1966

A mathematical study of Voigt viscoelastic Love wave propagation

David Nuse Peacock

A MATHEMATICAL STUDY OF VOIGT

VISCOELASTIC LOVE WAVE

PROPAGATION

BY

DAVID NUSE PEACOCK - 1943 -

A

THESIS

submitted to the faculty of

THE UNIVERSITY OF MISSOURI AT ROLLA

in partial fulfillment of the requirements for the

Degree of

MASTER OF SCIENCE IN GEOPHYSICAL ENGINEERING

Rolla, Missouri

1966

Approved by

Nuel B. Rupert (advisor) James C. Maxwell

George B. Clark Lee J. Bain

ABSTRACT

This research is a mathematical investigation of the propagation of a Love wave in a Voigt viscoelastic medium. A solution to the partial differential equation of motion is assumed and is shown to satisfy the three necessary boundary conditions. Velocity restrictions on the wave and the media are developed and are shown to be of the same form as those governing the elastic Love wave.

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Dr. Gerald B. Rupert for his continual encouragement and advice, without which this investigation would not have been possible. Special thanks are extended to the National Science Foundation for the financial support of the author's graduate study through a Traineeship grant.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION AND LITERATURE REVIEW

Although most wave equations assume propagation in an elastic medium, it is well known that many solids do not exactly obey the laws of the theory of elasticity. The purpose of this research, therefore, is to assume a non-elastic medium, that represented by a Voigt viscoelastic element, and investigate the conditions necessary for the propagation of a Love wave.

To the earthquake seismologist and to those concerned with predicting the effects of explosives in solids, the Love wave is one of the most important types of waves that have been observed. With accurate earthquake seismograms of Love waves, the thickness of the superficial layer of the earth (the crust) may be determined. On a smaller scale, in seismic exploration, knowledge of the thickness of the weathered surface layer is of primary importance.

A surface wave whose particle motion is horizontal and transverse to the direction of propagation is called a Love wave after A.E.H. Love, who proved its existence in an elastic medium (1), and demonstrated that it is propagated by multiple internal reflections within the low velocity superficial layer. See Figure 1. Love found that the wave could only exist if its phase velocity, V^*; was related to the velocities of normal shear waves, V_{s1} and V_{s2}, in the first and second medium, respectively, by

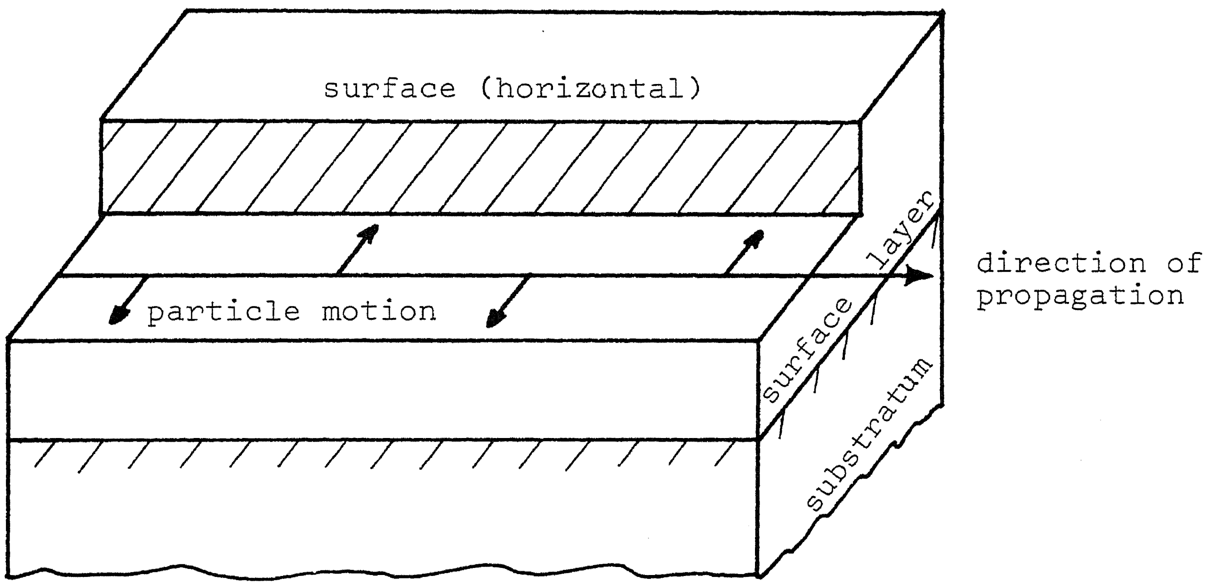

Figure 1. Love Wave Geometry

(after DOBRIN, 1960: Geophysical Prospecting, McGraw-Hill, p. 20.)

the inequality $V_{s1} < V^* V_{s2}$. The study of Love waves in an elastic double surface layer was undertaken by Stoneley and Tillotson (2), who assumed that the velocities of normal shear waves in the first, second, and third media were governed by $V_{s1} < V_{s2} < V_{s3}$, and then showed that there are two main cases which yield a solution. They are $V_{s1} < V^* V_{s2}$ and $V_{s2} < V^* V_{s3}$. Stoneley (3) has also treated the problem of the existence of a Love wave in the presence of three elastic surface layers, with the necessary conditions that $V_{s1} < V_{s2} < V_{s3} < V_{s4}$. He found three velocity conditions which yield solutions, namely $V_{s1} < V^* V_{s2}$, $V_{s2} < V^* V_{s3}$, and $V_{s3} < V^* V_{s4}$. Elastic Love waves exhibit what is commonly referred to as dispersion, a continual spreading out of the disturbance into trains of waves, each train propagating with its own group velocity. Numerous findings on dispersion curves, velocities measured from earthquake seismograms, and other characteristics of Love waves are to be found in textbooks and throughout seismic literature. Among the leading investigators are Jeffreys, Stoneley, Sezawa, Gutenberg, Byerly, and Wilson.

The investigation of viscoelastic wave propagation was initiated by Sezawa (4), who was concerned primarily with purely dilatational plane waves, and obtained his solution using Fourier integrals. An important contribution was made by Thompson (5), who developed a general theory of viscoelasticity by the complete application of the principle of virtual work to a strained and straining imperfectly elastic solid. He showed that any solution of the equations of motion which hold for forced or free vibrations, subject to given initial conditions of displacement and velocity, and subject to the boundary conditions, is a unique solution. Hardtwig (6) assumed the period of his plane shear waves to be complex, and the wave length to be imaginary. Rösler (7) let his complex shear modulus be $\mu' + \mu\frac{\partial}{\partial t}$ in operator form, calling μ his elastic constant and μ' his viscoelastic constant. This is in general disagreement with other work on the subject. The constants are obviously reversed since otherwise, the modulus does not degenerate to the elastic case for $\mu' = 0$. Sentis (8) employed a response time, τ, in his study of distortional viscoelastic waves, obtaining $V^2 = \frac{\mu}{\rho}(1 + \frac{\tau}{T})$ as an expression for the velocity, where μ and ρ follow the usual notation for elastic shear modulus and density, respectively, and T is an arbitrary coefficient.

The physical reasons most often discussed for the deviations from Hooke's law are creep along grain boundaries, diffusion of atoms, and thermoelastic heat flow. Lücke (9) studied in detail the effects of thermoelastic heat flow between neighboring grains in polycrystalline material and between the regions of successive rarefaction and compression in a compressional wave. He stated that pure shear waves exhibit no thermoelastic attenuation. Bland (10) presented an excellent treatise on viscoelasticity current to 1960, employing the operational calculus of Heaviside to obtain many of his solutions.

Kanai (11) treated the problem of Love wave propagation in a Voigt solid under the condition that there is a tangential resistance at the surface of discontinuity that is proportional to the relative tangential velocity. He presented a solution for the particle displacement of the first medium as

$$U_1 = (A\cos s_1 z + B\sin s_1 z)\exp\left[i(p_o t - fx)\right].$$

This research has shown that either or both of the constants A and B must be complex. Kanai has made no such statement. Furthermore, he assumed that p_o is complex and that f is purely real. In the undertaking of this problem, it is assumed that f should be complex and p_o should be purely real, following the arguments of Kolsky (12), Hunter (13), and Rupert (14), each of whom employed these conditions to obtain solutions for viscoelastic waves other than Love waves.

To the author's knowledge, no research has been done on the specific problem of Love wave propagation in a Voigt viscoelastic medium, other than the one paper mentioned above by Kanai, which appears to be in error.

CHAPTER II

MATHEMATICAL ANALYSIS OF VISCOELASTIC LOVE WAVES

In order to eliminate the necessity of using rather intricate mathematics, the following simplifying assumptions are made concerning the media:

1. Both strata are homogeneous isotropic solids, which extend to infinity in the positive and negative x and y directions (See figure 2).
2. The mass densities of both media are real, positive, finite parameters, and are not equal to zero.
3. All elastic and viscoelastic constants are real, positive, finite parameters, and are not equal to zero.
4. All initial effects of the disturbance have vanished.
5. No plastic deformation can occur.
6. All body forces are negligible.

A. DEVELOPMENT OF EQUATIONS OF MOTION

The general partial differential equation governing total wave displacement in a Voigt viscoelastic medium as given by Kolsky (12) is

$$\rho\frac{\partial^2(U,V,W)}{\partial t^2} = \left[(\lambda + \mu) + (\lambda' + \mu')\frac{\partial}{\partial t}\right]\left(\frac{\partial\Delta}{\partial x},\frac{\partial\Delta}{\partial y},\frac{\partial\Delta}{\partial z}\right) + (\mu + \mu'\frac{\partial}{\partial t})\nabla^2(U,V,W). \qquad (1)$$

Since it is presupposed for this problem that there is no dilatational wave motion, upon elimination of the terms involving the dilatation, Δ, this general equation immediately reduces to

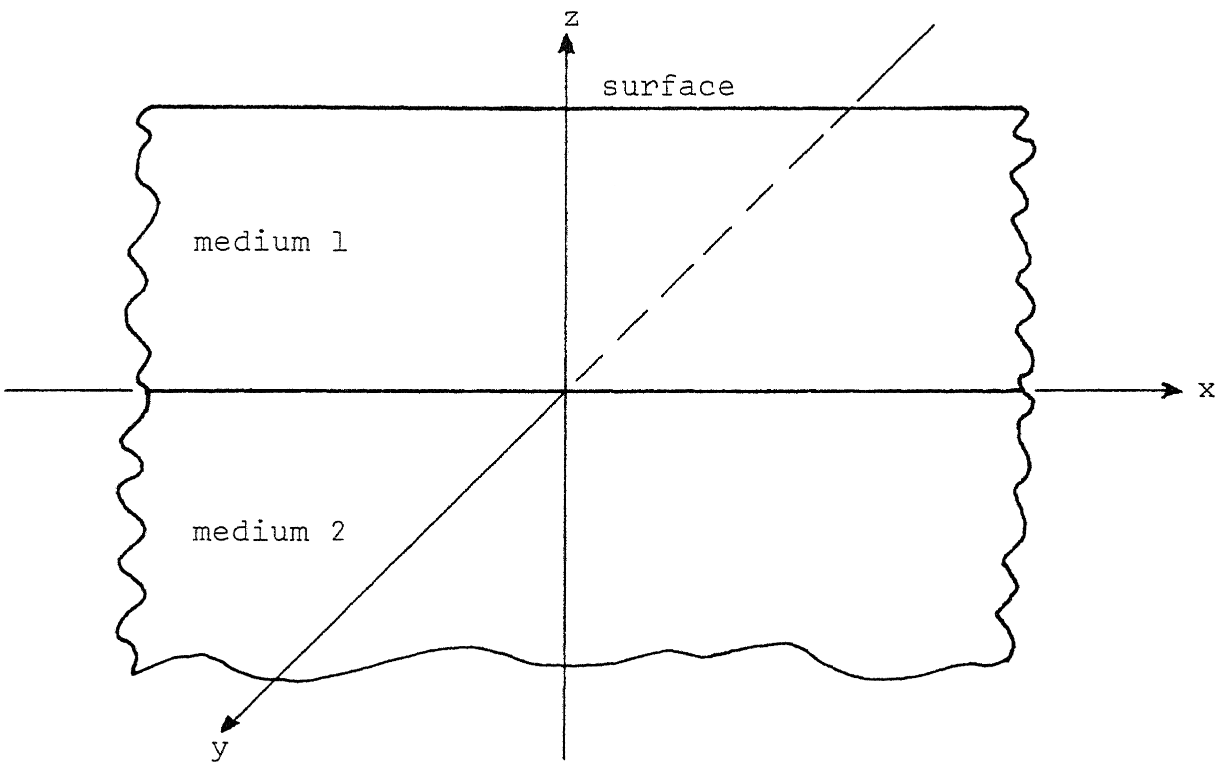

Figure 2. Love Wave Coordinate System.

$$(\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 (U,V,W) = \rho \frac{\partial^2 (U,V,W)}{\partial t^2} . \tag{2}$$

where U, V, and W represent the particle displacements in the x, y, and z directions, respectively. Because the particle motion is horizontal and transverse to the direction of propagation,

$$U = W \equiv 0, \tag{3}$$

which reduces equation (2) to

$$(\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 V = \rho \frac{\partial^2 V}{\partial t^2} . \tag{4}$$

If V is independent of y,

$$\frac{\partial^2 V}{\partial y^2} = 0, \tag{5}$$

which leads to

$$(\mu + \mu' \frac{\partial}{\partial t}) \left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} \right] = \rho \frac{\partial^2 V}{\partial t^2} . \tag{6}$$

Equation (6) is the partial differential equation of motion for an SH wave, i.e., a transverse wave whose particle motion is horizontal. This agrees with the general equation of Kanai (11).

In the preceding equations the quantity

$$(\mu + \mu' \frac{\partial}{\partial t}) \tag{7}$$

may be considered as the operator form of the complex shear modulus. It is seen that when $\mu' = 0$, for the case of no solid viscosity, equation (6) reduces to the classic equation for elastic shear wave propagation.

B. ASSUMED SOLUTION

In general, equation (6) will not be satisfied by solutions of the form $V = G(x - ct)$ or $V = G(x + ct)$ because of the presence of $\frac{\partial^3 V}{\partial x^2 \partial t}$ and $\frac{\partial^3 V}{\partial z^2 \partial t}$. Therefore, assuming an harmonic solution for the displacement

$$V = (A\cos mz + iB\sin mz)e^{i(pt - fx)} , \tag{8}$$

substituting into equation (6), and simplifying, one obtains

$$m^2 = \frac{\rho p^2}{\mu + i\mu' p} - f^2 . \tag{9}$$

In equation (8), the quantity

$$(\mu + i\mu' p) \tag{10}$$

is the complex shear modulus for an isotropic Voigt medium. For harmonic oscillations, where p is the coefficient of t, the use of the operator form of the modulus and the use of the complex form of the modulus both lead to the same result. Hence

$$(\mu + \mu' \frac{\partial}{\partial t}) = (\mu + i\mu' p). \tag{11}$$

Collecting real and imaginary terms in equation (9), one obtains

$$m^2 = \frac{\mu\rho p^2 - f^2\mu^2 - f^2\mu'^2 p^2}{\mu^2 + \mu'^2 p^2} - \frac{i\mu'\rho p^3}{\mu^2 + \mu'^2 p^2} . \tag{12}$$

By De Moivre's Theorem, there are exactly two distinct square roots of m. Letting

$$\frac{\mu\rho p^2 - f^2\mu^2 - f^2\mu'^2p^2}{\mu^2 + \mu'^2p^2} = F, \tag{13}$$

$$\frac{\mu'\rho p^3}{\mu^2 + \mu'^2p^2} = E, \tag{14}$$

and

$$\tan^{-1}\frac{E}{F} = \Theta \ , \tag{15}$$

the expressions for the roots are

$$m_1 = \sqrt[4]{E^2 + F^2}\ (\cos\frac{\Theta}{2} + i \sin\frac{\Theta}{2}), \tag{16}$$

and

$$m_2 = \sqrt[4]{E^2 + F^2}\ \left[\cos\ (\frac{\Theta}{2} + \pi) + i \sin\ (\frac{\Theta}{2} + \pi)\right], \tag{17}$$

or, by trigonometric reduction,

$$m_1 = \sqrt[4]{E^2 + F^2}(\cos\frac{\Theta}{2} + i \sin\frac{\Theta}{2}), \tag{18}$$

and

$$m_2 = -\sqrt[4]{E^2 + F^2}(\cos\frac{\Theta}{2} + i \sin\frac{\Theta}{2}) \ , \tag{19}$$

so that

$$m_2 = -m_1 \ . \tag{20}$$

Hereafter,

$$f_1,\ m_1,\ V_1,\ \mu_1,\ \mu_1'\ ,\ \rho_1,\ p_1$$

are assumed to be associated with the first medium, and

$$f_2,\ m_2,\ V_2,\ \mu_2,\ \mu_2',\ \rho_2,\ p_2$$

with the second medium.

Upon substitution of the subscripted parameters, the solution for the first medium is

$$V_1 = (A\cos m_1^2 z + iB\sin m_1^2 z)e^{i(p_1 t - f_1 x)} \quad . \tag{21}$$

With the proper changes in constants, the trigonometric terms of the general solution may be written as

$$A\cos mz + iB\sin mz = Ce^{imz} + De^{-imz} \quad . \tag{22}$$

Substituting the subscripted parameters, one obtains

$$V_2 = (Ce^{im_2^2 z} + De^{-im_2^2 z})e^{i(p_2 t - f_2 x)} \quad , \tag{23}$$

or

$$V_2 = (Ce^{im_1^2 z} + De^{-im_1^2 z})e^{i(p_2 t - f_2 x)} \quad . \tag{24}$$

Because m_1 is some complex number, equation (24) may be written as

$$V_2 = (Ce^{\,i(\delta + i\gamma)^2 z} + De^{-i(\delta + i\gamma)^2 z})e^{i(p_2 t - f_2 x)} \quad , \tag{25}$$

or more simply,

$$V_2 = (Ce^{-2\delta\gamma z + i(\delta^2-\gamma^2)z} + De^{2\delta\gamma z - i(\delta^2-\gamma^2)z})e^{i(p_2 t - f_2 x)}, \tag{26}$$

where

$$m_1 = \delta + i\gamma \ . \tag{27}$$

If $\delta\gamma$ is negative, V_2 must be restricted by saying

$$V_2 = Ce^{-2\delta\gamma z + i(\delta^2 - \gamma^2)z} \; e^{i(p_2 t - f_2 x)} \quad , \tag{28}$$

since $z < 0$ in the second medium, and the wave must be attenuated with depth. (If $\delta\gamma$ is positive, one may simply choose $De^{2\delta\gamma z - i(\delta^2-\gamma^2)z} \; e^{i(p_2 t - f_2 x)}$ as the solution.)

Following the arguments of Kolsky (12), it is assumed that

$$f_1 = f_1' + i\alpha_1 \quad , \tag{29}$$

so that, substituting in equation (21), the expression for the displacement becomes

$$V_1 = (A\cos m_1^2 z + iB\sin m_1^2 z)e^{\alpha_1 x + i(p_1 t - f_1' x)} \quad . \tag{30}$$

Similarly, let

$$f_2 = f_2' + i\alpha_2 \tag{31}$$

in equation (28) to obtain

$$V_2 = Ce^{\alpha_2 x + i(m_1^2 z + p_2 t - f_2' x)} \quad . \tag{32}$$

C. BOUNDARY CONDITIONS

The presence of three undetermined constants in the solutions necessitates the existence of three boundary conditions. The first of these is

$$V_1 = V_2 \quad ; \quad z = 0 \quad , \tag{33}$$

which states that the tangential displacements must be continuous at z = 0. To permit the media to behave otherwise would allow separation at the interface. This condition requires that

$$\alpha_1 = \alpha_2 \ , \tag{34}$$

$$p_1 = p_2 \ , \tag{35}$$

$$f_1' = f_2' \ , \tag{36}$$

and

$$A = C \quad . \tag{37}$$

The second boundary condition is

$$(\mu_1 + \mu_1'\frac{\partial}{\partial t})\frac{\partial V_1}{\partial z} = (\mu_2 + \mu_2'\frac{\partial}{\partial t})\frac{\partial V_2}{\partial z} \quad ; \quad z = 0. \tag{38}$$

This equation implies that the tangential stress must be continuous at z = 0, for the same reasons as those governing equation (33). Equation (38) follows the same format as the elastic boundary condition originally set forth by Love (1), and reiterated by Macelwane (15) and others. However, Kanai (11) employed

$$(\mu_1 + \mu_1'\frac{\partial}{\partial t})\frac{\partial V_1}{\partial z} = -K\frac{\partial}{\partial t}(V_1 - V_2) \tag{39}$$

and

$$(\mu_2 + \mu_2'\frac{\partial}{\partial t})\frac{\partial V_2}{\partial z} = -K\frac{\partial}{\partial t}(V_1 - V_2) , \tag{40}$$

giving as an explanation of these equations the statement that "there is a tangential resistance that is proportional to the relative tangential velocity." He also states that "the transversal components of stress are not (continuous)." From his statements, Kanai would allow separation of the media at the interface, and corresponding slippage of one layer upon the other. However, since the right hand sides of equations (39) and (40) are identical, the left hand sides may be equated to obtain equation (38), which appears to contradict the above quotations.

Substituting equations (30) and (32) into equation (38), performing the indicated operations, and simplifying, the result is

$$\frac{B}{C} = \frac{\mu_2 + i\mu_2' p_2}{\mu_1 + i\mu_1' p_1} = \frac{B}{A} \quad . \tag{41}$$

In general, this equation states that the relative amplitudes of the waves in the two media are a function of the parameters $p_1 = p_2$, which have units of reciprocal time. In order for equation (41) to be satisfied, either B or C, or both B and C must be complex except when $\mu_1 = \mu_2$ and $\mu_1' = \mu_2'$. Kanai made no such statement regarding the amplitudes.

The third boundary condition, which conforms to that of Kanai, is

$$(\mu_1 + \mu_1' \frac{\partial}{\partial t}) \frac{\partial V_1}{\partial z} = 0 \quad ; \quad z = S \quad . \tag{42}$$

This condition requires that there be no stress on the free surface, z = S, for all values of x and t. Equation (30), under this condition yields

$$(\mu_1 + i\mu_1' p_1) m_1^2 (-A\sin m_1^2 S + iB\cos m_1^2 S) = 0 \quad . \tag{43}$$

Since

$$(\mu_1 + i\mu_1' p_1) \neq 0 \quad , \tag{44}$$

and

$$m_1^2 \neq 0 \quad , \tag{45}$$

(otherwise the entire problem is trivial), equation (43) can only be satisfied if

$$(-A\sin m_1^2 S + iB\cos m_1^2 S) = 0 \quad . \tag{46}$$

Therefore

$$\tan m_1^2 S = \frac{iB}{A} \quad , \tag{47}$$

or, using equation (41),

$$\tan m_1^2 S = \frac{i(\mu_2 + i\mu_2' p_2)}{(\mu_1 + i\mu_1' p_1)} \quad . \tag{48}$$

The important result of this boundary condition is that for every real value of $p_1 = p_2$, there is a complex value of m_1 such that equation (48) is satisfied.

D. VELOCITY CONSIDERATIONS

By normal convention, the complex shear wave velocities of the two media are

$$V_{s1} = \left[\frac{\mu_1}{\rho_1}\left(1 + ip_1 \frac{\mu_1'}{\mu_1}\right)\right]^{1/2} \quad , \tag{49}$$

and

$$V_{s2} = \left[\frac{\mu_2}{\rho_2}\left(1 + ip_2 \frac{\mu_2'}{\mu_2}\right)\right]^{1/2} \quad . \tag{50}$$

Letting

$$\frac{\rho_1 p_1^2}{\mu_1 + i\mu_1' p_1} = k_1^2 \tag{51}$$

in equation (9), and rearranging, one obtains

$$\frac{1}{k_1^2} = \frac{\mu_1}{\rho_1 p_1^2}\left(1 + ip_1 \frac{\mu_1'}{\mu_1}\right) \quad . \tag{52}$$

Therefore,

$$k_1^2 = \frac{p_1^2}{V_{s1}^2} \quad , \tag{53}$$

and

$$k_2^2 = \frac{p_2^2}{V_{s2}^2} \quad . \tag{54}$$

However, from equation (35), $p_1 = p_2$, so that

$$k_2^2 = k_1^2 \frac{V_{s1}^2}{V_{s2}^2} \quad . \tag{55}$$

From equations (54) and (9) the relationship

$$-k_2^2 + f_2^2 = f_2^2 - \frac{p_2^2}{V_{s2}^2} = -m^2 \tag{56}$$

is found. Since

$$m_2^2 = m \quad , \tag{57}$$

equation (56) may be written as

$$m_2^4 = f_2^2 - k_1^2 \frac{V_{s1}^2}{V_{s2}^2} \quad , \tag{58}$$

after substitution of equation (55). By simultaneously adding and subtracting the quantity $f_2^2 \frac{V_{s1}^2}{V_{s2}^2}$, and since equations (34) and (36) imply that $f_1^2 = f_2^2$, equation (58) may be written as

$$m_2^4 = f_1^2\left(1 - \frac{V_{s1}^2}{V_{s2}^2}\right) - (k_1^2 - f_1^2)\frac{V_{s1}^2}{V_{s2}^2} \quad , \tag{59}$$

which becomes

$$m_2^4 = f_1^2\left(1 - \frac{V_{s1}^2}{V_{s2}^2}\right) - m_1^4 \frac{V_{s1}^2}{V_{s2}^2} \quad , \tag{60}$$

because $k_1^2 - f_1^2 = m_1^4$.

The fact that $|m^2| > 0$ directly implies that

$$\left| f_1^2 \left(1 - \frac{V_{s1}^2}{V_{s2}^2}\right) - m_1^4 \frac{V_{s1}^2}{V_{s2}^2} \right| > 0 \quad , \tag{61}$$

since $|m^2| = |m_2^4|$. A development is given below which shows that condition (61) is satisfied if $|V_{s1}| < |V_{s2}|$.

Beginning with equation (60), it follows immediately that

$$\left|\frac{m^2}{m^2}\right| = \left|\frac{f_1^2}{m^2}\left(1 - \frac{V_{s1}^{\ 2}}{V_{s2}^{\ 2}}\right) - \frac{V_{s1}^{\ 2}}{V_{s2}^{\ 2}}\right| \quad . \tag{62}$$

Hence the right hand side of equation (62) lies within the unit circle in the complex plane. Using a familiar triangle inequality, this condition is given by

$$|1| \; > \; \left|\frac{f_1^2}{m^2}\left(1 - \frac{V_{s1}^{\ 2}}{V_{s2}^{\ 2}}\right)\right| - \left|\frac{V_{s1}^{\ 2}}{V_{s2}^{\ 2}}\right| \quad , \tag{63}$$

which leads to

$$|1| + \left|\frac{V_{s1}^{\ 2}}{V_{s2}^{\ 2}}\right| > \left|\frac{f_1^2}{m^2}\right|\left(|1| - \left|\frac{V_{s1}^{\ 2}}{V_{s2}^{\ 2}}\right|\right) \quad . \tag{64}$$

Assuming that

$$|1| - \left|\frac{V_{s1}^{\ 2}}{V_{s2}^{\ 2}}\right| \tag{65}$$

is positive, (see Appendix B.), inequality (64) may be written as

$$|V_{s1}| < |V_{s2}| \quad . \tag{66}$$

Since this assumption does not lead to a contradiction or an absurdity, it is regarded as justifiable. Condition (66) is entirely logical because velocity is generally observed to increase with depth.

Restating equation (48) in the form

$$\tan m_1^2 S = \frac{i(\mu_2 + i\mu_2' P_2)}{(\mu_1 + i\mu_1' P_1)}\left[+\left(\frac{m^2}{m^2}\right)^{\frac{1}{2}}\right] \quad , \tag{67}$$

and substituting from equation (62), it follows that

$$\tan m_1^2 S = \frac{i(\mu_2 + i\mu_2' p_2)}{(\mu_1 + i\mu_1' p_1)} \left[\frac{f_1^2}{m_1^4}\left(1 - \frac{V_{s1}^2}{V_{s2}^2}\right) - \frac{V_{s1}^2}{V_{s2}^2}\right]^{1/2} , \quad (68)$$

which, because of inequality (61), implies that

$$|\tan m_1^2 S| \quad > \quad 0 \quad . \quad (69)$$

Expressing the Love wave phase velocity as

$$V^* = \frac{p_1}{f_1} \quad , \quad (70)$$

using equation (53) and the fact that $m_1^4 = k_1^2 - f_1^2$, the relationship

$$m_1^2 = p_1\left(\frac{1}{V_{s1}^2} - \frac{1}{V^{*2}}\right)^{1/2} \quad (71)$$

is obtained. Upon substitution of this expression into equation (68), the result is

$$\tan\left[p_1 S\left(\frac{1}{V_{s1}^2} - \frac{1}{V^{*2}}\right)^{\frac{1}{2}}\right] = \frac{i(\mu_2 + i\mu_2' p_2)}{(\mu_1 + i\mu_1' p_1)}$$

$$\left[\frac{p_1^2/V_{s1}^2}{\frac{p_1^2}{V_{s1}^2} - \frac{p_1^2}{V^{*2}}}\left(1 - \frac{V_{s1}^2}{V_{s2}^2}\right) - \frac{V_{s1}^2}{V_{s2}^2}\right]^{1/2} , \quad (72)$$

or more simply,

$$\tan\left[p_1 S\left(\frac{1}{V_{s1}^2} - \frac{1}{V^{*2}}\right)^{\frac{1}{2}}\right] = \frac{i(\mu_2 + i\mu_2' p_2)}{(\mu_1 + i\mu_1' p_1)}$$

$$\left[\frac{V_{s1}}{V_{s2}}\left(\frac{V_{s2}^2 - V^{*2}}{V^{*2} - V_{s1}^2}\right)^{1/2}\right] . \quad (73)$$

Equation (68) may also be written as

$$f_1 S = m_1^2 S \left[\frac{V_{s1}^2}{V_{s2}^2 - V_{s1}^2} - \frac{V_{s2}^2}{V_{s2}^2 - V_{s1}^2}\frac{(\mu_1 + i\mu_1' p_1)^2}{(i\mu_2 - \mu_2' p_2)^2}\tan^2 m_1^2 S\right]^{\frac{1}{2}} , (74)$$

which implies that there is a value of f_1 corresponding to any value of m_1^2. Thus as $|m_1^2 S|$ ranges from 0 to $\pi/2$, $|f_1 S|$ ranges from 0 to ∞. Therefore $\left|\frac{m_1^2}{f_1}\right|$ decreases as $|f_1|$ increases. But the wave length is

$$|\Lambda| = \left|\frac{2\pi}{f_1}\right| \quad , \tag{75}$$

so that $\left|\frac{m_1^2}{f_1}\right|$ decreases as $|\Lambda|$ decreases. Furthermore, equation (71) may be rearranged to yield

$$V^{*2} = V_{s1}^{2} \left(\frac{p_1^2}{p_1^2 - m_1^4 V_{s1}^{2}} \right) \quad , \tag{76}$$

which, after simplification utilizing equation (53), reduces to

$$V^{*2} = V_{s1}^{2} \left(1 + \frac{m_1^4}{f_1^2} \right) \quad . \tag{77}$$

As the real and imaginary parts of f_1 approach infinity, $|\Lambda|$ approaches zero, and the quantity $\frac{m_1^4}{f_1^2}$ may be neglected, so that the magnitude of the Love wave velocity approaches the magnitude of the shear wave velocity in the first medium as a limit.

Conversely, as $|\Lambda|$ increases toward infinity, $|f_1|$ approaches zero, and so do the real and imaginary parts of $m_1^2 S$. However, under these conditions, the term in equation (74) involving $\tan^2 m_1^2 S$ may be ignored so that

$$f_1 = m_1^2 \left(\frac{V_{s1}^{2}}{V_{s2}^{2} - V_{s1}^{2}} \right)^{1/2} \tag{78}$$

as a limit. Upon simplifying equation (78), one obtains

$$\frac{m_1^4}{f_1^2} = \frac{V_{s2}^{\;2}}{V_{s1}^{\;2}} - 1, \tag{79}$$

which, when substituted into equation (77), implies that $|V^*|$ approaches $|V_{s2}|$ under these conditions. Therefore, the statements following equations (77) and (79) may be combined with inequality (66) to obtain

$$|V_{s1}| < |V^*| < |V_{s2}| \quad . \tag{80}$$

CHAPTER III

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

This research has developed the conditions and equations governing the propagation of Love waves in an isotropic Voigt viscoelastic medium. To this end, a solution to the partial differential equation of motion has been assumed and has been shown to satisfy the three boundary conditions. Finally, velocity restrictions on the wave and the media have been considered and developed.

Comparing the viscoelastic solution to the known solution for elastic Love wave propagation, it is seen that both are of the same form, but that m and f must be complex in the viscoelastic case. Also, either or both of the amplitude constants must be complex in order to satisfy the second boundary condition, that of continuous tangential stress at the interface. The velocity restrictions on the viscoelastic Love wave are of the same form as those governing the elastic Love wave. However, in the Voigt solid, the restrictions involve the magnitudes of the velocities. Therefore, the use of complex velocities is permitted.

It is recommended that further research be undertaken to separate the real and imaginary parts of the relationships governing the velocities to determine whether restrictions on the real and imaginary velocity components can be made. It is also recommended that numerical values of the various parameters be employed to obtain families of dispersion curves such as those readily available for elastic conditions. Finally, the relationships should be developed which govern Love wave propagation in various other viscoelastic media, such as the Maxwell, the generalized Voigt, and the generalized Maxwell media. Initial efforts in this direction have been set forth using Fourier integrals by Bessonova (16).

APPENDIX A

TABLE OF NOMENCLATURE

(Numerical Subscript Denotes Medium)

A,B,C,D,	Arbitrary amplitude constants.
E	Imaginary part of m^2.
F	Real part of m^2.
G	Arbitrary function.
K	Arbitrary constant.
S	Surface.
T	Arbitrary coefficient.
U,V,W	Displacement in x, y, and z directions.
V^*	Love wave phase velocity.
V_s	Shear wave velocity.
c	Elastic plane wave velocity.
e	Base of natural logarithms.
f	2π/wavelength.
f'	Real part of f.
i	$\sqrt{-1}$.
k	p/shear wave velocity.
m	Coefficient of z.
m_1, m_2	Square roots of m.
p	2π/period.
t	Time.
x,y,z	Coordinate axes.
α	Imaginary part of f.
γ	Imaginary part of m_1.

δ	Real part of m_1.
Δ	Dilatation.
Θ	Phase angle of m.
λ	Elastic Lamé constant.
λ'	Viscoelastic Lamé constant.
Λ	Wavelength.
μ	Elastic Lamé constant.
μ'	Viscoelastic Lamé constant.
ρ	Density.
τ	Relaxation time.

APPENDIX B

JUSTIFICATION OF THE CONDITION $|V_{s1}| < |V_{s2}|$

Equation (62) may be written as

$$\left|V_{s2}^{2}\right| = \left|\frac{f_1^2}{m^2}(V_{s2}^{2} - V_{s1}^{2}) - V_{s1}^{2}\right| \quad . \qquad \text{(B-1)}$$

Equation (B-1) must be satisfied in all regions of the complex plane. Therefore it must be satisfied by values along the positive real axis, i.e., positive real numbers. Under these conditions, equation (B-1) degenerates to its elastic counterpart, which is identical in form. Considering all the quantities of equation (B-1) to be real and positive, one can now assume

$$V_{s2}^{2} < V_{s1}^{2} \quad . \qquad \text{(B-2)}$$

However, it is obvious that

$$\left|\frac{f_1^2}{m^2}(V_{s2}^{2} - V_{s1}^{2}) - V_{s1}^{2}\right| > V_{s1}^{2} \quad , \qquad \text{(B-3)}$$

which is an immediate contradiction of assumption (B-2). Conversely, if one assumes

$$V_{s2}^{2} > V_{s1}^{2} \quad , \qquad \text{(B-4)}$$

a contradiction such as condition (B-3) does not arise. Therefore, assumption (B-4) may be regarded as justifiable.

Since an elastic condition is a specific case of a viscoelastic condition, it seems reasonable that a viscoelastic counterpart of inequality (B-4) should hold in the general case under consideration, although this cannot be directly shown.

BIBLIOGRAPHY

1. LOVE, A.E.H. (1911): Some Problems of Geodynamics. Cambridge Univ. Press, p. 89-104 and 149-152.

2. STONELEY, R. and E. TILLOTSON (1928): The effect of a double surface layer on Love waves. Mon. Not. Roy. Astr. Soc. Geophy. Sup., 1, p. 521-527.

3. STONELEY, R. (1937): Love waves in a triple surface layer. Mon. Not. Roy. Astr. Soc. Geophy. Sup., 4, p. 43-50.

4. SEZAWA, K. (1927): On the decay of waves in visco-elastic solid bodies. Bul. Earthq. Res. Inst. (Japan), 3, p. 43-53.

5. THOMPSON, J.H.C. (1933): On the theory of visco-elasticity. Phil. Trans. Roy. Soc., Ser. A, 231, p. 339-407.

6. HARDTWIG, E. (1943): Über die Wellenausbreitung in einem visko-elastichen Medium (On wave propagation in a viscoelastic medium). Zeits. für Geophy., 18, p. 1-20.

7. RÖSLER, R. (1958): Betrachtungen zu den Spannungs-Dehnungs-Beziehungen nach Nakamura (Considerations of the stress-strain relations of Nakamura). Ger. Beitr. z. Geophy., 67, p. 32-48.

8. SENTIS, A. (1957): Sur la propagation des ondes dans un milieu visco-élastique (On the propagation of waves through a viscoelastic medium). Comptes Rendus, 244, p. 558-560.

9. LÜCKE, K. (1956): Ultrasonic attenuation caused by thermo-elastic heat flow. Jour. Appl. Phys., 27, p. 1433-1438.

10. BLAND, D.R. (1960): The Theory of Linear Viscoelasticity. International series of monographs on pure and applied mathematics, vol. 10, Pergamon Press, 125 p.

11. KANAI, K. (1961): A new problem concerning surface waves. Bul. Earthq. Res. Inst. (Japan), 39, p. 359-366.

12. KOLSKY, H. (1963): Stress Waves in Solids. Dover Publications. p. 99-129.

13. HUNTER, S.C. (1959): Viscoelastic Waves; Progress in Solid Mechanics, Vol. 1. (ed. by I.N. Sneddon and R. Hill), North-Holland Pub. Co., p. 1-57.

14. RUPERT, G.B. (1964): A study of plane and spherical compressional waves in a Voigt viscoelastic medium. Ph.D. thesis, Univ. of Missouri, Rolla, Mo. p. 37.

15. MACELWANE, J.B. (1932): Introduction to Theoretical Seismology, Part I - Geodynamics. St. Louis Univ. Press, p. 127-132.

16. BESSONOVA, E.N. (1963): O rasprostranenii prodol'nykh i poperechnykh ploskikh voln v besgranichnoi vyazkouprugoi srede Maksvella (Longitudinal and transverse plane waves in an infinite Maxwell medium); Problems of Theoretical Seismology and Physics of the Earth's Interior. Pub. for National Science Foundation by Israel Program for Scientific Translations, Jerusalem, p. 132-148.

VITA

The author was born March 16, 1943 in Washington, D. C. and is the son of Mr. Walter H. Peacock and Mrs. Helen Nuse Peacock. He received his elementary education at St. Peter's School and his high school education at Point Pleasant Beach High School, both in Point Pleasant Beach, New Jersey. In 1960 he enrolled in the University of Missouri School of Mines and Metallurgy, majoring in Geology. During his undergraduate study he held the New Jersey State Scholarship, the Mathis Memorial Scholarship, and the V. H. McNutt Geology Scholarship. He received his B.S. degree in Geology in May, 1964. In September, 1964, he enrolled in the Mining Engineering Department of the University of Missouri at Rolla as a graduate student in Geophysical Engineering, studying under a National Science Foundation Traineeship grant. He is a member of Sigma Gamma Epsilon honorary earth science fraternity and of Delta Sigma Phi social fraternity.