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A study of probability of error calculations for baseband multi-level digital transmission systems using the fixed threshold decision rule

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A STUDY OF PROBABILITY OF ERROR CALCULATIONS FOR BASEBAND

MULTI-LEVEL DIGITAL TRANSMISSION SYSTEMS USING THE

FIXED THRESHOLD DECISION RULE

BY

DON J. POPP -1936-

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A

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Approved by

_____ (advisor) _____

_____ _____

ABSTRACT

This study is an investigation into analytical techniques used in deriving probability of error expressions for baseband multi-level digital transmission systems. A fixed-level decision rule is used to solve this statistical detection problem together with some simple concepts from probability theory. To illustrate these techniques, probability of error expressions are derived for the general uncoded system, a simple coded system, and a complex coded system. Curves of probability of error versus signal-to-noise ratio are plotted for the various systems considered.

ACKNOWLEDGEMENT

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I. INTRODUCTION

Most present day digital data transmission systems use a binary method of operation. However, these systems are somewhat restricted in their capacity, ie., the number of information bits they can transmit in a specified period of time. Recently, in an effort to achieve higher speed digital transmission, multi-level techniques have been developed[1] and used[2]. Multi-level techniques make use of M discrete signal levels and therefore represent Log_2 M binary channels. That is to say; for a fixed peak power, an M-level signal has Log_2 M times greater capacity relative to a binary signal. This increase in capacity, however, is accomplished only at the expense of a higher signal-to-noise ratio in the system.

A much used figure of merit for any baseband digital transmission system is the expression of error probability as a function of signal and noise. Slepian[3] lists error probability as one of six important parameters used in comparing the performance of transmission systems. Since most literature on this subject refers to binary systems, there is a need to treat probability of error calculations for current M-level systems of interest.

II. TERMINOLOGY AND ASSUMPTIONS

Before discussing probability of error calculations
for multi-level systems, it is necessary to become familiar
with pertinent terminology and assumptions which seem to
be common to digital data techniques.

A typical baseband digital data transmission system
is described in the block diagram of Figure 1. Referrring
to Figure 1, a rectangular binary input is assumed as the
initial signal to be transmitted. This binary signal is
converted in the encoder to either an uncoded or coded
M-level signal. This M-level output signal from the
encoder can be described as being either polybinary or
polybipolar in character. A polybinary signal can gener-
ally be defined as a multi-level signal (where $M > 2$)
with upper peak value of A and lower peak value of 0.
There will always be a d.c. component present in a polybi-
nary signal except when the 0 level is present. A polybi-
polar signal, on the other hand, can be defined as a multi-
level signal (where $M > 2$) with upper peak value of
+A/2 and lower peak value of -A/2. These definitions do
not seem to be standard in the literature but are used by
some authors[4] to distinguish between binary and M-ary sig-
nals. For the sake of uniformity, only polybinary signals
will be considered in developing the probability of error
expressions, although Bennett and Davy[5], have indicated

INPUT

ENCODER

TRANSMITTING FILTER

DATA TRANSMISSION LINK

RECEIVING FILTER

DECISION THRESHOLD

DECODER

OUTPUT

NOISE
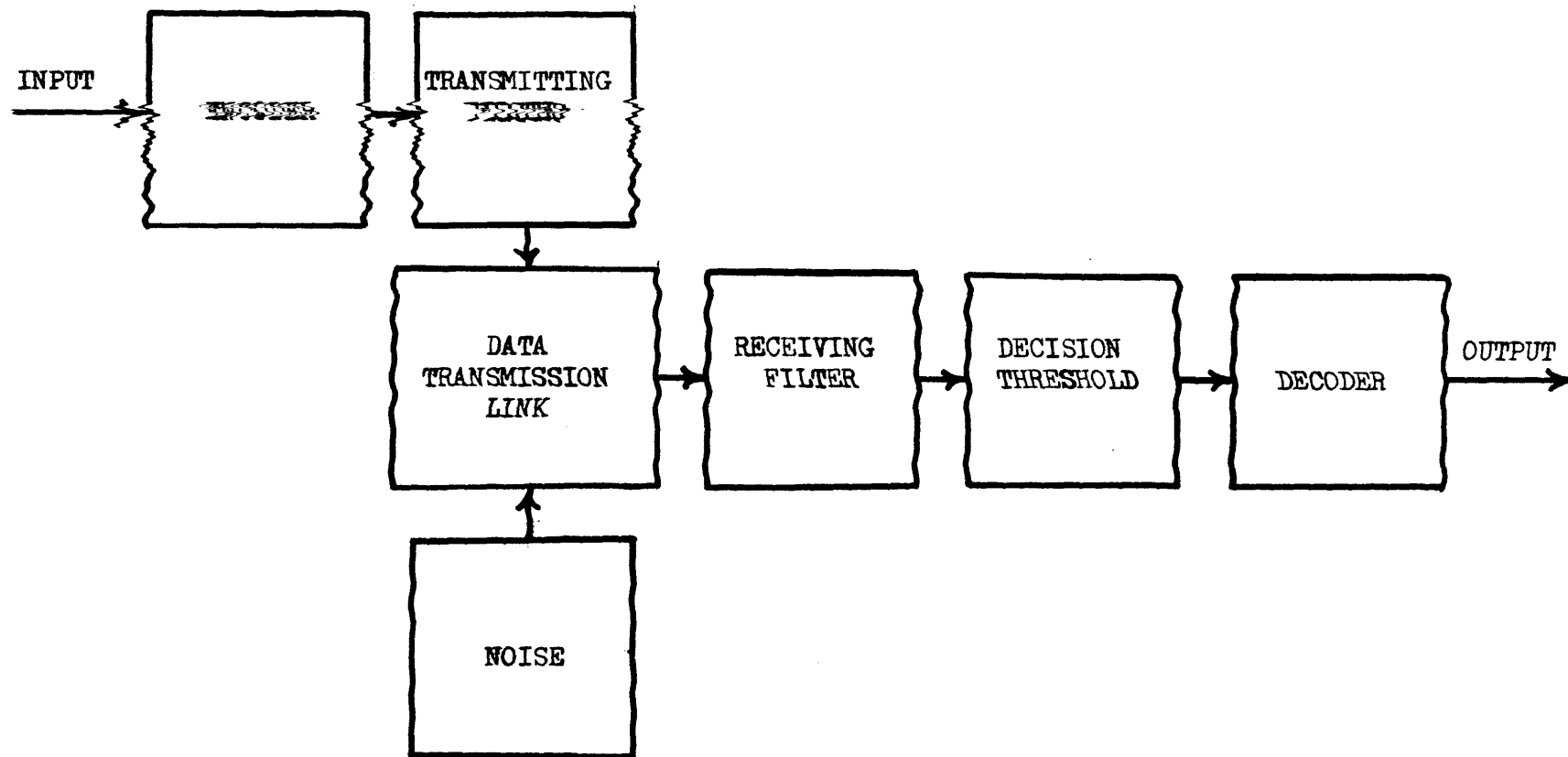
FIGURE 1   BASEBAND DIGITAL DATA TRANSMISSION SYSTEM

the resulting expressions for the two types of signals
are identical. Figure 2 and Figure 3 are typical of a
3-level polybipolar and polybinary rectangular digital
signal.

The output of the encoder in Figure 1, then, is
represented by these kinds of rectangular M-level pulse
type signals. In most practical data transmission sys-
tems, bandwidth is expensive and it is not economical to
attempt to preserve a rectangular wave shape. For this
reason, a transmitting filter is used to limit the signal
spectrum applied to the transmission link as indicated in
Figure 1. A receiving filter, also indicated in Figure 1,
serves to exclude noise and other interference picked up
by the transmission link.

It shall be assumed that a simple threshold type
detector is used in the receiver for the various systems
under consideration. For uniformity and simplicity, the
decision threshold level shall be chosen to be one half
the pulse height between signal levels. This seems to be
a frequent choice in the literature for idealized calcu-
lation purposes and it does make the derivations simpler.

When discussing transmission systems, it is important
to distinguish between modulation techniques and coding
techniques. The baseband digital signals referred to
here are coding techniques. An uncoded M-level signal is
a multi-level signal in which the probability of occurrence
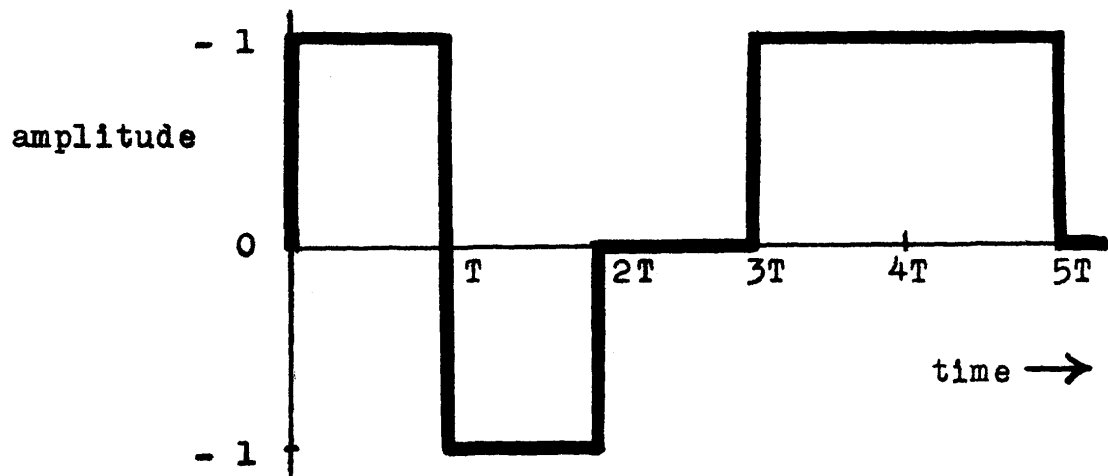
FIGURE 2   POLYBIPOLAR 3-LEVEL RECTANGULAR SIGNAL

FIGURE 3   POLYBINARY 3-LEVEL RECTANGULAR SIGNAL
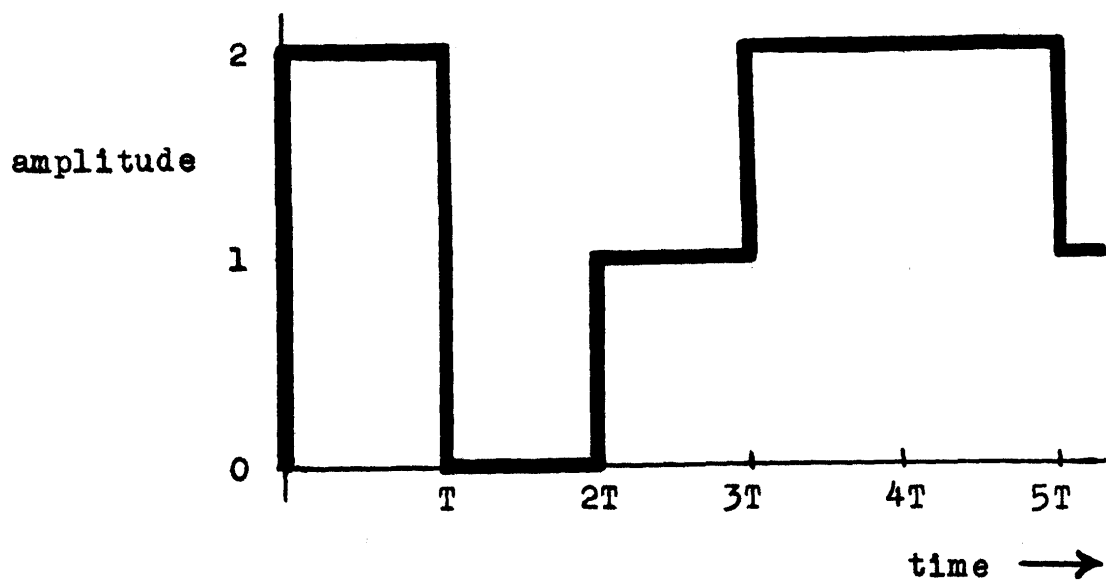
of any particular level is as likely as any other. By
the expression "coded" M-level signal is meant a multi-
level signal in which the probability of occurrence of
any particular level is not as likely as any of the
others, ie., the probability distribution is not uniform.

The baseband digital signals to be discussed can all
be used for direct wired transmission systems capable of
passing d.c. For radio transmission systems however, only
those baseband digital signals which contain no inherent
d.c. component are compatible (in the frequency domain)
with the various acceptable techniques for modulation.
The uncoded polybipolar signal is a simple example. Some
forms of the coded signals to be discussed can also be
used for radio transmission. In the block diagram of
Figure 1, therefore, all the means of transmission such
as modulators, demodulators, and transmission line or
medium, are included in the box labeled data transmission
link. The major concern in the forthcoming developments
shall be with the encoding generator and that part of the
receiver which contains the decision mechanism.

All transmission systems, irrespective of the modu-
lation technique used, are corrupted by the presence of
what is generally called "noise". By the term noise[6],
is meant any spurious or undesired disturbances that
tend to mask the transmitted signal. There are many
common sources of these disturbances and they can occur

almost anywhere in the transmission system. Since a simple threshold type decision mechanism has been assumed in the receiver, our major concern shall be for that additive noise which might effect the amplitude of the desired M-level signal prior to that signal being processed through the decision mechanism. Therefore, it shall be assumed that the transmitted signal is corrupted with additive noise. This is shown in Figure 1 as occurring in the transmission link, although in practice, this may not be the case.

Noise can generally be classified into three categories: man-made interference, impulse noise, and random gaussian noise. For ideal analysis purposes, random gaussian noise with zero average value and mean square value of σ^2, is generally assumed as the corrupting additive noise in the transmission system. This assumption will also be made here. Since the statistics of a gaussian probability density function are well known, a valuable tool exists for the evaluation of probability of error expressions for digital systems.

A further restriction is that baseband systems only are to be considered. If some technique of modulation is used to transmit the signal at radio frequencies, it is assumed that a linear detector in the receiver will restore the baseband signal to its original form. It has been shown[7], that a linear operation on a gaussian random

process yields a gaussian random process, therefore, we are able to make direct application of the gaussian probability density function in all derivations.

The following is a listing of the previous assumptions:

1. Assume the input signal is composed of a rectangular binary pulse train.

2. For derivations, assume the multi-level signal is polybinary in character.

3. Assume the received signal is corrupted with additive gaussian noise only.

4. Assume the decision mechanism in the receiver operates on a threshold basis only.

5. Consider baseband signals only.

Note that in making the above assumptions, general digital transmission systems are no longer being considered. The digital systems to be discussed are now somewhat restricted.

III. REVIEW OF LITERATURE

The general approach used in calculating the probability of error for multi-level digital systems does not seem to be discussed very thoroughly in the literature.

In his textbook on Information Theory, Abramson[8] presents an expression for finding the probability of error for binary signals. However, this author quickly moves on to other topics without giving the reader any insight as to how the expression might be used.

Schwartz[6], considers the binary signal in more detail. He formulates the problem assuming an uncoded binary transmitted signal corrupted with additive gaussian noise and then leaves it as an exercise for the reader. He does present the resulting expression, however:

Probability of Error = P(E)

$$P(E) = (1/2) \left[1 - \text{erf} \frac{A}{2\sigma\sqrt{2}} \right] \tag{1}$$

where: A is the peak amplitude of the signal.

σ^2 is the mean square noise power.

Authors Bennett and Davy in a recent book[5], also mention the binary uncoded signal and likewise write down the resulting expression which is identical to equation 1 above. In addition, Bennett and Davy consider a multi-level uncoded transmitted signal and again write

down the resulting expression (without derivation),
which is:

$$P(E) = \frac{M-1}{M} \left[1 - \text{erf} \left(\frac{A}{2(M-1)\sigma\sqrt{2}} \right) \right] \qquad (2)$$

They also include a plot of the probability of error
versus the ratio of average signal power to average noise
power for uncoded multi-level signals where $2 \leq M \leq 16$.

Recent periodical literature makes frequent use of
probability of error curves and expressions as figures
of merit for various multi-level systems. For example,
Shagena and Kvarda[9] submit a relatively simple coding
technique for an M-level signal and present the corres-
ponding probability of error expression and a set of curves
for the various M levels.

Lender[1,4,10,11], in a series of articles suggests
a rather complex coding technique for an M-level signal.
He presents a limited amount of discussion together with
his probability of error expression and a set of curves
showing probability of error versus normalized signal to
noise ratio.

In summing up this perusal of the literature, it is
difficult to determine the probability of error for a
given system on the basis of what any one author is
saying.

IV. OBJECTIVE

In the following discussion, a general analytical approach will be presented which can be used to determine the probability of error expression for any coded or uncoded multi-level digital signal of interest. Furthermore, this general approach will be illustrated by applying it to the four signals mentioned previously. These are:

1. The binary uncoded signal.
2. The M-level uncoded signal.
3. The M-level coded signal of Shagena and Kvarda.
4. The M-level coded signal of Lender.

In addition, probability of error curves for the above systems will be presented and discussed. A typical computer program written in the Fortran IV language will be presented.

V. FORMULATION OF THE GENERAL EXPRESSION

The general approach for determining the probability of error expression for an M-level signal is taken from introductory probability theory. (See Appendix I.) The desired expression is the probability of the event E which occurred when an experiment was performed. Let this probability be called P(E) in keeping with information theory[8] and probability theory[12] notation. Let the sample space of the experiment be divided into M mutually exclusive regions S_1, S_2, --- S_M. These regions represent the M possible causes of an experimental outcome which are of interest.

Next, let E be the event that occurred when the experiment was performed and consider the problem of calculating the probability that S_k was the cause of the occurrence of E (where $1 \leq k \leq M$). In other words, the sample point was one of the points inside S_k associated with the occurrence of E. From Appendix I, equation I-5, this conditional probability is given by:

$$P(S_k|E) = P(S_k,E)/P(E) \qquad (3)$$

From equation I-4:

$$P(S_k,E) = P(S_k|E) \ P(E) = P(E|S_k) \ P(S_k) \qquad (4)$$

Now, the event E can occur only in conjunction with one of the M possible events S_1, S_2, ---- S_M. Thus, E will occur if, and only if, one of the mutually exclusive events (S_1,E), (S_2,E), ---- (S_M,E) occurs. The addition rule for mutually exclusive events, as stated in Appendix I, gives as the probability of the event E:

$$P(E) = P(S_1,E) + P(S_2,E) + ------ + P(S_M,E) \tag{5}$$

Applying the last expression of equation 4 to each term on the right of equation 5 will result in:

$$P(E) = P(S_1)P(E|S_1) + P(S_2)P(E|S_2) + --------$$
$$---- + P(S_M)P(E|S_M) \tag{6}$$

$$P(E) = \sum_{k=1}^{k=M} P(S_k)\, P(E|S_k) \tag{7}$$

Re-define the variables in equation 7. Let E represent the event "ERROR" and let S_1, S_2, ---- S_M represent the S_k levels of the transmitted signal s(t). By changing the meaning of the variables and not the variables themselves, the general expression for calculating the probability of error for any M-level transmitted signal is given by equation 7.

VI. STATEMENT OF THE PROBLEM

A concise statement of the problem can now be presented. Consider an encoding generator as in Figure 4, which converts a continuous binary input message m(t) into a polybinary multi-level coded or uncoded message s(t). The message s(t) is applied to a "noisy" linear channel for transmission as in Figure 4. The output of the noisy channel is a data signal f(t) where, for all time:

$$f(t) = s'(t) + n(t) \tag{8}$$

Now:

s'(t) is a polybinary M-level message of peak voltage A which has been operated on in a linear manner in the channel.

n(t) is additive noise described by a gaussian random process with zero average value and mean square value σ^2.

The continuous data signal f(t) is applied to the decision mechanism in the receiver for appraisal. This decision mechanism will first synchronously gate or sample f(t) at periodic intervals of time and, on the basis of the amplitude of f(t) at these sample points, will decide which level of s(t) has been transmitted through the channel. The decision mechanism operates on a simple voltage threshold principle. This decision threshold

DECISION MECHANISM

```
m(t) →  ┌─────────────┐        ┌─────────────┐        ┌──────┬──────────┐
        │ MULTILEVEL  │  s(t)  │   CHANNEL   │  f(t)  │      ┊ THRESHOLD │  o(t) →
        │  ENCODER    │ ─────→ │  (LINEAR)   │ ─────→ │ GATE ┊ DETECTOR  │ ─────→
        │             │        │             │        │      ┊          │
        └─────────────┘        └─────────────┘        └──────┴──────────┘
                                      ↑
                                     n(t)
```

m(t) = binary pulse train   f(t) = s'(t) + n(t)
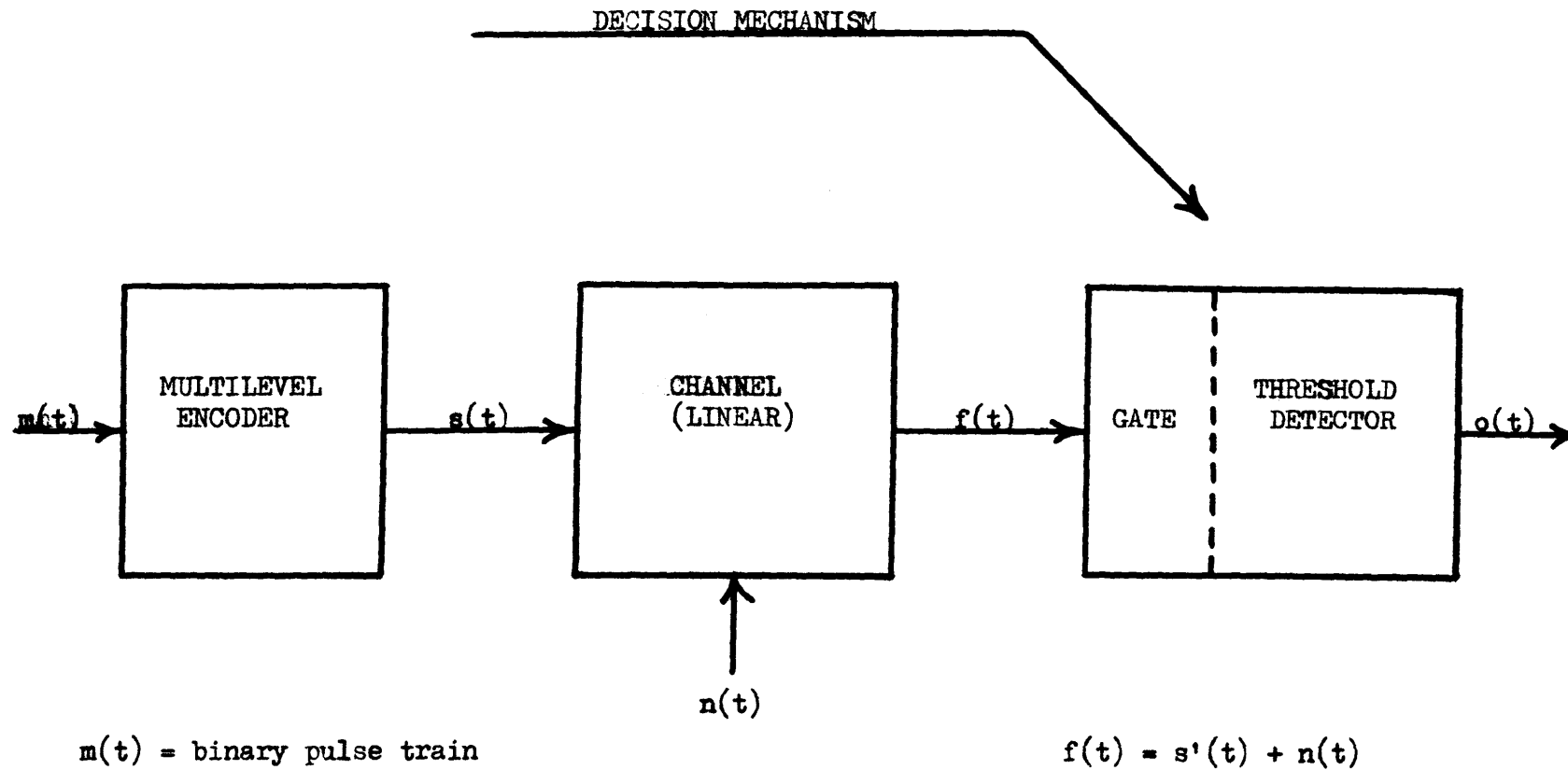
FIGURE 4   BLOCK DIAGRAM OF M-LEVEL SYSTEM

shall be chosen to lie halfway between the peak values of the M-levels present in s'(t) somewhat ideally. With no thought here of optimization, this decision criterion is a convenient choice as it leads to simpler derivations and expressions. Note that an M-level received signal s'(t) will require a decision mechanism with (M-1) decision thresholds.

Figure 5 represents a typical 5-level bandlimited signal. It illustrates the decision threshold voltages necessary for the detector. Let the decision threshold for level 1 be called D. Then, the decision threshold for level 2 will be 3D (as a consequence of selecting the decision voltage halfway between levels). etc. etc.

Now, the decision mechanism in the receiver must examine f(t) and decide which level of s(t) was transmitted. Let this decision process obey the following general rules:

If $f(t) \leq D$; receiver decides $s'(t) = $ level 1 $= S_1$.

If $D < f(t) \leq 3D$; receiver decides $s'(t) = $ level 2 $= S_2$.

If $3D < f(t) \leq 5D$; receiver decides $s'(t) = $ level 3 $= S_3$.

If $(S_k-D) < f(t) \leq (S_k+D)$; receiver decides level k $= S_k$.

If $(2M-3)D < f(t)$; receiver decides $s'(t) = $ level M $= S_M$.

where: $1 \leq k \leq M$ (9)

SIGNAL LEVEL

5

4

S_K

2

1

s'(t) AMPLITUDE

A

$\frac{3A}{4}$

$\frac{(K-1)A}{M-1}$

$\frac{A}{4}$

0

DECISION LEVEL

(2M-3)D

(S_K+D)

(S_K-D)

D
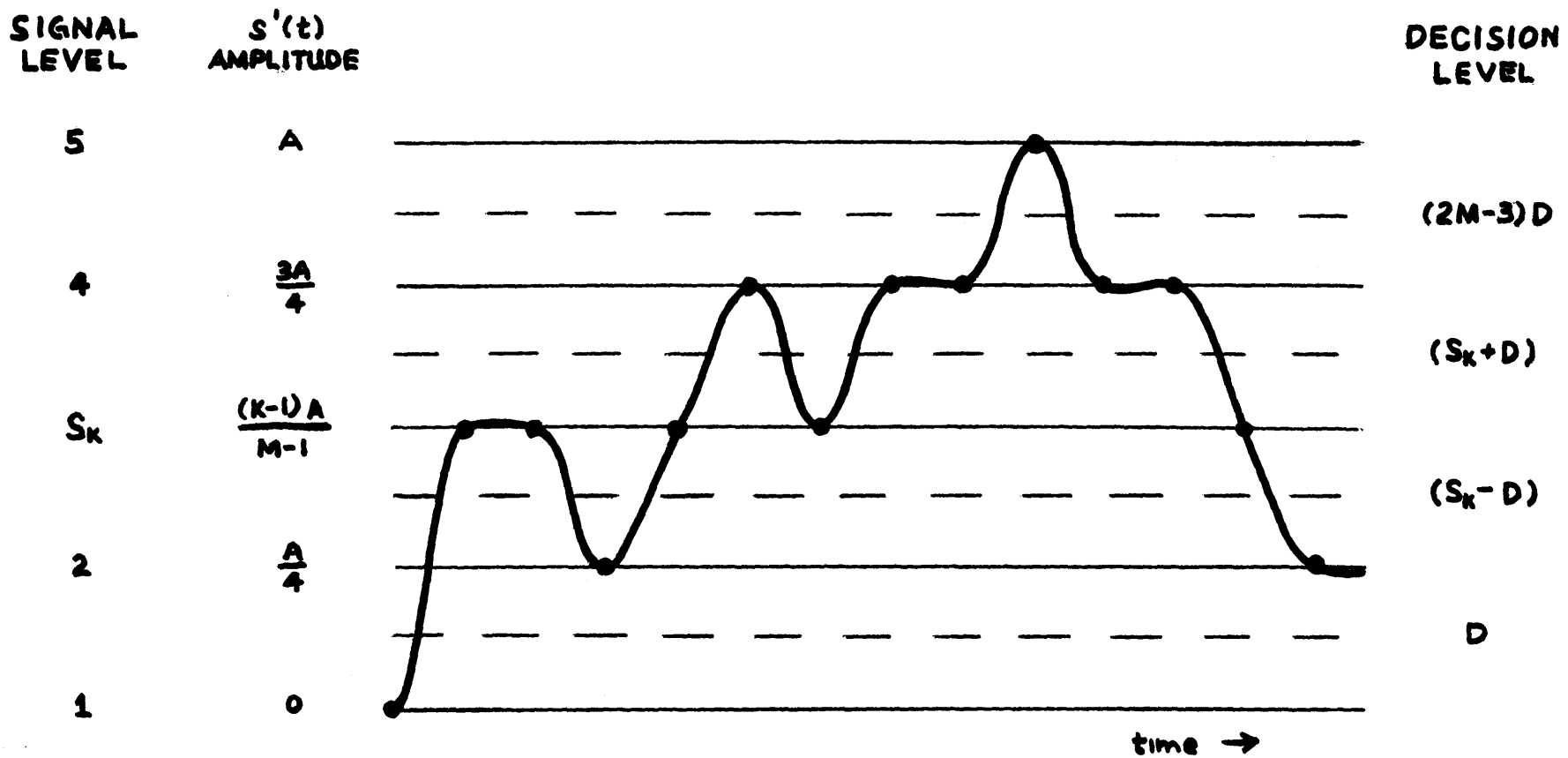
time →

Note: The decision level D has amplitude $\frac{A}{2(M-1)}$

For this signal, $1 \leq k \leq M$

FIGURE 5   TYPICAL VOLTAGE WAVEFORM OF BANDLIMITED 5-LEVEL SIGNAL

At any given instant of time, it is necessary that f(t) lie within a specific amplitude range for the receiver to make the correct decision. If, due to additive noise, f(t) does not lie within this specified amplitude range, then the receiver will make an error. For example, if s(t) is at level 5 of an M-level signal, then $7D < f(t) \leq 9D$ for the receiver to make the correct decision. Should $f(t) \leq 7D$ or $f(t) > 9D$, then the receiver will make an error.

To determine the probability that the receiver will make just such an error for a given system is the problem to be considered here. Its solution requires careful application of equation 7, using the known statistics of gaussian random noise together with the probabilities of the system under investigation. Re-stating equation 7:

$$P(\text{Error}) = P(E) = \sum_{k=1}^{k=M} P(S_k)\, P(E|S_k) \tag{10}$$

In short, this equation says that the total probability of an error, given that an M-level signal is transmitted, is the sum of the following:

$$P(E) = P\left[s(t) = S_1\right] P\left[\text{Error} \mid s(t) = S_1\right] +$$

$$+ P\left[s(t) = S_2\right] P\left[\text{Error} \mid s(t) = S_2\right] + \text{------}$$

$$\text{------} + P\left[s(t) = S_k\right] P\left[\text{Error} \mid s(t) = S_k\right] + \text{------}$$

$$- - - + P\left[s(t) = S_M\right] P\left[Error \mid s(t) = S_M\right] - - \qquad (11)$$

Note that the first probability expression after the summation sign in equation 10 is the average probability of occurrence of any of the k levels in a very long message train s(t) and is a function only of the particular signal being transmitted. For uncoded systems, this probability can be simply expressed. For coded systems however, a thorough understanding of the coding technique of the particular system is necessary before this quantity can be put into a concise form. Note also, the second probability expression in equation 10 is a conditional probability. This probability should read as follows:

$P(E \mid S_k)$ = The probability the receiver makes an erroneous decision given that a certain k level of s(t) was transmitted.

In general, the solution for these conditional probabilities for any given M-level system is not trivial. To proceed in a straightforward manner requires the use of some introductory concepts of continuous random signals. Indeed, if these concepts of random signals did not exist, it would be difficult to make any meaningful probability of error calculations[6].

Appendix II contains a limited review of the pertinent continuous random signal theory needed. The previous assumption that the error producing noise is a gaussian random process provides the statistical distribution of the noise which is given by equation II-3 in Appendix II as a probability density function and is re-stated here as:

$$p\left[n(t)\right] = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-n(t)^2/2\sigma^2} \tag{12}$$

Re-arranging equation 8.

$$n(t) = f(t) - s'(t) \tag{13}$$

For convenience, change the notation in equation 13 to read:

$$N = F - S \tag{14}$$

Now, substituting this into equation 12:

$$p(N) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-(F-S)^2/2\sigma^2} = p(F|S) \tag{15}$$

For an M-level signal S_k; where $1 \leq k \leq M$.

$$p(F|S_k) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-(F-S_k)^2/2\sigma^2} \tag{16}$$

As these are density functions, it is necessary to use the idea of a probability distribution function as defined in Appendix II to solve for actual probabilities. From the defining equation II-5:

$$P\left[e(t) \leq x_1\right] = \int_{-\infty}^{x_1} p(x)\, dx \qquad (17)$$

Using this expression and equation II-4 from Appendix II, note the following relationships are also true.

$$P\left[x_1 \leq e(t) \leq x_2\right] = \int_{x_1}^{x_2} p(x)\, dx \qquad (18)$$

$$P\left[x_2 \leq e(t)\right] = \int_{x_2}^{\infty} p(x)\, dx \qquad (19)$$

<u>Let:</u>

$$e(t) = F \quad \text{and} \quad p(x)\, dx = e^{-(F-S_k)^2/2\sigma^2}\, dF$$

$$x_1 = f_1$$

And equation 17 becomes:

$$P(F \leq f_1) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{f_1} e^{-(F-S_k)^2/2\sigma^2}\, dF \qquad (20)$$

<u>where:</u> $1 \leq k \leq M$

Likewise, substituting equation 16 into 18 and 19:

$$P(f_1 \leq F \leq f_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{f_1}^{f_2} e^{-(F-S_k)^2/2\sigma^2} \, dF \quad (21)$$

$$P(f_2 \leq F) = \frac{1}{\sigma\sqrt{2\pi}} \int_{f_2}^{\infty} e^{-(F-S_k)^2/2\sigma^2} \, dF \quad (22)$$

Thus, by appropriate use of equations 20 and 22, the probability of finding the noisy signal F within any specified amplitude range can be solved for, given that some S_k level of s(t) was transmitted. Using these functions, the conditional error probabilities of equation 10 can be determined.

It might be mentioned here that some authors prefer to talk of the probability of detection, whereas others speak only of the probability of error. There should be no confusion about this. When a particular level of a multi-level signal is transmitted, the receiver will either make a correct decision (that is, interpret the signal correctly.) or make an incorrect decision (make an error).

Let the probability the receiver makes a correct decision = P(RCD).

Let the probability the receiver makes an incorrect decision = P(RID).

Then, $P(RCD) + P(RID) = 1$, because some level of signal was sent. Or:

$$P(RCD) = 1 - P(RID) \tag{23}$$

$$P(RID) = 1 - P(RCD) \tag{24}$$

As an initial illustration of the general approach used for calculations, the previously discussed topics will be utilized and the probability of error for a binary uncoded transmitted signal will be determined.

VII. PROBABILITY OF ERROR OF A BINARY UNCODED SIGNAL

A. Forming the General Expression

For a binary uncoded signal s(t), there are two possible levels of signal which are transmitted. Call these two levels:

$$s(t) = S_1$$
$$s(t) = S_2$$

From equation 10, the general expression for calculating the probability of error P(E) is:

$$P(E) = P(S_1)P(E|S_1) + P(S_2)P(E|S_2) \tag{25}$$

where: The probability that level 1 was sent $= P(S_1)$.

The probability that level 2 was sent $= P(S_2)$.

The conditional probability the receiver makes an error given that S_1 was sent $= P(E|S_1)$.

The conditional probability the receiver makes an error given that S_2 was sent $= P(E|S_2)$.

Assume the detector threshold level between S_1 and S_2 to be D as shown in Figure 5. Assume also, that s(t) is a random pulse train. Then:

$$P(S_1) = 1/2 = P(S_2) \tag{26}$$

And:

$$P(E) = 1/2 \left[P(E|S_1) + P(E|S_2) \right] \tag{27}$$

To solve the conditional error probabilities of equation 27, consider the following:

Let R_1 refer to the event the receiver decides that $S = S_1$.

Let R_2 refer to the event the receiver decides that $S = S_2$.

From the previously discussed decision rules on page 16, the receiver decides the following:

S = level 1 when $F \leq D$.

S = level 2 when $F > D$.

B. Solving for $P(E|S_1)$

$$P(E|S_1) = P(R_2|S_1) = P\left[(F > D)|S_1\right]$$

Using equation 22:

$$P(E|S_1) = \frac{1}{\sigma\sqrt{2\pi}} \int_{D}^{\infty} e^{-(F-S_1)^2/2\sigma^2} \, dF$$

Change the variable of integration.

Let: $Q = \dfrac{F-S_1}{\sigma\sqrt{2}}$; Then: $dQ = \dfrac{dF}{\sigma\sqrt{2}}$

As: $F \longrightarrow \infty$; $Q \longrightarrow \infty$

As: $F \longrightarrow D$; $Q \longrightarrow \dfrac{D-S_1}{\sigma\sqrt{2}} = L$

Therefore:

$$P(E|S_1) = \frac{1}{\sqrt{\pi}} \int_{L}^{\infty} e^{-Q^2} dQ \qquad (28)$$

The integral of equation 28 is frequently called the complementary error function and its solution is tabulated in many handbooks[16] as the following:

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt = 1 - \text{erf } x \qquad (29)$$

Therefore:

$$P(E|S_1) = (1/2) \text{ erfc } \left(\frac{D-S_1}{\sigma\sqrt{2}}\right) \qquad (30)$$

C. Solving for $P(E|S_2)$

$$P(E|S_2) = P(R_1|S_2) = P\left[(F \leq D) \mid S_2\right]$$

Using equation 20:

$$P(E|S_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{D} e^{-(F-S_2)^2/2\sigma^2} dF$$

Change the variable of integration.

Let: $\quad Q = \dfrac{F-S_2}{\sigma\sqrt{2}} \quad ; \quad$ Then: $\quad dQ = \dfrac{dF}{\sigma\sqrt{2}}$

As: $\quad F \longrightarrow D \quad ; \quad Q \longrightarrow \dfrac{D-S_2}{\sigma\sqrt{2}} = T$

As: $\quad F \longrightarrow -\infty \quad ; \quad Q \longrightarrow -\infty$

Therefore:

$$P(E|S_2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{T} e^{-Q^2} \, dQ$$

However, this is not in an acceptable form. Change the variable of integration again.

Let: Q = -Z ;   Then: dQ = -dZ

As: Q $\longrightarrow$ -∞ ;   Z $\longrightarrow$ ∞

As: Q $\longrightarrow$ $\frac{D-S_2}{\sigma\sqrt{2}}$ Z $\longrightarrow$ $\frac{S_2-D}{\sigma\sqrt{2}}$ = L

Therefore:

$$P(E|S_2) = -\frac{1}{\sqrt{\pi}} \int_{\infty}^{L} e^{-Z^2} \, dZ$$

Inverting the limits:

$$P(E|S_2) = \frac{1}{\sqrt{\pi}} \int_{L}^{\infty} e^{-Z^2} \, dZ$$

Again, the form of the complementary error function given in equation 29 is recognized. Therefore:

$$P(E|S_2) = (1/2) \ \text{erfc} \left(\frac{S_2-D}{\sigma\sqrt{2}} \right) \tag{31}$$

D. Solving for P(E)

Substitution of equations 30 and 31 into equation 27 forms the final expression for the probability of error of an uncoded binary signal.

$$P(E) = (1/2) \left[(1/2) \text{ erfc} \left(\frac{D-S_1}{\sigma \sqrt{2}} \right) + (1/2) \text{ erfc} \left(\frac{S_2-D}{\sigma \sqrt{2}} \right) \right] \quad (32)$$

This expression can be simplified by substituting in our previous assumptions that the transmitted polybinary signal has peak amplitude A and the receiver threshold level is midway between the two signal levels. Thus:

$$S_1 = 0$$

$$S_2 = A$$

$$D = A/2$$

$$P(E) = (1/4) \text{ erfc} \left(\frac{(A/2)-0}{\sigma \sqrt{2}} \right) + (1/4) \text{ erfc} \left(\frac{A-(A/2)}{\sigma \sqrt{2}} \right)$$

$$P(E) = (1/2) \text{ erfc} \left(\frac{A}{2\sigma \sqrt{2}} \right) \quad (33)$$

per symbol or pulse in the binary signal.

Equation 33 above, is the widely used probability of error expression for a binary uncoded signal[5,6,15]. It agrees with equation 1. This same approach can be used to extend the above results and determine the probability of error for an M-level uncoded transmitted signal.

VIII.   PROBABILITY OF ERROR OF AN M-LEVEL UNCODED SIGNAL

A.   Forming the General Expression

For an uncoded M-level signal with received peak value of A, there are $A(k-1)/(M-1)$ possible voltage levels for s'(t) (where $1 \leq k \leq M$). Figure 5 indicates the appropriate terminology for describing the amplitude variations of the received signal f(t), the desired signal s'(t), and the voltage thresholds for the decision mechanism in the receiver. From equation 10, the general expression for P(E) is:

$$P(E) = \sum_{k=1}^{M} P(S_k) \, P(E|S_k)$$

where $P(S_k)$ and $P(E|S_k)$ are defined on pages 18 and 19.

For an uncoded M-level baseband signal with no correlation between levels, each of the k-levels (where $1 \leq k \leq M$) will appear with equal liklihood; therefore:

$$P(S_k) = \frac{1}{M} \qquad \text{for all } k$$

And:

$$P(E) = \frac{1}{M} \sum_{k=1}^{M} P(E|S_k) \tag{34}$$

The conditional probabilities $P(E|S_k)$ can be divided into three possibilities:

$P(E|S_1)$ = Probability of an error when the transmitted signal S is at level 1.

$P(E|S_M)$ = Probability of an error when the transmitted signal S is at level M.

$P(E|S_{k'})$ = Probability of an error when the transmitted signal S is at the k' level where $2 \leq k' \leq (M-1)$

Therefore:

$$P(E) = \frac{1}{M} \left[P(E|S_1) + P(E|S_M) + P(E|S_{k'}) \right] \qquad (35)$$

From the decision rules, the receiver decides the following: (See Figure 5)

$S \neq$ level 1 if $(F > D)$

$S \neq$ level 2 if $(D \geq F > 3D)$

$S \neq$ level 3 if $(3D \geq F > 5D)$

- - - - - - - - - - - - - -

$S \neq$ level k' if $\left[(S_{k'}-D) \geq F > (S_{k'}+D) \right]$

- - - - - - - - - - - - - -

$S \neq$ level M if $\left[F \leq (2M-3)D \right]$

B. Solving for $P(E|S_1)$

$$P(E|S_1) = P \left[(F > D) | S_1 \right]$$

Note that this is the same situation as solving for $P(E|S_1)$
for the binary case. Hence, that result, equation 30, can
be written down:

$$P(E|S_1) = (1/2) \text{ erfc } \left(\frac{D - S_1}{\sigma \sqrt{2}} \right)$$

Substituting into this expression with the appropriate
amplitudes:

$$S_1 = 0$$

$$D = \frac{A}{2(M-1)}$$

<u>Therefore:</u>

$$P(E|S_1) = (1/2) \text{ erfc } \left(\frac{A}{2(M-1)\sigma \sqrt{2}} \right) \qquad (36)$$

C. Solving for $P(E|S_M)$

$$P(E|S_M) = P\left[\left(F \le (2M-3)D \right) | S_M \right] \qquad (37)$$

If equation 20 is used to solve for this conditional pro-
bability, recognize that this is the same form as the solu-
tion for $P(E|S_2)$ for the binary uncoded signal. From
equation 31:

$$P(E|S_2) = (1/2) \text{ erfc } \left(\frac{S_2 - D}{\sigma \sqrt{2}} \right)$$

Now, however: S_2 is replaced by S_M

  D is replaced by $(2M-3)D$

$$P(E|S_M) = (1/2) \; erfc \; \left(\frac{S_M - (2M-3)D}{\sigma \sqrt{2}} \right)$$

Substituting into this expression with the appropriate amplitudes:

$$S_M = A$$

$$(2M-3)D = \frac{(2M-3)A}{2(M-1)}$$

Then:

$$P(E|S_M) = (1/2) \; erfc \; \left(\frac{A}{2(M-1)\sigma \sqrt{2}} \right) \tag{38}$$

D. Solving for $P(E|S_{k'})$

$$P(E|S_{k'}) = P \left[(S_{k'}-D \leq F < S_{k'}+D) \; | \; S_{k'} \right] \tag{39}$$

$$\text{for all } 2 \leq k' \leq (M-1)$$

One approach that could be used to solve for $P(E|S_{k'})$ would be to derive the conditional error probability for the k' level and then sum over all $2 \leq k' \leq (M-1)$ levels for the total $P(E|S_{k'})$.

$$P(E|S_{k'}) = \sum_{k'=2}^{M-1} \left[P(1) + P(2) \right] \tag{40}$$

where:

$$P(1) = P \left[(S_{k'}-D \leq F) \; | \; S_{k'} \right]$$

$$P(2) = P \left[(F < S_{k'}+D) \; | \; S_{k'} \right]$$

The solution of these conditional probabilities is similar to previous derivations. However, this derivation will be given in detail as it is for the general k' (interior) level.

1. Solving for P(1)

$$P(1) = P\left[(S_{k'}-D \leq F)\, |\, S_{k'}\right]$$

From equation 20:

$$P(1) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{S_{k'}-D} e^{-(F-S_{k'})^2/2\sigma^2}\, dF$$

Change the variable of integration.

Let: $\quad Q = \dfrac{F-S_{k'}}{\sigma\sqrt{2}}$; $\quad$ then $\quad dQ = \dfrac{dF}{\sigma\sqrt{2}}$

As: $\quad F \longrightarrow S_{k'}-D;$ $\quad Q \longrightarrow \dfrac{-D}{\sigma\sqrt{2}} = -L$

As: $\quad F \longrightarrow -\infty;$ $\quad Q \longrightarrow -\infty$

Then:

$$P(1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-L} e^{-Q^2}\, dQ$$

To put this integral into the form of the complementary error function, change the variable of integration again.

Let: $Z = -Q$;   then $dZ = -dQ$

As: $Q \longrightarrow -L$; $Z \longrightarrow L$

As: $Q \longrightarrow -\infty$; $Z \longrightarrow \infty$

$$P(1) = -\frac{1}{\sqrt{\pi}} \int_{\infty}^{L} e^{-Z^2} \, dZ$$

$$P(1) = \frac{1}{\sqrt{\pi}} \int_{L}^{\infty} e^{-Z^2} \, dZ \qquad (42)$$

$$P(1) = (1/2) \, \text{erfc} \, L = (1/2) \, \text{erfc} \left(\frac{D}{\sigma \sqrt{2}} \right)$$

$$P(1) = (1/2) \, \text{erfc} \left(\frac{\Delta}{2(M-1)\sigma \sqrt{2}} \right) \qquad (43)$$

2. Solving for P(2)

$$P(2) = P \left[(F > S_{k'} + D) \mid S_{k'} \right]$$

From equation 22:

$$P(2) = \frac{1}{\sigma \sqrt{2\pi}} \int_{S_{k'}+D}^{\infty} e^{-(F-S_{k'})^2/2\sigma^2} \, dF$$

Change the variable of integration.

Let: $Q = \frac{F - S_{k'}}{\sigma \sqrt{2}}$;   then $dQ = \frac{dF}{\sigma \sqrt{2}}$

As: $F \longrightarrow S_{k'} + D$; $Q \longrightarrow \frac{D}{\sigma \sqrt{2}} = L$

As: $F \longrightarrow \infty$; $Q \longrightarrow \infty$

Then:

$$P(2) = \frac{1}{\sqrt{\pi}} \int_{L}^{\infty} e^{-Q^2} \, dQ$$

$$P(2) = (1/2) \text{ erfc } L = (1/2) \text{ erfc } \left(\frac{D}{\sigma\sqrt{2}}\right)$$

$$P(2) = (1/2) \text{ erfc } \left(\frac{A}{2(M-1)\sigma\sqrt{2}}\right) \tag{44}$$

Now that P(1) and P(2) have been evaluated, the total expression for $P(E|S_{k'})$ can be formed by substituting into equation 40.

$$P(E|S_{k'}) = \sum_{k'=2}^{M-1} \text{ erfc } \left(\frac{A}{2(M-1)\sigma\sqrt{2}}\right) \tag{45}$$

Note that equation 45 is independent of k', therefore, replace the summation with multiplication by (M-2).

$$P(E|S_{k'}) = (M-2) \text{ erfc } \left(\frac{A}{2(M-1)\sigma\sqrt{2}}\right) \tag{46}$$

In comparing equation 45 with equations 36 and 38, note that the probability the receiver makes an error given the transmitted signal was an interior level (a k' level) turns out to be twice the probability of an error given the transmitted signal was an end level (that is, level 1 or level M).

E. Solving for P(E)

Substitution of equations 36, 38, and 46 into equation 35, forms the desired expression for the probability of error for an uncoded multi-level signal.

$$P(E) = \frac{1}{M}\left[\frac{1}{2}\,\text{erfc}\left(\frac{\Delta}{2(M-1)\sigma\sqrt{2}}\right) + \frac{1}{2}\,\text{erfc}\left(\frac{\Delta}{2(M-1)\sigma\sqrt{2}}\right) + \right.$$

$$\left. + (M-2)\,\text{erfc}\left(\frac{\Delta}{2(M-1)\sigma\sqrt{2}}\right)\right] \qquad (47)$$

$$P(E) = \frac{(M-1)}{M}\,\text{erfc}\left(\frac{\Delta}{2(M-1)\sigma\sqrt{2}}\right) \qquad (48)$$

per symbol or pulse in the multi-level signal.

Equation 48 agrees with Bennett and Davy[5] in their equation 7-53 which was presented as equation 2 of this paper.

Note that the derivation of P(E) for uncoded signals depended only on the fixed threshold levels.

IX CODED SYSTEMS

Consider now some coded multi-level systems found in recent literature. The reasons for using coded signals seem to vary with the respective systems. For example, one might use coding techniques to increase the data rate in an existing channel with a fixed bandwidth; such as Shagena and Kvarda[9] claim to have done with their simple coding technique. On the other hand, one might use coding techniques to re-distribute the spectral energy in the frequency domain of the baseband signal, such as Lender[4] has done with a complex coding technique. Without going into the relative merits of the many reasons for coding, let us investigate the approach which can be used to calculate the P(E) for these coded multi-level signals.

X. PROBABILITY OF ERROR FOR SYSTEM CODE ONE

Note that Code One refers to the multi-level coded signal of Shagena and Kvarda[9]. Now that coded signals are being discussed, it is necessary to consider more than just the fixed decision thresholds; a thorough understanding of the coding technique is necessary to determine the required channel probabilities.

A. Coding Technique for Code One

This coding technique concerns taking a binary pulse train and encoding it into an M-level signal such that each level is advanced one step for each character (bit) of the binary data that is in the ONE state and remains in the existing level for each binary character that is in the ZERO state. These levels advance in one direction until either level M or level 1 of the M-level signal is reached, at which point succeeding binary ONE characters cause the levels to step in the opposite direction. Hence the transitions are cyclic. With an M-level signal, the shortest complete cycle requires $2(M-1)$ binary bits in the ONE state. Figure 6 illustrates a rectangular base-band coded signal for a given input binary sequence with $M = 5$.

Input Binary Sequence

time

Code One M-level Signal (M=5)
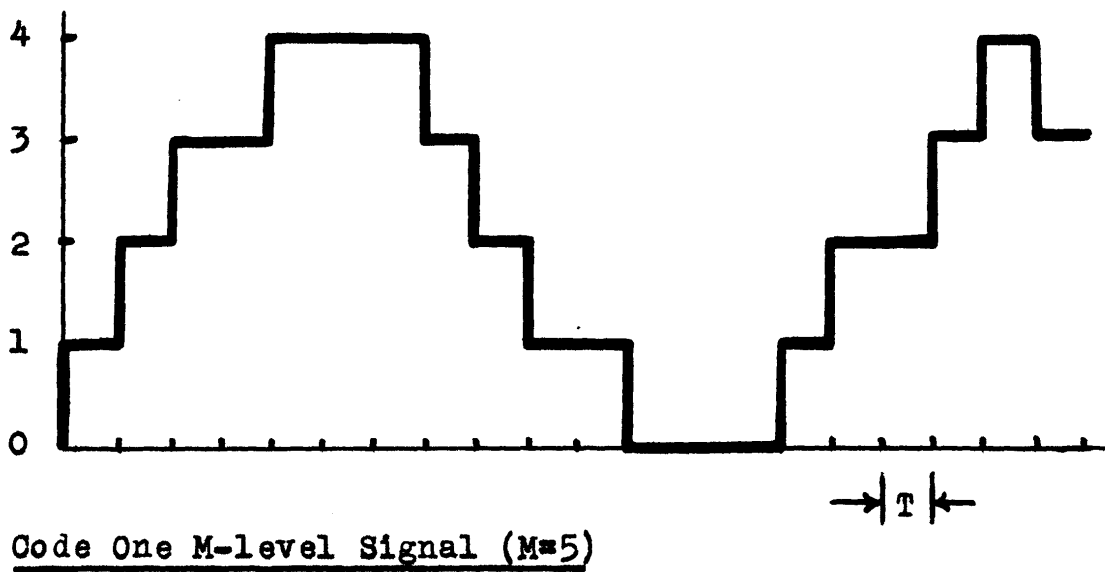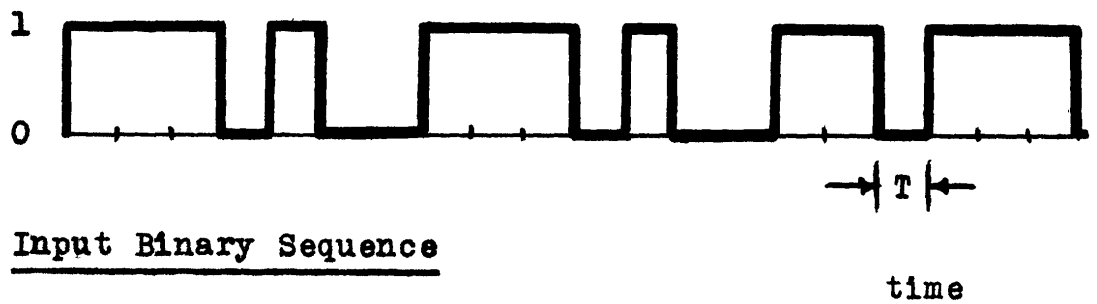
FIGURE 6   RECTANGULAR BASEBAND WAVEFORMS FOR CODE ONE

B. Forming the General Expression

For the code one M-level signal, with received peak value of A, there are M possible voltage levels of peak value $A(k-1)/(M-1)$ for $s'(t)$ where $1 \leq k \leq M$. This is the same as for the uncoded M-level signal. Again, Figure 5 indicates the appropriate terminology for describing the amplitude variations of the received signal $f(t)$ and the voltage thresholds for the detector. Using equation 10, the general expression for $P(E)$:

$$P(E) = \sum_{k=1}^{M} P(S_k) \, P(E|S_k)$$

where $P(S_k)$ and $P(E|S_k)$ are defined on pages 18 and 19.

First, consider the conditional probabilities $P(E|S_k)$, and observe that they can again be divided into three possibilities.

$P(E|S_1)$ = Probability of an error when S = level 1.

$P(E|S_M)$ = Probability of an error when S = level M.

$P(E|S_{k'})$ = Probability of an error when S = level k'.
where $2 \leq k' \leq (M-1)$

It appears the coding constraints imposed by code
one do not effect the conditional probabilities found for
the uncoded M-level system if a simple threshold detector
is assumed. As this is a previous assumption, observe that
the conditional probabilities for the code one system are
identical to the conditional probabilities solved pre-
viously for the uncoded M-level system. Therefore, equa-
tions 36, 38, and 46 apply also for the code one system
Thus:

$$P(E \mid S_1) = \frac{1}{2} \quad \text{erfc} \left(\frac{A}{2(M-1)\sigma\sqrt{2}} \right)$$

$$P(E \mid S_M) = \frac{1}{2} \quad \text{erfc} \left(\frac{A}{2(M-1)\sigma\sqrt{2}} \right)$$

$$P(E \mid S_{k'}) = (M-2) \quad \text{erfc} \left(\frac{A}{2(M-1)\sigma\sqrt{2}} \right)$$

The general expression for $P(E)$ now is: (49)

$$P(E) = \left[\frac{P(S_1)}{2} + \frac{P(S_M)}{2} + (M-2) \, P(S_{k'}) \right] \quad \text{erfc} \left(\frac{A}{2(M-1)\sigma\sqrt{2}} \right)$$

It is now necessary to determine $P(S_k)$ for the code one
system.

G. Solving for $P(S_k)$

Recall that $P(S_k)$ is the average probability of
occurrence of any of the k levels in a very long message
$s(t)$. It can no longer be assumed that each of the k levels

will appear with equal liklihood because of the coding
constraints imposed by the code one system. However, it
is assumed the two levels of the input binary pulse se-
quence to the encoder are equiprobable. That is:

$$P(0) = P(1) = 1/2 \tag{50}$$

One way to find $P(S_k)$ for the code one system is to
determine $P(S_k)$ for specific values of M and then arrive
at a general expression. First, let $M = 5$. From proba-
bility theory[12], there is the relation (whose form is
similar to equation 10):

$$P(a_i) = \sum_{\text{all } j} P(b_j) P(a_i | b_j) \tag{51}$$

With $M = 5$, $P(S_1)$ can be solved for by applying this
relation to a code one signal. Let $i = 1$; and $1 \leq j \leq 5$.

$$P(S_1) = P(S_1)P(S_1|S_1) + P(S_2)P(S_1|S_2) + P(S_3)P(S_1|S_3) +$$
$$+ P(S_4)P(S_1|S_4) + P(S_5)P(S_1|S_5)$$

From the system coding constraints, the following
conditional probabilities are realized:

$$P(S_1|S_1) = P(0) = 1/2$$
$$P(S_1|S_2) = P(1)/2 = 1/4$$
$$P(S_1|S_3) = 0$$
$$P(S_1|S_4) = 0$$
$$P(S_1|S_5) = 0$$

Therefore:

$$P(S_1) = \frac{1}{2} P(S_1) + \frac{1}{4} P(S_2)$$

$$P(S_1) = \frac{P(S_2)}{2} \tag{52}$$

With $M = 5$, $P(S_2)$ can be solved for by using relation 51 and let $i = 2$; $1 \leq j \leq 5$. Thus:

$$P(S_2) = P(S_1)P(S_2|S_1) + P(S_2)P(S_2|S_2) + P(S_3)P(S_2|S_3) +$$
$$+ P(S_4)P(S_2|S_4) + P(S_5)P(S_2|S_5)$$

From the system coding constraints, the following conditional probabilities are realized:

$$P(S_2|S_1) = P(1) = 1/2$$
$$P(S_2|S_2) = P(0) = 1/2$$
$$P(S_2|S_3) = P(1)/2 = 1/4$$
$$P(S_2|S_4) = 0$$
$$P(S_2|S_5) = 0$$

Therefore:

$$P(S_2) = \frac{1}{2} P(S_1) + \frac{1}{2} P(S_2) + \frac{1}{4} P(S_3)$$

Using equation 52, this reduces to:

$$P(S_2) = P(S_3) \tag{53}$$

Similarly, it can be shown that:

$$P(S_3) = P(S_4) \tag{54}$$

$$P(S_5) = P(S_4)/2 \tag{55}$$

Recall that some level S_k is always being transmitted:

$$\sum_{k=1}^{M} P(S_k) = 1$$

<u>Then</u>:

$$P(S_1) + P(S_2) + P(S_3) + P(S_4) + P(S_5) = 1$$

Substituting in equations 52,53,54, and 55 in terms of $P(S_2)$:

$$\frac{P(S_2)}{2} + P(S_2) + P(S_2) + P(S_2) + \frac{P(S_2)}{2} = 1$$

$$P(S_2) = P(S_3) = P(S_4) = 1/4 \tag{56}$$

$$P(S_1) = P(S_5) = 1/8 \tag{57}$$

Therefore, for the code one system with $M = 5$, the interior level probabilities are equal to 1/4, and the two extreme level probabilities are equal to 1/8.

Using similar operations, it can be shown that the following probabilities for $P(S_k)$ result for the code one system:

M	$P(S_k \cdot)$	$P(S_1)$	$P(S_M)$
3	1/2	1/4	1/4
4	1/3	1/6	1/6
5	1/4	1/8	1/8
6	1/5	1/10	1/10
7	1/6	1/12	1/12

These results can be generalized for an M-level code one system as:

$$P(S_k) \begin{cases} P(\text{level 1}) = \dfrac{1}{2(M-1)} \\[3ex] P(\text{level k}') = \dfrac{1}{(M-1)} \\[3ex] P(\text{level M}) = \dfrac{1}{2(M-1)} \end{cases} \tag{58}$$

D. Solving for P(E)

The final P(E) expression can now be found by substituting equation 58 into equation 49.

$$P(E) = \left[\frac{1}{4(M-1)} + \frac{1}{4(M-1)} + \frac{(M-2)}{(M-1)} \right] \text{erfc} \left(\frac{A}{2(M-1)\sigma\sqrt{2}} \right)$$

$$P(E) = \frac{2M-3}{2(M-1)} \text{erfc} \left(\frac{A}{2(M-1)\sigma\sqrt{2}} \right) \tag{59}$$

per symbol or pulse in the multi-level signal.

This then, is the expression for P(E) for the coded system of Shagena and Kvarda[9], and it agrees with the expression presented by these authors in their Addendum.

XI. PROBABILITY OF ERROR OF SYSTEM CODE TWO

Code two refers to the multi-level coded signal of Lender[1,4,10,11]. His investigations have been directed to the possibility of using descrete signaling levels that would be correlated in the process of generating such levels and yet, could be treated independently in the detection process. Unlike the general multi-level system, where each level in the signal might represent a specific binary sequence (e.g., 00, 01, 10, 11), each level in a correlated system represents only one binary digit: MARK or SPACE. Therefore, the term "correlated levels" as used by Lender implies that, in the coding process at the transmitter, each MARK or SPACE is associated with one of several pre-determined levels and the choice of a particular level depends upon the past history of the signal. However, at the receiver, each level can still be uniquely associated with MARK or SPACE without examining the past history of the waveform.

Using such techniques with correlated levels, Lender has achieved over-all frequency spectrum shaping. He has found it possible to re-distribute the spectral energy so as to concentrate most of it at low frequencies or, alternatively, to eliminate any energy at low frequencies.

A. Coding Technique for Code Two

This coding technique concerns taking a binary pulse train as an input signal and encoding it into an M-level signal in two separate steps.

For the first step, the input binary sequence a_n, with two signaling levels (MARK or SPACE), is converted into another binary sequence b_n, with two signaling levels (0 or 1), in such a manner that a group of (M-1) consecutive digits in sequence b_n represents a MARK in sequence a_n if the group of digits contains an odd number of binary 1's, otherwise the group of (M-1) consecutive digits represents a SPACE. The binary sequence b_n has exactly the same bit speed as the input sequence a_n. Note however, the symbols 1 and 0 in the new sequence b_n no longer represent MARK and SPACE in the original sequence a_n. For example; suppose M = 5 levels and the input sequence a_n is M M S M S M M S (where M and S stand for MARK and SPACE respectively). Then, a group of (M - 1) = 4 bits of sequence b_n represent each M or S. A possible sequence b_n is 0 0 0 1 0 1 1 0 1 1 0 . Note the first four bits in b_n (0001) represent M, the second through fifth (0010) represent M, the third through sixth (0101) represent S, and so on. See Figure 7 a and b for a baseband waveform representation of a similar example.

The second transformation step of code two involves
the conversion of the new binary sequence b_n into a coded
signal with M levels numbered consecutively from zero to
(M-1), starting at the bottom. This conversion is accom-
plished by forming the digit sum of successive groups of
(M-1) consecutive digits of sequence b_n. Since only the
binary 1's contribute to the digit sum, an odd-numbered
level of the multi-level signal, representing a MARK will
result if the number of 1's in a group of (M-1) digits of
sequence b_n is odd. Similarly, an even-numbered level of
the multi-level signal representing a SPACE will result
if the number of 1's in the group of (M-1) digits of se-
quence b_n is even. Using the example on the previous
page, 0001 and 0100 will result in level 1, each repre-
senting a MARK, and 0101 will result in level 2 repre-
senting a SPACE, etc.

One result of the level conversion process of code
two is that SPACES and MARKS of the input binary sequence
a_n correspond uniquely to the even and odd-numbered levels
respectively of the multi-level signal. Therefore, in
spite of the correlation properties which span over (M-1)
digits, each level of the multi-level signal can be inde-
pendently identified at the receiver as MARK or SPACE.
Figure 7 gives an example of this entire coding process
in a rectangular baseband waveform representation.

a.   Input Binary Sequence

b.   Converted Binary Sequence
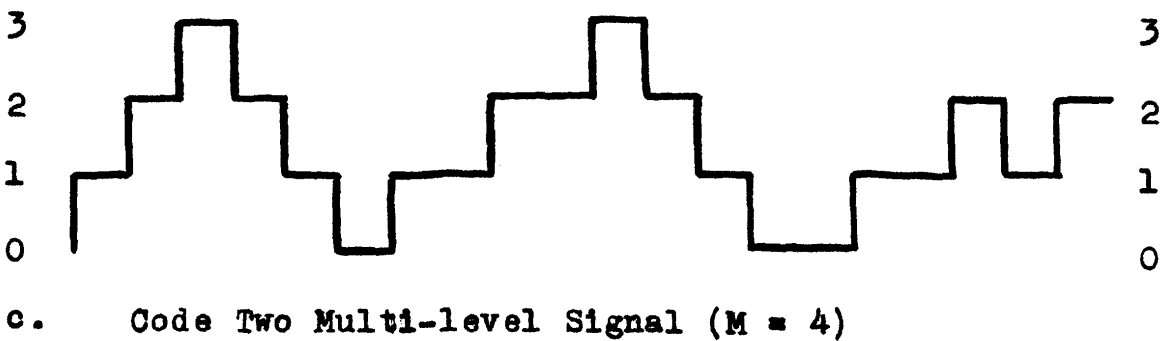
c.   Code Two Multi-level Signal (M = 4)
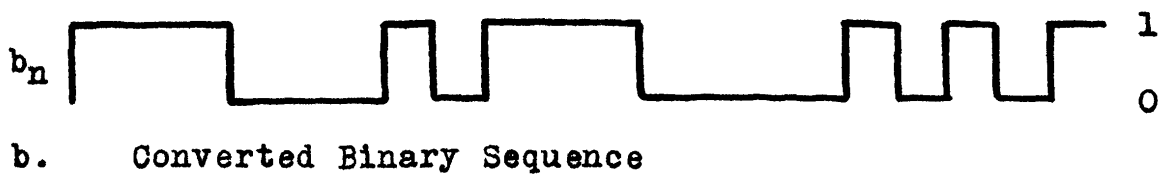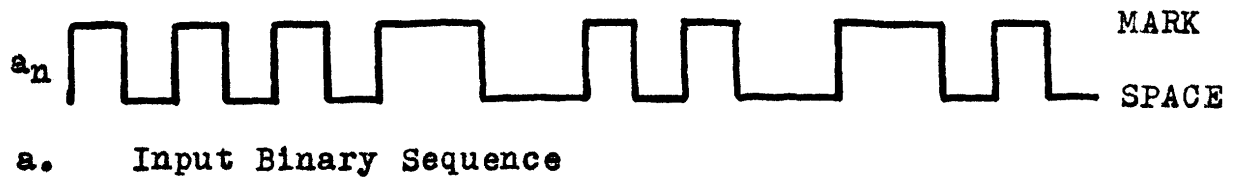
FIGURE 7

RECTANGULAR BASEBAND WAVEFORMS FOR CODE TWO

B. Forming the General Expression

For the code two multi-level signal with recieved peak value of A, there are M possible voltage levels of peak value $Ah/(M-1)$ for $s'(t)$ where $0 \leq h \leq M-1$. Note that the variable k has been replaced with the variable h, and that h has a slightly different range than k which was used previously. This variable change is introduced so that the resulting expression might more closely resemble Lender's. Figure 8 then, represents a typical 5-level band-limited signal with appropriate terminology for describing the received signal of code two. Note that it is similar to Figure 5 except for the different labeling on the signal and decision levels. Again, equation 10 is used for the general expression for $P(E)$. Thus:

$$P(E) = \sum_{h=0}^{M-1} P(S_h) \, P(E \mid S_h) \tag{60}$$

where $P(S_h)$ and $P(E \mid S_h)$ are essentially defined on pages 18 and 19 as $P(S_k)$ and $P(E \mid S_k)$.

C. Solving for $P(S_h)$

$P(S_h)$ is the average probability of occurrence of any of the h levels in a very long message $s(t)$ where $0 \leq h \leq (M-1)$. Code two is a relatively complex coding

SIGNAL LEVEL	S'(t) AMPLITUDE		DECISION LEVEL
4	A		$(2M-3)D$
3	$\dfrac{3A}{4}$		(S_K+D)
S_h	$\dfrac{Ah}{(M-1)}$		(S_K-D)
1	$\dfrac{A}{4}$		D
0	0		

time →
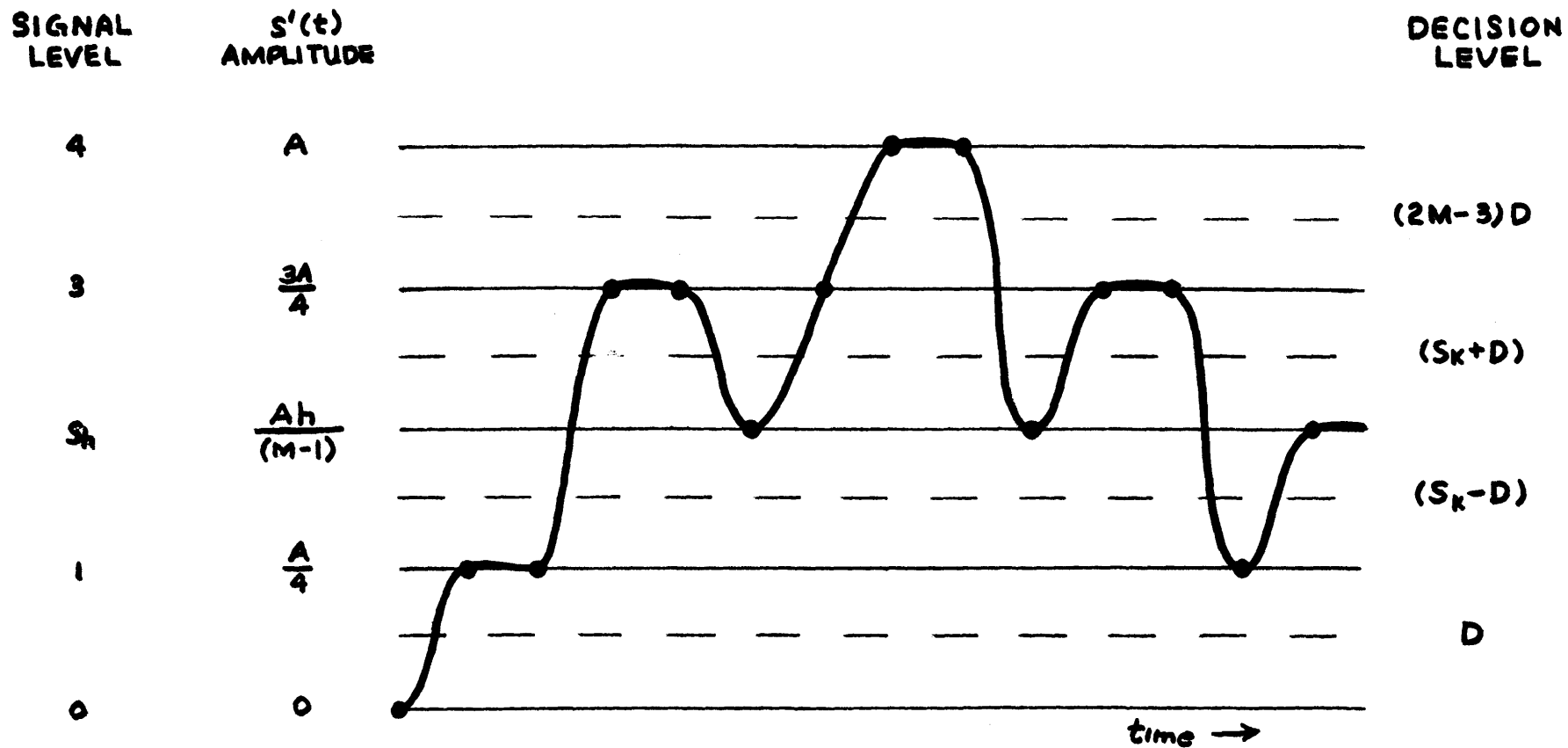
Note: For this signal, $0 \leq h \leq (M-1)$

FIGURE 8 TYPICAL VOLTAGE WAVEFORM OF BANDLIMITED 5-LEVEL SIGNAL

scheme and it cannot be assumed that each of the h levels of s(t) will appear with equal liklihood. In keeping with previous practice, the two levels of the input binary pulse sequence a_n shall be assumed to be equiprobable. That is:

$$P(MARK) = P(SPACE) = 1/2 \qquad (61)$$

As code two has a two-step encoding process, the derivation of $P(S_h)$ will also be made in two steps.

1. Solving for P(1) and P(0) in binary sequence b_n

Since the first step of the code two encoding process involves the transformation of the input binary sequence a_n into a binary sequence b_n, the solution for P(1) and P(0) in binary sequence b_n is sought first. Due to the complexity of the coding technique however, a single general expression for the probability of getting a 1 or 0 in sequence b_n does not seem to exist. However, given the conditions of equation 61 above, some definite conclusions about P(1) and P(0) in binary sequence b_n can be drawn.

Consider the following example of the first step in the encoding process:

For this example, let M = 4; therefore, (M-1) = 3. Assume the encoding process has been going on for some time when it is stopped at time $t = t_1$. Consider the situation of predicting the next symbol at $t = t_2$. Observe the following figure.
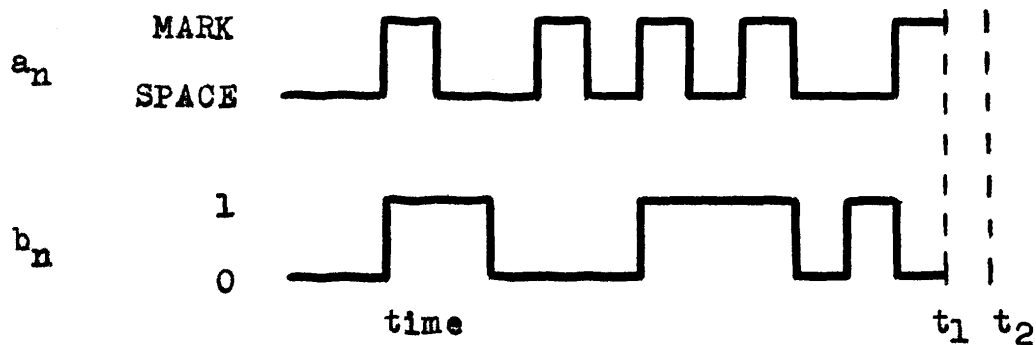
FIGURE 9. EXAMPLE OF FIRST SIGNAL TRANSFORMATION

For this specific example, with $M = 4$, and given the past history of a_n and b_n, the following can be said: From the encoding process, it is known that if the symbol at $t = t_2$ in sequence a_n is a MARK, then the corresponding symbol in sequence b_n will be a 0. Another way of saying the same thing is: If $P(MARK) = 1$ at $t = t_2$ in sequence a_n, then the $P(0) = 1$ for the corresponding symbol in sequence b_n. Conversely, it is known that if $P(SPACE) = 1$ at $t = t_2$ in sequence a_n, then $P(1) = 1$ for the corresponding symbol in sequence b_n. However, from equation 61, it is assumed $P(MARK) = P(SPACE)$ for all time in sequence a_n, therefore it follows that $P(0) = P(1)$ at $t = t_2$ in sequence b_n.

This conclusion apparently holds true, not only for the specific example considered above, but also for any other example with different past histories and different values of M. Simply stated this conclusion is: Given the probability of MARK and SPACE in sequence a_n is equiprobable, then the probability of 1 and 0 in sequence b_n is also equiprobable.

Therefore:

$$P(0) = P(1) = 1/2 \qquad (62)$$

2. Solving for $P(S_h)$ knowing $P(0)$ and $P(1)$

The second step of the code two encoding process involves the transformation of the binary sequence b_n composed of 1's and 0's into an M-level signal. This is done, as was discussed previously, by forming the digit sum of successive groups of (M-1) consecutive digits in sequence b_n. In other words, to get to the h level in the multi-level signal, there must be h 1's in the pulse sequence b_n (within M-1 digits). Now:

a. The total number of ways of getting h 1's in a series of (M-1) digits is $\binom{M-1}{h}$ where $0 \leq h \leq (M-1)$. The symbol $\binom{M-1}{h}$ is the binomial function[13,17].

b. The probability of getting any particular combination of 0's and 1's in (M-1) digits is:

$$P(1)^h P(0)^{M-1-h}$$

Therefore:

$$P(S_h) = P(1)^h P(0)^{M-1-h} \binom{M-1}{h}$$

Using Equation 62:

$$P(S_h) = (1/2)^h (1/2)^{M-1-h} \binom{M-1}{h}$$

And:

$$P(S_h) = (1/2)^{M-1} \binom{M-1}{h} \qquad (63)$$

This result agrees with Lender in his equation 32.

E. Solving for $P(E \mid S_h)$ for Code Two

Equation 60 can be expanded to be:

$$P(E) = \sum_{\substack{\text{all } h \\ \text{odd}}} P(S_h)\, P(E|S_h) + \sum_{\substack{\text{all } h \\ \text{even}}} P(S_h)\, P(E|S_h) \qquad (64)$$

Due to the unique character of code two, $P(E)$ must be derived separately for $(M-1) =$ ODD and $(M-1) =$ EVEN number of levels using equation 64. Having just determined $P(S_h)$; next define $P(E|S_h)$ for the following four cases:

$Y_{h1} = P(E|S_h)$; when $(M-1)$ is even and h is even.

$Y_{h2} = P(E|S_h)$; when $(M-1)$ is even and h is odd.

$Y_{h3} = P(E|S_h)$; when $(M-1)$ is odd and h is even.

$Y_{h4} = P(E|S_h)$; when $(M-1)$ is odd and h is odd.

Therefore equation 64 now becomes:

When (M-1) is even

$$P(E) = \sum_{\substack{\text{all } h \\ \text{even}}} P(S_h)\, Y_{h1} + \sum_{\substack{\text{all } h \\ \text{odd}}} P(S_h)\, Y_{h2} \qquad (65)$$

When (M-1) is odd

$$P(E) = \sum_{\substack{\text{all } h \\ \text{even}}} P(S_h)\ Y_{h3} \quad + \quad \sum_{\substack{\text{all } h \\ \text{odd}}} P(S_h)\ Y_{h4} \tag{66}$$

1. Solving for Y_{h1}

$Y_{h1} = P(E \mid S_h)$ when (M-1) is even; h is odd.

When the transmitted signal s(t) is at an even level (h = even), an error will occur only when the received signal f(t) occurs in the amplitude decision range of an odd level. Note that when considering level numbered h of height 2Dh, the noise has this height as an average value. (See Figure 8)

Call a typical odd level S_b where b = 1, 3, 5, 4 . . .
. . highest odd number in (M-1). When (M-1) is even, the highest odd number will be (M-2). The amplitude of S_b = 2D(2i-1), where i = 1, 2, 3, ... L.

Let: b = (2i-1)

Then: (2L-1) = (M-2)

Or: L = (M-1)/2 \hfill (67)

$$Y_{h1} = \sum_{i=1}^{L} P\left[(S_b - D) < F \le (S_b + D) \right] \tag{68}$$

$$Y_{hl} = \sum_{i=1}^{L} \quad P\left[(S_b - D) < F\right] - \quad P\left[(S_b + D) < F\right]$$

$$Y_{hl} = \sum_{i=1}^{L} \left[P(3) - P(4)\right]$$

a. Derivation of P(3)

$$P(3) = P\left[(S_b - D) < F\right]$$

Using equation 22.

$$P(3) = \frac{1}{\sigma\sqrt{2\pi}} \int_{S_b - D}^{\infty} e^{-(F - 2Dh)^2/2\sigma^2} \, dF$$

<u>Let:</u> $Q = \dfrac{F - 2Dh}{\sigma\sqrt{2}}$ <u>then:</u> $dQ = \dfrac{dF}{\sigma\sqrt{2}}$

<u>As:</u> $F \longrightarrow \infty$; $Q \longrightarrow \infty$

<u>As:</u> $F \longrightarrow S_b - D$; $Q \longrightarrow \dfrac{S_b - D - 2Dh}{\sigma\sqrt{2}}$

<u>But:</u> $\dfrac{S_b - D - 2Dh}{\sigma\sqrt{2}} = \dfrac{D(41 - 3 - 2h)}{\sigma\sqrt{2}} = Z_1$

$$P(3) = \frac{1}{\sqrt{\pi}} \int_{Z_1}^{\infty} e^{-Q^2} \, dQ = (1/2)\ \text{erfc}\ Z_1$$

$$P(3) = (1/2)\ \text{erfc}\ \left(\frac{D(41 - 3 - 2h)}{\sigma\sqrt{2}}\right) \qquad (69)$$

b. Derivation of P(4)

$$P(4) = P\left[(S_b + D) < F\right]$$

Using a similar development as was used in solving for P(3), it can be shown that:

$$P(4) = (1/2) \; \text{erfc} \; \left(\frac{D(4i-1-2h)}{\sigma\sqrt{2}}\right) \tag{70}$$

Let: $\quad X = \dfrac{D}{\sigma\sqrt{2}}$

Then:

$$Y_{hl} = (1/2) \sum_{i=1}^{L} \left[\text{erfc} \; X(4i-3-2h) \; - \; \text{erfc} \; X(4i-1-2h)\right] \tag{71}$$

Converting to the error function (erfc Z = 1 - erf Z):

$$Y_{hl} = (1/2) \sum_{i=1}^{\frac{M-1}{2}} \left[- \; \text{erf} \; X(4i-3-2h) \; + \; \text{erf} \; X(4i-1-2h)\right]$$

But: $\quad$ for h even

$$\sum_{i=1}^{\frac{M-1}{2}} \left[- \; \text{erf} \; X(4i-3-2h)\right] = \sum_{i=1}^{M-h-1} (-1)^i \; \text{erf} \; X(2i-1)$$

And:

$$\sum_{i=1}^{\frac{M-1}{2}} \text{erf} \; X(4i-1-2h) = \sum_{i=1}^{h} (-1)^i \; \text{erf} \; X(2i-1)$$

Therefore:

$$(72)$$

$$Y_{h1} = \frac{1}{2} \sum_{i=1}^{M-h-1} (-1)^i \, \text{erf } X(2i-1) + \frac{1}{2} \sum_{i=1}^{h} (-1)^i \, \text{erf } X(2i-1)$$

Then, let $Y_{h1} = Y_h$ for h even   (73)

2. Solving for Y_{h2}

$Y_{h2} = P(E \mid S_h)$ when (M-1) is even; h is odd.

When the transmitted signal s(t) is at an odd level, an error will occur only when the received signal f(t) occurs in the amplitude decision range of an even level.

Level 0 is an even level.

Level (M-1) is an even level.

Let S_e be all even levels in between.

$e = 2i$ for i = 1, 2, 3, ---- J

          where $J = \dfrac{(M-3)}{2}$

The amplitude of S_e is 4Di.

$$Y_{h2} = P\left[F \leq D \right] + P\left[F > (2M-3)D \right] + \sum_{i=1}^{J} \left(P\left[F > S_e - D \right] - \right.$$

$$\left. - P\left[F > S_e + D \right] \right) \qquad (74)$$

By making use of equations 20 and 22 and a similar development used in solving for Y_{h1}, it can be shown that:

$$P\left[F \leq D\right] = (1/2)\ \text{erfc}\ \left(\frac{D(2h-1)}{\sigma\sqrt{2}}\right)$$

$$P\left[F > (2M-3)D\right] = (1/2)\ \text{erfc}\ \left(\frac{D(2M-3-2h)}{\sigma\sqrt{2}}\right)$$

$$P\left[F > S_e-D\right] = (1/2)\ \text{erfc}\ \left(\frac{D(4i-1-2h)}{\sigma\sqrt{2}}\right)$$

$$P\left[F > S_e+D\right] = (1/2)\ \text{erfc}\ \left(\frac{D(4i+1-2h)}{\sigma\sqrt{2}}\right)$$

<u>Let:</u> $X = \dfrac{D}{\sigma\sqrt{2}}$

<u>Then:</u>

$$Y_{h2} = (1/2)\left\{ \text{erfc}\ X(2h-1)\ +\ \text{erfc}\ X(2M-3-2h)\ + \right.$$
$$\left. +\ \sum_{i=1}^{J}\left[\text{erfc}\ X(4i-1-2h)\ -\ \text{erfc}\ X(4i+1-2h)\right]\right\} \qquad (75)$$

Converting to the error function:

$$Y_{h2} = 1 + \frac{1}{2}\left\{ -\text{erf}\ X(2h-1)\ -\ \text{erf}\ X(2M-3-2h)\ + \right.$$
$$\left. +\ \sum_{i=1}^{\frac{M-3}{2}}\left[(-)\ \text{erf}\ X(4i-1-2h)\ +\ \text{erf}\ X(4i+1-2h)\right]\right\}$$

<u>But:</u>
<u>For h odd:</u>

$$\left[-\text{erf}\ X(2M-3-2h)\ -\ \sum_{i=1}^{\frac{M-3}{2}}\text{erf}\ X(4i-1-2h)\right] = \sum_{i=1}^{M-h-1}(-1)^i\ \text{erf}\ X(2i-1)$$

<u>And:</u>

$$\left[-\text{erf}\ X(2h-1)\ +\ \sum_{i=1}^{\frac{M-3}{2}}\text{erf}\ X(4i+1-2h)\right] = \sum_{i=1}^{h}(-1)^i\ \text{erf}\ X(2i-1)$$

Therefore:

(76)

$$Y_{h2} = 1 + \frac{1}{2} \sum_{i=1}^{M-h-1} (-1)^i \, \text{erf } X(2i-1) + \frac{1}{2} \sum_{i=1}^{h} (-1)^i \text{erf } X(2i-1)$$

Then: $Y_{h2} = 1 + Y_h$ for h odd   (77)

3. Solving for Y_{h3}

$Y_{h3} = P(E|S_h)$ when (M-1) is odd; h is even

Again, when the transmitted signal s(t) is at an even level, an error will occur only when the received signal f(t) occurs in the amplitude range of an odd level.

Level (M-1) is odd.

Call S_b all other odd levels where b = 1, 3, ---

--- highest odd level below (M-1).

S_b = (2i-1) where i = 1,2,3,--- R

Then: R = (M-2)/2

The amplitude of S_b = 2D(2i-1)

$$Y_{h3} = P\left[F > (2M-3)D \right] + \sum_{\text{all } i} \left(P\left[F > S_b - D \right] - P\left[F > S_b + D \right] \right)$$

(78)

By again making use of equations 20 and 22 and the previous developments, it can be shown that:

$$P\left[F > (2M-3)D \right] = (1/2) \, \text{erfc} \left(\frac{D(2M-3-2h)}{\sigma\sqrt{2}} \right)$$

$$P\left[F > S_b - D \right] = (1/2) \, \text{erfc} \left(\frac{D(4i-3-2h)}{\sigma\sqrt{2}} \right)$$

$$P\left[F > S_b + D\right] = (1/2)\ \text{erfc}\ \left(\frac{D(4i-1-2h)}{\sigma\sqrt{2}}\right)$$

<u>Let:</u> $X = \dfrac{D}{\sigma\sqrt{2}}$

<u>Then:</u>

$$Y_{h3} = \frac{1}{2}\left\{\text{erfc}\ X(2M-3-2h) + \sum_{i=1}^{R}\left[\text{erfc}\ X(4i-3-2h)\ \text{----}\right.\right.$$

$$\left.\left.\text{---}\ -\ \text{erfc}\ X(4i-1-2h)\right]\right\} \qquad (79)$$

Converting to the error function:

$$Y_{h3} = \frac{1}{2} + \frac{1}{2}\left\{(-)\text{erf}\ X(2M-3-2h) + \sum_{i=1}^{\frac{M-2}{2}}\left[\text{erf}\ X(4i-1-2h)\ -\right.\right.$$

$$\left.\left.-\ \text{erf}\ X(4i-3-2h)\right]\right\}$$

<u>But:</u> for h even

$$\left[-\text{erf}\ X(2M-3-2h) + \sum_{i=1}^{\frac{M-2}{2}}\text{erf}\ X(4i-1-2h)\right] = \sum_{i=1}^{M-h-1}(-1)^{i}\ \text{erf}\ X(2i-1)$$

<u>And:</u>

$$\sum_{i=1}^{\frac{M-2}{2}}(-)\text{erf}\ X(4i-3-2h) = \sum_{i=1}^{h}(-1)^{i}\ \text{erf}\ X(2i-1)$$

<u>Therefore:</u>

$$\hspace{8cm}(80)$$

$$Y_{h3} = \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{M-h-1}(-1)^{i}\text{erf}\ X(2i-1) + \frac{1}{2}\sum_{i=1}^{h}(-1)^{i}\text{erf}\ X(2i-1)$$

Then: $Y_{h3} = (1/2) + Y_h$ for h even   (81)

4. Solving for Y_{h4}

$Y_{h4} = P(E|S_h)$ when (M-1) is odd;   h is odd.

Again, an error will occur only when f(t) occurs in the amplitude decision range of an even level.

Level 0 is an even level.

Call S_e all other even levels where e = 2,4, ---

--- highest even level in (M-1).

e = 2i, where i = 1,2,3, --- G.

Then: G = (M-2)/2

The amplitude of S_e = 4Di.

$$Y_{h4} = P\left[F \leq D\right] + \sum_{\text{all i}} \left(P\left[F > S_e - D\right] - P\left[F > S_e + D\right]\right) \quad (82)$$

It can be shown that:

$$P\left[F \leq D\right] = (1/2)\ \text{erfc}\left(\frac{D(2h-1)}{\sigma\sqrt{2}}\right)$$

$$P\left[F > S_e - D\right] = (1/2)\ \text{erfc}\left(\frac{D(4i-1-2h)}{\sigma\sqrt{2}}\right)$$

$$P\left[F > S_e + D\right] = (1/2)\ \text{erfc}\left(\frac{D(4i+1-2h)}{\sigma\sqrt{2}}\right)$$

Let: $X = \dfrac{D}{\sigma\sqrt{2}}$

Then:

$$Y_{h4} = (1/2)\left\{\text{erfc}\ X(2h-1) + \sum_{i=1}^{G}\left[\text{erfc}\ X(4i-1-2h) \right.\right.$$
$$\left.\left. --- - \text{erfc}\ X(4i+1-2h)\right]\right\} \quad (83)$$

Converting Y_{h4} to the error function:

$$Y_{h4} = \frac{1}{2} + \frac{1}{2}\left\{ (-)\ \text{erf}\ X(2h-1)\ +\ \text{----}\ \right.$$

$$\left. \text{----} + \sum_{i=1}^{\frac{M-2}{2}} \left[\text{erf}\ X(4i+1-2h) - \text{erf}\ X(4i-1-2h)\right] \right\}$$

<u>But:</u> for h odd

$$\left[(-)\text{erf}\ X(2h-1) + \sum_{i=1}^{\frac{M-2}{2}} \text{erf}\ X(4i+1-2h)\right] = \sum_{i=1}^{M-h-1} (-1)^i \text{erf}\ X(2i-1)$$

<u>And:</u>

$$\sum_{i=1}^{\frac{M-2}{2}} (-1)\ \text{erf}\ X(4i-1-2h) = \sum_{i=1}^{h} (-1)^i\ \text{erf}\ X(2i-1)$$

<u>Therefore:</u>
$$\tag{84}$$

$$Y_{h4} = \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{M-h-1} (-1)^i \text{erf}\ X(2i-1) + \frac{1}{2}\sum_{i=1}^{h} (-1)^i \text{erf}\ X(2i-1)$$

Then: $Y_{h4} = (1/2) + Y_h$ for h odd.   (85)

E. Solving for P(E)

When (M-1) is even

Substituting equations 73 and 77 into equation 65:

$$P(E) = \sum_{\substack{\text{all h} \\ \text{even}}} P(S_h)\, Y_h \;+\; \sum_{\substack{\text{all h} \\ \text{odd}}} P(S_h) \left[1 + Y_h \right]$$

$$P(E) = \sum_{\substack{\text{all h} \\ \text{odd}}} P(S_h) \;+\; \sum_{\text{all h}} P(S_h)\, Y_h$$

Using equation 63, however:

$$\sum_{\substack{\text{all h} \\ \text{odd}}} P(S_h) = \sum_{\substack{\text{all h} \\ \text{odd}}} (1/2)^{M-1} \binom{M-1}{h} = \tfrac{1}{2}$$

Therefore:

$$P(E) = \tfrac{1}{2} + \sum_{h=0}^{M-1} P(S_h)\, Y_h \qquad \text{for (M-1) even.} \qquad (86)$$

When (M-1) is odd

Substituting equations 81 and 85 into equation 66:

$$P(E) = \sum_{\substack{\text{all h} \\ \text{even}}} P(S_h) \left[\tfrac{1}{2} + Y_h \right] \;+\; \sum_{\substack{\text{all h} \\ \text{odd}}} P(S_h) \left[\tfrac{1}{2} + Y_h \right]$$

$$P(E) = \tfrac{1}{2} \sum_{h=0}^{M-1} P(S_h) \;+\; \sum_{h=0}^{M-1} P(S_h)\, Y_h$$

Using equation 63, however:

$$\sum_{h=0}^{M-1} P(S_h) = \sum_{h=0}^{M-1} (1/2)^{M-1} \binom{M-1}{h} = 1$$

Therefore:

$$P(E) = \frac{1}{2} + \sum_{h=0}^{M-1} P(S_h) \, Y_h \qquad \text{for (M-1) odd.} \qquad (87)$$

Combining equations 86 and 87:

$$P(E) = \frac{1}{2} + \sum_{h=0}^{M-1} P(S_h) \, Y_h \qquad (88)$$

per symbol or pulse in the multi-level signal.

where:

$$P(S_h) = (1/2)^{M-1} \binom{M-1}{h}$$

$$(89)$$

$$Y_h = \frac{1}{2} \sum_{i=1}^{M-h-1} (-1)^i \, \text{erf } X(2i-1) + \frac{1}{2} \sum_{i=1}^{h} (-1)^i \, \text{erf } X(2i-1)$$

$$X = \frac{D}{\sigma \sqrt{2}} = \left(\frac{A}{2(M-1)\sigma \sqrt{2}} \right)$$

Equation 88 then is the expression for P(E) for the coded multi-level signal of Lender[4] and it agrees with his equations 36 and 37.

X. P(E) CURVES

The usual method of illustrating probability of error for digital data transmission systems is by plotting P(E) versus signal-to-noise ratio curves. These curves take many varied forms in the literature. Perhaps the most common form is the plot of P(E) as the ordinate on a logarithmic scale with S/N as the abscissa expressed in decibels. Figures 10, 12, 13, and 14 are examples of this form of P(E) curve.

A. Signal-to-Noise Ratio

It is seen from equations 48, 59, and 89 that the error probability of baseband systems depends on the ratio of the voltage amplitude A of the received pulse sample to the rms noise voltage σ. This ratio is generally expressed in terms of a ratio between signal power and noise power. Unfortunately, there seems to be a lack of uniform terminology in defining what should be meant by signal-to-noise power ratio.. For example, it is difficult to detect any common terminology for S/N between Bennett and Davy, Shagena and Kvarda, and Lender. As such, the published P(E) curves for their respective systems cannot directly be compared. To circumvent a lengthy discussion into the many variations possible for this ratio, it shall be assumed that S/N, as used hereafter, refers to the ratio of

average signal power to average noise power. This as-
sumption provides the following advantages:

1. Shannon's[18] well-known formula for channel
   capacity $\left[C = B \, LOG_2 \, (1 + S/N)\right]$ holds only for
   average signal power and additive gaussian noise.

2. With a sampling type receiver (page 14), and
   assuming no intersymbol interference, considera-
   tion of the shape of the received pulse is un-
   necessary.

3. With this common base, comparative curves of the
   three discussed systems can then be presented.

1. Average Signal Power

From an early technical paper on pulse code modula-
tion[19], average signal power is defined as the mean-square
value of the individual pulse peak amplitudes. Thus, using
familiar terminology:

$$S = \sum_{\substack{\text{all h} \\ \text{levels}}} P(S_h) \, P_h \qquad (90)$$

where: $P(S_h)$ = Probability of h level $(0 \leq h \leq M-1)$.

P_h = Peak power in h level.

a. For uncoded signal

The previous reference[19] adequately presents the
average signal power expressions for the uncoded signal,

therefore, no development will be given. The expressions are:

For polybinary signal

$$S = \frac{A^2(2M-1)}{6(M-1)} \qquad \text{for all M} \qquad (91)$$

For polybipolar signal

$$S = \frac{A^2(M-1)}{12(M-1)} \qquad \text{for all M} \qquad (92)$$

b. For code one signal

Shagena and Kvarda make no reference to the average signal power for their coded signal. Therefore, recalling page 45 and using equation 90:

$$P(S_h) = \frac{1}{2(M-1)} \qquad \text{for } h = 0 \text{ and } (M-1)$$

$$P(S_h) = \frac{1}{(M-1)} \qquad \text{for } 1 \leq h \leq (M-2)$$

$$S = \frac{P_0}{2(M-1)} + \frac{P_{M-1}}{2(M-1)} + \sum_{h=1}^{M-2} \frac{P_h}{(M-1)}$$

For polybinary signal of peak amplitude A

From Figure 8:

$$P_0 = 0 ; \qquad P_{M-1} = A^2 ; \qquad P_h = \frac{A^2 h^2}{(M-1)^2}$$

Therefore:

$$S = \frac{A^2}{2(M-1)} + \frac{A^2}{(M-1)^3} \sum_{h=1}^{M-2} h^2$$

But:

$$\sum_{h=1}^{M-2} h^2 = \left[\frac{(M-2)(2M-3)(M-1)}{6} \right] \quad \text{for all } M$$

Then:

$$S = \frac{A^2}{(M-1)} \left[\frac{1}{2} + \frac{(M-2)(2M-3)}{6(M-1)} \right]$$

$$S = \frac{A^2}{6(M-1)^2} \left[2M^2 - 4M + 3 \right] \tag{93}$$

For polybipolar signal of peak amplitude + A/2 and - A/2.

The polybipolar expression for average signal power
can be obtained very simply from the polybinary expression.
Recall Figures 2 and 3. As can be seen from these figures,
a polybipolar waveform is equivalent to a polybinary wave-
form with the mean value removed. For the example pre-
sented in Figures 2 and 3, this mean value is 1. There-
fore, the average signal power for a polybipolar signal
is equal to the average signal power of its equivalent
polybinary signal minus the square of the mean value.
For a polybinary signal of peak amplitude A, this mean
value is A/2.

Therefore:

$$S = A^2 \left[\frac{2M^2 - 4M + 3}{6(M-1)^2} - \frac{1}{4} \right]$$

Or:

$$S = A^2 \frac{M^2 - 2M + 3}{12(M-1)^2} \tag{94}$$

c. For code two signal

Lender[4] does present an expression for average signal power of his polybinary signal. It is:

For polybinary signal

$$S = \frac{A^2 M}{4(M-1)} \qquad \text{for all } M \tag{95}$$

For polybipolar signal

Using the procedure that was used on the code one signal for finding the polybipolar expression, then:

$$S = A^2 \left[\frac{M}{4(M-1)} - \frac{1}{4} \right] \tag{96}$$

$$S = \frac{A^2}{4(M-1)} \tag{97}$$

2. Average Noise Power

Average noise power N shall be defined[5,4] as the mean square noise power in a circuit with unit resistance. Thus:

$$N = \frac{\sigma^2}{1} = \sigma^2 \tag{98}$$

B. The Computer Program

A digital computer was used to obtain values for the P(E) curves. This was done for the following two reasons.

1. To evaluate the error function and complementary error function.

2. To perform the many calculations necessary when plotting families of curves.

The erf function and erfc function are easily evaluated on a digital computer using approximations provided for that purpose in handbooks[16]. The approximation used (equation 7.1.26)[16] provides an error $\leq 1.5 \times 10^{-7}$ for any $0 \leq X \leq \infty$, and is easily programmed with S/N in units of decibels as a variable. A typical program is presented in Figure 10, using the Fortran IV language.

C. P(E) Curves for the Three Systems

The P(E) curves for the three baseband systems can now be presented.

1. Curves for the uncoded signal

It is necessary to first solve the expression of average signal power of a polybipolar signal (equation 92) for A^2/σ^2. Thus:

$$\frac{A^2}{\sigma^2} \cong \frac{S}{N} \frac{12(M-1)}{(M-1)}$$

```
/JOB       GO                                    DJ POPP
/FTC       LIST
           C   C** 44452EEX038   POPP DON              02/17/67 RACS          0002
           C          EVALUATION OF THE COMPLEMENTARY ERROR
           C          FUNCTION IN SOLVING P(E) EXPRESSIONS
           C          FOR POLYBIPOLAR SYSTEMS
S.0001            WRITE(3,1)
S.0002            WRITE(3,2)
S.0003            A1 = .25482959
S.0004            A2 = -.28449674
S.0005            A3 = 1.42141374
S.0006            A4 = -1.45315203
S.0007            A5 = 1.06140543
S.0008            P = .3275911
S.0009            DO 10   M=2,12
S.0010            R = (M-1.)/M
S.0011            Z = 3./(2.*(M**2-1.))
S.0012            DO 10   K=3,35
S.0013            D = K/10.
S.0014            Y = 10.**D
S.0015            X = SQRT(Z*Y)
S.0016            T = 1./(1.+P*X)
S.0017            XX = X*X
S.0018            ERFCX = (A1*T+A2*T*T+A3*T**3+A4*T**4+A5*T**5)*EXP(-XX)
S.0019            PE = R*ERFCX
S.0020         10 WRITE(3,3) M,K,PE
S.0021            CALL EXIT
S.0022          1 FORMAT(//T10,'FOR UNCODED SYSTEM - AVERAGE POWER'//)
S.0023          2 FORMAT(T11,'M',T15,'S/N',T30,'P(E)'//)
S.0024          3 FORMAT(9X,I2,3X,I2,3X,E18.8)
S.0025            END
```

FIGURE 10   TYPICAL   PROGRAM   FOR COMPUTING   P(E)

Using equation 48, P(E) for the uncoded polybipolar signal becomes:

$$P(E) = \frac{(M-1)}{M} \ \text{erfc} \ \left(\frac{S}{N} \ \frac{3}{2(M^2-1)} \right) \tag{99}$$

Similarly, using equation 91 for the polybinary signal:

$$\frac{A^2}{\sigma^2} = \frac{S}{N} \ \frac{6(M-1)}{(2M-1)}$$

Using equation 48, P(E) for the uncoded polybinary signal is:

$$P(E) = \frac{(M-1)}{M} \ \text{erfc} \ \left(\frac{S}{N} \ \frac{3}{4(2M-1)(M-1)} \right) \tag{100}$$

These P(E) expressions were programmed for $2 \leq M \leq 12$. The corresponding P(E) curves were plotted for $M = 2, 4, 6,$ and 8, and are given in Figure 11 a and b. The program for the polybipolar uncoded signal is given in Figure 10 as a typical program for finding P(E).
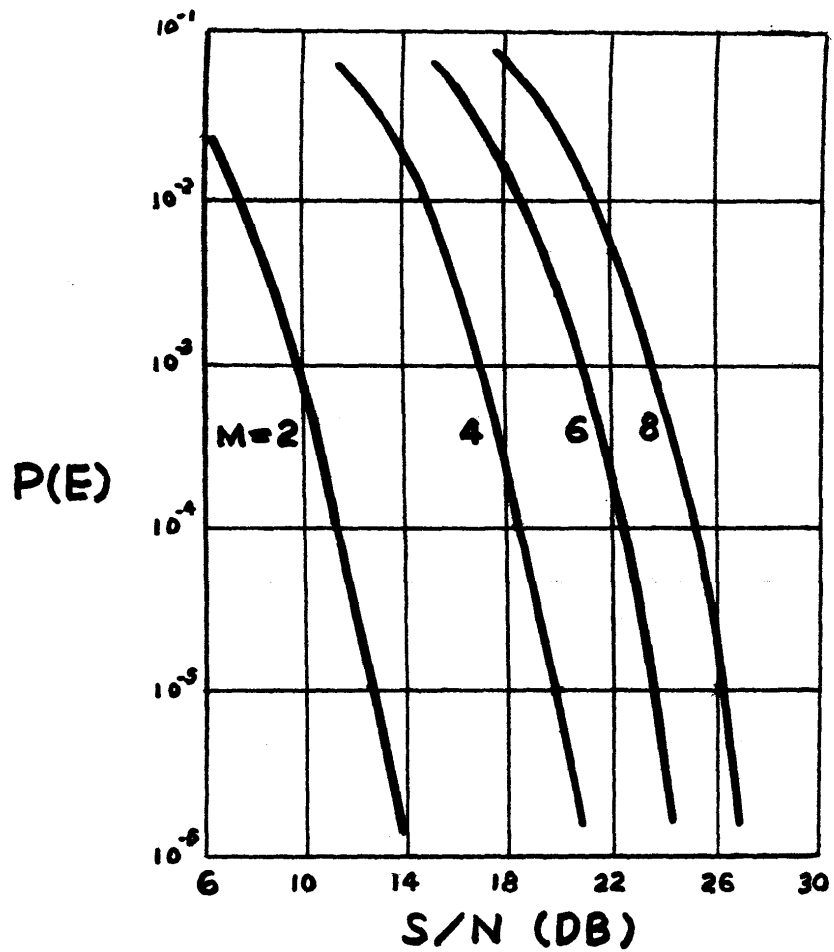
2. Curves for the code one signal
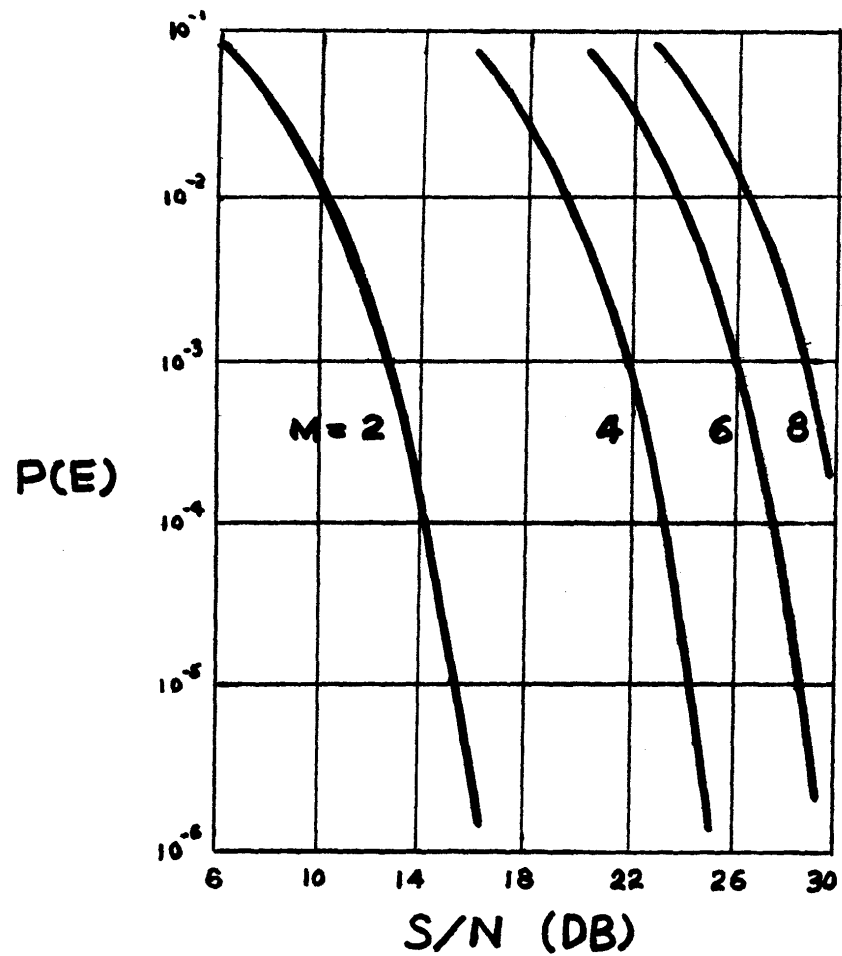
Using equations 94 and 98 for the polybipolar signal:

$$\frac{A^2}{\sigma^2} = \frac{S}{N} \ \frac{12(M-1)^2}{(M^2-2M+3)}$$

Substituting into equation 59, P(E) for the code one polybipolar signal is:

$$P(E) = \frac{2M-3}{2(M-1)} \ \text{erfc} \ \left(\frac{S}{N} \ \frac{3}{2(M^2-2M+3)} \right) \tag{101}$$

a. For Polybipolar Signals   b. For Polybinary Signals

FIGURE II   P(E) CURVES FOR UNCODED SIGNALS

For the polybinary signal, using equations 93 and 98:

$$\frac{A^2}{\sigma^2} = \frac{S}{N} \frac{6(M-1)^2}{(2M^2-4M+3)}$$

Substituting into equation 59, P(E) for the code one polybinary signal is:

$$P(E) = \frac{2M-3}{2(M-1)} \quad \text{erfc} \left(\frac{S}{N} \frac{3}{4(2M^2-4M+3)} \right) \tag{102}$$

The curves corresponding to these P(E) expressions for the code one signal are given in Figure 12 a and b.

3. Curves for the code two signal

For the polybipolar signal, using equations 97 and 98:

$$\frac{A^2}{\sigma^2} = \frac{S}{N} 4(M-1)$$

For the polybinary signal, using equations 95 and 98:

$$\frac{A^2}{\sigma^2} = \frac{S}{N} \frac{4(M-1)}{M}$$

Using equations 88 and 89, P(E) for the code two signal is:

$$P(E) = \frac{1}{2} + \sum_{h=0}^{M-1} (1/2)^{M-1} \binom{M-1}{h} Y_h$$
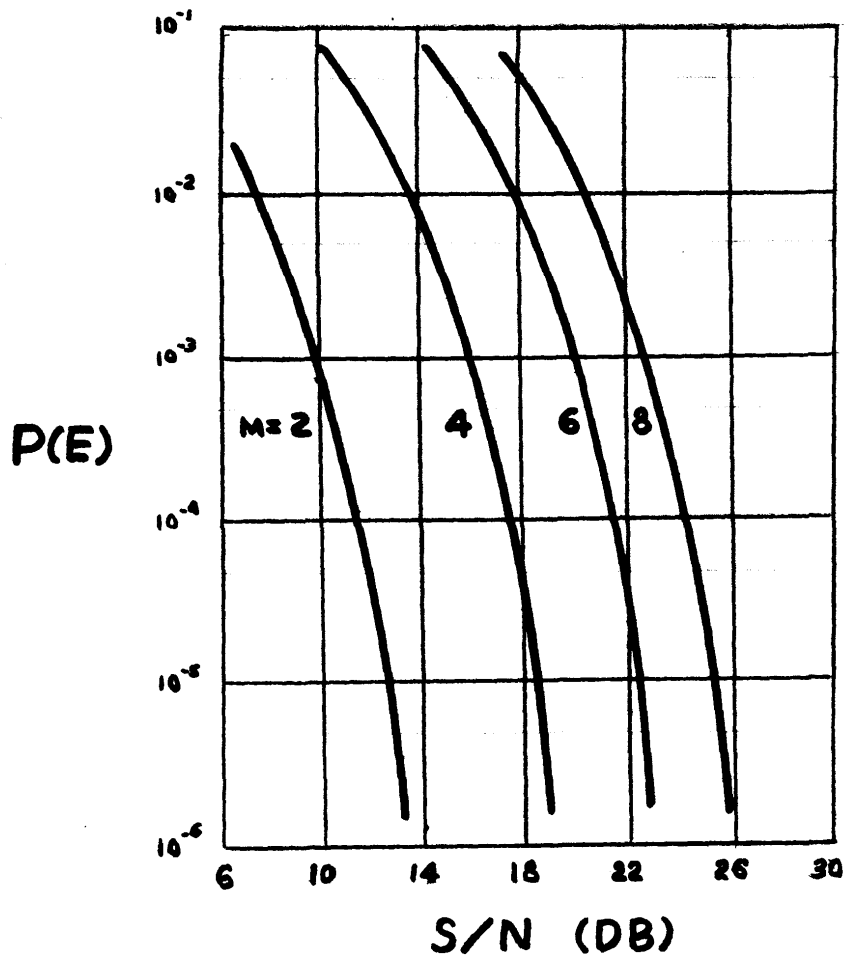
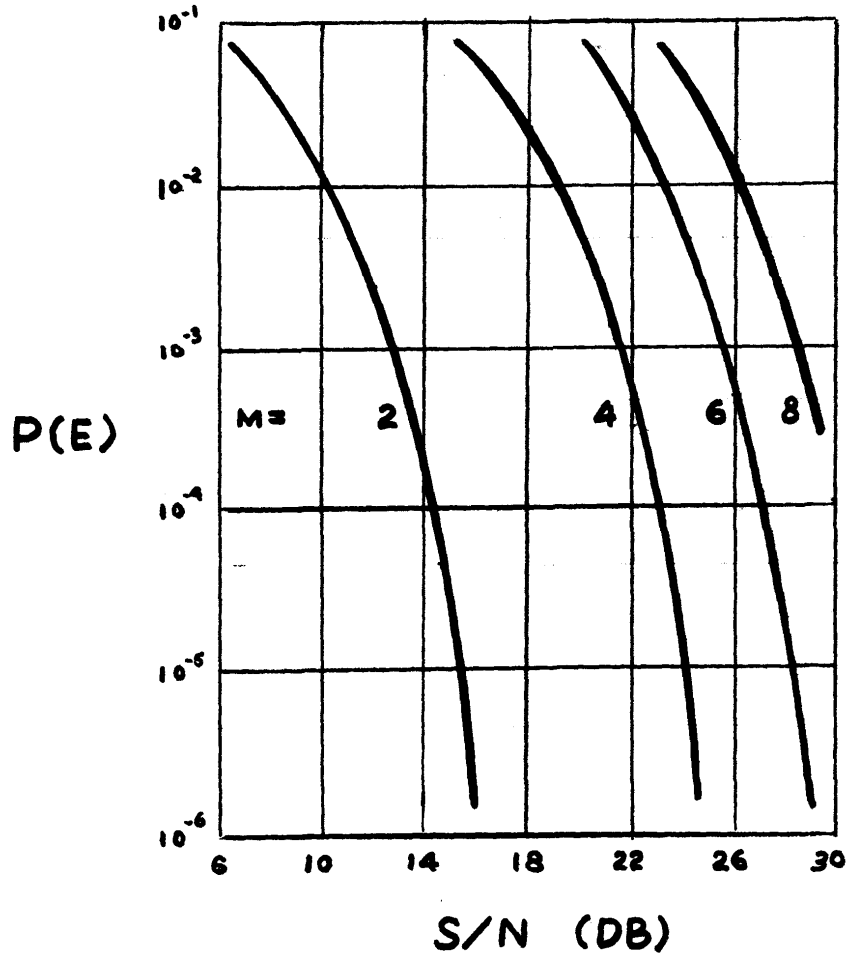where Y_h is defined in equation 89 and:

for polybipolar signal

$$X = \left(\frac{S}{N} \frac{1}{2(M-1)} \right)^{\frac{1}{2}}$$

for polybinary signal

$$X = \left(\frac{S}{N} \frac{1}{2M(M-1)} \right)^{\frac{1}{2}}$$

a. For Polybipolar Signals

b. For Polybinary Signals

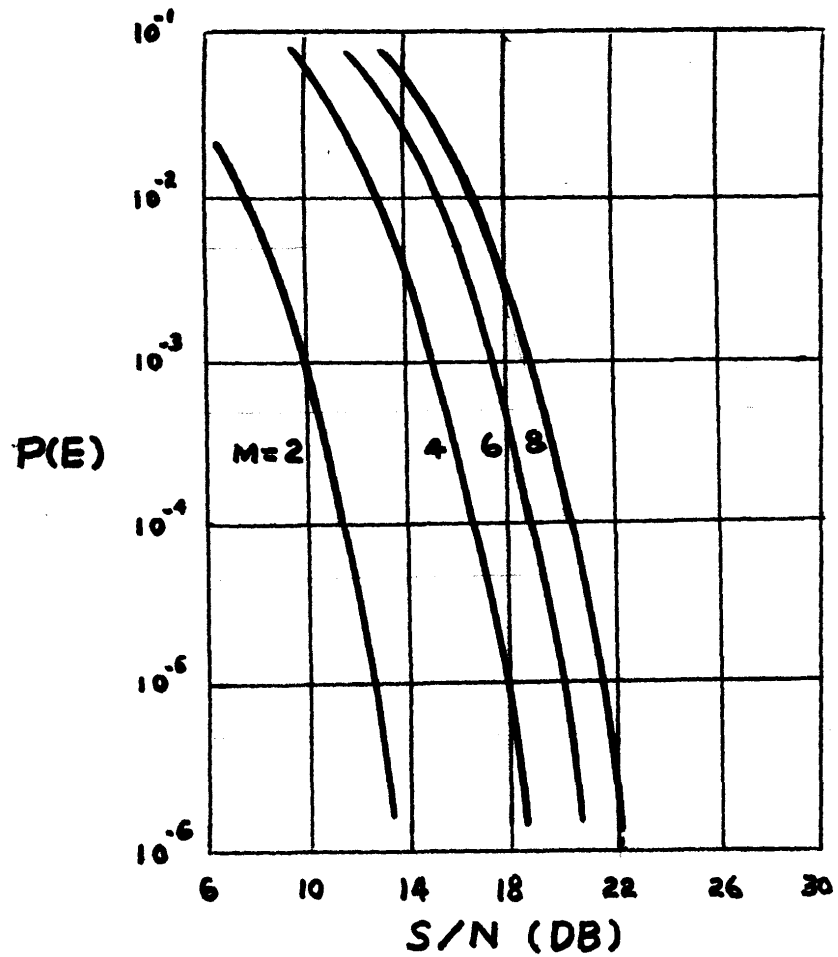FIGURE 12   P(E) CURVES FOR CODE ONE SIGNALS

The corresponding curves for P(E) of the code two signal are given in Figure 13 a and b.
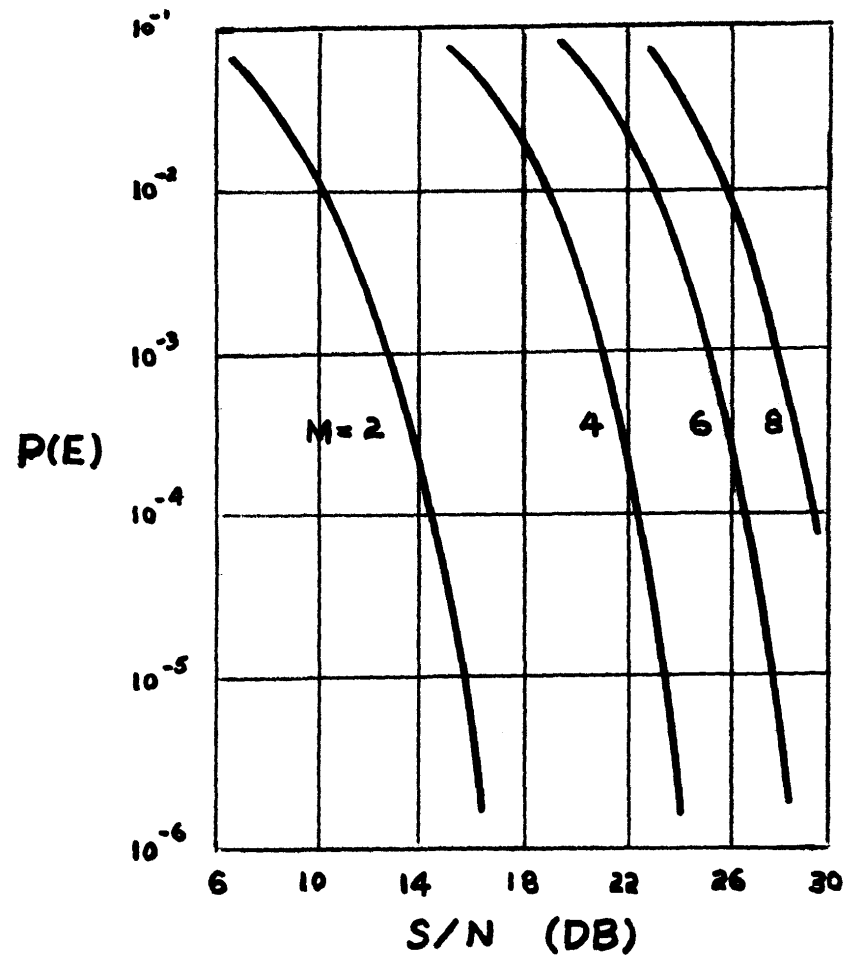
4. Comparative P(E) Curves

Comparative curves of the three systems for M = 4, 10 are given in Figures 14 and 15 with M = 2 also given for reference. Since the P(E) for the three systems are identical for the binary case, only one curve (that of the uncoded signal) is shown. As can be seen from Figure 15, in their polybinary form, the coded signals offer little improvement over the uncoded signal. In their polybipolar form as seen in Figure 14 however, the code two system shows a marked improvement over both the code one and uncoded systems for large M. The reason for this improvement is discussed in the next section.

D. The Polybinary versus Polybipolar Signal

Although derived P(E) expressions are identical for both polybinary and polybipolar signals[5], their corresponding P(E) curves are not. This difference is due to the dc component present in the polybinary average signal power expression. This dc component carries no signal information and, as the curves of Figures 11,12,13,14, and 15 indicate, requires the polybinary signal to have a higher S/N ratio for a given probability of error when compared to a polybipolar signal. Thus, the polybipolar signal is seen to have a useful advantage.

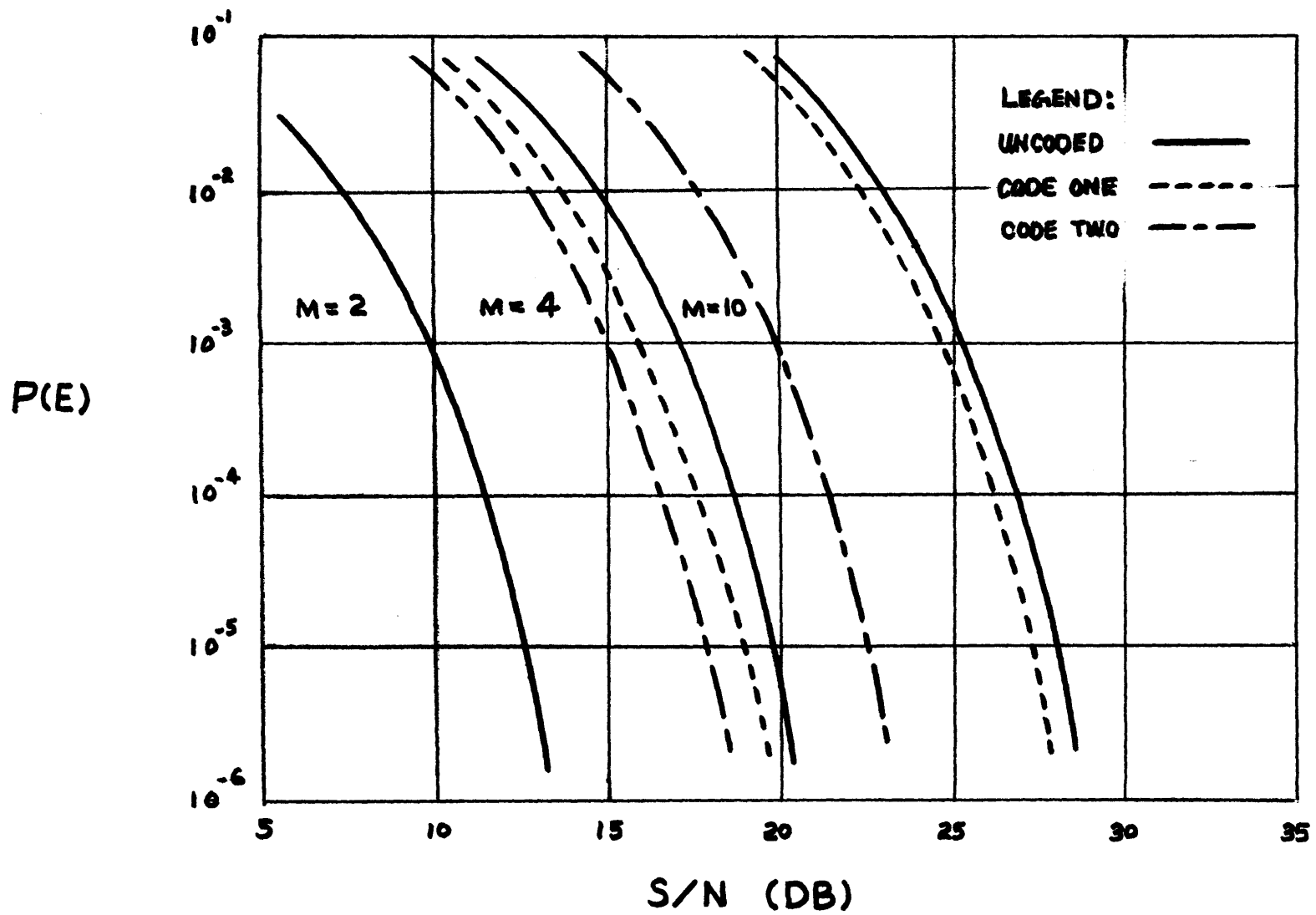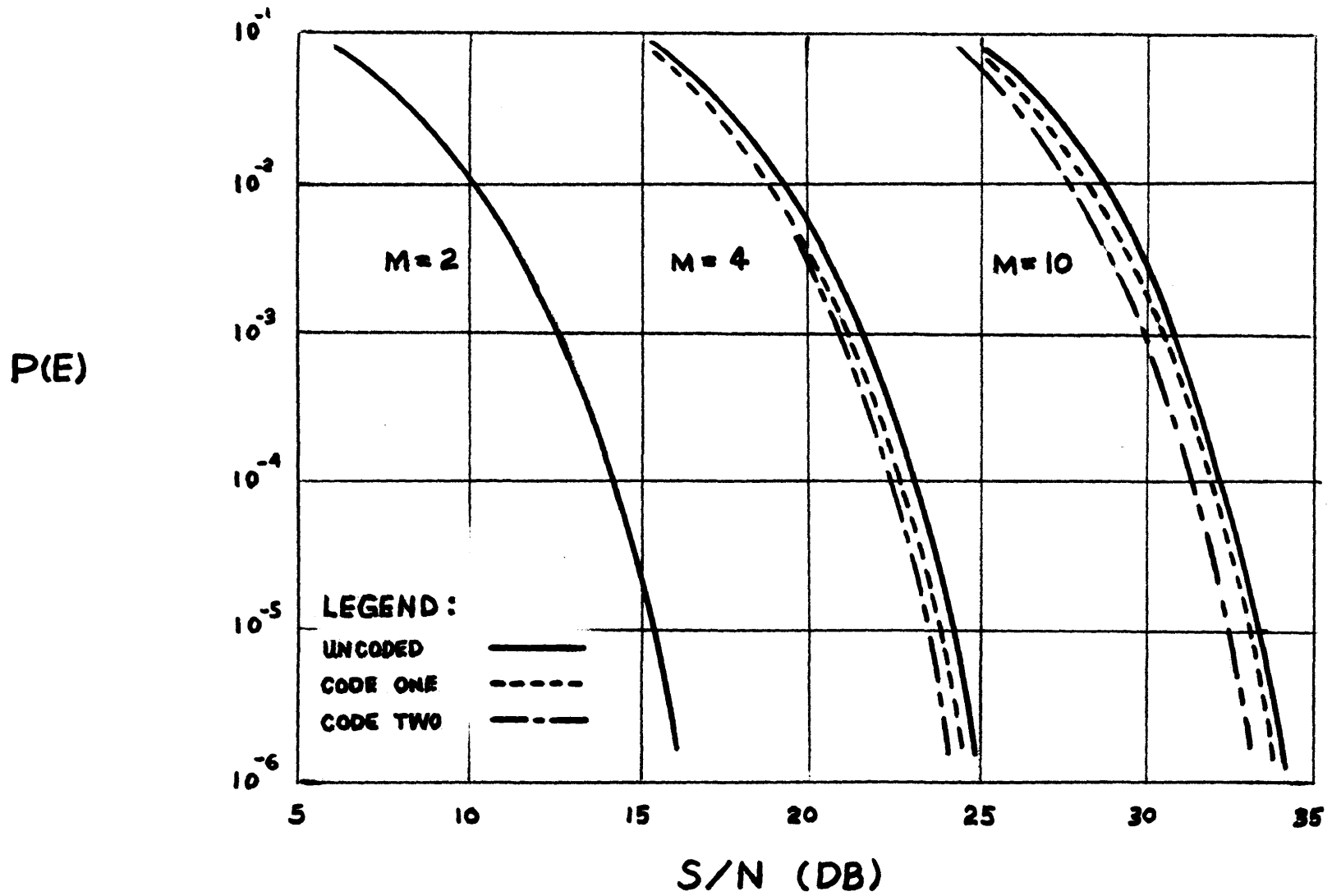a. For Polybipolar Signals

b. For Polybinary Signals

FIGURE 13   P(E) CURVES FOR CODE TWO SIGNALS

FIGURE 14   COMPARATIVE P(E) CURVES FOR POLYBIPOLAR SIGNALS

FIGURE 15   COMPARATIVE   P(E) CURVES FOR POLYBINARY SIGNALS

To investigate more fully the advantage of a polybi-
polar signal when discussing average signal power and
multi-level signals, consider the average signal power
expressions for the uncoded signal. From equation 91, for
the uncoded polybinary signal, let:

$$S = A^2 B \qquad \text{where:} \quad B = \frac{(2M-1)}{6(M-1)}$$

From equation 92, for the polybipolar signal, then:

$$S = A^2 C \qquad \text{where:} \quad C = \frac{(M+1)}{12(M-1)}$$

The following table compares these two multiplicative
terms B and C as functions of M.

M	B	C	Signal Power Differential in Decibels (B-C)
2	1/2	1/4	3.0
4	7/18	5/36	4.5
6	11/30	7/60	5.0
8	15/42	9/84	5.2
10	19/54	11/108	5.4
12	23/66	13/132	5.5

As can be seen, with increasing levels, the amount of
average power in a polybipolar signal decreases at a
faster rate than in a polybinary signal. This differential
approaches 6 db. for large M and, is reflected directly in
the P(E) curves for the uncoded signal in Figure 11.

A similar table can be formed for the code one
signal using the expressions for average signal power
found in equations 93 and 94.

M	Signal Power Differential in db.
2	3.0
4	5.4
6	5.8
8	5.9
10	5.9
12	6.0

For large M, the code one signal also approaches 6 db. as
the maximum signal power differential between polybinary
and polybipolar signals. The above values are also direct-
ly reflected in the corresponding curves of Figure 12.

Using equations 95 and 97, a similar table can be
constructed for the code two signal.

M	Signal Power Differential in db.
2	3.0
4	6.0
6	7.8
8	9.0
10	9.9
12	10.7

As the table for the code two signal indicates, the signal power differential continues to increase as M is increased. For M = 100 (a rather impractical case), the differential is 20 db. The differential values in the table are seen to correspond to the differences in the curves of Figure 13, and they fully account for the marked advantage shown for the code two polybipolar signal in Figure 14.

XI. CONCLUSIONS

A. Discussion

This study has illustrated that, after making numerous assumptions, comparative P(E) curves for various multi-level baseband digital systems can be presented. There are certain factors which should be kept in mind when making such comparisons.

For one thing, only baseband multi-level signals have been considered; therefore P(E) has been defined per symbol or pulse of the multi-level signal. In a data communication system using multi-level signals, each level frequently represents a specific binary sequence. (The code two system is an exception.) To determine P(E) for such a communication system, consideration must also be given to the specific assignments of binary codes to the various levels. The resulting P(E) would then be expressed per bit. In general, this is not a straightforward consideration. According to the literature[5], certain assumptions can be made which allow the use of the following approximation.

$$\frac{\text{average number of errors}}{\text{bit}} \approx \frac{P(E)}{(\log_2 M)}$$

If this approximation were used on the uncoded or code one signal however, the P(E) curves would not change appreciably due to the steepness of the curves.

The general utility of the P(E) expressions derived in this study is somewhat restricted by the assumption of additive gaussian noise. (This restriction limits the means of transmission of the the three baseband signals either to direct line or to those modulation methods which linearly translate the signal and noise back to baseband in the receiver.)

As mentioned by Shagena, Kvarda, and Lender, both code one and code two signals have interesting possibilities for providing error detecting functions. Further investigation into these possibilities should lead to a reduction in P(E) for the coded systems. It is somewhat of a paradox that these authors assume a simple threshold detector when discussing P(E) for their respective systems.

B. Summary

The analytical approach used to derive P(E) expressions for uncoded and coded multi-level digital signals has been presented. It has been shown that the derivation for a random uncoded multi-level signal is simply an extension of the derivation for a random binary signal. The same analytical approach can be used to determine P(E) expressions for both simple and complex coded multi-level signals. The development for coded signals is reasonably straightforward. However, the coding constraints require

the derivations to be more complicated and the resulting P(E) expressions more complex than was the case for random signals.

The transformation necessary to convert these P(E) expressions into comparative curves has also been presented. It has been shown that one of the two coded signals, in its polybipolar form, offers a distinct P(E) advantage as the number of levels in the signal are increased. These families of curves require many calculations and the use of a digital computer has been extremely helpful.

APPENDIX I

A convenient mathematical device for developing the
theory of probability as it applies to the outcomes of
experiments is the idea of a sample point and a sample
space.[12,13]

A.   DEFINITION

An event is simply a collection or "set" of sample
points; a simple event is comprised of only one
sample point. A set of sample points representing
the possible outcomes of an experiment is called the
sample space, or the event space of the experiment.

B.   DEFINITION

Two events are said to be mutually exclusive if the
occurrence of one precludes (and therefore excludes
from consideration) the other.

C.   DEFINITION

The union of two events is also an event. It is com-
prised of all the sample points which belong to either
or both of the two events. However, in forming the
union, no point is counted more than once. The
union of two events is symbolized by "A U B" where
A and B are two events.

D.   DEFINITION

The intersection of two events is also an event. It
is comprised of all points which are common to both

of the events from which it is formed. The inter-
section of two events is symbolized by "AB". Note
that this is not to be interpreted as the product
of A and B.

E. DEFINITION

The probability that an event A will occur is the
sum of the probabilities of the sample points that
are associated with the occurrence of A. Symboli-
cally, if P(A) denotes that the event A will occur
when the experiment is performed, then:

$$P(A) = \sum_A P_i \qquad (I-1)$$

where the sum is over the values of the probabilities
for all sample points corresponding to A.

F. THEOREM 1

Stated symbolically is:

$$P(A \cup B) = P(A) + P(B) - P(A,B) \qquad (I-2)$$

It frequently happens that the event A and the event
B have no sample points in common. When this happens,
the events A and B are said to be mutually exclusive.
Then:

$$P(A \cup B) = P(A) + P(B) \qquad (I-3)$$

G. THEOREM 2

Stated symbolically is:

$$P(A,B) = P(A) \ P(B|A) = P(B) \ P(A|B) \qquad (I-4)$$

where $P(A,B)$ is read as the probability of the
joint event AB.

H. DEFINITION

Conditional Probability stated symbolically is:

$$P(B|A) = P(A,B)/P(A) \qquad (I-5)$$

$$P(A|B) = P(B,A)/P(B) \qquad (I-6)$$

where $P(A|B)$ is read as the probability of the event
A given the knowledge of the occurrence of event B.

APPENDIX II

In general, a random signal or process or variable is one whose value at any given time is a function of chance[12,14,15]. There are a number of ways a random signal can be described. This discussion will be limited to the amplitude domain.

A. PROBABILITY DENSITY FUNCTION

There are two ways a random signal e(t) can be characterized in the amplitude domain. First, the probability density function can be defined as:

$$p(x) = \lim_{\Delta x \to 0} P\left[\frac{(x-\Delta x/2) \leq e(t) \leq (x+\Delta x/2)}{\Delta x}\right] \qquad (II-1)$$

Equation II-1 states the probability of finding a random signal e(t) in the small voltage interval Δx around a certain voltage x. See Figure 16.
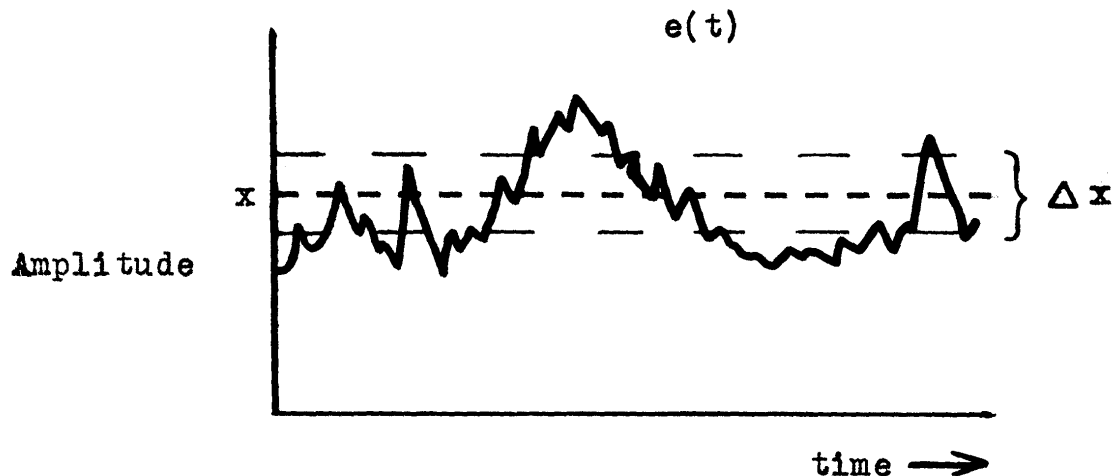

FIGURE 16   WAVEFORM OF A RANDOM SIGNAL

APPENDIX II

Theoretically,Δ x can be made infinitesimal, however,
in practice Δ x is made small in comparison with the
signal amplitude range.

Probability density functions assume many varied
forms or shapes. Perhaps, one of the most important
forms is the gaussian or normal density function which
is commonly used to characterize random noise. If N_1, N_2,
N_3, - - - - N_k are k independent random variables, each
distributed according to a given probability density
function; then the probability density function n(t),
(where n(t) = N_1 + N_2 + N_3 + - - + N_k) will approach the
gaussian density function for large k. Stated symbolical-
ly, the gaussian density function is:

$$p\left[n(t)\right] = \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\left[n(t)-N_0\right]^2/2\sigma^2} \qquad (II\text{-}2)$$

where: N_0 = average value = 0, for gaussian random noise.

σ^2 = mean square noise power of random noise

(on 1 ohm resistor).

therefore:

$$p\left[n(t)\right] = \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\left[n(t)\right]^2/2\sigma^2} \qquad (II\text{-}3)$$

APPENDIX II

It should be emphasized that p(x) is a density function and must be integrated over a finite range of values in order to yield a probability. Thus:

$$\int_{-\infty}^{-\infty} p(x) \, dx = 1 \tag{II-4}$$

B. PROBABILITY DISTRIBUTION FUNCTION

The second way a random signal can be characterized in the amplitude domain is by defining the probability distribution function as:

$$P\left[x_1\right] = P\left[e(t) \leq x_1\right] = \int_{-\infty}^{x_1} p(x) \, dx \tag{II-5}$$

This says the probability that the random signal $e(t)$ assumes a value less than or equal to some given value (e.g., x_1), is found with equation II-5. This expression, together with the probability density function of gaussian random noise (equation II-3), provides the necessary tools for calculating error probabilities.

BIBLIOGRAPHY

1. LENDER, A. (1966) Correlative Level Coding for Binary Data Transmission. I.E.E.E. Spectrum, February, Vol. 3, No. 2, p 104-115.

2. PIERCE, J.R. (1966) Information Rate of a Coaxial Cable with Various Modulation Systems. Bell Sys. Tech. Jour., Vol. 65, p 1197-1207.

3. SLEPIAN, D. (1963) Bounds On Communication. Bell Sys. Tech. Jour., Vol. 42, p 681-707.

4. LENDER, A. (1964) Correlative Digital Communication Techniques. I.E.E.E. Int. Conv. Record, Vol. 12, Pt. 5, p 45-53, March.

5. BENNETT, W.R. and J.R. DAVEY (1965) Data Transmission. McGraw-Hill, New York, p 98-118.

6. SCHWARTZ, M. (1959) Information, Transmission, Modulation, and Noise. McGraw-Hill, New York, p 197-199 and p 381-390.

7. DAVENPORT JR., W.B. and W.L. ROOT (1958) An Introduction to the Theory of Random Signals and Noise. McGraw-Hill, New York, p 154-157.

8. ABRAMSON, N. (1963) Information Theory and Coding. McGraw-Hill, New York, p 99-101 and p 150.

9. SHAGENA, J.L. and J.C. KVARDA (1964) A New Multi-level Coding Technique for Digital Communications. Pro. 1964 Int. Conv. Mil. Elec., p 326-331.

10. LENDER, A. (1963) The Duobinary Technique for High Speed Data Transmission. I.E.E.E. Trans. on Com. and Elec., Vol. 82, May, p 214-218.

11. LENDER, A. (1963) Faster Digital Communications with Duobinary Techniques. Electronics, March, p 61-65.

12. BETTEN, J.R. (1965) Notes On Random Signals. Lecture notes.

13. HOEL, W. Introduction to Mathematical Statistics. Wiley, New York, p 4-20.

14. AUTHOR UNKNOWN (1966) Analysis Techniques for Random Signals. Noratom-NEWS, January, p 2-3.

15. HANCOCK, J.C. (1961) An Introduction to the Principles of Communication Theory. McGraw-Hill, New York, p 89-93 and p 120.

16. ABRAMOWITZ, M. and I.A. STEGUN (1965) Handbook of Mathematical Functions. National Bureau of Standards, Washington D.C., p 297-299.

17. CONTE, S. (1965) Elementary Numerical Analysis. McGraw-Hill, New York, p 83.

18. SHANNON, C.E. (1948) A Mathematical Theory of Communication. Bell Sys. Tech. Jour., Vol. 27, p 379-423 and p 623-656.

19. OLIVER, B., PIERCE, J. and C. SHANNON (1948) The Philosophy of PCM. Proc. I.R.E., Nov., p 1324-1331.

VITA

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He was a communications engineer in the Research Division, Zenith Radio Corporation, Chicago, Illinois for five years. He has authored papers published in the I.E.E.E. Transactions on Broadcast and Television Receivers and the Journal of the Audio Engineering Society. He is a member of I.E.E.E. and Eta Kappa Nu.

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