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THE USE OF TURBULENCE ENERGY EQUATION IN BOUNDARY LAYER STUDY

BY 440

RO-CHI TAI, 1944

A

THESIS

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ABSTRACT

A turbulent boundary layer problem has been studied analytically and compared with an available experiment in the literature. Correlations of the experimental data were made to investigate the validity of the commonly used empirical relations on turbulent shear stresses. It was found that the model which related the local turbulent shear stress linearly with the local turbulent kinetic energy, as used by Bradshow et. al., appeared to be most reasonable. Combining this model with the expression of turbulent viscosity given by Boussinesq, it was then possible to introduce the turbulence-energy equation in addition to the governing equations of continuity and momentum. Consequently, the turbulent viscosity was able to be considered as one of the dependent variables to be solved for simultaneously with all other related flow parameters. Using the main-flow direction and the stream function as the two independent variables, the governing equations were reduced to two simultaneous parabolic-type partial differential equations through the von Mises transformation. The finite difference technique of Partankar was applied. The numerical solutions were obtained for the average velocity and the turbulent kinetic energy distributions. In comparison with the experimental results of Klebanoff in the fully developed region along a flat plate, very good agreement was reached on average velocity distribution. However, the turbulent kinetic energy distribution was not completely satisfactory, since the energy dissipation term of the turbulence-energy equation was not able to be expressed adequately due to the lack of sufficient experimental information. It is then concluded that the use of the turbulence-energy

equation in boundary layer study is possible to eliminate the uncertainty resulting with empirical models of the turbulent viscosity. However, further experimental investigations are needed to improve the understanding of the structure of turbulence.

ACKNOWLEDGEMENT

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NOMENCLATURE

Symbols	Meaning	Equation of first mention
a	Experimental constant	(2-4)
A,B,C,D	Coefficients in general partial differential equation	(4-9)
A_U, B_U, C_U	Coefficients in iteration equation for average velocity	(B-8)
A_T, B_T, C_T	Coefficients in iteration equation for	(B-9)
D_k	The dissipation of turbulent kinetic energy	(3-4)
F_1, F_2, F_3	Coefficients in the difference form of $\frac{\partial\phi}{\partial x}$	(B-1)
G	Coefficient in the difference form of $A\frac{\partial\phi}{\partial\omega}$	(B-2)
H_1, H_2, H_3	Coefficients in the difference form of $B\omega\frac{\partial\phi}{\partial\omega}$	(B-3)
I_1, I_2, I_3, I_4	Coefficients in the difference form of the convection term	(B-4)
I_5, I_6	Coefficients in the difference form of the diffusion term	(B-5)
J_k	Turbulent kinetic energy flux	(3-5)
k	Turbulent kinetic energy	(2-2)
ℓ_k	Have unit in length analogous to the Prandtl mixing length	(2-2)
ℓ_p	Prandtl mixing length	(2-1)

Symbol	Meaning	Equation of first mention
$\dot{M}$	Mass flow rate	
M_I	Mass-Flow rate across the I boundary	(4-6)
M_E	Mass-Flow rate across the E boundary	(4-6)
M_S	Mass-Flow rate across the wall	(B-23)
n	Exponent in the ϕ profile near a free boundary	(B-20)
p	Pressure	(B-24)
Pr	Prandtl Number	(B-24)
R	Coefficient in the difference form for the slip value of average velocity	(B-19)
R_ϕ	Coefficient in the difference form for slip value of ϕ	(B-21)
S_1, S_2, S_3, S_4	Coefficients in the difference form of coefficient D	(B-5)
U	The free stream velocity	——
u	Velocity in x-direction	(1-1)
v	Velocity in y-direction	(1-1)
w	Velocity in z-direction	(1-1)
x	Distance in the stream-wise direction	——
y	Distance in the Cross-Stream direction	——
z	Distance in the direction perpendicular to the flow plane	——
$\sigma_x, \sigma_y, \sigma_z$	Normal stress in the x,y,z, directions, respectively	(1-2)
σ_k	A constant in equation of conservation of turbulent kinetic energy, analogous to the turbulent Prandtl Number	(3-4)

Symbol	Meaning	Equation of first mention
τ	Shear stress	
$\tau_{xy}, \tau_{yz}, \tau_{zx}$	Shear stress in xy, yz, zx planes, respectively	(1-2)
τ_t	Shear stress in turbulence	(1-4)
τ_s	Shear stress at the wall	(B-24)
ρ	Density	(1-2)
ε	Turbulent viscosity	(1-4)
δ	Boundary layer thickness	
α	α=1 for two-dimensional, symmetrical flow α=0 for two-dimensional, plane flow	(3-1)
γ	Exponent of profile near the wall	(B-14)
β	Exponent of average velocity profile near the wall	(B-10)
ψ	Stream function	
ψ_I	Stream function at the I boundary	(4-5)
ψ_E	Stream function at the E boundary	(4-5)
ϕ	The flow parameter	(4-9)
ω	Dimensionless stream function	(4-5)

Subscripts

Symbols	Meaning	Equation of first mention
t	Fully turbulent condition	(1-4)
D	The down stream point on a portion of the grid	(B-1)
D+ D++ D- D--	Points near and at the same value of x at point D	(B-1)

Symbol	Meaning	Equation of first mention
E	The external boundary of the layer	(4-6)
I	The internal boundary of the layer	(4-6)
U	The upstream point on a portion of the grid	(B-1)
U+ U++ U- U	Points near and at the same value of x at point U	(B-1)

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CHAPTER I

INTRODUCTION

The concept of the boundary layer was introduced by Prandtl in 1904 to consider the viscous effect between a solid surface and its surrounding flow field. This concept made it possible for experimentally observed phenomena such as skin friction drag and aerodynamic heating to be evaluated by using the knowledge established in hydrodynamic theory.

The development of boundary layer theory can be found in the books of Schlichting (1)* and Pai (2). The available analytical methods generally predict the detailed flow field with reasonable success if the boundary layer is laminar. However, most of the problems encountered in engineering find that the boundary layer is generally turbulent. The momentary value of a turbulent velocity may be expressed as

$$u = \overline{u} + u'$$
$$v = \overline{v} + v' \qquad (1\text{-}1)$$
$$w = \overline{w} + w'$$

where u, v and w are the velocity components in the flow field in the x, y and z directions, respectively. The bar denotes the time average quantity and the prime denotes the fluctuating quantity. The existence of the fluctuating quantities adds some additional terms to the equation of motion. These terms are known as the Reynolds stresses with the stress tensors as:

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*Numbers in parentheses refer to listings under REFERENCES.

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} = - \begin{pmatrix} \rho\overline{u'^2} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{u'v'} & \rho\overline{v'^2} & \rho\overline{v'w'} \\ \rho\overline{u'w'} & \rho\overline{v'w'} & \rho\overline{w'^2} \end{pmatrix} \tag{1-2}$$

where ρ is the density of the fluid: $\overline{u'^2}$, $\overline{v'^2}$ and $\overline{w'^2}$ are the turbulence intensities in the x, y, and z directions, respectively; $\rho\overline{u'v'}$, $\rho\overline{u'w'}$, etc. are the turbulent shear stresses. The equation of motion in turbulent flow has six additional unknowns in comparison with the same equation for laminar flow. The major task in turbulent flow then is to establish the required relations for the additional unknowns so that a mathematical solution may be theoretically possible.

Experimental investigations to measure the Reynolds stresses were conducted by Klebanoff (3) in a fully developed turbulent flow along a flat plate. It was found that all the normal stresses are of the same order of magnitude, and that the shear stresses are of the order of the square of the magnitude of the normal stresses. Applying the order of magnitude analysis in the boundary layer to examine the equation of motion, the most significant term of the Reynolds stresses appears to be the shear stress term

$$\tau_{xy} = - \rho\overline{u'v'} \tag{1-3}$$

In order to make the turbulent boundary layer equation identical with the laminar boundary layer equation, J. Boussinesq (4) introduced a turbulent viscosity to define the turbulent shear stress as

$$\tau_t = \tau_{xy} = \varepsilon \frac{\partial \bar{u}}{\partial y} ; \tag{1-4}$$

where ε is the turbulent viscosity, $\frac{\partial \bar{u}}{\partial y}$ is the gradient of the time average velocity. Since Equation (1-4) is analogous to the Newton's law of viscosity, it is then possible to solve the turbulent boundary layer problems by applying the analytical method available for laminar boundary layer solutions, provided that the turbulent viscosity is known.

Detailed discussion of the turbulent viscosity models is given in Chapter II. Knowing that turbulence is a phenomenon which results from the history of the development of the entire flow field, it is necessary that the turbulent viscosity model considers the turbulence development. Examining the experimental result of Klebanoff (3) in comparison with the available turbulent viscosity models, the linear relation between the local turbulent shear stress and the local kinetic energy of turbulence, as suggested by Bradshow (5), appears to satisfy the requirement. Since the energy is conserved in a turbulent flow field, the historical effect of turbulence may then be taken into consideration by introducing the turbulence energy equation.

The current method of analyzing turbulent boundary layer problems is to select an empirical formula for the turbulent viscosity. However, a formula established from one engineering problem generally failed to provide meaningful solutions for different problems. Consequently, the engineers are not able to find an adequate turbulent viscosity model whenever a new situation arises. The approach of using the equation of conservation of turbulence energy made it possible that the turbulent viscosity may be solved for simultaneously with the other flow parameters. This study applies this new approach to investigate the development of turbulent boundary layer and compare the analytical solutions with the experimental results available in the literature.

CHAPTER II

THE TURBULENT VISCOSITY MODELS

The development of an appropriate turbulent viscosity model is essential in obtaining a meaningful solution for the turbulent boundary layer problems. Several models have been suggested in the last three decades. A comparison of the suggested models with the experimental data appears to be necessary in order to investigate the validity of each model in applying to engineering problems. The Reynolds stresses measured by Klebanoff (3) in a fully developed turbulent flow along a flat plate, provided the necessary information for comparison with the following three commonly used turbulent viscosity models.

A. Prandtl's Mixing Length Model

Prandtl (6) introduced the mixing length hypothesis in 1925 to relate the turbulent shear stress with the velocity gradient.

$$\tau_t = \rho \ell_p^{\,2} \left|\frac{\partial \bar{u}}{\partial y}\right| \frac{\partial \bar{u}}{\partial y} \; ; \tag{2-1}$$

where ℓ_p is the Prandtl mixing length which has the dimension of length and is to be determined experimentally. The experimental data of Klebanoff is shown in Figure 1 with $\frac{\tau}{\rho U^2}$ as the ordinate and $\left(\frac{\partial(\bar{u}/U)}{\partial(y/\delta)}\right)^2$ as the abscissa on a 3x3 cycle logarithmic plot, where U is the free stream velocity and δ is the boundary layer thickness. In order to satisfy Equation (2-1), a 45 degree straight line needs to be drawn in Figure 1 to pass as many data points as possible. For the best representation of Klebanoff's experimental data, the Prandtl mixing length can be written as:

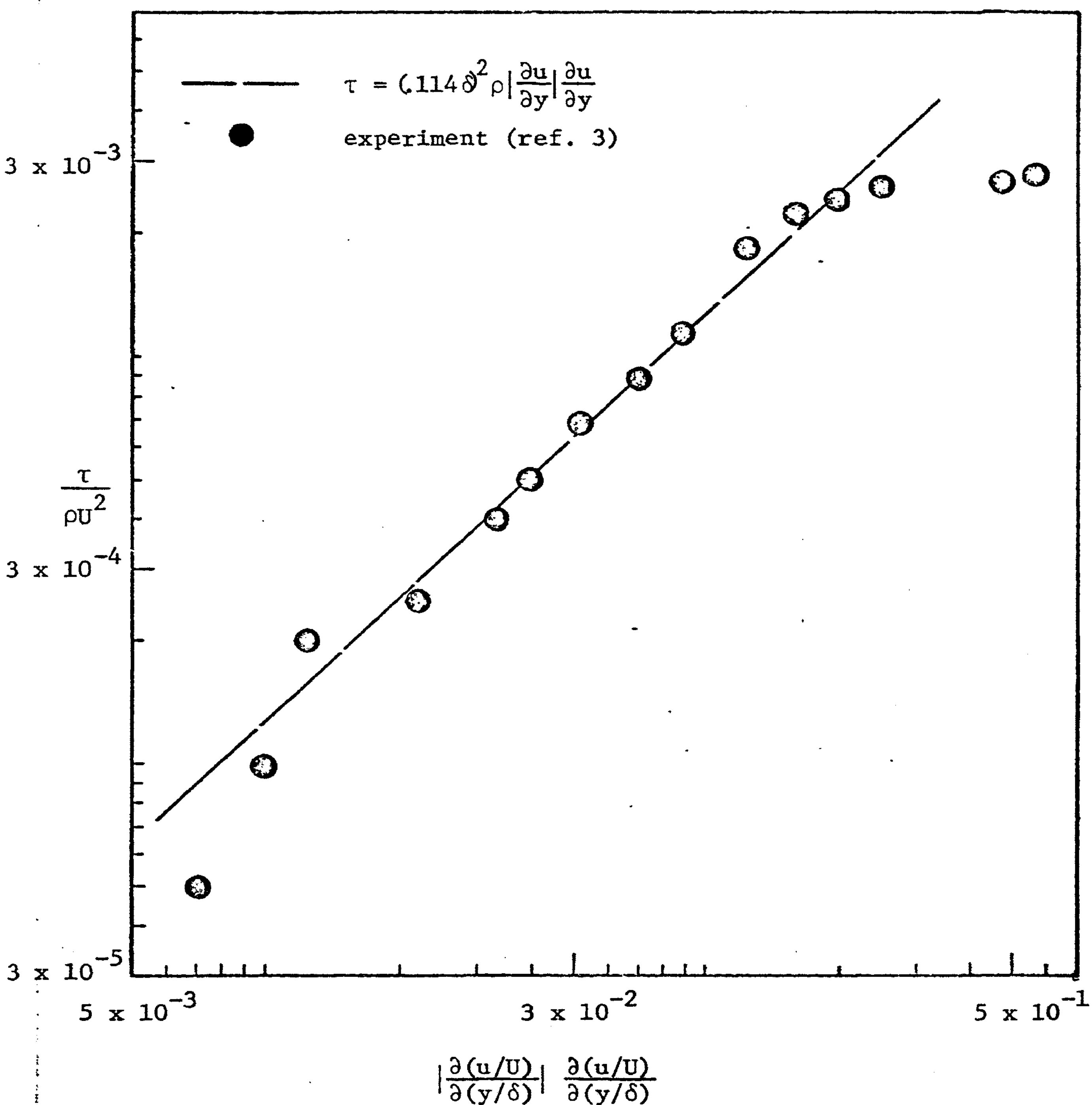

Figure 1. Prandtl Mixing Length Model

$$\ell_p = 0.114\delta$$

It can be seen that Equation (2-1) represents the data of Klebanoff reasonably well except at large values of the velocity gradient.

B. Kolmogorov's Model

Kolmogorov (7) made the suggestion in 1942 that the turbulent shear stress may be related with the turbulent kinetic energy as follows

$$\tau_t = \ell_k k^{\frac{1}{2}} \frac{\partial \overline{u}}{\partial y} \tag{2-2}$$

where k is the turbulent kinetic energy and is defined as

$$k = \tfrac{1}{2}[\overline{u'^2} + \overline{v'^2} + \overline{w'^2}] \tag{2-3}$$

ℓ_k is analogous to the Prandtl mixing length since it also has the dimension of length. The experimental data of Klebanoff is shown in Figure 2 as $\tau/\rho U^2$ versus $\frac{k^{\frac{1}{2}}}{U} \frac{\partial(\overline{u}/U)}{\partial(y/\delta)}$. For the best representation of Klebanoff's data, Equation (2-2) requires

$$\ell_k = 0.06\delta$$

It can be seen that Equation (2-2), like Equation (2-1), represents the Klebanoff's data well only in the region of small values of $\frac{k^{\frac{1}{2}}}{U}\frac{\partial(\overline{u}/U)}{\partial(y/\delta)}$.

C. Bradshow's Model

Bradshow, Ferris and Atwell (5) in 1967 used a linear relation between the local turbulent shear stress and the local kinetic energy of turbulence as

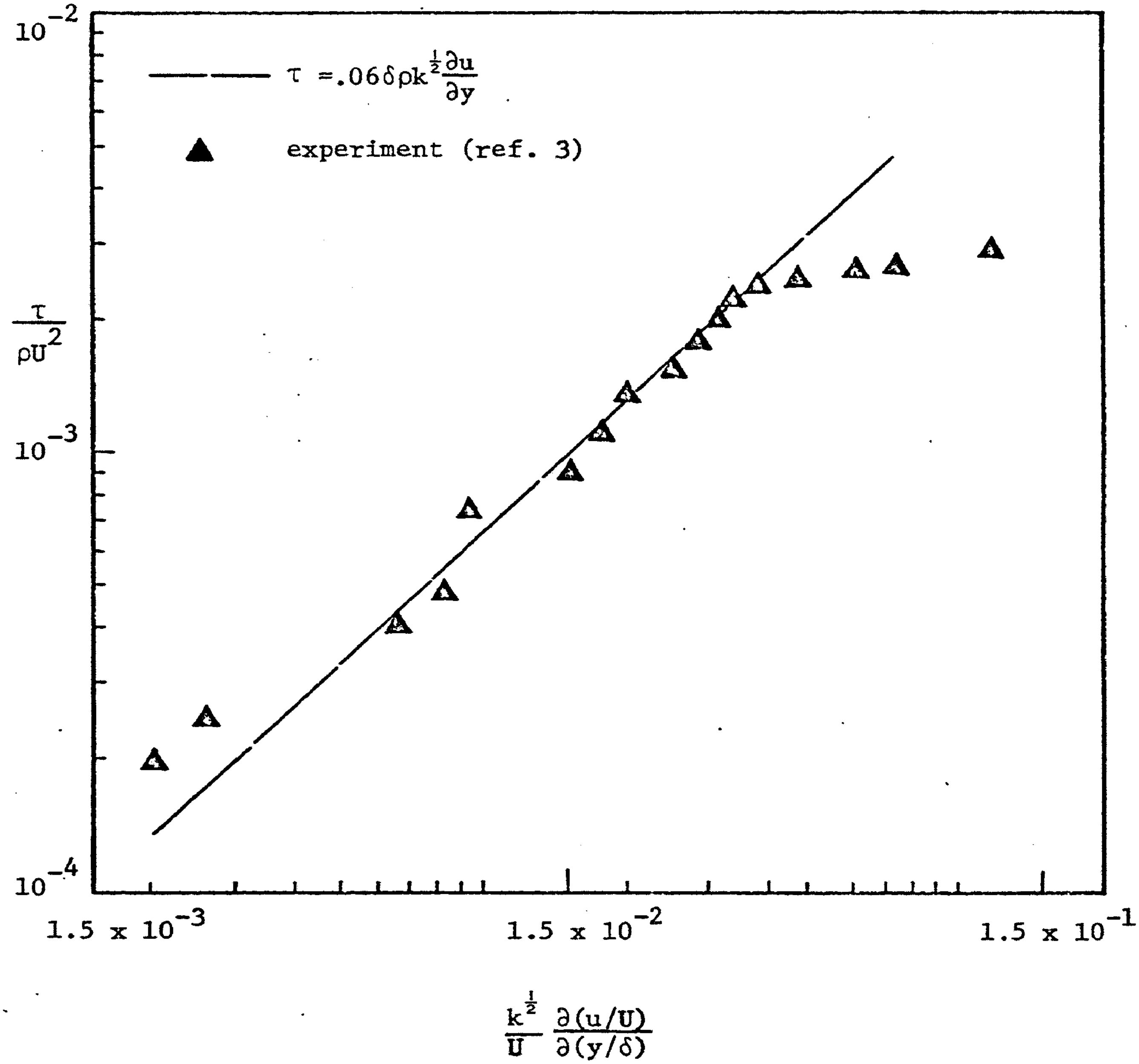

Figure 2. Kolmogorov's Model

$$\tau_t = a_1 \rho k, \qquad (2\text{-}4)$$

where a_1 is a constant which, unlike ℓ_p and ℓ_k, does not have the dimension of length. The experimental data of Klebanoff is shown in Figure 3 as $\tau/\rho U^2$ versus k/U^2. For the best representation of Klebanoff's experimental data, Equation (2-4) requires

$$a_1 = 0.3.$$

It is noted that Equation (2-4) represents a comparatively wider range of the measured data then both Equation (2-1) and Equation (2-2). Moreover, Lee and Harsha (8) found that this model is also valid in the region where flow similarity is reached.

The investigation of the turbulent viscosity models, leads into the following conclusions:

1. The turbulent viscosity, as defined by Boussinesq (4), is a very convenient way in expressing the equation of motion in a turbulent boundary layer. However, physically the turbulent and laminar viscosities have entirely different meanings because the laminar viscosity is a property of the fluid while the turbulent viscosity is a local phenomenon. Therefore, it seems impractical to expect that a simple expression of turbulent viscosity can be derived to represent all turbulent boundary layer problems.
2. The linear relation between the local turbulent shear stress and the local kinetic energy of turbulence appears to hold true in the flow field with or without flow similarities. With the linear relation, the Boussinesq's concept may then be used to relate the turbulent viscosity with the turbulent kinetic energy.

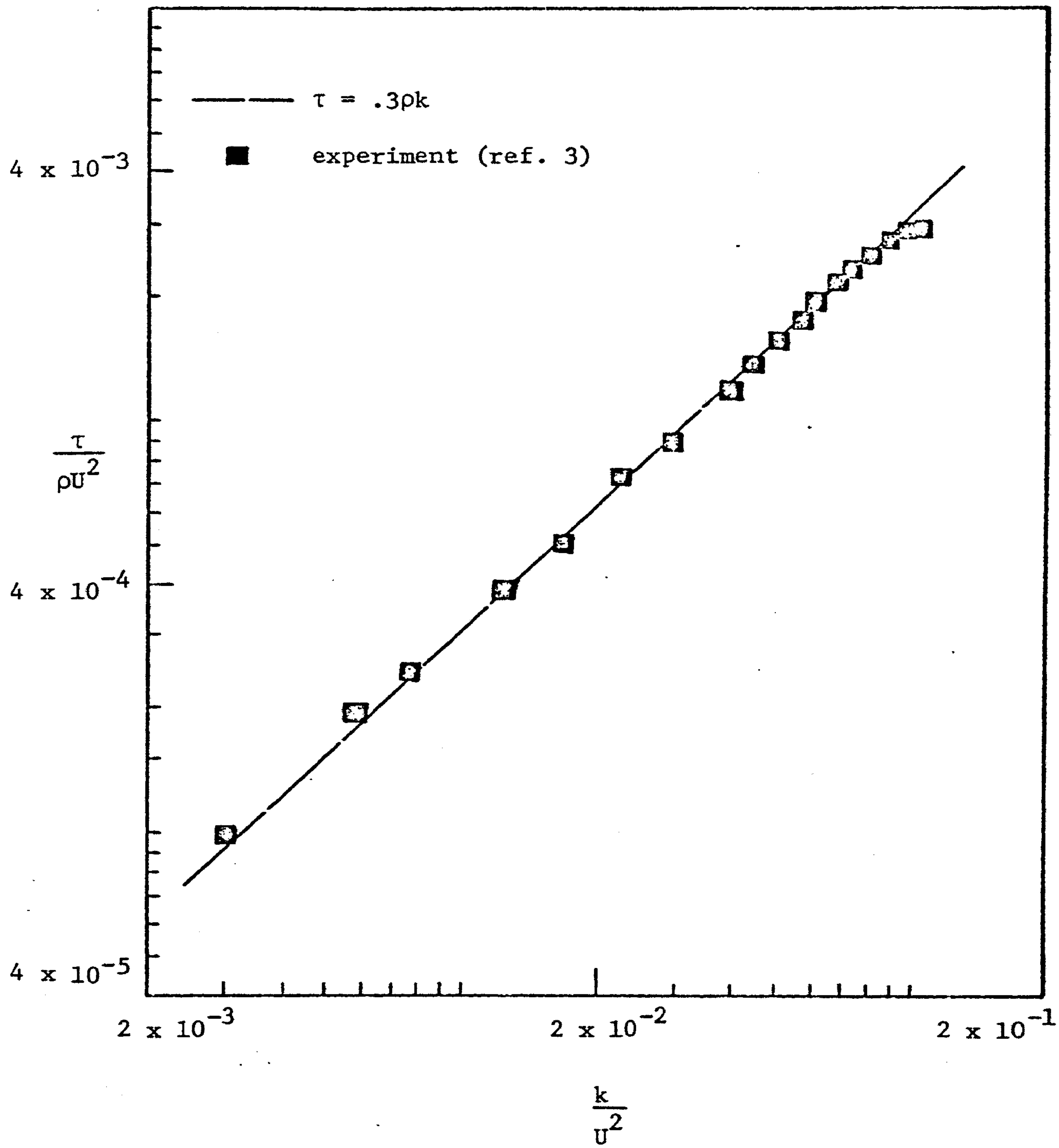

Figure 3. Bradshow's Model

3. Using the equation of conservation of turbulence energy, it is, thus possible to investigate the turbulent boundary layer problems by considering the turbulent viscosity as one of the dependent variables to be determined in a turbulent flow field.

CHAPTER III

THE GOVERNING EQUATIONS

For "steady", two-dimensional or axisymmetric flow, in a turbulent boundary layer, the governing partial differential equations may be expressed as follow:

A. Continuity

$$\frac{\partial \overline{\rho u} y^{\alpha}}{\partial x} + \frac{\partial \overline{\rho v} y^{\alpha}}{\partial y} = 0 \; ; \qquad (3\text{-}1)$$

where $\bar{u}$ and $\bar{v}$ are the time average velocities in the x and y directions, respectively. $\bar{\rho}$ is the time average density. The index α is equal to zero for two-dimensional flow and unity for axisymmetric flow. For the investigated two-dimensional incompressible flow,

$$\alpha = 0, \; \bar{\rho} = \text{constant}$$

B. Momentum

$$\overline{\rho u} \frac{\partial \bar{u}}{\partial x} + \overline{\rho v} \frac{\partial \bar{u}}{\partial y} = y^{-\alpha} \frac{\partial}{\partial y} [y^{\alpha} \varepsilon \frac{\partial \bar{u}}{\partial y}] - \frac{\partial \bar{p}}{\partial x} \; ; \qquad (3\text{-}2)$$

where $\bar{p}$ is the time average static pressure. For the investigated cases, the fluctuating pressure is negligible. $\frac{\partial \bar{p}}{\partial x}$ is zero for flow along a flat plate. ε is the turbulent viscosity which can be expressed through Boussinesq's definition of Equation (1-4) and Bradshow's relation of Equation (2-4), as

$$\varepsilon = \frac{\tau_t}{\frac{\partial \bar{u}}{\partial y}} = \frac{a_1 \rho k}{\frac{\partial \bar{u}}{\partial y}} \; . \qquad (3\text{-}3)$$

C. Turbulence Energy

$$\overline{\rho u} \frac{\partial k}{\partial x} + \overline{\rho v} \frac{\partial k}{\partial y} = y^{-\alpha} \frac{\partial}{\partial y} [y^{\alpha} \frac{\varepsilon}{\sigma k} \frac{\partial k}{\partial y}] + \varepsilon (\frac{\partial \bar{u}}{\partial y})^2 - D_k \; ; \qquad (3\text{-}4)$$

where $\frac{\varepsilon}{\sigma k}$ is the exchange coefficient of the turbulent kinetic energy flux and is defined as

$$\frac{\varepsilon}{\sigma k} = - \frac{J_k}{\frac{\partial k}{\partial y}} = \frac{\overline{(\rho v)'k}}{\frac{\partial k}{\partial y}} \tag{3-5}$$

with J_k as the turbulent kinetic energy flux. The parameter σ_k is analogous to the turbulent Prandtl number defined in the total energy equation when conductive heat transfer is being considered. Therefore, σ_k may be considered as the ratio of the frictional energy to the turbulent kinetic energy. The numerical value of σ_k and its effect on the flow field are being discussed in Appendix A. For the investigated cases $\sigma_k = 0.7$. The term D_k represents the dissipation of the turbulence energy. For isotropic turbulence, the expression of D_k is given in Townsend (9) and Hinze (10). However, for nonisotropic turbulence, as observed in all engineering problems, D_k is not yet being evaluated. Patankar and Spalding (11) expressed D_k by using dimensional analysis as

$$D_k = a\rho k^{3/2}/\ell \; ; \tag{3-6}$$

where "a" is a constant and ℓ is equivalent to the Prandtl mixing length. For the studied problems, ℓ is considered as proportional to the boundary layer thickness. The constant of proportionality and "a" are being combined to give a new constant, say a_2. The influence of a_2 on the solution of the flow field is also discussed in Appendix A. For the investigated cases, a_2 is assumed to be a function of y as

$$a_2 = 3.0 \frac{\partial \bar{u}}{\partial y} \Big/ \left|\frac{\partial \bar{u}}{\partial y}\right|_{max} \; ;$$

where $\left|\frac{\partial \bar{u}}{\partial y}\right|_{max}$ is the maximum velocity gradient at each x location.

In order to analyze boundary layer problems in an incompressible turbulent flow field with known pressure gradient, there are four unknowns: u, v, ε and k to be determined through four simultaneous equations: (3-1), (3-2), (3-3) and (3-4). Theoretically, the solutions are obtainable, if the required boundary conditions are prescribed.

The boundary conditions for the average velocities are:

$$\begin{aligned} &\bar{u} = \bar{v} = 0 \quad \text{at} \quad y = 0 \\ &\left.\begin{aligned} \bar{u} &= \bar{u}_\delta \\ \bar{v} &= \bar{v}_\delta \end{aligned}\right\} \quad \text{at} \quad y = \delta \end{aligned} \tag{3-8}$$

The boundary conditions for the turbulent kinetic energy are:

$$\begin{aligned} k &= 0 \quad \text{at } y = 0 \\ k &= k_\delta \quad \text{at } y = \delta \end{aligned} \tag{3-9}$$

CHAPTER IV

METHOD OF SOLUTIONS

The major difficulty in solving the governing equations of Chapter III lies in the non-linearity of the parabolic differential equations of (3-2) and (3-4). A numerical method on solving simultaneous parabolic equations was developed by Patankar (11). It is, therefore, possible to use Patankar's method in this study, if the governing equations may be transformed into a general form of parabolic differential equations.

A. The von Mises Transformation

von Mises transformed the physical coordinate system (x,y) to a streamline coordinate system (x,ψ) with the stream function defined to satisfy the continuity equation (3-1) as:

$$\bar{\rho}\ \bar{u}\ y^{\alpha} = \frac{\partial\psi}{\partial y}$$
$$\bar{\rho}\ \bar{v}\ y^{\alpha} = -\frac{\partial\psi}{\partial x} \qquad (4-1)$$

The partial differential in the x and y directions may be written as:

$$\left.\frac{\partial}{\partial x}\right)y = \left.\frac{\partial}{\partial x}\right)\psi - \rho v y^{\alpha} \left.\frac{\partial}{\partial \psi}\right)x$$
$$\left.\frac{\partial}{\partial y}\right)x = \rho u y^{\alpha} \left.\frac{\partial}{\partial \psi}\right)x \qquad (4-2)$$

Substituting Equation (4-2) into the momentum Equation (3-2) and the turbulent kinetic energy Equation (3-4), the momentum equation becomes

$$\frac{\partial\bar{u}}{\partial x} = \frac{\partial}{\partial\psi}\left(\varepsilon\bar{\rho}\bar{u}y^{2\alpha}\frac{\partial\bar{u}}{\partial\psi}\right) - \frac{1}{\bar{\rho}\bar{u}}\frac{\partial\bar{p}}{\partial x} \qquad (4-3)$$

The Turbulence Energy Equation becomes

$$\frac{\partial k}{\partial x} = \frac{\partial}{\partial \psi} \left(\frac{\varepsilon}{\sigma_k} \bar{\rho}\bar{u}y^{2\alpha} \frac{\partial k}{\partial \psi}\right) + \bar{\rho}\bar{u}y^{2\alpha} \frac{\partial \bar{u}}{\partial \psi} - \frac{D_k}{\bar{\rho}\bar{u}} \tag{4-4}$$

B. The Dimensionless Stream Function

In order to solve the parabolic equations of (4-3) and (4-4), two boundary conditions are required in the ψ direction, namely:

$\psi = \psi_I$ at the internal boundary, and

$\psi = \psi_E$ at the external boundary.

For the flat plate case, ψ_I is the stream line along the plate and ψ_E is the stream line intersecting the edge of the boundary layer at each x location. Since the interested flow field in boundary layer problems is between ψ_I and ψ_E, Patankar introduced a dimensionless stream function

$$\omega = \frac{\psi - \psi_I}{\psi_E - \psi_I} \tag{4-5}$$

Thus, the numerical solutions will be needed only in the range of $0<\omega<1$. In terms of the dimensionless stream function, the momentum equation can be written as

$$\frac{\partial \bar{u}}{\partial x} + \frac{y_I^\alpha M_I + \omega(y_E^\alpha M_E - y_I^\alpha M_I)}{\psi_E - \psi_I} \frac{\partial \bar{u}}{\partial \omega}$$

$$= \frac{\partial}{\partial \omega} \left(\frac{y^{2\alpha}\bar{\rho}\bar{u}\varepsilon}{(\psi_E - \psi_I)^2} \frac{\partial \bar{u}}{\partial \omega}\right) - \frac{1}{\bar{\rho}\bar{u}} \frac{\partial \bar{p}}{\partial x} \tag{4-6}$$

The turbulent kinetic energy equation becomes

$$\frac{\partial k}{\partial x} + \frac{y_I^\alpha M_I + \omega(y_E^\alpha M_E - y_I^\alpha M_I)}{\psi_E - \psi_I} \frac{\partial k}{\partial \omega}$$

$$= \frac{\partial}{\partial \omega} \left(\frac{y^{2\alpha}\bar{\rho}\bar{u}\varepsilon}{(\psi_E - \psi_I)^2 \sigma_k} \frac{\partial k}{\partial \omega}\right) + \frac{y^{2\alpha}\bar{\rho}\bar{u}\varepsilon}{(\psi_E - \psi_I)^2 \sigma_k} \left(\frac{\partial \bar{u}}{\partial \omega}\right)^2 - \frac{D_k}{\bar{\rho}\bar{u}} \tag{4-7}$$

where M_I and M_E are the mass flux at the internal and external boundaries, respectively, with

$$M_I = - y_I^{-\alpha} \frac{\partial \psi_I}{\partial x}$$
$$M_E = - y_E^{-\alpha} \frac{\partial \psi_E}{\partial x} \tag{4-8}$$

C. The Generalized Parabolic Equation

The generalized parabolic differential equation was given by Patankar (11) as

$$\frac{\partial \phi}{\partial x} + (A+B\omega) \frac{\partial \phi}{\partial \omega} = \frac{\partial}{\partial \omega} (C \frac{\partial \phi}{\partial \omega}) + D \tag{4-9}$$

In order to use Patankar's numerical method, the coefficients A, B, C, and D for the momentum equation (4-6) and the turbulent kinetic energy equation (4-7) are tabulated in Table I.

D. The Finite Difference Solution

In solving partial differential equations by finite difference method, it is necessary to consider the stability and convergence criteria. The solution of a finite difference equation is said to be stable, if any small error (such as a round-off error) introduced at some point in the computing process becomes smaller and smaller as the computation advances. It is said to be convergent if the solution of the finite difference equation approaches the exact solution of the differential equation. For parabolic type differential equation, Wu (13) showed that the convergence criterion is automatically satisfied if the stability criterion is satisfied. Crank and Nickolson (14) showed that an implicit finite difference equation is always stable if the parabolic differential equation is linear. Since the generalized parabolic equation (4-9) is

Table 1. The Coefficient of the Generalized Parabolic Equation

<table>
<tr><th>ϕ</th><th>u</th><th>k</th></tr>
<tr><td>A</td><td colspan="2">$\frac{y_I^{\alpha} m_I}{\psi_E - \psi_I}$</td></tr>
<tr><td>B</td><td colspan="2">$\frac{y_E^{\alpha} m_E - y_I^{\alpha} m_I}{\psi_E - \psi_I}$</td></tr>
<tr><td>C</td><td colspan="2">$\frac{y^{2\alpha} \rho u \varepsilon}{(\psi_E - \psi_I)^2}$</td></tr>
<tr><td>D</td><td>$-\frac{1}{\rho u} \frac{\partial p}{\partial x}$</td><td>$\frac{C}{\sigma_k} (\frac{\partial \overline{u}}{\partial \omega})^2 - \frac{D_k}{\rho u}$</td></tr>
</table>

non-linear only with respect to the independent variable ω, it may then be quasi-linearized by evaluating the coefficients of $(A+B\omega)$, C and D at the upstream location. The partial differentials in the implicit form become:

$$\frac{\partial\phi}{\partial x} = \frac{1}{3\Delta x}\left[(\phi_{j+1,k+1} - \phi_{j,k+1}) + (\phi_{j+1,k} - \phi_{j,k}) + (\phi_{j+1,k-1} - \phi_{j,k-1})\right]$$

$$\frac{\partial\phi}{\partial\omega} = \frac{1}{4(\Delta Y)}\left[(\phi_{j+1,k+1} - \phi_{j+1,k-1}) + (\phi_{j,k+1} - \phi_{j,k-1})\right]$$

$$\frac{\partial^2\phi}{\partial\omega^2} = \frac{1}{2(\Delta Y)^2}\left[(\phi_{j+1,k+1} - 2\phi_{j+1,k} + \phi_{j+1,k-1}) + (\phi_{j,k+1} - 2\phi_{j,k} + \phi_{j,k-1})\right]$$

(4-10)

where the subscripts j and j+1 designate the upstream and downstream locations in the x-direction, respectively. The subscripts k-1, k, and k+1 designate the successive locations in the y-direction from the internal to the external boundaries. Figure 4 shows the nomenclature used in the numerical solutions. The detailed programming technique was given in Patankar's Ph.D. dissertation (12) from the Imperial College, London, England, and is briefly outlined in Appendix B. The FORTRAN IV statement of the modified Patankar's program is given in Appendix C.

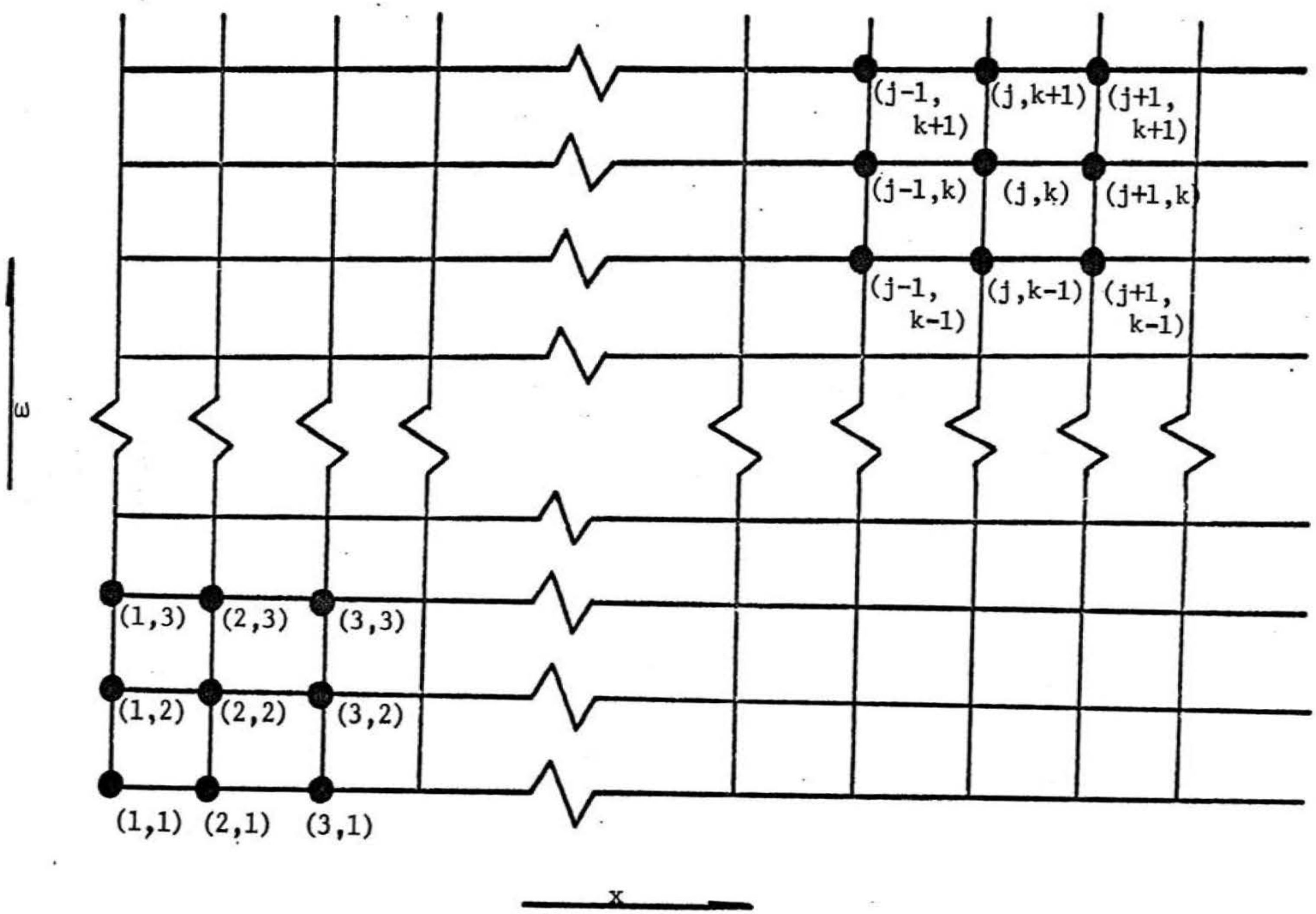

Figure 4. Flow field on x-ω coordinate

CHAPTER V

RESULTS AND DISCUSSION

The analytical solutions from the momentum and the turbulence energy equations were obtained by using the computer program of Appendix C. In order to verify the validity of this approach, an experiment conducted by Klebanoff (3) for fully developed turbulent flow along a flat plate was used for comparison. A schematic diagram of the velocity development is shown in Figure 5. The initial conditions for both average velocity and turbulent kinetic distributions were assumed to be linear between the values at the wall and the values at the free stream. The numerical solutions in the entire flow field were obtained through the step-by-step marching technique described in Appendix B. As the marching distance advances, the dimensionless profiles of the considered flow parameters should approach a unique distribution function; since the region where the dimensionless profiles remain unchanged is called a fully developed region. The experimental results of Klebanoff were comparable with the analytical solutions only in the fully developed region.

Klebanoff's experiment was conducted at the National Bureau of Standards in a 4½ foot wind tunnel. The turbulent level of the tunnel was 0.02 percent when local velocity was 30 feet per second, and 0.04 percent at 100 feet per second. The boundary layer was developed along a smooth, flat, aluminum plate 12 feet long, 4½ feet wide and ¼ inch thick with a symmetrical and pointed leading edge. The free stream velocity in this experiment was 50 feet per second. In order to obtain a condition of zero pressure gradient along the plate, the

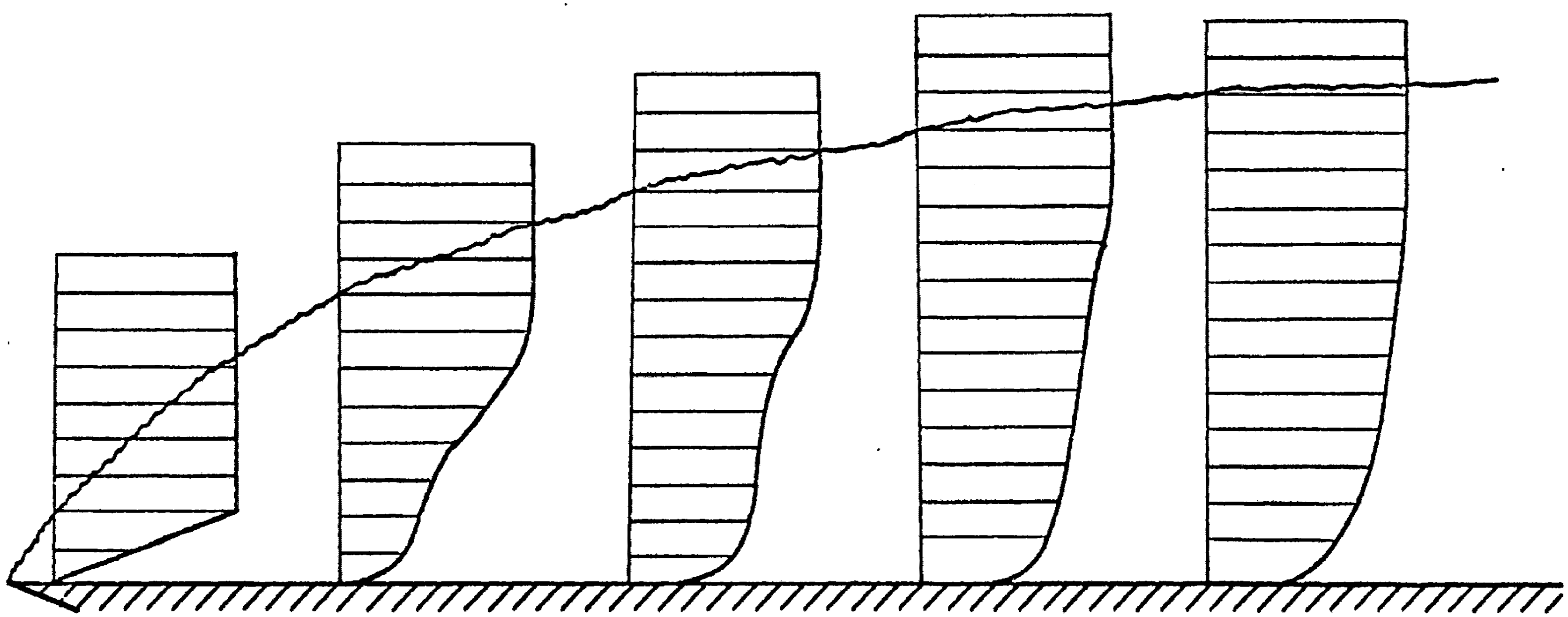

Figure 5. A Schematic Diagram of the Average Velocity Development along a Flat Plate

passage between the tunnel wall and the plate was made sufficiently divergent to offset the natural fall in pressure due to boundary layer growth. The average velocity was obtained by using pitot probes to measure the difference between the total and static pressures. The fluctuating quantities were obtained by using a constant current hot-wire anemometer to measure the various fluctuating components and their correlations.

Comparisons of the analytical solutions with Klebanoff's experimental results were made for average velocity, turbulent kinetic energy and turbulent shear stress.

A. Average Velocity Distribution

The development of the average velocity profile is shown in Figure 6. It is noted that the average velocities near the wall accelerate faster than those farther away from the wall. The dimensionless velocity distributions approach a unique profile as x increases. Comparison with Klebanoff's experimental results indicated that the computed velocity profiles agreed well with the measured profile in the fully developed region.

B. Turbulent Kinetic Energy Distribution

The development of the dimensionless turbulent kinetic energy distribution is shown in Figure 7. The analytical results indicated that turbulent kinetic energy increased first for smaller values of x then decreased slowly as x further increased. However, the analytical results in the fully developed region were somewhat less than the measure quantities. The disagreement in turbulent kinetic energy distributions may be the result of the uncertainty of either the effective

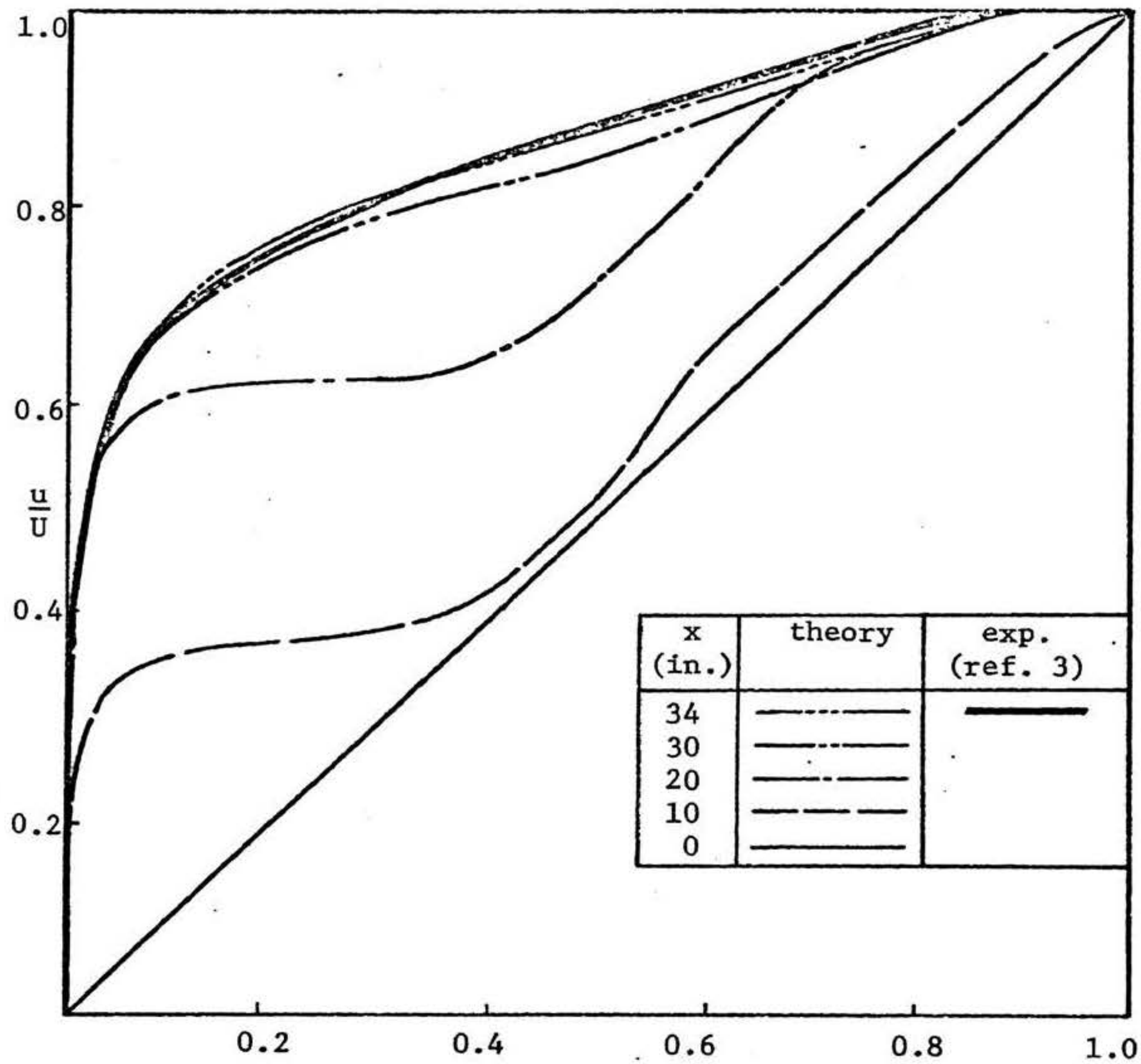

Figure 6. Average Velocity Distribution

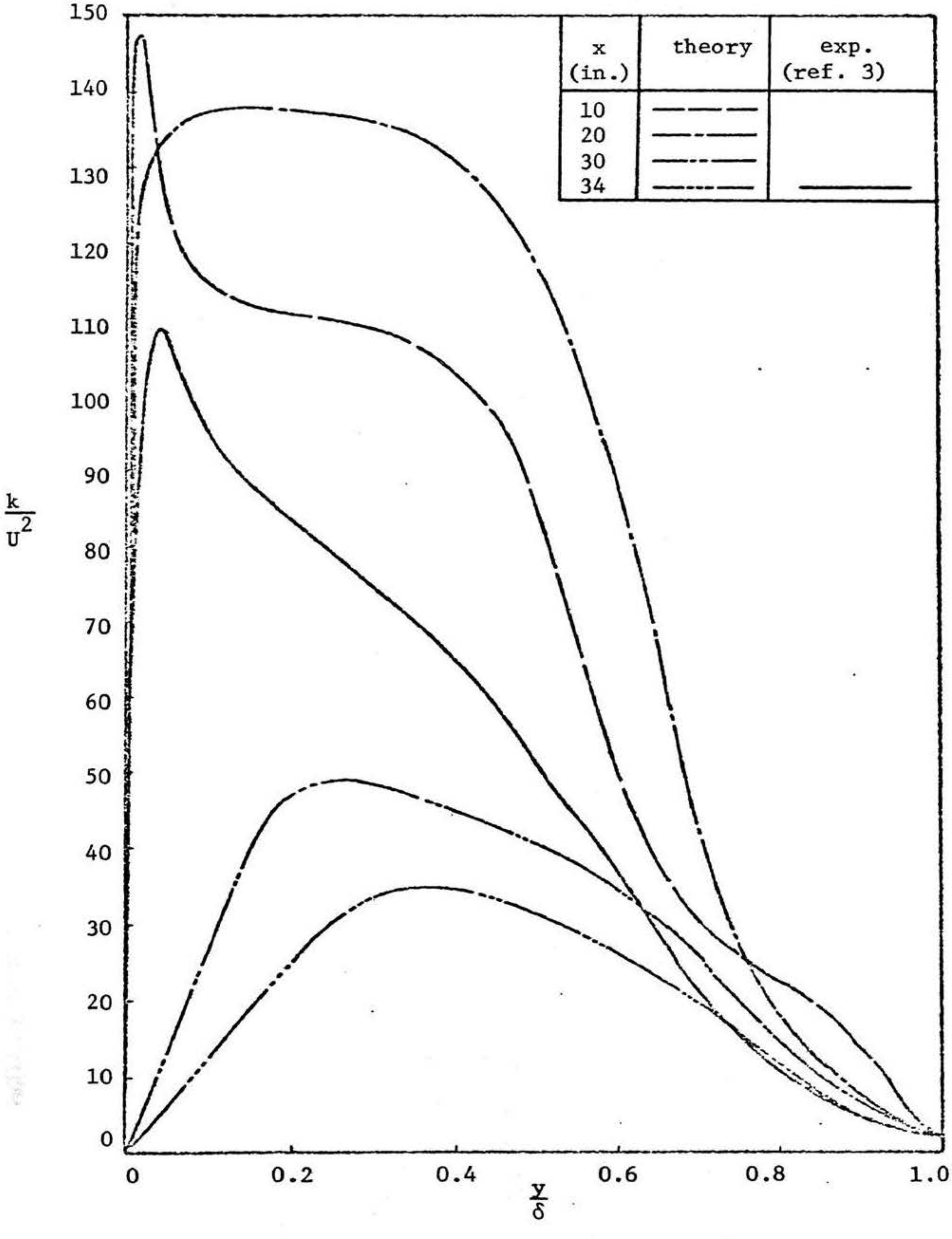

Figure 7. Turbulent Kinetic Energy Distribution

Prandtl number, σ_k, or the constant related to the dissipation, a_2. Further discussion of these two terms is given in Appendix A.

C. Shear Stress Distribution

The development of dimensionless turbulent shear stress is shown in Figure 8. The analytical results of the turbulent shear stress behave similarly as the turbulent kinetic energy with respect to the change in the x direction. The agreement with Klebanoff's data appears to be reasonably well except in the region near the wall. The effect of laminar shear along the wall was considered and is discussed in Appendix D.

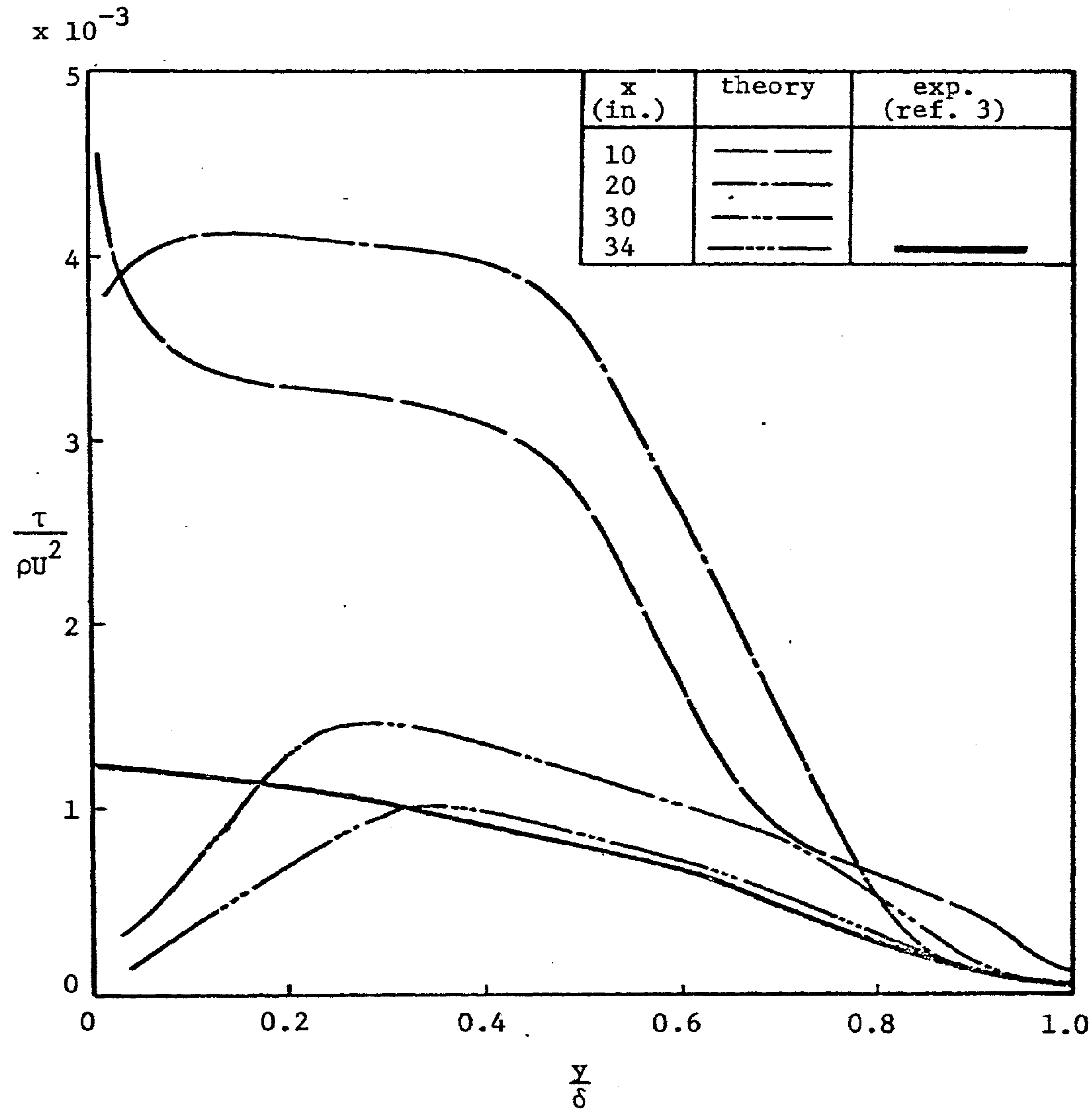

Figure 8. Turbulent Shear Stress Distribution

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The use of turbulence energy equation in analyzing boundary layer problems was conducted. By comparing with available experimental results in fully developed turbulent flow along flat plate, the following conclusions are reached:

1. The linear relation between local turbulent shear stress and local turbulent kinetic energy appears to be valid in the boundary layer region.
2. The turbulent viscosity may be treated as a dependent variable to be solved for simultaneously with the other related flow parameters, if the turbulence energy equation can be appropriately expressed.
3. The analytical solutions on average velocity distribution converge to that of the fully developed turbulent boundary layer.

The apparent success of using the turbulence energy approach in analyzing momentum transfer problems leads into the following suggestions:

1. Turbulent flow problems with heat and mass transfer in addition to momentum transfer may also be analyzed by the presented scheme as long as the governing equations can be expressed as the generalized parabolic differential equations.
2. The use of the turbulence energy equation, however, brings the necessity of further understanding of the turbulence structure in order to adequately express the terms such as turbulence

energy dissipation, effective transport coefficient of turbulence energy flux, etc. Detailed measurements of turbulence structure are needed especially in non-homogeneous and non-isotropic flow fields.

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APPENDIX A

THE EFFECT OF a_2 AND σ_k IN A TURBULENT FLOW FIELD

The dissipation term D_k in the turbulence energy equation was expressed by Patankar and Spalding (11) and Bradshow, et. al., (5) as

$$D_k = a\rho k^{3/2} \;/\; \ell_k \;; \tag{A-1}$$

where a is a constant. Since this model was originally recommended for a full developed turbulent flow field, certain modification is needed in order to apply it in the developing region. The presence of a wall is known to generate turbulence; it is, thus, reasonable to expect that the dissipation to be larger near the wall; i.e., the constant "a" may be considered as a function normal to the flow direction. The model

$$a = a_2 \left|\frac{\partial \bar{u}}{\partial y}\right| \;/\; \left|\frac{\partial \bar{u}}{\partial y}\right|_{max} \tag{A-2}$$

is to introduce the ratio of the local velocity gradient to the maximum gradient at that location as a control factor for the dissipation energy. The dissipation function then becomes

$$D_k = a_2 \; \rho \; k^{3/2} \left|\frac{\partial \bar{u}}{\partial y}\right| \;/\; \left|\frac{\partial \bar{u}}{\partial y}\right|_{max} \tag{A-3}$$

where a_2 is a constant. The effect of a_2 on the average velocity distribution in the fully developed region is shown in Figure A-1. It can be seen that the value of a_2 = 3.0 appeared to be a reasonable assumption.

The parameter σ_k appears in the turbulence energy equation, together with the turbulent viscosity ε in the diffusion term as ε/σ_k.

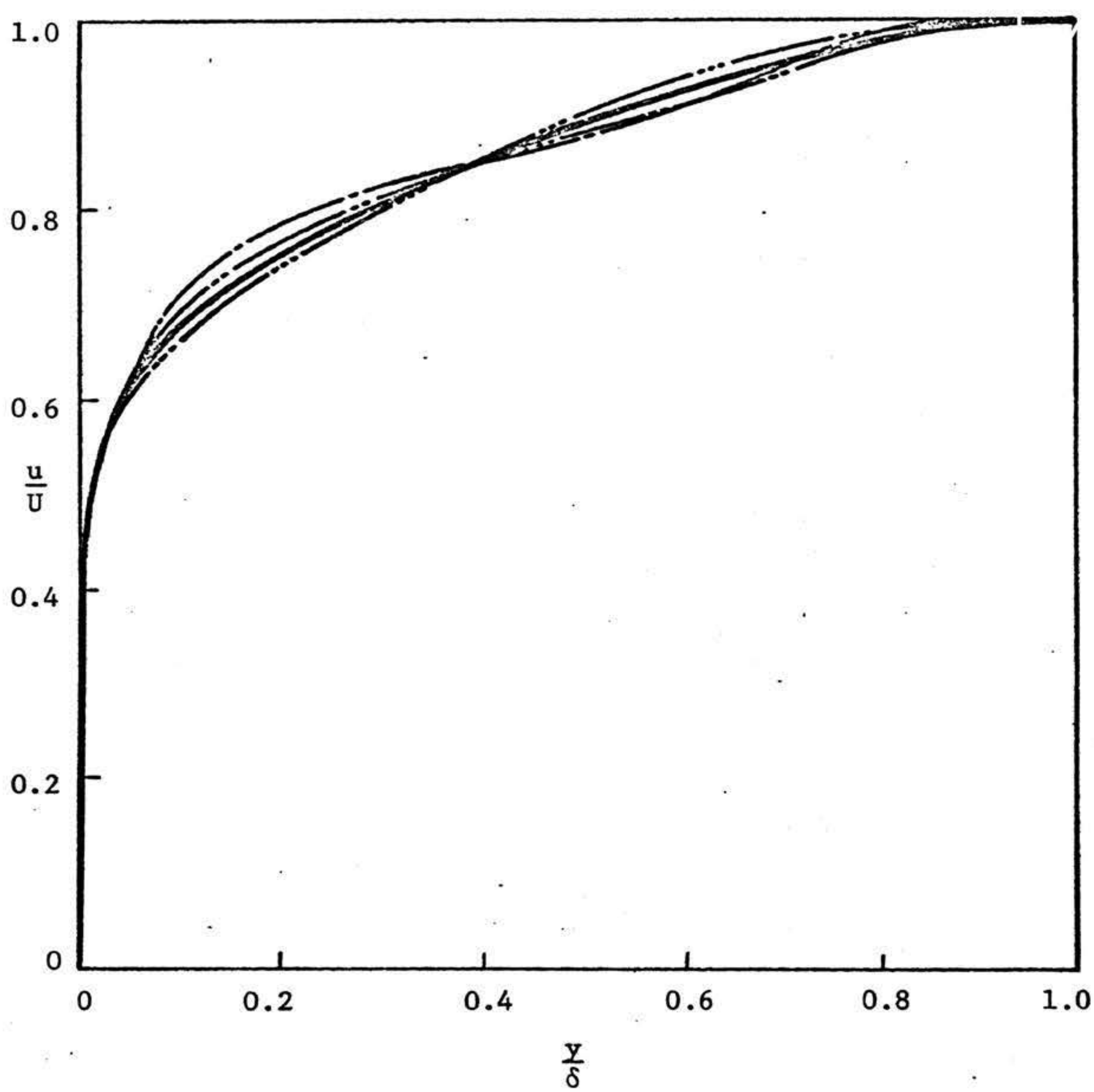

Figure A-1. The average velocity distributions affected by σ_k (for a_1=0.3, a_2=3.0)

Klebanoff's experiment
Analytical result for σ_k = 0.6
σ_k = 0.7
σ_k = 0.8

The quantity ε/σ_k is called the exchange coefficient for the turbulent kinetic energy flux in the y-direction, and is defined as

$$\frac{\varepsilon}{\sigma_k} = - \frac{J_k}{\frac{\partial k}{\partial y}} = \frac{\overline{((\rho v)'k)}}{\frac{\partial k}{\partial y}} \quad . \tag{A-4}$$

Comparing equation (A-4) with the definition of total energy flux or composition flux, as discussed by Dorrance (14), σ_k appears to have the same physical significance as Prandtl Number or Schmidt Number, respectively. In this study, due to lack of experimental evidence, σ_k is considered as constant.

The effect of σ_k on the average velocity distribution is shown in Figure A-1. It is noted that the average velocity distributions appear to be affected very insignificantly for a large range of σ_k. The value σ_k = .7 was selected because the Prandtl Number of air is known to be in this range.

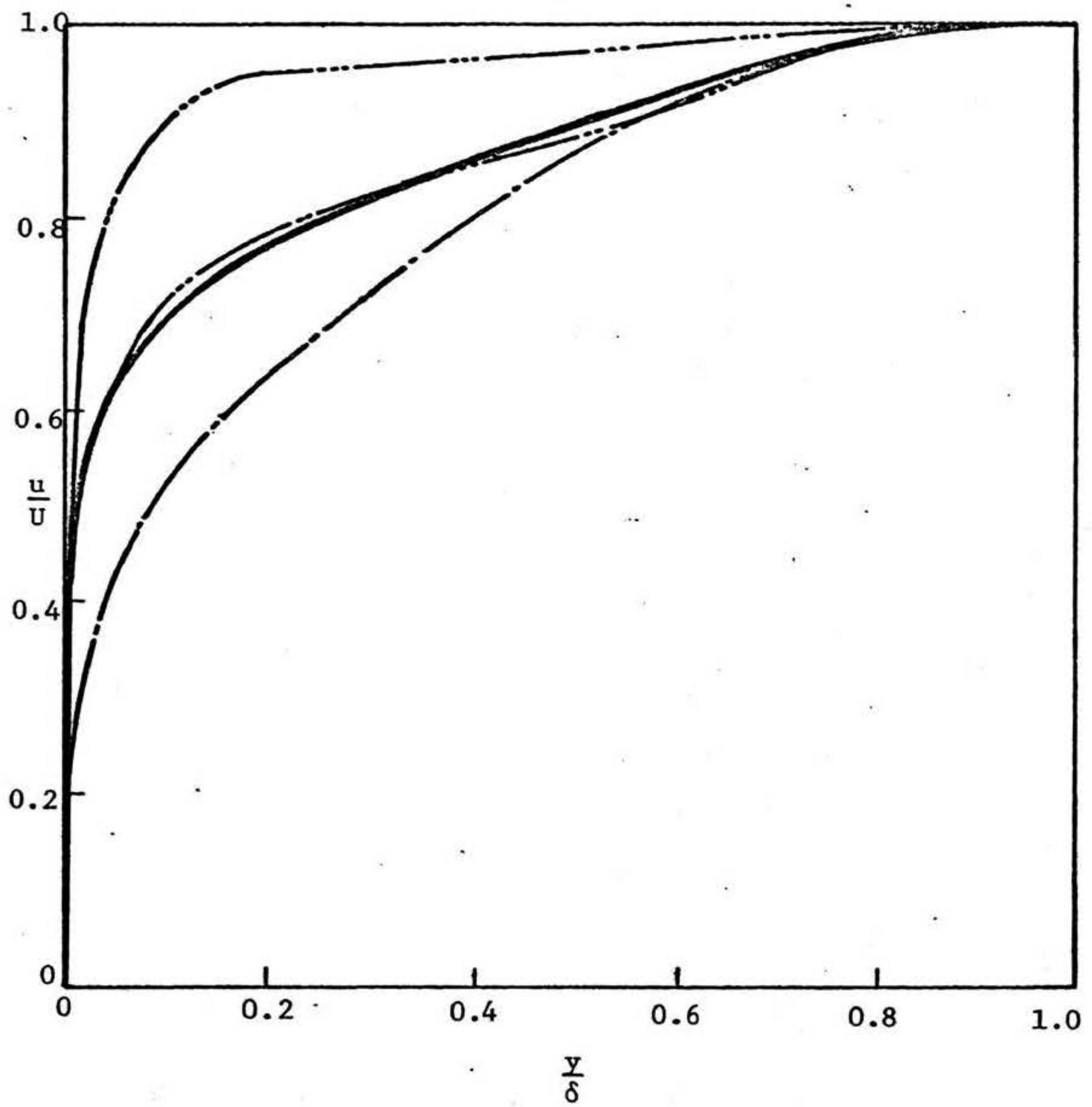

Figure A-2. The average velocity distributions affected by a_2 (for a_1=0.3, σ_k=0.7)

Klebanoff's experiment result
Analytical result a_2=1.0
a_2=3.0
a_2=5.0

APPENDIX B

OUTLINE OF THE PROCEDURE IN PATANKAR'S METHOD

Patankar (12) solved the generalized parabolic differential equation (4-9) by step-by-step numerical integration. The partial derivatives with respect to were evaluated in terms of the ϕ values at x_U, x_D or $\frac{1}{2}(x_U+x_D)$, where the subscript U and D designate upstream and downstream, respectively. Using the procedure of Crank and Nicholson (15), the stability criterion was satisfied without imposing limitations on step length in the x-direction.

For convenience, it is desired to have the resultant difference equations linear in ϕ. Therefore, the coefficients such as A, B, C in equation (4-9) will always be evaluated from the upstream values of ϕ to linearize the differential equation.

To obtain a finite-difference equation from equation (4-9), a miniature integral equation over the control volume can be formulated. The control volume is shown in Figure B-1. It is assumed that, in the ω direction, ϕ varies linearly with ω between the grid points. The variation in the x-direction will be considered to be stepwise. The values of ϕ for the interval from x_U to x_D, except at x_U, being uniform and equal to those at x_D.

The terms on the left-hand side of equation (4-9) can be expressed in the integration form.

$$\frac{\partial\phi}{\partial x} \approx F_1(\phi_{D+} - \phi_{U+}) + F_2(\phi_D - \phi_U) + F_3(\phi_{D-} - \phi_{U-}) \qquad \text{(B-1)}$$

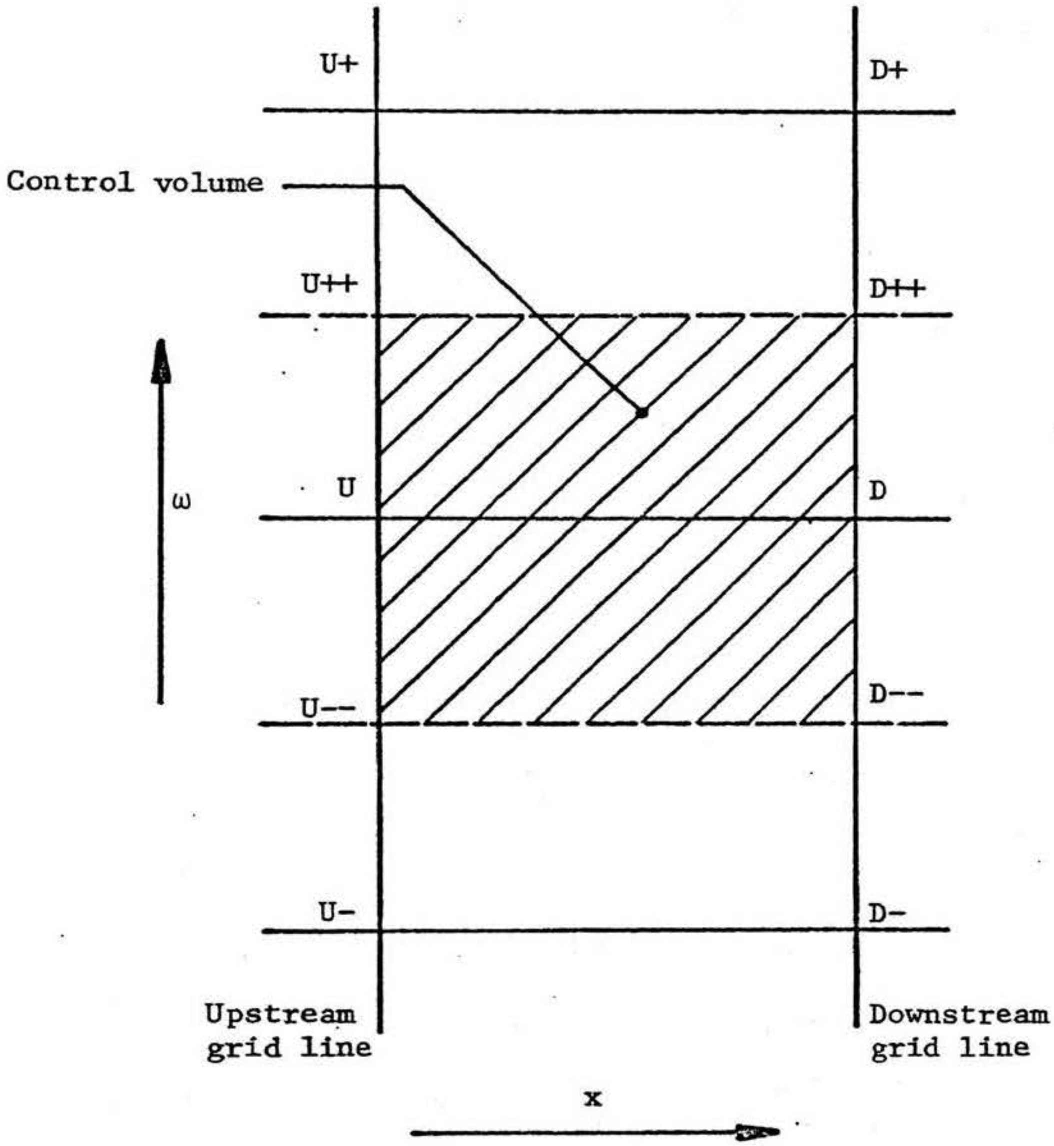

Figure B-1. The control volume for integration where

$$F_1 = \frac{\omega_{D+} - \omega_D}{4(x_D - x_U)(\omega_{D+} - \omega_{D-})} \tag{B-1A}$$

$$F_2 = \frac{3}{4(x_D - x_U)} \tag{B-1B}$$

$$F_3 = \frac{\omega_D - \omega_{D-}}{4(x_D - x_U)(\omega_{D+} - \omega_{D-})} \tag{B-1C}$$

$$a \frac{\partial \phi}{\partial \omega} \approx G(\phi_{D+} - \phi_{D-}) \tag{B-2}$$

where

$$G = \frac{a}{\omega_{D+} - \omega_{D-}} \tag{B-2A}$$

$$b\omega \frac{\partial \phi}{\partial \omega} \approx H_1 \phi_{D+} + H_2 \phi_D + H_3 \phi_{D-} \tag{B-3}$$

where

$$H_1 = \frac{b}{4} \left(\frac{\omega_{D+} + 3\omega_D}{\omega_{D+} - \omega_{D-}}\right) \tag{B-3A}$$

$$H_2 = -\frac{b}{4} \tag{B-3B}$$

$$H_3 = -\frac{b}{4} \left(\frac{\omega_{D-} + 3\omega_D}{\omega_{D+} - \omega_{D-}}\right) \tag{B-3C}$$

The complete convection term can be expressed as

$$\frac{\partial \phi}{\partial x} + (a + b\omega) \frac{\partial \phi}{\partial \omega} \approx I_1 \phi_{D+} + I_2 \phi_D + I_3 \phi_{D-} + I_4 \tag{B-4}$$

where

$$I_1 = F_1 + G + H_1 \tag{B-4A}$$

$$I_2 = F_2 + H_2 \tag{B-4B}$$

$$I_3 = F_3 - G + H_3 \tag{B-4C}$$

$$I_4 = -F_1 \phi_{U+} - F_2 \phi_U - F_3 \phi_{U-} \tag{B-4D}$$

The flux term of the right side of equation (4-9) can be expressed as

$$\frac{\partial}{\partial \omega}\left(C \frac{\partial \phi}{\partial \omega}\right) \approx I_5(\phi_{D+} - \phi_D) + I_6(\phi_D - \phi_{D-}) \tag{B-5}$$

where

$$I_5 = \frac{2C_{U++}}{(\omega_{D+} - \omega_{D-})(\omega_{D+} - \omega_D)} \tag{B-5A}$$

$$I_6 = \frac{2C_{U--}}{(\omega_{D+} - \omega_{D-})(\omega_D - \omega_{D-})} \tag{B-5B}$$

The coefficient D will be considered as uniform over the control volume and equal to that at downstream. Since D may not be linear in ϕ, D_D should be obtained from the following linearized formula

$$D_D \approx D_U + \left(\frac{\partial D}{\partial \phi}\right)_U (\phi_D - \phi_U) \ . \tag{B-6}$$

The coefficient D in the equation of conservation of momentum was assumed to vary linearly with ω between grid points. Knowing $D = -(d\bar{p}/d\bar{x})/\bar{\rho}\bar{U}$ in the equation of conservation of momentum, it can be expressed as

$$D \approx S_1 U_{D+} + S_2 U_D + S_3 U_{D-} + S_4 \tag{B-7}$$

where

$$S_1 = \frac{F_1}{\bar{\rho}_{U+}\bar{u}_{U+}^2} \frac{d\bar{p}}{dx}(x_D - x_U) \tag{B-7A}$$

$$S_2 = \frac{F_2}{\bar{\rho}_U \bar{u}_U^2} \frac{d\bar{p}}{dx}(x_D - x_U) \tag{B-7B}$$

$$S_3 = \frac{F_3}{\bar{\rho}_{U-}\bar{u}_{U-}^2} \frac{d\bar{p}}{dx}(x_D - x_U) \tag{B-7C}$$

$$S_4 = -2\,\frac{dp}{dx}(x_D - x_U)\left[\frac{F_1}{\bar{\rho}_{U+}\bar{u}_{U+}} + \frac{F_2}{\bar{\rho}_U \bar{u}_U} + \frac{F_3}{\bar{\rho}_{U-}\bar{u}_{U-}}\right] \tag{B-7D}$$

Substituting equations (B-4), (B-5) and (B-7) into equation (4-9), yields

$$\phi_D = A_U\phi_{D+} + B_U\phi_{D-} + C_U \quad \text{(B-8)}$$

where ϕ represents u and

$$A_U = \frac{I_5-I_1+S_1}{I_2+I_5-I_6-S_2} \quad \text{(B-8A)}$$

$$B_U = \frac{I_6-I_3+S_3}{I_2+I_5-I_6-S_2} \quad \text{(B-8B)}$$

$$C_U = \frac{S_4-I_4}{I_2+I_5+I_6-S_2} \quad \text{(B-8C)}$$

Substituting equations (B-4), (B-5) and (B-6) into equation (4-9), yields

$$\phi_D = A_T\phi_{D+} + B_T\phi_{D-} + C_T \quad \text{(B-9)}$$

where ϕ represents any flow parameter other than u, and

$$A_T = \frac{I_5-I_1}{I_2+I_5-I_6-(\partial D/\partial\phi)_U} \quad \text{(B-9A)}$$

$$B_T = \frac{-I_6-I_3}{I_2+I_5-I_6-(\partial D/\partial\phi)_U} \quad \text{(B-9B)}$$

$$C_T = \frac{D_U-(\partial D/\partial\phi)_U\phi_U-I_4}{I_2+I_5-I_6-(\partial D/\partial\phi)_U} \quad \text{(B-9C)}$$

In forming the finite-difference equation, the variation of ϕ, bewteen the grid points, is assumed to be linear in ω. But, near the wall, a straight line in u-ω plot, which passing through the true u value at the wall, would be poor representation of the reality in which the variations are much steeper. Patanker introduced a "slip" value of ϕ at the boundary such that the ϕ-ω line passing through the slip value rather than the true one.

The definition of slip value should be in conformity with the above requirement. In Figure B-2, the grid lines divide the interval from $\omega = 0$ to $\omega = 1$ into N strips. The subscripts 1 and 2 at the internal boundary and the subscripts N+3, N+2 at the external boundary denote the true and slip values at internal and external boundary boundary respectively. The subscript 2.5 refers to a line midway between the internal boundary and the grid line 3. Similarly, N+1.5 refers to a line midway between N=1 and the external boundary. The slip value ϕ_2 is defined as the one which enables us to obtain the correct slope and the value of ϕ at the point 2.5. Similar remarks apply to the slip value ϕ_{N+2}.

The correct values of the slope and value of ϕ at the point 2.5 and N+1.5 depends upon the nature of the boundary and on the flow properties.

Near a wall, we shall assume that the velocity profile is of the power-law type

$$u \quad \alpha |(y - y_1)|^{\beta} \tag{B-10}$$

Since by definition

$$(\omega - \omega_1) \ \alpha \int_{y_1}^{y} u \, dy \tag{B-11}$$

Thus

$$u \quad \alpha |(\omega - \omega_1)|^{\beta/(1+\beta)} \tag{B-12}$$

By matching the slope and the value of u at point 2.5, it yields

$$u_2 = \frac{1}{2+\beta} u_3 \tag{B-13}$$

For ϕ represent a value different from u, Patankar assumed a power-law profile for ϕ with power γ different than β

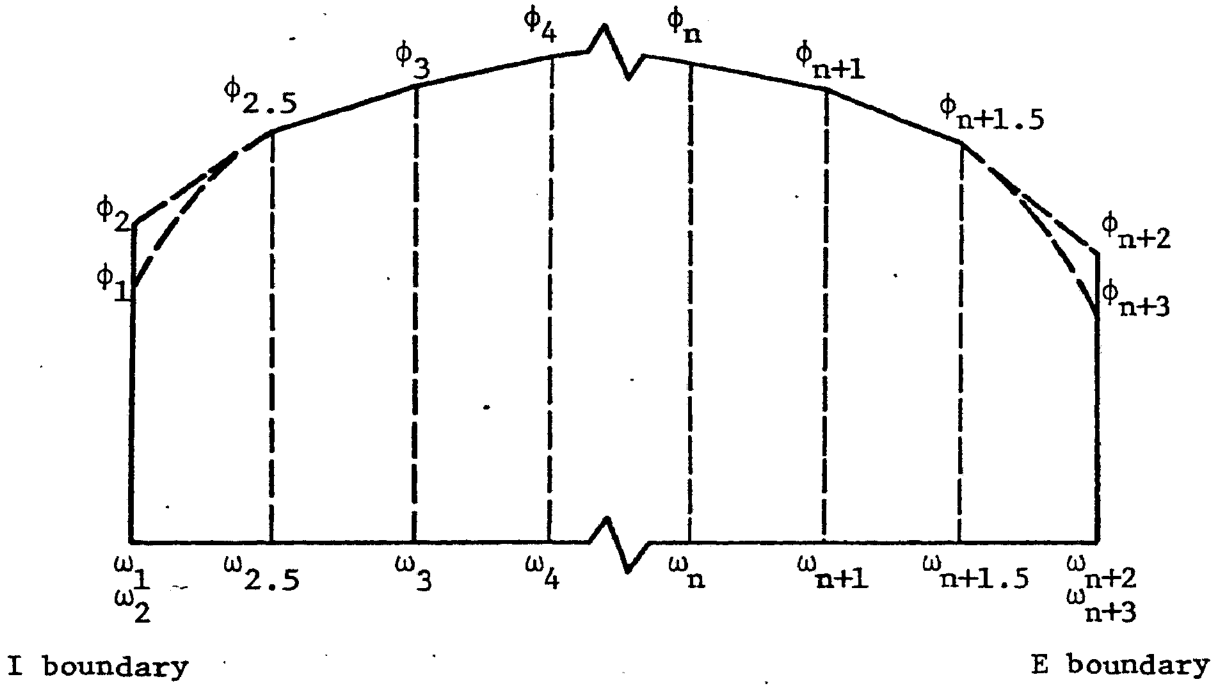

Figure B-2. The scheme of subscripts for the true and slip values of ϕ.

ϕ_2 and ϕ_{n+2} are slip values
ϕ_1 and ϕ_{n+3} are true values

$$(\phi - \phi_1) \; \alpha |(y - y_1)|^{\gamma} \qquad \text{(B-14)}$$

Substituting equations (B-10) and (B-12) into equation (B-14), yields

$$(\phi - \phi_1) \; \alpha |(\omega - \omega_1)|^{\gamma/1+\beta} \qquad \text{(B-15)}$$

Use of the slip-value definition then yields

$$\phi_2 = \phi_3 \left(\frac{1+\beta-\gamma}{1+\beta+\gamma}\right) + \phi_1 \left(\frac{2\gamma}{1+\beta+\gamma}\right) \qquad \text{(B-16)}$$

Replacing ϕ with y and γ with unity yields

$$y_2 = y_3 \left(\frac{\beta}{\beta+2}\right) + y_1 \left(\frac{2}{\beta+2}\right). \qquad \text{(B-17)}$$

In the region near a free boundary, a velocity profile can be shown parabolic in distance for turbulent flow. Therefore

$$(u - u_1) \; \alpha (y - y_1)^2 \qquad \text{(B-18)}$$

Application of the slip-value definition leads to

$$u_2 = u_3 R + u_1 (1-R) \qquad \text{(B-19)}$$

where

$$R = \frac{u_2 + u_3 - 8u_1}{5u_2 + 5u_3 + 8u_1} \qquad \text{(B-19A)}$$

Since R is a ratio of u's, it will very slowly, and hence its value calculated by using the upstream values of u can be conveniently used.

The profile for ϕ other than u will be taken as power-law type.

$$(\phi - \phi_1) \; \alpha |y - y_1|^n \qquad \text{(B-20)}$$

Use of the definition of slip value and of equation (B-20) yields

$$\phi_2 = \phi_3 R_\phi + \phi_1 (1-R_\phi) \qquad \text{(B-21)}$$

where

$$R_\phi = \frac{R+(2-n)/(2+n)}{1+R(2-n)/(2+n)} \tag{B-21A}$$

The slip value of y may then be obtained by setting n = 1, thus

$$y_2 = y_3 \left(\frac{3R+1}{3+R}\right) + y_1 \left(\frac{2(R-1)}{3+R}\right) \tag{B-22}$$

The values of β and γ were found by using the "Couette flow" concept and the van Driest's hypothesis (16), as

$$\beta = \frac{\mu(\tau_s + y\frac{dp}{dx} + uM_s)}{\varepsilon u} \tag{B-23}$$

$$\gamma = \frac{Pr(\tau_s + Mu)y}{\varepsilon u} \tag{B-24}$$

The value of n was found as twice of the Prandtl Number.

APPENDIX C

THE MODIFIED COMPUTER PROGRAM

The computer program of using the finite difference method described in Appendix B is presented in this Appendix. It is intended to provide the necessary guide lines for the user of this program.

A list of subroutines and their explanations is presented in the following:

MAIN The Main programs starts the computation and controls the sequence of operations. The choice of forward step is also made here.

COEFF This subroutine is used to obtain the coefficients A_U, B_U, C_U in equation (B-8) and A_T, B_T, C_T in equation (B-9).

SLIP The relations connecting the slip values to the neighboring true values have been expressed in the form of the corresponding coefficients in the subroutine SLIP.

SHEAR The shear stress is calculated in this subroutine by using the turbulent kinetic energy model.

SOLVE The subroutine SOLVE performs the mathematical operation of solving simultaneous equations of the type of equation (B-8) and equation (B-9).

READY After each integration, we obtain the values of u and other ϕ's for known values of ω. In the subroutine READY, the calculation of the corresponding normal distance y for every grid point is undertaken. This makes the stage ready for the performance of the next integration.

VEFF The subroutine VEFF is used to calculate the viscosity by taking use of the suggested model.

LENGTH The boundary layer thickness is calculated in this subroutine.

ENTRN The subroutine ENTRN supplies the mass flow rate.

WALL The purpose of the subroutine WALL is to evaluate the exponents β and γ for the region near a wall boundary. This subroutine derives its main information from two other subroutines, WF1 and WF2, which incorporate the wall flux relationships.

WF1 Subroutine WF1 provide the wall-flux relationship concerning the momentum transfer.

WF2 Similar to WF1, subroutine WF2 is relevant when represents a variable other than u.

SOURCE Source term in equation (4-9) is presented in this subroutine.

CONST The values of different constants including some fluid properties, mixing-length constants etc. are to be given by the user in the subroutine CONST.

DENSTY The purpose of the subroutine DENSTY is to evaluate the density at all the grid points as a function of the dependent variables.

RAD The subroutine RAD supplies the geometrical information regarding the problem.

PRE The specification of the pressure gradient is through the subroutine PRE.

MASS This subroutine is called only when a wall boundary is presented. Through this subroutine, the mass-transfer rate through the wall is supplied.

FBC When a wall boundary is presented, the boundary conditions for the fluid parameter other than average velocity are supplied in subroutine FBC.

BEGIN The initial profiles and other auxiliary quantities are to be specified in the subroutine BEGIN. A large portion of this subroutine is used to set up the slip values and calculate ω's.

OUTPUT The instructions for printing out the results are to be contained in the OUTPUT subroutine.

Several variable names in the input list are explained below.

KRAD KRAD = 1 means axisymmetrical flow
KRAD = 0 means plane flow

KIN specifies the type of internal boundary
KIN = 1 wall boundary
KIN = 2 free stream boundary
KIN = 3 symmetry-line boundary

KEX specifies the type of external boundary
KEX = 1 wall boundary
KEX = 2 free stream boundary
KEX = 3 symmetry-line boundary

NEQ number of partial differential equations to be solved

N number of grid points

KPRAN KPRAN = 0 use turbulence energy equation
KPRAN = 1 use Prandtl's mixing length hypothesis

XL values of x at which computation is to be terminated (in feet).

ASD1	a_1
ASD2	a_2
PREF(1)	σ_k
PREF(2)	effective Prandtl number
PREF(3)	effective Schmidt number

```
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
     1/B/BETA,GAMA(3),TAUI,TAUE,AJI(3),AJE(3),INDI(3),INDE(3)
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/C/SC(43),AU(43),BU(43),CU(43),A(3,43),B(3,43),C(3,43)
      COMMON/PR/UGU,UGD
      COMMON /L/AK,ALMG
      COMMON/AUXP/TEMPF(43),TEMP(43),PO(43),AMACH(43)
      COMMON/BAR/GABAR(43),RBAR(43)
      COMMON/AUXY/YY(43),XXU,RR1
      COMMON  /SHEAR/ SHEAR(43),SCSH(43)
      COMMON /ASD/ ASD1,ASD2
      COMMON/XPLOT/NPLOT
      COMMON /IDIN/ INDIC
      COMMON/DUD/DUDOM(43), DUDY(43), ADUDY(43), ADUDYM
      COMMON/DCON/DXC
      INDIC=0
      READ (5,8000) NCASE
 8000 FORMAT (I5)
   16 CONTINUE
      INDIC=INDIC+1
      X = 0.0
      INTG=0
      CALL CONST
      CALL BEGIN
      AMI=0.
      AME=0.
      GO TO 25
   15 CALL READY
   25 CONTINUE
      DO 102 I=4,N
      A2=((U(I+1)-U(I-1))/(OM(I+1)-OM(I-1))-(U(I)-U(I-1))/
     1(OM(I)-OM(I-1)))/(OM(I+1)-OM(I))
      A1=-(OM(I)+OM(I-1))*A2+(U(I)-U(I-1))/(OM(I)-OM(I-1))
  102 DUDOM(I)=A1+2.*A2*OM(I)
      DUDOM(   3)=(U(   1)-U(   4))/(OM(   1)-OM(   4))
      DUDOM(   2)=(U(   1)-U(   3))/(OM(   1)-OM(   3))
      DUDOM(1)=DUDOM(2)
      DUDOM(NP1)=(U(N   )-U(NP3))/(OM(N   )-OM(NP3))
      DUDOM(NP2)=(U(NP1)-U(NP3))/(OM(NP1)-OM(NP3))
      DUDOM(NP3)=0.
      INTG=INTG+1
      CALL LENGTH
      CALL SHEARS
      CALL ENTRN
C  CHOICE OF FORWARD STEP
      FRA=.05
      DXCN=.2+DXC
      DX=ABS(FRA    *PEI/(R(1)*AMI-R(NP3)*AME))
      IF (DX.GT.DXCN*Y(NP3))  DX=DXCN*Y(NP3)
      IF (DX.LT.0.) GO TO 85
      XD=XU+DX
   77 CONTINUE
C     CALCULATES CHANGE IN FREE STREAM VELOCITY
      UGD=U(NP3)
      UGU = UGD
      CALL PRE(XU,XD,DPDX)
      IF(KASE.EQ.2) GO TO 26
      IF(KIN.EQ.1)CALL MASS(XU,XD,AMI)
      IF(KEX.EQ.1)CALL MASS(XU,XD,AME)
      CONTINUE
      CALL WALL
   26 XXU=12.0*XU
      RR1=12.0*R(1)
      DO 90  I=1,NP3
   90 YY(I)=12.0*Y(I)
      CALL COEFF
      CALL OUTPUT
C SETTING UP VELOCITIES AT A FREE BOUNDARY
      IF(KEX.EQ.2)U(NP3)=SQRT(U(NP3)*U(NP3)-2.*(XD-XU)*DPDX/
     1RHO(NP3))
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      IF(KIN.EQ.2)U(1)=SQRT(U(1)*U(1)-2.*(XD-XU)*DPDX/RHO(1))
      CALL SOLVE(AU,BU,CU,U,NP3)
C SETTING UP VELOCITIES AT A SYMMETRY LINE
      IF(KEX.EQ.3)U(NP3)=.75*U(NP2)+.25*U(NP1)
   72 CONTINUE
      IF(NEQ.EQ.1) GO TO 30
      DO 45 J=1,NPH
      DO 46 I=2,NP2
      AU(I)=A(J,I)
      BU(I)=B(J,I)
   46 CU(I)=C(J,I)
      DO 47 I=1,NP3
   47 SC(I)=F(J,I)
      CALL SOLVE(AU,BU,CU,SC,NP3)
      DO 48 I=1,NP3
   48 F(J,I)=SC(I)
      IF(KASE.EQ.2) GO TO 81
C SETTING UP WALL VALUES OF F
      IF(KIN.EQ.1.AND.INDI(J).EQ.2)F(J,1)=((1.+BETA+GAMA(J))
     1*F(J,2)-(1.+BETA-GAMA(J))*F(J,3))*.5/GAMA(J)
      IF(KEX.EQ.1.AND.INDE(J).EQ.2)F(J,NP3)=((1.+BETA+GAMA(J
     1))*F(J,NP2)-(1.+BETA-GAMA(J))*F(J,NP1))*.5/GAMA(J)
C SETTING UP SYMMETRY-LINE VALUES OF F
   82 IF(KEX.EQ.3)F(J,NP3)=.75*F(J,NP2)+.25*F(J,NP1)
   45 CONTINUE
   30 XP=XU
      XU=XD
      IF (XU.GT.XPCG) NPLOT=5
      CALCULATION OF AUXILLARY PARAMETERS
      CALL DENSTY
      DO 60 I = 2,NP2
      AMACH(I)=U(I)/SQRT(GABAR(I)*RBAR(I)*32.2*TEMP(I))
   60 CONTINUE
      PEI=PEI+DX*(R(1)*AMI-R(NP3)*AME)
C THE TERMINATION CONDITION
      IF(INTG.EQ.151)GO TO 85
      IF(XU.LT.XL)GO TO 15
      IF(XU.GE.XL)GO TO 85
      GO TO 16
   85 CONTINUE
      IF (INDIC.NE.NCASE) GO TO 16
      STOP
      END
      SUBROUTINE COEFF
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
     1/B/BETA,GAMA(3),TAUI,TAUE,AJI(3),AJE(3),INDI(3),INDE(3)
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/C/SC(43),AU(43),BU(43),CU(43),A(3,43),B(3,43),C(3,43)
      COMMON /L/AK,ALMG
      COMMON/MXMN/RHUMX,RHUMN,RHU(43),AL
      COMMON  /SHEAR/ SHEAR(43),SCSH(43)
      COMMON/DUD/DUDOM(43), DUDY(43), ADUDY(43), ADUDYM
      COMMON /RUH/ RAAUH(43)
      DIMENSION G1(43),G2(43),G3(43),D(3,43),S1(43),S2(43),
     1S3(43)
C CALCULATION OF SMALL C 'S
      DO 99 I=2,NP1
      CALL VEFF(I,I+1,EMU)
      SC(I)=RAAUH(I)    *EMU/(PEI*PEI)
   99 CONTINUE
C THE CONVECTION TERM
      SA=R(1)*AMI/PEI
      SB=(R(NP3)*AME-R(1)*AMI)/PEI
      DX=XD-XU
      DO 71 I=3,NP1
      OMD=OM(I+1)-OM(I-1)
      P2=.25/DX
      P3=P2/OMD
      P1=(OM(I+1)-OM(I))*P3
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      P3=(OM(I)-OM(I-1))*P3
      P2=3.*P2
      Q=SA/OMD
      R2=-SB*.25
      R3=R2/OMD
      R1=-(OM(I+1)+3.*OM(I))*R3
      R3=(OM(I-1)+3.*OM(I))*R3
      G1(I)=P1+Q+R1
      G2(I)=P2+R2
      G3(I)=P3-Q+R3
      CU(I)=-P1*U(I+1)-P2*U(I)-P3*U(I-1)
C THE DIFFUSION TERM
      AU(I)=2./OMD
      BU(I)=SC(I-1)*AU(I)/(OM(I)-OM(I-1))
      AU(I)=SC(I)*AU(I)/(OM(I+1)-OM(I))
      IF(NEQ.EQ.1) GO TO 33
      DO 34 J=1,NPH
      C(J,I)=-P1*F(J,I+1)-P2*F(J,I)-P3*F(J,I-1)
      CALL SOURCE(J,I,CS,D(J,I))
      C(J,I)=-C(J,I)+CS-F(J,I)*D(J,I)
      A(J,I)=AU(I)/PREF(J)
      B(J,I)=BU(I)/PREF(J)
   34 CONTINUE
C SOURCE TERM FOR VELOCITY EQUATION
   33 PHI = 0.0
      S1(I) = (DPDX + PHI)*DX
      S2(I)=P2*S1(I)/(RHO(I)*U(I))
      S3(I)=P3*S1(I)/(RHO(I-1)*U(I-1))
      S1(I)=P1*S1(I)/(RHO(I+1)*U(I+1))
      CU(I)=-CU(I)-2.*(S1(I)+S2(I)+S3(I))
      S1(I)=S1(I)/U(I+1)
      S2(I)=S2(I)/U(I)
      S3(I)=S3(I)/U(I-1)
   71 CONTINUE
C COEFFICIENTS IN THE FINAL FORM
      DO 91 I=3,NP1
      RL=1./(G2(I)+AU(I)+BU(I)-S2(I))
      AU(I)=(AU(I)+S1(I)-G1(I))*RL
      BU(I)=(BU(I)+S3(I)-G3(I))*RL
   91 CU(I)=CU(I)*RL
      IF(NEQ.EQ.1) GO TO 76
      DO 92 J=1,NPH
      DO 92 I=3,NP1
      RL=1./(G2(I)+A(J,I)+B(J,I)-D(J,I))
      A(J,I)=(A(J,I)-G1(I))*RL
      B(J,I)=(B(J,I)-G3(I))*RL
   92 C(J,I)=C(J,I)*RL
   76 CALL SLIP
      RETURN
      END
      SUBROUTINE SHEARS
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/L1/YL,UMAX,UMIN,FR,YIP,YEM
      COMMON  /SHEAR/ SHEAR(43),SCSH(43)
      COMMON /ASD/ ASD1,ASD2
      COMMON/DUD/DUDOM(43), DUDY(43), ADUDY(43), ADUDYM
      COMMON /RUH/ RAAUH(43)
      COMMON/KJU/KMU
      ADUDYM=.0001
      DO 98   I=2,NP1
      RA=.5*(R(I+1)+R(I))
      RH=.5*(RHO(I+1)+RHO(I))
      UM=.5*(U(I+1)+U(I))
      RAAUH(I)=RA*RA*RH*UM
      SCSH(I)=RA*RH*UM/PEI
      DUDY(I)=DUDOM(I)*SCSH(I)
   98 CONTINUE
      SCSH(1)=R(1)*RHO(1)*U(2)/PEI
      SCSH(NP2)=R(NP2)*RHO(NP2)*U(NP2)/PEI
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      SCSH(NP3)=R(NP3)*RHO(NP3)*U(NP3)/PEI
      DUDY(   1)=DUDOM(   1)*SCSH(   1)
      DUDY(NP2)=DUDOM(NP2)*SCSH(NP2)
      DUDY(NP3)=DUDOM(NP3)*SCSH(NP3)
      DO 97 I=1,NP3
      ADUDY(I)=ABS(DUDY(I))
   97 CONTINUE
      DO 96 I=5,NP1
      IF (ADUDY(I).GT.ADUDYM) ADUDYM=ADUDY(I)
   96 CONTINUE
      DO 101 I=2,NP2
      IF (KPRAN.NE.0.OR.NEQ.LT.2) GO TO 35
   33 SHEAR(I)=ASD1*RHO(I)*F(1,I)*DUDY(I)/ABS(DUDY(I))
     1+0.0000036*DUDY(I)
      GO TO 101
   35 F(1,I)=0.
  101 CONTINUE
      CALL WALL
      SHEAR(NP3)=0.0
      RETURN
      END
      SUBROUTINE SLIP
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
     1/B/BETA,GAMA(3),TAUI,TAUE,AJI(3),AJE(3),INDI(3),INDE(3)
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
      COMMON /L/AK,ALMG
     1/C/SC(43),AU(43),BU(43),CU(43),A(3,43),B(3,43),C(3,43)
C SLIP COEFFICIENTS NEAR THE I BOUNDARY FOR VELOCITY EQUATION
      CU(2)=0.
      CU(NP2)=0.
      BU(2)=0.
      AU(2)=1./(1.+2.*BETA)
      SQ=84.*U(NP3)*U(NP3)-12.*U(NP3)*U(NP1)+9.*U(NP1)*U(NP1)
      AU(NP2)=8.*(2.*U(NP3)+U(NP1))/(2.*U(NP3)+7.*U(NP1)+
     1SQRT(SQ))
      BU(NP2)=1.-AU(NP2)
      IF(NEQ.EQ.1)RETURN
C SLIP COEFFICIENTS NEAR THE I BOUNDARY FOR OTHER EQUATIONS
      DO 54 J=1,NPH
      C(J,2)=0.
      C(J,NP2)=0.
      CALL FBC(XD,J,INDI(J),QI)
      IF(INDI(J).EQ.1) GO TO 61
      AJI(J)=QI
      A(J,2)=1.
      B(J,2)=0.
      C(J,2)=8.*(1.+2.*BETA)*PREF(J)*AJI(J)/(AK*AK*BETA*(1.+
     1BETA)*(1.+BETA)*(3.*RHO(2)+RHO(3))*U(3))
C SLIP COEFFICIENTS NEAR THE E BOUNDARY FOR OTHER EQUATIONS
      B(J,NP2)=(U(NP2)+U(NP1)-8.*U(NP3))/(5.*(U(NP2)+U(NP1))
     1+8.*U(NP3))
      GF=(1.-PREF(J))/(1.+PREF(J))
      B(J,NP2)=(B(J,NP2)+GF)/(1.+B(J,NP2)*GF)
   54 A(J,NP2)=1.-B(J,NP2)
      RETURN
      END
      SUBROUTINE SOLVE(A,B,C,F,NP3)
C     THIS SOLVES EQUATIONS OF THE FORM
C     F(I) = A(I)*F(I+1) + B(I)*F(I-1) + C(I)
      DIMENSION A(NP3),B(NP3),C(NP3),F(NP3)
      NP2=NP3-1
      B(2) = B(2)*F(1) + C(2)
      DO 48 I=3,NP2
      T = 1./(1.-B(I)*A(I-1))
      A(I) = A(I)*T
   48 B(I) = (B(I)*B(I-1) + C(I))*T
      DO 50 I=2,NP2
      J=NP2-I+2
   50 F(J)=A(J)*F(J+1)+B(J)
      RETURN
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      END
      SUBROUTINE READY
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
     1/B/BETA,GAMA(3),TAUI,TAUE,AJI(3),AJE(3),INDI(3),INDE(3)
      CALL DENSTY
      CALL RAD(XU,R(1),CSALFA)
C     Y NEAR THE I BOUNDARY
      IF (R(1).EQ.0.) KIN=3
      Y(2)=(1.+BETA)*OM(3)*4./((3.*RHO(2)+RHO(3))*(U(2)+U(3)
     1))
      Y(3)=Y(2)+.25*OM(3)*(1./(RHO(3)*U(3))+2./(RHO(3)*U(3)+
     1RHO(2)*U(2)))
C     Y 'S FOR INTERMEDIATE GRID POINTS
      DO 50 I=4,NP1
   50 Y(I)=Y(I-1)+.5*(OM(I)-OM(I-1))*(1./(RHO(I)*U(I))+1./
     1(RHO(I-1)*U(I-1)))
C     Y NEAR THE E BOUNDARY
      Y(NP2)=Y(NP1)+.25*(OM(NP2)-OM(NP1))*(1./(RHO(NP1)*
     1U(NP1))+2./(RHO(NP1)*U(NP1)+RHO(NP2)*U(NP2)))
      Y(NP3)=Y(NP2)+12.*(OM(NP2)-OM(NP1))/((RHO(NP1)+3.*
     1RHO(NP2))*(U(NP2)+U(NP1)+4.*U(NP3)))
      IF(CSALFA.EQ.0..OR.KRAD.EQ.0) GO TO 51
      DO 52 I=2,NP3
   52 Y(I)=2.*Y(I)*PEI/(R(1)+SQRT(R(1)*R(1)+2.*Y(I)*PEI*
     1CSALFA))
      GO TO 56
   51 DO 54 I=2,NP3
   54 Y(I)=PEI*Y(I)/R(1)
   56 Y(2)=2.*Y(2)-Y(3)
      Y(NP2)=2.*Y(NP2)-Y(NP1)
C     CALCULATION OF RADII
      DO 57 I=2,NP3
      IF(KRAD.EQ.0)R(I)=R(1)
      IF(KRAD.NE.0)R(I)=R(1)+Y(I)*CSALFA
   57 CONTINUE
      RETURN
      END
      SUBROUTINE VEFF(I,IP1,EMU)
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
      COMMON /L/AK,ALMG
     1/L1/YL,UMAX,UMIN,FR,YIP,YEM
      COMMON  /SHEAR/ SHEAR(43),SCSH(43)
      COMMON/MXMN/RHUMX,RHUMN,RHU(43),AL
      COMMON  /ASD/  ASD1,ASD2
      COMMON/DUD/DUDOM(43), DUDY(43), ADUDY(43), ADUDYM
      AL=ALMG*YL
      IF(R(1).EQ.0.) AL=1.28*ALMG*YL
      IF(KASE.EQ.2) GO TO 66
      IF(KIN.EQ.1)YM=(Y(I)+Y(IP1))*.5
      IF(KEX.EQ.1)YM= Y(NP3)-.5*(Y(I)+Y(IP1))
       IF(YM.LT.AL/AK)AL=AK*YM
   66 IF (KPRAN.EQ.0) GO TO 67
C     THIS SUBROUTINE USES THE MIXING-LENGTH HYPOTHESIS
      EMU=.5*(RHO(I)+RHO(IP1))*AL*AL*ABS(DUDOM(I)
     1    *SCSH(I)
      SHEAR(I)=EMU*DUDOM(I)*SCSH(I)
      RETURN
   67 EMU=ABS(SHEAR(I)/DUDY(I))
      RETURN
      END
      SUBROUTINE LENGTH
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
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   1/L1/YL,UMAX,UMIN,FR,YIP,YEM
    COMMON/MXMN/RHUMX,RHUMN,RHU(43),AL
    COMMON/UMUM/UMUZ(43),YMU
    COMMON/KJU/KMU
 SEARCH FOR MAX AND MIN RHU
    RHUMX=RHO(1)*U(1)
    RHUMN=RHO(1)*U(1)
    RHUMN=RHO(1)*U(1)
    DO 39 J=3,NP3
    RHU(J)=RHO(J)*U(J)
    IF (RHU(J).GT.RHUMX) RHUMX=RHU(J)
    IF (RHU(J).LT.RHUMN) RHUMN=RHU(J)
 39 CONTINUE
 SEARCH FOR THE MAXIMUM AND MINIMUM VELOCITIES
 40 UMAX=U(1)
    UMIN=U(1)
    DO 41 J=3,NP3
    IF(J.EQ.NP2)GO TO 41
    IF(U(J).GT.UMAX)UMAX=U(J)
    IF(U(J).LT.UMIN) GO TO 42
    GO TO 41
 42 UMIN=U(J)
    YMU=Y(J)
    KMU=J
 41 CONTINUE
    IF (U(1) .LT. U(NP3)) GO TO 411
    U10=.1*U(NP3)
    UTOL=ABS(UMIN-U(NP3))
    GO TO 412
411 U10=.1*U(1)
    UTOL=ABS(UMIN-U(1))
412 CONTINUE
    IF (UTOL.GT.U10) GO TO 61
    IF (U(1).EQ.U(NP3)) GO TO 61
    UM=.5*(U(1)+U(NP3))
    UMUZZ=ABS(U(1)-UM)
    DO 21 K=3,NP1
    UMUZ(K)=ABS(U(K)-UM)
    IF (UMUZ(K).LT.UMUZZ) UMU=U(K)
 21 CONTINUE
    DO 22  K=3,NP1
    IF (U(K).NE.UMU) GO TO 22
    KKU=K
    IF (U(KKU).EQ.UM) YMU=Y(KKU)
    IF (U(KKU).GT.UM) YMU=Y(KKU)+(UM-U(KKU))*(Y(KKU+1)-
   1Y(KKU))/(U(KKU+1)-U(KKU))
    IF (U(KKU).LT.UM) YMU=Y(KKU)+(UM-U(KKU))*(Y(KKU-1)-
   1Y(KKU))/(U(KKU-1)-U(KKU))
 22 CONTINUE
 61 CONTINUE
    DIF=ABS(UMAX-UMIN)*FR
 SEARCH NEAR THE I BOUNDARY
 43 YIP=0.
 SEARCH NEAR THE E BOUNDARY
 44 IF(KEX.NE.2) GO TO 45
    DO 211 I=1,NP3
    IF(U(I) .GE. .99*U(NP3)) GO TO 222
211 CONTINUE
222 YEM=Y(I)
    GO TO 46
223 U21=ABS(.5*(U(NP1)+U(NP2))-U(NP3))
    IF(U21.LT.DIF) GO TO 50
    YEM=SQRT(DIF/U21)*(.5*(Y(NP1)+Y(NP2))-Y(NP3))+Y(NP3)
    GO TO 46
 50 J=NP2
 51 J=J-1
    UJ1=U(J)-U(NP3)
    IF(ABS(UJ1).GE.DIF) GO TO 52
    GO TO 51
 52 A1=1.
    IF(UJ1.LT.0.)A1=-1.
    YEM=Y(J+1)+(Y(J)-Y(J+1))*(U(NP3)+A1*DIF-U(J+1))/(U(J)-
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  1U(J+1))
    GO TO 46
 45 YEM=Y(NP3)
 46 YL=YEM-YIP
    RETURN
    END
    SUBROUTINE FNTRN
    COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
   1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
    COMMON /L/AK,ALMG
   1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
   1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
   1/L1/YL,UMAX,UMIN,FR,YIP,YEM
    COMMON  /SHEAR/ SHEAR(43),SCSH(43)
    COMMON/DUD/DUDOM(43)
    COMMON /ASD/ ASD1,ASD2
    IF (KPRAN.NE.0.OR.NEQ.EQ.1) GO TO 822
    AME=-ABS((SHEAR(NP2)+SHEAR(NP1)-2.*SHEAR(NP3))/
   1(U(NP2)+U(NP1)-2.*U(NP3)))
    RETURN
822 AME=-8.*RHO(NP3)*((ALMG*YL)/(Y(NP1)+Y(NP2)-2.*Y(NP3)))
   1**2*ABS(U(NP1)+U(NP2)-2.*U(NP3))
    RETURN
    END
    SUBROUTINE WALL
    COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
   1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
   1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
   1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
   1/B/BETA,GAMA(3),TAUI,TAUE,AJI(3),AJE(3),INDI(3),INDE(3)
    COMMON   /SHEAR/ SHEAR(43),SCSH(43)
    COMMON/DUD/DUDOM(43), DUDY(43), ADUDY(43), ADUDYM
    COMMON /L/AK,ALMG
    COMMON /ASD/ ASD1,ASD2
 15 YI=.5*(Y(2)+Y(3))
    UI=.5*(U(2)+U(3))
    RH=.25*(3.*RHO(2)+RHO(3))
    RE=RH*UI*YI/VISCO( 1 )
    FP=DPDX*YI/(RH*UI*UI)
    AM=AMI/(RH*UI)
    CALL WF1(RE,FP,AM,S)
    BETA=SQRT(ABS(S+FP+AM))/AK
    TAUI=S*RH*UI*UI
    IF(NEQ.EQ.1) RETURN
CALCULATION OF GAMA 'S FOR THE I BOUNDARY
    DO 38 J=1,NPH
    CALL WF2(RE,FP,AM,PR(J),PREF(J),P(J),SF)
    GAMA(J)=(SF+AM)*PREF(J)/(AK*AK*BETA)
    IF(INDI(J).EQ.1)AJI(J)=SF*RH*UI*(2.*F(J,1)-F(J,2)-
   1F(J,3))*.5
 38 CONTINUE
    SHEAR(1)=ASD1*RHO(1)*F(1,1)*DUDY(1)/ABS(DUDY(1))
    RETURN
    END
    SUBROUTINE WF1(R,F,AM,S)
    COMMON /L/AK,ALMG
   1/WL/STO,AKS,RT,FT,AMT
    AKS=AK*AK
    RT=R*AKS
    ST=1./RT-.1561*RT**(-.45)+.08723*RT**(-.3)+.03713*RT**
   1(-.18)
    STO=ST
    IF(F.EQ.0.) GO TO 15
    FT=F/AKS
    FM=1.-4.*FT*RT/(585.+RT**2.5)**.4
    IF(FM.LT.0.)FM=0.
    ST=ST*FM**1.6
    GO TO 16
 15 IF(AM.EQ.0.) GO TO 16
    AMT=AM/AKS
    AMM=1.-AMT/(7.74*RT**(-1.17)+.956*RT**(-.25))
    ST=ST*AMM**4
```

```
16 S=ST*AKS
   RETURN
   END
   SUBROUTINE WF2(R,F,AM,  PR ,PRT,P,S)
   COMMON /L/AK,ALMG
  1/WL/STO,AKS,RT,FT,AMT
   ST1=STO/(1.+P*SQRT(STO))
   IF(F.EQ.0.) GO TO 15
   SSEP=1.725*RT**(-.3333)*(P+6.8)**(-1.165)
   FD=.25*FT*RT/(1.+.0625*RT)
   ST1=ST1*(1.-FD)+FD*SSEP
15 ST=ST1/PRT
   S=ST*AKS
   RETURN
   END
   SUBROUTINE SOURCE(J,I,CS,DS)
FOR CONSERVATION OF STAGNATION ENTHALPY
CAUTION- USE CONSISTENT UNITS
   THE DOT PRODUCT OF E WITH J IS NEGLECTED
   COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
 1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
  1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
  1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
  1/L1/YL,UMAX,UMIN,FR,YIP,YEM
  1/C/SC(43),AU(43),BU(43),CU(43),A(3,43),B(3,43),C(3,43)
   COMMON/ASD/ASD1,ASD2
   COMMON  /SHEAR/ SHEAR(43),SCSH(43)
   COMMON/DUD/DUDOM(43), DUDY(43), ADUDY(43), ADUDYM
   COMMON/AVDU/AVDUY
   IF  (J.GT.3) GO TO 12
   IF(J .EQ. 1) GO TO 13
12 CONTINUE
   CS = 0.0
   DS = 0.0
   GO TO 3
13 CS=SC(I)*DUDOM(I)**2
   IF (F(1,I).LE.0.)  GO TO 30
   DK=(ASD2*F(1,I)**1.5/YL)*DUDY(I)/ADUDYM
   GO TO 24
30 DK=0.
24 CONTINUE
   DS=0.
   CS=CS-DK/(.5*(U(I)+U(I+1)))
   CONTINUE
   RETURN
   END
   SUBROUTINE CONST
   COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
 1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
   COMMON /L/AK,ALMG
  1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
  1/L1/YL,UMAX,UMIN,FR,YIP,YEM
   COMMON /ASD/ASD1,ASD2
   AK=.435
   FR=.01
   PR(3) = 0.35
   PR(1)=1.
   PR(2)=.7
   AMU = 0.000012
   RETURN
   END
   SUBROUTINE DENSTY
   COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
  1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
  1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
  1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
   COMMON/AUXP/TEMPE(43),TEMP(43),PO(43),AMACH(43)
   COMMON/BAR/GABAR(43),RBAR(43)
   COMMON/TEM/TEMPT(43)
   PINF=14.7*144.
   DO 45 I=1,NP3
   CPF=.24*25000.
```

```
      F(3,I)=1.
      GABAR(I)=1.4
      RBAR(I)=53.35
      IF(NPH.LT.2)   F(2,I)=CPF*520.
   44 TEMP(I)=(F(2,I)-.5*U(I)*U(I))/CPF
      RHO(I)=PINF/(TEMP(I)*RBAR(I))
      TEMPT(I)=F(2,I)/CPF
   45 CONTINUE
      RETURN
      END
      FUNCTION VISCO(I)
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
      VISCO=AMU*(F(2,I)/F(2,NP3))**.76
      RETURN
      END
      SUBROUTINE RAD(X,R1,CSALFB)
      APPLICABLE TO AXISYMMETRIC MIXING LAYER AND JET
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
      COMMON/UMUM/UMUZ(43), YMU
      CSALFB=1.
      IF (KRAD.EQ.0) GO TO 18
      IF(X.EQ.0.) GO TO 15
      R1=R(1)*(R(1)-2.*AMI*(X-XP)/(RHO(1)*U(1)))
      IF(R1.LT.0.)R1=0.
      R1=SQRT(R1)
      RETURN
   15 RO=.25/12.
      R1=RO-YMU
      RETURN
   18 R1=1.
      RETURN
      END
      SUBROUTINE PRE(XU,XD,DPDX)
      COMMON /PR/UGU,UGD
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
    JGU AND UGD STAND FOR FREE-STREAM VELOCITIES AT XU AND XD
      DPDX=(UGU+UGD)*(UGU-UGD)*.5*RHO(NP3)/(XD-XU)
      RETURN
      END
      SUBROUTINE MASS(XU,XD,AM)
    APPLICABLE TO AN IMPERMEABLE-WALL SITUATION
      AM=0.
      RETURN
      END
      SUBROUTINE FBC(X,J,IND,AJFS)
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
    PRESCRIBES A STEP-RAMP WALL TEMPERATURE
      IF(J.NE.2) GO TO 2
      IND=1
      H MUST HAVE UNITS FT.FT/SEC.SEC
      AJFS= 0.24*1000.0*25000.0
      GO TO 3
      CONTINUE
      IND = 1
      AJFS = F(1,1)
      CONTINUE
      RETURN
      END
      SUBROUTINE BEGIN
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
     1/B/BETA,GAMA(3),TAUI,TAUE,AJI(3),AJE(3),INDI(3),INDE(3)
```

```
   1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
    COMMON/AUXP/TEMPE(43),TEMP(43),PO(43),AMACH(43)
    COMMON/BAR/GABAR(43),RBAR(43)
    COMMON /XPLOT/NPLOT
    COMMON /ASD/ ASD1,ASD2
    COMMON /L/AK,ALMG
    COMMON  /SHEAR/ SHEAR(43),SCSH(43)
    COMMON/DCON/DXC
PROBLEM SPECIFICATION
    READ (5,42) KRAD,KIN,KEX,NEQ,N,NPLOT,KPRAN,KSST
42  FORMAT (9I5)
    READ (5,43) XL,XPCG,ASD1,ASD2,ALMG,PREF(1),PREF(2),
   1PREF(3),DXC,SHS
43  FORMAT (10E5.0)
44  FORMAT (2E10.0)
    KASA=1
    XU=0.
    NPH=NEQ-1
    NP1=N+1
    NP2=N+2
    NP3=N+3
INITIAL VELOCITY PROFILE
45  CONTINUE
    READ (5,444) Y(1), (Y(I), I=3,NP1), Y(NP3)
    READ (5,444) U(1), (U(I), I=3,NP1), U(NP3)
    READ (5,444) F(1,1), (F(1,I), I=3,NP1), F(1,NP3)
44  FORMAT (7F10.5)
    Y(  1)=Y(  1)/12.
    DO 111 I=3,NP1
    Y(I)=Y(I)/12.
11  CONTINUE
    Y(NP3)=Y(NP3)/12.
46  CONTINUE
CALCULATION OF SLIP VELOCITIES AND DISTANCES
    BETA=.143
    U(2)=U(3)/(1.+2.*BETA)
    Y(2)=Y(3)*BETA/(2.+BETA)
    U11=U(NP1)*U(NP1)
    U13=U(NP1)*U(NP3)
    U33=U(NP3)*U(NP3)
    SQ=84.*U33-12.*U13+9.*U11
    U(NP2)=(16.*U33-4.*U13+U11)/(2.*(U(NP1)+U(NP3))+SQRT(
   1SQ))
    Y(NP2)=Y(NP3)-(Y(NP3)-Y(NP1))*(U(NP2)+U(NP1)-2.*U(NP3))
   1*.5/(U(NP2)+U(NP1)+U(NP3))
    IF(NEQ.EQ.1) GO TO 45
CALCULATION OF CORRESPONDING SLIP VALUES
    DO 88 J=1,NPH
    GAMA(J)=.143
    F(J,2)=F(J,1)+(F(J,3)-F(J,1))*(1.+BETA-GAMA(J))/(1.
   1+BETA+GAMA(J))
    G=(U(NP2)+U(NP1)-8.*U(NP3))/(5.*(U(NP2)+U(NP1))+8.*
   1U(NP3))
    GF=(1.-PREF(J))/(1.+PREF(J))
    GF=(G+GF)/(1.+G*GF)
88  F(J,NP2)=F(J,NP1)*GF+(1.-GF)*F(J,NP3)
45  CONTINUE
    CALL DENSTY
CALCULATION OF RADII
    CALL LENGTH
    CALL RAD(XU,R(1),CSALFA)
    IF(CSALFA.EQ.0..OR.KRAD.EQ.0) GO TO 27
    DO 28 I=2,NP3
28  R(I)=R(1)+Y(I)*CSALFA
    GO TO 29
27  DO 30 I=2,NP3
30  R(I)=R(1)
29  CONTINUE
CALCULATION OF OMEGA VALUES
    OM(1)=0.
    OM(2)=0.
    DO 49 I=3,NP2
```

```
      OM(I)=OM(I-1)+.5*(RHO(I)*U(I)*R(I)+RHO(I-1)*U(I-1)*
     1R(I-1))*(Y(I)-Y(I-1))
      PEI=OM(NP2)
      DO 59 I=3,NP1
      OM(I)=OM(I)/PEI
      OM(NP2)=1.0
      OM(NP3)=1.
      IF(NEQ.EQ.1)RETURN
      DO 69 J=1,NPH
      IF(KEX.EQ.1)INDE(J)=1
      IF(KIN.EQ.1) INDI(J)=1
      CONTINUE
      RETURN
      END
      SUBROUTINE OUTPUT
      COMMON /GEN/PEI,AMI,AME,DPDX,PREF(3),PR(3),P(3),DEN,
     1XL,DX,INTG,CSALFA,XPCG,AMU,XU,XD,XP
     1/V/U(43),F(3,43),R(43),RHO(43),OM(43),Y(43)
     ./C/SC(43),AU(43),BU(43),CU(43),A(3,43),B(3,43),C(3,43)
      COMMON /L/AK,ALMG
     1/L1/YL,UMAX,UMIN,FR,YIP,YEM
     1/I/N,NP1,NP2,NP3,NEQ,NPH,KEX,KIN,KASE,KRAD,KPRAN
     ./B/BETA,GAMA(3),TAUI,TAUE,AJI(3),AJE(3),INDI(3),INDE(3)
      COMMON/AUXP/TEMPE(43),TEMP(43),PO(43),AMACH(43)
      COMMON/AUXY/YY(43),XXU,RR1
      COMMON /XPLOT/NPLOT
      COMMON  /SHEAR/ SHEAR(43),SCSH(43)
      COMMON /IDIN/ INDIC
      COMMON/MXMN/RHUMX,RHUMN,RHU(43),AL
      COMMON/DUD/DUDOM(43), DUDY(43), ADUDY(43), ADUDYM
      COMMON  /ASD/  ASD1,ASD2
      COMMON/TEM/TEMPT(43)
      COMMON/UMUM/UMUZ(43),YMU
      DIMENSION BUFF(2000),XP1(45),XP2(45),XP3(45),XP4(45),
     .XP5(45),YYYY(45)
      IF(INTG.NE.1) GO TO 15
      WRITE(6,49)(OM(I),I=1,NP3)
      IKONT=NPLOT-1
      FORMAT(24H1THE VALUES OF OMEGA ARE/(11F10.4))
      CONTINUE
      UJUO=U(1)/U(NP3)
      RHJO=RHO(1)/RHO(NP3)
      TOJO=TEMPT(1)/TEMPT(NP3)
      DO 60  I=1,NP3
      AMACH(I)=SHEAR(I)/(RHO(I)*100.**2)
      IF (KRAD.EQ.0) GO TO 61
      TEMPE(I)=RR1+YY(I)
      GO TO 60
      TEMPE(I)=YY(I)-12.*YMU
      CONTINUE
      WRITE(6,51) XXU,RR1
      FORMAT('1    XU=   ',2PE11.2,'    RI =    ',2PE11.2,' IN')
      IF (KPRAN.NE.0.OR.NEQ.LT.2) GO TO 250
      WRITE (6,55) UJUO,RHJO,TOJO,PREF(1),PREF(2),PREF(3),
      ASD1,ASD2
      FORMAT(1H0,6HUJ/UO=F6.3,2X,8HRHJ/RHO=F6.3,2X,7HTOJ/TO=
      F6.3,2X,6HPREF1=F5.3,2X,6HPREF2=F5.3,2X,6HPREF3=F5.3,
      2X,5HASD1=F6.3,2X,5HASD2=F6.3)
      GO TO 251
      WRITE (6,50) UJUO,RHJO,TOJO,PREF(2),PREF(3),AL
      FORMAT(1H0,6HUJ/UO=F6.3,2X,8HRHJ/RHO=F6.3,2X,7HTOJ/TO=
       F6.3,2X,6HPREF2=F5.3,2X,6HPREF3=F5.3,2X,3HAL=F7.3)
      CONTINUE
      WRITE(6,54)
      WRITE(6,52)
      FORMAT( 1H0,8X,1HY,11X,1HU,11X,1HH,11X,2HCE,10X,1HT,
      11X,2HRY,7X,5HK. E.,10X,1HM,10X,3HRHO,6X,5HDU/DY,9X,
      ,2HIN,7X,6HFT/SEC,3X,12HFT**2/SEC**2,19X,1HR,10X,2HIN,
      2X,12HFT*FT/SEC**2,12X,8HLB/FT**3,4X,9HFT/SEC/FT/)
      FORMAT(1H  1P11E12.3   )
      FORMAT(1H0   )
      IKONT=IKONT+1
```

```
      IF (IKONT-NPLOT)101,100,100
      IKONT=0
      CONTINUE
      DO 10  J1=1,NP3
      J2=NP2-J1+2
      YYYY(J2)=YY(J2)/YY(NP3)
      WRITE(6,53) YY(J2),U(J2),F(2,J2),F(3,J2),TEMP(J2),
     1YYYY(J2),F(1,J2),AMACH(J2),RHO(J2),DUDY(J2)
      IF(FLOAT(INTG-1)/5..NE.FLOAT((INTG-1)/5))RETURN
      RETURN
      END
```

APPENDIX D

SOME CONSIDERATIONS ON THE BOUNDARY CONDITION

The turbulent viscosity model derived from the linear relation :ween the local turbulence shear stress and local turbulent kinetic :rgy is being further examined here since the relation fails to be ısonable at the wall boundary. It is obvious that the turbulent ıetic energy is zero at the wall because the fluctuating components : zero. However, the shear stress always exists between the solid l fluid surfaces. The use of laminar shear stress in addition to the ·bulent shear was also considered. The experimental data of Kle-ıoff were recalculated by assuming

$$\tau = a_1 \bar{\rho} k + \mu \frac{\partial \bar{u}}{\partial y} \tag{D-1}$$

:re μ is the dynamics viscosity of air. A dimensionless plot with ρU^2 as ordinate and $\frac{k}{U^2} \frac{\partial u/\partial y}{|\partial u/\partial y|} + \frac{\mu}{a_1 \rho U^2} \frac{\partial \bar{u}}{\partial y}$ as abscissa is shown in ;ure (D-1). The linear relation remains valid and $a_1 = .3$ is still .e. Analytical solutions for average velocity profiles were obtained using three shear stress models represented by equations (D-1), (2-1), . (2-3). Figure D-2 shows the comparison of the three solutions with .banoff's experimental data. It is noted that the difference is not ·y significant. However, the influence on turbulent kinetic energy substantial. The analytical results in Chapter V were obtained by .ng equation (D-1) to relate the local turbulent shear stress with local kinetic energy. The boundary condition at the wall for shear ess is also related to the roughness of the wall which was not able be evaluated from Klebanoff's experiment. This may partially result

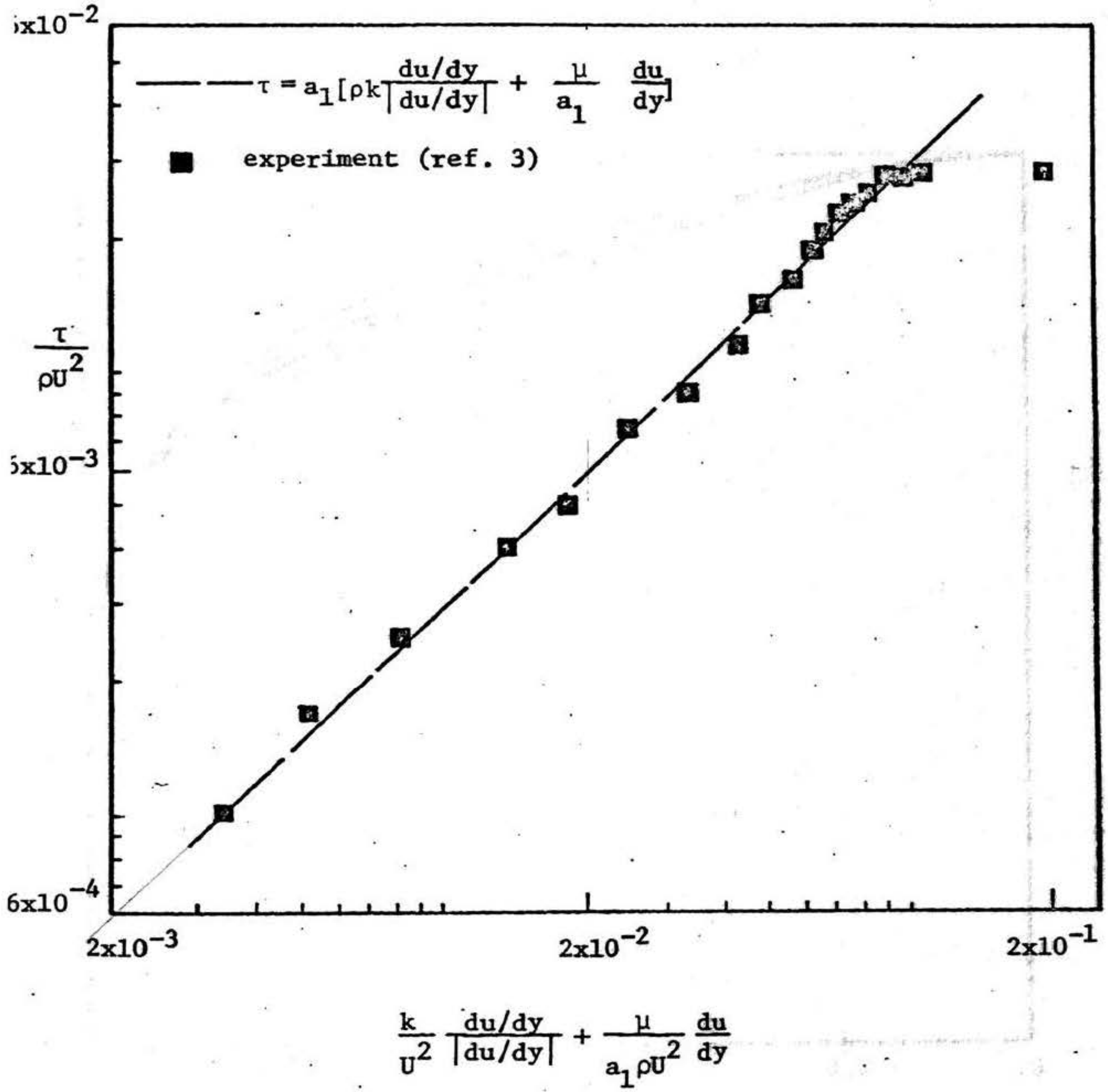

Figure D-1. Plot of Klebanoff's data by Using Equation (D-1)

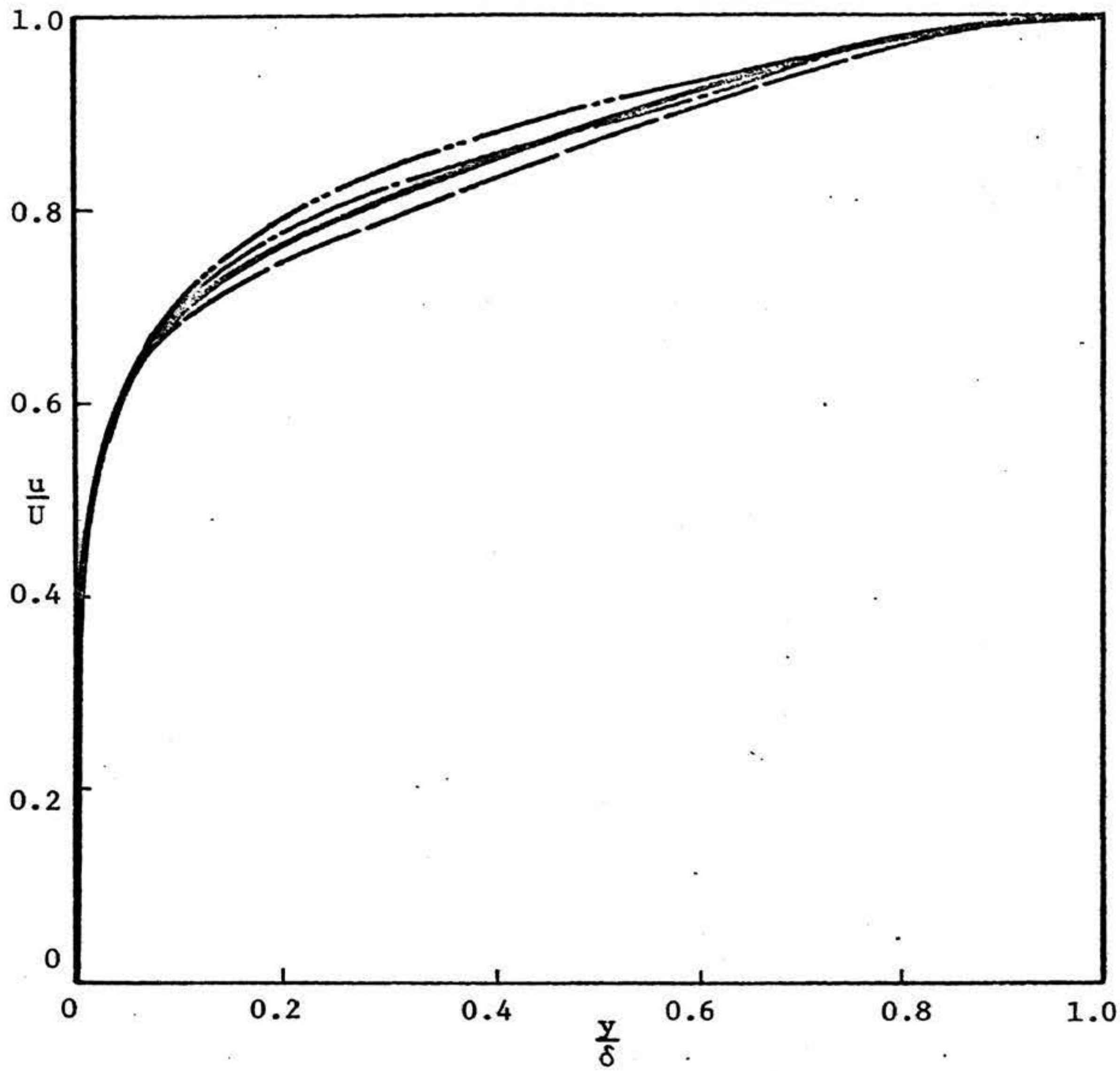

Figure D-2. The average velocity distributions

Klebanoff's experimental result
result of Bradshow's model
result of Equation (D-1) model (a_1=0.3, a_2=3.0, σ_k=0.7)
Prandtl's mixing length theory

o the deviation of the analyzed shear stress from the measured shear tress in the vicinity of the wall.

VITA

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