0.0	1.70	0.4882573E 00
0.0	1.80	0.4311965E 00
0.0	1.90	0.3722183E 00
0.0	2.00	0.3122274E 00
0.0	2.10	0.2525629E 00
0.0	2.20	0.1950455E 00
0.0	2.30	0.1418623E 00
0.0	2.40	0.9529376E-01
0.0	2.50	0.5731901E-01
0.0	2.60	0.2918218E-01
0.0	2.70	0.1103973E-01
0.0	2.80	0.1793347E-02
0.0	2.90	-0.8393787E-03
0.0	3.00	-0.2742163E-06

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.05	0.0	-0.3740923E-01
0.05	0.10	-0.3653624E-01
0.05	0.20	-0.3382948E-01
0.05	0.30	-0.2906998E-01
0.05	0.40	-0.2202803E-01
0.05	0.50	-0.1261451E-01
0.05	0.60	-0.1010124E-02
0.05	0.70	0.1227594E-01
0.05	0.80	0.2644348E-01
0.05	0.90	0.4052735E-01
0.05	1.00	0.5357810E-01
0.05	1.10	0.6483912E-01
0.05	1.20	0.7387155E-01
0.05	1.30	0.8058488E-01
0.05	1.40	0.8517075E-01
0.05	1.50	0.8796430E-01
0.05	1.60	0.8928239E-01
0.05	1.70	0.8929259E-01
0.05	1.80	0.8796012E-01
0.05	1.90	0.8508664E-01
0.05	2.00	0.8042687E-01
0.05	2.10	0.7384282E-01
0.05	2.20	0.6543571E-01
0.05	2.30	0.5560927E-01
0.05	2.40	0.4503622E-01
0.05	2.50	0.3453118E-01
0.05	2.60	0.2486643E-01
0.05	2.70	0.1658558E-01
0.05	2.80	0.9873092E-02
0.05	2.90	0.4520498E-02
0.05	3.00	0.4971404E-06

Solution A_6

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.2782915E 00
0.10	0.10	-0.2801309E 00
0.10	0.20	-0.2853352E 00
0.10	0.30	-0.2930262E 00
0.10	0.40	-0.3019327E 00
0.10	0.50	-0.3106307E 00
0.10	0.60	-0.3177900E 00
0.10	0.70	-0.3223748E 00
0.10	0.80	-0.3237538E 00
0.10	0.90	-0.3217108E 00
0.10	1.00	-0.3163663E 00
0.10	1.10	-0.3080480E 00
0.10	1.20	-0.2971569E 00
0.10	1.30	-0.2840649E 00
0.10	1.40	-0.2690704E 00
0.10	1.50	-0.2524130E 00
0.10	1.60	-0.2343246E 00
0.10	1.70	-0.2150905E 00
0.10	1.80	-0.1950878E 00
0.10	1.90	-0.1747818E 00
0.10	2.00	-0.1546767E 00
0.10	2.10	-0.1352371E 00
0.10	2.20	-0.1168083E 00
0.10	2.30	-0.9955740E-01
0.10	2.40	-0.8346111E-01
0.10	2.50	-0.6834328E-01
0.10	2.60	-0.5394832E-01
0.10	2.70	-0.4002703E-01
0.10	2.80	-0.2641167E-01
0.10	2.90	-0.1305088E-01
0.10	3.00	-0.1457302E-05

Solution A_7

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.9319854E 00
0.0	0.10	0.9318539E 00
0.0	0.20	0.9310628E 00
0.0	0.30	0.9285688E 00
0.0	0.40	0.9230354E 00
0.0	0.50	0.9132777E 00
0.0	0.60	0.8986008E 00
0.0	0.70	0.8789331E 00
0.0	0.80	0.8547071E 00
0.0	0.90	0.8265800E 00
0.0	1.00	0.7951236E 00
0.0	1.10	0.7606217E 00
0.0	1.20	0.7230517E 00
0.0	1.30	0.6822239E 00
0.0	1.40	0.6379912E 00
0.0	1.50	0.5904119E 00
0.0	1.60	0.5397883E 00
0.0	1.70	0.4865812E 00
0.0	1.80	0.4312818E 00
0.0	1.90	0.3743439E 00
0.0	2.00	0.3162621E 00
0.0	2.10	0.2577894E 00
0.0	2.20	0.2002069E 00
0.0	2.30	0.1454767E 00
0.0	2.40	0.9614515E-01
0.0	2.50	0.5493687E-01
0.0	2.60	0.2409892E-01
0.0	2.70	0.4693806E-02
0.0	2.80	-0.3913723E-02
0.0	2.90	-0.4177775E-02
0.0	3.00	-0.6959053E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	-0.1767437E-01
0.05	0.10	-0.1835791E-01
0.05	0.20	-0.1991001E-01
0.05	0.30	-0.2101332E-01
0.05	0.40	-0.2000417E-01
0.05	0.50	-0.1546466E-01
0.05	0.60	-0.6724626E-02
0.05	0.70	0.5898375E-02
0.05	0.80	0.2119013E-01
0.05	0.90	0.3740679E-01
0.05	1.00	0.5277152E-01
0.05	1.10	0.6591511E-01
0.05	1.20	0.7612371E-01
0.05	1.30	0.8333254E-01
0.05	1.40	0.8791566E-01
0.05	1.50	0.9038603E-01
0.05	1.60	0.9113860E-01
0.05	1.70	0.9032810E-01
0.05	1.80	0.8790737E-01
0.05	1.90	0.8377320E-01
0.05	2.00	0.7793391E-01
0.05	2.10	0.7061362E-01
0.05	2.20	0.6224665E-01
0.05	2.30	0.5337605E-01
0.05	2.40	0.4451016E-01
0.05	2.50	0.3600300E-01
0.05	2.60	0.2800714E-01
0.05	2.70	0.2050643E-01
0.05	2.80	0.1339922E-01
0.05	2.90	0.6583136E-02
0.05	3.00	0.7576825E-06

Solution A_7

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.10	0.0	-0.2886260E 00
0.10	0.10	-0.2896503E 00
0.10	0.20	-0.2926244E 00
0.10	0.30	-0.2972452E 00
0.10	0.40	-0.3029925E 00
0.10	0.50	-0.3091381E 00
0.10	0.60	-0.3147975E 00
0.10	0.70	-0.3190350E 00
0.10	0.80	-0.3210028E 00
0.10	0.90	-0.3200765E 00
0.10	1.00	-0.3159438E 00
0.10	1.10	-0.3086115E 00
0.10	1.20	-0.2983363E 00
0.10	1.30	-0.2855037E 00
0.10	1.40	-0.2705078E 00
0.10	1.50	-0.2536811E 00
0.10	1.60	-0.2352966E 00
0.10	1.70	-0.2156327E 00
0.10	1.80	-0.1950601E 00
0.10	1.90	-0.1740940E 00
0.10	2.00	-0.1533712E 00
0.10	2.10	-0.1335460E 00
0.10	2.20	-0.1151382E 00
0.10	2.30	-0.9838790E-01
0.10	2.40	-0.8318555E-01
0.10	2.50	-0.6911397E-01
0.10	2.60	-0.5559301E-01
0.10	2.70	-0.4208026E-01
0.10	2.80	-0.2825820E-01
0.10	2.90	-0.1413102E-01
0.10	3.00	-0.1593739E-05

Solution A_8

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.8931431E 00
0.0	0.10	0.8934000E 00
0.0	0.20	0.8940127E 00
0.0	0.30	0.8945212E 00
0.0	0.40	0.8942003E 00
0.0	0.50	0.8921223E 00
0.0	0.60	0.8872319E 00
0.0	0.70	0.8784397E 00
0.0	0.80	0.8647137E 00
0.0	0.90	0.8451727E 00
0.0	1.00	0.8191683E 00
0.0	1.10	0.7863503E 00
0.0	1.20	0.7467139E 00
0.0	1.30	0.7006181E 00
0.0	1.40	0.6487810E 00
0.0	1.50	0.5922459E 00
0.0	1.60	0.5323272E 00
0.0	1.70	0.4705313E 00
0.0	1.80	0.4084679E 00
0.0	1.90	0.3477501E 00
0.0	2.00	0.2898939E 00
0.0	2.10	0.2362239E 00
0.0	2.20	0.1877970E 00
0.0	2.30	0.1453358E 00
0.0	2.40	0.1091913E 00
0.0	2.50	0.7933241E-01
0.0	2.60	0.5535704E-01
0.0	2.70	0.3652928E-01
0.0	2.80	0.2184407E-01
0.0	2.90	0.1010163E-01
0.0	3.00	0.1034739E-05

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	0.1453605E 00
0.05	0.10	0.1453632E 00
0.05	0.20	0.1453301E 00
0.05	0.30	0.1451416E 00
0.05	0.40	0.1446103E 00
0.05	0.50	0.1434994E 00
0.05	0.60	0.1415443E 00
0.05	0.70	0.1384788E 00
0.05	0.80	0.1340622E 00
0.05	0.90	0.1281046E 00
0.05	1.00	0.1204906E 00
0.05	1.10	0.1111956E 00
0.05	1.20	0.1002983E 00
0.05	1.30	0.8798373E-01
0.05	1.40	0.7453841E-01
0.05	1.50	0.6033859E-01
0.05	1.60	0.4583079E-01
0.05	1.70	0.3150601E-01
0.05	1.80	0.1787094E-01
0.05	1.90	0.5416218E-02
0.05	2.00	-0.5414531E-02
0.05	2.10	-0.1425400E-01
0.05	2.20	-0.2083442E-01
0.05	2.30	-0.2500628E-01
0.05	2.40	-0.2674869E-01
0.05	2.50	-0.2617095E-01
0.05	2.60	-0.2350643E-01
0.05	2.70	-0.1909849E-01
0.05	2.80	-0.1337814E-01
0.05	2.90	-0.6838027E-02
0.05	3.00	-0.7956940E-06

Solution A_8

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3098634E 00
0.10	0.10	-0.3100009E 00
0.10	0.20	-0.3103778E 00
0.10	0.30	-0.3108897E 00
0.10	0.40	-0.3113711E 00
0.10	0.50	-0.3116060E 00
0.10	0.60	-0.3113437E 00
0.10	0.70	-0.3103155E 00
0.10	0.80	-0.3082539E 00
0.10	0.90	-0.3049102E 00
0.10	1.00	-0.3000723E 00
0.10	1.10	-0.2935780E 00
0.10	1.20	-0.2853277E 00
0.10	1.30	-0.2752894E 00
0.10	1.40	-0.2635018E 00
0.10	1.50	-0.2500720E 00
0.10	1.60	-0.2351680E 00
0.10	1.70	-0.2190083E 00
0.10	1.80	-0.2018487E 00
0.10	1.90	-0.1839664E 00
0.10	2.00	-0.1656440E 00
0.10	2.10	-0.1471534E 00
0.10	2.20	-0.1287426E 00
0.10	2.30	-0.1106230E 00
0.10	2.40	-0.9296101E-01
0.10	2.50	-0.7587498E-01
0.10	2.60	-0.5943320E-01
0.10	2.70	-0.4365675E-01
0.10	2.80	-0.2852795E-01
0.10	2.90	-0.1399844E-01
0.10	3.00	-0.1551748E-05

Solution A_9

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.8514726E 00
0.0	0.10	0.8518972E 00
0.0	0.20	0.8530821E 00
0.0	0.30	0.8547593E 00
0.0	0.40	0.8564771E 00
0.0	0.50	0.8576000E 00
0.0	0.60	0.8573105E 00
0.0	0.70	0.8546329E 00
0.0	0.80	0.8484679E 00
0.0	0.90	0.8376559E 00
0.0	1.00	0.8210581E 00
0.0	1.10	0.7976568E 00
0.0	1.20	0.7666672E 00
0.0	1.30	0.7276450E 00
0.0	1.40	0.6805812E 00
0.0	1.50	0.6259676E 00
0.0	1.60	0.5648257E 00
0.0	1.70	0.4986818E 00
0.0	1.80	0.4294979E 00
0.0	1.90	0.3595472E 00
0.0	2.00	0.2912529E 00
0.0	2.10	0.2269974E 00
0.0	2.20	0.1689304E 00
0.0	2.30	0.1187773E 00
0.0	2.40	0.7768732E-01
0.0	2.50	0.4613028E-01
0.0	2.60	0.2384952E-01
0.0	2.70	0.9883802E-02
0.0	2.80	0.2660286E-02
0.0	2.90	0.1376420E-03
0.0	3.00	-0.1191247E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	0.1726855E 00
0.05	0.10	0.1725783E 00
0.05	0.20	0.1721700E 00
0.05	0.30	0.1712152E 00
0.05	0.40	0.1693469E 00
0.05	0.50	0.1661372E 00
0.05	0.60	0.1611650E 00
0.05	0.70	0.1540900E 00
0.05	0.80	0.1447152E 00
0.05	0.90	0.1330337E 00
0.05	1.00	0.1192512E 00
0.05	1.10	0.1037814E 00
0.05	1.20	0.8721393E-01
0.05	1.30	0.7026088E-01
0.05	1.40	0.5368559E-01
0.05	1.50	0.3822574E-01
0.05	1.60	0.2452013E-01
0.05	1.70	0.1304642E-01
0.05	1.80	0.4080653E-02
0.05	1.90	-0.2319687E-02
0.05	2.00	-0.6305695E-02
0.05	2.10	-0.8203849E-02
0.05	2.20	-0.8462787E-02
0.05	2.30	-0.7590715E-02
0.05	2.40	-0.6090183E-02
0.05	2.50	-0.4398856E-02
0.05	2.60	-0.2845576E-02
0.05	2.70	-0.1625881E-02
0.05	2.80	-0.7984948E-03
0.05	2.90	-0.3042046E-03
0.05	3.00	-0.3905475E-07

Solution A_9

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3273795E 00
0.10	0.10	-0.3274465E 00
0.10	0.20	-0.3275830E 00
0.10	0.30	-0.3276036E 00
0.10	0.40	-0.3272280E 00
0.10	0.50	-0.3261174E 00
0.10	0.60	-0.3239210E 00
0.10	0.70	-0.3203227E 00
0.10	0.80	-0.3150827E 00
0.10	0.90	-0.3080699E 00
0.10	1.00	-0.2992778E 00
0.10	1.10	-0.2888253E 00
0.10	1.20	-0.2769402E 00
0.10	1.30	-0.2639285E 00
0.10	1.40	-0.2501345E 00
0.10	1.50	-0.2358970E 00
0.10	1.60	-0.2215073E 00
0.10	1.70	-0.2071751E 00
0.10	1.80	-0.1930087E 00
0.10	1.90	-0.1790074E 00
0.10	2.00	-0.1650727E 00
0.10	2.10	-0.1510316E 00
0.10	2.20	-0.1366731E 00
0.10	2.30	-0.1217868E 00
0.10	2.40	-0.1062037E 00
0.10	2.50	-0.8983141E-01
0.10	2.60	-0.7267731E-01
0.10	2.70	-0.5485717E-01
0.10	2.80	-0.3659184E-01
0.10	2.90	-0.1818680E-01
0.10	3.00	-0.2036773E-05

Solution A_{10}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.8686016E 00
0.0	0.10	0.8688809E 00
0.0	0.20	0.8695535E 00
0.0	0.30	0.8701519E 00
0.0	0.40	0.8699781E 00
0.0	0.50	0.8682096E 00
0.0	0.60	0.8640040E 00
0.0	0.70	0.8565936E 00
0.0	0.80	0.8453354E 00
0.0	0.90	0.8297219E 00
0.0	1.00	0.8093528E 00
0.0	1.10	0.7838846E 00
0.0	1.20	0.7529885E 00
0.0	1.30	0.7163336E 00
0.0	1.40	0.6736209E 00
0.0	1.50	0.6246690E 00
0.0	1.60	0.5695450E 00
0.0	1.70	0.5087095E 00
0.0	1.80	0.4431479E 00
0.0	1.90	0.3744400E 00
0.0	2.00	0.3047433E 00
0.0	2.10	0.2366656E 00
0.0	2.20	0.1730392E 00
0.0	2.30	0.1166052E 00
0.0	2.40	0.6966823E-01
0.0	2.50	0.3377095E-01
0.0	2.60	0.9440139E-02
0.0	2.70	-0.3940426E-02
0.0	2.80	-0.8099109E-02
0.0	2.90	-0.5695913E-02
0.0	3.00	-0.8111454E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	0.1623606E 00
0.05	0.10	0.1623410E 00
0.05	0.20	0.1622415E 00
0.05	0.30	0.1619369E 00
0.05	0.40	0.1612089E 00
0.05	0.50	0.1597419E 00
0.05	0.60	0.1571303E 00
0.05	0.70	0.1529080E 00
0.05	0.80	0.1466033E 00
0.05	0.90	0.1378160E 00
0.05	1.00	0.1263067E 00
0.05	1.10	0.1120828E 00
0.05	1.20	0.9545898E-01
0.05	1.30	0.7707900E-01
0.05	1.40	0.5788103E-01
0.05	1.50	0.3900852E-01
0.05	1.60	0.2167543E-01
0.05	1.70	0.7002003E-02
0.05	1.80	-0.4147142E-02
0.05	1.90	-0.1129661E-01
0.05	2.00	-0.1443734E-01
0.05	2.10	-0.1403154E-01
0.05	2.20	-0.1093944E-01
0.05	2.30	-0.6281462E-02
0.05	2.40	-0.1256537E-02
0.05	2.50	0.3050952E-02
0.05	2.60	0.5839936E-02
0.05	2.70	0.6706916E-02
0.05	2.80	0.5686916E-02
0.05	2.90	0.3212075E-02
0.05	3.00	0.3780731E-06

Solution A_{10}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3224669E 00
0.10	0.10	-0.3225754E 00
0.10	0.20	-0.3228589E 00
0.10	0.30	-0.3231889E 00
0.10	0.40	-0.3233559E 00
0.10	0.50	-0.3230745E 00
0.10	0.60	-0.3220013E 00
0.10	0.70	-0.3197603E 00
0.10	0.80	-0.3159811E 00
0.10	0.90	-0.3103453E 00
0.10	1.00	-0.3026348E 00
0.10	1.10	-0.2927750E 00
0.10	1.20	-0.2808632E 00
0.10	1.30	-0.2671726E 00
0.10	1.40	-0.2521307E 00
0.10	1.50	-0.2362694E 00
0.10	1.60	-0.2201537E 00
0.10	1.70	-0.2042991E 00
0.10	1.80	-0.1890938E 00
0.10	1.90	-0.1747361E 00
0.10	2.00	-0.1612036E 00
0.10	2.10	-0.1482587E 00
0.10	2.20	-0.1354946E 00
0.10	2.30	-0.1224096E 00
0.10	2.40	-0.1085035E 00
0.10	2.50	-0.9337604E-01
0.10	2.60	-0.7680988E-01
0.10	2.70	-0.5882197E-01
0.10	2.80	-0.3967763E-01
0.10	2.90	-0.1985986E-01
0.10	3.00	-0.2235244E-05

Solution A_{11}

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.0	0.0	0.8602529E 00
0.0	0.10	0.8606509E 00
0.0	0.20	0.8617611E 00
0.0	0.30	0.8633131E 00
0.0	0.40	0.8648013E 00
0.0	0.50	0.8654464E 00
0.0	0.60	0.8641950E 00
0.0	0.70	0.8597996E 00
0.0	0.80	0.8509650E 00
0.0	0.90	0.8365431E 00
0.0	1.00	0.8157201E 00
0.0	1.10	0.7881380E 00
0.0	1.20	0.7539154E 00
0.0	1.30	0.7135502E 00
0.0	1.40	0.6677424E 00
0.0	1.50	0.6171972E 00
0.0	1.60	0.5624854E 00
0.0	1.70	0.5040185E 00
0.0	1.80	0.4421692E 00
0.0	1.90	0.3774992E 00
0.0	2.00	0.3110304E 00
0.0	2.10	0.2444469E 00
0.0	2.20	0.1801452E 00
0.0	2.30	0.1210516E 00
0.0	2.40	0.7022768E-01
0.0	2.50	0.3032443E-01
0.0	2.60	0.3004458E-02
0.0	2.70	-0.1151678E-01
0.0	2.80	-0.1467873E-01
0.0	2.90	-0.9470560E-02
0.0	3.00	-0.1259736E-05

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.05	0.0	0.1675628E 00
0.05	0.10	0.1674691E 00
0.05	0.20	0.1670970E 00
0.05	0.30	0.1661982E 00
0.05	0.40	0.1644346E 00
0.05	0.50	0.1614637E 00
0.05	0.60	0.1570112E 00
0.05	0.70	0.1509103E 00
0.05	0.80	0.1430953E 00
0.05	0.90	0.1335655E 00
0.05	1.00	0.1223390E 00
0.05	1.10	0.1094324E 00
0.05	1.20	0.9488136E-01
0.05	1.30	0.7881331E-01
0.05	1.40	0.6154399E-01
0.05	1.50	0.4366432E-01
0.05	1.60	0.2607439E-01
0.05	1.70	0.9925008E-02
0.05	1.80	-0.3537298E-02
0.05	1.90	-0.1320288E-01
0.05	2.00	-0.1835490E-01
0.05	2.10	-0.1888019E-01
0.05	2.20	-0.1536731E-01
0.05	2.30	-0.9052146E-02
0.05	2.40	-0.1605146E-02
0.05	2.50	0.5198531E-02
0.05	2.60	0.9850115E-02
0.05	2.70	0.1142786E-01
0.05	2.80	0.9786785E-02
0.05	2.90	0.5564116E-02
0.05	3.00	0.6575976E-06

Solution A_{11}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3252814E 00
0.10	0.10	-0.3253500E 00
0.10	0.20	-0.3254859E 00
0.10	0.30	-0.3254944E 00
0.10	0.40	-0.3251011E 00
0.10	0.50	-0.3240060E 00
0.10	0.60	-0.3219368E 00
0.10	0.70	-0.3186793E 00
0.10	0.80	-0.3140831E 00
0.10	0.90	-0.3080456E 00
0.10	1.00	-0.3004881E 00
0.10	1.10	-0.2913410E 00
0.10	1.20	-0.2805507E 00
0.10	1.30	-0.2681109E 00
0.10	1.40	-0.2541124E 00
0.10	1.50	-0.2387884E 00
0.10	1.60	-0.2225337E 00
0.10	1.70	-0.2058805E 00
0.10	1.80	-0.1894236E 00
0.10	1.90	-0.1737046E 00
0.10	2.00	-0.1590839E 00
0.10	2.10	-0.1456354E 00
0.10	2.20	-0.1330988E 00
0.10	2.30	-0.1209105E 00
0.10	2.40	-0.1083149E 00
0.10	2.50	-0.9453791E-01
0.10	2.60	-0.7897949E-01
0.10	2.70	-0.6137620E-01
0.10	2.80	-0.4189584E-01
0.10	2.90	-0.2113242E-01
0.10	3.00	-0.2386478E-05

Solution A_{12}

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.0	0.0	0.7477986E 00
0.0	0.10	0.7512725E 00
0.0	0.20	0.7612512E 00
0.0	0.30	0.7764639E 00
0.0	0.40	0.7949674E 00
0.0	0.50	0.8143817E 00
0.0	0.60	0.8321536E 00
0.0	0.70	0.8458214E 00
0.0	0.80	0.8532283E 00
0.0	0.90	0.8526695E 00
0.0	1.00	0.8429599E 00
0.0	1.10	0.8234314E 00
0.0	1.20	0.7938857E 00
0.0	1.30	0.7545242E 00
0.0	1.40	0.7058896E 00
0.0	1.50	0.6488332E 00
0.0	1.60	0.5845183E 00
0.0	1.70	0.5144431E 00
0.0	1.80	0.4404706E 00
0.0	1.90	0.3648282E 00
0.0	2.00	0.2900558E 00
0.0	2.10	0.2188814E 00
0.0	2.20	0.1540301E 00
0.0	2.30	0.9796667E-01
0.0	2.40	0.5262124E-01
0.0	2.50	0.1913566E-01
0.0	2.60	-0.2328254E-02
0.0	2.70	-0.1267271E-01
0.0	2.80	-0.1381179E-01
0.0	2.90	-0.8493539E-02
0.0	3.00	-0.1104644E-05

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.05	0.0	0.1350543E 00
0.05	0.10	0.1360539E 00
0.05	0.20	0.1388375E 00
0.05	0.30	0.1427958E 00
0.05	0.40	0.1470203E 00
0.05	0.50	0.1504499E 00
0.05	0.60	0.1520336E 00
0.05	0.70	0.1508916E 00
0.05	0.80	0.1464431E 00
0.05	0.90	0.1384833E 00
0.05	1.00	0.1272010E 00
0.05	1.10	0.1131369E 00
0.05	1.20	0.9709352E-01
0.05	1.30	0.8001733E-01
0.05	1.40	0.6287187E-01
0.05	1.50	0.4652539E-01
0.05	1.60	0.3166808E-01
0.05	1.70	0.1876738E-01
0.05	1.80	0.8063473E-02
0.05	1.90	-0.4028135E-03
0.05	2.00	-0.6740190E-02
0.05	2.10	-0.1114977E-01
0.05	2.20	-0.1387123E-01
0.05	2.30	-0.1514446E-01
0.05	2.40	-0.1518857E-01
0.05	2.50	-0.1419726E-01
0.05	2.60	-0.1234749E-01
0.05	2.70	-0.9813357E-02
0.05	2.80	-0.6777465E-02
0.05	2.90	-0.3437119E-02
0.05	3.00	-0.3967749E-06

Solution A_{12}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.3091411E 00
0.10	0.10	-0.3097445E 00
0.10	0.20	-0.3114280E 00
0.10	0.30	-0.3138298E 00
0.10	0.40	-0.3164072E 00
0.10	0.50	-0.3185144E 00
0.10	0.60	-0.3194953E 00
0.10	0.70	-0.3187726E 00
0.10	0.80	-0.3159201E 00
0.10	0.90	-0.3107110E 00
0.10	1.00	-0.3031307E 00
0.10	1.10	-0.2933595E 00
0.10	1.20	-0.2817284E 00
0.10	1.30	-0.2686570E 00
0.10	1.40	-0.2545867E 00
0.10	1.50	-0.2399194E 00
0.10	1.60	-0.2249715E 00
0.10	1.70	-0.2099478E 00
0.10	1.80	-0.1949389E 00
0.10	1.90	-0.1799363E 00
0.10	2.00	-0.1648621E 00
0.10	2.10	-0.1496038E 00
0.10	2.20	-0.1340517E 00
0.10	2.30	-0.1181257E 00
0.10	2.40	-0.1017940E 00
0.10	2.50	-0.8508247E-01
0.10	2.60	-0.6807202E-01
0.10	2.70	-0.5088904E-01
0.10	2.80	-0.3369407E-01
0.10	2.90	-0.1666840E-01
0.10	3.00	-0.1863400E-05

Solution A_{13}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.6806533E 00
0.0	0.10	0.6845590E 00
0.0	0.20	0.6960596E 00
0.0	0.30	0.7144734E 00
0.0	0.40	0.7385688E 00
0.0	0.50	0.7664883E 00
0.0	0.60	0.7957305E 00
0.0	0.70	0.8232595E 00
0.0	0.80	0.8457433E 00
0.0	0.90	0.8599055E 00
0.0	1.00	0.8629193E 00
0.0	1.10	0.8527604E 00
0.0	1.20	0.8284426E 00
0.0	1.30	0.7900783E 00
0.0	1.40	0.7387723E 00
0.0	1.50	0.6763839E 00
0.0	1.60	0.6052409E 00
0.0	1.70	0.5278785E 00
0.0	1.80	0.4468749E 00
0.0	1.90	0.3647935E 00
0.0	2.00	0.2842115E 00
0.0	2.10	0.2077663E 00
0.0	2.20	0.1381574E 00
0.0	2.30	0.7802260E-01
0.0	2.40	0.2969188E-01
0.0	2.50	-0.5156517E-02
0.0	2.60	-0.2586310E-01
0.0	2.70	-0.3306919E-01
0.0	2.80	-0.2881271E-01
0.0	2.90	-0.1639425E-01
0.0	3.00	0.2620593E-05

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	0.1435881E 00
0.05	0.10	0.1444355E 00
0.05	0.20	0.1467624E 00
0.05	0.30	0.1499625E 00
0.05	0.40	0.1531431E 00
0.05	0.50	0.1552808E 00
0.05	0.60	0.1553921E 00
0.05	0.70	0.1526949E 00
0.05	0.80	0.1467267E 00
0.05	0.90	0.1374058E 00
0.05	1.00	0.1250278E 00
0.05	1.10	0.1102061E 00
0.05	1.20	0.9376955E-01
0.05	1.30	0.7664388E-01
0.05	1.40	0.5973423E-01
0.05	1.50	0.4382919E-01
0.05	1.60	0.2953879E-01
0.05	1.70	0.1726508E-01
0.05	1.80	0.7206496E-02
0.05	1.90	-0.6157265E-03
0.05	2.00	-0.6317344E-02
0.05	2.10	-0.1010767E-01
0.05	2.20	-0.1224808E-01
0.05	2.30	-0.1301961E-01
0.05	2.40	-0.1269876E-01
0.05	2.50	-0.1154102E-01
0.05	2.60	-0.9772364E-02
0.05	2.70	-0.7586181E-02
0.05	2.80	-0.5143378E-02
0.05	2.90	-0.2577233E-02
0.05	3.00	0.1194909E-05

Solution A_{13}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	-0.2743527E 00
0.10	0.10	-0.2753136E 00
0.10	0.20	-0.2781471E 00
0.10	0.30	-0.2826896E 00
0.10	0.40	-0.2886217E 00
0.10	0.50	-0.2954249E 00
0.10	0.60	-0.3023686E 00
0.10	0.70	-0.3085436E 00
0.10	0.80	-0.3129581E 00
0.10	0.90	-0.3146760E 00
0.10	1.00	-0.3129719E 00
0.10	1.10	-0.3074605E 00
0.10	1.20	-0.2981659E 00
0.10	1.30	-0.2855054E 00
0.10	1.40	-0.2701942E 00
0.10	1.50	-0.2530890E 00
0.10	1.60	-0.2350168E 00
0.10	1.70	-0.2166283E 00
0.10	1.80	-0.1983172E 00
0.10	1.90	-0.1802174E 00
0.10	2.00	-0.1622733E 00
0.10	2.10	-0.1443530E 00
0.10	2.20	-0.1263679E 00
0.10	2.30	-0.1083538E 00
0.10	2.40	-0.9049523E-01
0.10	2.50	-0.7308722E-01
0.10	2.60	-0.5644911E-01
0.10	2.70	-0.4082290E-01
0.10	2.80	-0.2629691E-01
0.10	2.90	-0.1277439E-01
0.10	3.00	-0.2490921E-05

Solution A_{14}

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.0	0.0	0.1061953E 01
0.0	0.10	0.1054136E 01
0.0	0.20	0.1032212E 01
0.0	0.30	0.1000298E 01
0.0	0.40	0.9639000E 00
0.0	0.50	0.9283088E 00
0.0	0.60	0.8971221E 00
0.0	0.70	0.8713292E 00
0.0	0.80	0.8492203E 00
0.0	0.90	0.8271360E 00
0.0	1.00	0.8007810E 00
0.0	1.10	0.7666753E 00
0.0	1.20	0.7232764E 00
0.0	1.30	0.6714280E 00
0.0	1.40	0.6140248E 00
0.0	1.50	0.5550347E 00
0.0	1.60	0.4982320E 00
0.0	1.70	0.4460697E 00
0.0	1.80	0.3990694E 00
0.0	1.90	0.3559067E 00
0.0	2.00	0.3141478E 00
0.0	2.10	0.2713555E 00
0.0	2.20	0.2261825E 00
0.0	2.30	0.1790509E 00
0.0	2.40	0.1321940E 00
0.0	2.50	0.8903575E-01
0.0	2.60	0.5310523E-01
0.0	2.70	0.2685147E-01
0.0	2.80	0.1075710E-01
0.0	2.90	0.3031876E-02
0.0	3.00	0.4768306E-05

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.05	0.0	0.5730752E 00
0.05	0.10	0.5698819E 00
0.05	0.20	0.5607352E 00
0.05	0.30	0.5468408E 00
0.05	0.40	0.5299202E 00
0.05	0.50	0.5118552E 00
0.05	0.60	0.4943156E 00
0.05	0.70	0.4784628E 00
0.05	0.80	0.4647843E 00
0.05	0.90	0.4530931E 00
0.05	1.00	0.4426697E 00
0.05	1.10	0.4324974E 00
0.05	1.20	0.4215273E 00
0.05	1.30	0.4088969E 00
0.05	1.40	0.3940670E 00
0.05	1.50	0.3768502E 00
0.05	1.60	0.3573546E 00
0.05	1.70	0.3358705E 00
0.05	1.80	0.3127517E 00
0.05	1.90	0.2883257E 00
0.05	2.00	0.2628522E 00
0.05	2.10	0.2365333E 00
0.05	2.20	0.2095560E 00
0.05	2.30	0.1821322E 00
0.05	2.40	0.1545265E 00
0.05	2.50	0.1270475E 00
0.05	2.60	0.1000080E 00
0.05	2.70	0.7367086E-01
0.05	2.80	0.4820545E-01
0.05	2.90	0.2366378E-01
0.05	3.00	-0.2189063E-06

Solution A_{14}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	0.3018270E 00
0.10	0.10	0.3007016E 00
0.10	0.20	0.2974447E 00
0.10	0.30	0.2923880E 00
0.10	0.40	0.2860136E 00
0.10	0.50	0.2788642E 00
0.10	0.60	0.2714558E 00
0.10	0.70	0.2642123E 00
0.10	0.80	0.2574283E 00
0.10	0.90	0.2512724E 00
0.10	1.00	0.2458097E 00
0.10	1.10	0.2410311E 00
0.10	1.20	0.2368761E 00
0.10	1.30	0.2332305E 00
0.10	1.40	0.2299080E 00
0.10	1.50	0.2266213E 00
0.10	1.60	0.2229640E 00
0.10	1.70	0.2184182E 00
0.10	1.80	0.2124016E 00
0.10	1.90	0.2043459E 00
0.10	2.00	0.1937981E 00
0.10	2.10	0.1805155E 00
0.10	2.20	0.1645299E 00
0.10	2.30	0.1461564E 00
0.10	2.40	0.1259463E 00
0.10	2.50	0.1045887E 00
0.10	2.60	0.8278441E-01
0.10	2.70	0.6112636E-01
0.10	2.80	0.4001684E-01
0.10	2.90	0.1964120E-01
0.10	3.00	0.3600652E-05

Solution A_{15}

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.0	0.0	0.1032290E 01
0.0	0.10	0.1022141E 01
0.0	0.20	0.9974625E 00
0.0	0.30	0.9709415E 00
0.0	0.40	0.9522028E 00
0.0	0.50	0.9407894E 00
0.0	0.60	0.9276912E 00
0.0	0.70	0.9036440E 00
0.0	0.80	0.8668205E 00
0.0	0.90	0.8235559E 00
0.0	1.00	0.7820981E 00
0.0	1.10	0.7453707E 00
0.0	1.20	0.7091372E 00
0.0	1.30	0.6667285E 00
0.0	1.40	0.6156045E 00
0.0	1.50	0.5597032E 00
0.0	1.60	0.5057415E 00
0.0	1.70	0.4572285E 00
0.0	1.80	0.4118355E 00
0.0	1.90	0.3643556E 00
0.0	2.00	0.3122214E 00
0.0	2.10	0.2583028E 00
0.0	2.20	0.2084855E 00
0.0	2.30	0.1665151E 00
0.0	2.40	0.1311044E 00
0.0	2.50	0.9801471E-01
0.0	2.60	0.6499946E-01
0.0	2.70	0.3474782E-01
0.0	2.80	0.1282530E-01
0.0	2.90	0.2308320E-02
0.0	3.00	-0.7742767E-07

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.05	0.0	-0.5765131E 00
0.05	0.10	-0.5777606E 00
0.05	0.20	-0.5806465E 00
0.05	0.30	-0.5829863E 00
0.05	0.40	-0.5822052E 00
0.05	0.50	-0.5763215E 00
0.05	0.60	-0.5645138E 00
0.05	0.70	-0.5471320E 00
0.05	0.80	-0.5253151E 00
0.05	0.90	-0.5005305E 00
0.05	1.00	-0.4742200E 00
0.05	1.10	-0.4475856E 00
0.05	1.20	-0.4214643E 00
0.05	1.30	-0.3962772E 00
0.05	1.40	-0.3721123E 00
0.05	1.50	-0.3489298E 00
0.05	1.60	-0.3267882E 00
0.05	1.70	-0.3058952E 00
0.05	1.80	-0.2864137E 00
0.05	1.90	-0.2681372E 00
0.05	2.00	-0.2503144E 00
0.05	2.10	-0.2318208E 00
0.05	2.20	-0.2116213E 00
0.05	2.30	-0.1892213E 00
0.05	2.40	-0.1647934E 00
0.05	2.50	-0.1389146E 00
0.05	2.60	-0.1121308E 00
0.05	2.70	-0.8470076E-01
0.05	2.80	-0.5669000E-01
0.05	2.90	-0.2828107E-01
0.05	3.00	-0.3159470E-05

Solution A_{15}

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.10	0.0	0.2406955E 00
0.10	0.10	0.2412781E 00
0.10	0.20	0.2433871E 00
0.10	0.30	0.2476892E 00
0.10	0.40	0.2542723E 00
0.10	0.50	0.2621217E 00
0.10	0.60	0.2694713E 00
0.10	0.70	0.2748206E 00
0.10	0.80	0.2778107E 00
0.10	0.90	0.2792657E 00
0.10	1.00	0.2803704E 00
0.10	1.10	0.2816480E 00
0.10	1.20	0.2825143E 00
0.10	1.30	0.2816816E 00
0.10	1.40	0.2780098E 00
0.10	1.50	0.2711055E 00
0.10	1.60	0.2612813E 00
0.10	1.70	0.2490504E 00
0.10	1.80	0.2346814E 00
0.10	1.90	0.2181745E 00
0.10	2.00	0.1995864E 00
0.10	2.10	0.1793197E 00
0.10	2.20	0.1580824E 00
0.10	2.30	0.1365660E 00
0.10	2.40	0.1151757E 00
0.10	2.50	0.9404796E-01
0.10	2.60	0.7330501E-01
0.10	2.70	0.5325002E-01
0.10	2.80	0.3428260E-01
0.10	2.90	0.1660425E-01
0.10	3.00	0.1829228E-05

Solution A_{16}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.1043404E 01
0.0	0.10	0.1034257E 01
0.0	0.20	0.1010118E 01
0.0	0.30	0.9791583E 00
0.0	0.40	0.9501919E 00
0.0	0.50	0.9282673E 00
0.0	0.60	0.9124335E 00
0.0	0.70	0.8968234E 00
0.0	0.80	0.8742777E 00
0.0	0.90	0.8404122E 00
0.0	1.00	0.7958541E 00
0.0	1.10	0.7455279E 00
0.0	1.20	0.6955946E 00
0.0	1.30	0.6499416E 00
0.0	1.40	0.6082769E 00
0.0	1.50	0.5668288E 00
0.0	1.60	0.5210824E 00
0.0	1.70	0.4688072E 00
0.0	1.80	0.4115578E 00
0.0	1.90	0.3538142E 00
0.0	2.00	0.3003988E 00
0.0	2.10	0.2538143E 00
0.0	2.20	0.2131502E 00
0.0	2.30	0.1751744E 00
0.0	2.40	0.1368824E 00
0.0	2.50	0.9787190E-01
0.0	2.60	0.6101776E-01
0.0	2.70	0.3097793E-01
0.0	2.80	0.1138783E-01
0.0	2.90	0.2391053E-02
0.0	3.00	-0.1364638E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	0.3768965E 00
0.05	0.10	0.3750266E 00
0.05	0.20	0.3703920E 00
0.05	0.30	0.3653657E 00
0.05	0.40	0.3623770E 00
0.05	0.50	0.3625400E 00
0.05	0.60	0.3650219E 00
0.05	0.70	0.3674976E 00
0.05	0.80	0.3674050E 00
0.05	0.90	0.3632855E 00
0.05	1.00	0.3554536E 00
0.05	1.10	0.3456489E 00
0.05	1.20	0.3359182E 00
0.05	1.30	0.3273908E 00
0.05	1.40	0.3196562E 00
0.05	1.50	0.3110630E 00
0.05	1.60	0.2997153E 00
0.05	1.70	0.2845473E 00
0.05	1.80	0.2658455E 00
0.05	1.90	0.2449445E 00
0.05	2.00	0.2233348E 00
0.05	2.10	0.2017679E 00
0.05	2.20	0.1799197E 00
0.05	2.30	0.1568123E 00
0.05	2.40	0.1317162E 00
0.05	2.50	0.1049482E 00
0.05	2.60	0.7803828E-01
0.05	2.70	0.5311765E-01
0.05	2.80	0.3185946E-01
0.05	2.90	0.1457575E-01
0.05	3.00	0.1498955E-05

Solution A_{16}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	0.2529876E-01
0.10	0.10	0.2664276E-01
0.10	0.20	0.3030907E-01
0.10	0.30	0.3533233E-01
0.10	0.40	0.4048713E-01
0.10	0.50	0.4469374E-01
0.10	0.60	0.4732773E-01
0.10	0.70	0.4832932E-01
0.10	0.80	0.4810025E-01
0.10	0.90	0.4727143E-01
0.10	1.00	0.4645792E-01
0.10	1.10	0.4610131E-01
0.10	1.20	0.4643280E-01
0.10	1.30	0.4752073E-01
0.10	1.40	0.4933729E-01
0.10	1.50	0.5178600E-01
0.10	1.60	0.5468050E-01
0.10	1.70	0.5770610E-01
0.10	1.80	0.6041649E-01
0.10	1.90	0.6229320E-01
0.10	2.00	0.6285793E-01
0.10	2.10	0.6179180E-01
0.10	2.20	0.5900251E-01
0.10	2.30	0.5461339E-01
0.10	2.40	0.4888297E-01
0.10	2.50	0.4210228E-01
0.10	2.60	0.3452111E-01
0.10	2.70	0.2633304E-01
0.10	2.80	0.1770971E-01
0.10	2.90	0.8849647E-02
0.10	3.00	0.1006133E-05

Solution A_{17}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.9485748E 00
0.0	0.10	0.9526629E 00
0.0	0.20	0.9626361E 00
0.0	0.30	0.9725540E 00
0.0	0.40	0.9752364E 00
0.0	0.50	0.9651240E 00
0.0	0.60	0.9403933E 00
0.0	0.70	0.9034818E 00
0.0	0.80	0.8598448E 00
0.0	0.90	0.8155606E 00
0.0	1.00	0.7748408E 00
0.0	1.10	0.7385193E 00
0.0	1.20	0.7041129E 00
0.0	1.30	0.6673385E 00
0.0	1.40	0.6243336E 00
0.0	1.50	0.5735565E 00
0.0	1.60	0.5165273E 00
0.0	1.70	0.4571346E 00
0.0	1.80	0.3999002E 00
0.0	1.90	0.3480452E 00
0.0	2.00	0.3022737E 00
0.0	2.10	0.2608122E 00
0.0	2.20	0.2206556E 00
0.0	2.30	0.1793800E 00
0.0	2.40	0.1366499E 00
0.0	2.50	0.9468365E-01
0.0	2.60	0.5742903E-01
0.0	2.70	0.2881545E-01
0.0	2.80	0.1087777E-01
0.0	2.90	0.2596461E-02
0.0	3.00	-0.7945994E-07

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	0.4855346E 00
0.05	0.10	0.4826443E 00
0.05	0.20	0.4748253E 00
0.05	0.30	0.4643111E 00
0.05	0.40	0.4538841E 00
0.05	0.50	0.4458601E 00
0.05	0.60	0.4412931E 00
0.05	0.70	0.4396994E 00
0.05	0.80	0.4393729E 00
0.05	0.90	0.4381412E 00
0.05	1.00	0.4342244E 00
0.05	1.10	0.4268476E 00
0.05	1.20	0.4163876E 00
0.05	1.30	0.4040340E 00
0.05	1.40	0.3911582E 00
0.05	1.50	0.3786846E 00
0.05	1.60	0.3667330E 00
0.05	1.70	0.3546569E 00
0.05	1.80	0.3414145E 00
0.05	1.90	0.3260623E 00
0.05	2.00	0.3081286E 00
0.05	2.10	0.2876963E 00
0.05	2.20	0.2651845E 00
0.05	2.30	0.2409601E 00
0.05	2.40	0.2150034E 00
0.05	2.50	0.1868173E 00
0.05	2.60	0.1556402E 00
0.05	2.70	0.1208840E 00
0.05	2.80	0.8259434E-01
0.05	2.90	0.4171286E-01
0.05	3.00	0.4763288E-05

Solution A_{17}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	0.4202600E 00
0.10	0.10	0.4216609E 00
0.10	0.20	0.4250659E 00
0.10	0.30	0.4283853E 00
0.10	0.40	0.4290210E 00
0.10	0.50	0.4248256E 00
0.10	0.60	0.4148524E 00
0.10	0.70	0.3996271E 00
0.10	0.80	0.3808488E 00
0.10	0.90	0.3606823E 00
0.10	1.00	0.3409385E 00
0.10	1.10	0.3224817E 00
0.10	1.20	0.3050755E 00
0.10	1.30	0.2876903E 00
0.10	1.40	0.2691003E 00
0.10	1.50	0.2484908E 00
0.10	1.60	0.2258223E 00
0.10	1.70	0.2018206E 00
0.10	1.80	0.1776387E 00
0.10	1.90	0.1543794E 00
0.10	2.00	0.1327051E 00
0.10	2.10	0.1127096E 00
0.10	2.20	0.9408033E-01
0.10	2.30	0.7644111E-01
0.10	2.40	0.5968936E-01
0.10	2.50	0.4414802E-01
0.10	2.60	0.3045413E-01
0.10	2.70	0.1923945E-01
0.10	2.80	0.1076154E-01
0.10	2.90	0.4665483E-02
0.10	3.00	0.4195030E-06

Solution A_{18}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.1093078E 01
0.0	0.10	0.1087113E 01
0.0	0.20	0.1069686E 01
0.0	0.30	0.1042134E 01
0.0	0.40	0.1006497E 01
0.0	0.50	0.9652557E 00
0.0	0.60	0.9210113E 00
0.0	0.70	0.8761772E 00
0.0	0.80	0.8326819E 00
0.0	0.90	0.7917606E 00
0.0	1.00	0.7538438E 00
0.0	1.10	0.7185633E 00
0.0	1.20	0.6848729E 00
0.0	1.30	0.6512639E 00
0.0	1.40	0.6160438E 00
0.0	1.50	0.5776392E 00
0.0	1.60	0.5348791E 00
0.0	1.70	0.4872148E 00
0.0	1.80	0.4348525E 00
0.0	1.90	0.3787709E 00
0.0	2.00	0.3206279E 00
0.0	2.10	0.2625659E 00
0.0	2.20	0.2069485E 00
0.0	2.30	0.1560543E 00
0.0	2.40	0.1117813E 00
0.0	2.50	0.7540083E-01
0.0	2.60	0.4738627E-01
0.0	2.70	0.2734840E-01
0.0	2.80	0.1408564E-01
0.0	2.90	0.5732890E-02
0.0	3.00	0.4678741E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	-0.5777692E 00
0.05	0.10	-0.5747580E 00
0.05	0.20	-0.5659713E 00
0.05	0.30	-0.5521170E 00
0.05	0.40	-0.5342702E 00
0.05	0.50	-0.5137315E 00
0.05	0.60	-0.4918552E 00
0.05	0.70	-0.4698777E 00
0.05	0.80	-0.4487604E 00
0.05	0.90	-0.4290791E 00
0.05	1.00	-0.4109671E 00
0.05	1.10	-0.3941236E 00
0.05	1.20	-0.3778862E 00
0.05	1.30	-0.3613529E 00
0.05	1.40	-0.3435413E 00
0.05	1.50	-0.3235570E 00
0.05	1.60	-0.3007514E 00
0.05	1.70	-0.2748417E 00
0.05	1.80	-0.2459828E 00
0.05	1.90	-0.2147719E 00
0.05	2.00	-0.1821929E 00
0.05	2.10	-0.1495017E 00
0.05	2.20	-0.1180761E 00
0.05	2.30	-0.8924001E-01
0.05	2.40	-0.6409597E-01
0.05	2.50	-0.4338538E-01
0.05	2.60	-0.2739185E-01
0.05	2.70	-0.1590588E-01
0.05	2.80	-0.8253779E-02
0.05	2.90	-0.3383444E-02
0.05	3.00	-0.2843793E-06

Solution A_{18}

TIME	RAD.DISTANCE	DEFLECTION(ω/ω_{max})
0.10	0.0	0.1310496E 00
0.10	0.10	0.1306212E 00
0.10	0.20	0.1293961E 00
0.10	0.30	0.1275454E 00
0.10	0.40	0.1253217E 00
0.10	0.50	0.1230171E 00
0.10	0.60	0.1209115E 00
0.10	0.70	0.1192236E 00
0.10	0.80	0.1180667E 00
0.10	0.90	0.1174219E 00
0.10	1.00	0.1171301E 00
0.10	1.10	0.1169035E 00
0.10	1.20	0.1163605E 00
0.10	1.30	0.1150738E 00
0.10	1.40	0.1126297E 00
0.10	1.50	0.1086873E 00
0.10	1.60	0.1030322E 00
0.10	1.70	0.9561425E-01
0.10	1.80	0.8656538E-01
0.10	1.90	0.7619381E-01
0.10	2.00	0.6495529E-01
0.10	2.10	0.5340455E-01
0.10	2.20	0.4213310E-01
0.10	2.30	0.3170133E-01
0.10	2.40	0.2257359E-01
0.10	2.50	0.1506547E-01
0.10	2.60	0.9308655E-02
0.10	2.70	0.5239185E-02
0.10	2.80	0.2610345E-02
0.10	2.90	0.1027288E-02
0.10	3.00	0.8735373E-07

Solution A_{19}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.0	0.0	0.9241962E 00
0.0	0.10	0.9267311E 00
0.0	0.20	0.9334573E 00
0.0	0.30	0.9419267E 00
0.0	0.40	0.9486398E 00
0.0	0.50	0.9497773E 00
0.0	0.60	0.9419979E 00
0.0	0.70	0.9231243E 00
0.0	0.80	0.8925678E 00
0.0	0.90	0.8514180E 00
0.0	1.00	0.8021746E 00
0.0	1.10	0.7481955E 00
0.0	1.20	0.6929894E 00
0.0	1.30	0.6395091E 00
0.0	1.40	0.5896080E 00
0.0	1.50	0.5437719E 00
0.0	1.60	0.5011811E 00
0.0	1.70	0.4600746E 00
0.0	1.80	0.4183307E 00
0.0	1.90	0.3741152E 00
0.0	2.00	0.3264502E 00
0.0	2.10	0.2755652E 00
0.0	2.20	0.2229563E 00
0.0	2.30	0.1711259E 00
0.0	2.40	0.1230757E 00
0.0	2.50	0.8166397E-01
0.0	2.60	0.4896603E-01
0.0	2.70	0.2579106E-01
0.0	2.80	0.1146136E-01
0.0	2.90	0.3897760E-02
0.0	3.00	0.2808877E-06

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.05	0.0	0.3338708E 00
0.05	0.10	0.3353090E 00
0.05	0.20	0.3395268E 00
0.05	0.30	0.3462366E 00
0.05	0.40	0.3549682E 00
0.05	0.50	0.3650858E 00
0.05	0.60	0.3758174E 00
0.05	0.70	0.3862989E 00
0.05	0.80	0.3956298E 00
0.05	0.90	0.4029439E 00
0.05	1.00	0.4074810E 00
0.05	1.10	0.4086552E 00
0.05	1.20	0.4061068E 00
0.05	1.30	0.3997294E 00
0.05	1.40	0.3896660E 00
0.05	1.50	0.3762721E 00
0.05	1.60	0.3600512E 00
0.05	1.70	0.3415716E 00
0.05	1.80	0.3213807E 00
0.05	1.90	0.2999278E 00
0.05	2.00	0.2775102E 00
0.05	2.10	0.2542548E 00
0.05	2.20	0.2301359E 00
0.05	2.30	0.2050219E 00
0.05	2.40	0.1787474E 00
0.05	2.50	0.1511939E 00
0.05	2.60	0.1223578E 00
0.05	2.70	0.9240091E-01
0.05	2.80	0.6166700E-01
0.05	2.90	0.3066156E-01
0.05	3.00	0.3452427E-05

Solution A_{19}

TIME	RAD.DISTANCE	DEFLECTION (ω/ω_{max})
0.10	0.0	0.3386165E 00
0.10	0.10	0.3378915E 00
0.10	0.20	0.3356043E 00
0.10	0.30	0.3314503E 00
0.10	0.40	0.3250223E 00
0.10	0.50	0.3159353E 00
0.10	0.60	0.3039581E 00
0.10	0.70	0.2891189E 00
0.10	0.80	0.2717592E 00
0.10	0.90	0.2525219E 00
0.10	1.00	0.2322723E 00
0.10	1.10	0.2119687E 00
0.10	1.20	0.1925099E 00
0.10	1.30	0.1745907E 00
0.10	1.40	0.1585962E 00
0.10	1.50	0.1445596E 00
0.10	1.60	0.1321902E 00
0.10	1.70	0.1209632E 00
0.10	1.80	0.1102572E 00
0.10	1.90	0.9950191E-01
0.10	2.00	0.8830774E-01
0.10	2.10	0.7655019E-01
0.10	2.20	0.6439114E-01
0.10	2.30	0.5223051E-01
0.10	2.40	0.4060764E-01
0.10	2.50	0.3007097E-01
0.10	2.60	0.2104505E-01
0.10	2.70	0.1372881E-01
0.10	2.80	0.8044794E-02
0.10	2.90	0.3648899E-02
0.10	3.00	0.3670432E-06