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Design of a sequential detector

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DESIGN OF A SEQUENTIAL DETECTOR

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ABSTRACT

The communication engineer has in the past treated the detection problem from a limited point of view. Detectors have been designed to reach a decision based on a specified number of samples. This approach has proved valuable, but it offers no provisions for an early decision if a conclusion becomes obvious early in the sampling. Neither is there an opportunity for additional sampling if necessary to reach a definite conclusion. Because of these limitations it is desirable to find more general detectors.

This study looks at a specific statistic (the sign test) and designs a sequential detector to utilize this statistic. The sequential detector is a type of detector where the decision to terminate depends only on the previous samples. This detector is also of the nonparametric class in that a complete knowledge of the signal and noise is not necessary for its operation.

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CHAPTER I

INTRODUCTION

An important problem in the field of communications is the detection of signals in the presence of noise. The detection problem is concerned with determining the presence or absence of signal in a background of noise. The radar problem in which a target is detected by determining the presence of a radar return signal is an example of signal detection.

In the past most of the work in signal detection has been limited to cases of a fixed number of samples on which the decision is based. This method offers no provisions to obtain additional data if the sample is not sufficiently suggestive of a decision. Neither is there opportunity to cut the sampling short if a conclusion becomes obvious early in the process of obtaining the sample.

Thus, there is a need for a class of detectors in which the number of samples required is not predetermined. This type of detector, where the decision to terminate depends only on the previous samples, is called a sequential detector. There are two types of sequential detectors, parametric and nonparametric. Parametric detection deals with signals of known deterministic form and noise of known statistical form. Nonparametric detection deals with signals and noise, the statistical forms of which are not completely known. Nonparametric detectors therefore have the advantage of requiring less statistical information than parametric detectors.

In this paper a nonparametric sequential detector will be investigated. The detector will be designed to detect signals which change the d-c level of the noise. After this theory is developed the sequential detector will be used to detect a frequency modulated (FM) signal in the presence of noise. Since many FM detection problems are restricted to messages expressed in binary coded form (Binary Frequency Shift Keyed) or r-ary coded form (Multiple Frequency Shift Keyed), attention here will be limited to these forms.

CHAPTER II

FORMULATION OF THE DETECTION PROBLEM

The function of the detector is to determine whether an observed waveform x(t) consists of signal plus noise or noise alone. The detection decision problem can be expressed as a statistical hypothesis testing problem (4). There are two alternatives in statistical hypothesis testing. These two alternatives may be represented as the null hypothesis, the samples are from noise alone, versus the alternate hypothesis, the samples are from signal plus noise.

The detector considered in this paper is characterized by three mutually exclusive zones for decision. The three zones will be specified as:

S_1: the zone of preference for acceptance

S_2: the zone of indifference

S_3: the zone of preference for rejection.

The decision is based on the fact that some function of the sample falls in one of the three zones. The boundaries on the zones are predetermined by the number of errors which can be allowed. The detector can make two types of mutually exclusive errors:

1. signal is not present and the detector decides signal is present; this will be called a type I error and the probability of such an error will be denoted by α,

2. signal is present and the detector decides there is no signal present; this will be called a type II error and the probability of such an error will be denoted by β.

In this paper α will also be called the false alarm probability and β will be called the false dismissal probability.

A sequential test of a statistical hypothesis makes a certain calculation after each observation, or group of observations, and decides on one of three courses of action, depending on the outcome of the calculations. After each calculation the detector either accepts the null hypothesis, accepts the alternate hypothesis, or decides there is insufficient data to accept either of the hypotheses with predetermined degrees of confidence. If the null hypothesis or alternate hypothesis is accepted, the experiment is terminated. If neither hypothesis is accepted the detector takes another sample and on the basis of the sample of size two decides again on one of the three courses of action. The detector continues in this manner until either the null hypothesis or alternate hypothesis is accepted.

It can be seen from the preceding discussion that the number of samples required for the termination of a sequential test with the acceptance of the null or alternate hypothesis is a random variable. In a sequential test both the false alarm probability and false dismissal probability are specified. The sequential test which minimizes the average number of samples required for termination is considered optimum.

A. Wald (1) has developed the sequential probability ratio test, which is regarded as the optimum sequential test. It is considered optimum in the sense that the average number of samples required to reach a decision is a minimum. To discuss this test, let $f(x,\theta)$ be the distribution of a random variable x to be tested. The null hypothesis, H_0, is $\theta=\theta_0$. The alternate hypothesis, H_1, is $\theta=\theta_1$. Successive samples on x are given by $x_1, x_2, \ldots. x_n, \ldots$, etc.

Assuming the samples are independent, the probability of a sample $x_1, \ldots. x_n$, when the null hypothesis is true, is given by

$$p_{0n}= f(x_1,\theta_0)\ldots\ldots\ldots f(x_n,\theta_0)$$

When the alternate hypothesis is true the probability of a sample $x_1, \ldots\ldots x_n$ is given by

$$p_{1n}= f(x_1,\theta_1)\ldots\ldots\ldots f(x_n,\theta_1)$$

To form the sequential probability ratio test let

$$\lambda = p_{1n}/ p_{0n}$$

Two positive constants A and B are chosen. The test is carried out with the desired false alarm probability, α, and false dismissal probability, β, if the constants are chosen in a specific manner. By choosing

$$A = \frac{1-\beta}{\alpha}$$

$$B = \frac{\beta}{1-\alpha}$$

it is guarantied that the test will have the predetermined degrees of reliability.

The test is carried out by computing λ. If $\lambda \geq A$ the alternate hypothesis is accepted. If $\lambda \leq B$ the null hypothesis is accepted. If $B < \lambda < A$ an additional sample is taken. This process is continued until a decision is reached.

An important characteristic used in gauging the performance of a sequential test is the average sample number function (ASN function). The ASN function gives the average number of samples required for the termination of a sequential test.

The signal to noise ratio used in this paper will be defined to be the ratio of the r.m.s. value of the signal to the r.m.s. value of the noise.

In the following chapters $f(x)$ is defined to be the probability density function on x, i.e., the probability of x falling between x and $x + \Delta x$ is $f(x)\Delta x$. The function $b(x)$ is defined as the discrete probability distribution on x, i.e., the probability of x taking on the value x_k is $b(x_k)$.

In all cases considered in this paper the noise will be assumed to be gaussian with mean zero and variance

σ_0^2. If f(x) has gaussian distribution with mean zero and variance σ_0^2 then it will be abbreviated as $f(x) \sim G(0, \sigma_0^2)$.

CHAPTER III

THE SEQUENTIAL SIGN DETECTOR

3.1 Definition

As stated before, the signal detection problem is a problem of hypothesis testing. Therefore it is necessary to develop a hypothesis about the observed distributions which may be tested. The detector used in this paper is the sequential sign detector. This detector is useful for any signal that changes the d-c level of the noise. The sign detector determines whether two sets of observations can reasonably be thought to have come from the same distribution.

The sign test is a test which relies upon the sign of the difference between a pair of observations. Since this test depends only on the sign of the difference, and puts no significance on the magnitude of the difference, some useful information is lost. It has been shown that the asymptotic relative efficiency of the sign detector with respect to the t-detector is .636. The t-detector is considered the optimum parametric detector for detecting signals which change the d-c level of the noise. Thus although there are tests which are more efficient than the sign test, the sequential sign detector is developed here because of the extreme simplicity of the test and the relative ease with which it can be implemented. Another very important attribute of the sign detector is its ability to be useful even if the noise is not strictly gaussian.

Assumo any pair of values $(y(t_0),x(t_0))$ to be randomly drawn. The sample $y(t_0)$ is drawn from what is known to be noise alone and the sample $x(t_0)$ is drawn from the input to be tested. The test depends on the difference, $y(t_0)-x(t_0)$, between each sample pair.

If the signal is not present the samples from $y(t)$ and the samples from $x(t)$ will both be drawn from the same distribution, noise alone. Since the noise distribution is assumed to be gaussian with mean zero and variance σ_0^2, the distribution of the difference, $y(t_i)-x(t_i)$ for $i=1,2,\ldots n$, will be gaussian with mean zero and variance $2\sigma_0^2$. Therefore if signal is not present it is reasonable to expect approximately the same number of plus and minus signs from a group of n observations.

If a signal is present which changes the d-c level of the noise, the distribution of the difference will still be gaussian but with a mean shifted from zero. If the signal causes a positive shift in the mean it would be reasonable to expect a greater number of minus signs than plus signs from a group of n observations. As the shift in the mean becomes greater, the percentage of minus signs that can be expected will also increase.

Thus if a signal is not present, the plus and minus signs would be expected to have a dichotomous distribution with the probability of either sign equal to one-half. Therefore in n independent trials in which a plus or minus sign is determined for each pair and for which the probability of a negative (positive) sign on each trial is $p=\frac{1}{2}$, the probability of k negative (positive) signs is given by the binomial distribution

$$b(k) = b(k;n,p=\tfrac{1}{2}) = \frac{n}{n!\,(n-k)!}\,(\tfrac{1}{2})^n \qquad (3.1)$$

If signal is present, the distribution of negative signs would be expected to take the form

$$b(k) = b(k;n,p\neq\tfrac{1}{2}) = \frac{n}{n!\,(n-k)!}\,(p)^k(1-p)^{n-k} \qquad (3.2)$$

where p is the probability of a minus sign.

A point has now been reached where the hypothesis to be tested can be determined. The hypothesis will deal with the parameter p in the binomial distribution. The null hypothesis, H_0, to be tested is $p = p_0 = \frac{1}{2}$, noise alone is present. The alternate hypothesis, H_1, to be tested is $p = p_1 \neq \frac{1}{2}$, signal plus noise is present.

The sign test is somewhat difficult to apply for sequential testing unless the observations are taken in groups. The reason for this difficulty stems from the fact that when an additional observation is taken, the number of negative (positive) signs can either stay the same or increase by only one. Thus, there is a heavy dependence between observations. When observations are taken in groups the number of negative (positive) signs is random from group to group and each sample is independent. In the rest of this paper observations will be taken in groups of n and each group will be called a sample.

A method (15) has been developed by Tsao which in many cases is easier to apply than Wald's sequential probability ratio test. This method was developed for continuous probability density functions. Even though the sign test has a binomial distribution as its underlying distribution, this method is applicable if the observations are taken in groups. By taking the observations in groups of sufficient size, the binomial distribution may be approximated by the normal distribution function which is continuous.

The sequential procedure used in this paper takes observation pairs in groups of size n. The detector then determines the number of minus signs which result from taking the difference $y(t_i)-x(t_i)$ for $i=1,2,\ldots n$. There will be n of these differences in each sample, one for each observation pair. The range on the number of minus signs occuring will be from zero to n. These n+1 possible values $(0,1,2,\ldots n)$ are divided into three zones. That is, all samples with less than K_1 minus signs occuring will be placed in one zone. All samples with more than K_1 and less than K_2 minus signs occuring will be placed in another zone. All samples with more than K_2 minus signs occuring will be placed in a third zone.

Random samples of n observation pairs are drawn successively. A stage of the procedure will occur after each sample has been placed in one of the three zones. At each stage the number of samples falling in each of the three zones will be counted. Denote by m_i the number of samples falling in the zone S_i for $i = 1,2,3$ after the mth sample is taken. Let a and r be two predetermined positive integers. The detector will continue to take samples (groups of observations) as long as $m_1 < a$ and $m_3 < r$. That is, if the number of samples in zone S_1 is less than some predetermined constant, a, and the number of samples in zone S_3 is less than some predetermined constant, r, another sample will be taken. The number of minus signs occuring in this sample will be determined and the sample will be placed in one of the three zones. The number of samples in each zone will again be determined and checked against the predetermined constants. The detector will accept the null hypothesis if, at any stage, $m_1 = a$; it will accept the alternate hypothesis if, at any stage, $m_3 = r$. As long as $m_1 < a$ and $m_3 < r$ additional samples will be taken. As soon as one of the hypothesis is accepted the experiment is terminated.

Consider the following quantities:

$$A = \sum_{S_1} b(k)$$

$$I = \sum_{S_2} b(k)$$

$$R = \sum_{S_3} b(k) \qquad (3.3)$$

where S_1, S_2, and S_3 represent the summation over all sample points in zones 1,2, and 3 respectively. These quantities will be designated by A_i, I_i and R_i if b(k) is replaced by $b_i(k)$ for $i = 0$ or 1. From this point on, $b_0(k)$ will be used to designate the distribution of k under the null hypothesis, and $b_1(k)$ will denote the distribution of k under the alternate hypothesis.

Using the definitions of the three zones given in Eq. (3.3), the probability density function of the sample size may be derived. It will be stated here without proof as (15)

$$g_f(m; a, r, S_1, S_3) = \sum_{x=0}^{x=a-1} \frac{(m-1)!}{(r-1)!\,x!\,(m-r-x)!} R^r A^x I^{m-r-x}$$

$$+ \sum_{x=0}^{x=r-1} \frac{(m-1)!}{(a-1)!\,x!\,(m-a-x)!} A^a R^x I^{m-a-x} \qquad (3.4)$$

This is obtained from the multinomial distribution which is a generalization of the binomial distribution.

Using the probability density function of the sample size given in Eq. (3.4), the moment generating function is defined as $\sum e^{tm} p(m)$. Therefore the moment generating function of the sample size is given by

$$M_f(t;a,r,S_1,S_3) = \sum_{m=r}^{\infty} \sum_{x=0}^{a-1} \frac{(m-1)!}{(r-1)!\,x!\,(m-x-r)!} R^r A^x I^{m-r-x} e^{mt}$$

$$+ \sum_{m=a}^{\infty} \sum_{x=0}^{r-1} \frac{(m-1)!}{(a-1)!\,x!\,(m-x-a)!} A^a R^x I^{m-r-x} e^{mt} \qquad (3.5)$$

A new function will now be defined as

$$h(p,q,B,C,D) = \sum_{m=q}^{\infty} \sum_{x=0}^{p-1} \frac{(m-1)!}{(q-1)!\,x!\,(m-q-x)!} B^q C^x D^m$$

Eq. (3.5) may be written

$$M_f(t;a,r,S_1,S_3) = \sum_{m=r}^{\infty} \sum_{x=0}^{a-1} \frac{(m-1)!}{(r-1)!\,x!\,(m-r-x)!} \left(\frac{R}{I}\right)^r \left(\frac{A}{I}\right)^x \left(Ie^t\right)^m$$

$$+ \sum_{m=a}^{\infty} \sum_{x=0}^{r-1} \frac{(m-1)!}{(a-1)!\,x!\,(m-a-x)!} \left(\frac{A}{I}\right)^a \left(\frac{R}{I}\right)^x \left(Ie^t\right)^m$$

or

$$M_f(t;a,r,S_1,S_3) = h(a,r,R/I,A/I,Ie^t) + h(r,a,A/I,R/I,Ie^t) \qquad (3.6)$$

At this point it is possible to make use of a lemma of advanced calculus (17) which states that

$$h(p,q,B,C,D) = \left(\frac{BD}{1-D-CD}\right)^q$$

$$\cdot \left[1 - \int_0^{CD/(1-D)} \frac{(p+q-1)!}{(p-1)!\,(q-1)!} Z^{p-1} (1-Z)^{q-1}\, dZ\right] \qquad (3.7)$$

Substituting Eq. (3.7) into Eq. (3.6), the moment generating function of the sample size may be written as

$$M_f(t;a,r,S_1,S_3) = \left(\frac{Re^t}{1-(1-R)e^t}\right)^r$$

$$\cdot \left[1- \int_0^{Ae^t/(1-Ie^t)} \frac{(a+r-1)!}{(a-1)!\,(r-1)!}\, Z^{a-1}(1-Z)^{r-1}\, dZ\right]$$

$$+ \left(\frac{Ae^t}{1-(1-A)e^t}\right)^a$$

$$\cdot \left[1- \int_0^{Re^t/(1-Ie^t)} \frac{(a+r-1)!}{(a-1)!\,(r-1)!}\, Z^{r-1}(1-Z)^{a-1}\, dZ\right] \qquad (3.8)$$

At this point it will be convenient to define the power function. The power function of a test gives the probability of rejecting the null hypothesis when the null hypothesis is false. Thus for the alternate hypothesis the power function gives the probability of accepting H_1 when the parameter p in the binomial distribution is equal to p_1. The power function may be written as

$$\varphi(b;a,r,S_1,S_3) = \sum_{m=r}^{\infty}\sum_{x=0}^{a-1} \frac{(m-1)!}{(r-1)!\,x!\,(m-r-x)!}\, R^r A^x I^{m-r-x} \qquad (3.9)$$

By rewriting the above equation we have

$$\varphi(b;a,r,S_1,S_3) = \sum_{m=r}^{\infty}\sum_{x=0}^{a-1} \frac{(m-1)}{(r-1)\;x\;(m-r-x)} \left(\frac{R}{I}\right)^r \left(\frac{A}{I}\right)^x (I)^m \qquad (3.10)$$

Thus the power function can be written as

$$\varphi(b;a,r,S_1,S_3) = h(a,r,R/I,A/I,I)$$

The power function is now in a form to which the lemma discussed earlier may be applied. Substituting the results of Eq. (3.10) into Eq. (3.7) yields

$$\varphi(b;a,r,S_1,S_3) = \left[\frac{(R/I)I}{1-I-A}\right]$$

$$\cdot\left[1-\int_0^{\frac{A}{1-I}} \frac{(a+r-1)!}{(a-1)!\,(r-1)!} Z^{a-1}(1-Z)^{r-1}\,dZ\right] \quad (3.11)$$

This equation may be simplified by realizing that the quantity $(1-I-A) = R$, and also by rewriting the upper limit on the integral as

$$\frac{A}{1-I} = \frac{A}{R+A} = 1-\frac{R}{R+A} = 1-\frac{1}{1+A/R} = 1-\beta'$$

This substitution allows Eq. (3.11) to be written as

$$\varphi(b;a,r,S_1,S_3) = 1-\int_0^{1-\beta'} \frac{(a+r-1)!}{(a-1)!\,(r-1)!} Z^{a-1}(1-Z)^{r-1}\,dZ$$

or

$$\varphi(b;a,r,S_1,S_3) = \int_0^{\beta'} \frac{(a+r-1)!}{(a-1)!\,(r-1)!} Z^{a-1}(1-Z)^{r-1}\,dZ$$

where

$$\beta' = \beta(b;S_1,S_3) = \frac{1}{1+A/R}$$

For a given false alarm probability α and false dismissal probability β, the problem now becomes one of determining zones S_1 and S_3 such that the power function fulfills the following conditions:

$$\varphi(b_0; a, r, S_1, S_3) = \alpha$$
$$\varphi(b_1; a, r, S_1, S_3) = 1 - \beta$$

Since the binomial distribution is a discrete distribution it may not be possible to find values of φ which exactly equal α or $1-\beta$. For this reason any value of $\varphi(b; a, r, S_1, S_3)$ which is less than or equal to α or greater than or equal to $1-\beta$ is considered satisfactory. This will assure that the false alarm probability is no greater than α and the false dismissal probability is no greater than β. Satisfying these conditions will assure that the detector will decide signal is present no more than 100α percent of the time when in fact signal is present. It will also assure that the detector will decide signal is absent no more than 100β percent of the time when signal is actually present.

All sets of parameters (a, r, S_1, S_3) which satisfy the above conditions are acceptable to use as detector parameters. However, some sets of parameters can be considered to be better than others. The optimum set of parameters will be that set which, for a given α and β, requires the fewest number of samples on the average.

An expression must now be developed which will give the ASN. The ASN is the expected number of samples required for termination of the sequential test. It is known that the derivative of the moment generating function evaluated at $t = 0$ yields the expected value of the sample size. Taking the derivitive of Eq. (3.8) and evaluating the result at $t = 0$ gives

$$\mathrm{ASN} = \frac{r}{R}\left[\varphi(b;a,r,S_1,S_3) - \binom{r+a-1}{r} \beta'^{\,r} (1-\beta')^{a}\right]$$

$$+ \frac{a}{A}\left[1-\varphi(b;a,r,S_1,S_3) - \binom{r+a-1}{a} \beta'^{\,r} (1-\beta')^{a}\right]$$

Using the above expression, the ASN can be computed for each set of acceptable parameters. The set which has the smallest ASN will be used as the detector parameter.

These are the basic results which describe the sequential test that will be used in this paper. The remainder of this paper will show how these results may be applied to the sign test and to the detection of signals in noise.

The detector will be designed for a constant signal. Then it will be shown how other types of signals may be detected without altering the detector. The general procedure just discussed will be applied to each case. Specific methods will be developed for determining the quantities A_i and R_i. All other equations will be the same as stated in the previous discussion.

In each case to be considered, the noise is assumed to be gaussian with mean zero and variance σ_0^2. The signal to noise ratio, θ, is defined to be the ratio of the r.m.s. signal to the r.m.s. noise.

3.2 Detection of a Signal Which Changes the d-c Level of the Noise

Consider a constant signal, C. The r.m.s. value of a constant is just that constant and the r.m.s. value of a gaussian distribution is σ_0. Knowing this, the signal to noise ratio may be written as

$$\theta = \frac{C}{\sigma_0}$$

This signal to noise ratio can be used to determine the value of the parameter p_1 in the alternate hypothesis. The value of the parameter p_1 will depend on the difference in the mean of the distribution of signal plus noise and the mean of the distribution of noise alone. It is known that adding a constant to a gaussian distribution shifts the mean of the distribution by that constant. Therefore the distribution of signal plus noise will be gaussian with mean C and variance σ_0^2. The amount of shift caused by adding a constant signal can be found from the signal to noise ratio. It is shown in appendix A that the relationship between p_1 and θ is given by

$$p_1 = 1 - P(Z \leq -|\theta|/\sqrt{2})$$

where Z is distributed G(0,1).

Since the signal, C, can be either positive or negative, the detector must be able to detect shifts in the mean in both the positive and negative direction. Since C can be either positive or negative this will be called a two-sided test. The detector takes n observations, $y(t_i)$, from noise alone, the observations being $Y_1, Y_2, \ldots Y_n$ where $Y_i = y(t_i)$, $i = 1, 2, \ldots n$. The detector also takes n observations $x(t_j)$ from the input to be tested. These observations are designated X_1, $X_2, \ldots X_n$ ($X_j = x(t_j)$, $j = 1, 2, \ldots n$). These 2n observations are paired and the difference $(Y_i - X_j)$, $i = j$, is determined. If signal is present, it is reasonable to expect that either a large or small number of minus signs will occur. This sample space of (n+1) points can be divided into five zones. These zones are shown in figure 1.

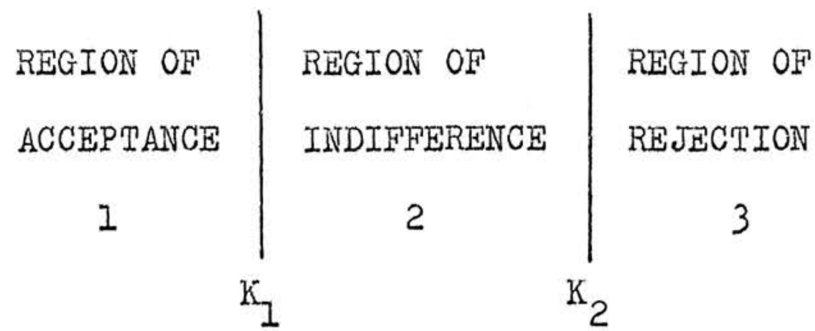
The zones of rejection will be zones of too few or too many minus signs, while the zone of acceptance will be a zone of approximately n/2 minus signs. To simplify this problem into three zones, consider only the sign which occurs the greater number of times. Then the sample space consists of (n/2 + 1) points if n is even or (n + 1)/2 points if n is odd. This sample space may

REGION OF REJECTION	REGION OF INDIFFERENCE	REGION OF ACCEPTANCE	REGION OF INDIFFERENCE	REGION OF REJECTION
1	2	3	4	5

K_1 K_2 K_3 K_4

SAMPLE SPACE FOR TWO-SIDED TEST

FIGURE 1



REDUCED SAMPLE SPACE FOR TWO-SIDED TEST

FIGURE 2

be divided into three zones as shown in figure 2.

Let N_p be the number of plus signs which occur and let N_m be the number of minus signs which occur. Then let K equal the maximum of N_m and N_p. All samples for which $K \leq K_1'$ will be classified into zone 1′ and all samples for which $K \geq K_2'$ will be classified in zone 3′. All other samples will be classified in zone 2′.

Referring back to Fig. 1, the null hypothesis would be rejected if $N_m \leq K_1$ or if $N_m \geq K_4$. Zone 1 would consist of all sample points from 0 to K_1 and zone 5 would consist of all sample points from K_4 to n. Therefore when the sample space is reduced to three zones it is important that zone 3 include all sample points included in zone 1 and zone 5 of the original sample space. Thus

$$R_i = \sum_{k=K_2'}^{n} b_i(k) = \sum_{k=0}^{K_1} b_i(k) + \sum_{k=K_4}^{n} b_i(k)$$

A sample will be in zone 1 only if there are K_1 or less minus signs occuring in the sample. This is equivalent to having $(n-K_1)$ or more plus signs occuring in the sample. Let P′ equal the probability of a minus sign occuring. Then the probability of a plus sign occuring is just 1-P′.

Therefore the summation over the lower zone may be written as

$$\sum_{x=0}^{K_1}\binom{n}{x}(P')^x(1-P')^{n-x} = \sum_{x=n-K_1}^{n}\binom{n}{x}(P')^{n-x}(1-P')^x$$

Thus the equation for R_i becomes

$$R_i = \sum_{x=n-K_1}^{n}\binom{n}{x}(P')^{n-x}(1-P')^x + \sum_{x=K_4}^{n}\binom{n}{x}(P')^x(1-P')^{n-x} \tag{3.13}$$

At this point it will be argued that the lower limits on the two summations must be equal. It is desired that both positive and negative signals be detected with the same false alarm and false dismissal probabilities. In order to detect both positive and negative signals with the same probabilities it is necessary that the rejection zone for negative signals and the rejection zone for positive signals be the same size. If a positive signal is recieved there is a certain probability of getting a minus sign for each observation, depending on the magnitude of the signal. If the signal is negative, and of the same magnitude, the probability of getting a positive sign is exactly the same as the probability of getting a minus sign when a positive signal is present. Since the zones of rejection are of equal size and the probabilities of being in each zone are identical they must contain the same number of sample points. Therefore $(n-K_1)$ must equal K_4 and Eq. (3.13) may be written

$$R_i = \sum_{x=K_4}^{n}\binom{n}{x}(P')^{n-x}(1-P')^x + \sum_{x=K_4}^{n}\binom{n}{x}(P')^x(1-P')^{n-x} \tag{3.14}$$

For $i = 0$, $P' = \frac{1}{2}$ and Eq. (3.14) reduces to

$$R_0 = 2\sum_{x=K_4}^{n}\binom{n}{x}(\tfrac{1}{2})^n$$

When $i = 1$, $P = P_1 \neq \frac{1}{2}$ and Eq. (3.14) becomes

$$R_1 = \sum_{x=K_4}^{n}\binom{n}{x}(p_1)^{n-x}(1-p_1)^x + \sum_{x=K_4}^{n}\binom{n}{x}(p_1)^x(1-p_1)^{n-x}$$

Determining A_i is slightly more difficult. Zone 3 will consist of an even number of sample points if the total number of observations in a sample group is odd. However if the total number of observations in a sample group is even, there will be an odd number of samples in zone 3. Therefore, when the number of observations in a sample group is odd, A_i is given by

$$A_i = \sum_{K_2}^{K_3} b_i(k) = \sum_{x=K_2}^{\frac{n-1}{2}} b_i(k) + \sum_{\frac{n+1}{2}}^{K_3} b_i(k)$$

In the same manner as before, it can be argued that the number of sample points between K_2 and $(n-1)/2$ and the number of sample points between $(n+1)/2$ and K_3 are equal. The above equation can then be written

$$A_i = \sum_{x=\frac{n+1}{2}}^{K_3}\binom{n}{x}(P')^x(1-P')^{n-x} + \sum_{x=\frac{n+1}{2}}^{K_3}\binom{n}{x}(P')^{n-x}(1-P')^x \qquad (3.15)$$

When $i = 0$, $P' = \frac{1}{2}$ and Eq. (3.15) becomes

$$A_0 = 2\sum_{\frac{n+1}{2}}^{K_3}\binom{n}{x}(\tfrac{1}{2})^n$$

When $i = 1$, $P' = p_1 \neq \frac{1}{2}$ and Eq. (3.15) becomes

$$A_i = \sum_{x=\frac{n+1}{2}}^{K_3} \binom{n}{x} (p_1)^x (1-p_1)^{n-x} + \sum_{x=\frac{n+1}{2}}^{K_3} \binom{n}{x} (p_1)^{n-x} (i-p_1)^x$$

The above equations for A_i hold only when the number of observations in a sample is odd. When the total number of observations in a sample is even, A_i is given by

$$A_i = \sum_{K_2}^{K_3} b_i(k) = \sum_{K_2}^{n/2} b_i(k) + \sum_{n/2+1}^{K_3} b_i(k)$$

The above equation can be written as

$$A_i = \sum_{K_2}^{n/2-1} b_i(k) \quad + \quad b_i(\tfrac{n}{2}) + \sum_{n/2+1}^{K_3} b_i(k)$$

Noting again that the summations must be equal and the number of sample points in each summation are the same, both summations can be written with the same index.

$$A_i = b_i(\tfrac{n}{2}) + \sum_{\frac{n}{2}+1}^{K_3} \binom{n}{x} (P')^x (1-P')^{n-x} + \sum_{\frac{n}{2}+1}^{K_3} \binom{n}{x} (P')^x (1-P')^{n-x}$$

When $i = 0$, $P' = \frac{1}{2}$ and A_o may be written as

$$A_0 = \frac{n!}{\frac{n}{2}!\,\frac{n}{2}!} (\tfrac{1}{2})^n + 2 \sum_{x=\frac{n}{2}+1}^{K_3'} \binom{n}{x} (\tfrac{1}{2})^n$$

When $i = 1$, $P = p_1 \neq \frac{1}{2}$ and A_i may be written as

$$A_1 = b_1(\tfrac{n}{2}) + \sum_{x=\frac{n}{2}+1}^{K_1'} \binom{n}{x} (p_1)^x (1-p_1)^{n-x} + \sum_{x=\frac{n}{2}+1}^{K_1'} \binom{n}{x} (p_1)^{n-x} (1-p_1)^x$$

or

$$A_1 = \sum_{x=\frac{n}{2}+1}^{K_1'} \binom{n}{x} (p_1)^x (1-p_1)^{n-x} + \sum_{x=\frac{n}{2}+1}^{K_1'} \binom{n}{x} (1-p_1)^x (p_1)^{n-x}$$

These equations are valid only when the number of observations in each sample is even.

If it could be guarantied that C would always be positive, the detector could be designed to be more efficient. When the signal is always positive it becomes unnecessary to consider a negative shift in the mean. If only the number of minus signs which occur are considered, it is necessary to consider only alternate hypothesis parameters, p_1, which are greater than .50. That is, if the signal caused only a shift in the positive direction, the probability of getting less than 50 percent minus signs when signal is present is very small. This test in which the alternate hypothesis can be only greater than .50 will be called a one-sided test.

For the one-sided test let K equal the number of minus signs occuring from the difference $(Y_i - X_j)$ when $i = j$. The null hypothesis to be tested is $p_0 = \frac{1}{2}$, noise alone, versus the alternate hypothesis that $p_1 > \frac{1}{2}$, signal plus noise. The sample space of n+1 sample points (0,1,2,...n) can be divided into three zones. These zones are shown in figure 3.

Zone 1, the zone of acceptance, consists of all sample points from 0 to K_1. Therefore A_i can be written

$$A_0 = \sum_{x=0}^{K_1} \binom{n}{x} (\tfrac{1}{2})^n$$

REGION OF ACCEPTANCE	REGION OF INDIFFERENCE	REGION OF REJECTION
1	2	3

K_1 (boundary between regions 1 and 2)　　K_2 (boundary between regions 2 and 3)

SAMPLE SPACE FOR ONE-SIDED TEST

FIGURE 3

when $i = 0$. When $i = 1$, A_i is written as

$$A_1 = \sum_{x=0}^{K_1} \binom{n}{x} (p_1)^x (1-p_1)^{n-x}$$

Zone 3, the zone of rejection, consists of all sample points from K_2 to n. Thus when $i = 0$, R_i may be written

$$R_0 = \sum_{x=K_2}^{n} \binom{n}{x} (\tfrac{1}{2})^n$$

and when $i = 1$, R_1 may be written

$$R_1 = \sum_{x=K_2}^{n} \binom{n}{x} (p_1)^x (1-p_1)^{n-x}$$

It has been shown how the sequential sign detector may be used to detect a constant signal in gaussian noise. However this detector would certainly be very limited if this were the only type of signal it could detect. Fortunately, many other type signals can be detected by testing for a shift in mean. In the next section a more general type of signal will be considered.

θ	P_1
1.00	.76
.75	.70
.50	.63
.35	.60
.25	.57
.177	.55
.108	.53

RELATIONSHIP OF P_1 AND θ
FOR A CONSTANT SIGNAL

TABLE I

$\alpha = .10 \qquad \beta = .10$

P_1	a	r	K_1	K_2	N
.70	2	2	5	9	12
.65	2	3	5	9	13
.60	3	4	6	10	15
.57	4	5	8	13	20
.55	5	6	10	17	26
.53	5	4	7	22	28

OPTIMUM VALUES OF a, r, S_1, and S_3 FOR VARIOUS VALUES OF THE ALTERNATIVE HYPOTHESIS, USING THE ONE-SIDED TEST

TABLE II

$\alpha = .05 \qquad \beta = .05$

P_1	a	r	K_1	K_2	N
.70	2	2	5	9	12
.65	2	3	7	12	17
.60	3	4	12	17	27
.57	5	5	12	18	28
.55	5	6	9	19	27
.53	6	6	6	23	28

OPTIMUM VALUES OF a, r, S_1 and S_3 FOR VARIOUS VALUES OF THE ALTERNATIVE HYPOTHESIS, USING THE ONE-SIDED TEST

TABLE III

$\alpha = .01 \qquad \beta = .01$

P_1	a	r	K_1	K_2	N
.70	2	2	5	9	12
.65	2	3	11	18	26
.60	5	4	11	18	26
.57	6	6	11	20	29
.55	5	6	6	21	26
.53	6	6	6	23	30

OPTIMUM VALUES OF a, r, S_1 and S_3 FOR VARIOUS VALUES OF THE ALTERNATIVE HYPOTHESIS, USING THE ONE-SIDED TEST

TABLE IV

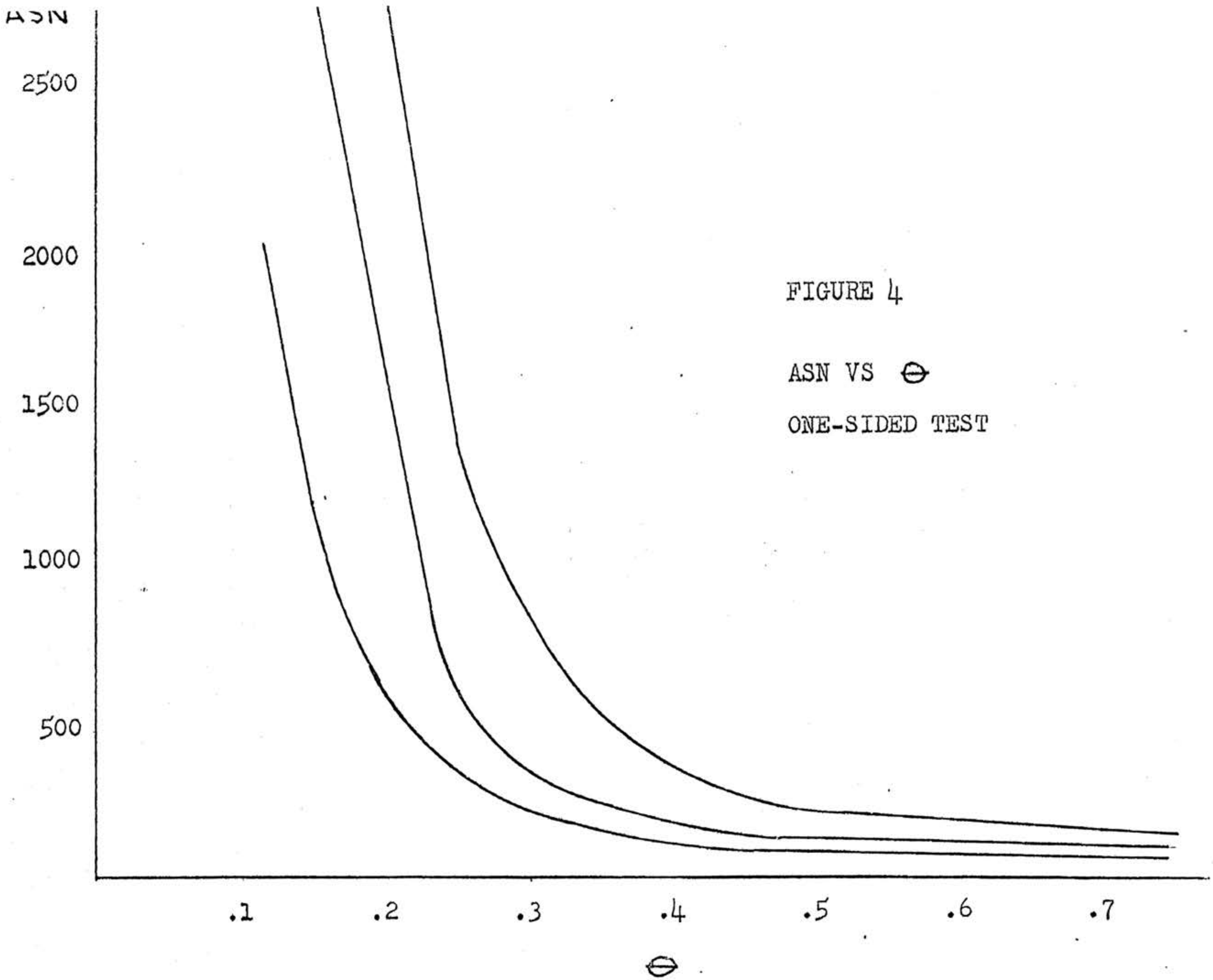


FIGURE 4

ASN VS θ

ONE-SIDED TEST

CHAPTER IV

DETECTION OF A FM SIGNAL

Up to this point the problem of detecting a signal has been discussed only for the very special case of a constant signal. In this chapter the sequential sign detector will be used to detect a frequency modulated signal. The noise will be assumed to be gaussian, although this is not a necessary condition for the detector to operate. The problem is for the detector to decide with a given false alarm and false dismissal probability, whether the FM signal is present or if noise alone is present.

It is desired to detect the presece or absence of a general message, m(t). This message frequency modulates the carrier, which is a cosine wave, to give

$$s(t) = A\cos\left[w_c t + \int_0^t m(z)dz\right] \qquad (4.1)$$

where A is the amplitude and w_c is the carrier frequency. The above equation is the basic form for the frequency modulated carrier. It is also necessary to consider amplitude and phase fading to make the problem more general. Thus the equation (4.1) may be rewritten

$$s(t) = a(t)\cos\left[w_c t + \int_0^t m(z)dz + \phi(t)\right] \qquad (4.2)$$

where a(t) is the amplitude fading term and $\phi(t)$ is the phase fading term.

The messages will be restricted to those which can be expressed in a binary coded form (Binary Frequency Shift Keying) or in r-ary coded form (Multiple Frequency Shift Keying). Attention is restricted to these type messages since many FM problems are frequently of this form. In Binary Frequency Shift Keying (B.F.S.K.) the message is of the form

$$\int_0^t m(z)dz = 0$$

or

$$\int_0^t m(z)dz = \Delta w_c t$$

and in the Multiple Frequency Shift Keying (M.F.S.K.) the message is of the form

$$\int_0^t m(z)dz = 0$$

or

$$\int_0^t m(z)dz = \Delta_1 w_c t$$

or

$$\int_0^t m(z)dz = \Delta_2 w_c t$$

or

$$\vdots$$

$$\int_0^t m(z)dz = \Delta_r w_c t$$

With the restrictions applied, the modulated carrier for the B.F.S.K. or M.F.S.K. message can be written

$$s(t) = a(t)\cos\left[w_j t + \phi(t)\right] \qquad (4.3)$$

where

$$w_j = w_c + \Delta_j w_c \qquad j \;\; 0,1,2,\ldots r$$

and Δ_0 equals zero. It is possible to detect a message of the form given in Eq. (4.3) by centering a matched filter at each frequency and following each filter with a sequential sign detector. The output of the matched filter will be gaussian if the input is uncorrelated white gaussian noise. If the filter is matched to the signal, the output, at a specific time, is equal to the total energy in the signal. The matched filter is a linear filter. Therefore the output, if signal plus noise is present, will be a waveform of the same form as if noise alone was present, but shifted by an amount equal to the energy of the signal.

Thus there is no change necessary in the design of the sequential sign detector developed for a constant signal. When following a matched filter, the detector must only detect a shift in mean of the output waveform.

$\alpha = \beta = .10$			$P_1 = .60$	
SIGNAL PRESENT			SIGNAL ABSENT	
DECISION	NUMBER OF OBSERVATIONS	TRIAL	DECISION	NUMBER OF OBSERVATIONS
YES	105	1	NO	225
YES	90	2	NO	210
YES	60	3	NO	165
YES	105	4	NO	210
YES	120	5	NO	150
YES	120	6	NO	195
YES	135	7	NO	180
YES	90	8	NO	90
YES	120	9	NO	60
YES	105	10	NO	165
YES	90	11	NO	90
YES	75	12	NO	135
YES	120	13	NO	240
YES	180	14	NO	210
NO	180	15	NO	225
YES	90	16	NO	90
YES	180	17	NO	75
YES	150	18	NO	60
YES	105	19	NO	120
YES	225	20	YES	270
YES	90	21	NO	150
YES	90	22	NO	120
YES	150	23	YES	165
YES	90	24	NO	315
YES	135	25	NO	75

USING SEQUENTIAL SIGN DETECTOR TO DETECT
AN FM SIGNAL

TABLE V

TABLE V

cont.

SIGNAL PRESENT			SIGNAL ABSENT	
DECISION	NUMBER OF OBSERVATIONS	TRIAL	DECISION	NUMBER OF OBSERVATIONS
YES	135	26	NO	135
YES	90	27	NO	135
YES	90	28	NO	75
YES	105	29	NO	105
YES	90	30	NO	75
NO	240	31	NO	105
YES	105	32	NO	120
YES	105	33	NO	105
YES	165	34	NO	90
YES	240	35	NO	75
YES	90	36	NO	225
YES	165	37	NO	75
YES	150	38	YES	105
NO	120	39	NO	90
YES	120	40	NO	195
YES	165	41	NO	210
YES	75	42	NO	60
YES	135	43	NO	150
YES	120	44	NO	90
YES	105	45	YES	225
YES	135	46	NO	135
NO	210	47	NO	270
YES	75	48	NO	75
YES	120	49	NO	195
YES	90	50	NO	135

$\alpha = \beta = .10$			$P_1 = .57$	
SIGNAL PRESENT			SIGNAL ABSENT	
DECISION	NUMBER OF OBSERVATIONS	TRIAL	DECISION	NUMBER OF OBSERVATIONS
YES	400	1	NO	180
YES	260	2	NO	420
YES	500	3	NO	220
YES	320	4	NO	220
YES	200	5	NO	400
YES	200	6	NO	400
YES	260	7	NO	160
YES	380	8	NO	100
YES	140	9	NO	160
YES	340	10	NO	260
YES	340	11	YES	580
YES	280	12	YES	400
YES	200	13	NO	620
YES	300	14	NO	200
YES	440	15	NO	160
YES	320	16	NO	220
YES	180	17	NO	280
YES	300	18	NO	380
YES	200	19	NO	440
YES	260	20	NO	300
YES	380	21	NO	220
YES	160	22	NO	400
NO	360	23	YES	200
YES	260	24	NO	160
YES	180	25	NO	120

USING SEQUENTIAL SIGN DETECTOR TO DETECT

AN FM SIGNAL

TABLE VI

TABLE VI

cont.

SIGNAL PRESENT			SIGNAL ABSENT	
DECISION	NUMBER OF OBSERVATIONS	TRIAL	DECISION	NUMBER OF OBSERVATIONS
YES	300	26	NO	440
YES	220	27	NO	480
YES	260	28	NO	160
YES	320	29	NO	280
NO	340	30	NO	380
YES	260	31	NO	240
YES	220	32	NO	280
YES	220	33	NO	240
YES	300	34	NO	120
YES	280	35	YES	320
NO	280	36	NO	400
YES	240	37	NO	280
YES	440	38	NO	200
NO	320	39	NO	180
YES	540	40	NO	540
YES	320	41	NO	520
YES	420	42	NO	420
YES	240	43	NO	200
YES	100	44	YES	520
YES	280	45	NO	300
YES	220	46	NO	160
YES	340	47	NO	200
YES	300	48	NO	300
YES	260	49	NO	360
YES	440	50	NO	300

CHAPTER V

SUMMARY AND CONCLUSIONS

In this paper the design of a sequential detector, which utilizes the statistical sign test, has been developed. It was shown that this detector could be used to detect signals which changed the d-c level of the noise. It also appears that this detector can be used to detect other classes of signals if appropriate filters precede the detector.

Specific detectors for special cases, such as gaussian noise, can be designed; those detectors require fewer samples than the sequential sign detector. However these detectors are incapable of efficient detection for cases other than the one for which they are designed. By using the sign test, the necessity of redesigning the detector is eliminated when the noise statistics change. Thus the sequential sign detector can effectively detect signals in an unlimited number of special cases without extreme loss in efficiency. This ability to detect a signal in a variety of special cases plus the ease with which the detector may be implemented are what make the sequential sign detector so attractive.

In using the sequential sign detector to detect a signal buried in noise, it was found that the average sample size depends not only on α and β but also on the signal to noise ratio, θ. Theoretically this detector can be used to detect signals with extremely low ($<<1$) signal to noise ratios. However it was found that as the signal to noise ratio became low ($<<1$) the average sample size became very large. This can be seen from figure 4. The results shown in Tables V and VI demonstrate that the sequential sign detector can efficiently detect the presence or absence of a signal in a background of noise with prespecified probability of error.

APPENDIX A

DETERMINATION OF P_1 FOR A CONSTANT SIGNAL IN NORMAL $(0, \sigma_0^2)$ NOISE

Let p_1 denote the probability of an observation X_i being greater than an observation from Y_i. That is, p_1 is the probability of obtaining a minus sign for any particular sample.

$$p_1 = P(Y < X) = P(Y-X < 0)$$

Since the distribution of the noise is known and the signal is a constant, the distribution of signal plus noise is also known.

$$Y \sim G(0, \sigma_0^2)$$

$$X \sim G(C, \sigma_0^2)$$

Since both distributions are gaussian, the distribution of the difference is also gaussian.

$$(X-Y) \sim G(C, 2\sigma_0^2)$$

$$P(X-Y > 0) = P\left(\frac{X-Y-C}{\sqrt{2}\,\sigma_0} > \frac{-C}{\sqrt{2}\,\sigma_0}\right)$$

$$= P\left(Z > \frac{-C}{\sqrt{2}\,\sigma_0}\right)$$

where $Z \sim G(0,1)$.

$$P(X-Y > 0) = P\left(Z > \frac{-C}{\sqrt{2}\,\sigma_0}\right) = 1-P\left(Z \leq \frac{-C}{\sqrt{2}\,\sigma_0}\right)$$

It is known that the signal to noise ratio, θ, for a constant signal in gaussian noise is

$$\theta = \frac{C}{\sigma_0}$$

Thus the value of the alternate hypothesis parameter, p_1, in terms of the signal to noise ratio is given by

$$p_1 = 1 - P\left(Z \leq \frac{-\theta}{\sqrt{2}} \right)$$

When the two sided test is used the value of p_1 may be less than .50. This would indicate a negative shift in the mean and the detector would actually use the number of positive signs observed for the test. Since the positive sign is being tested p_1 is given by

$$p_1' = 1 - p_1$$
$$p_1' = 1 - 1 - P\left(Z \leq \frac{-\theta}{\sqrt{2}}\right) = P\left(Z \leq \frac{-\theta}{2}\right)$$

Since the signal to noise ratio would now be negative, the above equation can be written

$$p_1' = P\left(Z \leq \frac{-(-\theta)}{\sqrt{2}} \right) = P\left(Z \leq \frac{\theta}{\sqrt{2}} \right)$$

but

$$P\left(Z \leq \frac{\theta}{\sqrt{2}} \right) = 1 - P\left(Z \leq \frac{-\theta}{\sqrt{2}} \right)$$

Therefore p_1 may be written

$$p_1 = 1 - P\left(Z \leq \frac{-|\theta|}{\sqrt{2}} \right)$$

The expression for p_1 is now in a form such that it can be used for both the one-sided and two-sided tests.

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