Spring 2019

Conflict Free Connectivity and the Conflict-Free-Connection Number of Graphs

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CONFLICT FREE CONNECTIVITY AND THE

CONFLICT-FREE-CONNECTION NUMBER OF GRAPHS

by

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(Under the Direction of Hua Wang)

ABSTRACT

We explore a relatively new concept in edge-colored graphs called conflict-free connectivity. A conflict-free path is an (edge-) colored path that has an edge with a color that appears only once. Conflict-free connectivity is the maximal number of internally disjoint conflict-free paths between all pairs of vertices in a graph. We also define the *c-conflict-free-connection* of a graph G. This is the maximum conflict-free connectivity of G over all c-colorings of the edges of G. In this thesis we will briefly survey the works related to conflict-free connectivity. In addition, we will use the probabilistic method to achieve a bound on the c-conflict-free connection number of complete graphs.

INDEX WORDS: Graph theory, Edge coloring, Conflict-Free, Connectivity

2009 Mathematics Subject Classification: 15A15, 41A10

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B.S., Georgia Southern University, 2016

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial

Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

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Electronic Version Approved:
May 2019

DEDICATION

This thesis is dedicated to the many hours I spent procrastinating hoping it would write itself. In addition, I would like to dedicate this thesis to Kim, because without her I probably would not have considered this degree.

ACKNOWLEDGMENTS

I wish to acknowledge my family, for reasons that only they may know. I also wish to acknowledge office 2016, for jokes, mutual ideas, and long-lasting memories.

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LIST OF SYMBOLS

K_n	Complete Graph on n vertices
C_n	Cycle with n vertices
P_n	Path with n vertices
S_n	A Star Graph on n vertices
$S(l_1, ..., l_k)$	A star-like tree with k paths
$V(G)$	Vertex set of a graph G
$E(G)$	Edge set of a Graph G
CF	Conflict-Free
P(event)	probability of an event
Bin(N, p)	Binomial with N trials and probability p of success

CHAPTER 1

INTRODUCTION

1.1 Background

An edge-colored graph G is called *rainbow connected* if any two vertices are connected by a path whose edges have pairwise distinct colors. An edge-coloring of a graph G is called *proper* if any two adjacent edges in this coloring receive different colors. The concept of rainbow connection was introduced by Chartrand et al. in 2008 [4]. These two concepts served as the inspiration for conflict-free (edge-) coloring of graphs. In 2003, Even et al. [9] introduced a version of conflict-free (vertex-) coloring involving hyper-graphs, where a hyper-graph H is a pair $H = (V, E)$ where V is the set of vertices and E is the set of nonempty subsets of V called edges. The coloring was created to solve a problem involving cellular networks. Seven years later, in 2010, Bar-Noy et al. [2] was one of the first to define the concept of a conflict-free (edge-) coloring also involving hyper-graphs. They defined a conflict-free coloring as follows.

Definition 1.1 ([2]). *A proper coloring is called conflict-free if each (hyper-) edge contains a color used only once on the (hyper-) edge.*

We use a similar definition. However instead of looking at conflict-free colorings of the entire graph we only consider conflict-free paths within them. A path in an edge-colored graph G is called *conflict-free*, denoted by CF, if there exists a color that appears exactly once in the path. Czap et al. [6] introduced the concept of conflict-free connection of graphs on the basis of the aforementioned hyper-graph version. Li et al. [10] created a counterpart to this called the conflict-free vertex-connection of graphs. In this paper, we take this one step further by defining conflict-free connectivity (using edge-colorings). The *conflict-free connectivity* is the maximum number k such that every pair of vertices has at least k internally disjoint CF paths between them. This will be the main emphasis of our results later on, where we will seek to find the maximal conflict-free connectivity of G. A formal definition is offered below.

Definition 1.2. *The* c-conflict-free-connection *of a graph G, denoted by* $CFC_c(G)$*, is the maximum conflict-free connectivity of G over all c-colorings of the edges of G.*

To better understand the definitions above, we will first start by defining helpful terminologies.

1.2 TERMINOLOGIES

Definition 1.3 ([12])**.** *A* simple *graph* G *with* n vertices *and* m edges *consists of a vertex set* $V(G) = \{v_1, v_2, \ldots, v_n\}$ *and an edge set* $E(G) = \{e_1, e_2, \ldots, e_m\}$ *where each edge is a distinct unordered pair of vertices. We write* uv *for the edge* $\{u, v\}$*. If* $uv \in E(G)$*, then* u *and* v *are called* adjacent*. The vertices contained in an edge are its* endpoints*; and the vertices which are endpoints of an edge are said to be incident with that edge. The* degree *of a vertex* v *is the number of edges incident with* v*.*

Once we know what exactly a graph is we can than start classifying graphs by their structure. Four of the most common graphs of these are Paths, Cycles, Stars, and Complete graphs.

Definition 1.4 ([12])**.** *A* path *in a graph is a finite or infinite sequence of edges which connect a sequence of vertices which are all distinct from one another. A path with* n *vertices is denoted by* P_n

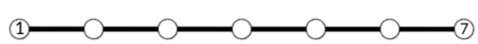

Figure 1.1: A P_7

Definition 1.5 ([12]). *A* complete graph *is a simple graph in which every pair of vertices forms an edge. A complete graph with n vertices is denoted by K_n.*

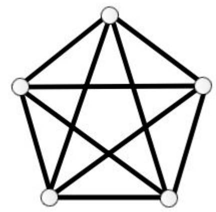

Figure 1.2: A K_5

Definition 1.6 ([12]). *A* cycle *is a closed path of length at least 3 with no repeated edges and whose "endpoint" is the only repeated vertex. A cycle with n vertices is denoted by C_n.*

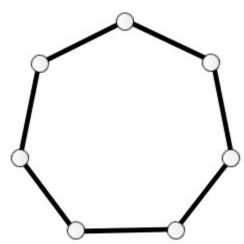

Figure 1.3: A C_7

Definition 1.7 ([12]). *A* star *is a graph in which every vertex is connected to a single vertex called the* center. *A star with n vertices is denoted by S_n.*

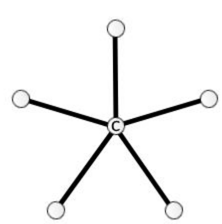

Figure 1.4: A S_6

Definition 1.8. *A* starlike tree *is a collection of paths (not necessarily the same length) which have an endpoint that is shared by all paths in the graph. A starlike tree with k paths of length l is often denoted by* $S(l_1, l_2, ..., l_k)$.

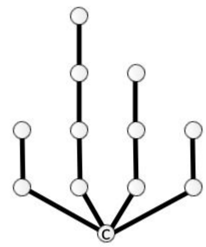

Figure 1.5: An $S(3, 5, 4, 3)$

It is interesting to note that other common graph structures such as Trees, are of similar nature to Paths and Stars. Furthermore we have graphs like Brooms and Wheels, that are simply combinations of a path/star, and cycle/star respectively. All of the graph structures mentioned so far usually fall in a category of graphs that have “nice” structure. Graphs not in this category are usually arbitrary, a conglomeration of graphs, hyper-graphs, digraphs, or disconnected graphs, etc. Here we care only about graphs with “nice” structure that are connected.

Definition 1.9 ([12])**.** *A graph G is* connected *if it has a u, v-path for each pair $u.v \in V(G)$. Otherwise, G is disconnected.*

Definition 1.10 ([12])**.** *A graph G is said to be* k-connected *if there does not exist a set of $k - 1$ vertices whose removal disconnects the graph.*

Definition 1.11 ([12])**.** *An* independent set *in a graph G is a vertex subset $S \subseteq V(G)$ such that the induced* subgraph $G[S]$ *has no edges.*

All of the above definitions serve to build an understanding of how a graph is constructed. However, graph theory goes beyond just looking at dots and lines. It also takes us back to a simpler time of crayolas and art projects. That is, besides just creating or looking at graphs, often times to describe certain properties of a graph it is best to color a portion of it or even all of it to look at a certain feature. The beauty of this is that it often brings color, or insight, into current problems, as well as introducing new problems like conflict-free connectivity for instance. In this paper, we only look at edge colored graphs.

Definition 1.12 ([12])**.** *An* edge coloring *of a graph G is an assignment of colors (which are elements of some set) to the edges of G. A graph G colored with k colors would be called a k-coloring of G*

Definition 1.13. *A graph G is considered to be* rainbow colored *if all edges in G have* distinct *colors.*

How exactly the edges of a graph are colored, while generally random, is often left up to the ingenuity of the mathematician that is working on it. There are cases where a very particular number of colors is needed or even a set number of colors given a set amount of edges. In Chapters 3 and 4 you will see that we will completely color our graphs with a set amount of colors. However, how it is colored is done randomly. We will see more on this in the following chapters.

In Chapter 2, we will explore conflict-free graphs and conflict-free connectivity in order to better understand what we hope to accomplish in Chapters 3 and 4. We will focus on works by Xueliang Li and others [3, 5, 6, 8]. Finally in Chapters 3 and 4 we will provide proofs of our main results, namely Theorem 1.14 and Theorem 1.15 below.

Theorem 1.14. *Let n be the number of vertices of a complete graph K_n. Then for $n \geq 3$ and $t > \frac{\sqrt{n\ ln(\frac{7n^2}{5})}}{4}$ we get the following bounds on the 2-conflict-free-connection number of K_n,*

$$\frac{3n}{4} - t \leq CFC_2(K_n) \leq \frac{3n-5}{4}.$$

In Theorem 1.14 we consider a very specific case of 2-coloring complete graphs and examine the bounds of the c-conflict-free-connection number of them. Not only is the upper bound stronger than the general bound for $c \geq 3$ colors, this also sheds some light on the structure of the proof of our more general statement below.

Theorem 1.15. *Let n be the number of vertices of a complete graph K_n colored with c colors such that $n \geq c$, then for $c \geq 3$, and $t > \dfrac{\sqrt{n \ln(\frac{7n^2(c+1)}{15})}}{4c} - \dfrac{n(c-2)}{2c^2}$ we achieve the following bounds for the c-conflict-free connection number of K_n*

$$\frac{(2c-1)n}{2c} - t \leq CFC_c(K_n) \leq \frac{(2c-1)n - c + 1}{2c}$$

CHAPTER 2

CONFLICT-FREE CONNECTION OF GRAPHS

In this Chapter, we will look at many of the characteristics of conflict-free colorings and more importantly, the conflict-free connection of graphs. First, let us start with the basics. In Chapter 1, we mentioned three definitions which are provided here for completeness.

Definition 2.1. *A path in an edge-colored graph G is called* conflict-free*, denoted by CF, if there exists a color that appears exactly once in the path.*

An example of a conflict-free path can be seen below.

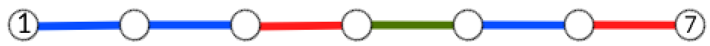

Figure 2.1: A conflict-free P_7 with the unique color green.

Definition 2.2. *The* conflict-free connectivity *is the maximum number k such that every pair of vertices has at least k internally disjoint CF paths between them.*

Definition 2.3. *The* c-conflict-free-connection *of a graph G, denoted by $CFC_c(G)$, is the maximum conflict-free connectivity of G over all c-colorings of the edges of G.*

We also add two more definitions for better clarity on conflict-free connected graphs.

Definition 2.4. *A graph G is conflict-free connected (with respect to the edge-coloring) if for every pair of vertices of G, there is a conflict-free path connecting them.*

Definition 2.5 ([6])**.** *For a connected graph G, the conflict-free connection number of G, denoted by $cfc(G)$, is defined as the minimum number of colors that are required to make G conflict-free connected.*

From Definition 2.5 we can construct a more general version of the conflict-free-connection number. We call this new term the the t-conflict-free connection number.

Definition 2.6. *For a connected graph G, the t-conflict-free-connection number of G, denoted by $cfc_t(G)$, is defined as the minimum number of colors needed for a coloring of G to exist so that there are at least t CF paths between every pair of vertices.*

Looking at Figure 2.1, we can clearly see that in this coloring, there exists sub-paths of the P_7 that are not CF. Consider the first two blue edges for example, Is there such a coloring of this path such that every pair of vertices is CF? The answer is yes. An example is pictured below.

Figure 2.2: A CF P_7 with every pair of vertices having a CF-path.

From the above definitions, we can also construct other colored graphs that are CF. For instance, a CF-cycle and a CF-star.

Figure 2.3 is not hard to construct. It is simply a CF-path where we create an extra edge connecting the first vertex and last vertex of said path. In addition, we can clearly see that Figure 2.3 is conflict-free connected. What about the conflict-free *connectivity* of Cycles? Well by Definition 2.2 the maximal number is two because there are only two possible paths between every pair of vertices. However in order to obtain this value, what

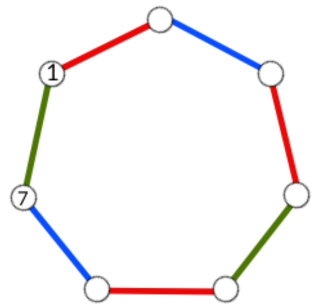

Figure 2.3: A conflict-free C_7

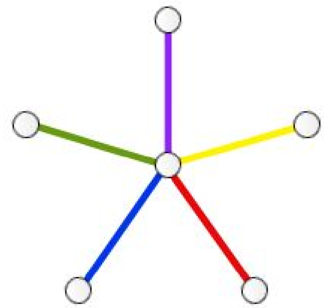

Figure 2.4: A conflict-free S_6

is the minimal number of colors needed? To answer this question we use the following theorem provided by Czap et.al.

Theorem 2.7 ([6]). $cfc(P_n) = \lceil \log_2(n) \rceil$.

This theorem tells us that the minimum number of colors needed to guarantee that every pair of vertices in a P_n has a CF path between them is $\lceil \log_2(n) \rceil$. From this theorem we can obtain several interesting results.

Proposition 2.8. *There minimum number of colors c such that* $CFC_c(C_k) = 2$ *is* $\lceil \log_2 \left(\frac{n}{2} + 1\right) \rceil \leq c \leq \lceil \log_2(n) \rceil + 1$.

Proof. By Theorem 2.7 we can easily obtain an upper bound. Since a cycle is simply a path with an extra edge as mentioned before. We can state that the $cfc(C_n) \leq \lceil \log_2(n) \rceil + 1$ where all we need is one extra color for the new edge.

For the lower bound pick a vertex v then following along the edges select the vertex with the maximal number of edges from v, say u. The number of edges between u and v form a path, call it P, with $\lfloor \frac{n}{2} \rfloor + 1$ vertices. Seeing that we now have a path, applying Theorem 2.7 gives us $cfc(P) = \lceil \log_2(\frac{n}{2} + 1) \rceil$. We can guarantee that the other half of the cycle is also cfc, because it forms a path of shorter length than P. Thus, we need at least $\lceil \log_2(\frac{n}{2} + 1) \rceil$ colors proving the result. $\square$

Proposition 2.9. *The c-conflict-free connection of a star with k vertices is equal to 1 if and only if the number of colors used is greater than or equal to $k-1$. In other words,*

$$CFC_c(S_k) = 1 \Longleftrightarrow c = k - 1.$$

Proof. Suppose by way of contradiction that we can color a S_k with less than $k-1$ colors and it remain conflict-free connected. Call the center vertex s. Pick two vertices that are not the center of the star, say u and v. Color the edges, su and sv with 2 of our colors. Pick a new vertex u_1 that is also not the center. We could color u_1 in one of our previous two colors, however this would result in one of the paths between u_1 and u or u_1 and v to not be conflict-free. So we must color the edge su_1 a new color. Repeating on in this fashion reveals that for each of the $k-1$ vertices the edge between s and that vertex must be a unique color in order to remain conflict-free with every other vertex, a contradiction. In the other direction, if we color an S_K with $k-1$ colors it will then be conflict-free trivially. □

Proposition 2.10. *The c-conflict-free connection of a star-like tree with n vertices is equal to 1 if the number of colors used is greater than or equal to $k + \lceil \log_2(l-1) \rceil$. Where k is the number of paths from the center, and l is the length of the longest path. In other words,*

$$CFC_c(S(l_1, ..., l_k)) = 1 \textit{ if } c \geq k + \lceil \log_2(l-1) \rceil.$$

Proof. Let G be a coloring of a Star-like Tree $S(l_1, ..., l_k)$. Let u be the center vertex of G. Let the length of the longest path be l. By Proposition 2.9 we have that the number of colors needed to guarantee that each pair of vertices in the star part (any vertex that is of distance 1 from u) of G is k since there are k vertices that are connected to the "center" vertex. Color the star-part in k colors. Then, by Theorem 2.7 we also have that the necessary number of colors needed to color our longest path is $\lceil log_2(l) \rceil$. Now, one of the colors needed to color our longest path is already present in the star part of G. To avoid re-coloring any edge, we reduce the length of our longest path, and subsequently every other path, by 1. This means that our longest path is actually $l-1$.

Finally, color every path, excluding the edge connected to u, from the list of $\lceil log_2(l-1)\rceil$ colors. Now choose two vertices, say u_1 and u_2. If u_1 and u_2 lie on the same path then we know there exists a coloring using $\lceil log_2(l-1)\rceil$ colors that guarantees they are CF. If u_1 and u_2 lie on different paths then the path between them is guaranteed to be CF by the coloring of the star-part of G. If either u_1, u_2 or both lie on the star-part of G we are also guaranteed a CF-Path by the way it is colored. Therefore the minimal number of colors needed is the number of colors needed to color the star-part of G plus the number of colors need to color our longest path of length $l-1$. This completes the proof. □

Before moving on to our main results, we now look at a brief survey of other works on the topic.

2.1 Other Works

While the following Lemmas and Theorems are not specifically used in this paper, they are worth mentioning as important results related to the subject of conflict-free.

Lemma 2.1.1 ([6]). *If G is a 2-connected and non-complete graph, then $cfc(G) = 2$*

Lemma 2.1.2 ([6]). *If $cfc(G) = 2$ for a graph G with cut-edges, then $C(G)$ is a linear forest whose every component has at most three edges.*

Theorem 2.11 ([6]). *If G is a connected graph and $C(G)$ is a linear forest whose every component is of order 2, then $cfc(G) = 2$.*

Let G be a connected graph and $h(G) = \max\{cfc(K) : K$ is a component of $C(G)\}$.

Theorem 2.12 ([6]). *If G is a connected graph with cut-edges, then $h(G) \leq cfc(G) \leq h(G) + 1$. Moreover, these bounds are tight.*

Theorem 2.13 ([3]). *Let G be a connected non-complete graph of order $n \geq 25$. If $C(G)$ induces a linear forest and $\delta(G) \geq \dfrac{n-4}{5}$, then $cfc(G) = 2$.*

Theorem 2.14 ([3]). *Let G be a connected non-complete graph of order n with $4 \geq 8$. If $C(G)$ induces a linear forest and $\delta(G) \geq 2$, then $cfc(G) = 2$.*

Theorem 2.15 ([3]). *Let G be a connected non-complete graph of order $n \geq 33$. If $C(G)$ induces a linear forest and deg(x)+deg(y) $\geq \dfrac{2n-9}{5}$ for each pair of two non-adjacent vertices x and y of $V(G)$, then $cfc(G) = 2$*

Theorem 2.16 ([8]). *Let G be a connected claw-free graph. Then G must belong to one of the following four cases:*

i. G is complete;

ii. G is non-complete and 2-edge-connected;

iii. $C(G)$ has at least two components K satisfying $cfc(K) = \lceil log_2(p+1) \rceil$

iv. $C(G)$ has only one component K satisfying $cfc(K) = \lceil log_2(p+1) \rceil$.

Theorem 2.17 ([8]). . *Let G be a connected claw-free graph of order n $\geq 2. Then, we have$*

i. $cfc(G) = 1$ if G is complete;

ii. $cfc(G) = 2$ if G is non-complete and 2-edge-connected, or $p = 1$ and $n \geq 3$;

iii. $cfc(G) = \lceil log_2(p+1) \rceil + 1$, if $C(G)$ has at least two components K satisfying $cfc(K) = \lceil log2(p+1 \rceil$; otherwise, $cfc(G) = \lceil log_2(p+1) \rceil$, where $p \geq 2$.

CHAPTER 3

2-COLORED CONFLICT-FREE-CONNECTION OF COMPLETE GRAPHS

It would be nice to have an exact answer to every question, but an answer is not always immediately obtainable. However, if the probability of there being no outcome is less than 1, then that is good enough. In other words, probability, when used in graph theory, is often used as a tool to conclude that the outcome we seek does exist. This is a necessary tool that can help obtain bounds on problems where we might not even know where to start. This method is often known as the Probabilistic Method. Its use in Graph Theory was popularized by Paul Erdős [1]. In this Chapter and Chapter 4, we use this method to complete the proofs of our main results. First, however, we must define some key probability terms that will be used throughout the proof.

3.1 Preliminaries

Definition 3.1. *A* probability *is the extent to which an event is likely to occur, measured by the ratio of the favorable cases to the whole number of cases possible. The probability of an event is denoted* ***P****(event).*

Definition 3.2. *A* binomial distribution *is a two-possible-outcome event, repeated a certain number of independent times. The distribution has as a variable k which denotes the number of successes. The other required parameters are N, the number of independent trials, and p, the probability of success on each trial. A binomial distribution with N trials and probability p is denoted* $\mathrm{Bin}(N, p)$

The binomial distribution with parameters $N =$ the number of trials and $k =$ the number of successes is calculated by the following formula:

$$\binom{N}{k} p^k (1-p)^{(N-k)}$$

Even though this formula isn't explicitly used in either proof, it is important to note that from the binomial distribution we can obtain the following Lemma.

Lemma 3.1.1 ([11]). *Let X be a binomial random variable with n number of trials and probability of success p, denoted by $X \sim Bin(n, p)$. Then for $p = \frac{1}{2}$ and r in the support of X,*

$$\boldsymbol{P}(X \leq r) \leq \frac{14}{15} \exp\Big(-\frac{16(\frac{n}{2} - r)^2}{n}\Big)$$

This lemma and the following theorem are both crucial to the proof of both Theorem 1.14 and Theorem 1.15.

Theorem 3.3 ([7]). *Every c-coloring of K_n contains two vertices u and v such that between u and v, there are at most $\frac{n(c-1)-1}{c}$ rainbow paths of length at most 2.*

Each of these things allows us to construct one half of the bound easily. Lemma 3.1.1 will be used to construct the lower bound and Theorem 3.3 will be used to construct the upper bound.

3.2 Bounding 2-colored conflict-free-connection numbers

Proof of Theorem 1.14. For the upper bound, let G_n be a 2-coloring of K_n and let u and v be the two identified vertices in G_n provided by Theorem 3.3. Note that the edge uv is one of the rainbow paths from u to v. If we let A be the internal vertices of the rainbow paths of length 2 from u to v, then each vertex of A represents a CF-path from u to v. If we further let $B = G_n \setminus (A \cup \{u, v\})$, then any CF-paths of length at least 2 that are internally disjoint from A and that go from u to v must use at least two vertices of B each. Since $|A| \leq \frac{n-1}{2} - 1$, there are at most

$$|A| + \frac{|B|}{2} + 1 \leq \left(\frac{n-1}{2} - 1\right) + \frac{\left(\frac{n-1}{2} - 1\right)}{2} + 1 \leq \frac{3n-5}{4}$$

CF-paths between u and v.

For the lower bound we randomly 2-color the edges of a copy of K_n. Now choose two vertices in K_n, call them u and v. Let B_u and R_u be the vertex sets of $K_n - \{u, v\}$ that have blue edges and red edges to u respectively. Let A_{uv} be the event that there are fewer than $\frac{3n}{4} - t$ CF paths from u to v. We now break up A_{uv} into three separate events; C_u, D_{uv}, and F_{uv}.

Let C_u be the event that either $|B_u| < \frac{n}{2} - t$ or $|R_u| < \frac{n}{2} - t$. Let D_{uv} be the event that v has less than $\frac{n}{4} - 2t$ red edges to B_u and F_{uv} be the event that v has less than $\frac{n}{4} - 2t$ blue edges to R_u. This means ${C_u}^c \wedge {D_{uv}}^c \wedge {F_{uv}}^c$ implies ${A_{uv}}^c$, so

$$\mathbf{P}(A_{uv}) \leq \mathbf{P}(C_u) + \mathbf{P}(D_{uv}) + \mathbf{P}(F_{uv}). \tag{3.1}$$

Now we associate C_u, D_{uv}, and F_{uv} with random variables $X_C, X_D,$ and X_F respectively so

$$\begin{aligned} X_C &\sim \text{Bin}\left(n, \frac{1}{2}\right) \\ X_D &\sim \text{Bin}\left(\frac{n}{2} - t, \frac{1}{2}\right) \\ X_F &\sim \text{Bin}\left(\frac{n}{2} - t, \frac{1}{2}\right) \end{aligned}$$

Then by applying Lemma 3.1.1, using $r = \frac{n}{4} - 2t$ for X_D and X_F and $r = \frac{n}{2} - t$ for X_C we get:

$$\mathbf{P}\left(X_D \leq \frac{n}{4} - 2t\right) + \mathbf{P}\left(X_F \leq \frac{n}{4} - 2t\right) \leq \frac{28}{15} \exp\left(-\frac{72t^2}{n - 2t}\right)$$

and

$$\mathbf{P}\left(X_C \leq \frac{n}{2} - t\right) \leq \frac{14}{15} \exp\left(-\frac{16t^2}{n}\right)$$

Then using (1) and the above inequalities we get

$$\begin{aligned}
\mathbf{P}(A_{uv}) &\leq \mathbf{P}(C_u) + \mathbf{P}(D_{uv}) + \mathbf{P}(F_{uv}) \\
&\leq \frac{28}{15}\exp\Big(-\frac{72t^2}{n-2t}\Big) + \frac{14}{15}\exp\Big(-\frac{16t^2}{n}\Big) \\
&\leq \frac{28}{15}\exp\Big(-\frac{16t^2}{n}\Big) + \frac{14}{15}\exp\Big(-\frac{16t^2}{n}\Big) \\
&\leq \frac{14}{15}\cdot 3\exp\Big(-\frac{16t^2}{n}\Big) = \frac{14}{5}\exp\Big(-\frac{(4t)^2}{n}\Big).
\end{aligned}$$

Thus, we have shown that $\mathbf{P}(A_{uv}) \leq \frac{14}{5}\exp\Big(-\frac{(4t)^2}{n}\Big)$. It now suffices to conclude that $\mathbf{P}(\cup_{u,v\in V(G)}A_{uv}) < 1$ because then there exists a coloring where A_{uv} does not hold for any pair of vertices. In other words, it suffices to show that the sum of all events, for some values of t, between each set of two vertices is less than 1. This comes directly from the way t was defined. That is, we have:

$$t > \frac{\sqrt{n\ln\Big(\frac{7n^2}{5}\Big)}}{4}$$

Getting the natural logarithm expression by itself we obtain:

$$\Rightarrow \frac{(4t)^2}{n} > -\ln\Big(\frac{5}{7n^2}\Big)$$

Then we take the exponential of both sides:

$$\Rightarrow \exp\Big(-\frac{(4t)^2}{n}\Big) < \frac{5}{7n^2}$$

Next we multiply both sides by $\dfrac{7n^2}{5}$

$$\Rightarrow \frac{7n^2}{5}\exp\Big(-\frac{(4t)^2}{n}\Big) < 1$$

Then since $\binom{n}{2} \sim \frac{n^2}{2}$ we finally obtain:

$$\binom{n}{2}\frac{14}{5}\exp\Big(-\frac{(4t)^2}{n}\Big) < 1.$$

This means that $\mathbf{P}(\cup_{u,v\in V(G)}A_{uv}) < 1$ completing the proof. $\square$

CHAPTER 4

GENERAL CONFLICT-FREE-CONNECTION NUMBERS OF COMPLETE GRAPHS

The proof of Theorem 1.15 follows similarly from that of Theorem 1.14. Specifically we can obtain an upper bound in the same manner. The lower bound also follows similarly, but it requires the construction of many more events.

Proof of Theorem 1.15. For the upper bound, let G_n be a c-coloring of K_n and let u and v be the two identified vertices in G_n provided by Theorem 3.3. Note that the edge uv is one of the rainbow paths from u to v. If we let A be the internal vertices of the rainbow paths of length 2 from u to v, then each vertex of A represents a CF-path from u to v. If we further let $B = G_n \setminus (A \cup \{u, v\})$, then any CF-paths of length at least 2 that are internally disjoint from A and that go from u to v must use at least two vertices of B each. Since $|A| \leq \frac{n(c-1)-1}{c} - 1$, there are at most

$$|A| + \frac{|B|}{2} + 1 \leq \left(\frac{n(c-1)-1}{c} - 1\right) + \frac{\left(\frac{n(c-1)-1}{c} - 1\right)}{2} + 1 \leq \frac{(2c-1)n - c + 1}{2c}$$

CF-paths between u and v.

For the lower bound we randomly c-color the edges of a copy of K_n. Now choose two vertices in K_n, call them u and v. Let W_{a_i}, where a_i represents the color $i \in (1, ..., c)$, be the vertex sets of $K_n - \{u, v\}$ that have a_i edges to u respectively. Let A_{uv} be the event that there are fewer than $\frac{(2c-1)n}{2c} - t$ CF paths from u to v. We now break up A_{uv} into $c + 1$ separate events; C_u, $D_{1,uv}$, . . . , $D_{c,uv}$.

Let C_u be the event that any one of $|W_{a_i}| < \frac{n}{c} - t$. Let $D_{i,uv}$ be the event that v has less than $\frac{n}{2c} - ct\ a_j\ (j \neq i)$ edges to W_{a_i}. This means ${C_u}^c \wedge {D_{1,uv}}^c \wedge ... \wedge {D_{c,uv}}^c$ implies ${A_{uv}}^c$, so

$$\mathbf{P}(A_{uv}) \leq \mathbf{P}(C_u) + \mathbf{P}(D_{1,uv}) + ... + \mathbf{P}(D_{c,uv}). \tag{4.1}$$

Now we associate C_u and the $D_{i,uv}$ events, with random variables X_C, X_{Di}, respectively so

$$X_C \sim \text{Bin}\left(n, \frac{1}{2}\right)$$
$$X_{Di} \sim \text{Bin}\left(\frac{n}{c} - t, \frac{1}{2}\right)$$

Then by applying Lemma 3.1.1, using $r = \dfrac{n}{2c} - ct$ for X_{Di} and $r = \dfrac{n}{c} - t$ for X_C we get:

$$\mathbf{P}\left(X_{D1} \leq \frac{n}{2c} - ct\right) + ... + \mathbf{P}\left(X_{Dc} \leq \frac{n}{c} - ct\right) \leq \frac{14c}{15}\exp\left(-\frac{c(2n(\frac{c-1}{c}) + 4ct)^2}{n - ct}\right)$$

and

$$\mathbf{P}\left(X_C \leq \frac{n}{c} - t\right) \leq \frac{14}{15}\exp\left(-\frac{16(\frac{n(c-2)}{2c} + t)^2}{n}\right)$$

Then using (1) and the above inequalities we get

$$\begin{aligned}
&\mathbf{P}(A_{uv}) \leq \mathbf{P}(C_u) + \mathbf{P}(D_{1,uv}) + ... + \mathbf{P}(D_{c,uv}) \\
&\leq \frac{14c}{15}\exp\left(-\frac{c(2n(\frac{c-1}{c}) + 4ct)^2}{n - ct}\right) + \frac{14}{15}\exp\left(-\frac{16(\frac{n(c-2)}{2c} + t)^2}{n}\right) \\
&\leq \frac{14c}{15}\exp\left(-\frac{16((\frac{n(c-2)}{2c}) + ct)^2}{n}\right) + \frac{14}{15}\exp\left(-\frac{16(\frac{n(c-2)}{2c} + t)^2}{n}\right) \\
&= \frac{14(c+1)}{15}\exp\left(-\frac{16((\frac{n(c-2)}{2c}) + ct)^2}{n}\right).
\end{aligned}$$

Thus, we have shown that $\mathbf{P}(A_{uv}) \leq \frac{14(c+1)}{15}\exp\left(-\frac{16((\frac{n(c-2)}{2c})+ct)^2}{n}\right)$. It now suffices to conclude that $\mathbf{P}(\cup_{u,v\in V(G)}A_{uv}) < 1$ because then there exists a coloring where A_{uv} does not hold for any pair of vertices. In other words, it suffices to show that the sum of all events, for some values of t, between each set of two vertices is less than 1. This comes directly from the way t was defined. That is we have:

$$t > \frac{\sqrt{n \ln(\frac{7n^2(c+1)}{15})}}{4c} - \frac{n(c-2)}{2c^2}$$

Getting the natural logarithm expression by itself we obtain:

$$\Rightarrow \frac{16(\frac{n(c-2)}{2c} + ct)^2}{n} > -\ln\Big(\frac{15}{7n^2(c+1)}\Big)$$

Then we take the exponential of both sides:

$$\Rightarrow \exp\Big(-\frac{16(\frac{n(c-2)}{2c} + ct)^2}{n}\Big) < \frac{15}{7n^2(c+1)}$$

Next we multiply both sides by $\dfrac{7n^2(c+1)}{15}$:

$$\Rightarrow \frac{7n^2(c+1)}{15} \exp\Big(-\ \frac{16(\frac{n(c-2)}{2c} + ct)^2}{n}\Big) < 1$$

Then since $\binom{n}{2} \sim \frac{n^2}{2}$ we finally obtain:

$$\binom{n}{2}\frac{14(c+1)}{15} \exp\Big(-\ \frac{16(\frac{n(c-2)}{2c} + ct)^2}{n}\Big) < 1.$$

This means that $\mathbf{P}(\cup_{u,v \in V(G)} A_{uv}) < 1$ completing the proof. $\square$

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