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Transient response of a vibration isolation system

Hemendra Shantilal Acharya

TRANSIENT RESPONSE OF A VIBRATION ISOLATION SYSTEM

BY

HEMENDRA SHANTILAL ACHARYA, 1943-

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THESIS

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Wm S Gatley (advisor) R. D. Rocke

Floyd M. Cunningham

ABSTRACT

A practical procedure for investigating the performance of a vibration isolation system under transient conditions is presented. For this investigation, an induction motor with an unbalanced rotor is studied during the period when it accelerates to its operating speed from rest.

Using Newton's second law of motion, equations of motion are derived, first neglecting and then considering the effect of "inertia torque". This torque is produced by the inertia force resulting from vertical acceleration of the unbalanced mass. The equations are solved on a digital computer using the Runge-Kutta method of order 4. The results obtained are compared with those obtained using Simpson's and Runge-Kutta methods of order 4 of the Continuous System Modeling Program. For the case when there is no external load, an attempt was made to obtain the responses of the system by the Convolution Integral Solution of the K. A. Foss method.

A study of steady state and transient analyses for "inertia" and "no inertia" cases is carried out. From the results obtained, graphs are plotted and guidelines useful for design of vibration isolators are given.

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I. INTRODUCTION

In this thesis, a practical procedure is developed for vibration isolation analysis during the transient period.

Generally, most machines and structures experience vibration for two reasons: (1) due to changes in the relative positions of the elements and (2) due to forces generated within the structures involved. Our discussion is restricted to the latter case.

Shaking forces produced during the operation of machinery can be categorized as follows:

1. Forces due to the inertia of unbalanced rotating and reciprocating members.
2. Forces due to operation of the machinery itself.

All of these forces may be transmitted to the structure upon which the machine is mounted. The supporting structure thus experiences vibration that depends upon the nature of the forces and on the characteristics of the structure. These vibrations may affect the operation of other machinery mounted on the same structure, or in some instances, may cause structural failures due to cyclic fatigue.

To avoid such harmful effects, it becomes necessary to eliminate or isolate the forces producing the unwanted vibration. This can be achieved in two ways: (1) by dynamic balancing of the forces which cause the vibration and (2) by isolating the support from such forces. It has been found that dynamic balancing of forces is not practical in many cases. In its simplest form, isolating a machine or particular component means mounting the machine or component upon properly designed isolators, so that forces transmitted to the supporting structure are minimized.

Often the vibration is caused by an oscillatory force which exists at a constant frequency. This condition is designated "steady state", because an identical pattern of vibration amplitude is repeated during each cycle. Sometimes, however, one observes a different pattern of vibration which is not periodic. This condition is usually designated "transient" and is produced by a suddenly applied force or by a force which changes with time.

In the steady state case when the exciting force is harmonic, the steady state vibration takes place at the frequency of the excitation. During the "transient" period, additional vibrations at one or more of the resonant frequencies may be superimposed upon the vibration at the excitation frequency. If the forcing frequency varies with time, dangerously large amplitudes may result when the excitation frequency approaches one of the resonant frequencies of the system.

To investigate the performance of vibration isolation systems during the transient period, a system in which an electric motor is mounted on isolators is studied. The supporting structure (floor) is represented by a mass and spring combination and the isolator is represented by a linear spring in parallel with a viscous damper. Excitation of the system is caused by unbalanced rotating masses within the motor rotor or the machinery driven by the motor.

In the following sections, the equations of motion for the system are derived and solved by first neglecting and then considering the effect of inertia torque produced by the vertical acceleration of unbalanced mass.

II. REVIEW OF LITERATURE

Vibration isolation of machinery is treated to some extent in almost every text or reference dealing with vibration analysis. In addition, numerous technical papers on the general subject of vibration isolation have been published in recent years.

S. Timoshenko and D. H. Young [1]* describe the theory of free and forced vibrations of conservative systems, giving special attention to the theory of vibration isolation.

Paul A. Crafton [7] has discussed the isolation of a machine from steady state sinusoidal components of motion. The machine has an independent force acting on it and the foundation has a motion that is independent of the force input. Crafton has assumed that the force and motion input functions are sinusoidal with time. He has also described a feedback system for isolation from discontinuous inputs. He has achieved isolation as far as the steady state component of motion of the machine is concerned, for cases of with and without damping.

R. T. Lowe [8] has discussed control of vibration through isolation of forces or motions. He has discussed only undamped vibration. For this, he uses a graph of the fraction of force and displacement transmitted by the system (i.e., transmissibility) versus the ratio of the forcing frequency to the undamped natural frequency to select a particular frequency ratio for which the value of transmissibility is less than unity. This frequency ratio is used to

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*Numbers in brackets refer to list of references at end of thesis.

determine the system parameters.

R. Plunkett [9], by citing examples of a turbine rotor and an automatic washing machine, has analyzed steady state vibration. He defines the steady state dynamic characteristics of a given system in terms of mechanical impedence, which is the ratio of an applied sinusoidal force to the resulting vibration velocity. The term mobility is defined as the inverse of mechanical impedence. By simple analysis, Plunkett derives the transmissibility ratio in terms of mobility and shows that to have effective isolation, the isolator should have high mobility compared to the mobilities of the machine and foundation. He has discussed steady state isolation with viscous damping.

J. C. Snowdon [10] presents an analysis of natural or synthetic rubbers used as damped resilient springs between an absorber mass and the principal mass. He has studied the steady state case with viscous damping.

G. J. Andrews' paper [11] is mainly concerned with the rigid-body-on-resilient-mounts problem. The author has derived the equations of motion for a rigid body of arbitrary shape, supported at three or more noncolinear points by resilient mounts which have damping. His solution (for the steady state case) is given in programs suitable for solution by a high speed digital computer.

J. E. Ruzica and R. D. Cavanaugh [12] have described an elastically supported damper system, which eliminates the damping force at high frequencies. They have plotted absolute and relative transmissibility values for zero and infinite damping. Optimum damping is determined by differentiating the transmissibility equation with respect to frequency ratio and equating the result to zero.

Hanley's paper [13] explains the dynamics underlying displacement-excited motions and provides curves whereby the dynamic forces can be calculated.

Carter and Liu [14] have analyzed a dynamic vibration absorber for the case where both the main and absorber springs have nonlinearities. A one term approximation solution is assumed for the motion of the two masses and the resulting amplitude equation is solved using a graphical procedure.

K. A. Foss [15] has developed a method for solving non-classically damped multi-degree of freedom systems. He transforms the original system into 2N space in order to uncouple the equations of motion (see chapter IVB). The solution is in matrix form.

B. B. Patel [16] has developed computer programs for solving vibration problems by the Foss method. These programs give eigenvalues, eigenvectors and the transient response of the system. Patel has also obtained the complete response of the system subjected to a sinusoidal force. His programs can be used for any system provided the Convolution Integral is evaluated by hand. In this thesis, the complex Convolution Integral solution is obtained as computer output, after the integral has been separated into real and imaginary parts.

It was found from the literature surveyed that relatively little research has been published concerning the transient response of vibration isolation systems. However, in certain systems it is necessary to analyze the transient response in order to avoid potentially harmful or annoying effects of large amplitudes, which may occur during the period when unbalanced rotating machine elements are being accelerated to their operating speeds.

III. DERIVATION OF EQUATIONS OF MOTION

A. Description of System

The system analyzed in this thesis consists of an electric motor isolated from a resilient supporting structure by pads whose behavior is approximated by a spring-viscous damper combination. A model of the system is shown in Figure 1, where

M_1 = Effective mass of floor, lb. sec.^{2} per in.

M_2 = Mass of motor, lb. sec.^{2} per in.

m = Equivalent unbalanced mass located at radius R from centerline of rotor, lb. sec.^{2} per in.

K_1 = Equivalent stiffness of floor, lb. per in.

K_2 = Combined stiffness of vibration isolators, lb. per in.

C = Equivalent viscous damping coefficient of vibration isolators, lb. per in. per sec.

Several simplifying assumptions have been made to reduce the complexity of the equations of motion. However, the simplified model selected for analysis is a reasonable approximation to many vibration isolation problems encountered in practice. Furthermore, the conclusions drawn from analysis of the simplified system are valid for more complex systems. The assumptions are:

1. Masses representing the machinery and floor are rigid.
2. Each isolator can be represented by an ideal massless spring in parallel with a viscous damper.
3. Masses representing machinery and floor are constrained to have translation motion in the vertical direction only.
4. The building structure is rigid.

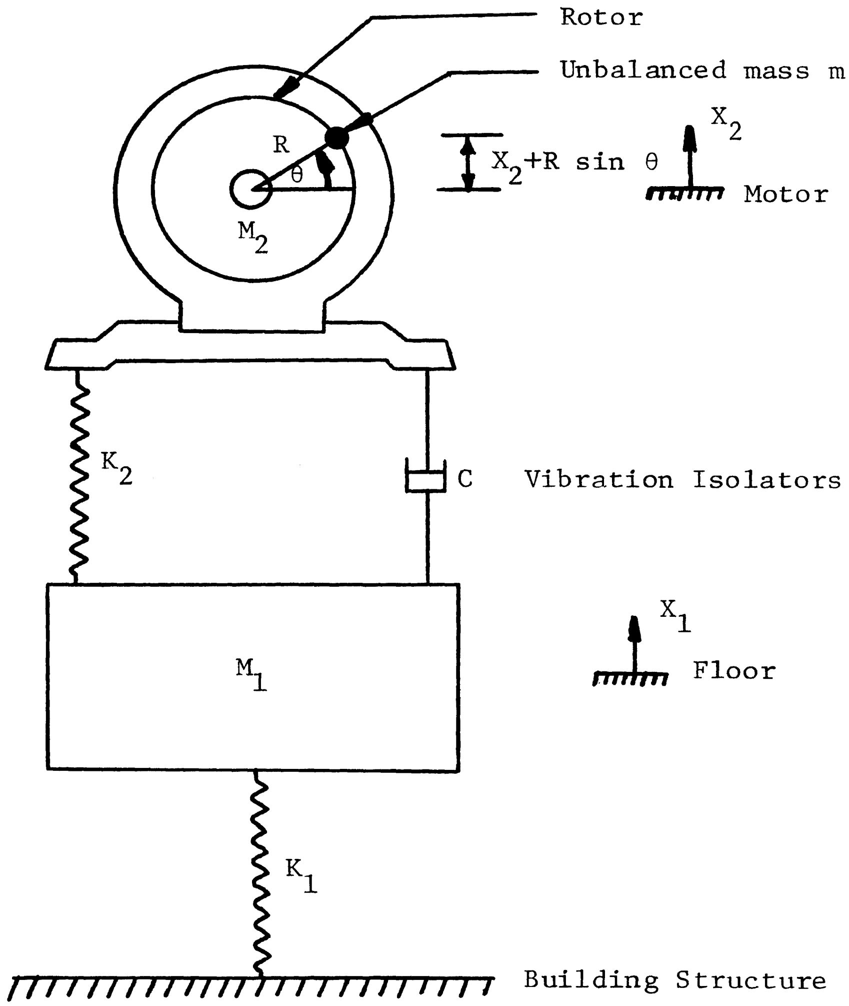

Figure 1. Schematic Diagram of System

B. Derivation of Equations Neglecting Torque due to Vertical Acceleration of Unbalanced Mass

The equations of linear motion for the system can be derived by using Newton's second law of motion, which states that the time rate of change of linear momentum in any direction is equal to the external force applied in that direction. In this case the external force is produced by acceleration of the rotor from rest.

The angular velocity of a typical induction motor (with no external load) increases exponentially from rest to its final value, as described by the following function [17]:

$$\omega = \omega_o(1-e^{-t/t_o}), \tag{1}$$

where

ω = Angular velocity of rotor at any time t, rad. per sec.

ω_o = Maximum (steady state) angular velocity of rotor, rad. per sec.

t_o = "Mechanical" time constant, sec.

If the voltage applied to a motor increases in proportion to time, such as occurs when accelerating an adjustable-voltage drive system, the speed versus time relationship is

$$\omega = \frac{k_a}{k_v}[t-t_o(1-e^{-t/t_o})], \tag{2}$$

where

k_a = Constant denoting increase in applied voltage, v per sec.

k_v = Motor voltage-rotation constant, v-sec. per rad.

Inspection of equations (1) and (2) shows that an infinite time must elapse before the transient condition completely subsides, and steady state operation occurs. In most practical cases, however, any transient component will decrease to negligible proportions after a few seconds.

In this system, vibration isolation mounts are provided between the motor and floor to reduce transmission of vertical shaking forces to the floor. The shaking forces are produced by rotation of the unbalanced mass, m. Free body diagrams for the system are shown in Figure 2.

Using Newton's second law of motion, the equations of motion are:

$$M_1\ddot{X}_1 = - K_1X_1 + K_2(X_2-X_1) + C(\dot{X}_2-\dot{X}_1) \tag{3}$$

$$(M_2-m)\ddot{X}_2+m\frac{d^2}{dt^2}(X_2+R\sin\theta) = -K_2(X_2-X_1) - C(\dot{X}_2-\dot{X}_1), \tag{4}$$

where

X_1 = Displacement of floor at any time t, in.

X_2 = Displacement of motor at any time t, in.

θ = Angular displacement of rotor at any time t, rad.

Rearranging equation (4), we get

$$M_2\ddot{X}_2 + mR(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta) = -K_2(X_2-X_1) - C(\dot{X}_2-\dot{X}_1) \tag{5}$$

We can write equations (3) and (5) as follows:

$$M_1\ddot{X}_1 + C\dot{X}_1 - C\dot{X}_2 + (K_1+K_2)X_1 - K_2X_2 = 0 \tag{6}$$

$$M_2\ddot{X}_2 - C\dot{X}_1 + C\dot{X}_2 - K_2X_1 + K_2X_2 = -mR(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta). \tag{7}$$

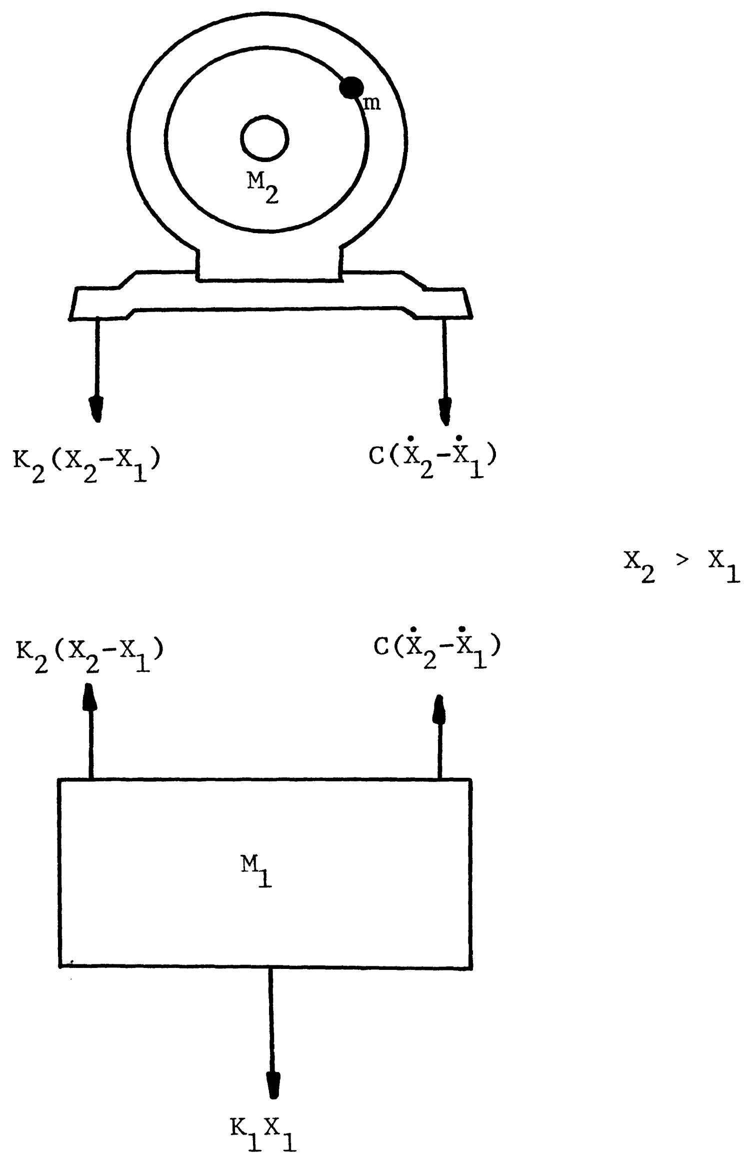

Figure 2. Free Body Diagram of Forces Acting on System

The term on the right hand side of equation (7) represents the shaking force produced by rotation of the unbalanced mass m, which is located at the distance R from the centerline of the rotor. Let us describe this force by the term f(t).

For a motor without an external load, the angular position of the rotor at any time can be obtained from equation (1) or (2). However, when the motor is coupled to an external load, $\theta(t)$ must be obtained by solving the equation

$$T_o = I_o \ddot{\theta}, \tag{8}$$

where

I_o = Moment of inertia of the rotor about center of rotation excluding unbalanced mass m.

T_o = Driving (electromotive) torque less the total load torque about center of rotation.

In this thesis, T_o is specified as a function of ω (see Appendix B). Use of this relation in equation (8) permits the latter to be solved for $\theta(t)$, which is inserted in equation (7). Equations (6) and (7) can then be solved on a digital computer for the displacements $X_1(t)$, $X_2(t)$ and $\theta(t)$ as discussed in Chapter IV.

Equations (6) and (7) can be written in matrix form as follows:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} C & -C \\ -C & C \end{bmatrix} \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} \tag{9}$$

Let us define

$$[M] = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

$$[C] = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}$$

$$\{X\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\{f(t)\} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

Now we can write equation (9) as follows:

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{f(t)\} \tag{10}$$

C. <u>Derivation of Equations Considering Torque due to Vertical Acceleration of Unbalanced Mass</u>

Figure 3(a) shows the shaking forces and Figure 3(b) shows the torques acting on the system when we take into consideration the effect of "inertia torque" created by vertical acceleration $\ddot{X}_2$ of the unbalanced mass m. The net torque available for accelerating the rotor is thus the electromotive torque less the total load torque, which includes both the external load torque and the "inertia torque".

The equations of motion considering the effect of "inertia

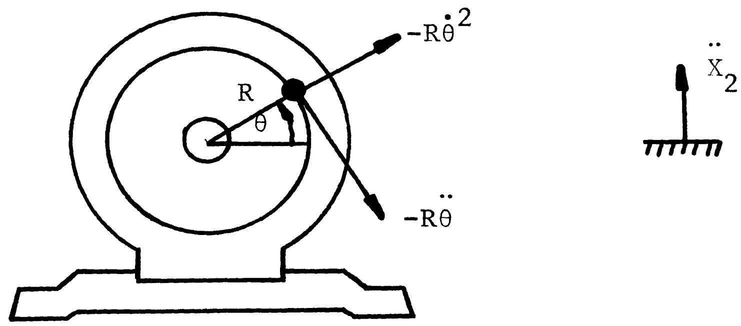

Figure 3(a)

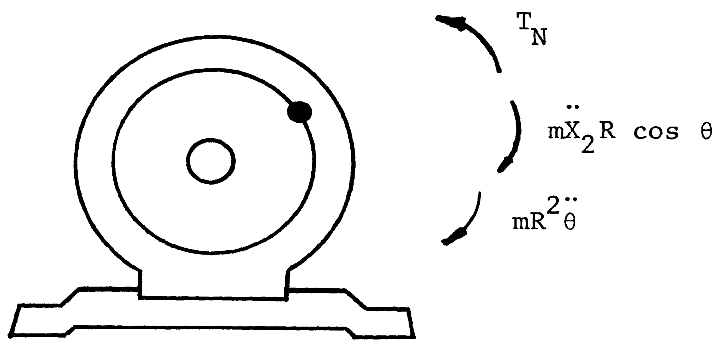

Figure 3(b)

Figure 3. Accelerations and Torques Acting on System

torque" are:

$$M_1\ddot{X}_1 = -K_1X_1 + K_2(X_2-X_1) + C(\dot{X}_2-\dot{X}_1) \tag{11}$$

$$(M_2-m)\ddot{X}_2 + m\frac{d^2}{dt^2}(X_2+R\sin\theta) = -K_2(X_2-X_1) - C(\dot{X}_2-\dot{X}_1) \tag{12}$$

$$I_o\ddot{\theta} = -mR^2\ddot{\theta} - m\ddot{X}_2R\cos\theta + T_N, \tag{13}$$

where T_N is obtained as shown in Appendix (B).

Equations (11) and (12) are identical to equations (3) and (4). However, equation (13) now includes the torque due to vertical acceleration $\ddot{X}_2$ of the motor and is no longer uncoupled from the equations describing vertical motion of the system.

IV. NUMERICAL SOLUTION OF EQUATIONS OF MOTION

A. Runge-Kutta Method

The purpose of the Runge-Kutta method is to provide an approximate means for integrating a system of first order ordinary differential equations with given initial values. It is a fourth order integration procedure which is stable and self starting, that is, only the functional values at a single previous point are required to obtain succeeding functional values. It requires four derivative evaluations per step.

The system of first order ordinary differential equations appears as follows:

$$Y' = \frac{dY}{dX} = F(X,Y)$$

where $Y(X_o) = Y_o$ is the initial condition.

Thus, starting at X_o with $Y(X_o) = Y_o$, resulting vector $Y_4 = Y(X_o+h)$ is computed by the following formulas:

$$K_1 = h \cdot F(X_o, Y_o)$$

$$K_2 = h \cdot F(X_o+h/2, Y_o+K_{1/2})$$

$$K_3 = h \cdot F(X_o+h/2, Y_o+K_{2/2})$$

$$K_4 = h \cdot F(X_o+h, Y_o+K_3)$$

where h is the step size.

The vector Y_4 is calculated as follows:

$$Y_1 = Y_0 + 1/2(K_1 - 2Q_0)^*$$

$$Q_1 = Q_0 + 3[1/2(K_1 - 2Q_0)] - 1/2K_1$$

$$Y_2 = Y_1 + (1-\sqrt{1/2})(K_2 - Q_1)$$

$$Q_2 = Q_1 + 3[(1-\sqrt{1/2})(K_2 - Q_1)] - (1-\sqrt{1/2})K_2$$

$$Y_3 = Y_2 + (1-\sqrt{1/2})(K_3 - Q_2)$$

$$Q_3 = Q_2 + 3[(1+\sqrt{1/2})(K_3 - Q_2)] - (1+\sqrt{1/2})K_3$$

$$Y_4 = Y_3 + 1/6(K_4 - 2Q_3)$$

$$Q_4 = Q_3 + 3[1/6(K_4 - 2Q_3)] - 1/2K_4$$

The error of the Runge-Kutta method of order 4 due to truncation is of the order $(h)^5$[18].

As our equations of motion are of the second order it is necessary to convert them to an equivalent set of first order equations before applying this method.

This is done as follows:

$$\ddot{X}_1 = \frac{d^2X_1}{dt^2};$$

$$\ddot{X}_2 = \frac{d^2X_2}{dt^2};$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2};$$

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*Quantities Q_i are used for determining the roundoff error in vectors Y_i. The initial value Q_0 is zero.

are converted as

$$\ddot{X}_1 = \frac{d\dot{X}_1}{dt} \quad \text{and} \quad \dot{X}_1 = \frac{dX_1}{dt} ;$$

$$\ddot{X}_2 = \frac{d\dot{X}_2}{dt} \quad \text{and} \quad \dot{X}_2 = \frac{dX_2}{dt} ;$$

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} \quad \text{and} \quad \dot{\theta} = \frac{d\theta}{dt} ;$$

Thus, we obtain six first order equations from three second order equations. In our problem the six first order equations are obtained as follows:

From equation (11)

$$\ddot{X}_1 = -\frac{(K_1+K_2)}{M_1} X_1 + \frac{K_2}{M_1} X_2 + \frac{C}{M_1} (\dot{X}_2-\dot{X}_1) \tag{14}$$

From equation (12)

$$\ddot{X}_2 = -\frac{mR}{M_2} \ddot{\theta} \cos\theta + \frac{mR}{M_2} \dot{\theta}^2 \sin\theta - \frac{K_2}{M_2} (X_2-X_1) - \frac{C}{M_2} (\dot{X}_2-\dot{X}_1) \tag{15}$$

From equation (13)

$$\ddot{\theta} = \frac{-m\ddot{X}_2 R \cos\theta + T_N}{(I_o+mR^2)} \tag{16}$$

Substituting equation (16) into (15) we get

$$\ddot{X}_2 = -\frac{mR}{M_2} \left[\frac{-m\ddot{X}_2 R \cos\theta + T_N}{(I_o+mR^2)} \right] \cos\theta + \frac{mR}{M_2} \dot{\theta}^2 \sin\theta - \frac{K_2}{M_2}(X_2-X_1)$$

$$- \frac{C}{M_2} (\dot{X}_2-\dot{X}_1) \tag{17}$$

or

$$\frac{M_2(I_o+mR^2)-m^2R^2\cos^2\theta}{M_2(I_o+mR^2)}\ddot{X}_2 = -\frac{mRT_N\cos\theta}{M_2(I_o+mR^2)} - \frac{K_2}{M_2}(X_2-X_1)$$

$$-\frac{C}{M_2}(\dot{X}_2-\dot{X}_1) + \frac{mR}{M_2}\dot{\theta}^2\sin\theta$$

Let us define

$$DEN = M_2(I_o+mR^2) - m^2R^2\cos^2\theta.$$

Thus, the six equations are:

1. $$\frac{dX_1}{dt} = \dot{X}_1 \tag{18}$$

2. $$\frac{d\dot{X}_1}{dt} = -\frac{(K_1+K_2)}{M_1}X_1 + \frac{K_2}{M_1}X_2 + \frac{C}{M_1}(\dot{X}_2-\dot{X}_1) \tag{19}$$

3. $$\frac{dX_2}{dt} = \dot{X}_2 \tag{20}$$

4. $$\frac{d\dot{X}_2}{dt} = -\frac{mRT_N\cos\theta}{DEN} - \frac{K_2(I_o+mR^2)}{DEN}(X_2-X_1) - \frac{C(I_o+mR^2)}{DEN}(\dot{X}_2-\dot{X}_1)$$
$$+\frac{mR(I_o+mR^2)}{M_2\cdot DEN}\dot{\theta}^2\sin\theta \tag{21}$$

5. $$\frac{d\theta}{dt} = \dot{\theta} \tag{22}$$

6. $$\frac{d\dot{\theta}}{dt} = \frac{-mR\cos\theta\,\ddot{X}_2 + T_N}{I_o+mR^2} \tag{23}$$

The initial conditions used are zero displacements and velocity for $X_1(t)$, $X_2(t)$ and $\theta(t)$. For the computer program the responses are defined as follows:

$$Y(1) = X_1$$

$$Y(2) = \frac{dX_1}{dt}$$

$$Y(3) = X_2$$

$$Y(4) = \frac{dX_2}{dt}$$

$$Y(5) = \theta$$

$$Y(6) = \frac{d\theta}{dt}$$

The derivatives (DER) of above are defined as:

DER Y(1) = Y(2)

DER Y(2) = Equation (19)

DER Y(3) = Y(4)

DER Y(4) = Equation (21)

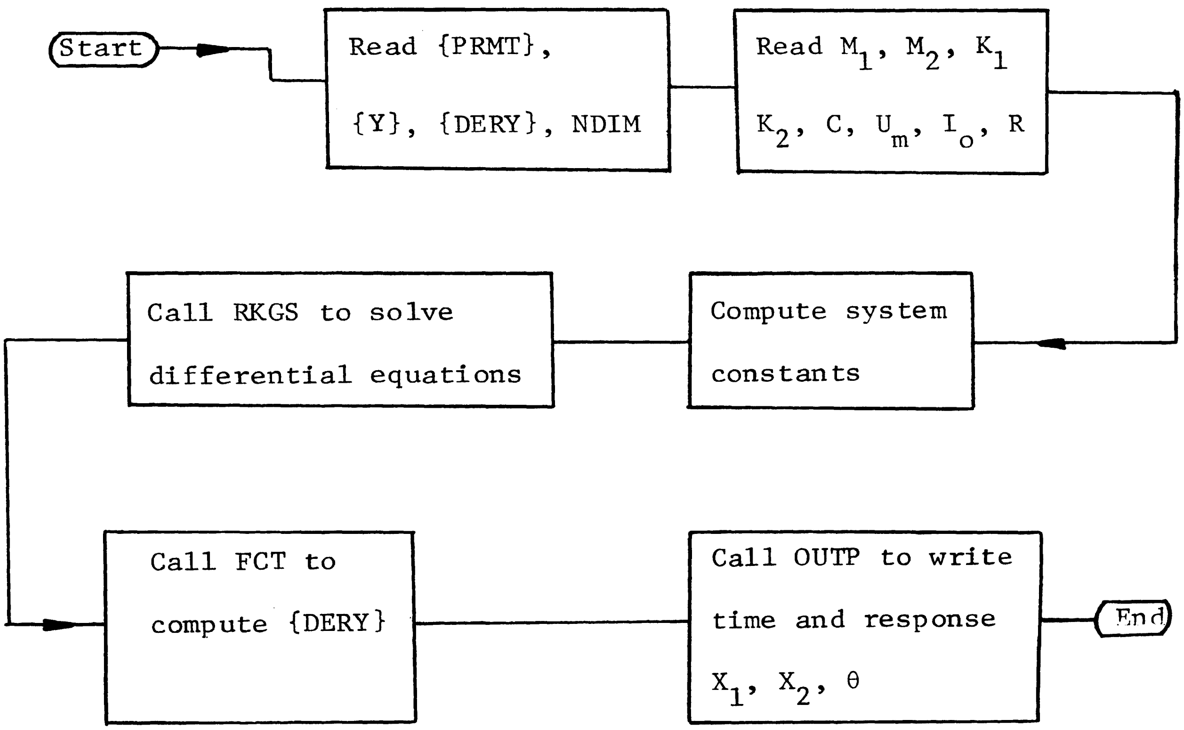
DER Y(5) = Y(6)

DER Y(6) = Equation (23)

The subroutine RKGS is used to integrate these six derivatives to obtain Y(1), Y(3) and Y(5); i.e., $X_1(t)$, $X_2(t)$ and $\theta(t)$ respectively. Figure 4 shows a block diagram for these calculations.

B. Foss Method

K. A. Foss has developed a method to solve for the free and harmonically forced responses of non-classically damped systems. (Non-classically damped systems are those in which the undamped natural modes can successfully diagonalize the [M] and [K] matrices but fail to diagonalize the [C] matrix. For non-classically damped systems, $[C][K] \neq [K][C]$ [20]). The method of K. A. Foss transforms the N original



Here

{PRMT} = An output and input vector which specifies the parameters of the interval and of accuracy.

{Y} = Input vector of initial values.

NDIM = Number of equations in the system.

Figure 4. Block Diagram for Runge-Kutta Method.

system coordinates into 2N space, in which the equations of motion of the system can be uncoupled.

In case of a linear damped system, the equations of motion are

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{f(t)\} \tag{24}$$

Foss defines new coordinates and forcing function such that

$$\{Z\} = \begin{Bmatrix} \{\dot{X}\} \\ \{X\} \end{Bmatrix}$$

$$\{F(t)\} = \begin{Bmatrix} \{0\} \\ \{f(t)\} \end{Bmatrix}$$

both {Z} and {F(t)} are column vectors of order 2N x 1

Foss also defines the following set of matrices of order 2N x 2N

$$[R] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}$$

$$[S] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}$$

With these definitions, the original equations are reduced to

$$[R]\{\dot{Z}\} + [S]\{Z\} = \{F(t)\} \tag{25}$$

The homogeneous solution is obtained from

$$[R]\{\dot{Z}\} + [S]\{Z\} = \{0\} \tag{26}$$

The assumed solution is

$$\{Z(t)\} = \begin{Bmatrix} \{\dot{X}\} \\ \{X\} \end{Bmatrix} = e^{\alpha t}\{\Phi\} = e^{\alpha t} \begin{Bmatrix} \alpha\{\phi\} \\ \{\phi\} \end{Bmatrix} , \tag{27}$$

where $1/\alpha$ = Eigenvalue

Substituting (27) into (26) we get

$$[\alpha[R]+[S]]\{\Phi\} = \{0\} \tag{28}$$

Premultiplying by $[S]^{-1}$ and dividing through by α, this equation becomes

$$[[S]^{-1}[R] + \frac{1}{\alpha}[I]]\{\Phi\} = \{0\} \tag{29}$$

Here [I] is an identity matrix of order 2N x 2N

By matrix operations it can be shown that

$$[S]^{-1} = \begin{bmatrix} -[M]^{-1} & [0] \\ [0] & [K]^{-1} \end{bmatrix}$$

and

$$[S]^{-1}[R] = \begin{bmatrix} -[M]^{-1} & [0] \\ [0] & [K]^{-1} \end{bmatrix} \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}$$

$$= \begin{bmatrix} [0] & -[I] \\ [K]^{-1}[M] & [K]^{-1}[C] \end{bmatrix}$$

Note that here [I] is an identity matrix of order N x N

Substituting $[S]^{-1}[R]$ in equation (29) we get

$$\begin{bmatrix} [0] & -[I] \\ [K]^{-1}[M] & [K]^{-1}[C] \end{bmatrix} \{\Phi\} + \frac{1}{\alpha}[I]\{\Phi\} = \{0\} \tag{30}$$

Let us define

$$[U] = \begin{bmatrix} [0] & [I] \\ -[K]^{-1}[M] & -[K]^{-1}[C] \end{bmatrix}$$

Then equation (30) becomes

$$[U]\{\Phi\} - \frac{1}{\alpha}[I]\{\Phi\} = \{0\}$$

or

$$[[U] - \frac{1}{\alpha}[I]]\{\Phi\} = \{0\} \tag{31}$$

The eigenvalue problem (31) has a non-trivial solution if and only if the characteristic determinant vanishes.

Thus

$$[U] - \frac{1}{\alpha}[I] = 0 \tag{32}$$

The solution of equation (32) will yield 2N eignevalues, $1/\alpha_n$ ($n = 1,2,\ldots,2N$). For a stable system each α_n is either real and negative or complex with a negative real part. The complex eigenvalues must occur as complex conjugate pairs [16]. Each complex conjugate pair of eignevalues gives corresponding complex conjugate modal columns.

Thus for 2N eigenvalues there exist 2N eigenvectors of the form

$$\{\Phi\}_n = \begin{Bmatrix} \alpha_n\{\phi\}_n \\ \{\phi\}_n \end{Bmatrix} \qquad n = 1,2,\ldots,2N$$

Thus for each eigenvalue we can compute the set of eigenvectors

$$[\phi] = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_N \end{Bmatrix}_n \qquad n = 1,2,\ldots,2N$$

For distinct eigenvalues $1/\alpha_n$, the orthogonality relations [16] for the system in 2N space are

$$\{\Phi\}^T_{(m)}[R]\{\Phi\}_{(n)} = 0 \qquad \text{when } n \neq m \tag{33}$$

$$\{\Phi\}^T_{(m)}[S]\{\Phi\}_{(n)} = 0 \qquad \text{when } n \neq m \tag{34}$$

1. Forced Vibration Response

A particular solution of the equation (25) can be obtained by expanding {Z} into a modal series.

Thus

$$\{Z\} = \sum_{n=1}^{2N} \{\Phi\}_{(n)} \xi_n(t), \tag{35}$$

where $\xi_n(t)$ = uncoupled system coordinate

Substituting (35) into (25) yields

$$\sum_{n=1}^{2N} [R]\{\Phi\}_{(n)} \dot{\xi}_n + \sum_{n=1}^{2N} [S]\{\Phi\}_{(n)} \xi_n = \{F(t)\} \tag{36}$$

Premultiplying this equation by $\{\Phi\}^T_{(m)}$ gives

$$\sum_{n=1}^{2N} \{\Phi\}^T_{(m)}[R]\{\Phi\}_{(n)} \dot{\xi}_n + \sum_{n=1}^{2N} \{\Phi\}^T_{(m)}[S]\{\Phi\}_{(n)} \xi_n = \{\Phi\}^T_{(m)}\{F(t)\} \tag{37}$$

$$m = 1,2,\ldots,2N$$

The orthogonality conditions (33) and (34) simplify equation (37) to

$$\{\Phi\}^T_{(n)}[R]\{\Phi\}_{(n)}\dot{\xi}_n + \{\Phi\}^T_{(n)}[S]\{\Phi\}_{(n)}\xi_n = \{\Phi\}^T_{(n)}\{F(t)\} \tag{38}$$

Let

$$R_n = \{\Phi\}^T_{(n)}[R]\{\Phi\}_{(n)}$$

Thus from equation (36) when n = m, we have

$$\{\Phi\}^T_{(n)}[S]\{\Phi\}_{(n)} = -\alpha_n\{\Phi\}^T_{(n)}[R]\{\Phi\}_{(n)}$$

or

$$\{\Phi\}^T_{(n)}[S]\{\Phi\}_{(n)} = -\alpha_n \cdot R_n$$

Let

$$F_n(t) = \{\Phi\}^T_{(n)}\{F(t)\}$$

so equation (38) reduces to

$$R_n\dot{\xi}_n - \alpha_n R_n \xi_n = F_n(t) \tag{39}$$

Dividing equation (39) by R_n, we get

$$\dot{\xi}_n - \alpha_n\xi_n = \frac{F_n(t)}{R_n} \tag{40}$$

Equation (40) is an uncoupled equation in 2N space. The Convolution Integral gives us the complete solution of equation (39) for zero initial conditions.

$$\xi_n(t) = \frac{1}{R_n}\int_o^t e^{\alpha_n(t-\tau)}F_n(\tau)d\tau \tag{41}$$

Substituting $\xi_n(t)$ into equation (35), we have

$$\{Z\} = \sum_{n=1}^{2N} \{\Phi\}_{(n)} \frac{1}{R_n} \int_o^t e^{\alpha_n (t-\tau)} F_n(\tau)\, d\tau \tag{42}$$

From our coordinate transformation, it is seen that the lower half of the column vector $\{Z\}$ gives us the system response in terms of our original generalized coordinates. Thus

$$\{X_i\} = \sum_{n=1}^{2N} \frac{1}{R_n} \{\phi_i\}_{(n)} \int_o^t e^{\alpha_n (t-\tau)} F_n(\tau)\, d\tau \tag{43}$$

2. <u>Solution in Matrix Form</u>

We know that

$$F_n(\tau) = \{\Phi\}^T_{(n)} \{F(\tau)\},$$

or

$$F_n(\tau) = \{\Phi\}^T_{(n)} \begin{Bmatrix} \{0\} \\ \{f(\tau)\} \end{Bmatrix},$$

or

$$F_n(\tau) = \{\Phi\}^T_{(n)} \{f(\tau)\}$$

and

$$R_n = \{\Phi\}^T_{(n)} [R] \{\Phi\}_{(n)} \quad .$$

Let $[{}^{\ulcorner}R^{\dagger}_{\lrcorner}] = [{}^{\ulcorner}R_{n\lrcorner}]$ $\qquad n = 1,2,\ldots,2N.$

By matrix operations, we can write

$$\left[\ulcorner R^{\dagger} \lrcorner\right]^{-1} = \left[\ulcorner 1/R_n \lrcorner\right]$$

Thus, equation (43) can be written in the form:

$$\{X_i\} = [\phi]\left[\ulcorner R^{\dagger} \lrcorner\right]^{-1}\left\{\int_0^t \left[\ulcorner e^{\alpha_n(t-\tau)} \lrcorner\right][\phi]^T\{f(\tau)\}d\tau\right\} \tag{44}$$

Equation (44) gives the total response of the system to the forcing function $\{f(t)\}$, for zero initial conditions.

The Convolution Integral defined by equation (34) contains certain complex elements. In order to evaluate the integrals on a digital computer, it was necessary to separate real and imaginary parts as follows:

From expansion of equation (44):

$$f(\tau) = \begin{Bmatrix} 0 \\ F(\tau) \end{Bmatrix}$$

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \end{bmatrix},$$

where $\phi_{11},\ldots,\phi_{24}$ are elements of 2N eigenvectors.

$$\left[\ulcorner e^{\alpha_n(t-\tau)} \lrcorner\right] = \begin{bmatrix} e^{\alpha_1(t-\tau)} & 0 & 0 & 0 \\ 0 & e^{\alpha_2(t-\tau)} & 0 & 0 \\ 0 & 0 & e^{\alpha_3(t-\tau)} & 0 \\ 0 & 0 & 0 & e^{\alpha_4(t-\tau)} \end{bmatrix}$$

$$\left[\ulcorner R^{\dagger} \lrcorner\right]^{-1} = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix}^{-1}$$

The inverse of this matrix is

$$\begin{bmatrix} 1/R_1 & 0 & 0 & 0 \\ 0 & 1/R_2 & 0 & 0 \\ 0 & 0 & 1/R_3 & 0 \\ 0 & 0 & 0 & 1/R_4 \end{bmatrix}$$

Carrying out the matrix multiplications,

$$[\phi]^T\{F(\tau)\} = \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \\ \phi_{13} & \phi_{23} \\ \phi_{14} & \phi_{24} \end{bmatrix} \qquad \begin{Bmatrix} 0 \\ f(\tau) \end{Bmatrix}$$

$$\begin{Bmatrix} \phi_{21} f(\tau) \\ \phi_{22} f(\tau) \\ \phi_{23} f(\tau) \\ \phi_{24} f(\tau) \end{Bmatrix}$$

$$[\overline{e^{\alpha_n(t-\tau)}}][\phi]^T\{F(\tau)\} = \begin{Bmatrix} e^{\alpha_1(t-\tau)} \phi_{21} f(\tau) \\ e^{\alpha_2(t-\tau)} \phi_{22} f(\tau) \\ e^{\alpha_3(t-\tau)} \phi_{23} f(\tau) \\ e^{\alpha_4(t-\tau)} \phi_{24} f(\tau) \end{Bmatrix}$$

$$\{\int_0^t [\overline{\ e^{\alpha_n(t-\tau)}\ }][\phi]^T\{F(\tau)\}d\tau\} = \begin{Bmatrix} \int_o^t e^{\alpha_1(t-\tau)} \phi_{21} f(\tau)d\tau \\ \int_o^t e^{\alpha_2(t-\tau)} \phi_{22} f(\tau)d\tau \\ \int_o^t e^{\alpha_3(t-\tau)} \phi_{23} f(\tau)d\tau \\ \int_o^t e^{\alpha_4(t-\tau)} \phi_{24} f(\tau)d\tau \end{Bmatrix}$$

$$[\overline{\ R^{\dagger}\ }]^{-1}\{\int_0^t [\overline{\ e^{\alpha_n(t-\tau)}\ }][\phi]^T\{F(\tau)\}d\tau\} = \begin{Bmatrix} 1/R_1 \int_0^t e^{\alpha_1(t-\tau)} \phi_{21} f(\tau)d\tau \\ 1/R_2 \int_0^t e^{\alpha_2(t-\tau)} \phi_{22} f(\tau)d\tau \\ 1/R_3 \int_0^t e^{\alpha_3(t-\tau)} \phi_{23} f(\tau)d\tau \\ 1/R_4 \int_0^t e^{\alpha_4(t-\tau)} \phi_{24} f(\tau)d\tau \end{Bmatrix}$$

Let us define this column by $\{D\}$.

Thus

$$[\phi][\overline{\ R^{\dagger}\ }]^{-1}\{\int_0^t [\overline{\ e^{\alpha_n(t-\tau)}\ }][\phi]^T\{F(\tau)\}d\tau\} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \end{bmatrix} \{D\}$$

The system response for zero initial conditions is

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [\phi]\{D\}$$

Substituting values for $[\phi]\{D\}$ and expanding we get values of X_1 and X_2 as follows:

$$X_1 = \frac{\phi_{11}\phi_{21}}{R_1}\int_0^t e^{\alpha_1(t-\tau)} f(\tau)d\tau + \frac{\phi_{12}\phi_{22}}{R_2}\int_0^t e^{\alpha_2(t-\tau)} f(\tau)d\tau$$
$$+ \frac{\phi_{13}\phi_{23}}{R_3}\int_0^t e^{\alpha_1(t-\tau)} f(\tau)d\tau + \frac{\phi_{14}\phi_{24}}{R_4}\int_0^t e^{\alpha_4(t-\tau)} f(\tau)d\tau \quad (45)$$

and

$$X_2 = \frac{\phi_{21}^2}{R_1}\int_0^t e^{\alpha_1(t-\tau)} f(\tau)d\tau + \frac{\phi_{22}^2}{R_2}\int_0^t e^{\alpha_2(t-\tau)} f(\tau)d\tau$$
$$+ \frac{\phi_{23}^2}{R_3}\int_0^t e^{\alpha_3(t-\tau)} f(\tau)d\tau + \frac{\phi_{24}^2}{R_4}\int_0^t e^{\alpha_4(t-\tau)} f(\tau)d\tau \quad (46)$$

In summation form,

$$X_1 = \sum_{n=1}^{4} \frac{\phi_{1n}\phi_{2n}}{R_n} \int_0^t e^{\alpha_n(t-\tau)} f(\tau)d\tau \quad (47)$$

and

$$X_2 = \sum_{n=1}^{4} \frac{\phi_{2n}^2}{R_n} \int_0^t e^{\alpha_n(t-\tau)} f(\tau)d\tau \quad (48)$$

The forcing function (τ) is obtained as follows:

from equation (7),

$$f(\tau) = - mR[\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta]$$

For the case where the motor voltage reaches its maximum value instantaneously, and no external load is applied, the angular velocity as given by equation (1) is:

$$\dot{\theta}(t) = \omega_0(1-e^{-t/t_0}) .$$

Thus

$$\ddot{\theta}(t) = \frac{\omega_0}{t_0} e^{-t/t_0} , \quad \text{and}$$

$$\theta(t) = \omega_0 t + \omega_0 t_0 e^{-t/t_0} + C$$

at time $t = 0$, $\theta = 0$, from which

$$C = -\omega_o t_0 .$$

Therefore

$$\theta = \omega_0 t + \omega_0 t_0 e^{-t/t_0} - \omega_0 t_0 \tag{49}$$

and

$$f(\tau) = -mR\frac{\omega_0}{t_0} e^{-\tau/t_0} \cos \theta + mR\omega_0^2(1-e^{-\tau/t_0})^2 \sin \theta, \tag{50}$$

where θ is given by equation (49).

Returning to equation (47) and (48), let us define:

$$\begin{aligned} \alpha_n &= A_n + iB_n \\ \phi_{1n} &= P_{1n} + iQ_{in} \\ \phi_{2n} &= P_{2n} + iQ_{2n} \\ R_n &= RR_n + iRI_n \end{aligned} \qquad n = 1,\ldots,4 \tag{51}$$

Calculation of R_n proceeds as follows:

as defined earlier,

$$R_n = \{\Phi\}_n^T[R]\{\Phi\}_n, \text{ from which}$$

$$R_n = \begin{Bmatrix} \alpha_n\phi_{1n} \\ \alpha_n\phi_{2n} \\ \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & M_1 & 0 \\ 0 & 0 & 0 & M_2 \\ M_1 & 0 & C & -C \\ 0 & M_2 & -C & C \end{bmatrix} \begin{Bmatrix} \alpha_n\phi_{1n} \\ \alpha_n\phi_{2n} \\ \phi_{1n} \\ \phi_{2n} \end{Bmatrix}$$

After matrix multiplication, substituting α_n, ϕ_{1n} and ϕ_{2n} from equation (51), and separating real (RR_n) and Imaginary (RI_n) parts, we get

$$\begin{aligned} RR_n = {} & 2A_n\{M_1(P_{1n}^2-Q_{1n}^2) + M_2(P_{2n}^2-Q_{2n}^2)\} \\ & -4B_n(M_1P_{1n}Q_{1n} + M_2P_{2n}Q_{2n}) \\ & +C\{(P_{1n}-P_{2n})^2 - (Q_{1n}-Q_{2n})^2\} \end{aligned}$$

and

$$\begin{aligned} RI_n = {} & 2B_n\{M_1(P_{1n}^2-Q_{1n}^2) + M_2(P_{2n}^2-Q_{2n}^2)\} \\ & +4A_n(M_1P_{1n}Q_{1n} + M_2P_{2n}Q_{2n}) \\ & + 2C(P_{1n}-P_{2n})(Q_{1n}-Q_{2n}) \end{aligned}$$

We calculate $\frac{\phi_{1n}\phi_{2n}}{R_n}$ by expanding it into real and imaginary parts. Thus

$$\frac{\phi_{1n}\phi_{2n}}{R_n} = FR1(n) + iFI1(n),$$

where

$$FR1(n) = \frac{PP_{1n}RR_n + QQ_{1n}RI_n}{RR_n^2 + RI_n^2}$$

and

$$FI1(n) = \frac{QQ_{1n}RR_n - RI_nPP_{1n}}{RR_n^2 + RI_n^2}$$

with

$$PP_{1n} = P_{1n}P_{2n} - Q_{1n}Q_{2n}$$

and

$$QQ_{1n} = Q_{1n}P_{2n} + P_{1n}Q_{2n}$$

In similar fashion, we can calculate $\frac{\phi_{2n}^2}{R_n}$. After expanding it in real and imaginary parts, we get

$$\frac{\phi_{2n}^2}{R_n} = FR2(n) + iFI2(n),$$

where

$$FR2(n) = \frac{PP_{2n}RR_n + QQ_{2n}RI_n}{RR_n^2 + RI_n^2}$$

and

$$FI2(n) = \frac{QQ_{2n}RR_n - PP_{2n}RI_n}{RR_n^2 + RI_n^2}$$

with

$$PP_{2n} = (P_{2n}-Q_{2n})^2$$

and

$$QQ_{2n} = 2P_{2n}Q_{2n}$$

To calculate $\int_0^t e^{\alpha_n(t-\tau)} F(\tau)d\tau$, we substitute α_n and $F(\tau)$ from equations (51) and (50) and use the relation $e^{iX} = \cos X + i \sin X$. From this we obtain two real and two imaginary integrals for evaluation for each eigenvalue as follows:

$$\text{Real } 1 = \frac{-mR\omega_0}{t_0}\int_0^t e^{A_n(t-\tau)}\{\cos B_n(t-\tau)\}e^{-\tau/t_0}\cos\omega\tau d\tau$$

$$\text{Real } 2 = mR\omega_0^2\int_0^t e^{A_n(t-\tau)}\{\cos B_n(t-\tau)\}(1-e^{-\tau/t_0})^2\sin\omega\tau d\tau$$

$$\text{Imaginary } 1 = \frac{-mR\omega}{t_0}\int_0^t e^{A_n(t-\tau)}\{\cos B_n(t-\tau)\}e^{-\tau/t_0}\sin\omega\tau d\tau$$

$$\text{Imaginary } 2 = mR\omega^2\int_0^t e^{[_n(t-\tau)}\{\sin B_n(t-\tau)\}(1-e^{-\tau/t_0})^2\sin\omega\tau d\tau$$

To evaluate these integrals, subroutine QSF from the IBM Scientific Subroutine Package (version III) was used. As the subroutine requires the values of the function at equidistant points, the main program evaluates the four integrals for each eigenvalue (equations (45) and (46))at points 0.005 sec. apart. Each integrand is defined at 201 points (to obtain the response for 1 sec.) for each of the four eigenvalues. Thus each integrand has three dimensions; i.e., 1) number of integral, 2) number of eigenvalue and 3) number of point (which indicates time) at which integral is to be evaluated.

To make use of subroutine QSF, it was necessary to transform the three-dimensional values for each integrand into one dimension, corresponding to a specific time. For this, subroutine MERREL [19] was developed. This subroutine converts any three-dimensional integrand $Y(I_2, M, J)$ to a one-dimensional value YY(J), where

I_2 = Number representing integral, $I_2 = 1,2,\ldots,4$

M = Number representing eigenvalue, $M = 1,\ldots,4$

J = Number representing time interval, J = 1,2,...,201

Subroutine QSF is now used to calculate ZZ(J), integrated value of YY(J), up to time J. Subroutine MERREL now transforms the one-dimensional value ZZ(J) to corresponding three-dimensional values $Z(I_2, M, J)$.

The real and imaginary parts of $\int_0^t e^{\alpha_n(t-\tau)} F(\tau)d\tau$ are defined as ZR(t) and ZI(t) respectively. After substituting in equations (47) and (48) all the elements as calculated above, we get the real (X1R(t) and X2R(t)) and imaginary (X1I(t) and X2I(t)) parts of responses $X_1(t)$ and $X_2(t)$. From this the responses are calculated as follows:

$$X(t) = \sqrt{[XR(t)]^2 + [XI(t)]^2} \sin\left(\omega\tau + \tan^{-1}\frac{XI(t)}{XR(t)}\right).$$

A block diagram of these calculations is shown in Figure 5.

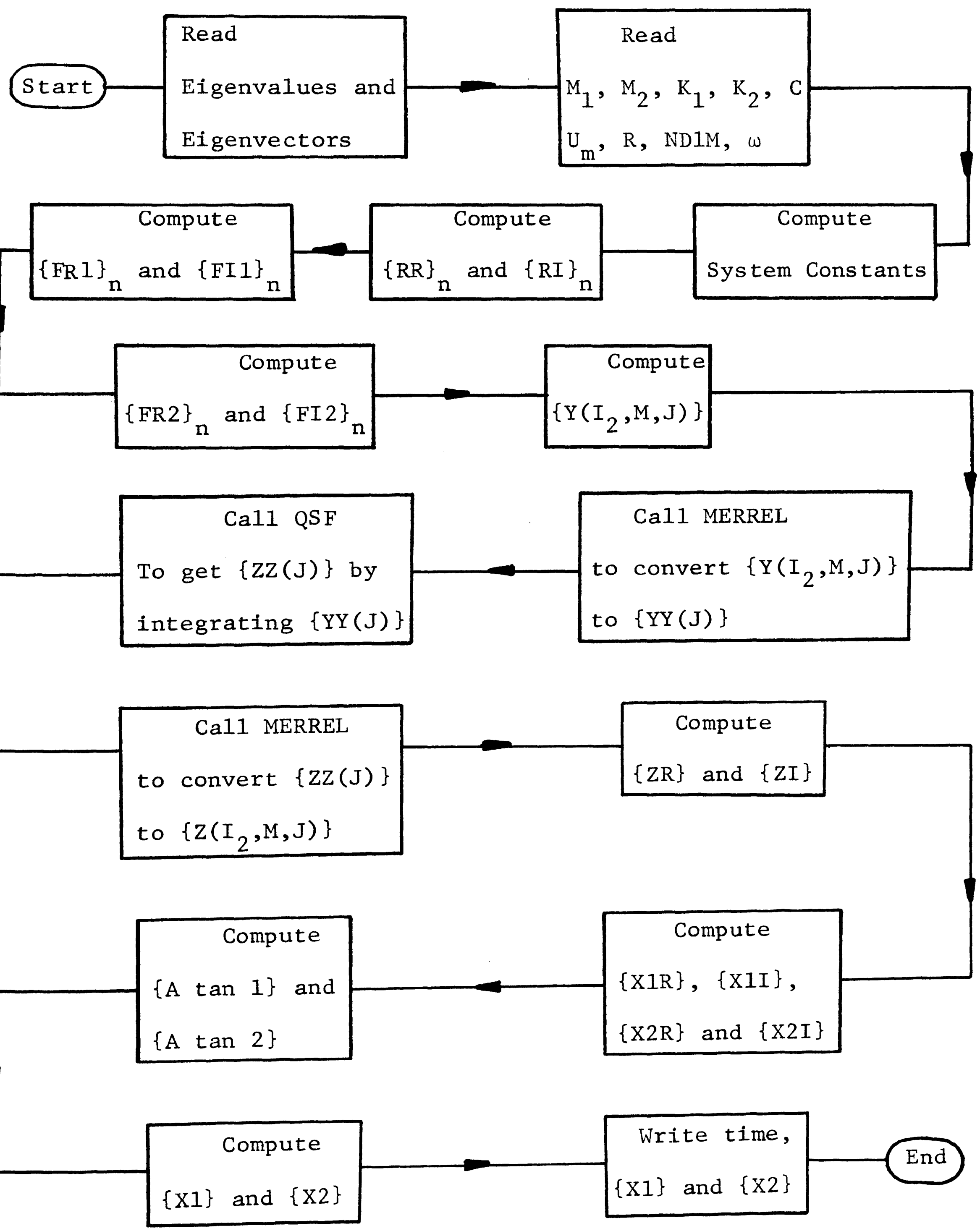


Figure 5. Block Diagram for Convolution Integral Solution

C. Integration of Convolution Integral by Subroutine QSF

For this purpose, the subroutine QSF from the IBM Scientific Subroutine Package (version III) is used.

The vector of integral values is given by

$$Z_i = A(X_i) = \int_a^{X_i} Y(X)\,dX \qquad i = 1,2,\ldots,n$$

with $X_i = a+(i-1)h$

h = step size

and a = Lower limit of integration.

For a table of function values Y_i $(i = 1,2,\ldots,n)$, given at equidistant points $X_i = a + (i-1)h$ $(i=1,2,\ldots,n)$, combination of Simpson's rule and Newton's 3/8 rule is used.

The formulas used are given below where Z_j are integral values and Y_j are function values at point j.

1. Simpson's Rule:

$$Z_j = Z_{j-2} + \frac{h}{3}(Y_{j-2}+4Y_{j-1}+Y_j)$$

2. Newton's 3/8 Rule:

$$Z_j = Z_{j-3} + 3/8h(Y_{j-3}+3Y_{j-2}+3Y_{j-1}+Y_j)$$

Combination of the above two gives:

$$Z_j = Z_{j-5} + \frac{h}{3}(Y_{j-5}+3.875Y_{j-4}+2.625Y_{j-3}+2.625Y_{j-2}+3.875Y_{j-1}+Y_j)$$

This formula is used to evaluate the integral $\int_0^t e^{\alpha_n(t-\tau)} F(\tau)\,d\tau$.

V. APPLICATION TO PRACTICAL VIBRATION ISOLATION SYSTEM

As indicated earlier the primary objectives of this thesis are to develop practical methods for calculating the transient responses of systems with vibration isolation, and to apply these methods to typical cases. In the examples that follow, transient and steady state responses are compared, and the influence of inertia torque on system behavior is illustrated.

A. System Selected for Analysis

The specific system selected for analysis is shown in Figure (6).

The motor is a 4 pole, 100 H.P., 1750 rpm, class 'B' induction motor, a type which is commonly used because of its simplicity and ruggedness. The performance of an induction motor is specified in terms of many parameters, of which the speed and torque characteristics are of prime importance. Appendix (B) shows the development of the equation for the net torque as a function of rotor speed. Again, note that the net torque is the accelerating electromotive torque less the load torque of the driven machinery.

To get a clear picture of system behavior, various cases were studied. The value of damping was taken as zero or as $0.5C_c$, where C_c is critical damping for subsystem consisting of the motor and isolators. The unbalanced weights are taken as follows:

1. Rotor unbalance simulated by 1/2 lb. at 6 in. radius. (This is equivalent to 1/4 lb. at 12 in. radius).
2. Driven machine unbalance simulated by 30 lbs. and 50 lbs. weights acting at 12 in. radius.

$M_1 g = 12000$ lbs.

$M_2 g = 1000$ lbs.

$K_1 = 5000$ lb/in

$K_2 = 2000$ lb/in

$C = 71.63$ lb/in/sec

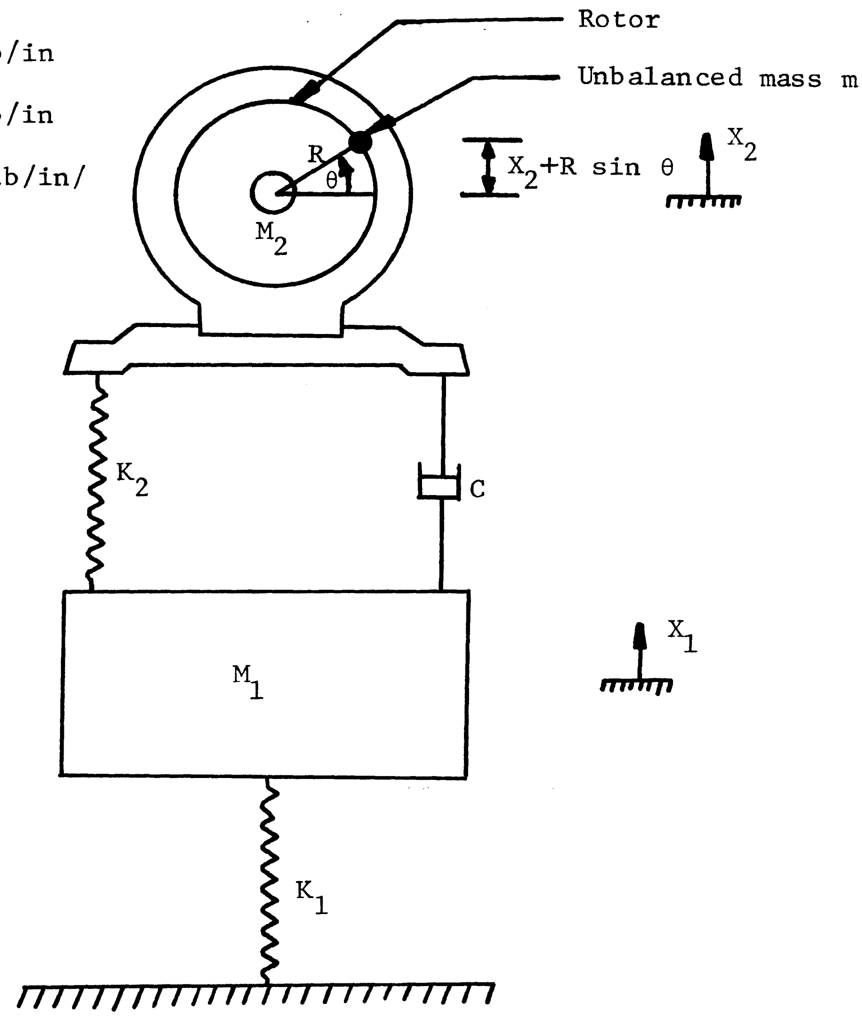


Figure 6. System Selected for Analysis

All the cases are summarized as shown in Figure (7).

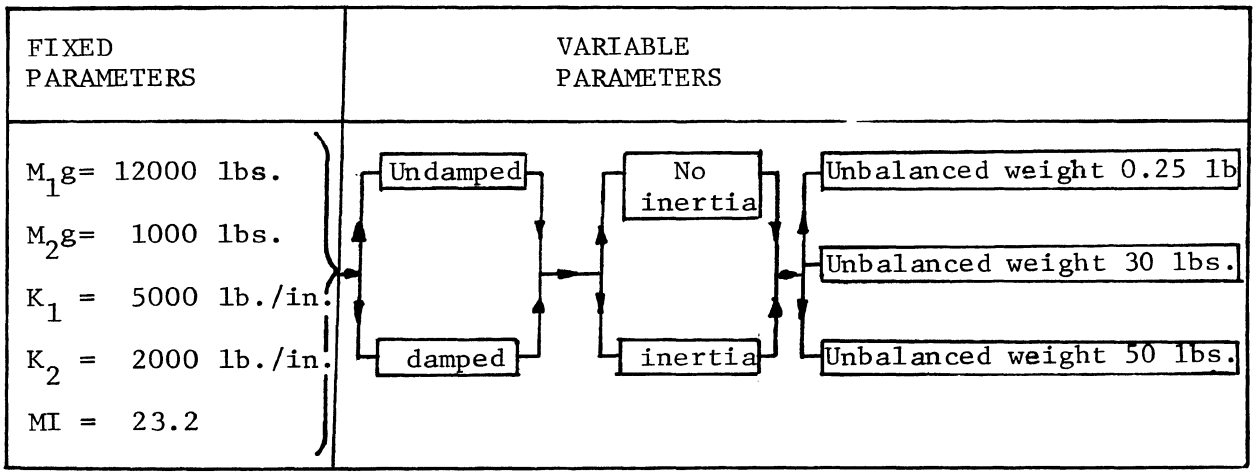


Figure 7. List of Cases Studied

B. Results

The results obtained by solving the differential equation of motion using the Runge-Kutta method of order 4 are shown in Tables I, II and III.

For a small unbalanced weight (0.25 lbs.) there is no difference in results for the cases when the inertia torque is considered or neglected (Table I). This is true for damped as well as undamped vibration. The values of transient amplitudes are slightly greater than the steady state values. (Note that the "transient period" is the time interval during which the motor accelerates from rest to its operating speed, and the "steady state period" is the period after the motor reaches a constant operating speed.) Because of low inertia, the motor reaches its constant (steady state) angular velocity in about three seconds, for all cases. Because there is no difference in amplitudes for any of the cases summarized in Table I, the force transmissibility is the same. (The "force transmissibility" is the ratio of the force transmitted by spring K to the unbalanced shaking force.)

For an unbalanced mass weighing 30 lbs., we observe significant differences in transient as well as steady state responses, for cases without damping, when the inertia torque is first considered and then neglected (Table II). However, when damping is present, the inertia torque has no effect on system response. This is because damping reduces the vertical displacement and minimizes the inertia torque produced by the vertical acceleration of the unbalanced mass. The transient amplitudes are greater than the steady state amplitudes, but when damping is present the difference is significantly less. The motor reaches its steady state speed (181.28 rad./sec.) in about 4 seconds when the effect of inertia torque is not considered. When inertia torque is included, the motor reaches 99% of its steady state value in about 4.8 seconds but varies thereafter about 178 rad./sec. and 181 rad./sec. in damped and undamped cases respectively. The force transmissibilities are identical for the damped inertia and no inertia cases. However, for the undamped cases force transmissibility is greater when the inertia torque is considered. To show the differences in amplitudes for various cases, graphs are plotted as follows:

1. Figure (8) - Transient responses of the motor for inertia and no inertia damped as well as undamped cases. The time interval was taken between 1.35 sec. to 1.6 sec., when maximum transient amplitudes occur.
2. Figure (9) - Transient responses of the floor are plotted

for the same interval (1.35 sec. to 1.6 sec.) for inertia and no inertia damped as well as undamped cases. The scales are identical to those in Figure (8).

3. Figure (10) - shows the graphs of steady state responses of motor and floor for the damped inertia case during the time 9.75 to 10.0 seconds. Note that the scales are different from those of previous graphs. It can be seen that the response of the floor consists of superposition of the forcing frequency (29 cps) and the lower resonant frequency (2 cps) of the system. The latter frequency is part of the transient response of the system and will disappear eventually. However, a long time interval is required for its elimination because the frequency is low and no damping is present in the floor.

For a relatively large unbalanced weight (50 lbs.), an entirely different picture is seen (see Table III). We get large differences in the amplitudes between the inertia and no inertia, damped as well as undamped cases. The differences between transient and steady state amplitudes are significant even for the case without inertia torque. For the undamped case with inertia torque, the motor never reaches its design speed, instead, the speed fluctuates about the second natural frequency, i.e., 29 rad./sec. This occurs because of the large inertia torque which prevents the motor from reaching its operating speed. The maximum transient undamped amplitude of the motor when inertia torque is considered is 15" and that of floor is about 2". For the no inertia undamped case, the amplitudes are 5" and 0.54" respectively. These results indicate the pronounced effect of inertia torque when the unbalanced mass is very large. In the damped case when inertia torque is considered, the motor reaches 176 rad./sec. in about 14 seconds. However, this increase is not continuous, because of the interaction between the inertia torque and the accelerating torque. The force transmissibility at steady state, when the effect of inertia torque is not considered, is about 30% larger than that for the corresponding case where the unbalance weight is 0.25 lb. The difference in transmissibility is due to the presence of the transient response at 2 cps.* When damping is present, this response will disappear eventually and the force transmissibility will decrease. Values for transmissibilities shown in Tables are obtained from an average of peak values at 5 and 10 seconds.

C. Confirmation of Results

In order to confirm the results obtained by the Runge-Kutta method of order four, the Continuous System Modeling Program (CSMP) was used. This system is available at the Computer Center, University of Missouri-Rolla, and is mainly used for solving coupled linear and non-linear differential equations. A choice of integration scheme is available at the option of the user. For the purpose of comparison, Simpson's, Hammings, and Runge-Kutta method of order four were used. The results obtained were found comparable, with difference occurring

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*For all undamped cases, when inertia torque is not considered, the force transmissibility is 0.00011, provided responses at resonant frequencies are not included.

only after the third decimal place.

For the case when the motor is operated without any external load and an unbalance weight of 1/2 lb., the results obtained using the Foss method, i.e., the Convolution Integral solution, were not comparable to the results of either the Runge-Kutta method or any method of the CSMP. The error is suspected either in mathematics of the Convolution Integral Solution or in the program itself. It is necessary to mention here that if a procedure suitable for evaluating equation (34) is developed, the Foss method will give the correct responses of the system. (The damped natural frequencies 24.96 rad./sec. and 12.07 rad./sec. were obtained by the Foss method.)

TABLE I

Results for 0.25 lb. Unbalanced Mass

Unbalanced Mass		Steady State								Transient			
		Speed rad/sec		Elapsed Time-Sec		Max ampl-in		Force Transmi.*		Speed+ rad/sec		Max ampl-in	
0.25 lbs		$C=0.$	$C=0.5C_c$	$C=0.$	$C=0.5C_c$	$C=0.$	$C=0.5C_c$	$C=0.$	$C=0.5C_c$	$C=0.$	$C=0.5C_c$	$C=0.$	$C=0.5C_c$
Inertia	Motor	181.28	181.28	2.85	2.85	0.0168	0.0028	0.037	0.0078	45.6	40.2	0.0180	0.0039
	Floor	—	—	—	—	0.0019	0.0004			—	—	0.002	0.0009
No Inertia	Motor	181.28	181.28	2.85	2.85	0.0168	0.0025	0.037	0.0078	45.6	40.2	0.0180	0.0039
	Floor	—	—	—	—	0.0019	0.0004			—	—	0.002	0.0009

$$^{*}\text{Force transmissibility} = \frac{\text{Force transmitted to floor}}{\text{Force acting on motor}} = \frac{K_1 X_1}{mR\omega^2}$$

+This is the speed at which maximum transient amplitude occurs.

TABLE II

Results for 30 lbs. Unbalanced Mass

Unbalanced Mass 30 lbs		Steady State								Transient			
		Speed rad/sec		Elapsed Time-Sec		Max ampl-in		Force Transmi.		Speed rad/sec		Max ampl-in	
		C=0.	C=0.5C_c	C=0.	C=0.5C_c	C=0.	C=0.5C_c	C=0.	C=0.5C_c	C=0.	C=0.5C_c	C=0.	C=0.5C_c
Inertia	Motor	179.5	179.5	4.88	4.88	2.70	0.35	0.052	0.0098	35.9	51.8	3.31	0.491
	Floor	—	—	—	—	0.32	0.06			—	—	0.346	0.118
No Inertia	Motor	181.28	181.28	4.2	4.2	2.2	0.350	0.040	0.0098	42.8	49.8	2.62	0.491
	Floor	—	—	—	—	0.25	0.06			—	—	0.30	0.117

TABLE III

Results for 50 lbs. Unbalanced Mass

<table>
<tr><th colspan="2" rowspan="3">Unbalanced Mass
50 lbs</th><th colspan="8">Steady State</th><th colspan="4">Transient</th></tr>
<tr><th colspan="2">Speed rad/sec</th><th colspan="2">Elapsed Time-Sec</th><th colspan="2">Max ampl-in</th><th colspan="2">Force Transmi.</th><th colspan="2">Speed rad/sec</th><th colspan="2">Max ampl-in</th></tr>
<tr><th>C-0.</th><th>C-0.5C_c</th><th>C=0.</th><th>C=0.5C_c</th><th>C=0.</th><th>C=0.5C_c</th><th>C=0.</th><th>C=0.5C_c</th><th>C=0.</th><th>C=0.5C_c</th><th>C=0.</th><th>C=0.5C_c</th></tr>
<tr><td rowspan="2">Inertia</td><td>Motor</td><td colspan="6">Motor is not reaching steady state speed even up to 10 sec. instead varies from* 22 rad./sec. to 34 rad./sec. for undamped case</td><td rowspan="2"></td><td rowspan="2"></td><td>29.9</td><td></td><td>15.2</td><td>0.86</td></tr>
<tr><td>Floor</td><td>—</td><td>—</td><td>—</td><td>—</td><td></td><td></td><td>—</td><td>—</td><td>1.8</td><td>0.24</td></tr>
<tr><td rowspan="2">No Inertia</td><td>Motor</td><td>181.28</td><td>181.28</td><td>5.1</td><td>5.1</td><td>4.1</td><td>0.701</td><td rowspan="2">0.048</td><td rowspan="2">0.012</td><td>40.</td><td>40.</td><td>5.00</td><td>0.85</td></tr>
<tr><td>Floor</td><td>—</td><td>—</td><td>—</td><td>—</td><td>0.49</td><td>0.126</td><td>—</td><td>—</td><td>0.54</td><td>0.23</td></tr>
</table>

*For damped case motor reaches 174 rad./sec. in about 14 sec. It then varies from 174 rad./sec. to 177 rad./sec.

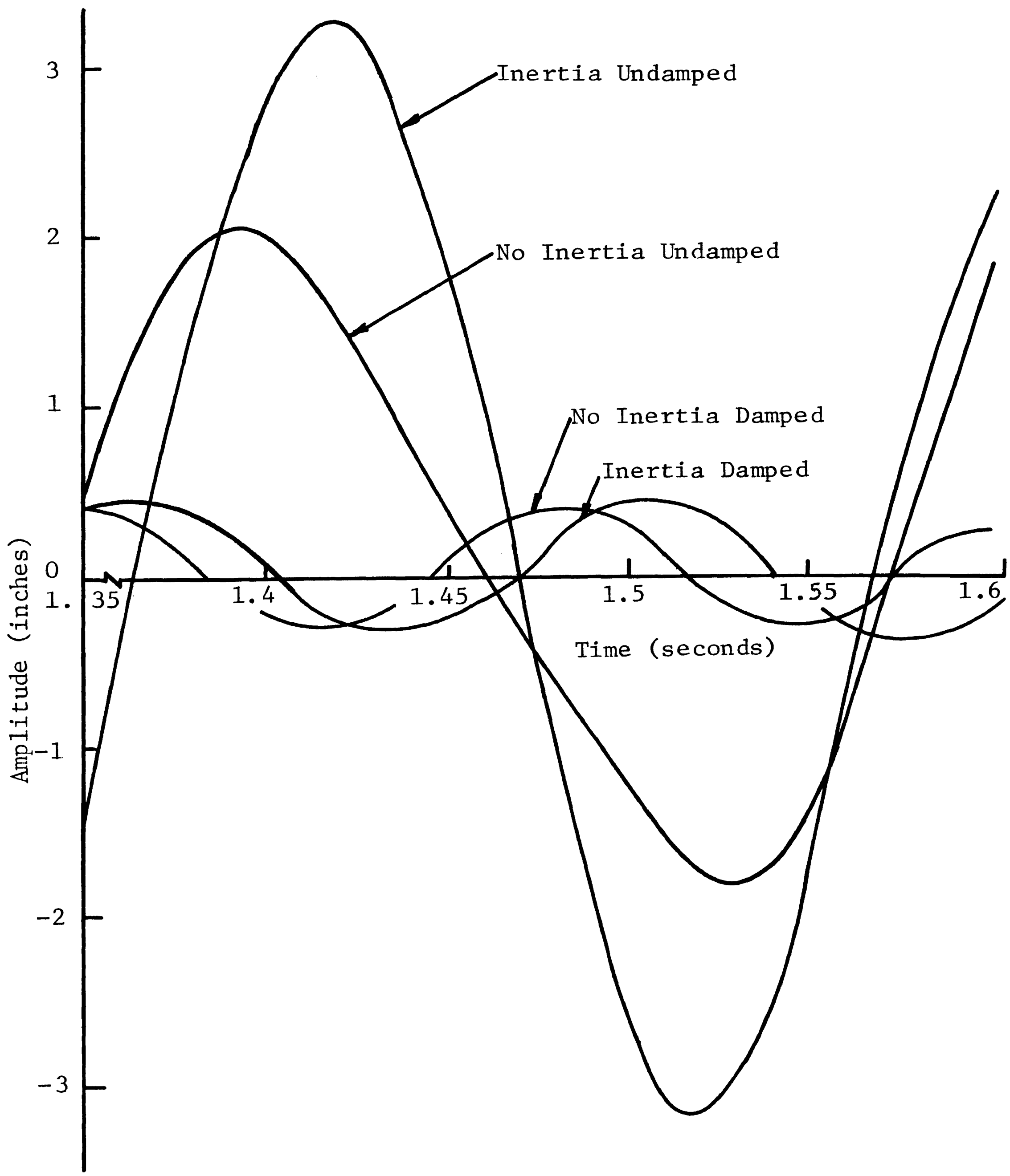

Figure 8. Transient Responses of the Motor

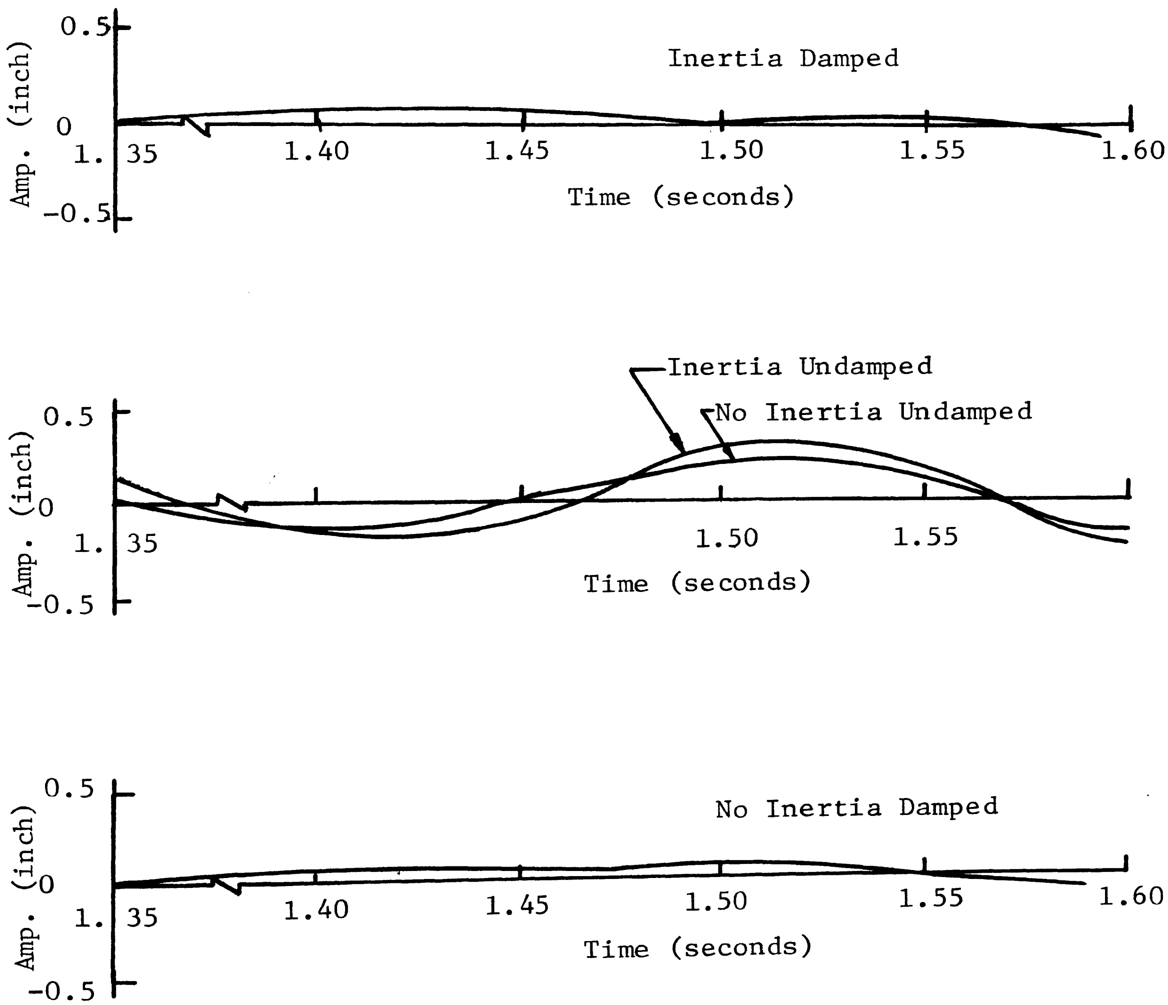

Figure 9. Transient Responses of Floor

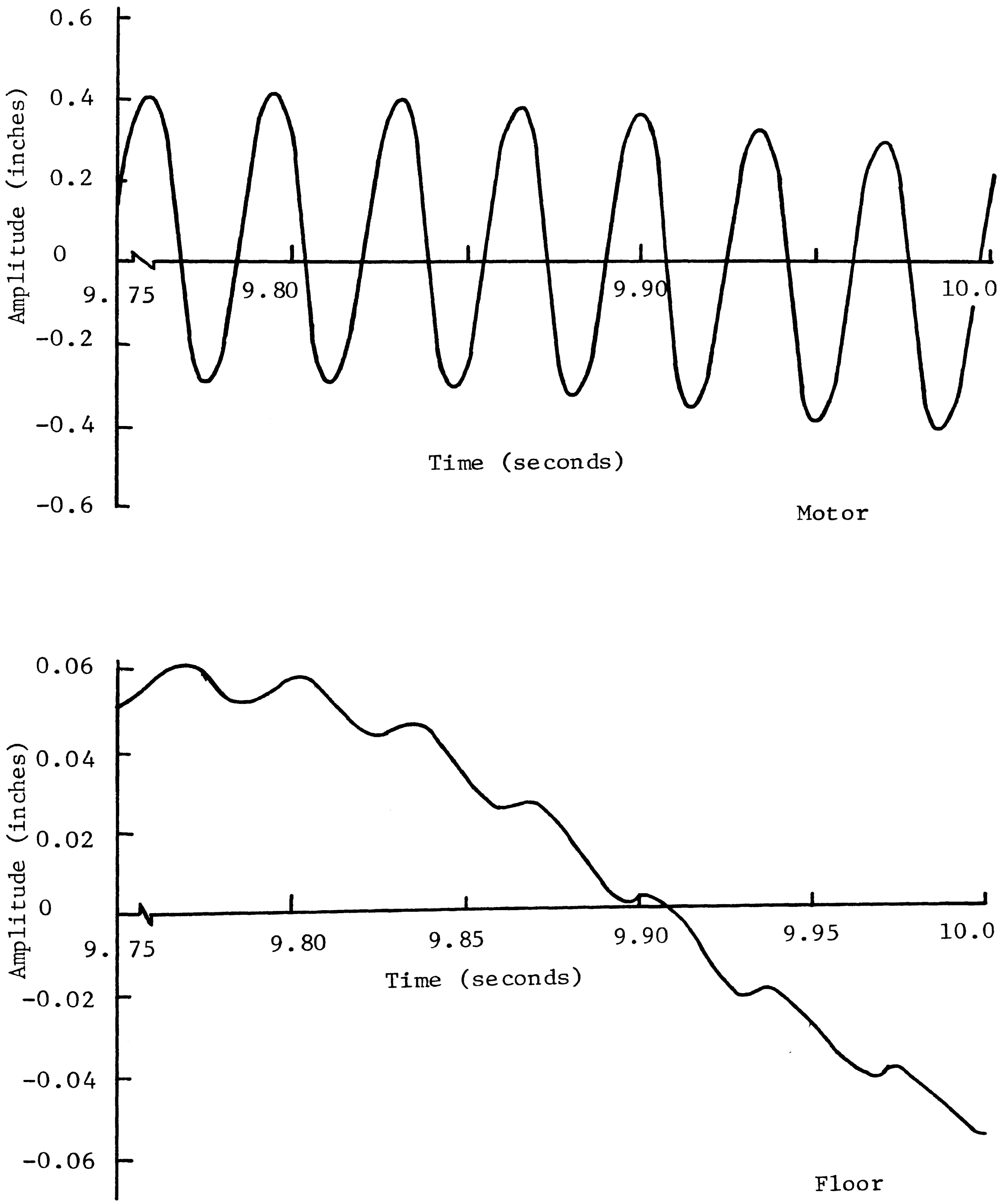

Figure 10. Steady State Responses

VI. CONCLUSIONS

Generally, the design of vibration isolators is based on the steady state response of the system for the following reasons:

1. The transient period usually lasts for only a few seconds and is followed by steady state vibration.

2. It is assumed that no annoying or potentially destructive behavior occurs during the transient period, during which the system accelerates to its operating speed.

However, these reasons are not always valid. As shown in Chapter V, the system may remain in the transient condition for an indefinite time, during which large amplitudes and forces can occur. The following guidelines are suggested for determining whether transient response, inertia torque, or damping should be considered in the design or analysis of vibration isolators.

1. For relatively small unbalanced forces, the effect of inertia torque can be neglected provided the system reaches its steady state speed within a few seconds. In this case, the increases in amplitudes during the transient period, as compared to the steady state values, are relatively small for all values of damping.

2. When large unbalanced forces are present and the system has little or no damping, the transient response (with inertia torque included) must be calculated in order to obtain maximum amplitudes, forces, and terminal speed. For example, transient analysis of the system with a 30 lbs. unbalance weight showed that the motor reaches 179 rad./sec.

in 4.8 sec. and then varies from 179 rad./sec. to 181 rad./sec. For a 50 lbs. unbalance weight without damping, the motor never reaches its steady state speed but fluctuates around its resonance frequency, i.e., 29 rad./sec. with a maximum amplitude of 15 inches. Thus a transient analysis may dictate the use of stops or other measures to limit system amplitudes. Note also that failure to include the inertia torque in this case leads to the erroneous conclusion that the motor reaches its operating speed in 5.1 sec. with only a small increase in maximum amplitude during the transient period. Finally, when large unbalanced forces occur in a system having significant damping, a transient analysis (with inertia torque included) may be needed to determine the time required to reach terminal speed. For example, the results for a 50 lbs. unbalance weight show that the motor takes about 14 sec. to reach a speed of 176 rad./sec.

3. Assumption of vertical motion is justified in cases where the system is constrained to move only in vertical direction or where lateral and rotational stiffnesses are large. In the latter case, however, horizontal and rotational motions may have to be considered when vertical amplitudes become large; the case in which the unbalance weight was 50 lbs. is an example for this.

VII. APPENDICES

A. CALCULATION OF NATURAL FREQUENCIES FOR UNDAMPED SYSTEM

The equations of motion in matrix form are as follows:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix} \tag{A-1}$$

Assuming the motion of every point in the system to be harmonic, let

$$X_1 = A_1 \sin \omega t$$
$$X_2 = A_2 \sin \omega t.$$

Substituting these assumed solutions into the differential equations (A-1), we get

$$(K_1+K_2-M_1\omega^2)A_1 - K_2A_2 = 0$$
$$-K_2A_1 + (K_2-M_2\omega^2)A_2 = 0$$

These equations are satisfied for any A_1 and A_2 only if the following determinant is zero:

$$\begin{vmatrix} K_1+K_2-M_1\omega^2 & -K_2 \\ -K_2 & K_2-M_2\omega^2 \end{vmatrix} = 0$$

Multiplying out the determinant, we obtain the frequency equation

$$\omega^4 - \left[\frac{K_1+K_2}{M_1} + \frac{K_2}{M_2}\right]\omega^2 + \frac{K_1K_2}{M_1M_2} = 0$$

Substituting values of K_1, K_2, M_1, and M_2 we get ω_1^2 and ω_2^2. Neglecting negative signs as being of no physical significance, we arrive at two natural frequencies ω_1 = 12.04 rad./sec. and ω_2 = 29.18 rad./sec.

These values agree with those obtained by the Foss Method. The damped natural frequencies (obtained by the Foss Method) are 12.07 rad./sec. and 24.96 rad./sec.

B. DERIVATION OF EQUATION FOR ACCELERATING TORQUE

a. Calculation of motor torque [5]

For an induction motor, the torque-slip relation is expressed by the ratios T/T_{max} and S/S_{maxT},

where

T = Motor torque

T_{max} = Maximum internal or breakdown torque

$$S = \text{Slip} = \frac{\left(\begin{matrix}\text{Synchronous}\\ \text{Speed}\end{matrix} - \begin{matrix}\text{Rotor}\\ \text{Speed}\end{matrix}\right) 100}{\begin{matrix}\text{Synchronous}\\ \text{Speed}\end{matrix}}$$

S_{maxT} = Slip at T_{max}.

The relation between T/T_{max} and S/S_{maxT} is given by the following expression:

$$\frac{T}{T_{max}} = \frac{1 + \sqrt{Q^2+1}}{1 + 1/2\sqrt{Q^2+1}\left(\frac{S}{S_{maxT}} + \frac{S_{maxT}}{S}\right)} , \qquad \text{(B-1)}$$

where

$$Q = \frac{\text{Reactance}}{\text{Resistance}}$$

Most induction motors will fall in the region between Q = 3 and Q = 7. The value of Q is taken as 5 in this thesis.

For a class 'B' design motor, T_{max} is 2.15 times the full load torque (see Figure B-1) which is obtained from

$$\text{H.P.} = \frac{\text{R.P.M.} \times T_{fl}}{5250}$$

where T_{fl} = Full load torque.

From this relation

$$T_{fl} = 3600 \text{ lb. in.}$$

(As mentioned in Chapter V, the motor is 4-pole, 60 cps, 100 H.P., 1750 R.P.M.)

Therefore,

$$T_{max} = 7740 \text{ lb. in.}$$

Figure B-1 shows the torque-speed relation for a class 'B' type motor. It shows that maximum torque is 215% of full load torque and slip at the maximum torque (S_{maxT}) is 12% of synchronous speed.

The synchronous speed is obtained from

$$\omega_{sync} = \frac{120 \text{ x Frequency}}{\text{No. of poles}}$$

$$= \frac{120 \text{ x } 60}{4}$$

$$= 60\ \pi \text{ rad./sec.}$$

Therefore, $$S = \frac{60\ \pi - \dot{\theta}}{60\ \pi}\ 100, \tag{B-2}$$

where $\dot{\theta}$ = speed of the rotor, rad./sec.

Substituting T_{max}, Q, S and S_{maxT} in equation (B-1), we get

$$T = \frac{47214}{1 + 2.55\left(\frac{60\pi - \dot{\theta}}{7.2\pi} + \frac{7.2\pi}{60\pi - \dot{\theta}}\right)} \tag{B-3}$$

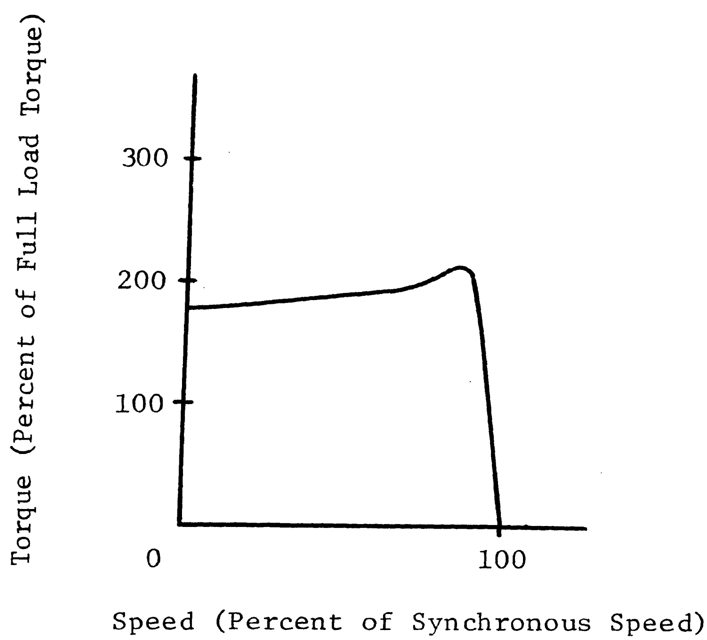

Figure B-1. Torque-Speed Curve for an Induction Motor

b. Calculation of Load Torque [6]

To obtain the load torque function in terms of speed, one of the load curves (see Figure B-2) was interpolated by using the divided difference polynomial mehtod [18].

From the given curve, for three different values of speed (S_o=0, S_1=73.3 and S_2=98.4), the values of load torque T(S) obtained were $T_o(S)$ = 30, $T_1(S)$ = 65 and $T_2(S)$ = 100, respectively. Note that speed is percent of synchronous speed and the load torque is percent of full load torque.

According to divided difference, polynomial method

$$T(S) = T_o(S)+(S-S_o)\delta f_1+(S-S_o)(S-S_1)\delta^2 f_1+(S-S_o)(S-S_1)(S-S_2)\delta^3 f_1+\ldots \quad (B-4)$$

Here, the divided difference operator S is defined by

$$\delta f_i = \frac{T_i(S)-T_{i-1}(S)}{S_i-S_{i-1}}$$

Higher differences are then defined by

$$\delta^k f_i = \frac{\delta^{k-1} f_{i+1} - \delta^{k-1} f_i}{S_{k+1}-S_{k-2}}$$

In our problem,

$$\delta f_i = \frac{T_2(S)-T_1(S)}{S_1-S_0}$$

$$= \frac{65-30}{73.3-0}$$

$$= 0.4775$$

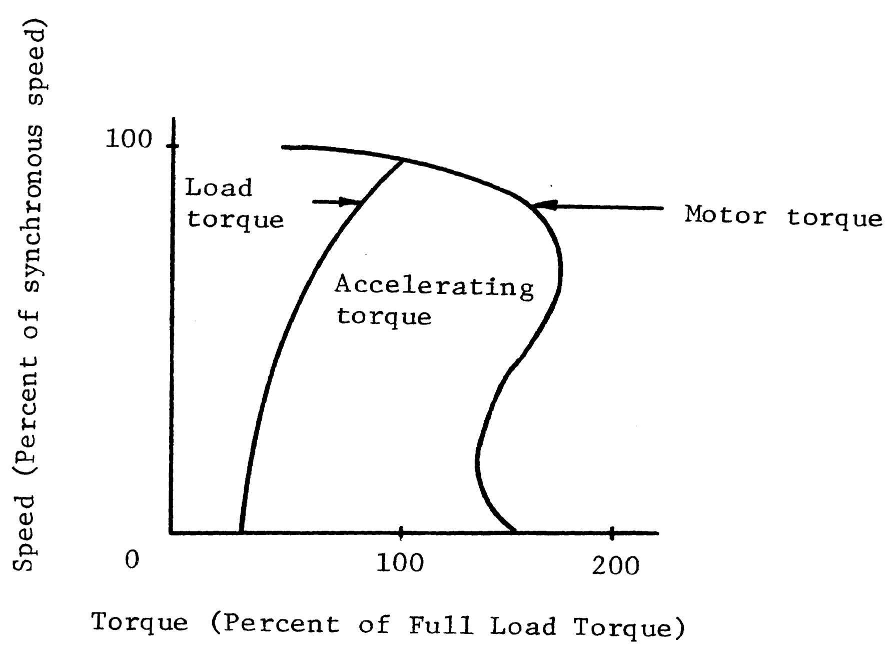

Figure B-2. Torques Acting on an Induction Motor

$$\delta f_2 = \frac{T_2(S)-T_1(S)}{S_2-S_1}$$

$$= \frac{100-65}{98.4-73.3}$$

$$= 1.394$$

$$\delta^2 f_1 = \frac{\delta f_2 - \delta f_1}{S_3 - S_0}$$

$$= \frac{1.394-0.4775}{98.4-0}$$

$$= 0.0093$$

Substituting all the necessary values in equation (B-3), we get

$$T(S) = 30+(S-0)0.4775+(S-0)(S-73.3)0.0093$$

Simplifying,

$$T(S) = 0.0093S^2-0.2042S+30.0$$

As mentioned earlier, T(S) is percent of full load torque and can also be written as

$$T(S) = \frac{\text{Load torque}}{\text{Full load torque}} \times 100$$

From this,

$$\text{Load torque} = \frac{\text{Full load torque}}{100} \times T(S)$$

$$= \frac{3600}{100}(0.0093S^2-0.2042S+30.0)$$

$$= 0.3348S^2-7.3512S+1080.0 \qquad \text{(B-5)}$$

Here the value of slip (S) is given by equation (B-2).

The net accelerating torque (T_N) developed by the motor is then the difference of motor torque given by (B-3) and load torque given by (B-5). This value of T_N is then used in equation (13) in Chapter III to solve for $\theta(t)$.

C. CALCULATION OF THE MOMENT OF INERTIA OF THE ROTOR

The moment of inertia of the rotor about its center of rotation is calculated as follows:

The weight of the rotor is assumed 500 lbs.

The radius of the rotor is assumed 6 in.

The moment of inertia of, the rotor about its center of rotation is given by

$$M.I = \frac{1}{2} MR^2$$

where M = Mass of the rotor, lb. sec.^{2} per in.

R = Radius of the rotor, in.

Therefore,

$$M.I = \frac{1}{2} \times \frac{500}{386.4} \times (6)^2$$

$$= 23.291 \text{ lb. in. sec.}^2$$

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IX. VITA

Hemendra Shantilal Acharya was born on October 21, 1943, at Halvad, Gujarat State, India. He graduated from C.V.O.S. High School, Bombay, India in 1960. He received a B.S. in Mechanical Engineering from the Gujarat University, Ahmedabad, India in 1965.

Immediately thereafter he was employed by Mukand Iron and Steel Works Limited, Bombay, in the Maintenance and Development Department.

In January 1969, he enrolled in the University of Missouri-Rolla for work toward his Master's Degree in Mechanical Engineering.