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The constant impedance tapered lossless transmission line

Yu Kuo Chen

THE CONSTANT IMPEDANCE TAPERED
LOSSLESS TRANSMISSION LINE

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Approved by

E. C. Bertnolli (advisor) J R Betten

S J Pagano C A Johnson

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I. INTRODUCTION

Considerable work has been done on the theory and development of the distributed RC network which is a special case of the general distributed network type of transmission line.

Sir William Thompson[1] analyzed telegraph cables, assuming the RC line as a model. Oliver Heaviside[2] made numerous contributions to transmission line theory. The solution of a two-wire transmission line with constant parameters R and C is obtainable by direct solution of the telegraphist's equations which are developed from the equivalent circuit of an incremental length of the network[3]. The sinusoidal steady-state solutions of certain tapered RC lines have been exhibited[4,5,6]. The exact and numerical analyses of RC lines have also been exhibited[7].

The general line has per unit length parameters of resistance R, inductance L, capacitance C, and leakage conductance G. If R and G are negligible, a distributed lossless network results. In this thesis, lossless tapered lines that have identical taper functions for L and C are considered. The uniform LC line is just a special case of this tapered transmission line.

II. THE TRANSFER MATRIX OF THE TAPERED LOSSLESS LINE

An incremental length of a tapered LC transmission line is represented by the equivalent circuit of Fig. 1b. L(x) and C(x) are the inductance and capacitance per unit length of the structure, x is distance along the line measured from the input terminals, as shown in Fig. 1a, s is the Laplace transform variable, and Δx is the incremental length. The parameters L(x) and C(x) are not constant, but are functions of position x along the line. Here an asymmetric L-section is used as the equivalent circuit, since a tapered line is asymmetric on an incremental basis.

The equilibrium equations of Fig. 1b are given by equations (1) and (2).

$$\frac{V(x+\Delta x, s) - V(x,s)}{\Delta x} = -s\,L(x)\,i(x,s) \tag{1}$$

$$\frac{i(x+\Delta x, s) - i(x,s)}{\Delta x} = -s\,C(x)\,V(x+\Delta x, s)\,. \tag{2}$$

Upon taking the limit as $\Delta x \to 0$, equations (1) and (2) become partial differential equations (3) and (4) with variable coefficients.

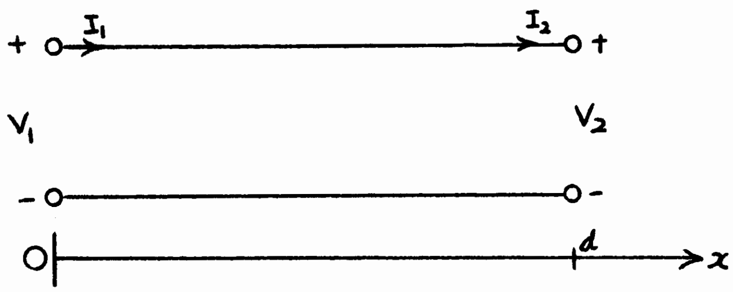

Fig. 1a. The transmission line.

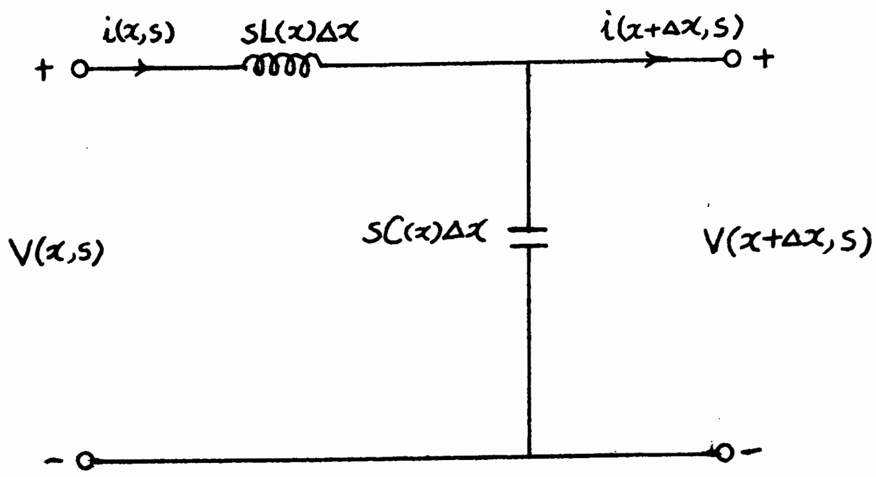

Fig. 1b. Equiavlent circuit of the tapered LC line of length Δx.

$$\frac{\partial V(x,s)}{\partial x} = -s\,L(x)\,i(x,s) \quad , \tag{3}$$

$$\frac{\partial i(x,s)}{\partial x} = -s\,C(x)\,V(x,s) \; . \tag{4}$$

These two equations are similar to those of the tapered RC line, except that R(x) is replaced by sL(x).

Equations (3) and (4) may be condensed and expressed into one matrix equation,

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x,s) \\ i(x,s) \end{bmatrix} = -\begin{bmatrix} 0 & sL(x) \\ sC(x) & 0 \end{bmatrix}\begin{bmatrix} V(x,s) \\ i(x,s) \end{bmatrix} . \tag{5}$$

Letting

$$K(x,s) = \begin{bmatrix} 0 & sL(x) \\ sC(x) & 0 \end{bmatrix} ,$$

equation (5) becomes

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x,s) \\ i(x,s) \end{bmatrix} = -K(x,s)\begin{bmatrix} V(x,s) \\ i(x,s) \end{bmatrix} . \tag{6}$$

Substituting d-y for x gives y=d and y=0 respectively for the source end and the load end of the transmission line. Equation (6) becomes

$$\frac{\partial}{\partial y}\begin{bmatrix} V(d-y,s) \\ i(d-y,s) \end{bmatrix} = K(d-y,s)\begin{bmatrix} V(d-y,s) \\ i(d-y,s) \end{bmatrix} . \tag{7}$$

Integrating both sides with respect to y gives

$$\begin{bmatrix} V(d-y,s) \\ i(d-y,s) \end{bmatrix} - \begin{bmatrix} V(d,s) \\ i(d,s) \end{bmatrix} = \int_0^y K(d-y_1,s)\begin{bmatrix} V(d-y_1,s) \\ i(d-y_1,s) \end{bmatrix} dy_1$$

or

$$\begin{bmatrix} V(d-y,s) \\ i(d-y,s) \end{bmatrix} = \begin{bmatrix} V(d,s) \\ i(d,s) \end{bmatrix} + \int_0^y K(d-y_1,s)\begin{bmatrix} V(d-y_1,s) \\ i(d-y_1,s) \end{bmatrix} dy_1 . \tag{8}$$

Equation (8) gives

$$\begin{bmatrix} V(d-y_1,s) \\ i(d-y_1,s) \end{bmatrix} = \begin{bmatrix} V(d,s) \\ i(d,s) \end{bmatrix} + \int_0^{y_1} K(d-y_2,s)\begin{bmatrix} V(d-y_2,s) \\ i(d-y_2,s) \end{bmatrix} dy_2 , \tag{9}$$

where y_1 and y_2 of equations (8) and (9) respectively are dummy variables of integration. Substitution of equation (9) into equation (8) gives

$$\begin{bmatrix} V(d-y,s) \\ i(d-y,s) \end{bmatrix} = \begin{bmatrix} V(d,s) \\ i(d,s) \end{bmatrix} + \left[\int_0^y K(d-y_1,s)\,dy_1\right]\begin{bmatrix} V(d,s) \\ i(d,s) \end{bmatrix}$$

$$+\int_0^y K(d-y_1,s)\int_0^{y_1} K(d-y_2,s)\begin{bmatrix} V(d-y_2,s) \\ i(d-y_2,s) \end{bmatrix} dy_2 d_1 \quad . \tag{10}$$

Repeating the substitution process, the following well known solution[8] is obtained:

$$\begin{bmatrix} V(d-y,s) \\ i(d-y,s) \end{bmatrix} = \left[\Omega_0^y[K(d-y,s]\right] \begin{bmatrix} V(d,s) \\ i(d,s) \end{bmatrix} , \tag{11}$$

where $\Omega_0^y[K(d-y,s)]$ is the matrizant of K(d-y,s), and $\Omega_0^y[K(y,s)]$ is defined as

$$\Omega_0^y[K(y,s)] = I + \int_0^y K(y_1,s)dy_1 + \int_0^y K(y_1,s)\int_0^{y_1} K(y_2,s)dy_2\,dy_1$$
$$+ \int_0^y K(y_1,s)\int_0^{y_1} K(y_2,s)\int_0^{y_2} K(y_3,s)dy_3\,dy_2\,dy_1 + \cdots . \tag{12}$$

I is an identity matrix of the same order as that of matrix K(y,s). If K(y,s) has finite elements for all values of y concerned, this series can be shown to be absolutely and uniformly convergent[8]. Hence, it can be accepted as a genuine solution. For y=d, equation (11) becomes

$$\begin{bmatrix} V(o,s) \\ i(o,s) \end{bmatrix} = \left[\Omega_0^d[K(d-y,s)]\right] \begin{bmatrix} V(d,s) \\ i(d,s) \end{bmatrix}$$

or

$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \left[\Omega_0^d[K(x,s)]\right] \begin{bmatrix} V_2(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2(s) \\ I_2(s) \end{bmatrix} . \tag{13}$$

The terminal variables are shown in Fig. 2, and $\Omega_0^d[K(x,s)]$ is the transfer matrix of the tapered line of length d. A,B,C, and D are transmission parameters of the network.

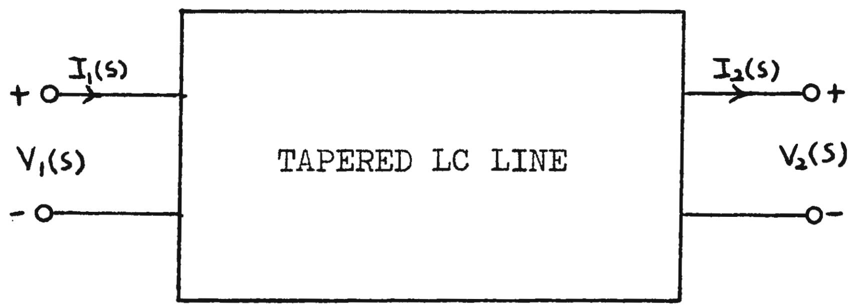

Fig. 2. The tapered LC line of length d.

III. THE CONSTANT IMPEDANCE TAPERED-LOSSLESS LINE

In this thesis, lines that have identical functions for L(x) and C(x), except for a constant factor, are considered. Let

$$L(x)=L_o f(x) \; ,$$

and
$$C(x)=C_o f(x) \; ,$$

where L_o and C_o are constants, and f(x) is a function of x only. Equation (5) becomes

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x,s) \\ i(x,s) \end{bmatrix} = -sf(x)\begin{bmatrix} 0 & L_o \\ C_o & 0 \end{bmatrix}\begin{bmatrix} V(x,s) \\ i(x,s) \end{bmatrix} , \tag{14}$$

and

$$K(x,s)=sf(x)\begin{bmatrix} 0 & L_o \\ C_o & 0 \end{bmatrix}$$

$$=sf(x)M \tag{15}$$

where

$$M=\begin{bmatrix} 0 & L_o \\ C_o & 0 \end{bmatrix} .$$

In general, f(x) can be a large class of functions of x, but in this thesis, two cases will be examined. First, let f(x) be the function e^{ax}, and second, the function x^a.

A. CASE 1. $f(x)=e^{ax}$

If $f(x)=e^{ax}$ (a denotes the taper constant), equation (15) becomes $K(x,s)=se^{axM}$. Using equations (12) and (13) for the line's transfer matrix gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \Omega_0^d[K(x,s)]$$

$$= I + \int_0^d sMe^{ax}dx + \int_0^d sMe^{ax}\int_0^x sMe^{ax_1}dx_1\,dx$$

$$+ \int_0^d sMe^{ax}\int_0^x sMe^{ax_1}\int_0^{x_1} sMe^{ax_2}dx_2\,dx_1\,dx + \cdots$$

$$= I + \frac{sM(e^{ad}-1)}{a} - \frac{s^2M^2(e^{ad}-1)^2}{2!\,a^2}$$

$$+ \frac{s^3M^3(e^{ad}-1)^3}{3!\,a^3} + \cdots + \frac{s^nM^n(e^{ad}-1)^n}{n!\,a^n} + \cdots$$

$$= e^{s(e^{ad}-1)M/a}$$

$$= \begin{bmatrix} \cosh \dfrac{s(e^{ad}-1)\sqrt{L_0C_0}}{a} & \sqrt{\dfrac{L_0}{C_0}}\sinh \dfrac{s(e^{ad}-1)\sqrt{L_0C_0}}{a} \\ \sqrt{\dfrac{C_0}{L_0}}\sinh \dfrac{s(e^{ad}-1)\sqrt{L_0C_0}}{a} & \cosh \dfrac{s(e^{ad}-1)\sqrt{L_0C_0}}{a} \end{bmatrix}. \quad (16)$$

Let $s(e^{ad}-1)\sqrt{L_0\,C_0}/a=\theta$, then equation (13) gives

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sqrt{\frac{L_0}{C_0}}\sinh\theta \\ \sqrt{\frac{C_0}{L_0}}\sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} . \tag{17}$$

Equation (17) shows that the transmission parameters

$$A = \cosh\theta$$
$$B = \sqrt{\frac{L_0}{C_0}}\sinh\theta$$
$$C = \sqrt{\frac{C_0}{L_0}}\sinh\theta$$
$$D = \cosh\theta$$

satisfy the reciprocity relation AD−BC=1. This network is also symmetric, since A=D. When an impedance Z_L is connected to the load end of the line, the voltage gain is

$$\begin{aligned} V.G. &= \frac{V_2}{V_1} \\ &= \frac{1}{\cosh\theta + \sqrt{\frac{L_0}{C_0}} \cdot \frac{1}{Z_L}\sinh\theta} . \end{aligned} \tag{18}$$

Now, if Z_L is resistive and equal to the value $\sqrt{L_0/C_0}$, the voltage gain becomes

$$\begin{aligned} V.G. &= \frac{1}{\cosh\theta + \sinh\theta} \\ &= e^{-\theta} . \end{aligned} \tag{19}$$

If a sinusoidal excitation is applied to the source end of the line, then $s=j\omega$, where ω is the radian frequency of the source, and

$$V.G. = e^{-j\omega(e^{ad}-1)\sqrt{L_0C_0}/a}$$

$$= 1 \underline{/ -\omega (e^{ad} - 1) \sqrt{L_o C_o} / a} \quad . \tag{20}$$

An interesting result has been reached. The magnitude of the ratio V_2/V_1 is unity for all frequencies at any distance from the source if this special kind of transmission line is used. It is an all-pass network. For a specific length d and specific value a, the delay in phase of the response, with respect to that of the excitation is linearly proportional to the frequency of the input voltage. This linearity is shown in Fig. 3, where $N = (e^{ad} - 1) \sqrt{L_o C_o} / a$. The delay in phase is $\phi_d = N\omega$.

For the purpose of comparison and simplification, make $\sqrt{L_o C_o}\,\omega = 1$, and the delay in phase becomes

$$\phi_d = \frac{e^{ad} - 1}{a} \quad . \tag{21}$$

The delay in phase is a function of the distance d. Therefore, when a sinusoidal voltage is applied at the source end of this special transmission line, a load voltage is obtained of the same magnitude, with a delay in phase which can be adjusted by varying the distance d. The relations between ϕ_d and d for the specific values a=±1, and a=0 are shown in Fig. 4. For the case a=±1, the relations can easily be plotted from equation (21). For the case a=0 (i.e., f(x)=1), $L(x)=L_o$, and $C(x)=C_o$. This is the case of a uniformly distributed LC line, which is similar to the uniformly distributed RC line, and the transfer matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \Omega_0^d \left[K(x,s)\right]$$

$$= e^{sMd}$$

$$= \begin{bmatrix} \cosh s\sqrt{L_0C_0}\, d & \sqrt{\frac{L_0}{C_0}} \sinh s\sqrt{L_0C_0}\, d \\ \sqrt{\frac{C_0}{L_0}} \sinh s\sqrt{L_0C_0}\, d & \cosh s\sqrt{L_0C_0}\, d \end{bmatrix} . \tag{22}$$

Or, taking the limit of θ as $a \to 0$, gives

$$\lim_{a \to 0} \theta = \lim_{a \to 0} \frac{s(e^{ad}-1)\sqrt{L_0C_0}}{a} = s\sqrt{L_0C_0}\, d$$

Substituting this value into equation (17), the same result is obtained as in equation (22).

For a sinusoidal input voltage, $s=j\omega$, the delay in phase for a=0 is $\phi_d = \sqrt{L_0C_0}\,\omega d$. Making $\sqrt{L_0C_0}\,\omega = 1$,

$$\phi_d = d . \tag{23}$$

Equation (23) denotes a direct relation between the phase delay and the distance d from the source end. From the curves in Fig. 4, it can be seen that for the case a=-1, the resultant network will not provide a 2π radian delay in phase, no matter how long the line is made. In order to have a variation of 2π radian delay in phase, the length of the transmission line has to be 6.283 normalized units for a=0, and 1.986 normalized units for a=1.

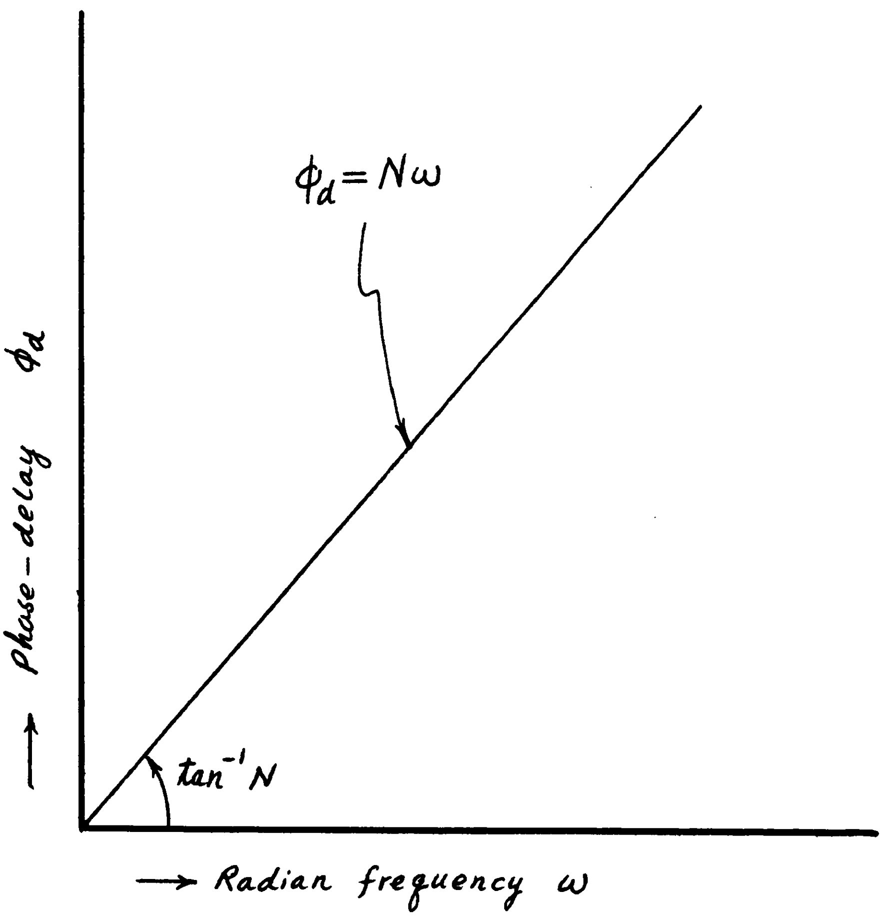

Fig. 3. Phase-frequency response of the tapered LC transmission line.

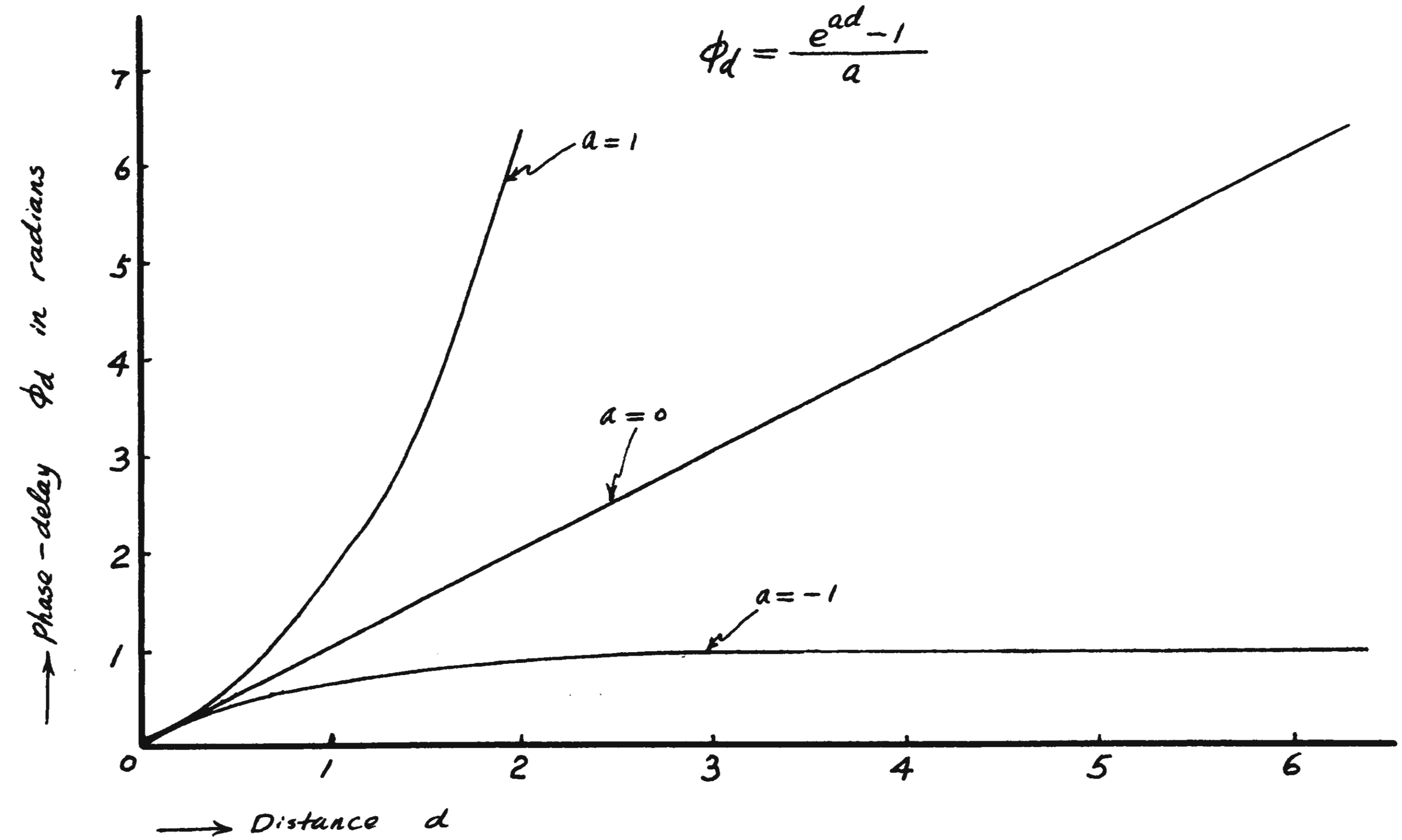

Fig. 4. Phase-distance response of the lossless transmission line, for $L(x) = L_0 e^{ax}$, and $C(x) = C_0 e^{ax}$.

The input impedance for these transmission lines of length d when terminated by Z_L is

$$Z_{in} = \frac{A + B/Z_L}{C + D/Z_L}$$

$$= Z_L \frac{\cosh\theta + \sqrt{\frac{L_0}{C_0}} \cdot \frac{1}{Z_L} \sinh\theta}{\sqrt{\frac{C_0}{L_0}} \cdot Z_L \sinh\theta + \cosh\theta} \quad . \tag{24}$$

If $Z_L=\sqrt{L_0/C_0}$, then

$$Z_{in} = Z_L = \sqrt{\frac{L_0}{C_0}} = Z_0 \quad . \tag{25}$$

It is noted that the input impedance is independent of the length of the line, so long as the load is resistive and equal to $\sqrt{L_0/C_0}$. The quantity Z_0 is the characteristic impedance of the transmission line.

B. CASE 2. $f(x)=x^a$

Here, a denotes the taper constant. When $f(x)=x^a$, then

$$L(x)=L_0x^a \ ,$$

$$C(x)=C_0x^a \ .$$

The transfer matrix of this transmission line of length d can be obtained from equation (13).

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \Omega_0^d[K(x,s)]$$

$$= I + \int_0^d sMx^a dx + \int_0^d sMx^a \int_0^x sMx_1^a dx_1 dx + \cdots$$

$$= I + \frac{sM\,d^{a+1}}{a+1} + \frac{s^2M^2 d^{2(a+1)}}{2!\,(a+1)^2} + \frac{s^3M^3 d^{3(a+1)}}{3!\,(a+1)^3}$$

$$+ \cdots + \frac{s^nM^n d^{n(a+1)}}{n!\,(a+1)^n} + \cdots$$

$$= e^{\frac{sMd^{a+1}}{a+1}}$$

$$= \begin{bmatrix} \cosh s\sqrt{L_oC_o}\, d^{a+1}/a+1 & \sqrt{\frac{L_o}{C_o}}\sinh s\sqrt{L_oC_o}\, d^{a+1}/a+1 \\ \sqrt{\frac{C_o}{L_o}}\sinh s\sqrt{L_oC_o}\, d^{a+1}/a+1 & \cosh s\sqrt{L_oC_o}\, d^{a+1}/a+1 \end{bmatrix} . \tag{26}$$

Letting

$$\theta = s\sqrt{L_oC_o}\, d^{a+1}/a+1 \quad ,$$

the transfer matrix for the network is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sqrt{\frac{L_o}{C_o}}\sinh\theta \\ \sqrt{\frac{C_o}{L_o}}\sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} . \tag{27}$$

It is noted that the parameter a cannot be -1. When an impedance Z_L is placed at the load end of the line, the voltage gain of this loaded network is

$$V.G. = \frac{1}{\cosh\theta + \sqrt{\frac{L_o}{C_o}}\cdot\frac{1}{Z_L}\sinh\theta} . \tag{28}$$

As in case 1, let $Z_L = \sqrt{L_o/C_o}$, and equation (28) becomes

$$V.G.=e^{-\theta}.$$

Equation (29) has the same form as equation (19). The only difference is that θ is a different function of x in the two cases. If a sinusoidal excitation is applied to the line, then $s=j\omega$, where ω is the radian frequency of the applied voltage, and equation (29) becomes

$$V.G. = e^{-j\omega\sqrt{L_o C_o}\, d^{a+1}/a+1}$$

$$= 1 \,\underline{/-\omega\sqrt{L_o C_o}\, d^{a+1}/a+1} \quad . \qquad (30)$$

Again, unity magnitude is obtained for the voltage gain. For a specific length d and specific value of a, the delay in phase is also linearly proportional to the frequency of the input voltage as in case 1. Letting $\sqrt{L_o C_o}\, d^{a+1}/a+1 = N$, the delay in phase, $\phi_d = N\omega$, with respect to the input voltage, is the same as shown in Fig. 3. These two properties denote that this kind of transmission line can be used as an all-pass phase shifting network. For the purpose of simplification and comparison, make $\sqrt{L_o C_o}\,\omega = 1$. Then,

$$\phi_d = \frac{d^{a+1}}{a+1} \qquad (31)$$

In equation (31), it is obvious that the delay in phase is rather simply related to the distance d. The relations between ϕ_d and d are shown in Fig. 5 for $-2/3 \le a \le 2$. When a=0, the case of a uniform LC line results. The length of the line required to have a 2π radian delay in phase

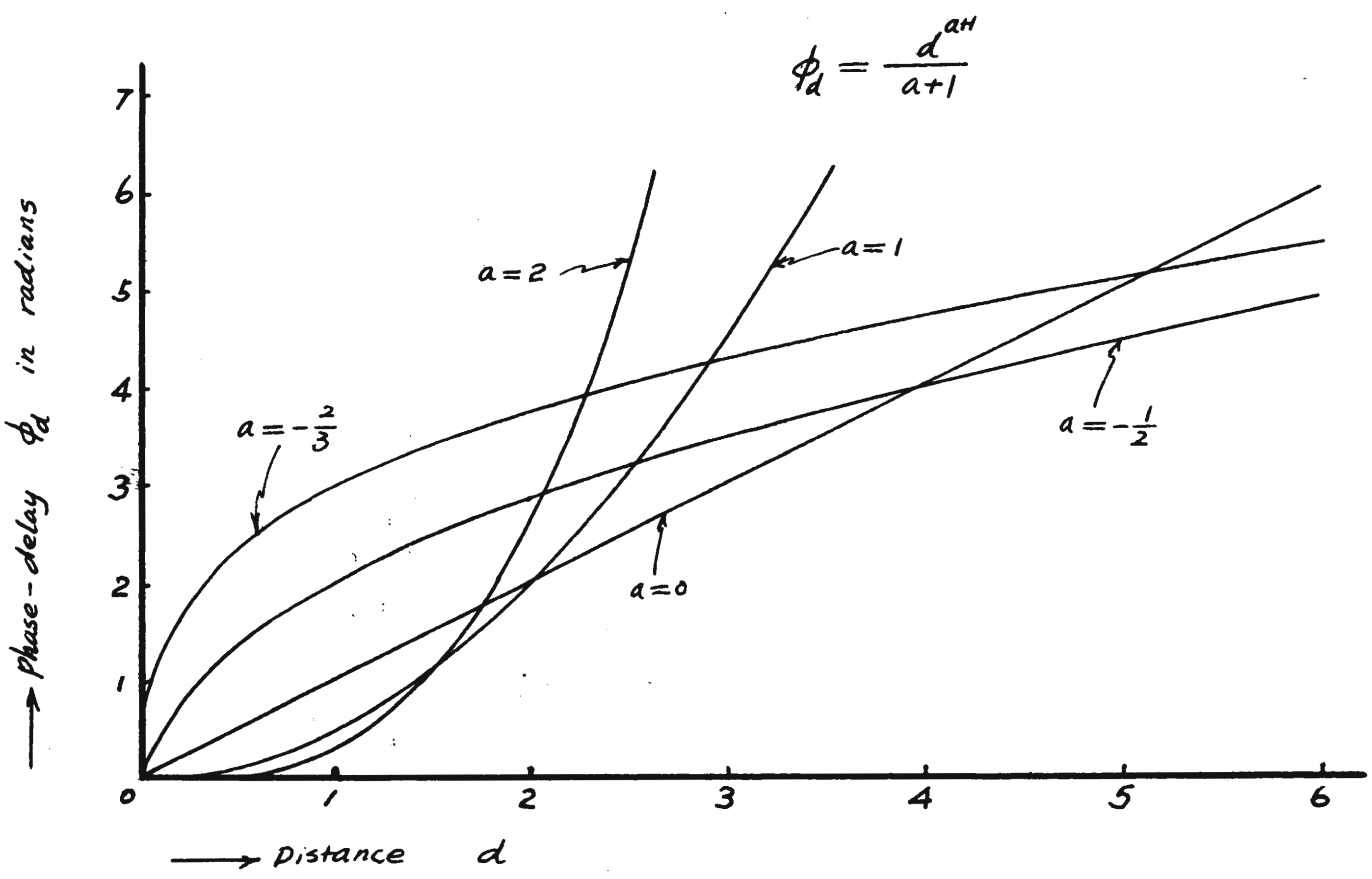

Fig. 5. Phase-distance response of the lossless transmission line, for $L(x) = L_0 x^a$, and $C(x) = C_0 x^a$.

Table 1. Relations of a and d of the lossless transmission line , for $L(x)=L_o x^a$, and $C(x)=C_o x^a$.

a	d
0	6.283
1	3.545
2	2.661
3	2.239
4	1.992

is shown in Table 1 for different values of a. The quantity d is in normalized units. The larger the value of a, the shorter will be the length of line needed to make a 2π radian delay in phase.

The input impedance of this family of transmission lines when terminated by $Z_L=\sqrt{L_o/C_o}$ is

$$Z_{in}=Z_L=\sqrt{L_o/C_o}\ . \tag{32}$$

Again, the input impedance is equal to the line's characteristic impedance, $\sqrt{L_o/C_o}$, and does not depend upon how far away the load is placed from the source end, so long as the load is resistive and equal to $\sqrt{L_o/C_o}$.

IV. CONSTRUCTION OF THE TAPERED LC LINE

In case 1, the parameters of the line have to be

$$L(x)=L_o e^{ax} , \tag{33-1}$$

and

$$C(x)=C_o e^{ax} . \tag{33-2}$$

Therefore, when this type of line is built, these two conditions must be satisfied. The line constants L and C of a coaxial cable[9] are

$$L = \frac{\mu}{2\pi} \ln \frac{R}{r} \quad \text{henrys/meter} \tag{34-1}$$

$$C = \frac{2\pi\epsilon}{\ln R/r} \quad \text{farads/meter} \tag{34-2}$$

where μ is the relative permeability, and ϵ is the relative dielectric constant or permitivity of the medium. The quantities R and r are respectively the radius of the outer and inner cylinders of the coaxial cable, as shown in Fig. 6. In order to build a coaxial cable having parameters L(x) and C(x) for case 1, use equation (33) and equation (34), and obtain

$$R = r\, e^{\frac{2\pi L_o}{\mu} e^{ax}} , \tag{35}$$

and

$$\epsilon = \frac{L_o C_o}{\mu} e^{2ax} . \tag{36}$$

When the radius of the inner conductor r is fixed, the radius of the outer cylinder R has to be varied along the cable as expressed in equation (35). By assuming the permeability to be constant, the dielectric constant ϵ

must be varied as stated in equation (36). The variation of ϵ can be approximately accomplished by using sections of different media whose dielecric constants vary according to equation (36) along the cable.

If a parallel-strip line, as shown in Fig. 7, is being used, its parameters[9] are

$$L = \mu \frac{d}{b} \qquad \text{henrys/meter} \qquad (37\text{-}1)$$

$$C = \epsilon \frac{b}{d} \qquad \text{farads/meter} \qquad (37\text{-}2)$$

when $d \ll b$. Using equations (37) and (33), one obtains

$$b = \frac{\mu d}{L_0} e^{-ax} \qquad (38)$$

and

$$\epsilon = \frac{L_0 C_0}{\mu} e^{2ax} \, . \qquad (39)$$

When d is fixed, b has to be varied as stated in equation (38). Equation (39) and equation (36) are exactly the same.

In case 2, the parameters of the line have to be

$$L(x)=L_0 x^a \, , \qquad (40\text{-}1)$$

and

$$C(x)=C_0 x^a \, . \qquad (40\text{-}2)$$

Therefore, when a coaxial cable is being used, the following conditions must be satisfied:

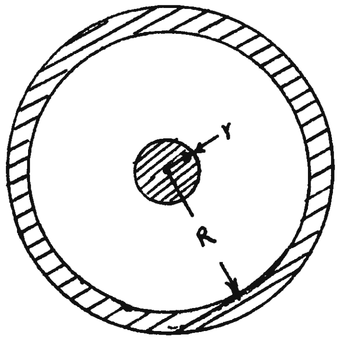

Fig. 6. Coaxial cable cross-section.

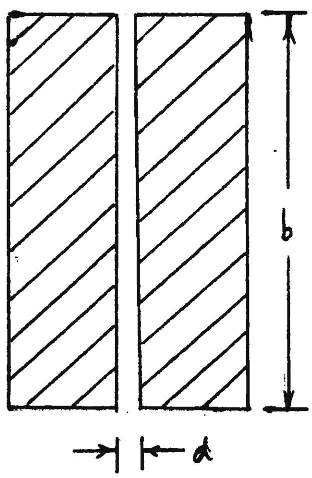

Fig. 7. Parallel-strip lines cross-section.

$$R = re^{\frac{2\pi L_0}{\mu}x^a} \tag{41}$$

$$\epsilon = \frac{L_0 C_0}{\mu} x^{2a} \tag{42}$$

Equation (42) is similar to equation (36), except that ϵ varies as x^{2a}.

Similarly, if a parallel-strip line is being used, the conditions are

$$b = \frac{\mu d}{L_0} x^{-a} \tag{43}$$

and

$$\epsilon = \frac{L_0 C_0}{\mu} x^{2a}. \tag{44}$$

Equation (44) is exactly the same as equation (42).

V. CONCLUSIONS

The tapered LC lines of the special types discussed in the previous sections have two important properties when terminated by their characteristic impedance.

A. The magnitude of the voltage gain is unity.

B. The phase of the response with respect to that of the excitation is a function of x and the taper constant a.

Having these two properties, these tapered LC lines may be considered as ideal phase-delay networks. For the same amount of delay in phase, these lines of certain taper constants are physically shorter than the uniformly distributed LC line that provides the same phase shift.

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VITA

The author was born on February 20, 1938, in Fukien, China. He received his primary and secondary education in Taiwan, Republic of China. He received a Bachelor of Science Degree in Electrical Engineering from the National Taiwan University in June, 1961.

In September 1965, the author came to the United States of America, and began his graduate studies in Electrical Engineering at the University of Missouri at Rolla.