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TRIANGULAR SIGNAL STABILIZATION
OF NONLINEAR SYSTEMS

BY

KAMALEDDIN YADAVAR NIKRAVESH, 1944

A THESIS

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Approved by

_______________ (Advisor) _______________

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ABSTRACT

Many nonlinear systems display self-sustained oscillations which are often undesirable. The stabilizing effect of a high frequency input signal on an oscillating system with one nonlinearity is determined by the characteristics of the nonlinear element in the system, the linear portion of the system and the amplitude of the signal.

This investigation has been concerned with the effect of a triangular wave stabilizing signal on these self oscillations. The equivalent gains for several common nonlinearities are derived. The pseudo describing function introduced by Oldenburger and Boyer[10,11] for sinusoidal stabilization has been extended to the triangular wave case, and it is shown that the pseudo describing function for an odd nonlinearity is real.

The pseudo describing function is used in an analysis similar to describing function analysis in order to predict the existence and amplitude of the self oscillation of a triangular wave stabilized, closed loop, nonlinear system. The experimental results are in close agreement with the predictions of the theory.

ACKNOWLEDGEMENTS

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I. INTRODUCTION

It is frequently observed that an external periodic excitation with frequency ω not in any rational ratio to the frequency of the existing oscillation extinguishes, or quenches, the existing oscillation. In some other cases, however, a potentially possible, but not yet existing oscillation is excited by providing such an asynchronous frequency.[1,2] The term asynchronous merely emphasizes the lack of any rational ratio between the frequencies of the self oscillation and the externally applied signal.

It is obvious that these phenomena differ essentially from the phenomenon of subharmonic resonance in which the existence of a rational ratio between the two frequencies is essential.

As far as known these effects were observed for the first time by Groszkowski[3], but their theory was developed later by the Russian physicists L. Mandelstorm and N. Papalexi, and still later by Kobsareff and others[1].

The theory of the Russian authors uses the so-called "equivalent parameters" introduced in the theory of equivalent linearization, and, in fact, is a further extension of this thoery when there are two frequencies present instead of one. Minorsky[2] traced out these effects directly from the differential equations without making use of the somewhat artificial tool of equivalent linearization.

A similar phenomenon occuring in relay systems was investigated by J. C. Lozier[4] and L. A. MacColl.[5]

One method of treating this problem is the use of the two sinusoidal input describing function. An early approach for calculating the two sinusoidal input describing function was based on the use of the double Fourier series. This work was done by Bennett[6], and Kalb and Bennett.[7] West, Dousce, and Lively[8] used a similar approach to describe the sinusoidal input-output relationship of some nonlinear elements when the input to the nonlinear element is the sum of two sinusoidal signals with different integer frequencies and a phase shift between the two signals. In this case, single Fourier integrals can be used, but two additional parameters, the ratio of the two frequencies and the phase shift, are introduced.

In their analysis of signal stabilized systems, Oldenburger and Liu[9] defined an equivalent gain as the limit of the ratio of the average value of the output to the average value of the input, as the average value of the input goes to zero. In this case the input to the nonlinear element is the sum of a sinusoidal signal and a constant (dc) bias. They proposed that analysis of signal stabilized systems be accomplished by replacing the nonlinear element by its equivalent gain.

The concept of equivalent gain introduced by Oldenburger and Liu was extended by Olderburger and Boyer.[10,11] In this extension they did not assume that the bias approached zero. The same concept was also used by Gibson.[12]

The calculation of the two sinusoidal input describing function by integral representation was introduced by Gibson and Sridhar[13] for the case of non-harmonically-related input sinusoids, using techniques of random process theory. The same problem has been studied by Gelb, and Vander Velde.[14] They suggested a power-series expansion as a solution.

Another approach for this problem is given by Hsu and Meyer.[15] Their approach involves both linear and nonlinear characteristics of the system. This approach is more exact than that of Oldenburger and Boyer.[10,11] However, Gibson and Sridhar[13] checked Boyer's approximation and showed that it is a reasonable approximation under the assumption of widely differing frequencies.

In order to analyze the effect of the extra signal, a nonlinear element with the extra signal is replaced by an equivalent nonlinear element without the extra signal. This concept was introduced by Oldenburger.[16] Since the injection of the extra signal has the effect of altering nonlinear characteristics in closed-loop systems, it is reasonable to expect that limit cycles in nonlinear systems can be turned on, altered, turned off, and, in general, controlled by proper choice of the extra signal. The use of the external signal, or "dither", to quench limit cycles is referred to as signal stabilization. It has been extensively investigated by Oldenburger and his students,[9,10,11,16,17,18] among the first to discover this phenomenon experimentally and, subsequently, to provide analytical justification. The most practical approach seems to be the use of the equivalent gain concept as introduced by Oldenburger and Boyer.[10,11] The equivalent gain is used to derive an equivalent nonlinear element. When the input to the nonlinear element is the sum of two sinusoidal signals with greatly different frequencies the equivalent nonlinear characteristic can be used to approximate the low frequency sinusoidal input-output relationship of the nonlinear element. This approximation, called "pseudo describing function" by Oldenburger and Boyer[10,11] is used in stability analysis of nonlinear systems.

A general signal stabilized system is shown in Figure 1.1. For simplicity it is assumed that the stabilizing signal is brought directly into the nonlinear element. The linear portions of the closed loop system are $G_1(s)$, $G_2(s)$, $G_3(s)$. The output of the nonlinearity is $m(t)$ and the output of the system is $c(t)$. The input of the nonlinear element from the system is assumed to be of the form $E \sin \omega t$ and will be called the input

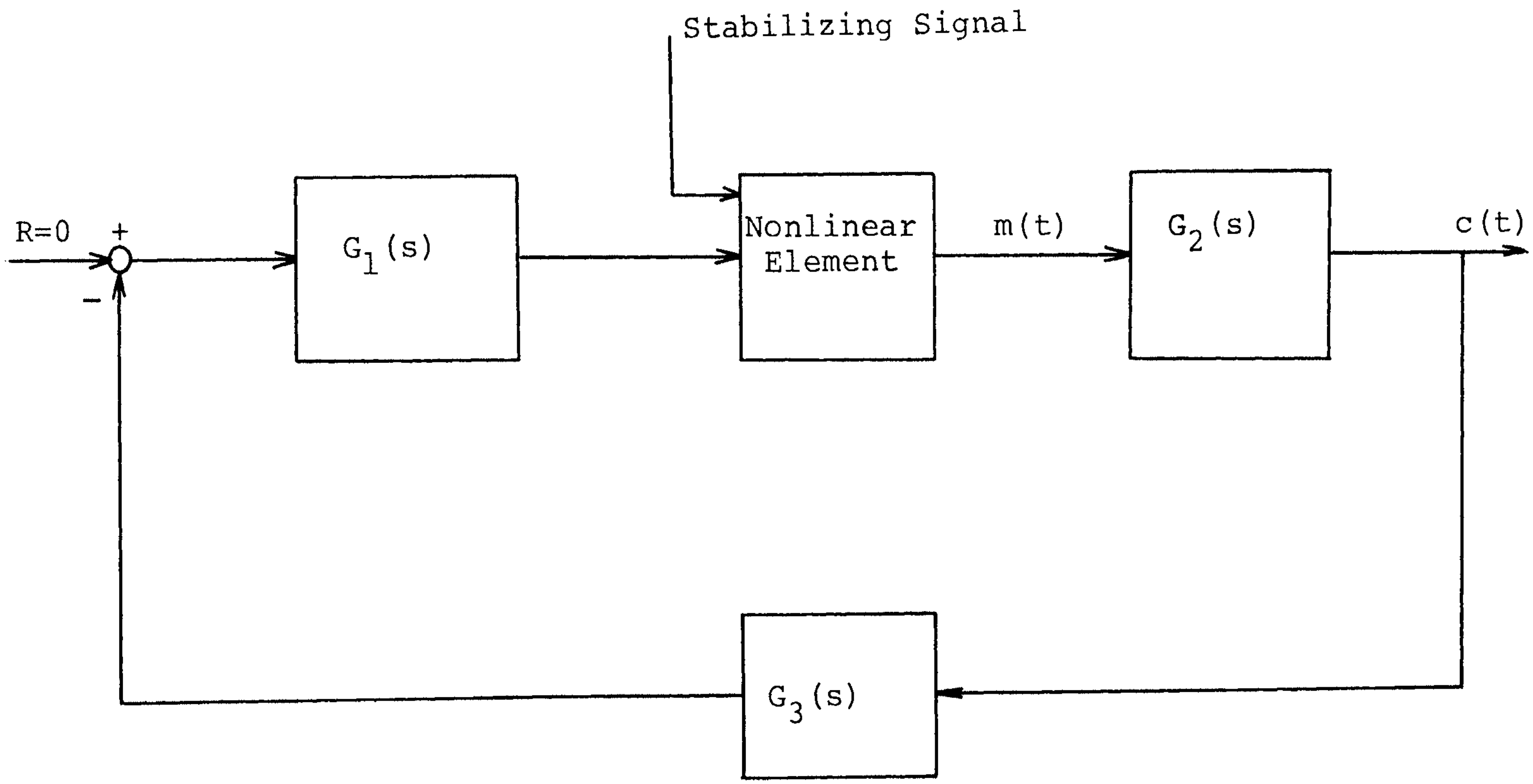


Figure 1.1. Nonlinear System With Stabilizing Signal.

fundamental, since it has the same frequency ω as the fundamental component of the output of the system $c(t)$. It is also assumed that β is larger than ω.

The stabilizing signal can be a sinusoidal signal, a triangular wave, a pulse train[19], random noise, or other appropriate repetitive signals with a frequency greater than or equal to 10ω or containing components of this size, and sufficient amplitude. A reasonable portion of the time must be spent away from saturation of the bounded element. Thus a square wave will not work.[17]

By the use of such signals the input-output characteristics of nonlinear elements may often be transformed so that a given nonlinearity behaves as if it were quite different. In many cases the introduction of an extra signal is easily accomplished. This is especially true in electrical systems where one almost always has available power sources of frequency 60Hz or 400Hz. Also, noise is always present in physical systems, modifying the input-output characteristics of nonlinearities so that they behave differently from what might otherwise be expected. The effect of undesired extra signals on the performance of a nonlinear system becomes particularly important when, as in the control of missles and satellites, one is concerned with threshold signals.

Physical systems always involve time lags. The systems to which the stabilization technique described here applies are systems with two or more lags, where one of the lags does not dominate the others, and at the same time the controller is based on the use of a linear control function.

A great deal of work has been done by Oldenburger and his students for sinusoidal signal stabilization, but few results have been obtained for triangular wave signal stabilization. In this study, an effort will be made to extend the theory of signal stabilization to include the triangular dither signal. The following chapter will illustrate briefly the work which has been done concerning sinusoidal signal stabilization.

II. SINUSOIDAL SIGNAL STABILIZATION

A. Introduction

Self-oscillation of some nonlinear systems can be reduced or eliminated by injecting an additional high frequency sinusoidal signal, as well as some other periodic signals, at the input to the nonlinear element. The additional signal is called the stabilizing signal because of its stabilizing effect on these systems.

This chapter will be devoted to a review of sinusiodal signal stabilization in order that a basis is established for the consideration of triangular wave stabilization in Chapter III. The treatment follows that of Oldenburger and his students.[9,10,11,16,17,18]

B. Equivalent Gain

As was discussed in Chapter I, the calculation of an exact two sinusoidal input describing function is a difficult task involving double Fourier series methods or other rather difficult integral calculations. However, Oldenburger and Boyer[10,11] have developed an approximate dual-input describing function that has been shown to be quite accurate when the frequency of the second signal is greater than ten times the frequency of the signal of interest. No harmonic relation is needed. The assumed form of the input signal to the nonlinearity is

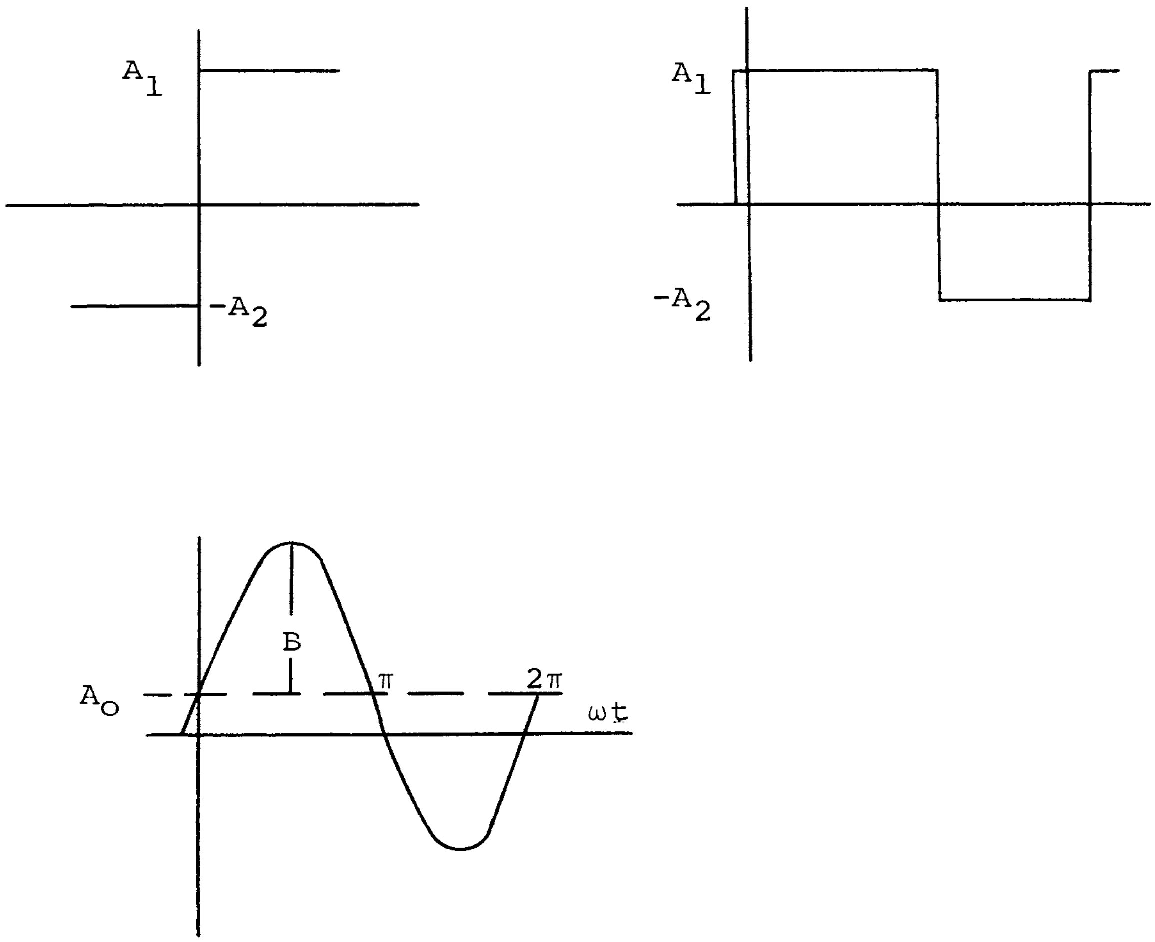


Figure 2.1. A Control System With Asymmetrical Relay.

$$e = E \sin \omega t + B \sin \beta t, \tag{2.1}$$

where $\beta \gg \omega$. This separation of the two frequency components allows the magnitude of the low-frequency wave to be assumed constant over any one cycle of the high-frequency wave with little error.

Instead of "dual-input describing function", Oldenburger and Boyer used the new term "pseudo describing function". This new term was introduced instead of using the existing terms, "dual-input describing function" or "modified describing function", because the methods used to calculate these functions imply that the ratio between the frequencies of the sinusoidal signal is an integer.[8] The pseudo describing function requires a different assumption, namely, a large frequency ratio. Although the pseudo describing function is valid only for a frequency ratio greater than about ten, signal stabilization can, and often does, occur at lower frequency ratios. However, when lower frequency ratios are used the stabilizing signal is often seen at the output of the system, which may be just as undesirable as the original limit cycle.

The equivalent gain is defined as the ratio of the average value of the output to the average value of the input when the input to the nonlinear element is a sinusoidal signal with a constant bias. This gain is a function of the bias. Let A_o denote the bias, and $B \sin \beta t$ the sinusoidal signal. The input e(t) to the nonlinear element is of the form

$$e(t) = A_o + B \sin \beta t, \qquad (2.2)$$

and the output is then

$$m(t) = A_v + B' \sin \beta t + B'' \sin \beta t \cdots \qquad (2.3)$$

Therefore,

$$g(A_o, B) = \frac{\text{average value of output}}{\text{average value of input}} = \frac{A_v}{A_o} . \qquad (2.4)$$

As an example, consider an asymmetric relay. The input-output characteristic of such a relay is shown in Figure 2.1. The output of the relay reaches the value A_1 or $-A_2$ when the input is positive or negative, respectively. The input to the nonlinear element is given by equation 2.2. One period of the corresponding output is given by[15]

$$m(t) = \begin{cases} A_1, & -\sin^{-1}\frac{A_o}{B} \leq \omega t < \pi + \sin^{-1}\frac{A_o}{B} \\ -A_2, & \pi + \sin^{-1}\frac{A_o}{B} \leq \omega t < 2\pi - \sin^{-1}\frac{A_o}{B} . \end{cases} \qquad (2.5)$$

By using Fourier series analysis, the dc value of the output is obtained as

$$A_v = \frac{1}{2\pi}\int_0^{2\pi} m(t)\, d(\omega t). \qquad (2.6)$$

Thus,

$$A_v = \frac{A_1}{\pi}(\pi + \sin^{-1}\frac{A_o}{B}) - \frac{A_2}{2\pi}(\pi - 2\sin^{-1}\frac{A_o}{B})$$

$$= \frac{1}{2}(A_1 - A_2) + \frac{1}{\pi}(A_1 \sin^{-1}\frac{A_o}{B} + A_2 \sin^{-1}\frac{A_o}{B}). \qquad (2.7)$$

If the relay is symmetric, or $A_1 = A_2 = M$ equation 2.7 becomes

$$A_v = \frac{2M}{\pi}\sin^{-1}\frac{A_o}{B}. \qquad (2.8)$$

Therefore, the equivalent gain is

$$g(A_o, B) = \frac{2M}{\pi A_o}\sin^{-1}\frac{A_o}{B}. \qquad (2.9)$$

From this example it is clear that the calculation of the equivalent dc gain requires only a direct application of Fourier analysis. The results for several common nonlinearities, have been calculated by Boyer.[11]

The equivalent gain may be used to develop an equivalent nonlinear element. To simplify system computations, the equivalent nonlinear element can be substituted for the extra sinusoidal input signal and the existing nonlinear element. The accuracy of the approach is comparable to the accuracy of the describing function analysis of systems unexcited by stabilizing inputs.

C. Equivalent Nonlinear Element

The equivalent nonlinear element can be used to explain how the stabilizing signal affects the characteristics of the nonlinear element for simultaneous low frequency inputs. This view can be used even when the loop is closed, since, if the describing function applies at all, the high-frequency components will certainly be attenuated around the loop. The equivalent nonlinear element allows some simplifying approximation in the stability and transient analysis of signal stabilized systems. Assume that the input to a given nonlinearity is of the form in equation 2.1. If $\beta >> \omega$, the value of the ω component may be considered constant over a cycle of the β component; i.e., equation 2.2 may be used to approximate the input. Thus, for each cycle of β one may compute an average value of the output by multiplying the proper A_o by the value of $g(A_o,B)$. This results in a stepped or quantized wave. The width of each step corresponds to one period of sinusoidal input.

As the frequency, β, of the sinusoidal portion of the input increases, the duration of the staircase output diminishes and the staircase output wave may be assumed to approach a smooth curve. This smooth output is called the representative output wave.

For a slowly varying input, $f(t)$, the nonlinearity behaves as if the component $B \sin \beta t$ were absent and the nonlinearity were replaced by an equivalent nonlinearity for which the output is the input multiplied by the equivalent gain. This equivalent gain is a function of $f(t)$ and thus of time t. The equivalent nonlinear element can also be obtained directly from the given nonlinear characteristic and an input of the form of equation 2.2. It is in reality a plot of A_v versus A_o for the given nonlinearity, with B as a parameter. The equivalent nonlinear element for the limiter and the relay are given in Figures 2.2 and 2.3. From these figures, one can obtain the following important results: At any instant the output of the equivalent nonlinear element is less than or equal to the output of the actual nonlinear element. In general, if the input to a nonlinear element is e and the output a differentiable function $f(e)$, the introduction of an extra sinusoidal signal will decrease the slope $f'(e)$ at $e = 0$, if

$$e > 0, \text{ for } f''(e) < 0, \tag{2.10a}$$

and

$$e < 0, \text{ for } f''(e) > 0. \tag{2.10b}$$

Similarly, if

$$f''(e) > 0, \text{ for } e > 0, \tag{2.11a}$$

$$f''(e) < 0, \text{ for } e < 0, \tag{2.11b}$$

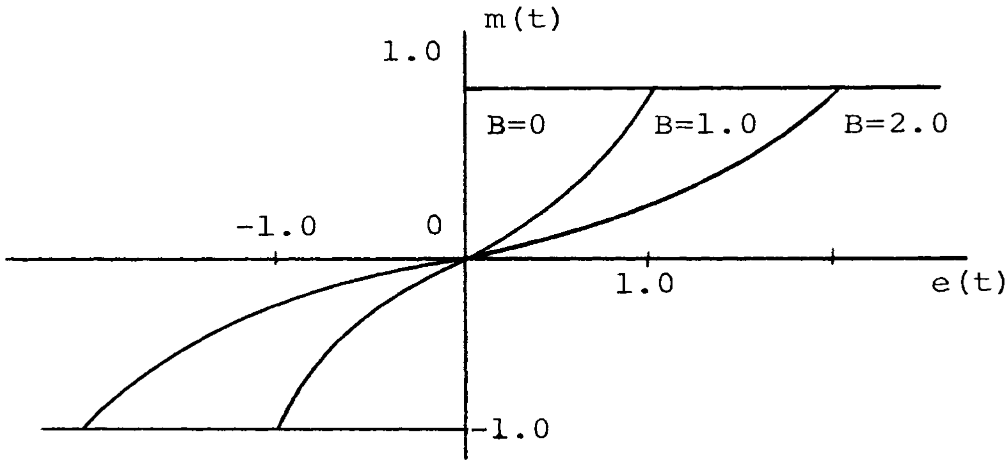


Figure 2.2. Equivalent Nonlinear Element for the Relay.

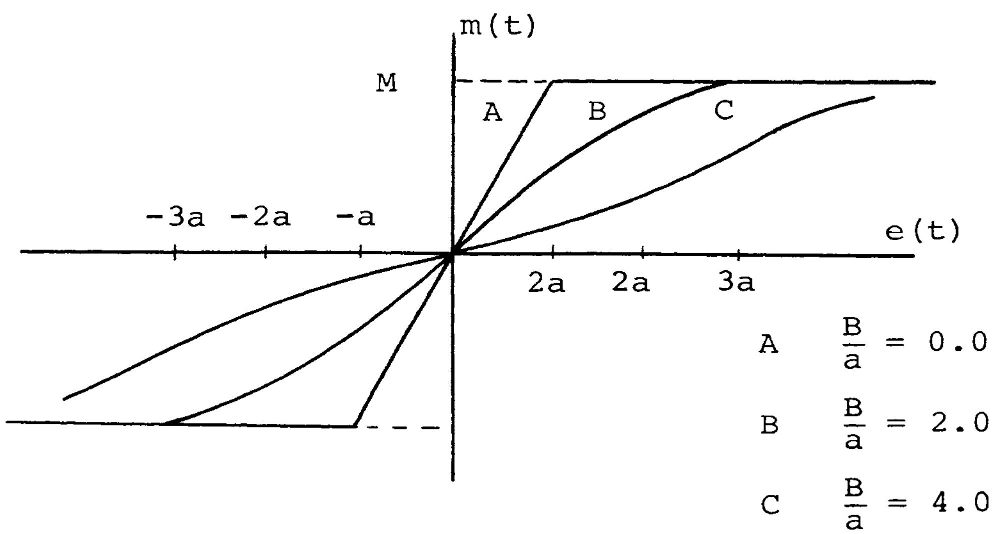


Figure 2.3. Equivalent Nonlinear Element for the Limiter.

then the introduction of an extra sinusoidal input signal will raise the gain at $e = 0$.[17]

D. Pseudo Describing Function

The conventional describing function is used in the stability analysis of nonlinear systems with no external inputs. With an external input, the modified describing function may be defined as the ratio of the fundamental component of the output, to the low frequency component of the input. The input to the nonlinear element is the sum of two sinusoidal signals with different frequencies. Practical application usually requires the use of a digital computer or other computational aid to compute this quantity. An approximation of the modified describing function is the pseudo describing function. It is defined as the ratio of the fundamental component of the representative output wave to the low frequency component of the input. This output wave is shown in Figure 2.4. Again the input to the nonlinearity is the sum of two sinusoidal signals. Experimental and theoretical work has shown that the ratio of these two frequencies should be at least 10, but not necessarily an integer.[10]

The pseudo describing function is an approximation because two assumptions are necessary for the derivation:

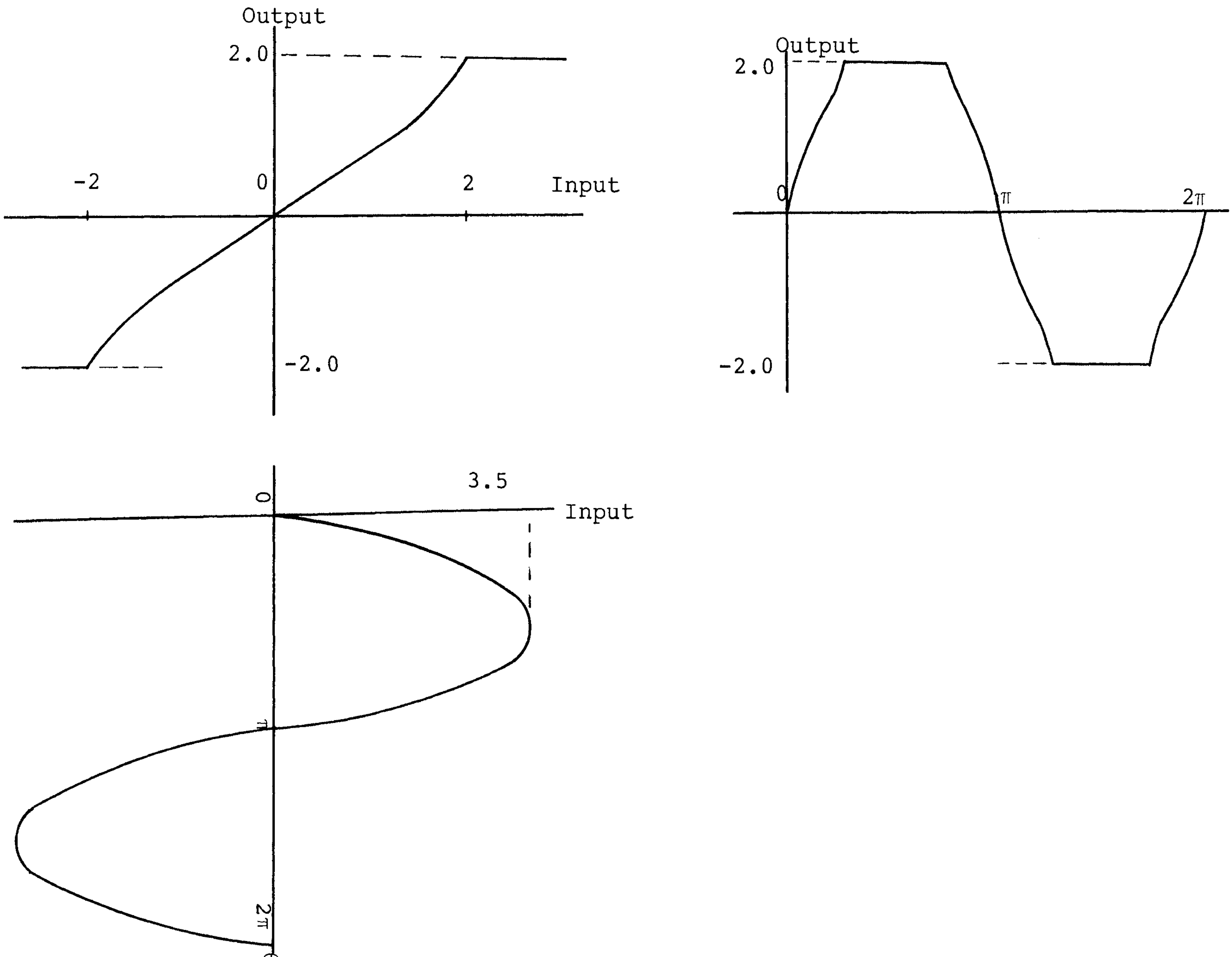


Figure 2.4. Representative Output Waveform for the Relay With Sinusoidal Dither.

1) the low frequency component of the input can be considered a constant for any period of the stabilizing signal,

2) the representative output waveform contains the amplitude and phase of the fundamental component of the actual output of the nonlinear element.

The fundamental component of the representative output can be found by well known graphical and trigonometric methods.[20]

If the fundamental input component is given as $E \sin \omega t$, and the representative output wavefore is symmetric about $t = 0$ and $t = \frac{\pi}{\omega}$, the phase shift of the pseudo describing function will be zero. The above symmetry will exist if the output of the equivalent nonlinear characteristic is an odd function of the input. This condition will be satisfied if the equivalent gain is an odd function of A_o, the dc value of the input. The equivalent gain will be an odd function of A_o when the output of the actual nonlinear element is an odd function of the input. Thus the phase shift of the pseudo describing function will be zero if the nonlinear element has been idealized with a nonlinearity whose output is an odd function of the input. This applies to nonlinearities both with and without memory.[11]

The pseudo describing function has been calculated and plotted for the relay, limiter, dead band, relay with dead band, relay with hysteresis, and the absquare by Boyer.[11] These are summarized in several texts and are not given here.[12,14]

The describing function is a gain associated with the low frequency input component when the amplitude B of the high frequency component of the input vanishes. Thus it is reasonable that the pseudo describing function approximates the describing function when the amplitude A of the low frequency input component dominates B, i.e., A >> B.

E. Pseudo Describing Function Analysis

The pseudo describing function may be used to determine the possibility of limit cycle operation for some closed loop nonlinear systems when a sinusoidal stabilizing signal is present in the system. If the input to the nonlinear element is of the form

$$e(t) = E \sin \omega t + B \sin(\omega t + \theta), \tag{2.12}$$

then by the use of the pseudo describing function one can determine the over-all loop gain for the self oscillation signal, $E \sin \omega t$.

Pseudo describing function stability analysis is an extension of describing function analysis and is subject to the restrictions usually associated with describing function analysis, except that in this case the input to the nonlinearity is the sum of two sinusoidal signals with greatly different frequencies. Pseudo describing function stability analysis will be fully considered in the next chapter in connection with triangular wave signal stabilization.

III. TRIANGULAR WAVE SIGNAL STABILIZATION

A. Introduction

As was mentioned in the previous chapters, the stabilizing signal may be one of a wide variety of periodic waveforms, provided that its frequency is large compared to the self-oscillation frequency and that its amplitude is large enough that it causes the nonlinearity to enter its saturation region, if the nonlinearity exhibits saturation. A further requirement is that the waveform not be such that the nonlinearity is always in saturation.

In the previous chapter the stabilizing effect of a sinusoidal signal was considered. This problem was studied initially by Oldenburger and Boyer.[10,11] Pulse train signal stabilization has been considered by Korolov.[19] Very little work concerning triangular wave stabilization has been done. The equivalent gain for the limiter was derived by Oldenburger and Nakada.[18] The pseudo describing function for the limiter with hysteresis was developed by Ochiai and Oldenburger.[16] Both of these studies were carried out using a triangular wave dither signal, but other nonlinearities were not considered. In this chapter the effect of a triangular wave dither on systems containing a more general class of nonlinearities will be considered.

A comparison of the effects of applying triangular wave and sinusoidal dither to the ideal relay is given in Table 3.1. The linearization effects of these dithers in the neighborhood of the origin are quite comparable. In the following section these effects will be considered mathematically.

B. Comparison of the Effects of Sinusoidal and Triangular Wave Dithers

The comparison of the output of the nonlinearity due to sinusoidal and triangular wave dither may be developed with the aid of Fourier analysis. The output of the ideal relay to either a sinusoidal or triangular wave input with the same frequency is exactly the same. The outputs of the limiter with sinusoidal and triangular wave inputs are given in Figure 3.1 and 3.2 respectively. Let a sine wave input to the limiter of Figure 3.1 be given by

$$e(t)_s = B_1 \sin \omega t. \tag{3.1}$$

The amplitude of the fundamental sinusoidal component of the output can easily be obtained by use of the Fourier series expansion of the output as

$$m_s = B_1 \left(\frac{M}{a}\right) \frac{2}{\pi} \left\{ \sin^{-1} \frac{a}{B_1} + \frac{a}{B_1} \left(1 - \left(\frac{a}{B_1}\right)^2\right)^{1/2} \right\}. \tag{3.2}$$

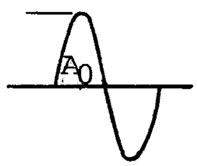
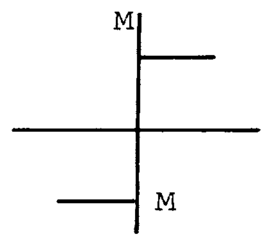
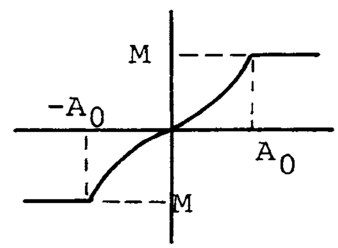
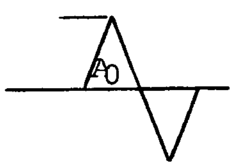
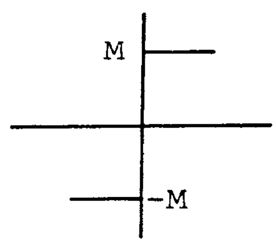
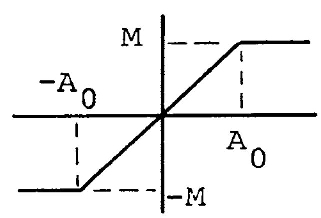
Dither	Nonlinear Element	Equivalent Nonlinear Element
		
		

TABLE 3.1

IDEAL RELAY WITH TWO TYPES OF DITHER AND THE ASSOCIATED

EQUIVALENT NONLINEAR CHARACTERISTICS

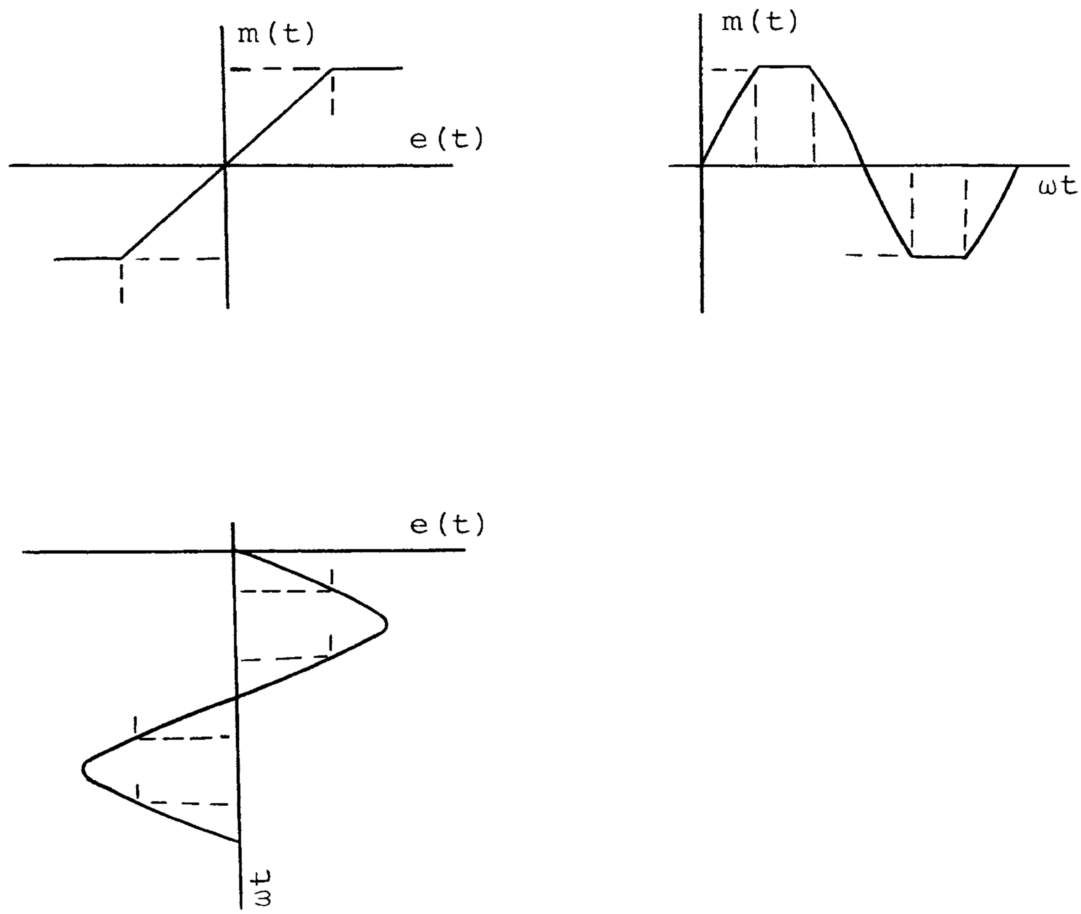


Figure 3.1. Output Waveform for Sinusoidal Dither Input.

If $(\frac{a}{B_1})$ is small then it is possible to use the Taylor series expansion of $\sin^{-1}\frac{a}{B_1}$ and $(1 - (\frac{a}{B_1})^2)^{1/2}$ to obtain the approximate output expression:

$$m_s = \frac{2M}{\pi}\frac{B_1}{a}\left\{\frac{a}{B_1} + \frac{1}{6}(\frac{a}{B_1})^3 + \ldots + \frac{a}{B_1}[1 - \frac{1}{2}(\frac{a}{B_1})^2 + \ldots]\right\},$$

or

$$m_s \simeq \frac{2M}{\pi}\left\{2 - \frac{1}{3}(\frac{a}{B_1})^2\right\}. \qquad (3.3)$$

Now, let a triangular wave input to the limiter of Figure 3.2 have amplitude B_2. Then the Fourier series expansion of the dither signal is

$$e(t)_t = \frac{8B_2}{\pi^2}(\sin\omega t - \frac{\sin 3\omega t}{3^2} + \frac{\sin 5\omega t}{5^2} - \ldots). \qquad (3.4)$$

The output expression can then by shown to be

$$m_t = \frac{8B_2}{\pi^2}(\frac{M}{a})[\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cdot(\frac{a}{B_2})\sin n\theta]. \qquad (3.5)$$

The amplitude of the fundamental component of the output signal, m_t, is given by

$$m_t \simeq \frac{8B_2}{2}(\frac{M}{a})\sin\frac{\pi}{2}(\frac{a}{B_2}). \qquad (3.6a)$$

for $(\frac{a}{B_2})$ sufficiently small, m_t may be approximated as

$$m_t = M\left\{\frac{4}{\pi} - \frac{\pi}{6}(\frac{a}{B_2})^2\right\}. \qquad (3.6b)$$

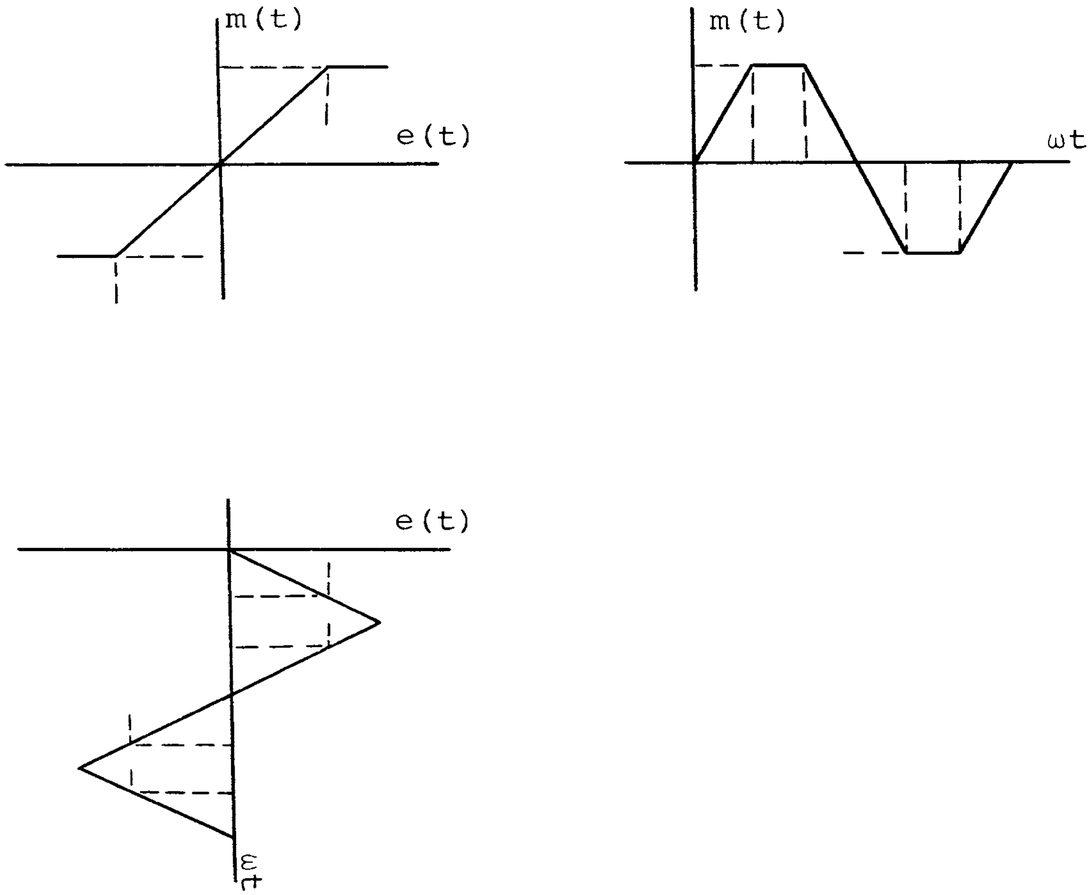


Figure 3.2. Output Waveform for Triangular Dither Input

By considering equations 3.3 and 3.6b, it is evident that if

$$B_2 = \frac{\pi}{2} B_1, \qquad (3.7)$$

the output of the limiter to either sinusoidal or triangular-wave input is approximately the same.

For other nonlinearities the same procedure might be exployed to show the similarity of the effect of these two dither signals. Therefore, for analyzing triangular wave stabilization, the method utilized in Chapter II for the sinusoidal case will be used here.

C. Equivalent Gain

The classical describing function technique cannot be used to analyze the stabilizing effect of an additional dither signal because the input to the nonlinear element is not a single sine wave. A signal stabilized system was shown in Figure 1.1. It was assumed that the system sustained an oscillation with frequency ω. Therefore, in the absence of the stabilizing signal, the input of the nonlinear element is assumed to be of the form $E \sin \omega t$. The linear portions of the system must be effective enough filters so that only the fundamental component of the output of the system is propogated to the input of the nonlinearity. This condition is the standard assumption of describing function analysis and is satisfied in a great many physical situations.

The stabilizing signal may be introduced at any point in the system as long as the total input to the nonlinear element is of the form

$$e(t) = f(\beta,B) + E \sin \omega t, \qquad (3.8)$$

where, $f(\beta,B)$ is the triangular wave stabilizing signal with amplitude B and frequency $\beta >> \omega$. In other words, the sinusoidal input signal, $E \sin \omega t$, varies slowly compared to the triangular-wave, as shown in Figure 3.3. During each period of the triangular wave dither, the sinusoidal input is nearly a constant, and can be approximated by a staircase function $F(t)$. The width of each step of $F(t)$ corresponds to one period of the triangular wave dither input, so that the actual input can be approximated by

$$e(t) = f(\beta,B) + F(t), \qquad (3.9)$$

as shown in Figure 3.4. Let the value of $F(t)$ for one step be denoted by A_o. Then the input for the period of this step may be approximated by

$$e(t) = f(\beta,B) + A_o. \qquad (3.10)$$

Thus, the input consists of a triangular wave, $f(\beta,B)$, and a bias of value A_o. Because of this bias, the output should contain some non-zero, dc value, A_v. The equivalent gain is defined as

$$g(A_o,B) = \frac{\text{average value of the output}}{\text{average value of the input}} = \frac{A_v}{A_o}\ . \qquad (3.11)$$

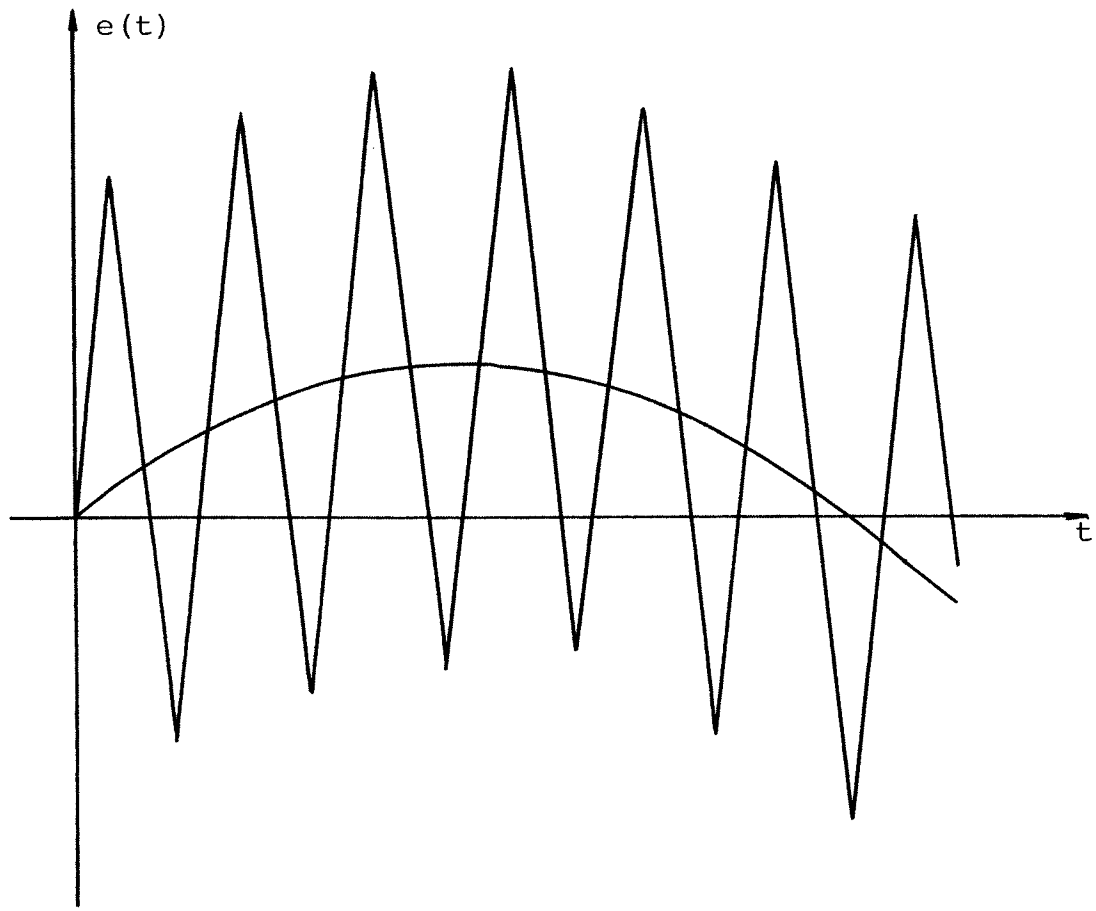


Figure 3.3. Input to the Nonlinear Element of the Form $e(t)=E\sin\omega t+f(\beta,B)$

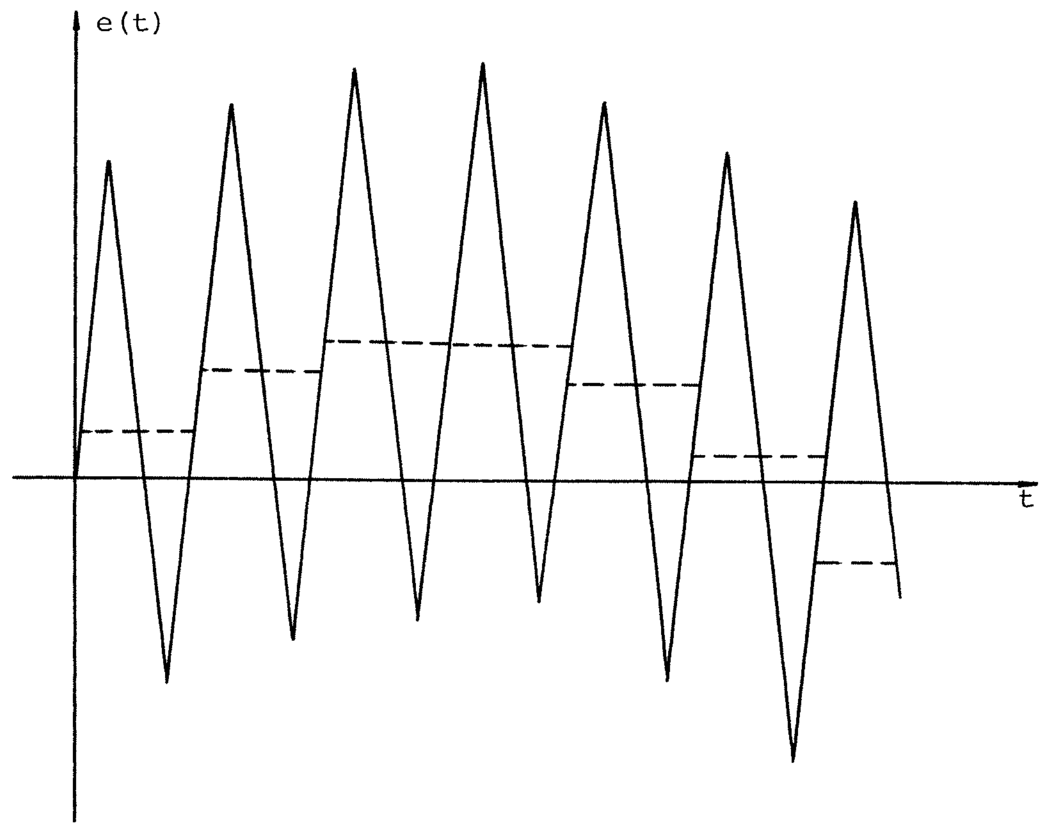


Figure 3.4. Approximate Input of the Form $e(t) = f(\beta,B) + F(\omega,E)$.

This equivalent gain is applied to signal stabilization by letting the input of the nonlinear element be the sum of the stabilizing signal and a bias A_o. It is assumed that the average value of the stabilizing signal is zero. The triangular-wave stabilizing signal satisfies this assumption. It is now necessary to derive expressions for the equivalent gain, $g(A_o,B)$, of several different nonlinear elements for a triangular wave stabilizing signal.

Consider a symmetrical nonlinear element as shown in Figure 3.5. Let e(t) given by equation 3.10 represent the total input to the nonlinearity. Thus, the input may be represented as

$$e(t) = \frac{2B}{\pi}\beta t + A_o , \qquad 0 \leq \beta t \leq \frac{\pi}{2} ,$$

$$e(t) = -\frac{2B}{\pi}\beta t + 2B + A_o, \quad \frac{\pi}{2} \leq \beta t \leq \frac{3\pi}{2} ,$$

$$e(t) = \frac{2B}{\pi}\beta t - 4B + A_o, \quad \frac{3}{2} \leq \beta t \leq 2\pi. \tag{3.11}$$

The output of the nonlinear element will be (see Figure 3.6)

$$\begin{aligned} m(t) &= f(e), & \theta_1 < \beta t < \theta_2, \\ m(t) &= M, & \theta_2 < \beta t < \theta_3, \\ m(t) &= f(-e) = -f(e), & \theta_3 < \beta t < \theta_5, \\ m(t) &= -M, & \theta_5 < \beta t < \theta_6, \\ m(t) &= f(e), & \theta_6 < \beta t < \theta_7. \end{aligned} \tag{3.12}$$

Now θ_1 can be found from equation 3.11 as follows

$$e(t) = b = \frac{2B}{\pi} \theta_1 + A_o,$$

$$\theta_1 = \left(\frac{b-A_o}{2B}\right)\pi. \tag{3.13a}$$

The θ_i, $i=2,3,\ldots,7$ are found by exactly the same procedure as θ_1, and the results are

$$\theta_2 = \left(\frac{c-A_o}{2B}\right)\pi,$$

$$\theta_3 = \left(\frac{2B + A_o - a}{2B}\right)\pi,$$

$$\theta_4 = \left(\frac{2B + A_o + b}{2B}\right)\pi,$$

$$\theta_5 = \left(\frac{2B + A_o + c}{2B}\right)\pi,$$

$$\theta_6 = \left(\frac{4B - A_o - a}{2B}\right)\pi,$$

$$\theta_7 = \left(\frac{4B - A_o + b}{2B}\right)\pi. \tag{3.13b}$$

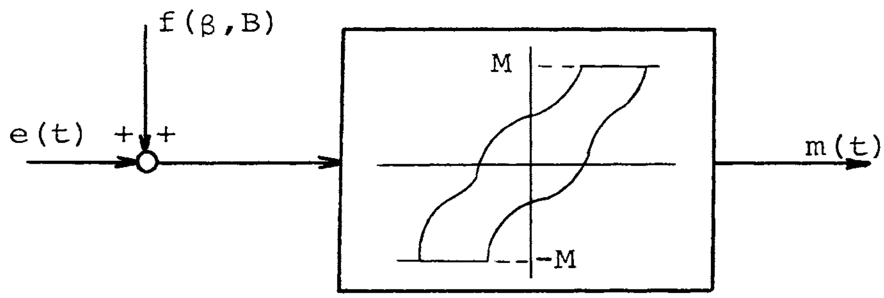


Figure 3.5. A Symmetric Saturating Nonlinear Element.

The average value of the output may be found by calculating the algebraic sum of the areas, S_i, i=1,2,...6, shown in Figure 3.6. This yields

$$A_v = \frac{1}{2\pi}\int_0^{2\pi} m(t)d(\beta t) = \frac{1}{2\pi}\sum_{i=1}^{n} S_i. \tag{3.14}$$

Note that $S_1 = -S_4$, due to the summetry properties assumed for the nonlinearity. The same argument is true for S_3 and S_6. Thus,

$$\begin{aligned} A_v &= \frac{1}{2\pi}\int_0^{2\pi} m(t)dt = \frac{1}{2\pi}[S_2 + (-S_5)] \\ &= \frac{1}{2}[M(\theta_3 - \theta_2) - M(\theta_6 - \theta_5)] \\ &= \frac{M}{2}[\theta_3 - \theta_2 - \theta_6 + \theta_5] = \frac{MA_o}{B}, \end{aligned} \tag{3.15}$$

or

$$g(A_o,B) = \frac{M}{B} \tag{3.16}$$

which is quite an interesting result. This consideration proves that the equivalent gain $g(A_o,B)$ for a symmetrical nonlinear element depends on the saturation level, and the triangular-wave amplitude, B. All of these are constants. The following theorem may then be stated.

a) Equivalent Gain Theorem: The application of a triangular-wave dither signal to a symmetrical saturating nonlinearity results in a constant equivalent gain.

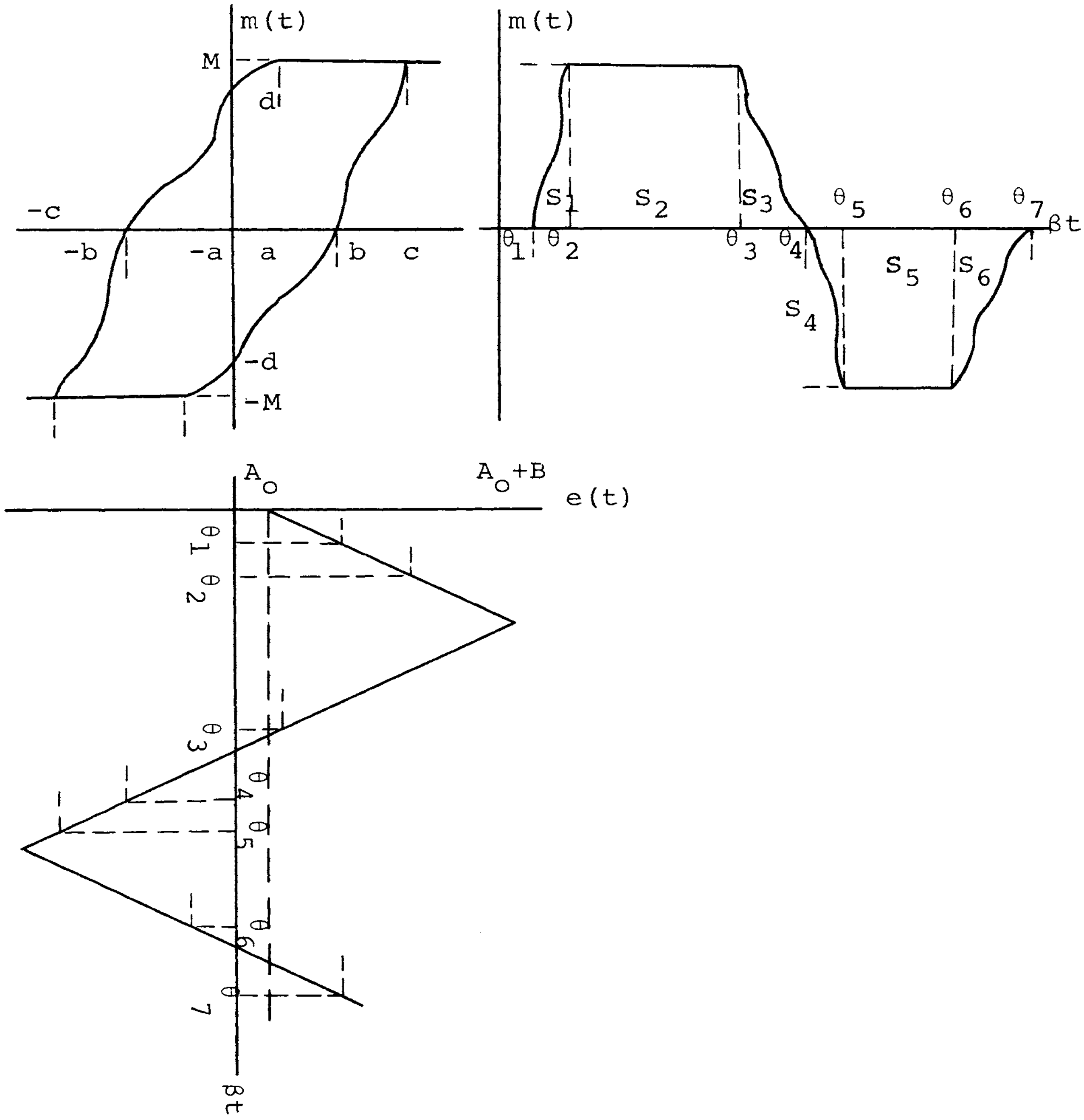


Figure 3.6. Equivalent Gain for a General Symmetric Saturating Nonlinear Element.

It is interesting to consider another somewhat general example.

b) Example 1: Equivalent Gain of the Gain-Changing Nonlinearity with Dead-band

Figure 3.7 shows the characteristic of the gain changing nonlinearity with dead-band, its input which is of the form given by equations 3.10 and 3.11, and also its corresponding output. The input-output characteristic equations of such a nonlinearity are given by

$$\begin{aligned}
m(t) &= 0 && |e(t)|<a, \\
m(t) &= n[e(t)-a], && a\le e(t)\le a+b, \\
m(t) &= p[e(t)-(a+b)] + bn, && e(t) > a+b, \\
m(t) &= n[e(t) + a], && -(a+b) \le e(t)< -a, \\
m(t) &= p[e(t) + (a+b)]-bn, && e(t)< -(a+b).
\end{aligned} \quad (3.17)$$

The basic angles of the discontinuity in the output are easily obtained as

$$\begin{aligned}
\theta_1 &= \left(\frac{a + b - A_o}{2B}\right)\pi, \\
\theta_2 &= \left(\frac{2B + A_o - (a+b)}{2B}\right)\pi, \\
\theta_3 &= \left(\frac{2B + A_o + (a+b)}{2B}\right)\pi, \\
\theta_4 &= \left(\frac{4B - A_o - (a+b)}{2B}\right)\pi.
\end{aligned} \quad (3.18)$$

The maximum and minimum amplitudes of the output are obtained by the use of equations 3.17 and the maximum and minimum amplitudes of the input which are given by

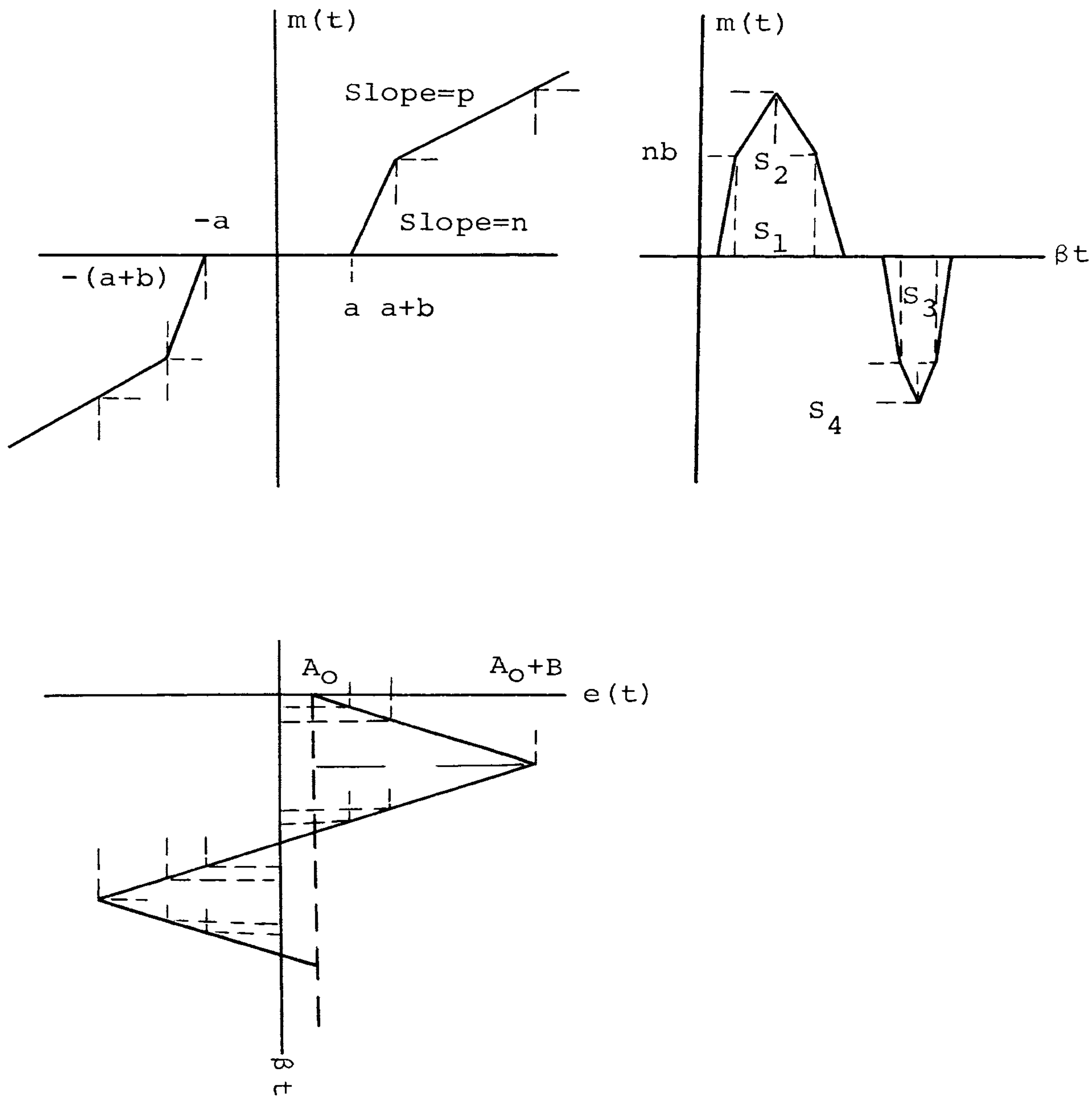


Figure 3.7. Gain Changing Nonlinear Element With Dead Band.

$$E_{max} = A_o + B,$$

$$E_{min} = A_o - B. \tag{3.19}$$

The maximum and minimum values of the output are:

$$M_{max} = P[A_o + B - (a+b)] + bn,$$

$$M_{min} = P[A_o - B + (a+b)] - bn. \tag{3.20}$$

The average value of the output, A_v, is given by equation 3.14. Here the S_i are the areas represented in Figure 3.7.

$$S_1 + S_3 = nb(\theta_2 - \theta_1) + (-nb)(\theta_4 - \theta_3) = \frac{2\pi n\ A_o b}{B}$$

$$S_2 + S_4 = \frac{1}{2}(\theta_2 - \theta_1)(M_{max}-nb) + \frac{1}{2}(\theta_4 - \theta_3)(M_{min}+nb)$$

$$= \frac{2P\pi A_o}{B}[B-(a+b)].$$

Thus,

$$A_v = \frac{1}{2\pi}\sum_{i=1}^{4} S_i = \frac{n\ A_o b}{B} + \frac{P\ A_o}{B}[B-(a+b)]. \tag{3.21}$$

The equivalent gain is then given by

$$g(A_o,B) = \frac{A_v}{A_o} = \frac{nb}{B} + \frac{P}{B}\ [B-(a+b)]. \tag{3.22}$$

The equivalent gain for several other nonlinearities can be obtained from equation 3.22 by proper choice of parameters n, P, a or b. For example, by letting a = 0 in equation 3.22 the equivalent gain expression for the gain changing nonlinearity is obtained. Another example of this case is the preload nonlinearity. By letting a + b = 0 in equation

3.22, the equivalent gain expression for the preload nonlinearity is obtained. Note that nb = M must be substituted in the new expression. The equivalent gain expression for the dead band is obtained by letting P=n in equation 3.22. One more example is the relay with dead-band and gain which is obtained by letting nb = m $\neq$ 0, and b = 0.

Now, consider the equivalent gain for the backlash nonlinear element, since it can not be obtained from the equivalent gain theorem or from the results of the previous example.

c) Example 2: Equivalent Gain of the Backlash Element

Figure 3.8 shows the characteristic of the backlash element, its input which is of the form given by equations 3.10 and 3.11, and also its corresponding output. The basic angles of the discontinuity in the output are easily obtained as

$$\theta_1 = \left(\frac{a-A_o}{2B}\right)\pi,$$

$$\theta_2 = \frac{\pi}{2},$$

$$\theta_3 = \left(\frac{B+2a}{2B}\right)\pi,$$

$$\theta_4 = \left(\frac{2B+A_o+a}{2B}\right)\pi,$$

$$\theta_5 = \frac{3\pi}{2},$$

$$\theta_6 = \left(\frac{3B+2a}{2B}\right)\pi,$$

$$\theta_7 = \left(\frac{4B-A_o+a}{2B}\right)\pi. \qquad (3.23)$$

The maximum and minimum amplitudes of the output occur whenever the input reaches the maximum and minimum amplitudes given by equation 3.19. These two values are as follows,

$$M_{max} = n(A_o+B)-na,$$

$$M_{min} = n(A_o-B)+na. \tag{3.24}$$

The average value of the output, A_v, is given by Equation 3.14. The S_i are represented in Figure 3.8

$$S_1 = \frac{1}{2} M_{max}[(\theta_4 - \theta_1) + (\theta_3 - \theta_2)] = \frac{n\pi}{2B}[(A_o+B)^2 - a^2],$$

$$S_2 = \frac{1}{2} M_{min}[(\theta_7 - \theta_4) + (\theta_6 - \theta_5)] = -\frac{n\pi}{2B}[(A_o-B)^2-a^2].$$

Thus

$$A_v = \frac{1}{2\pi} \sum_{i=1}^{2} S_i = n A_o. \tag{3.25}$$

The equivalent gain is then given by,

$$g(A_o,B) = \frac{A_v}{A_o} = n. \tag{3.26}$$

The equivalent gain expressions for several common nonlinearities are represented in Table 3.2. Those characteristics which are obtainable from the equivalent gain theorem are indicated by asterisks, and those which follow from Example are indicated by daggers.

The equivalent gain may be used to develop an equivalent nonlinear element. To simplify system computations, the equivalent nonlinear element can be

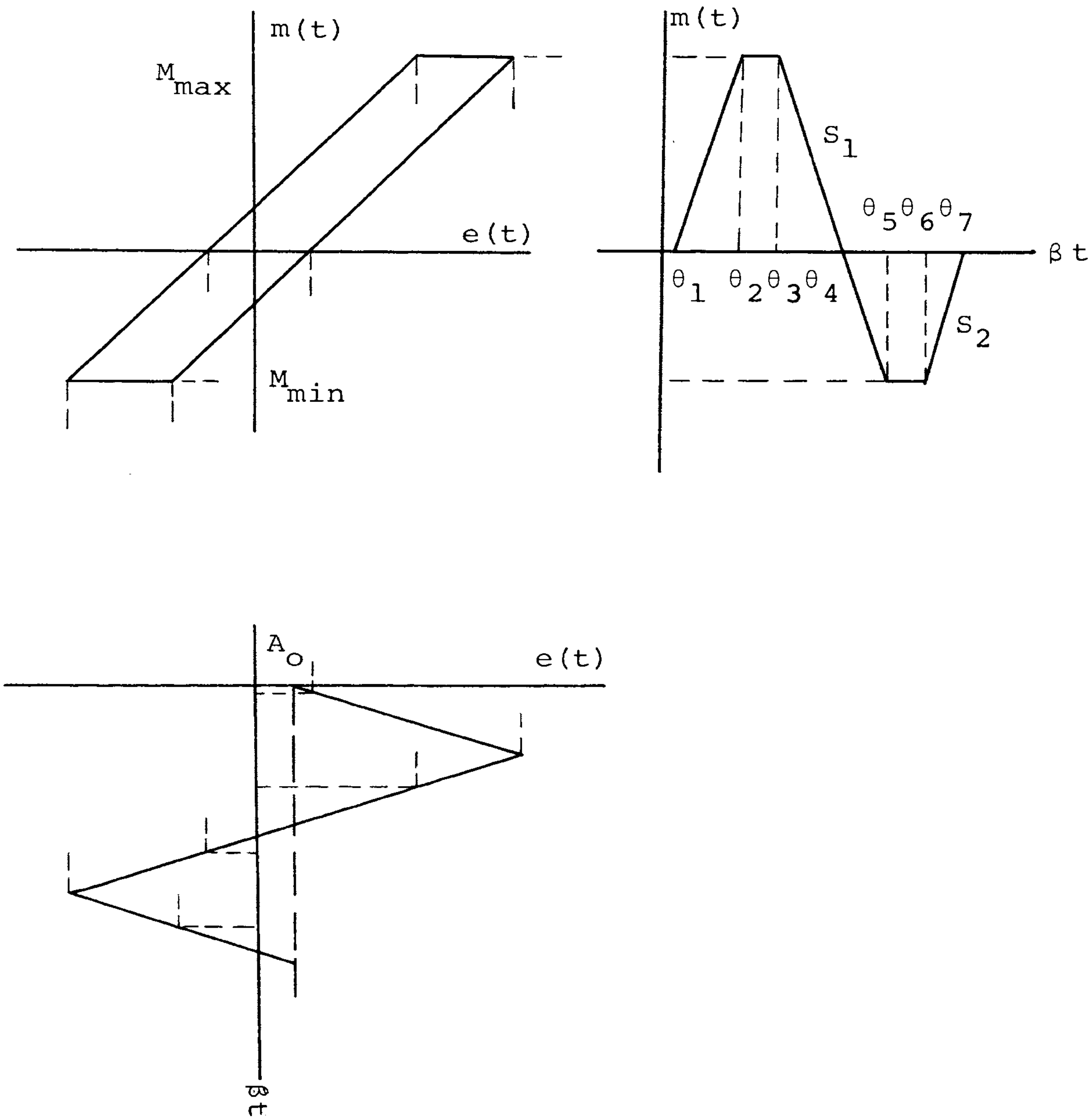


Figure 3.8. Backlash Nonlinear Element Characteristic.

substituted for the extra sinusoidal input signal and the existing nonlinear element. The accuracy of the approach is comparable to the accuracy of describing function analysis of systems unexcited by stabilizing inputs.

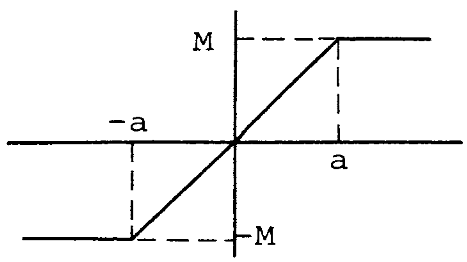
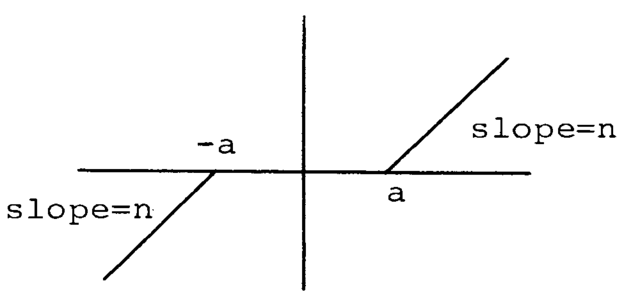
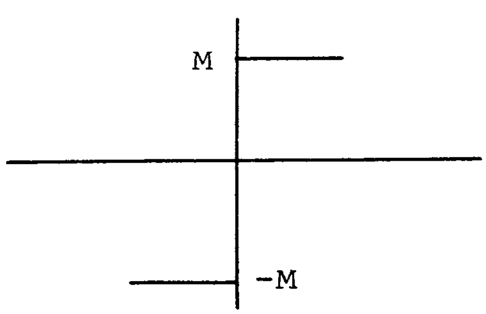
D. Equivalent Nonlinear Element

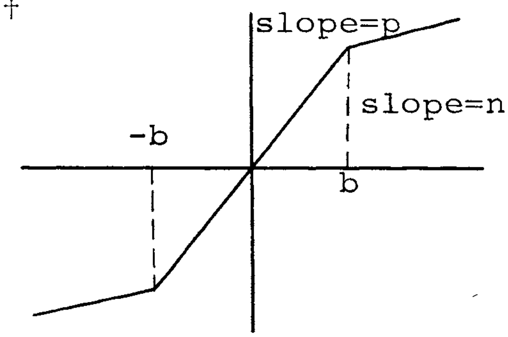
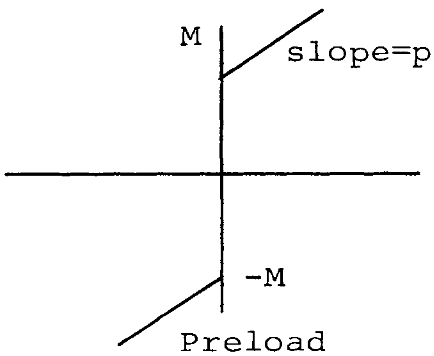
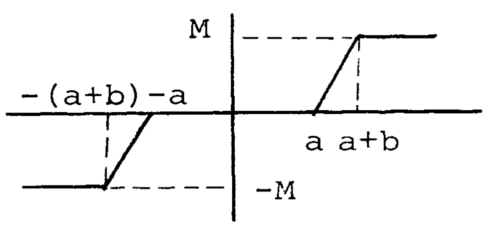
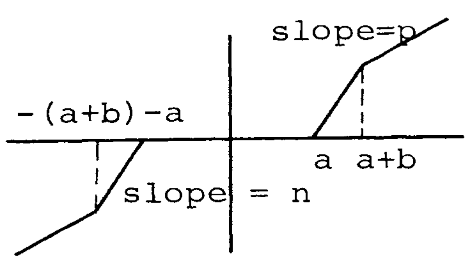
As was mentioned in the previous section, the input to the nonlinear element is the sum of two signals with greatly different frequencies. As the frequency ratio is increased, the low frequency component of the input signal approaches a constant for any given stabilizing signal period. This was given by equation 3.10. If the low frequency component of the input signal is considered a constant over each stabilizing signal period, the equivalent gain can be applied to the input signal by letting the instantaneous values of the low frequency component of the input signal equal A_0, and the instantaneous values of a representative output equal the average value of the output, A_v, determined by the equivalent gain calculations of section C. Thus, a plot of the average value of the output, A_v, versus the average value of the input, A_o, with B as a parameter represents an altered input-output characteristic because the low frequency component of the input signal is not actually a constant for any period of stabilizing signal. However, this

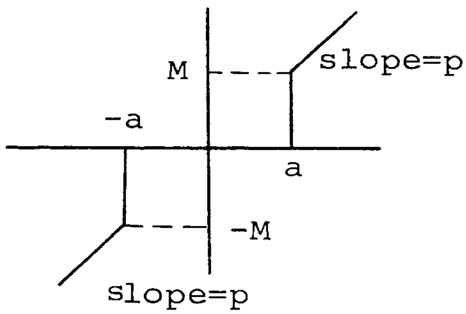
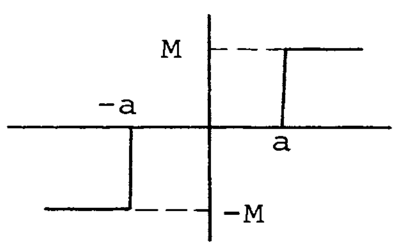
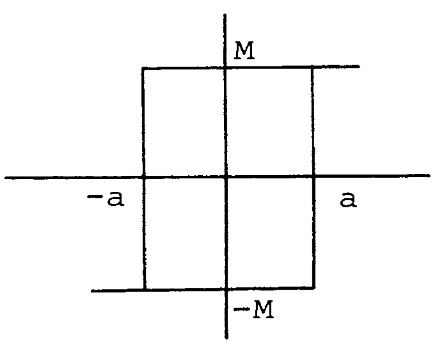
TABLE 3.2

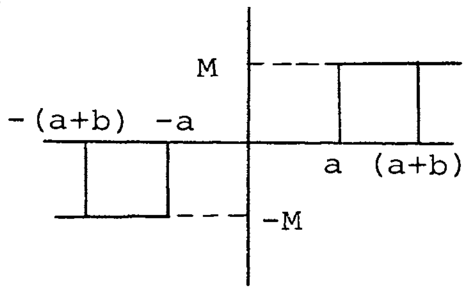
EQUIVALENT GAIN FOR TRIANGULAR-WAVE SIGNAL STABILIZATION

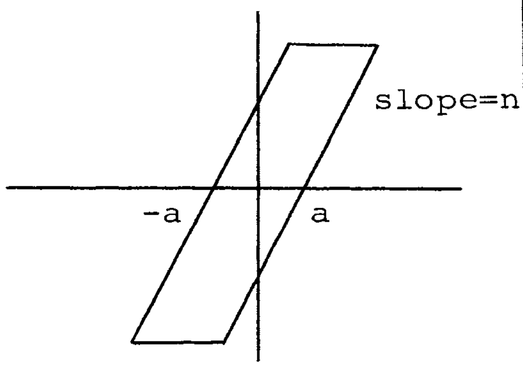
B = Amplitude of the Triangular Wave Input

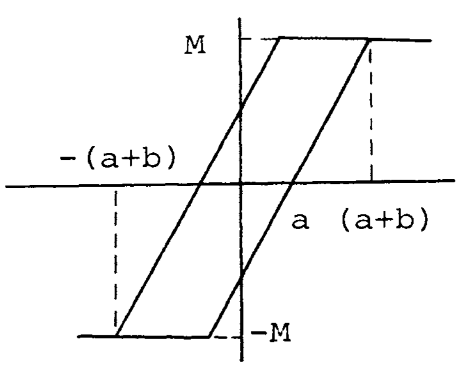
Nonlinear Element	Equivalent Gain
*†  Limiter	$g(A_o,B) = \frac{M}{B}$
†  Dead Band	$g(A_o,B) = \frac{n}{B}\ (B-a)$
*†  Relay	$g(A_o,B) = \frac{M}{B}$

Nonlinear Element	Equivalent Gain
†  Gain Changing	$g(A_o,B) = \frac{nb}{B} + \frac{p}{B}(B-b)$
†  Preload	$g(A_o,B) = \frac{M}{B} + p$
*†  Limiter With Dead-Band	$g(A_o,B) = \frac{M}{B}$
†  Gain Changing With Dead-Band	$g(A_o,B) = \frac{nb}{B} + \frac{p}{B}[B-(a+b)]$

<table>
<tr><th>Nonlinear Element</th><th>Equivalent Gain</th></tr>
<tr><td>†

Relay With Dead-Band and Gain</td><td>$g(A_o,B) = \frac{M}{B} + \frac{p}{B}(B-a)$</td></tr>
<tr><td>*†

Relay With Dead-Band</td><td>$g(A_o,B) = \frac{M}{B}$</td></tr>
<tr><td>*

Relay With Hysteresis</td><td>$g(A_o,B) = \frac{M}{B}$</td></tr>
</table>

<table>
<tr><th>Nonlinear Element</th><th>Equivalent Gain</th></tr>
<tr><td>*


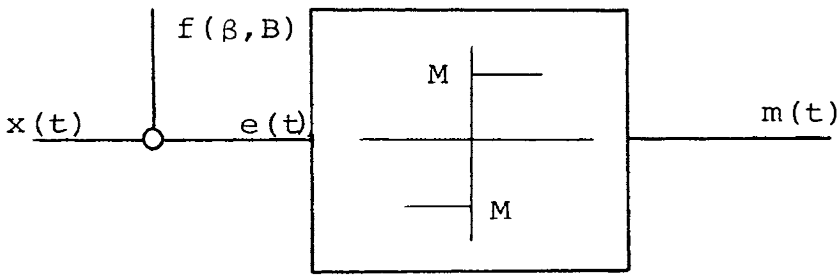
Relay With Dead-Band and Hysteresis</td><td>$g(A_o,B) = \frac{M}{B}$</td></tr>
<tr><td>

Backlash</td><td>$g(A_o,B) = n$</td></tr>
<tr><td>*


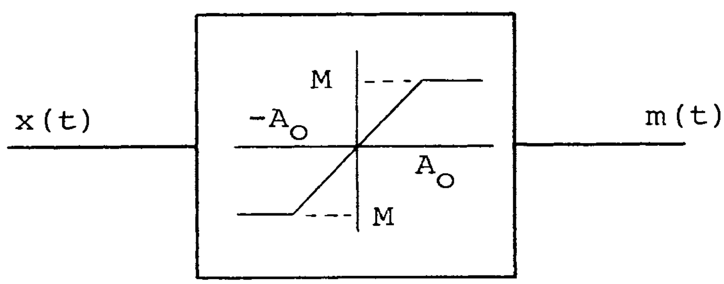
Limiter with Hysteresis</td><td>$g(A_o,B) = \frac{M}{B}$</td></tr>
</table>

approximation approaches the actual altered characteristic as the frequency ratio of the input signals is increased.

When the input to the actual nonlinear element is of the form given by equation 3.9, a representative output waveform can be found by considering F(t) as the input to the altered characteristic and plotting the resulting output. This output will be called the representative output wave. The altered characteristic of the relay of Figure 3.9a is plotted in Figure 3.9b. The representative output wave for the relay is shown in Figure 3.10, from which it is obvious that at any instant the output of the equivalent nonlinear element is less than or equal to the output of the actual nonlinearity. In general, if the input to the nonlinear element is e and the output a differentiable function f(e), the introduction of an extra triangular signal with proper frequency and amplitude will decrease the slope $f'(e)$ at $e = 0$ to a lower value than in the original nonlinearity. Different equivalent nonlinearities may usually be obtained from a given nonlinear element by varying the amplitude B of the dither signal. However, the equivalent gains and the equivalent nonlinearity, or altered characteristic, of the nonlinear elements which satisfy the conditions of the theorem in the previous section are identical. The equivalent gain and the equivalent nonlinear element for the backlash



a) Relay Nonlinearity



b) Equivalent Nonlinearity

Figure 3.9. Equivalent Relay Characteristic.

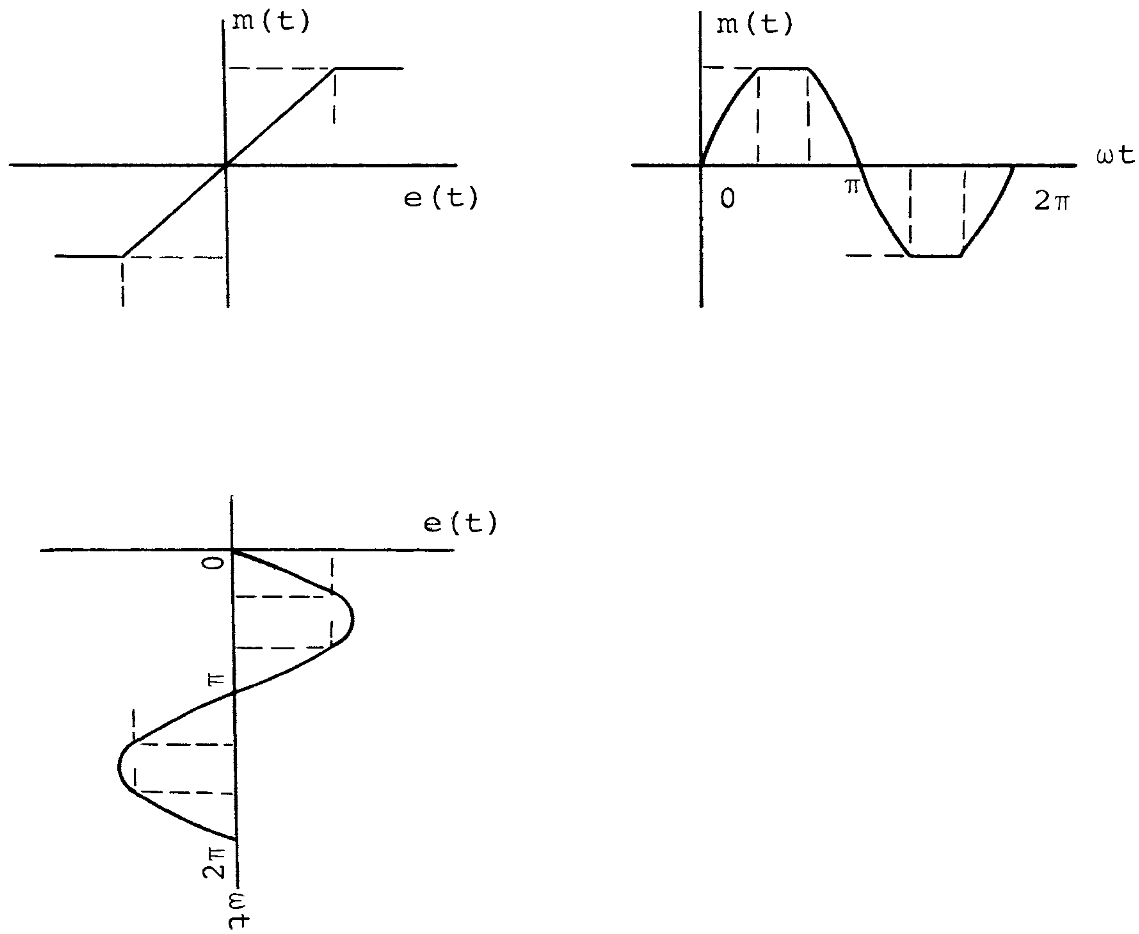


Figure 3.10. Representative Output Waveform for the Relay With Triangular Wave Dither.

element are independent of the amplitude B of the dither signal if $B > a$, where 2a is the width of the backlash loop. For the backlash element, when the input e(t) switches direction (from increasing to decreasing, or vice-versa), the input must change by 2a before the output will follow the input.

Although the representative output wave is not the actual output of the nonlinear element when the sum of a sinusoidal signal and a triangular wave dither are introduced to the nonlinearity, it is assumed that the representative output wave contains the amplitude and phase of the fundamental component of the actual output of the nonlinear element. By using the equivalent nonlinear element concept it is possible to define the pseudo describing function.

E. Pseudo Describing Function

As was mentioned in the previous chapter, the conventional describing function is used in the stability analysis of nonlinear systems with no external inputs. With an external input, the modified describing function can be defined. This is the ratio of the fundamental component of the output to the low frequency component of the input. Practical application usually requires the use of a digital computer or other computational aid to compute this quantity.

The representative output wave derived in the preceding section can be used to approximate the modified describing function. This approximation is called the pseudo describing function. The pseudo describing function is defined as the ratio of the amplitude of the fundamental component of the representative output wave to the amplitude of the fundamental component of the input.[10,11] The input to the nonlinear element is in general the sum of two signals with greatly different frequencies, and the lowest of the two input frequencies in the fundamental frequency.

The pseudo describing function is an approximation because two assumptions are necessary for its derivation:

1) the low frequency component of the input can be considered a constant for any period of the stabilizing signal, and
2) the representative output waveform contains the amplitude and phase of the fundamental component of the actual output of the nonlinear element.

Since mathematical difficulties were encountered with the direct calculation of the pseudo describing function, Ochiai and Oldenburger introduced the concepts of the space derivative of the nonlinearity and the second describing function.[16] This approach will be reviewed here, since it will be used later in this section.

The input-output relationship of commonly occuring nonlinear elements is usually represented by a curve in two dimensional space. The derivative of the output of the element with respect to the input of the element is called the space derivative of the nonlinearity (SDN). For piecewise linear elements the SDN is sectionally constant with respect to the input. Use of this property of the SDN substantially reduces the computational work required for obtaining the conventional and pseudo describing functions.

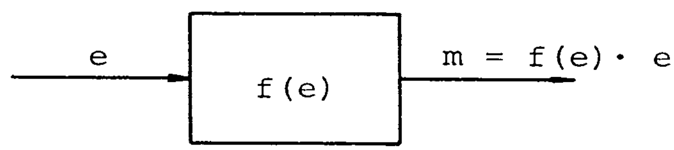
1. The SDN and the Second Describing Function

With reference to Figure 3.11, the relationship between the input e to a nonlinear element and the time derivative $\frac{dm}{dt}$ of the output m is as follows:

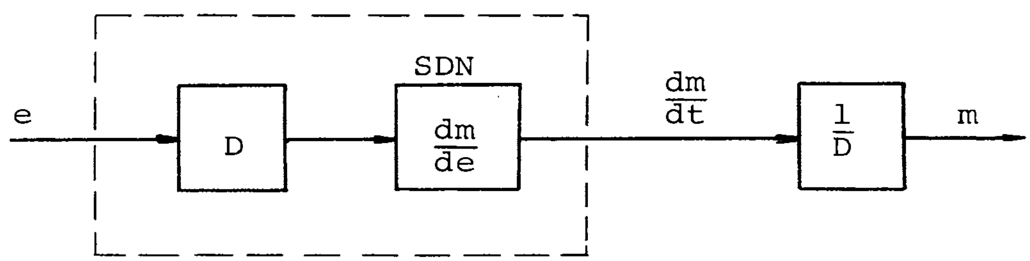
$$\frac{dm}{dt} = \frac{dm}{de} \cdot \frac{de}{dt} = \frac{dm}{de} De, \qquad (3.27)$$

where $D = \frac{d}{dt}$. As shown in the figure, the nonlinear element is divided into the SDN, or $\frac{dm}{de}$, and two linear elements. It is convenient to think of e as the input to a nonlinear element with the characteristic $\frac{dm}{de}$ D and output $\frac{dm}{dt}$, instead of as the input to a nonlinear element with the characteristic f(e) and output m.

If the describing function of the given nonlinearity with the characteristic $m = f(e) \cdot e$ of Figure 3.11a is



a) Nonlinear Element



b) Equivalent SDN

Figure 3.11. Nonlinearity Described by SDN

called the first describing function, N_1, of this nonlinear element, it is possible to define the describing function of the second nonlinearity with the characteristic $\frac{dm}{dt} = \frac{dm}{de}$ De of Figure 3.11b to the second describing function, N_2, of the given nonlinear element.

2. Pseudo Describing Function and Equivalent SDN

Consider a time invariant nonlinear element, the input of which is a triangular signal $f(\beta,B)$ with a constant bias A_o, shown in Figure 3.12. Let A_v denote the average of the output of the nonlinear element for one period of the triangular signal. In section D the given nonlinearity was replaced by the equivalent nonlinearity characterized by the equivalent gain $\frac{A_v}{A_o}$, as is represented in Figure 3.13. Let $A_o = E \sin \omega t$, where the frequency β of the triangular wave signal is large compared to ω. This means that $A_o = E \sin \omega t$ may be treated as a constant value for one period of $f(\beta,B)$.

The pseudo describing function is the describing function of the equivalent nonlinearity. That is,

$$P(E,B) = \frac{1}{\pi E}\left[\int_0^{2\pi} A_v \sin \omega t \, d(\omega t) + j\int_0^{2\pi} A_v \cos \omega t \, d(\omega t)\right]. \tag{3.28}$$

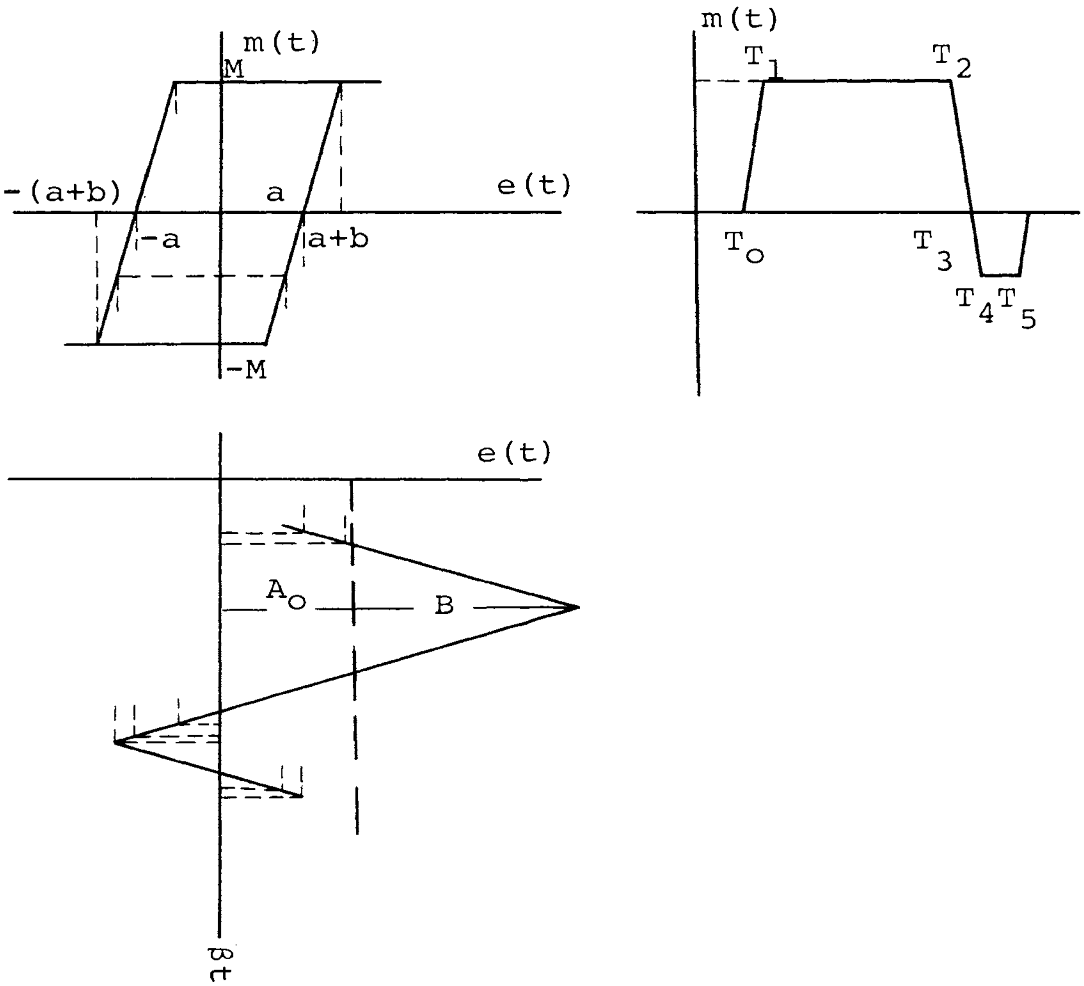


Figure 3.12. Triangular Signal With a Constant Bias.

The concept of the SDN may be used to simplify the computation of the integrals in equation 3.28.

The describing function of a nonlinear element with the characteristic $\frac{dA_v}{dt}$, or equivalently the second pseudo describing fucntion, $P_2(E,B)$, of a nonlinearity with the characteristic A_v, is introduced as follows,

$$P_2(E,B) = \frac{1}{\pi E}\left[\int_0^{2\pi} \frac{dA_v}{dt} \sin \omega t \, d(\omega t) + j\int_0^{2\pi} \frac{dA_v}{dt} \cos \omega t \, d(\omega t)\right], \tag{3.29}$$

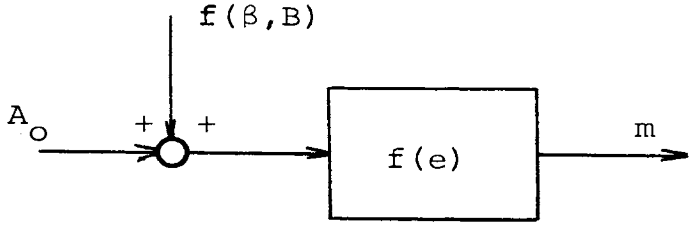
where $\frac{dA_v}{dt}$ is sectionally continuous. It is noted that the relationship of Figures 3.13b and 3.13c is comparable to that of Figures 3.11a and 3.11b. In equation 3.29

$$\frac{dA_v}{dt} = \frac{dA_v}{dA_o} \cdot \frac{dA_o}{dt} \tag{3.30}$$

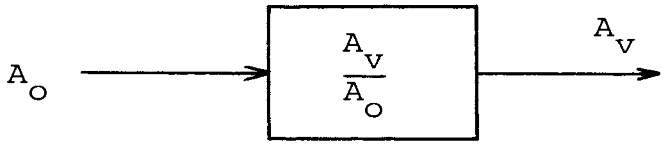
Using partial integration techniques in equation 3.20 and comparing the result with equation 3.28 yields the following relation:

$$P_2(E,B) = j\omega P(E,B) \tag{3.31}$$

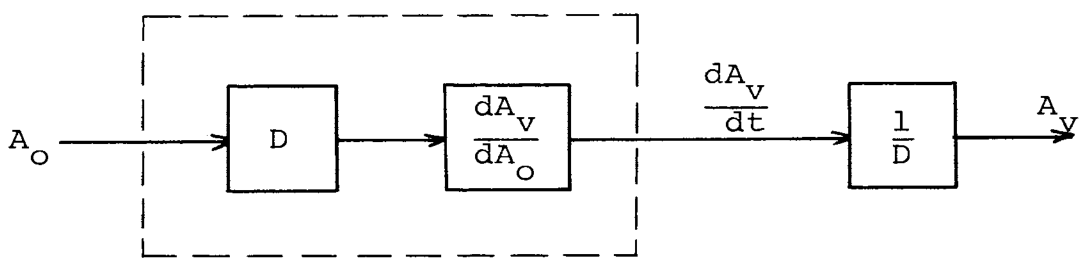
For an extra sinusoidal input it is generally very difficult to carry out the integrations in equation 3.29 analytically. Often, however, it is possible to develop an expression in closed form for the pseudo describing function if a triangular wave dither is used.



a) Given Nonlinearity



b) Equivalent Nonlinearity



c) Nonlinearity with Characteristic Equivalent SDN

Figure 3.13. Nonlinearity Described by Various Characteristics.

By use of the following theorem, one can simplify the expression for the pseudo describing function given by equation 3.28.

a. Phase Shift Theorem: The phase shift of the pseudo describing function of a symmetrical nonlinearity is zero.

The proof of this useful theorem may be developed as follows. From equations 3.29 and 3.31 it is noted that the theorem is true if it can be shown that $\frac{dA_v}{dt}$ is an even function of ωt. But, for a symmetrical nonlinearity A_v is an odd function of the input to the nonlinearity, A_o. Hence, $\frac{dA_v}{dA_o}$ is an even function of A_o. However, $A_o = E \sin \omega t$ is an odd function of ωt, and thus $\frac{dA_v}{dA_o}$ is an even function of ωt. Since the product of two even functions is even, from equation 3.30 $\frac{dA_v}{dt}$ is even in ωt. Thus, the first integral in equation 3.29 vanishes, $P_2(E,B)$ is real, and the theorem is proved.

The procedure for obtaining the pseudo describing function by the use of the second describing function is illustrated in the following example.

B. Example: For a limiter with hysteresis, shown in Figure 3.12, the equivalent SDN is as follows

$$\frac{dA_v}{dA_0} = \begin{cases} \frac{M}{A} & 0 < A_o < |B-(a+b)| , \\ \frac{M}{2Bb}(B+b-A_o), & |B-(a+b)| < A_o < B - a+b, \\ 0 & b - a+b < A_o, \end{cases} \qquad (3.32)$$

where

$$A = \begin{cases} B & , B \geq a+b \\ b & , a \leq B < a+b. \end{cases} \qquad (3.33)$$

$B \geq a+b$ is assumed here. That is, in order to utilize the effect of the extra signal, the total amplitude of this signal is assumed to be equal to or greater than the width of the hysteresis band. As will be seen below only $\frac{dA_v}{dA_o}$ for the case $A_o > 0$ is needed for deriving the second describing function $P_2(E,B)$. By use of the phase shift theorem $P_2(E,B)$ may be obtained as follows,

$$P_2(E,B) = j\,\frac{4\omega}{\pi E}\Big[\int_0^{\omega t_1} \frac{M}{A}(E\cos\omega t)\cos\omega t\, d(\omega t)$$

$$+ \int_{\omega t_1}^{\omega t_2} \frac{M}{2bB}(B+b-E\sin\omega t)(E\cos\omega t)\cos\omega t\, d(\omega t)$$

$$+ \int_{\omega t_2}^{\pi/2} 0\,(E\cos\omega t)\cos\omega t\, d(\omega t)$$

$$= j\,\frac{M\omega}{\pi bB}\Big\{(B+b)(\omega t_2 + \tfrac{1}{2}\sin 2\,\omega t_2) - [\mathrm{sgn}(B - a+b)]\ x$$

$$(B-b)(\omega t_1 + \tfrac{1}{2}\sin 2\,\omega t_1) + \tfrac{2}{3}E(\cos^3\omega t_2 - \cos^3\omega t_1)\Big\}$$

where

$$\omega t_1 = \frac{\pi}{2}, \qquad \omega t_2 = \frac{\pi}{2}, \; 0 \leq E \leq |B-(a+b)|,$$

$$\omega t_2 = \sin^{-1} \frac{|B-(a+b)|}{E}, \omega t_2 = \frac{\pi}{2}, \; |B-(a+b)| < E \leq B-a+b,$$

$$\omega t_2 = \sin^{-1} \frac{|B-(a+b)|}{E}, \omega t_2 = \sin^{-1} \frac{B-a+b}{E}, \; B-a+b < E, \tag{3.35}$$

and

$$\text{sgn}(B-a-b) = \begin{cases} 1 & , B-a-b > 0, \\ -1 & , B-a-b \leq 0. \end{cases} \tag{3.36}$$

Substitution of $P_2(E,B)$ from equation 3.34 into equation 3.31 gives the following expression for the pseudo describing function:

$$P(E,B) = \frac{M}{\pi bB} [(B+b)(\omega t_2 + \frac{1}{2} \sin 2 \omega t_2)$$

$$- [\text{sgn}(B-a-b)](B-b)(\omega t_1 + \frac{1}{2} \sin 2 \omega t_1)$$

$$+ \frac{2}{3} E(\cos^3 \omega t_2 - \cos^3 \omega t_1)]. \tag{3.37}$$

The complete derivation of the equivalent SDN for the limiter with hysteresis is found in Appendix A.

By letting a = 0 in equations 3.35, 3.36 and 3.37 the pseudo describing function for the limiter is obtained.

One can obtain the pseudo describing function for the relay with hysteresis by applying l' Hopital's rule to equation 3.37, but the direct application of equation 3.29 is more straightforward. In this case,

$$\frac{dA_v}{dA_o} = \begin{cases} \frac{M}{B}, & 0 < A_o < B-a, \\ 0, & A_o > B-a. \end{cases} \qquad (3.38)$$

$B \geq a$ is assumed here. Thus,

$$P_2(E,B) = j\,\frac{4\omega}{\pi E}\left[\int_0^{\omega t_1} \frac{M}{B}(E \cos \omega t) \cos \omega t \; d(\omega t)\right.$$

$$\left. + \int_{\omega t_1}^{\pi/2} 0(E \cos \omega t) \cos \omega t \; d(\omega t)\right],$$

or

$$P(E,B) = \frac{M}{\pi B}(2\omega t_1 + \sin 2\omega t_1), \qquad (3.39)$$

where

$$\omega t_1 = \frac{\pi}{2}, \qquad 0 \leq E \leq B-a,$$

$$\omega t_1 = \sin^{-1}\frac{B-a}{E}, \qquad E > B-a. \qquad (3.40)$$

Letting $a = 0$ in equation 3.39 and 3.40, results in the pseudo describing function expression for the relay.

From these examples it is clear that the calculation of the pseudo describing function is straightforward if a triangular wave dither is used. Note that the number of terms in equation 3.29 is increased if the nonlinearity has several piecewise linear segments. For example, for the gain-changing element with dead-band there are five terms in the pseudo describing function expression.

By normalizing the input, output, and dimensions of the nonlinearity with respect to one parameter of the nonlinear element, it is sometimes possible to plot the pseudo describing function for different noramlized stabilizing signal amplitudes, B/a, on one graph. This is true only for dimensionless nonlinear elements. In Figure 3.14 the normalized pseudo describing function, P(E,B)/M, of the relay is plotted versus the amplitude E of the fundamental component of the input to the nonlinearity, with the amplitude of the stabilizing signal as a parameter. In Figures 3.15 and 3.16 the normalized pseudo describing function of the limiter and the limiter with hysteresis, respectively, are plotted versus the normalized amplitude, $\frac{E}{a}$, of the fundamental component of the input, with the normalized amplitude, B/a, of the stabilizing signal as a parameter.

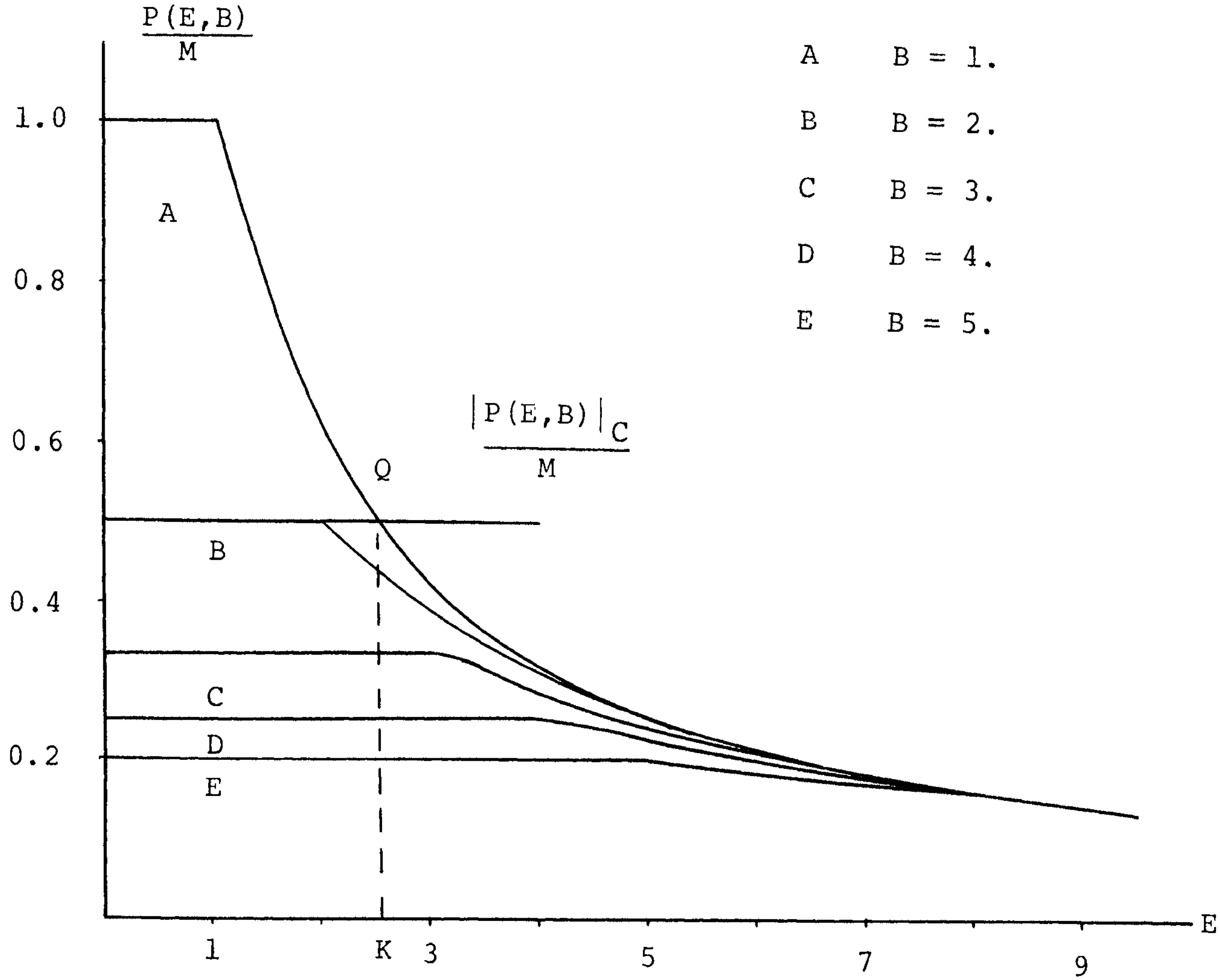


Figure 3.14. Normalized Pseudo Describing Function for the Relay.

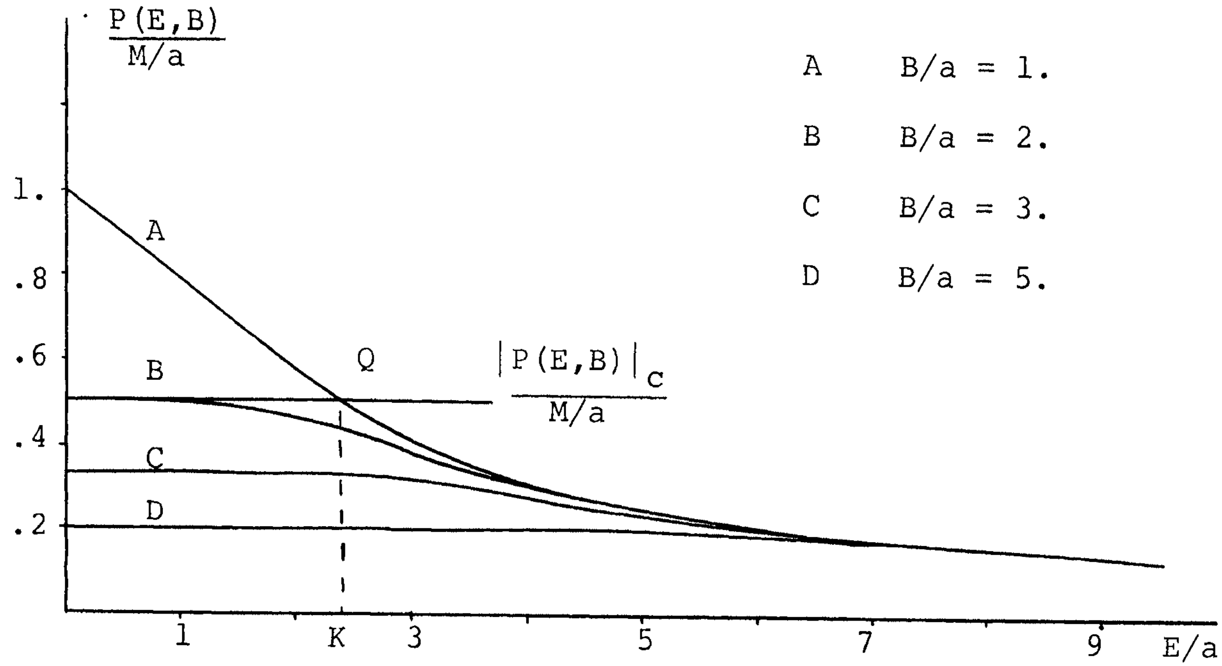


Figure 3.15. Normalized Pseudo Describing Function for the Limiter.

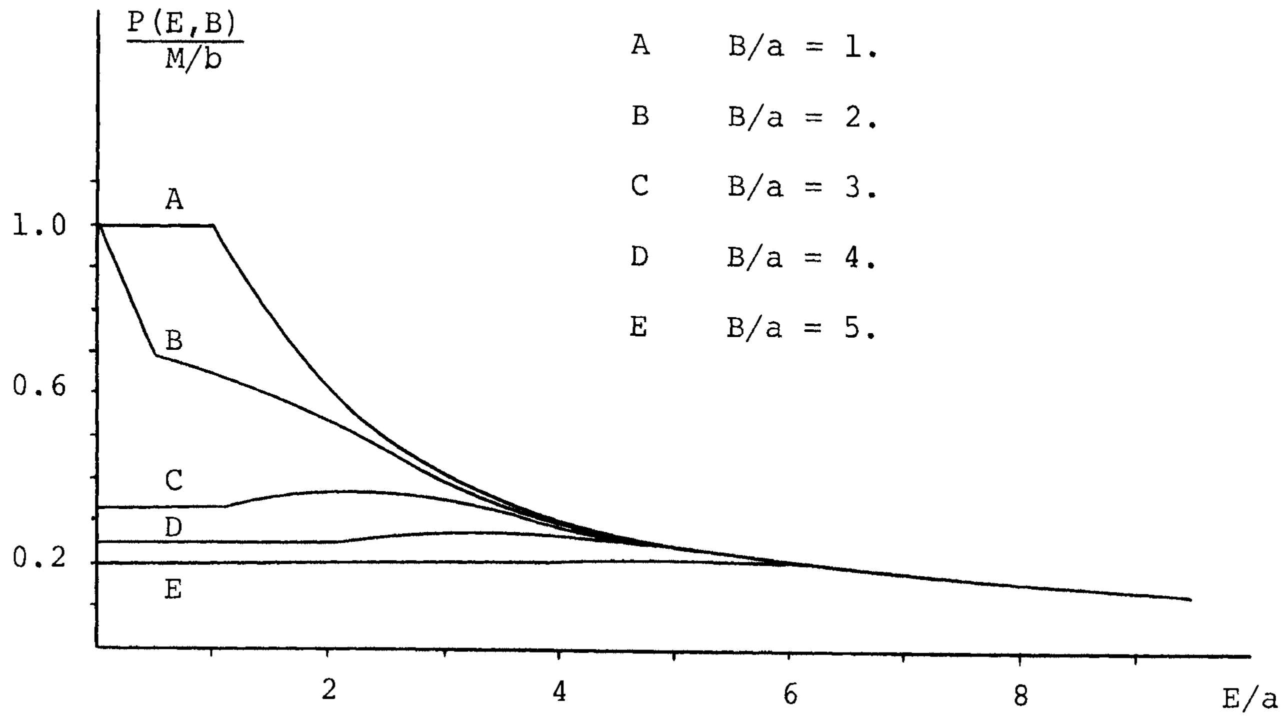


Figure 3.16. Normalized Pseudo Describing Function for Limiter With Hysteresis.

F. Pseudo Describing Function Analysis

As was mentioned in the previous chapter, the pseudo describing function may be used to determine the possibility of limit cycle operation for some closed loop nonlinear systems when a dither signal is present in the system. The basic assumptions of pseudo describing function analysis are as follows:

1. The nonlinearities of the system can be lumped into one time invariant and frequency insensitive nonlinear element,
2. The frequency of the dither signal must be on the order of 10 or more times the frequency of possible limit cycles, and
3. the system must contain sufficient low pass filtering so that the input to the nonlinear element can be approximated by an input of the form

$$e(t) = f(\beta,B) + E \sin(\omega t), \tag{3.41}$$

when the limit cycle exists.

From the describing function technique it is well known that for the system to have a limit cycle,

$$G(j\omega) = -\frac{1}{N(E,\omega)}, \tag{3.42}$$

where $G(j\omega)$ is the complex representation of the overall linear system transfer function and $N(E,\omega)$ is the describing function for the nonlinear element in the system. Assume that the general system shown in Figure 1.1 satisfies the pseudo describing function requirements. Let $P(E,B)$, denote the pseudo describing function, ω_o the self oscillation frequency, and $G(j\omega)$ the overall transfer function of the linear portion of the system as given by

$$G(j\omega) = G_1(j\omega)\ G_2(j\omega)\ G_3(j\omega). \qquad (3.43)$$

The pseudo describing function, $P(E,B)$, is substituted for the describing function in equation 3.42. A limit cycle can be sustained if for some value of B, A, and ω_o.

$$G(j\omega_o) = -\frac{1}{P(E,B)}. \qquad (3.44)$$

Equation 3.44 yields the limit cycle amplitude E and frequency ω_o at the input to the nonlinear element when the amplitude B of the extra signal is given. It frequently happens that the nonlinearity is symmetrical. If this is the case, phase shift theorem guarantees that the pseudo describing function is real. Then equation 3.44 can be split into two simultaneous equations,

$$|G(j\omega_o)| = \left|\frac{1}{P(E,B)}\right| \qquad (3.45)$$

and

$$\arg G(j\omega_o) = \arg\left(-\frac{1}{P(E,B)}\right) = (1 \pm 2k)\pi \qquad (3.46)$$

where ω_o is the frequency of oscillation which should be used to determine the stabilizing signal frequency and k is an integer. The frequency ω_o is found from equation

3.46 directly or with the aid of Bode or Nyquist diagrams of the linear portion of the system. Several solutions to equation 3.47 may exist, each representing a possible self oscillation frequency.

Define the critical value $|P(E,B)|_c$ of the pseudo describing function to be the nonlinear gain necessary to sustain a limit cycle. By equation 3.45,

$$|P(E,B)|_c = \frac{1}{G(j\omega_o)} . \qquad (3.47)$$

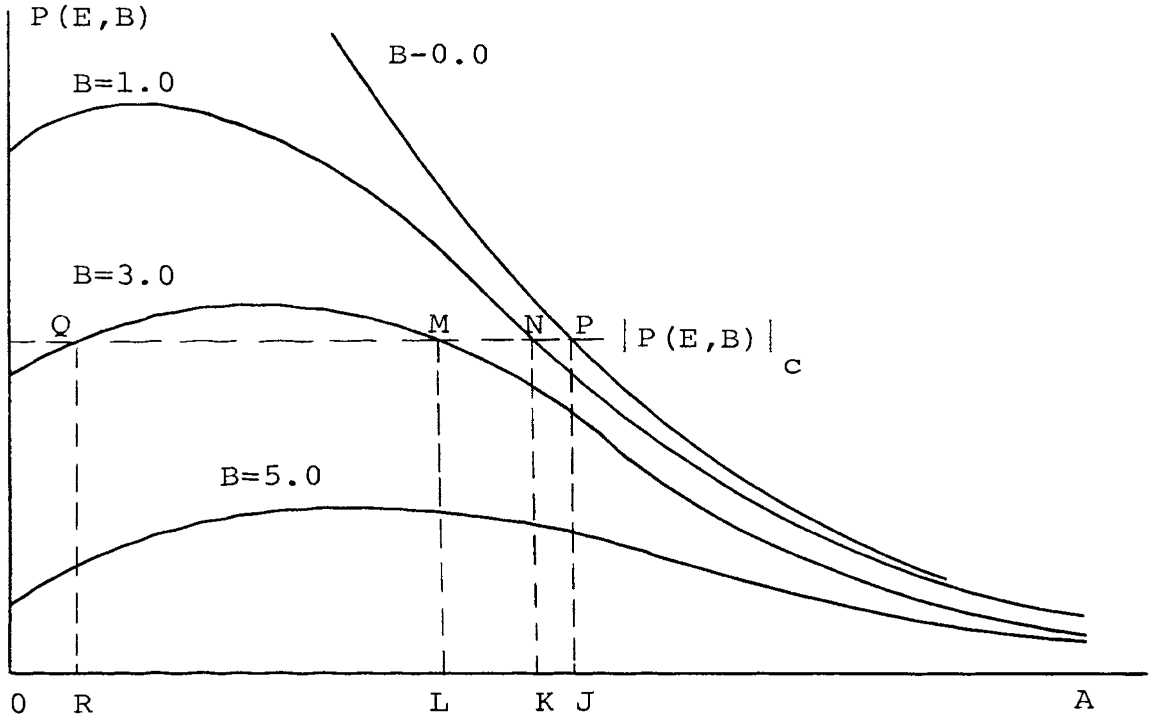
Since $G(j\omega_o)$ is known from equation 3.46, $|P(E,B)|_c$ can be calculated from equation 3.47.

The self-oscillation amplitude is determined by constructing on the pseudo describing function curves for the nonlinearity a horizontal line representing the critical value of the pseudo describing function. The self-oscillation amplitude is read from the intersection of the $|P(E,B)|_c$ line and the pseudo describing function curve for the stabilizing signal amplitude being investigated. Thus the self-oscillation amplitude will usually be a function of the stabilizing signal amplitude. Figure 3.17 shows a procedure for finding the self-oscillation amplitude.

The $|P(E,B)|_c$ line may not intersect the pseudo describing function curve for a particular stabilizing signal B_o. If the entire pseudo describing function curve for B_o lies below the $P(E,B)_c$ line, a stabilizing signal with amplitude B_o will completely remove the self-oscillation of frequency ω_o. If it lies above the $|P(E,B)|_c$ line, a stabilizing signal with amplitude B_o will make the system unstable. Therefore, the stabilizing signal does not always have a stabilizing effect on the system.

If the system can sustain an oscillation, it is necessary to determine if the resultant limit cycle is stable or unstable. If the limit cycle is stable, the oscillation amplitude will return to its original value after the system has received any small disturbance. If it is unstable, a small disturbance may either finally decrease in amplitude to zero or increase until the output becomes unbounded.

The stability of the limit cycle can be determined from the slope of the pseudo describing function curve at its intersection with the $|P(E,B)|_c$ line. If the slope is negative or positive the limit cycle is stable or unstable, respectively. If the slope is zero, the limit cycle is semi-stable. Thus N, M, and P in Figure 3.17 represents stable limit cycles.



Hypothetical pseudo describing function curve

Points M, N and P represent stable limit cycles.

Point Q represents an unstable limit cycle with an amplitude R when B = 3.0

Figure 3.17. Determination of Possible Limit Cycle From the Pseudo Describing Function Curves.

IV. EXPERIMENTAL RESULTS

A. Introduction

In this chapter a specific system is examined, both theoretically and by a simulation, in order to illustrate the methods discussed in previous chapters. A comparison of the system behavior under triangular wave stabilization with that under sinusoidal stabilization is made.

B. Example of Signal Stabilization

The system to be considered is shown in Figure 4.1. The linear part is obviously low pass. It is assumed that the input to the relay is the sum of the fundamental sinusoidal component and a stabilizing signal whose frequency is at least ten times that of the sinusoid. The assumption that a sinusoid exists is a valid one, because this system exhibits self oscillation for all values of K. The frequency and amplitude of this self oscillation may be determined by an application of classical describing function techniques. The results for $K = 0.4$ are,[14]

$$\omega_o = 1, \; E \simeq 2.55 \tag{4.1}$$

where E is the amplitude of the oscillation.

In order to determine how to stabilize the system it is necessary to consider the methods discussed in Chapter III. For K = 0.4, the critical value of the pseudo describing function, $|P(E,B)|_c$, is determined by substituting the linear element transfer function and equation 4.1 into equation 3.45. This yields,

$$|P(E,B)|_c = 5.0$$

The normalized critical value of the pseudo describing function for this particular system is then,

$$\frac{|P(E,B)|_c}{M} = \frac{5.0}{10.0} = 0.5 \qquad (4.2)$$

The normalized pseudo describing function curves for the relay are shown in Figure 3.14, which also shows the normalized critical pseudo describing function line for this example. Consider the case where B = 1.0. The intersection of the normalized pseudo describing function curve for B = 1.0 and $\frac{|P(E,B)|_c}{M} = 0.5$ at point Q represents a possible limit cycle. As seen from the projection of the intersection at point Q to the point K on the abscissa, the amplitude of the self-oscillation will be 2.54 units.

The critical line does not intersect the normalized pseudo describing function curve for B > 2.0. Therefore, if the high frequency input amplitude B is greater than 2.0 units, no limit cycle will exist.

The pseudo describing function analysis presented in the previous chapter was checked on an IBM 360/50 digital computer using CSMP language. The dither signal is a high frequency triangular wave (β = 40. rad./sec.). The self oscillation amplitude for a stabilizing signal amplitude of B = 1.0 was obtained as shown in Figure 4.2. The differences between the experimental results and the corresponding theoretical calculations are seen to be within the tolerances normally associated with describing function methods.

The stabilizing effect of triangular wave dither with amplitude B = 3.0 is shown in Figure 4.3. Note that the sampling period used in the simulation must be small with respect to the period of the dither signal. Otherwise the stabilizing signal may appear to have a lower frequency, and may fail to stabilize the system.

For comparison, the transient responses of the system under triangular wave dither and sinusoidal dither are shown in Figures 4.4 and 4.5, respectively. The input used is a unit step.

A comparison of the equivalent gain for sinusoidal and triangular wave dithers (Table 3.1) would seem to indicate that the stabilizing effect of the triangular wave dither would be faster than that of sinusoidal dither, because of the higher gain which is associated with the triangular wave equivalent gain for small signals. But the simulation does not show this to be the case. A

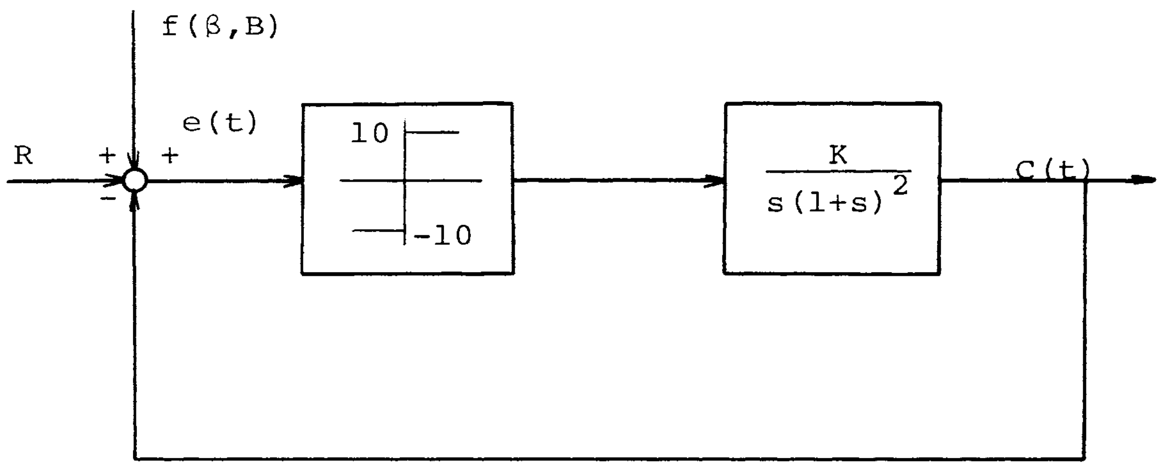


Figure 4.1. Relay System With Stabilizing Signal.

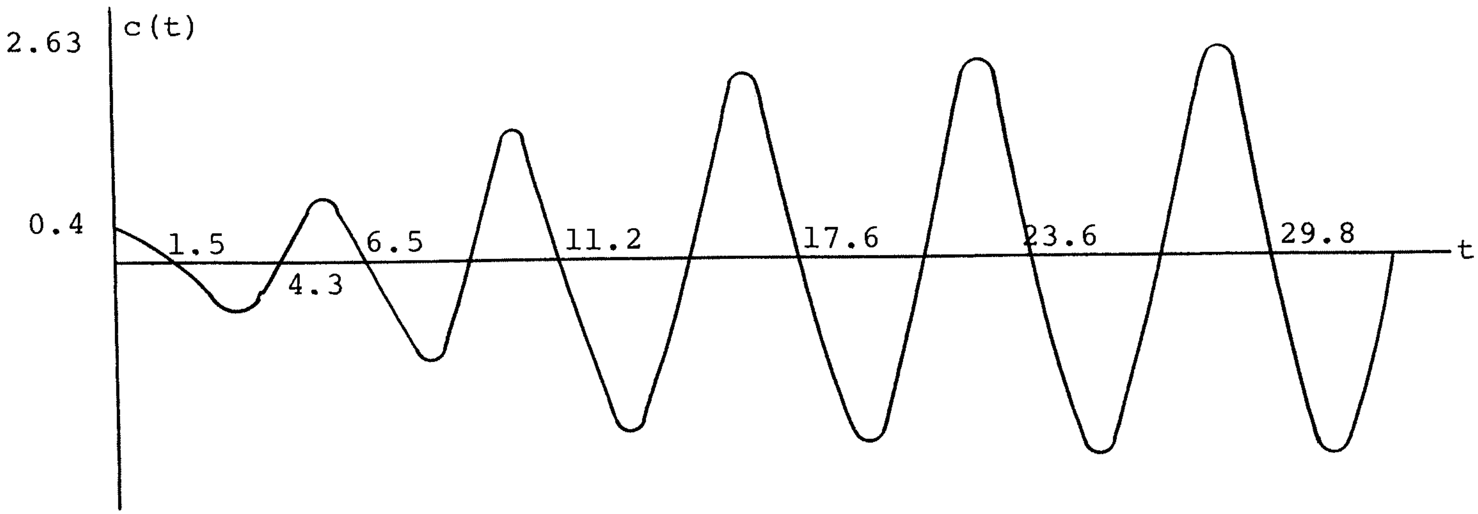


Figure 4.2. Self-oscillation of the System of Figure 4.1 With Small Triangular Wave Dither.

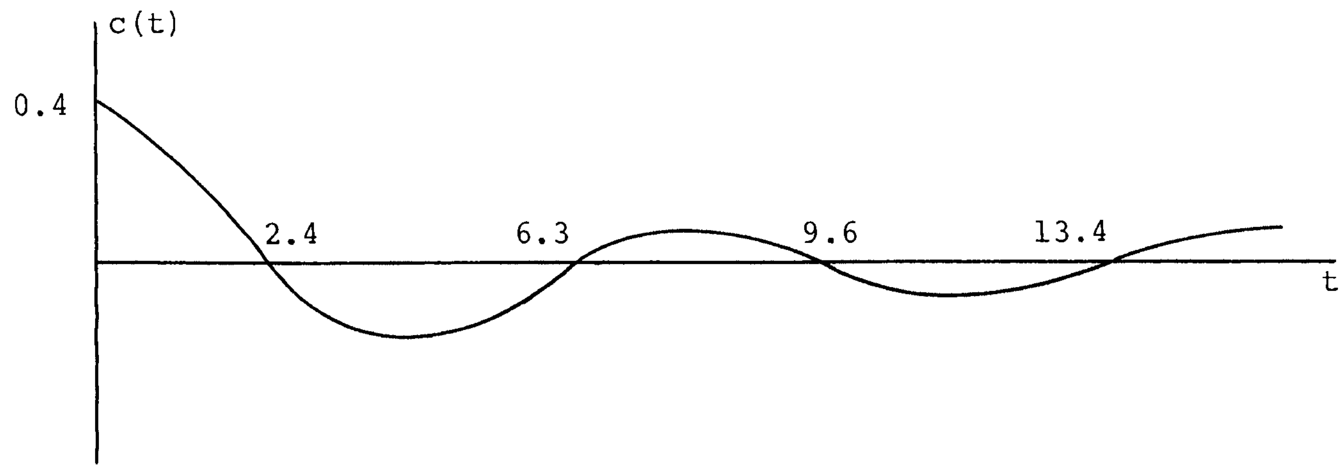


Figure 4.3. Triangular Wave Quenching of the System of Figure 4.1.

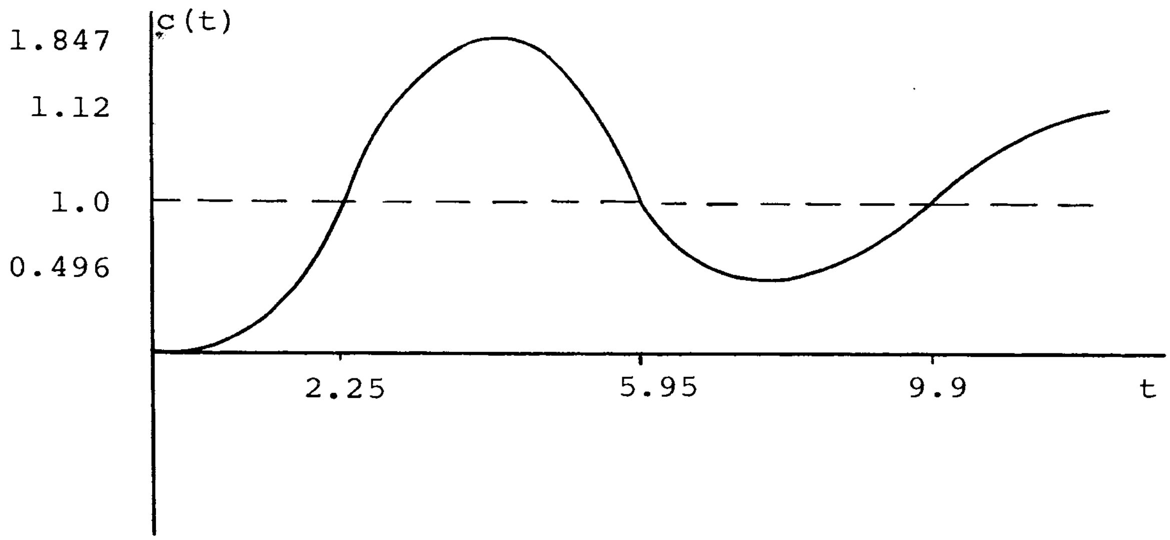


Figure 4.4. Transient Response With Triangular Wave.

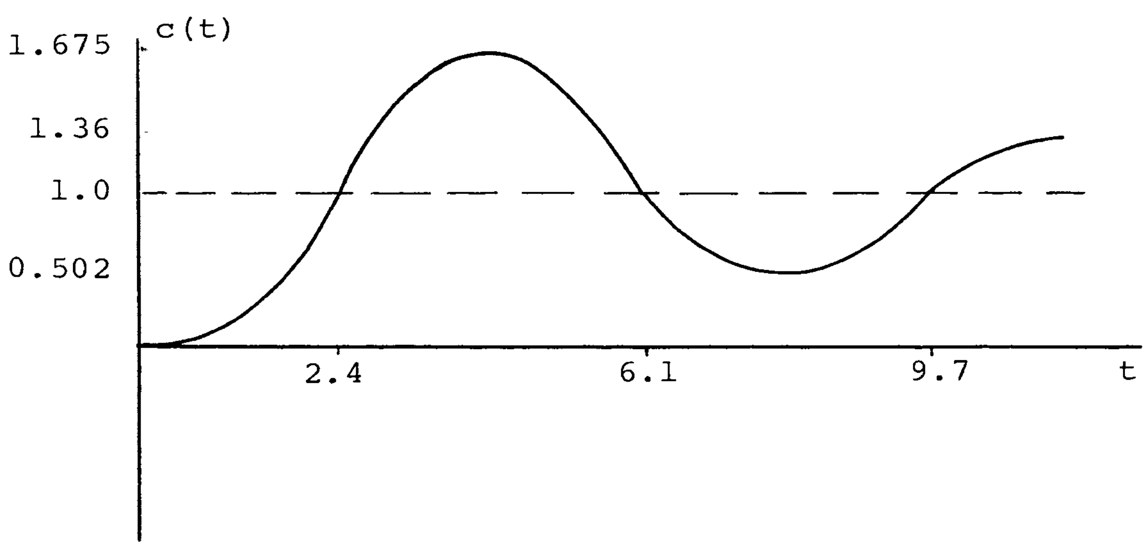


Figure 4.5. Transient Response With Sinusoidal Signal.

closer examination shows that the amplitude of the self-oscillation is large enough that the system does not operate in the small signal mode. Therefore, it is not possible to state general results relating to the transient response by considering only the equivalent nonlinearity. Another possible source of error may be in the digital synthesis of the triangular wave, since a small dc component was introduced due to a finite sampling period.

V. CONCULSIONS

Many nonlinear systems display self-sustained oscillations which are often undesirable. The self oscillation of a physical system may often be removed by the introduction of an appropriate stabilizing signal which may change the open loop gain in a nonlinear manner.

The stabilizing effect of a high frequency input signal on an oscillating system with one nonlinearity is determined by the characteristics of the nonlinear element in the system, the linear portion of the system, and the amplitude of the signal.

This investigation has been concerned with the effect of a triangular wave stabilizing signal on these self oscillations. The equivalent gains for several common nonlinearities are derived. The pseudo describing function introduced by Oldenburger and Boyer[10,11] for sinusoidal stabilization has been extended to the triangular wave case, and it is shown that the pseudo describing function for an odd nonlinearity is real.

The pseudo describing function is used in an analysis similar to describing function analysis in order to predict the existence and amplitude of the self oscillation of a triangular wave stabilized, closed loop, nonlinear system. The experimental results are in close agreement with the predictions of the theory.

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VITA

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He enrolled at the University of Missouri-Rolla as a graduate student in January, 1970.

APPENDIX

DERIVATION OF THE EQUIVALENT SDN FOR THE LIMITER WITH HYSTERESIS

DERIVATION OF THE EQUIVALENT SDN

The derivation of the first part of equations 3.32 and 3.33 was given in Table 3.2. The derivation of the case in which A_o and B satisfy the following inequalities

$$0 < A_o < |B - (a+b)|, \qquad (A.1)$$

$$a \leq B < a+b, \qquad (A.2)$$

is as follows. Consider Figure A.1 which illustrates both equations A.1 and A.2 simultaneously. The input-output characteristic of the limiter with hysteresis is given by,

$$\begin{aligned}
m(t) &= -M & -\infty < e(t) < 0,\\
m(t) &= \frac{M}{b}[e(t)-a], & 0 < e(t) < a+b,\\
m(t) &= M, & a+b < e(t) < \infty,\\
m(t) &= \frac{M}{b}[e(t)+a], & -(a+b) < e(t) < 0,\\
m(t) &= -M, & -\infty < E(t) < -(a+b). \qquad (A.3)
\end{aligned}$$

The input, given by

$$e(t) = f(\beta, B) + A_o, \qquad (A.4)$$

may be represented as follows,

$$\begin{aligned}
e(t) &= \frac{2B}{\pi}\beta t + A_o, & 0 \leq \beta t \leq \frac{\pi}{2},\\
e(t) &= -\frac{2B}{\pi} t + 2B + A_o, & \frac{\pi}{2} \leq \beta t \leq \frac{3\pi}{2},\\
e(t) &= \frac{2B}{\pi}\beta t - 4B + A_o, & \frac{3\pi}{2} \leq \beta t \leq 2\pi. \qquad (A.5)
\end{aligned}$$

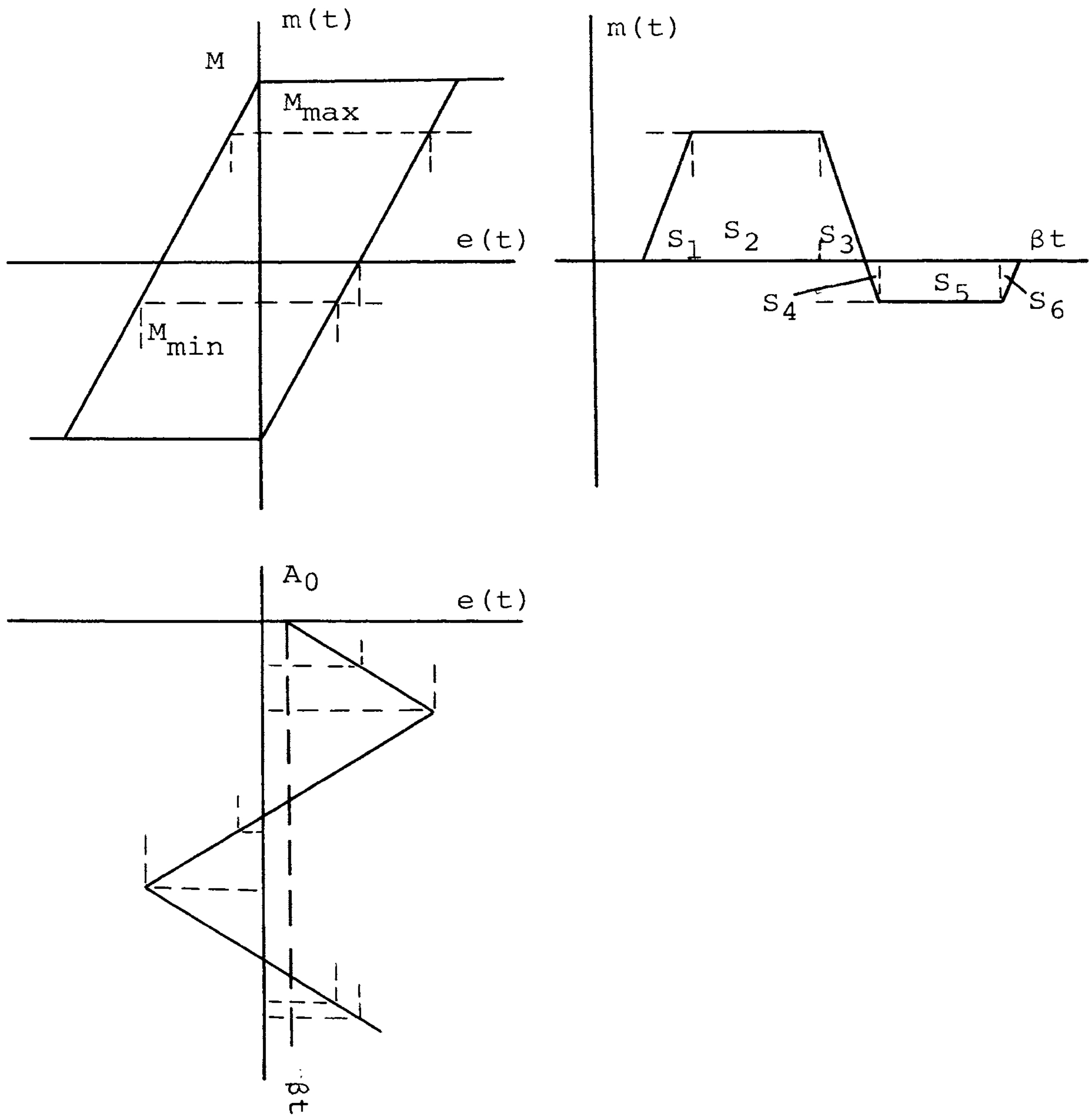

Figure A.1. Triangular Signal with $0 \leq A_o \leq |B-(a+b)|$ and $a \leq B \leq a+b$.

The maximum and minimum amplitudes of the output will be obtained with the aid of equations A.3 and the maximum and minimum amplitudes

$$E_m = A_o \pm B \tag{A.6}$$

of the input of the nonlinearity as follows:

$$M_{max} = \frac{M}{b}(A_o + B-a),$$

$$M_{min} = \frac{M}{b}(A_o - B+a). \tag{A.7}$$

The basic angles θ_i, $i = 1,2,\ldots,7$ of the discontinuities in the output can be obtained with the aid of equations A.3, A.5, and A.7 as,

$$\theta_1 = \left(\frac{a-A_o}{2B}\right)\pi,$$

$$\theta_2 = \frac{\pi}{2},$$

$$\theta_3 = \left(\frac{B+2a}{2B}\right)\pi,$$

$$\theta_4 = \left(\frac{2B+A_o+a}{2B}\right)\pi,$$

$$\theta_5 = \frac{3\pi}{2},$$

$$\theta_6 = \left(\frac{3B+2a}{2B}\right)\pi,$$

$$\theta_7 = \left(\frac{4B+a-A_o}{2B}\right)\pi. \tag{A.8}$$

The average value of the output, A_v, is given by

$$A_v = \frac{1}{2\pi}\int_0^{2\pi} m(\beta t)\ d(\beta t) = \frac{1}{2\pi}\sum_{i=1}^{6} S_i, \qquad (A.9)$$

where the S_i are the areas represented in Figure A.1, given by

$$S_1 = S_3 = \left(\frac{\theta_2 - \theta_1}{2}\right) M_{max} = \frac{M\pi}{4bB}(B + A_o - a)^2,$$

$$S_2 = (\theta_3 - \theta_2) M_{max} = \frac{Ma\pi}{bB}(B + A_o - a)$$

$$S_4 = S_6 = \frac{1}{2}(\theta_5 - \theta_4) M_{min} = -\frac{M\pi}{4bB}(B - A_o - a)^2,$$

$$S_5 = (\theta_6 - \theta_5)\cdot M_{min} = -\frac{Ma\pi}{bB}(B - A_o - a). \qquad (A.10)$$

Then

$$A_v = \frac{1}{2\pi}\sum_{i=1}^{6} S_i = \frac{M\ A_o}{b}. \qquad (A.11)$$

The equivalent SDN is then given by

$$\frac{dA_v}{dA_o} = \frac{M}{b}\ \ldots, \qquad (A.12)$$

for

$0 < A_o < [B - (a+b)],$

$a \le B < a+b.$

In the other case where,

$$|B - (a+b)| < A_o < B - a+b, \qquad (A.13)$$

equation 3.32 may be derived by the same method. In figure A.2, which is actually Figure 3.12, the output of the nonlinearity with duration $t_6 - t_o = \frac{2\pi}{\beta}$ is shown. Note that the duration of the output does not have any effect on the final result since the waveform is periodic. By definition, the average output A_v is given by the following equation:

$$A_v = \frac{\beta}{2\pi} (S_1 - S_2). \tag{A.14}$$

Where S_1 and S_2 represent the areas of the trapezoids $T_0T_1T_2T_3$ and $T_3T_4T_5T_6$, respectively. Therefore the equivalent SDN is

$$\frac{dA_v}{dA_o} = \frac{\beta}{2\pi} \left(\frac{dS_1}{dA_o} - \frac{dS_1}{dA_o}\right). \tag{A.15}$$

The values of S_1 and S_2 are obtained by considering Figure A.2. They are

$$S_1 = \frac{\pi M}{2B\beta}(2B + 2A_o - b),$$

and

$$S_2 = \frac{\pi M}{2Bb\beta} [(B - A_o)^2 - a^2]. \tag{A.16}$$

Substituting for S_1 and S_2 from equation A.16 into equation A.15 yields

$$\frac{dA_v}{dA_o} = \frac{M}{2bB} (B + b - A_o). \tag{A.17}$$

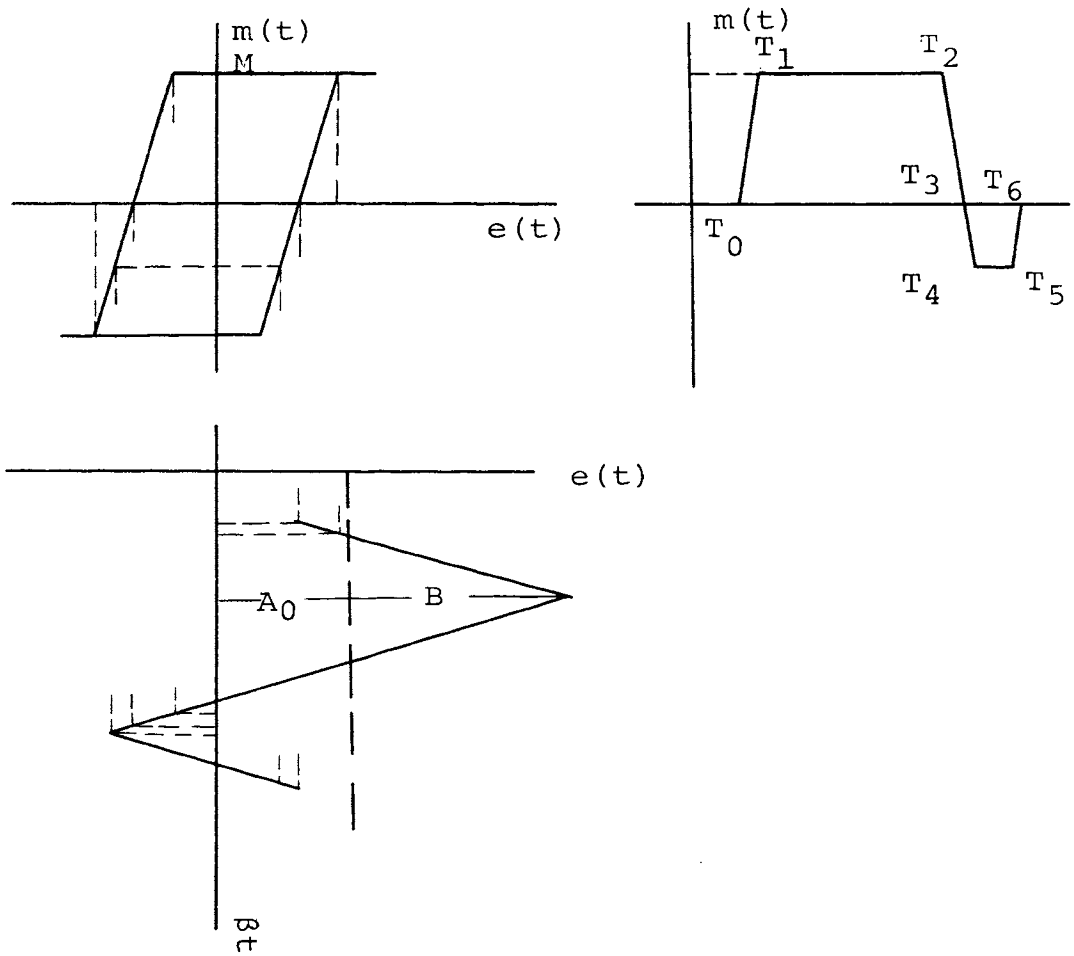

Figure A.2: Triangular Signal With B - (a+b) $\leq A_o \leq$ B - a+b.

The last case given in equation 3.32, where

$$B - a+b < A_o, \tag{A.18}$$

may be derived by the same procedure. But, note that in this case the triangular wave dither never goes to the negative linear region of the nonlinearity, therefore

$$A_v = M,$$

or

$$\frac{dA_v}{dA_o} = 0. \tag{A.19}$$

The input of the nonlinear element given by equation 3.10 is symmetric about $t = \frac{\pi}{2\omega}$, and $\frac{dA_v}{dt}$ is even in ωt. Using these properties and the results of the phase shift theorem allows equation 3.29 to be written as

$$P_2(E,B) = j \frac{4\omega}{\pi E} \int_0^{2\pi} \frac{dA_v}{dt} \cos \omega t \, d(\omega t). \tag{A.20}$$

For the limiter with hysteresis, the second pseudo describing function consists of three terms, since there are three expressions for $\frac{dA_v}{dt}$ depending on the size of A_o. Thus

$$P_2(E,B) = j\,\frac{4\omega}{\pi E}\,\Big[\int_0^{\omega t_1} \frac{M}{A}\,(E\cos\omega t)\,\cos\omega t\;d(\omega t)$$

$$+ \int_{\omega t_1}^{\omega t_2} \frac{M}{2bB}\,(B + b - E\sin\omega t)(E\cos\omega t)\,\cos\omega t\;d(\omega t)$$

$$+ \int_{\omega t_2}^{\pi/2} 0\,(E\cos\omega t)\,\cos\omega t\;d(\omega t)\Big], \qquad \text{(A.21)}$$

which is equation 3.34.

The limits of the integrals of equation A.21 depend on the amplitude of the self oscillation. Consider Figure A.3, which represents the triangular input for two cases, when

$$B > a+b,$$

and

$$B < a+b. \qquad \text{(A.22)}$$

Depending on the amplitude of the fundamental component, three distinct regions of operation may be distinguished. Region 1: When the triangular wave either goes into saturation on both sides or never goes into saturation. Which of these occurs depends on which condition of equation A.22 is met. To satisfy both cases simultaneously requires that

$$0 \le E \le |B - (a+b)|. \qquad \text{(A.23)}$$

In this case there is only one expression for $\frac{dA_v}{dt}$, which

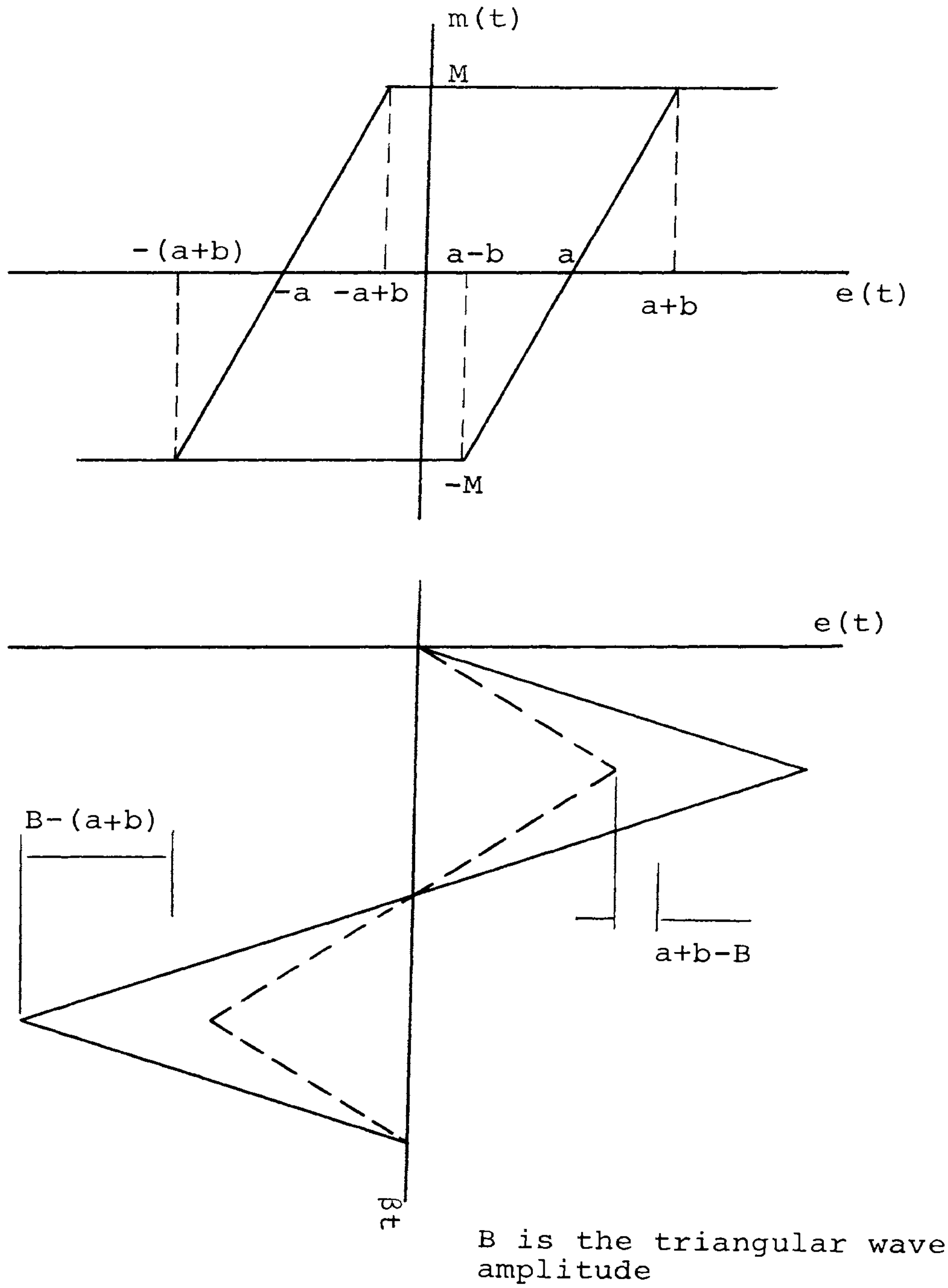

Figure A.3. Two Different Cases of Triangular Signal Input.

is given by the first term in equation A.21. By considering equation A.20, the following limits are obtained,

$$\omega t_1 = \omega t_2 = \frac{\pi}{2} \,. \tag{A.24}$$

Region 2: When the triangular wave is in saturation for $e(t) > 0$, and is in the linear region for $e(t) < 0$. This occurs as soon as E is large enough that the condition of region 1 is no longer met. Thus, the lower bound of this region is

$$E \geq |B - (a+b)| \,. \tag{A.25}$$

Note that in this region the tip of the triangular wave does not leave the negative linear region, so $E - B \leq - a+b$ or $E \leq B - a+b$. Thus this region is bounded as

$$|B - (a+b)| < E \leq B - a+b, \tag{A.26}$$

where E is the maximum dc value. Thus, the lower limit is obtained from,

$$E \sin \omega t_1 = |B - (a+b)| \,. \tag{A.27}$$

In this case there are two expressions for $\frac{dA_v}{dt}$, which are given by the first two terms in equation A.21. By considering equations A.20 and A.27, the following limits are obtained:

$$\omega t_1 = \sin^{-1} \frac{|B - (a+b)|}{E} ,$$

$$\omega t_2 = \frac{\pi}{2} . \tag{A.28}$$

Region 3: When E is large enough that the tip of the triangular wave does not reach the negative linear region, or $E - B \geq - a+b$. Thus,

$$E \geq B - a+b, \tag{A.29}$$

represents the boundary of this open region. In this case there are three expressions for $\frac{dA_V}{dt}$ which are given by equation 3.32. By considering equation A.20, the following limits are obtained:

$$\omega t_1 = \sin^{-1} \frac{|B - (a+b)|}{E} , \tag{A.30}$$

$$\omega t_2 = \sin^{-1} \frac{B - a+b}{E} .$$

By carrying out the integrations in equation A.21, and considering the limits which were obtained above, equation 3.34 is obtained.