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A NEW PERFORMANCE INDEX FOR

SENSITIVITY AND ENERGY OPTIMIZATION

MOHAMED ZAFER DAJANI ~ /942

A

THESIS

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ABSTRACT

The principal aim of this thesis is to present a new index of performance which when minimized will simultaneously optimize the system energy and sensitivity function of a multivariable linear control system.

With this performance index it is usually desirable to discriminate between the sensitivity factor and the energy term by weighting them differently. Two types of weighting factors are presented.

Since it is not easy to minimize that performance index for all types of problems, further research is suggested to find methods of solution for such problems.

Illustrative examples demonstrate the optimization process and the type of assumptions needed for solving such a problem.

ACKNOWLEDGEMENT

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INTRODUCTION

A. STATEMENT OF THE PROBLEM

In a practical multivariable optimal linear control system one usually encounters the problem of small variations of plant parameters, or uncontrolled small perturbations in the signals flowing from the plant to the controller or vice-versa. In order to estimate the effect of these unpredicted changes on the system's performance, a sensitivity coefficient or, equally well, a sensitivity function was defined in the early literature [6] as the ratio of the normalized transmission function variation to the normalized parameter variation, or,

$$S_P^T = \frac{\Delta T/T}{\Delta P/P}$$

where,

S_P^T = the sensitivity coefficient

T = the system's nominal transmission function

ΔT = the change in the system's transmission function

P = the system's parameter under investigation

ΔP = the change of the system's parameter. The system's dynamics may be represented by the following state variable equation

$$\dot{\underline{x}} = \underline{F}(\underline{x}, \underline{m}, \underline{q}, t) \qquad (1)$$

where,

$\underline{x}$ = the system's state vector

$\dot{\underline{x}}$ = the time derivative of the state vector

$\underline{m}$ = the system's control vector

$\underline{q}$ = the system's parameter vector

t = the time.

Equation (1) may be rewritten as

$$\underline{x}' = \underline{G}\ (\dot{\underline{x}},\ \underline{m},\ \underline{q},\ t) \tag{2}$$

Expanding $\underline{G}$ around its operating point, $\underline{q}_o$, by Taylor's expansion and neglecting all the partial derivatives except those with respect to $\underline{q}$

$$\underline{G}\ (\dot{\underline{x}},\ \underline{m},\ \underline{q},\ t) = \underline{G}\ (\dot{\underline{x}},\ \underline{m},\ \underline{q}_o,\ t) + \frac{\partial \underline{x}}{\partial \underline{q}}\ (\underline{q}-\underline{q}_o) +$$

$$\frac{\partial^2 \underline{x}}{\partial \underline{q}^2}\ \frac{(\underline{q}-\underline{q}_o)^2}{2!} + \ldots \tag{3}$$

The term $\frac{\partial \underline{x}}{\partial \underline{q}}$ is the Jacobian matrix and its columns $\frac{\partial \underline{x}}{\partial q_i}$ are defined as the sensitivity coefficient vectors $\underline{V}_i$, or the sensitivity function vectors $\underline{V}_i$. The second partial derivative is a higher order sensitivity coefficient but has little practical use, because it will usually be negligible in comparison with the first partial derivative.

The time domain sensitivity functions can be found from the sensitivity equation. This equation is developed as follows. Equation (1) may be rewritten as

$$\underline{g}\ (\dot{\underline{x}},\ \underline{x},\ \underline{m},\ \underline{q},\ t) = 0 \tag{4}$$

Assuming $\underline{m}(t)$ is independent of $\underline{q}$, the partial derivative of equation (4) with respect to $\underline{q}$, gives

$$\frac{\partial \underline{g}}{\partial \underline{q}} + \frac{\partial \underline{g}}{\partial \underline{x}}\ \frac{\partial \underline{x}}{\partial \underline{q}} + \frac{\partial \underline{g}}{\partial \dot{\underline{x}}}\ \frac{\partial \dot{\underline{x}}}{\partial \underline{q}} = 0 \tag{5}$$

Since

$$\frac{\partial \dot{\underline{x}}}{\partial \underline{q}} = \frac{d}{dt}\left(\frac{\partial \underline{x}}{\partial \underline{q}}\right) = \dot{\underline{v}}$$

equation (5) can be written as

$$\frac{\partial \underline{g}}{\partial \dot{\underline{x}}} \cdot \dot{\underline{v}} + \frac{\partial \underline{g}}{\partial \underline{x}} \cdot \underline{v} = -\frac{\partial \underline{g}}{\partial \underline{q}} \tag{6}$$

Equation (6) contains the sensitivity coefficient and its first derivatives with respect to time, hence, it is called the sensitivity equation. It has been pointed out by Tomovic [19] that the sensitivity equation (6) is always a linear differential equation in $(\underline{v})$. This results from the fact that the coefficients of $(\dot{\underline{v}})$ and $(\underline{v})$ are independent of $(\underline{v})$ and its derivatives.

In sensitivity analysis of optimal control systems, it may be necessary to investigate the variations of the performance index due to the variations in the parameters of the system. For such cases a method of numerical computation was proposed [5]. This method has been further developed, and useful results obtained for linear control systems whereby a performance index of the quadratic form has been considered [18, 9]. Furthermore, a normalized sensitivity function of the performance index has been suggested to be a design criterion for optimization of the systems with incompletely specified or variable plant characteristics by using the time domain sensitivity [9] definition for one parameter only as either:

$$I = \int_{o}^{T} |\underline{v}_i(t)| \, dt \tag{7}$$

or

$$I = \int_{o}^{T} \underline{v}_i^2(t) \, dt \tag{8}$$

where

$$\underline{v}_i^2 = [v_1^2, v_2^2, \dots, v_k^2]^T$$

The unifying characteristic in the references discussed above is that the sensitivity analysis of the optimal control system is performed after their optimal input control is synthesized.

In this thesis, using the time domain sensitivity techniques, a rather general index of optimality is sought wherein both sensitivity and optimality characteristics are incorporated. This enables the synthesis of the optimal control action to be performed simultaneously with respect to sensitivity and performance criteria required for system optimization.

B. LITERATURE REVIEW

The sensitivity problem, its definition and solution, have passed through several steps of development.

The first one to define and use the sensitivity term was Bode in 1945 [6]. He used it as a tool for feed-back amplifier design. Since then many scientific applications have made it necessary to have a measure of change of some system behaviour arising from disturbances and parameter variations. Several aspects of sensitivity have been investigated, such as the root locus sensitivity due to parameter variations of the system by Ur [20], the pole-zero sensitivity by Kuo [7], and sensitivity of large multiple loop control system by Ness and Imad [13].

A wide step of advancement was achieved through the application of the state space representation of linear systems in the formulation of the measure of the system's sensitivity by Porter [10]. An important feature of such an approach is the ability to handle within a common framework the free and forced response sensitivity problems of discrete, continuous, and composite systems.

The use of higher order terms of Taylor's expansion of the system equation to define the sensitivity function was first used by Chang [15]. However, most authors disregard the second and higher order derivatives in the expansion. This fact brought about a big dispute between Thau and Sinha [8, 1], from one side, Witsenhausen and Athans [18, 17] from the opposite side. The latter have proved the equality between open and closed loop quadratic performance index sensitivity using only the first derivative term; the former have numerically shown that the closed loop sensitivity is always less than the open loop sensitivity.

In the investigation of the minimization of sensitivity as one of the ultimate goals of design, it was observed that the system stability and sensitivity are mutually contradictory. This fact was stressed by Siljak and Burzio [16] who used the concept of a parameter plane to check for some stability constraints during the sensitivity minimization. However, their approach was limited due to being able to only adjust two parameters simultaneously, and that the form of the system must be defined before-hand. Many authors use the procedure of optimizing the sensitivity functions by conventional methods [5, 16] and then check the other system criteria, such as stability, rise time, etc. Optimization of the sensitivity functions alone have been treated by many authors [10, 18].

For the system described by equation (1), this thesis seeks to extend the previous work by finding an optimal control vector that will force the system from its initial to its specified final states with minimum energy and optimum sensitivity. The formulation of a new performance index is sought, which will, in effect, help to optimize the system behaviour with respect to energy and sensitivity.

SENSITIVITY MINIMIZATION IN OPTIMAL CONTROL SYSTEMS

A. STATEMENT OF THE PROBLEM

The system to be considered here is a linear time invariant multivariable system of degree n represented by the vector differential equation (I). Equation (I) may be represented as

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{m} \tag{9}$$

where $\underline{A}$ is the Jacobian matrix so that,

$$a_{ij} = \frac{\partial F_i}{\partial x_j} \tag{10}$$

and,

$$b_{ij} = \frac{\partial F_i}{\partial x_j}$$

The solution of equation (9) is:

$$\underline{x}(t) = \underline{Q}(t)\,\underline{x}(0) + \int_0^t \underline{Q}(t-T)\underline{B}\,\underline{m}(T) \cdot dT \tag{12}$$

where,

$\underline{Q}(t)$ is the state transition matrix given by $\underline{Q}(t) = \text{Exp}[\underline{A}t]$

The schematic representation of equation (9) is shown in Figure I.

EVALUATION OF THE STATE CHANGE DUE TO A PARAMETER CHANGE.

The sensitivity of the state variables to variations of the parameters can be represented by the change of state $d\underline{x}(t)$ due to a change of a parameter q_k by using perturbation techniques as follows,

$$\dot{\underline{x}} = (\underline{A} + E\underline{D})\underline{x} \tag{13}$$

where, $E\underline{D}$ represents the change in the system matrix and E is a small constant. Then equation (13) may be written as,
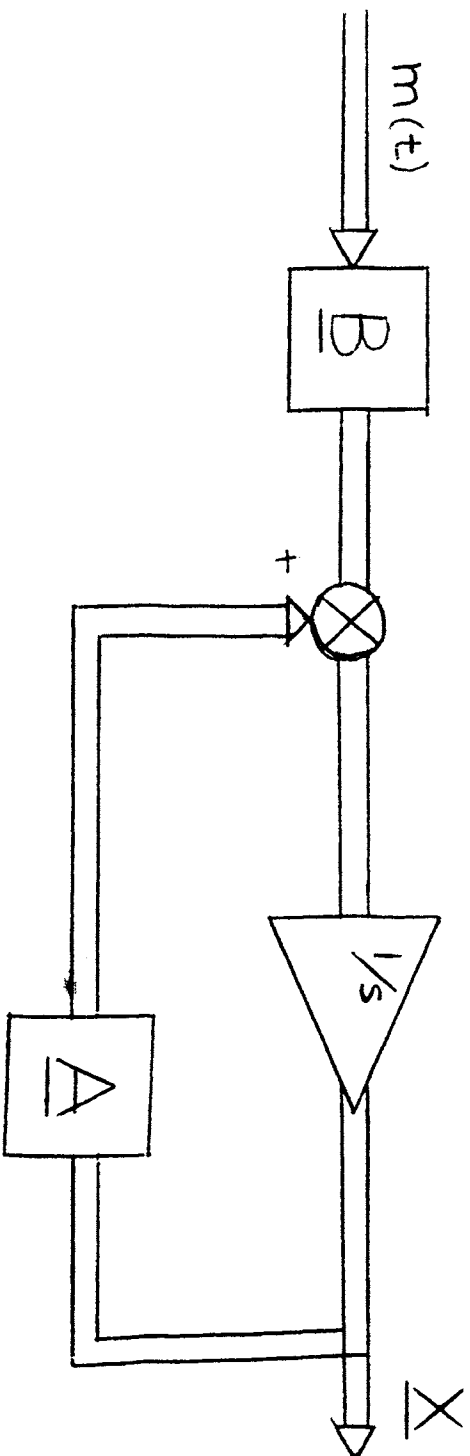
Figure (1)

SCHEMATIC REPRESENTATION OF EQUATION (9)

$$\dot{\underline{x}} = \underline{A}\, \underline{x} + E\, \underline{D}\, \underline{x} \tag{14}$$

Due to the similarity between equation (14) and (9), it follows

that the solution of equation (14) is similar to equation (12),

thus,

$$\underline{x}(t) = \underline{Q}(t)\, \underline{x}(0) + E \int_o^t \underline{Q}(t-T)\underline{D}\, \underline{x}(T)\, dT \tag{15}$$

The approximate solution of equation (15) may be obtained by using

the approximation,

$$\underline{x}(t) = \underline{Q}(t)\, \underline{x}(0)$$

Then

$$\underline{x}(T) = \underline{Q}(T)\, \underline{x}(0) \tag{16}$$

and equation (15) becomes approximately

$$\underline{x}(t) = \underline{Q}(t)\, \underline{x}(0) + E \int_o^t \underline{Q}(t-T)\underline{D}\, \underline{Q}(T)\, \underline{x}(0)\, dT \tag{17}$$

Therefore, the change in the state vector due to the change in the

system matrix is,

$$\Delta \underline{x}(t) = E \int_o^t \underline{Q}(t-T)\underline{D}\, \underline{Q}(T)\, \underline{x}(0)\, dT \tag{18}$$

For a stable system, the elements of $\underline{Q}(t)$ decay exponentially with

time and thus equation (18) converges to zero as time increases

without limit. If the variation of the system parameters is such

that identical perturbation in each coefficient of the system

matrix $\underline{A}$ occurs, then, equation (13) becomes

$$\dot{\underline{x}} = (\underline{A} + E\underline{A})\underline{x} \tag{19}$$

Thus the change in the state vector is,

$$\underline{x}(t) = E \int_o^T \underline{Q}(t-T)\underline{A}\, \underline{Q}(T)\, \underline{x}(0)\, dT \tag{20}$$

Furthermore, the $\underline{A}$ matrix may be written through a suitable linear

transformation in the following manner,

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & -a_{nn} \end{bmatrix} \tag{21}$$

If the system was initially displaced so that,

$$\underline{x}^T(0) = [1, 0, 0, \ldots, 0]$$

then equation (20) becomes

$$\Delta\underline{x}(t) = E \int_o^T \underline{Q}(t-T) \cdot \begin{bmatrix} Q_{21}(T) \\ Q_{31}(T) \\ \cdot \\ \cdots \\ \cdot \\ Q_{n-1,1}(T) \\ R(T) \end{bmatrix} dT \tag{22}$$

where,

$$R(T) = -a_{n1}Q_{11} - a_{n2}Q_{21} - \cdots \cdots - a_{n,n-1}Q_{n-1,1} - a_{nn}Q_{n1}$$

Therefore, the change in the state vector can be evaluated for a change in the system parameter matrix by utilizing equation (18) or equation (22) as the case might require.

THE SENSITIVITY COEFFICIENTS

The variations in the state variables due to a small variation of parameter q_i may be written as,

$$\underline{V}_i = \frac{\partial \underline{x}}{\partial q_i} \tag{23}$$

The above definition of the sensitivity coefficient $\underline{V}_i$ is frequently used in modern literature. In the case where K parameters vary, then the variation in the state variables due to the $\underline{q}$ vector may be

evaluated as the partial derivative of the state vector $\underline{x}$ with respect to the parameter vector $\underline{q}$, where,

$$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix} \tag{24}$$

then the change of the state vector $\underline{x}$ due to $\underline{q}$ is,

$$\frac{\partial \underline{x}}{\partial \underline{q}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial q_1} & \cdots & \dfrac{\partial x_1}{\partial q_k} \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial q_1} & \cdots & \dfrac{\partial x_n}{\partial q_k} \end{bmatrix} \tag{25}$$

or,

$$= \begin{bmatrix} \dfrac{\partial \underline{x}}{\partial q_1} , & \dfrac{\partial \underline{x}}{\partial q_2} , & \cdots & \dfrac{\partial \underline{x}}{\partial q_k} \end{bmatrix} \tag{26}$$

From equation (23), equation (26) may be written as,

$$\frac{\partial \underline{x}}{\partial \underline{q}} = \begin{bmatrix} \underline{V}_1 , & \underline{V}_2 , & \cdots & , & \underline{V}_k \end{bmatrix} \tag{27}$$

The sensitivity coefficients $(\underline{V}_1 , \underline{V}_2 , \cdots , \underline{V}_k)$ may be obtained by perturbation techniques [10], or the time domain method [9] as in equation (27). The last method, however, is the one which will be considered in this thesis for its simplicity in mathematics and its physically significant results.

SENSITIVITY COEFFICIENTS IN THE TIME DOMAIN

The system's vector differential equation (1) may be rewritten

as,

$$\dot{\underline{x}} - \underline{F} (\underline{x}, \underline{m}, \underline{q}, t) = 0$$

or,

$$g (\dot{\underline{x}}, \underline{x}, \underline{m}, \underline{q}, t) = \underline{0} \qquad (28)$$

Equation (28) is a system of linear differential equations of the form:

$$g_i (\dot{\underline{x}}, \underline{x}, \underline{m}, \underline{q}, t) = 0 \qquad i = 1, 2, \ldots , n \qquad (29)$$

Now it is desired to obtain the variation in each state variable with respect to the parameter q_k; that is $\dfrac{\partial x_i}{\partial q_k}$. Assuming $\underline{m}(t)$

is independent of $\underline{q}$ then the partial derivative of equation (29) with respect to the parameter q_k is:

$$\sum_{j=1}^{m} \frac{\partial g_i}{\partial \dot{x}_j} \cdot \frac{\partial \dot{x}_j}{\partial q_k} + \sum_{j=1}^{m} \frac{\partial g_i}{\partial x_j} \times \frac{\partial x_j}{\partial q_k} + \frac{\partial g_i}{\partial q_k} = 0 \qquad (30)$$

But from equation (28) and (29) we can see that:

$$\frac{\partial g_i}{\partial \dot{x}_j} = \begin{cases} 0 \quad \text{when} \quad i \neq j \\ \\ 1 \quad \text{when} \quad i = j \end{cases} \qquad (31)$$

Then equation (30) may be modified to the following form,

$$\frac{\partial \dot{x}_i}{\partial q_k} + \sum_{j}^{n} \frac{\partial g_i}{\partial x_j} \frac{\partial x_j}{\partial q_k} + \frac{\partial g_i}{\partial q_k} = 0 \qquad (32)$$

For different values of j and i equation (32) may be written in matrix form as;

$$
\begin{bmatrix} \dfrac{\partial \dot{x}_1}{\partial q_k} \\[4pt] \dfrac{\partial \dot{x}_2}{\partial q_k} \\[2pt] \cdot \\ \cdots \\ \cdot \\ \dfrac{\partial \dot{x}_n}{\partial q_k} \end{bmatrix}
+
\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} & \cdot & \cdot & \cdot & \dfrac{\partial g_1}{\partial x_n} \\[6pt] \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} & \cdot & \cdot & \cdot & \dfrac{\partial g_2}{\partial x_n} \\[2pt] \cdot \\ \cdots \\ \cdot \\ \dfrac{\partial g_n}{\partial x_1} & & & \cdot & \cdot & \cdot & \dfrac{\partial g_n}{\partial x_n} \end{bmatrix}
\cdot
\begin{bmatrix} \dfrac{\partial x_1}{\partial q_k} \\[4pt] \dfrac{\partial x_2}{\partial q_k} \\[2pt] \cdot \\ \cdots \\ \cdot \\ \dfrac{\partial x_n}{\partial q_k} \end{bmatrix}
+
\begin{bmatrix} \dfrac{\partial g_1}{\partial q_k} \\[4pt] \dfrac{\partial g_2}{\partial q_k} \\[2pt] \cdot \\ \cdots \\ \cdot \\ \dfrac{\partial g_n}{\partial q_k} \end{bmatrix}
= \underline{0}
\tag{33}
$$

Or in vector form, equation (33) may be rewritten as

$$
\frac{\partial \underline{\dot{x}}}{\partial q_k} + (\underline{\nabla}\,\underline{g}) \quad \frac{\partial \underline{x}}{\partial q_k} + \frac{\partial \underline{g}}{\partial q_k} = \underline{0}
\tag{34}
$$

Where $(\underline{\nabla}\underline{g})$ is the gradient of $\underline{g}$ with respect to $\underline{x}$, and is also the

Jacobian matrix $(\underline{+L})$, and $\dfrac{\partial \underline{x}}{\partial q_k}$ is the sensitivity coefficient vector

$(\underline{V})$ as defined before. Equation (34) may be written as:

$$
\underline{\dot{V}} = -\underline{L}\,\underline{V} + \underline{Z}
\tag{35}
$$

Where,

$$
\underline{Z} = \frac{-\partial \underline{g}}{\partial q_k}
\tag{36}
$$

Since equation (35) is of the same form as the linearized system

equation (9), the solution of equation (35) is of the same form of

the solution of equation (9), that is,

$$
\underline{V}(t) = \underline{Q}(t)\,\underline{V}(0) + \int_0^t \underline{Q}(t-T)\,\underline{Z}(T)\,dT
\tag{37}
$$

where,

$$
\underline{Q}(t) = \mathrm{Exp}(-\underline{L}t)
\tag{38}
$$

Therefore, we have the variation in the state vector due to a

change in a single parameter as

$$
\frac{\partial \underline{x}(t)}{\partial q_k(t)} = \underline{V}_k(t)
\tag{39}
$$

The forcing function $\underline{Z}(t)$ in the integrand of equation (37) relates the solution of the parameter sensitivity vector $\underline{V}$ to the original system state vector, $\underline{x}(t)$. If the parameter appears in only one of the set of equations (29), for example at $i=n$, then

$$\underline{Z}(T) = (0, \ldots, 0, \frac{\partial g_n}{\partial q_k}) \tag{40}$$

In general, $\underline{Z}$ is comprised of several of the system state variables $x_1, x_2, \ldots, x_m$. In any case, the explicit solution of equation (35) requires the availability of several, if not all, of the state variables. To evaluate the resulting change in each state variable for a linear system with parameter variations, the system matrix $(\underline{A})$ will be written in the form of equation (21). For example, if the parameter that is varying is $-a_{nl}=q_l$, then we have

$$\underline{Z}^T = [0, 0, \ldots, x_l]$$

If the parameter varies a small amount so that it assumed a new value of $-a_{nl}(1+E)$, and if $\underline{V}(0) = 0$ (which is a reasonable assumption, since for any number of systems under comparison it just changes the datum of comparison by making $\underline{V}(0) = 0$), then from equation (37) it follows that,

$$\underline{V}(t) = \int_0^t \underline{Q}(t-T) \, \underline{Z}(T) \, dT \tag{41}$$

By substituting the value of $\underline{Z}(T)$ from equation (40) into (41) it follows that,

$$\underline{V}(t) = \int_0^t \begin{array}{c} Q_{1n}(t-T) \\ Q_{2n}(t-T) \\ \vdots \\ Q_{nm}(t-T) \end{array} \cdot x_l(T) \, dT \tag{42}$$

However, if $\underline{x}^T(0) = [1, 0, 0, \ldots, 0]$, then from equation (16) it follows that

$$x_i(T) \cong Q_{11}(T)$$

thus,

$$V_i(t) \cong \int_o^t Q_{in}(t-T) \; Q_{11}(T) \; dT \qquad\qquad (43)$$

Equation (43) is the time domain expression of the sensitivity

coefficients.

B. THE COMBINED PERFORMANCE INDEX

In order to compare one system design with another, some numerical measure of system sensitivity is necessary. It was shown in equation (43) that the sensitivity coefficients $V_i(t)$ are functions of time and thus cannot readily serve as an index. In general, an integral function of the sensitivity time response $\underline{V}_i(t)$ is useful because it will give the combined effect as a number for any specified period of time. Therefore, a general index may be written as:

$$I = \int_0^T K(\underline{V}_1, \underline{V}_2, \ldots\ldots, \underline{V}_k) \, dt \qquad (44)$$

For practical purposes the sum of weighted quadratics or the absolute value of sensitivity coefficients is useful for writing the sensitivity index I as,

$$I = \int_0^T (\underline{V}_1^2 + a_1\underline{V}_2^2 + \ldots + a_{k-1}\underline{V}_k^2) \cdot dt \qquad (45)$$

Or,

$$I = \int_0^T (|\underline{V}_1| + a_1|\underline{V}_2| + \ldots + a_{k-1}|\underline{V}_k|) \cdot dt \qquad (46)$$

where $a_1, a_2, \ldots\ldots, a_{k-1}$ are weighting factors [9].

The optimal control problem is normally taken as the problem of minimizing a given performance index J under the constraints on both the control vector $\underline{m}$ and the state vector $\underline{x}$, where,

$$J = \int_0^T f(\underline{x}, \underline{m}, t) \cdot dt \qquad (47)$$

Then the sensitivity of the performance index J due to a change in the parameters is evaluated and defined as the sensitivity of

the optimal system , thus the sensitivity of the optimal system is

by definition $\dfrac{\partial J(m)}{\partial q_k}$.

In contrast to the above idea, it is the purpose of this thesis

to include the effects of parameter variations in the performance

index. Then the minimization of the new performance index results

in an optimal system which will be optimal with respect to the

performance criteria and sensitivity. A general form of the

performance index useful for this purpose is

$$J = \int_0^T f(\underline{x}, \underline{m}, \underline{V}_1, \underline{V}_2, \ldots , \underline{V}_k, t) \cdot dt \qquad (48)$$

The optimal control law $\underline{m}^*(t)$, which may minimize the index J,

may be obtained, in some special cases, by using the modern

optimization techniques [4, 11]. Then the optimal control law is

written as:

$$m^*(t) = (\underline{x}, \underline{V}_1, \underline{V}_2, \ldots , \underline{V}_k, t) \qquad (49)$$

Or,

$$m^*(t) = (\underline{x}, \underline{q}, t) \qquad (50)$$

For practical purposes, however, the optimal control problem

including the effect of sensitivity may be formulated in several

ways. One standard approach is to use a formulation in terms of

quadratic functions of the variables. Then the optimal control

problem becomes the minimization of the performance index J where,

$$J = \int_0^T (a_1 \underline{x}^2 + a_2 \underline{m}^2 + a_3 \underline{V}_1^2 + \ldots + a_{k+2} \underline{V}_k^2) \cdot dt \qquad (51)$$

In practice, it may be desirable to emphasize the importance of some of the

sensitivity coefficients over the others. Thus a proposed weighting

scheme for the sensitivity coefficients in the performance index is

shown in the following equation as,

$$J = \int_o^T [a_1\underline{x}^2 + a_2\underline{m}^2 + a_3(\underline{w}^T\underline{V}_1)^2 + \ldots a_{k+2}(\underline{w}^T\underline{V}_k)^2] \cdot dt \quad (52)$$

where $\underline{w}^T$ is the transpose of the weighting vector $\underline{w}$ where,

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \quad (53)$$

The nature of the weighting vectors will be discussed later in the examples and the conclusion.

In the case where only one parameter is to be considered in the optimal control problem, a new state vector $(\underline{y})$ can be defined so that the sensitivity vector$(\underline{V})$ can augment the system state variables, then

$$\underline{y} = \begin{bmatrix} \underline{x} \\ \underline{V}_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_m \\ \dfrac{\partial x_1}{\partial q_1} \\ \cdot \\ \cdot \\ \dfrac{\partial x_n}{\partial q_1} \end{bmatrix} \quad (54)$$

Then the optimal control problem may be formulated as the minimization of the index J, where

$$\quad (55)$$

$$J = \int_o^{T_f} g(\underline{y}, \underline{m}, t) \, dt$$

and T_f is the final time at which the control force will stop. A quadratic formulation of this index can then be written as,

$$\quad (56)$$

$$J = \int_o^{T_f} (\underline{y}^2 + a\underline{m}^2) \cdot dt$$

The optimal control law, $\underline{m}^*(t)$, which will minimize the index J

results in a system which is optimal with respect to both performance

criterion.

In the following chapter the principle of performance index

sensitivity will be presented, and some illustrative examples will

be solved to show the applicability of that new combined performance

index.

DISCUSSION

In this chapter, illustrative examples are presented to indicate the usage and effect of the proposed new index in control problems. Through the presentation of these examples, the time domain sensitivity derivation and the value of particular types of weighting factors will be shown. The concept of performance index sensitivity will be presented to distinguish it from our approach that will optimize the sensitivity and the other performance criteria of the system simultaneously.

A. PERFORMANCE INDEX SENSITIVITY

The sensitivity problem in optimal control theory is concerned with the variations in performance index caused by variations in plant parameters. Assume that the performance index J is given by

$$J = \int_{0}^{T_f} F(\underline{x}, \underline{m}) \, dt \qquad (57)$$

where T_f is the final time at which the control force stops, and $\underline{x} = \underline{x}(\underline{q}, t)$. The plant (controlled object) output $\underline{x}(t)$ is related to the plant input (control input) $\underline{m}(t)$ by a vector differential equation

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{m}(t), \underline{q}] \qquad (58)$$

where $\underline{q}$ is the plant parameter vector of dimension k, $\underline{x}$ and $\underline{m}$ are column vectors of dimension (n) and (L) respectively. The optimal closed-loop control law, denoted by $m^*(t)$, which minimizes J (assuming that J can be minimized) may be obtained at least in principle from any currently available optimization techniques and is generally of the form

$$\underline{m}^*(t) = \underline{R}\,[\underline{x}(t),\ \underline{q},\ t] \qquad (59)$$

Unfortunately in actual practice the plant parameter vector $\underline{q}$

which appears in equation (57) seldom corresponds to the value of

$\underline{q}$ used in the controller. This is due to such things as component

inaccuracies, environmental effects, aging, etc. The problem then

is to determine the effect of such variations on (J). Generally,

the controller components are less subject to variations than plant

components; hence, it is assumed that $\underline{q}$ in equation (58) remains

fixed at a nominal value $\underline{q}_0$ while $\underline{q}$ in equation (57) may vary

arbitrarily. However, in any optimization interval $[t_0,\ T_f]\ \underline{q}$

is assumed to be a constant vector. The closed loop system dynamics

are then described by:

$$\underline{\dot{x}}(t) = f[\underline{x},\ \underline{R}(\underline{x}(t),\ \underline{q}_0,\ t),\ \underline{q}] \qquad (60)$$

with a corresponding performance index value $J(\underline{q}_0,\ \underline{q})$. Optimal

operation, however, requires that $\underline{q} = \underline{q}_0$ and that the minimum

value of (J) is given by $J(\underline{q}_0,\ \underline{q}_0)$. Variations in J due to plant

parameters variations may then be represented by the difference

ΔJ where,

$$\Delta J = J(\underline{q}_0,\ \underline{q}) - J(\underline{q}_0,\ \underline{q}_0) \qquad (61)$$

For inifinitesimal parameter variations one can evaluate ΔJ as

follows:

$$\Delta J = dJ = \frac{\partial J}{\partial q_1}\ dq_1 + \frac{\partial J}{\partial q_2}\ dq_2 + \ldots\ldots + \frac{\partial J}{\partial q_k}\ dq_k \qquad (62)$$

Equation (62) may be written in vector form

$$\Delta J = \frac{\partial J}{\partial \underline{q}}\ \cdot\ d\underline{q}$$

where $\frac{\partial J}{\partial \underline{q}}$ is defined to be a row vector with components $\frac{\partial J}{\partial q_i}$,

and $d\underline{q}$ is a column vector with components $d\underline{q} = (q_i - q_{0i})$. The

term $\frac{\partial J}{\partial \underline{q}}$ is referred to here as the performance index sensitivity

vector and is evaluated at the point $\underline{q} = \underline{q}_o$. In terms of J given by

equation (57), $\frac{\partial J}{\partial \underline{q}}$ becomes

$$\frac{\partial J}{\partial \underline{q}} = \int_o^{Tf} \frac{\partial F}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial \underline{q}} \, dt \qquad (64)$$

where $\frac{\partial F}{\partial \underline{x}}$ is a row vector with components $\frac{\partial F}{\partial x_i}$, and $\frac{\partial \underline{x}}{\partial \underline{q}}$ is an

(n x k) matrix in the following form,

$$\frac{\partial \underline{x}}{\partial \underline{q}} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \cdot & \cdot & \cdot & \frac{\partial x_1}{\partial q_k} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \frac{\partial x_n}{\partial q_1} & \cdot & \cdot & \cdot & \frac{\partial x_m}{\partial q_k} \end{bmatrix} \qquad (65)$$

The various derivatives which appear as components of $\frac{\partial \underline{x}}{\partial \underline{q}}$ are

also evaluated at the point $\underline{q} = \underline{q}_o$. The value of $\frac{\partial \underline{x}}{\partial \underline{q}}$ is determined

from the equation (58) as,

$$\frac{d}{dt}\left(\frac{\partial \underline{x}}{\partial \underline{q}}\right) = \frac{\partial \underline{f}}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial \underline{q}} + \frac{\partial \underline{f}}{\partial \underline{q}} \qquad (66)$$

where,

$$\frac{\partial \underline{f}}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \frac{\partial f_n}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad (67)$$

$$\frac{\partial \underline{f}}{\partial \underline{q}} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial q_k} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \frac{\partial f_n}{\partial q_1} & \cdot & \cdot & \cdot & \frac{\partial f_n}{\partial q_k} \end{bmatrix} \qquad (68)$$

and both $\frac{\partial f}{\partial \underline{x}}$ and $\frac{\partial f}{\partial \underline{q}}$ are evaluated along the nominally optimal

trajectory $(\underline{q} = \underline{q}_o)$. The usual boundary conditions for equation (66)

are

$$\frac{\partial \underline{x}}{\partial \underline{q}}\Bigg|_{t=0} = \underline{0} \qquad (69)$$

where $\underline{0}$ represents an (n x k) zero matrix. Since equation (66) is

linear, though time varying, a solution of equation (66) may be

written explicitly in terms of the transition matrix $\underline{0}(t,T)$ for

the system:

$$\frac{d}{dt}\left(\frac{\partial \underline{x}}{\partial \underline{q}}\right) = \frac{\partial f}{\partial \underline{x}}\Bigg|_{q=q_o} \cdot \frac{\partial \underline{x}}{\partial \underline{q}} \qquad (70)$$

Thus one can write the solution of equation (70) as:

$$\frac{\partial \underline{x}}{\partial \underline{q}} = {}_0\!\int^T f \;\; \underline{0}(t,T) \;\; \frac{\partial f}{\partial \underline{q}} \; dT \qquad (71)$$

which completes the evaluation of $\frac{\partial J}{\partial \underline{q}}$ given by equation (64).

In general, the solution of equation (66) and the integration in

equation (64) are sufficiently complex as to require computer

solution. It should be noted that if the components of

$$\frac{\partial J(\underline{q}_o \, , \, \underline{q})}{\partial \underline{q}}\Bigg|_{q=\underline{q}_o} \qquad (72)$$

do not all vanish; then a certain plant parameter variation may

actually yield a smaller value of (J) than the previously optimal

value $J(\underline{q}_o \, , \, \underline{q}_o)$. Therefore, by the very definition of an optimal

controller, it follows that all the components of

$$\frac{\partial J(\underline{q}_o \, , \, \underline{q})}{\partial \underline{q}_o}\Bigg|_{q=\underline{q}_o} \qquad (73)$$

must always vanish.

B. ILLUSTRATIVE EXAMPLES

Example I:

Consider a simple second order linear system with a gain of (K) represented by the following transfer function

$$\frac{y(s)}{m(s)} = \frac{K}{s^2} \tag{74}$$

It is required to find the optimal control function, $m^*(t)$, that will take the system from its initial state to its final state, in a specified final time ($T_f = 10$ sec.), with the optimal performance index in the form of

$$J = {}_0\int^{T_f} [m^2(t) + m^2(t)\ \underline{v}^2(t)]\ dt$$

where $m^2(t)$ represents energy put into the system, and $m^2(t)$ is taken to be the weighting factor of $\underline{v}^2(t)$.

Writing the state equations of the system in the following form:

$$\dot{x}_1 = x_2 \tag{75}$$

$$\dot{x}_2 = km \tag{76}$$

then it is possible to write the system's matrix equation as follows:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ K \end{bmatrix} m \tag{77}$$

Hence the $\underline{A}$ matrix of the system will be equal to

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{78}$$

The initial and final states will be assumed to be $\underline{x}(0) = [1, 1]^T$ and $\underline{x}(T_f) = \underline{0}$ respectively. Since the system's matrix (A) is known, by [12], the state transition matrix of the system may be written as

$$\underline{Q}(t) = \begin{bmatrix} u(t) & t \\ & \\ 0 & u(t) \end{bmatrix} \qquad \text{where } u(t) \text{ is the unit step function.} \qquad (79)$$

To find the sensitivity coefficient in the time domain, will need the use of equation (41). In this case we are considering the sensitivity of the system to small variations in the element (a_{12}) of $(\underline{A})$ which has the nominal value of one in this example. It is also assumed that the initial value of the sensitivity coefficient is equal to zero just for the sake of simplicity of development, but it becomes clear that to compare two systems it is essential that the initial value of the sensitivity coefficient must be the same for a fair comparison. From equation (40) it follows that

$$\underline{Z}(T) = \begin{bmatrix} x_2 \\ \\ 0 \end{bmatrix} \qquad (80)$$

Then using equation (41) to get $\underline{V}(t)$

$$\underline{V}(t) = \int_0^t \begin{bmatrix} u(t-T) & (t-T) \\ & \\ 0 & u(t-T) \end{bmatrix} \begin{bmatrix} x_2(T) \\ \\ 0 \end{bmatrix} dT \qquad (81)$$

By using the approximate value of $x_2(T)$ from equation (81) it follows that

$$\underline{x}(T) = \underline{Q}(T)\,\underline{x}(0) \qquad (82)$$

therefore,

$$x_2(T) = u(T) \qquad (83)$$

Substituting the value of $x_2(T)$ from equation (83) into equation (81) and integrating from zero to (t)

$$\underline{V}(t) = \begin{bmatrix} t \\ \\ 0 \end{bmatrix} \qquad (84)$$

therefore,

$$v_1(t) = \frac{x_1}{a_{12}} = t \qquad (85)$$

and,

$$V_2(t) = \frac{x_2}{a_{12}} = 0 \qquad (86)$$

At this step of minimization it is required to form the Hamiltonian of the system and the costate vector $\underline{P}(t)$ as

$$\underline{P}(t) = \begin{bmatrix} P_1(t) \\ \\ P_2(t) \end{bmatrix} \qquad (87)$$

The Hamiltonian, H, will then be equal to

$$H = m^2(t) + m^2(t)\,\underline{v}^2(t) + P_1 x_2 + P_2 Km(t) \qquad (88)$$

The necessary conditions for the existence of an optimal control is that the first and second derivatives of the Hamiltonian of the system with respect to the control function $\underline{m}(t)$ are zero and a positive quantity, respectively [11]. Therefore,

$$\frac{\partial H}{\partial m} = 2m(1+t^2) + KP_2(t) = 0 \qquad (89)$$

and

$$\frac{\partial^2 H}{\partial m^2} = 2(1+t^2) \qquad (90)$$

From equation (90), it is clear that the second derivative will be always a positive quantity, therefore we can assure the existence of the optimal control law, which can be found from equation (89) as

$$m^*(t) = \frac{-KP_2(t)}{2(1+t^2)} \qquad (91)$$

To determine the optimal control function it is required to find the function $P_2(t)$, this may be accomplished by using Lagrange equations for a conservative system [11] as follows:

$$\frac{\partial H}{\partial \underline{x}} = -\underline{\dot{P}} \tag{92}$$

and,

$$\frac{\partial H}{\partial \underline{P}} = \underline{\dot{x}} \tag{93}$$

Applying equation (92) on the Hamiltonian in equation (88) one may then get,

$$\frac{\partial H}{\partial x_1} = -\dot{P} = 0 \tag{94}$$

and,

$$\frac{\partial H}{\partial x_2} = -\dot{P}_2 = P_1 \tag{95}$$

Solving equations (94) and (95) then finding $P_1(t)$ and $P_2(t)$ as,

$$P_1(t) = b_1 \tag{96}$$

and,

$$P_2(t) = -b_1 t + b_2 \tag{97}$$

where b_1 and b_2 are the initial values of $P_1(t)$ and $P_2(t)$ respectively, and may be determined by using the final conditions at $t = T_f = 10$ seconds. From equations (91) and (97), we may write $m^*(t)$ as

$$m^*(t) = \frac{b_1 t - b_2}{2(1+t^2)} \tag{98}$$

Solving, now, for the state variable using equation (98) and the systems state equations, we get

$$x_2^*(t) = \int_0^t Km^*(T) \, dT + x_2(0)$$

then,

$$x_2^*(t) = \int_0^t \frac{b_1 T - a_2}{2(1+T^2)} \, dT + 1$$

therefore,

$$x_2^*(t) = \frac{b_1 K}{4} \ln(1+t^2) - \frac{b_2 K}{2} \tan^{-1}(t) + 1 \qquad (99)$$

and,

$$x_1^*(t) = \int_0^t x_1^*(T) \, dT + x_1(0)$$

then,

$$x_1^*(t) = \frac{b_1 k}{4} \int_0^t \ln(1+T^2) \, dT - \frac{b_2 k}{2} \int_0^t \tan^{-1}(T) \, dT + \int_0^t 1 \, dt + 1$$

From integration tables the above integrals may be found, and then,

$$x_1^*(t) = \frac{b_1 k}{4} \left[t \ln(1+t^2) - 2t + 2\tan^{-1}(t) \right]$$

$$- \frac{b_2 k}{2} \left[t \tan^{-1}(t) - 1/2 \ln(1+t^2) \right] + t + 1 \qquad (100)$$

Now, applying the final time conditions on $x_1(t)$ and $x_2(t)$ in equations (99) and (100), we then will get the following two equations, with two unknowns namely, b_1 and b_2 as follows,

$$ \qquad (101)$$

$$5.8006 \, b_1 - 6.22 \, Kb_2 = -11$$

and,

$$ \qquad (102)$$

$$1.1538 \, b_1 - 0.7374 \, b_2 k = -1$$

Hence; solving for b_1 and b_2 in equations (101) and (102) we get,

$$ \qquad (103)$$

$$b_1 = \frac{0.5975}{k}$$

and,

$$ \qquad (104)$$

$$b_2 = \frac{2.38}{k}$$

Thus, by substituting the values of b_1 and b_2 into equation (98) we get the expression for the optimal control $m^*(t)$ as

$$m^*(t) = \frac{0.5975t - 2.38}{2(1+t^2)} \qquad (105)$$

Equation (105) represents the optimal control function that will force the system from its initial state to its final state in 10 seconds, with minimal performance index which will in turn optimize the weighted input energy and sensitivity to any small fluctuation of the parameter (a_{12}) of matrix $\underline{A}$. This control function tends to go to zero as the time tends to increase, and it has the shape indicated in Figure 2.
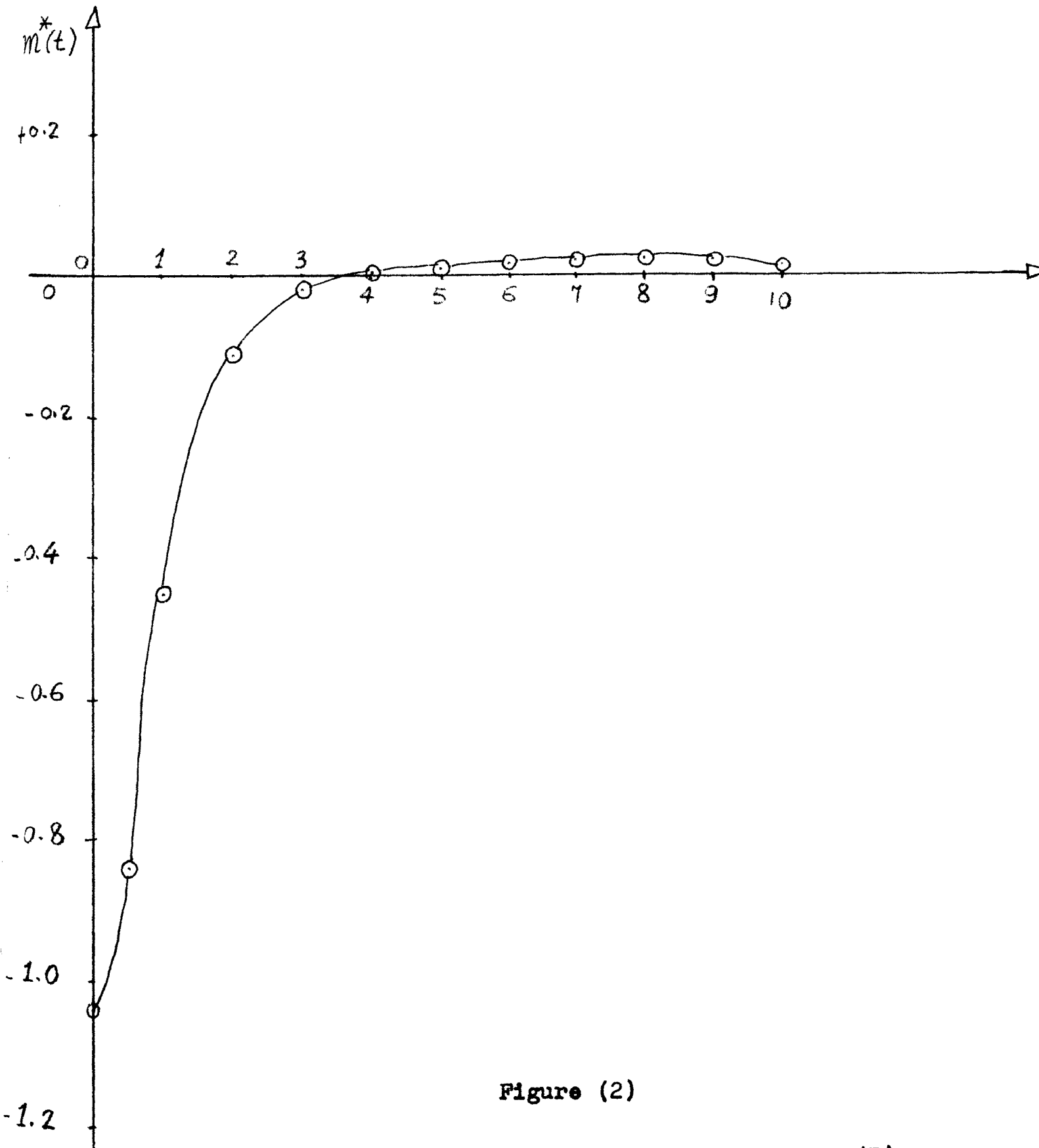
Figure (2)

THE OPTIMAL CONTROL FORCE FOR EXAMPLE (I)

k = 1

Example II

Consider the same system of Example I, but with a different performance index in which the sensitivity term is weighted by powers of the time (t), it is required to find the optimal control function that will take the system from its initial states to its final state in 10 seconds, and that will minimize the performance index

$$J_2 = \int_0^{T_f} [m^2(t) + t^2(v^2(t)] \, dt \qquad (106)$$

The same development for $\underline{V}(t)$ will be used and the system state vector equation, however the system's Hamiltonian will be changed to,

$$H = m^2(t) + t^4 + x_2 P_1 + P_2 Km(t) \qquad (107)$$

Applying the necessary conditions of optimality, then

$$m^*(t) = \frac{-P_2 k}{2} \qquad (108)$$

Now, using Lagrange's equations to find the costate variables $P_1(t)$ and $P_2(t)$, then

$$\frac{\partial H}{\partial x_1} = -\dot{P}_1 = 0 \qquad (109)$$

and,

$$\frac{\partial H}{\partial x_2} = -\dot{P}_2 = P_1 \qquad (110)$$

Solving equations (109) and (110) simultaneously it follows:

$$P_1 = b_1 \qquad (111)$$

and,

$$P_2 = -b_1 t + b_2 \qquad (112)$$

Substituting the value of P_2 from equation (112) into equation (108) therefore,

$$m^*(t) = \frac{K}{2} (b_1 t - a_2) \tag{113}$$

Using this value of $m^*(t)$ in the original state equations (75) and (76) and integrating,

$$x_2^*(t) = \frac{k^2}{4} b_1 t^2 - \frac{k^2 b_2}{2} t + 1 \tag{114}$$

and,

$$x_1^*(t) = \frac{k^2 b_1}{12 t^3} - \frac{k^2}{4} + t + 1 \tag{115}$$

Now, applying the final conditions on $x_1(t)$ and $x_2(t)$, then write the following two equations in two unknowns, b_1 and b_2 as follows

$$25 k^2 b_1 - 5 k^2 b_2 = -1 \tag{116}$$

and,

$$\frac{1000}{12 k^2 b_1} - 25 k^2 b_2 = -11 \tag{117}$$

Solving equations (116) and (117) for b_1 and b_2, thus,

$$b_1 = \frac{0.144}{k^2} \tag{118}$$

and,

$$b_2 = \frac{0.92}{k^2} \tag{119}$$

Substituting these values of b_1 and b_2 into equation (113) then

$$m^*(t) = (0.072t - 0.46)/k \tag{120}$$

From equation (120), it is clear that the optimum control function is proportional to time, and inversely proportional to the gain of the system. A picture of the control function may be seen in Figure 3.

Example III

Consider the same system of Examples I and II. Then by using a performance index without the sensitivity term, find the optimal control that will minimize this performance index, and take the system from its initial to its final state in 10 seconds with minimum energy. The performance index will be of the following form:

$$J = \int_0^{T_f} m^2(t) \, dt \qquad (121)$$

Forming the Hamiltonian of the system as

$$H = m^2(t) + x_2 P_1 + km(t)P_2 \qquad (122)$$

applying the necessary conditions for the existence of an optimal control we deduce that the optimal control function $m^*(t)$ is

$$m^*(t) = -\frac{k}{2} P_2(t) \qquad (123)$$

Now, using Lagrange's equations,

$$\frac{\partial H}{\partial x_1} = -\dot{P}_1 = 0 \qquad (124)$$

and,

$$\frac{\partial H}{\partial x_2} = -\dot{P}_2 = P_1 \qquad (125)$$

Equations (124) and (125) are identical to equations (109) and (110) in Example II, then, they must have the same solution. Therefore,

$$P_1(t) = b_1 \qquad (126)$$

and

$$P_2(t) = -b_1 t + b_2 \qquad (127)$$

Since equations (123) and (108) are identical, then the optimal control functions for both Examples II and III are the same, therefore,

$$m^*(t) = \frac{1}{k}(0.072t - 0.46) \tag{128}$$

From equation (128) and (120), it is clear that the value of the optimal control function is not affected with or without the presence of a sensitivity term in the performance index, whenever the term was weighted with a weighting factor of the type (t^n) Since the optimal control law was obtained by the necessary conditions of optimality, in which the partial derivative of the Hamiltonian H with respect to the control function was taken, then if the sensitivity term was not weighted with the control function, then it is clear that the effect of the sensitivity term in the optimization process will be lost.
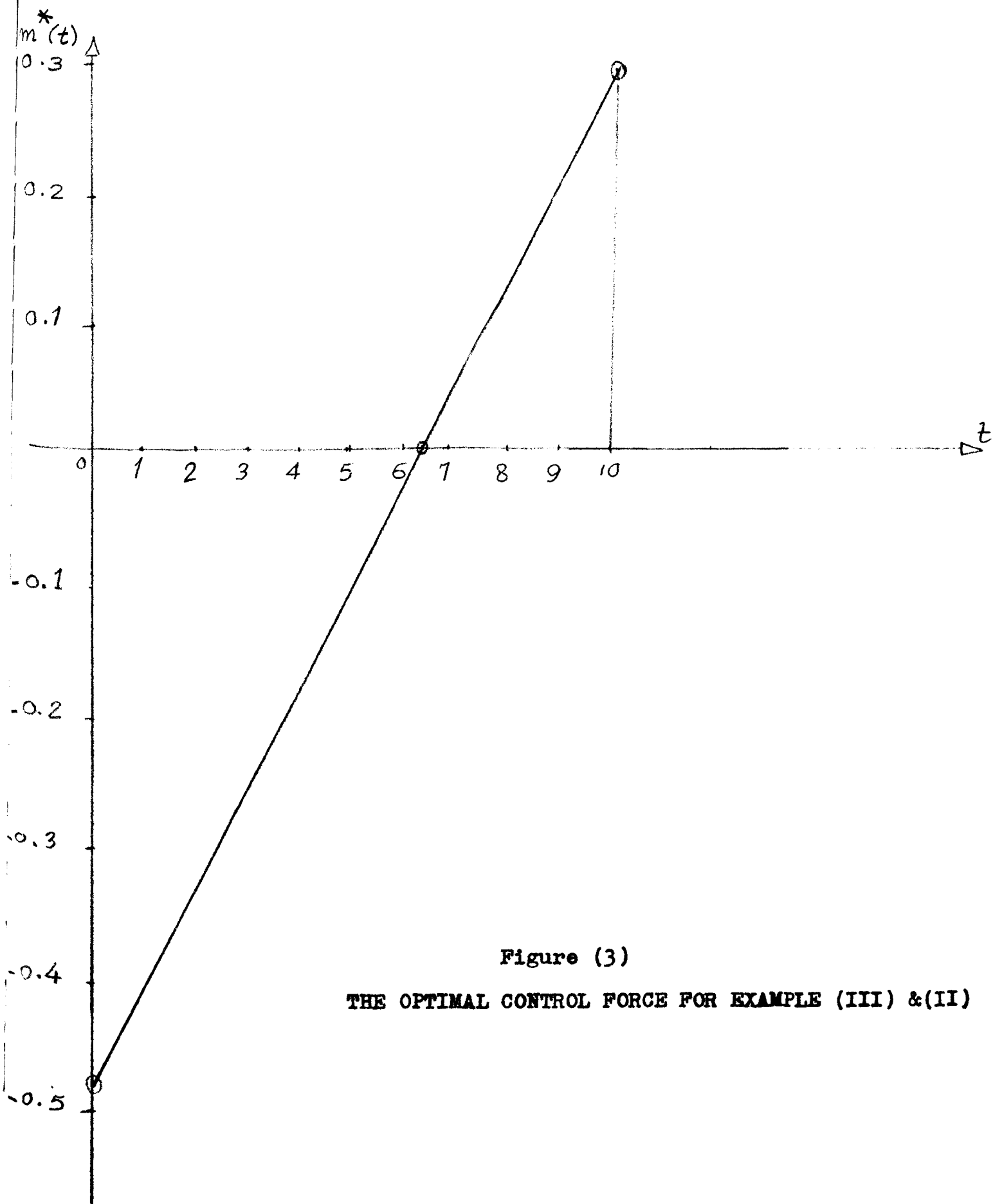
Figure (3)

THE OPTIMAL CONTROL FORCE FOR EXAMPLE (III) &(II)

Chapter IV

CONCLUSION

For a multivariable linear optimal control system, the sensitivity coefficients play an important role in the system's performance and it is always favorable to minimize these coefficients. Another significant requirement in such a system is the minimization of some performance criteria, such as the input energy, the fuel consumption, the time taken to reach the final target, etc.

It was noticed in the literature that the optimization processes for a certain system for two or more of its performance criteria were performed in two or more optimization processes respectively.

In this thesis, however, a new performance index was proposed, which is capable of optimizing the system with respect to two different functions of the parameters (the sensitivity coefficients and the energy) in just one process of optimization.

A few simple illustrative examples were presented to show the optimization process, the generation of the time domain sensitivity coefficients, and the significance of the choice of weighting factors. It was noted that this proposed performance index has value only when the weighting factors of the sensitivity coefficients are not in the form of (t^n) where, $n = 1, 2 \cdots$

It is believed that the significant point of this thesis is the idea of optimizing the control system by using a performance index containing more than one performance criterion.

The practical significance of such an index is the ability it gives to reduce the effort and time that may be required to optimize

a specific industrial plant with respect to its parameter variations
and any other operation criteria.

Further research is suggested to apply this index to more
complicated systems, and to extend the idea of optimizing the system
simultaneously with respect to any number of system criteria that
may be desirable to optimize simultaneously. The choice of the
weighting factors and their effect deserves a good amount of research.
The use of a hybrid computation scheme may help in trying to solve
such complicated problems.

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VITA

Mohamed Zafer Dajani was born on December 12, 1942 in Jaffa, Palestine. He received his high school diploma with an honour average, and with the first prize in Mathematics. His undergraduate education was in Cairo University, where he was granted a full scholarship through all his undergraduate years. He received a Bachelor of Science in Electrical Engineering in July, 1964.

He was then employed by the Kuwaiti Government for two years as an Assistant District Engineer. His job was mainly to plan, analyze, execute and maintain the L.V. and H.V. systems in his district.

In September of 1966, he pursued his graduate program in the field of Electrical Engineering at the University of Missouri at Rolla. He is a member of Kappa Mu Epsilon and IEEE.