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Numerical calculation of the currents in bent wire antennas

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NUMERICAL CALCULATION OF THE CURRENTS IN BENT WIRE ANTENNAS

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ABSTRACT

The current distribution for bent dipole antennas is numerically calculated by using a digital computer. A delta-gap at the center is assumed for the drive. Several different bend-angles are considered including the cases of the straight wire and the inverted-L. The total length of the wire considered is $\lambda/2$ while the ratio of the bent portion over the vertical portion is varied. The method of solution is based on Mei's integral equation and the method of moments. The use of Mei's integral equation eliminates the need for the application of the boundary condition at the point of bending. For the basis functions, both the rectangular pulses and the piecewise sinusoids are used in order to compare the convergence. The test functions are the Dirac delta functions.

ACKNOWLEDGMENTS

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I. INTRODUCTION

In treating wire antenna problems, the entire evaluation of the electromagnetic field clearly depends on a prior knowledge of the distribution of current along the wire. This is why the determination of current in an antenna is of the utmost importance.

In the late 1920's, a new phase began for antenna theory when the importance of the finite lengths of radiating element was recognized. However, no attempt was made to determine the actual distributions of current analytically. Instead, a convenient sinusoidal distribution was assumed, based in part on measured values and in part on its adequacy in special cases.

However, the correct method for determining the properties of a wire antenna is to find the current distribution generating the electromagnetic fields that actually satisfy the boundary conditions. In order to have a mathematically tractable problem, it is frequently assumed that the perfectly conducting wire is electrically thin enough so that all the current is directed axially and is driven at the center across an infinitesimally narrow gap by a generator.

Taking consideration of the boundary conditions and the above assumptions, approximate solutions for the current distribution were obtained in 1937 with the work of L. V. King [1] and in 1938 with that of E. Hallen [2]. The latter derived an integral equation for the axial current of a straight dipole using essentially the retarded-potential method of H. C. Pocklington [3]. Since then R. W. P. King, et al [4] have used the iteration method and Fourier series expansion to calculate the currents. In 1965, K. K. Mei [5] introduced a Hallen-type integral equation for an arbitrarily curved wire and investigated some special antennas using this integral equation and a numerical method which may be termed as the subsection expansion point matching method. This method is a special case of a general method called the method of moments (see Appendix A). In this method, an increase of number of the match points along the wire, where the equation is satisfied, improves the accuracy of the solution if the procedures converge.

T. L. Simpson [6] obtained a solution for the current distribution and driving point admittance of top-loaded antennas such as inverted-L and T type antenna by using coupled integral equations. His solution requires separate sets of equations for each straight wire. The uses of these antennas are found where the height required to achieve self-resonance is prohibitive at long wavelength and it is desired to reduce the effect of the ground plane for the ground-to-ground communications.

This paper obtains the current distribution of bent thin wire antennas by using Mei's integral equation for an arbitrarily shaped wire and the method of moments. Unlike other methods, the use of Mei's form eliminates the need for the application of the boundary condition at the point of bending. The method of moments leads to a set of linear simultaneous equations for solving the unknown coefficients in the expansion of currents. Currents are calculated for several bend-angles and ratios of vertical portion over the bent portion of the wire.

II. FORMULATION OF THE PROBLEM

Consider an arbitrarily curved thin wire of length L as shown in Figure 1 which is situated in a homogeneous, isotropic, nonmagnetic, and linear medium, characterized by the free space permittivity ε_o and the free space permeability μ_o. The wire is sufficiently thin so that only the axial component of current need be considered.

Mei's integral equation [5] for such a curved wire is

$$\int_L I(s')\ \pi(s,s')\ ds' = C \cos ks - \frac{j4\pi\omega\varepsilon_o}{k}\int_o^s E_\xi^i(\xi)\sin k(s-\xi)d\xi \tag{1}$$

where

$$\pi(s,s') = G(s,s')(\hat{s}\cdot\hat{s}') - \int_o^s \{\frac{\partial G(\xi,s')}{\partial s'} + \frac{\partial}{\partial \xi}[G(\xi,s')(\hat{\xi}\cdot\hat{s})]\} \cdot \cos k(s-\xi)d\xi. \tag{2}$$

ω is the angular velocity, $G(s,s')$ is the free space Green's function,

$$G(s,s') = \frac{e^{-jkR}}{R}, \tag{3}$$

$k = 2\frac{\pi}{\lambda}$ is the free space wave number, s and s' are the arc length measured from the coordinate origin somewhere along the wire, $R = R(s,s')$ is the distance between the source point s' and the observation point s, and $\hat{s}$ and $\hat{s}'$ are axial unit vectors at s and s', respectively. The constant c is to be determined by boundary condition.

The derivation given by Mei is based on an auxiliary function which he defines and assumes at the outset the currents at end points vanish. We will give in this chapter an alternative derivation which we believe is simpler and more general in that the currents at end points do not have to be zero. We will first derive the Pocklington-type integral equation for the curved wire which will be subsequently transformed to the Mei's equation by our method.

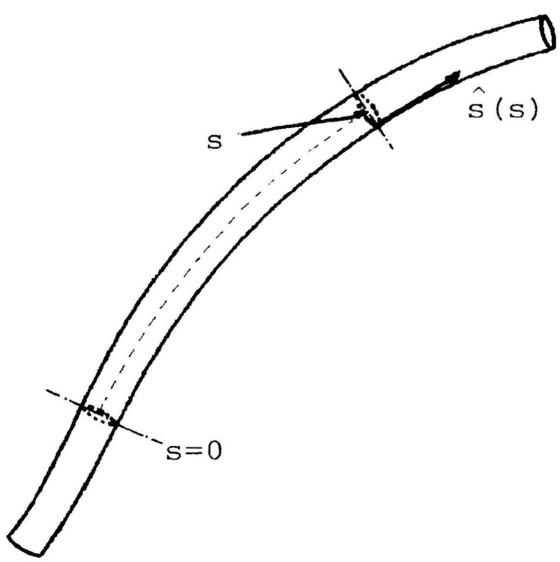

Figure 1. An Arbitrarily Curved Wire

A. DERIVATION OF POCKLINGTON-TYPE INTEGRAL EQUATION

The magnetic vector potential at a point on the wire can be given by

$$\vec{A}(s) = \frac{\mu_o}{4\pi} \int_L I(s')\ G(s,s')\ \vec{ds}' \ . \tag{4}$$

On projecting $\vec{A}(s)$ to the s-direction, the s-component of $\vec{A}(s)$ becomes

$$A_s(s) = \frac{\mu_o}{4\pi} \int_L I(s')G(s,s')(\hat{s}\cdot\hat{s}')ds' \ . \tag{5}$$

Substituting the Lorentz condition

$$\nabla\cdot\vec{A} + j\omega\mu_o\varepsilon_o\phi = 0 \tag{6}$$

into the usual definition of $\vec{E}$ in terms of potentials

$$\vec{E} = -j\omega\vec{A} - \nabla\phi \tag{7}$$

gives in our case

$$\vec{E}(s) = -j\omega\vec{A}(s) + \frac{1}{j\omega\mu_o\varepsilon_o}\nabla[\nabla\cdot\vec{A}(s)] \tag{8}$$

where the time dependence of $e^{j\omega t}$ is understood. The tangential electric field then becomes

$$E_s(s) = -j\omega A_s(s) + \frac{1}{j\omega\mu_o\varepsilon_o} \nabla_s[\nabla\cdot\vec{A}(s)] \tag{9}$$

where ∇_s is the directional derivative in the s-direction.

Taking the divergence of (4) yields

$$\nabla\cdot\vec{A}(s) = \nabla\cdot[\frac{\mu_o}{4\pi}\int_L I(s')\ G(s,s')\ d\vec{s}']$$

$$= \frac{\mu_o}{4\pi}\int_L I(s')\ \nabla\cdot[G(s,s')d\vec{s}']$$

$$= \frac{\mu_o}{4\pi}\int_L I(s')\ \nabla\ G(s,s')\cdot d\vec{s}'\ . \tag{10}$$

The last step is based on the vector identity, $\nabla\cdot(\phi\vec{A}) = \nabla\phi\cdot\vec{A} + \phi\nabla\cdot\vec{A}$. Noting the symmetry property of G or $\nabla G = -\nabla' G$, we have

$$\nabla\cdot\vec{A}(s) = -\frac{\mu_o}{4\pi}\int_L I(s')\ \nabla' G(s,s')\cdot d\vec{s}'$$

$$= -\frac{\mu_o}{4\pi}\int_L I(s')\ \frac{\partial G(s,s')}{\partial s'}\ ds'\ . \tag{11}$$

Thus

$$\nabla_s[\nabla\cdot\vec{A}(s)] = -\frac{\mu_o}{4\pi}\int_L I(s')\ \frac{\partial^2 G}{\partial s\partial s'}\ ds'\ . \tag{12}$$

Putting (5) and (12) into (9) and rearranging gives

$$\int_L I(s')\left[\frac{\partial^2}{\partial s \partial s'} G(s,s') - k^2 G(s,s')(\hat{s}\cdot\hat{s}')\right] ds' = -j4\pi\omega\varepsilon_o E_s(s). \tag{13}$$

For the total tangential electric field on the perfectly conducting wire to vanish, it is required that

$$E_s(s) + E_s^i(s) = 0$$

where $E_s^i(s)$ is the s-component of the impressed field of the source if the antenna is transmitting, or it is the incident electric field when the antenna is receiving.

Replacing $E_s(s)$ by $-E_s^i(s)$, we finally obtain the Pocklington-type integral equation for an arbitrarily curved wire.

$$\int_L I(s')\left[\frac{\partial}{\partial s \partial s'} G(s,s') - k^2 G(s,s')(\hat{s}\cdot\hat{s}')\right] ds' = j4\pi\omega\varepsilon_o E_s^i(s). \tag{14}$$

B. DERIVATION OF MEI'S INTEGRAL EQUATION*

Beginning with the Pocklington-type integral equation (14), let's change the variable s to ξ, multiply both sides by $\sin k(s-\xi)$, and take the integration $\int_o^s d\xi$

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*This derivation is due to my advisor, Dr. B. K. Park

$$\int_o^s \int_L I(s')\left[\frac{\partial^2}{\partial\xi\partial s'} G(\xi,s') - k^2 G(\xi,s')(\hat{\xi}\cdot\hat{s}')\right] \sin k(s-\xi)ds'd\xi$$

$$= \int_o^s j4\pi\omega\varepsilon_o E_\xi^i(\xi) \sin k(s-\xi)d\xi. \qquad (15)$$

Applying integration by parts to the first and the second terms of the left hand side of (13) gives

$$\int_o^s \frac{\partial}{\partial\xi} G(\xi,s') \sin k(s-\xi)d\xi$$

$$= [G(\xi,s') \sin k(s-\xi)]_o^s + k\int_o^s G(\xi,s') \cos k(s-\xi)d\xi$$

$$= -G(o,s) \sin ks + k\int_o^s G(\xi,s') \cos k(s-\xi)d\xi, \qquad (16)$$

and

$$\int_o^s G(\xi,s')(\hat{\xi}\cdot\hat{s}') \sin k(s-\xi)d\xi$$

$$= \frac{1}{k}\int_o^s G(\xi,s')(\hat{\xi}\cdot\hat{s}') \frac{d}{d\xi}\cos k(s-\xi)d\xi$$

$$= \frac{1}{k} [G(\xi,s')(\hat{\xi}\cdot\hat{s}') \cos k(s-\xi)]_o^s$$

$$- \frac{1}{k} \int_0^s \frac{\partial}{\partial \xi} [G(\xi,s')(\hat{\xi}\cdot\hat{s}')] \cos k(s-\xi)d\xi$$

$$= \frac{1}{k} G(s,s')(\hat{s}\cdot\hat{s}') - \frac{1}{k} G(o,s')(\hat{o}\cdot\hat{s}') \cos ks$$

$$- \frac{1}{k} \int_0^s \frac{\partial}{\partial \xi} [G(\xi,s')(\hat{\xi}\cdot\hat{s}')] \cos k(s-\xi)d\xi. \tag{17}$$

Substituting equation (16) and (17) into equation (15), we have

$$\int_L I(s') \, \pi(s,s')ds'$$

$$= D \sin ks + C \cos ks - \frac{j4\pi\omega\varepsilon_o}{k} \int_0^s E_\xi^i(\xi) \sin k(s-\xi)d\xi \tag{18}$$

where

$$\pi(s,s') = G(s,s')(\hat{s}\cdot\hat{s}') - \int_0^s \left\{\frac{\partial G(\xi,s')}{\partial s'} + \frac{\partial}{\partial \xi}[G(\xi,s')(\hat{\xi}\cdot\hat{s}')]\right\} \cdot$$

$$\cos k(s-\xi)d\xi \tag{19}$$

$$C = \int_L I(s') \, G(o,s')(\hat{o}\cdot\hat{s}')ds' \tag{20}$$

$$D = -\frac{1}{k}\int_L I(s')\,\frac{\partial G(o,s')}{\partial s'}ds' . \qquad (21)$$

Following Mei we leave out the first term of the right hand side of equation (18), D sinks, thus obtaining the usual Mei's integral equation.

As shown above, our derivation does not require the introduction of an auxiliary scalar function as an intermediate tool which is inevitable in Mei's derivation.

C. CALCULATION OF CURRENTS BY THE METHOD OF MOMENTS

The general form of the bent wire antenna under study is illustrated in Figure 2. It consists of a vertical element of length H and a bent element of length L which forms an angle with the vertical element. The structure is excited by a delta-gap generator of EMF strength $\frac{V}{2}$ just above the ground plane which is assumed to be an infinite and perfectly conducting plane. Based on the image theory, this structure can be considered as a symmetric center-driven dipole antenna as illustrated in Figure 3.

The impressed electric field can then be given as

$$E^i_\xi\ (\xi) = (\frac{V}{2})\ \delta(\xi) \qquad (22)$$

where $\delta(\xi)$ is the Dirac delta function. Hence the right hand side of equation (1) becomes

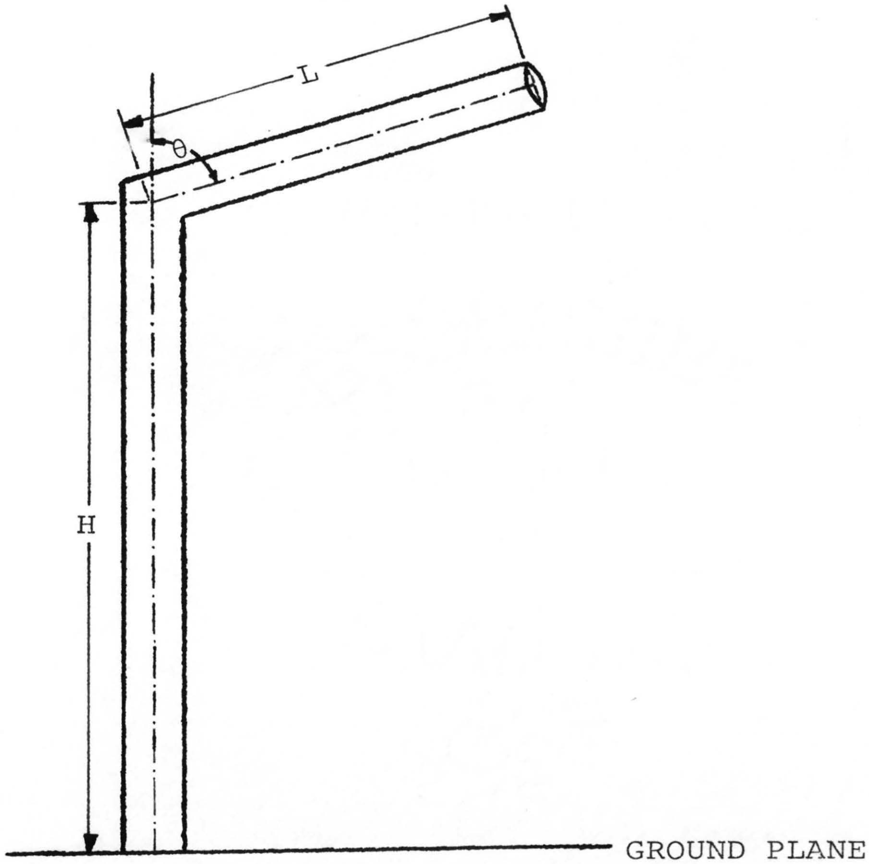

Figure 2. A Bent Wire on a Ground Plane

$$-j\left(\frac{4\pi}{Z_o}\right) \int_o^s E_\xi^i(\xi) \sin k(s-\xi)d\xi$$

$$= -j\left(\frac{2\pi V}{Z_o}\right) \sin ks \tag{23}$$

where $Z_o = 120\pi$ is the free space wave impedence, and equation (1) becomes

$$\int_L I(s')\ \pi(s,s')ds'$$

$$= c \cos ks - j(\frac{V}{60}) \sin ks. \tag{24}$$

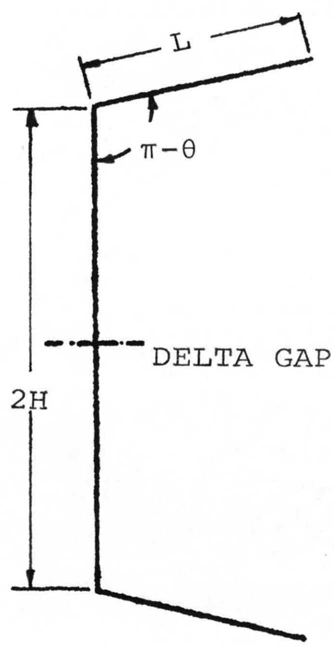

Figure 3. A Symmetric Center-Driven Dipole Antenna

We now seek a solution for the current I(s') which satisfies the integral equation (24) and the requirement that the currents be zero at the ends of wire. According to the method of moments (see Appendix) the first step is to assume that I(s') can be expressed as a linear combination of a finite number of expression functions $I_n(s')$ in the form

$$I(s') = \sum_n C_n I_n(s') \qquad (25)$$

where C_n's are unknown coefficients to be determined.

Inserting the equation (25) into (24) and interchanging the order of summation and integration gives

$$\sum_n C_n \int_L I_n(s') \pi(s,s')ds'$$

$$= C \cos ks + j(\frac{V}{60}) \sin ks. \qquad (26)$$

By enforcing this integral equation to satisfy at a specified number of match points along the wire, we obtain a set of linear simultaneous equations to solve for unknown coefficients C_n. This step amounts to choosing Dirac delta functions for test functions in the general method of moments. If we choose the number of expansion terms in equation (25) to be M+N where M and N are defined in the case of pulse expansion functions by

$$\Delta s' = \frac{H}{(M-1)} = \frac{L}{N} .$$

This is illustrated in Figure 4. In the case of piecewise sinusoidal expansion functions, the definitions of M and N are slightly different from the above definition and are given by

$$\Delta s' = \frac{H}{(M-0.5)} = \frac{L}{(N-0.5)}$$

which is shown in Figure 6. Since the expansion functions are constrained to satisfy the condition that the currents at wire ends be zero a priori, we set C_{M+N} to zero. Since at the same time C must take on the correct value associated with this condition, we treat C as another unknown in the system of linear equations. With $C_{M+N} = 0$ and C assuming the role of another unknown, the system has M+N unknowns. Thus to render the system solvable, we choose the number of match points to be equal to the number of unknowns M+N and we locate them as shown in Figures 4 and 6.

We then have the following set of simultaneous linear equations

$$\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1,M+N-1} & Q_1 \\ P_{21} & P_{22} & \cdots & P_{2,M+N-1} & Q_2 \\ \vdots & \vdots & & \vdots & \vdots \\ P_{M+N,1} & P_{M+N,2} & \cdots & P_{M+N,M+N-1} & Q_{M+N} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{M+N-1} \\ C \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{M+N} \end{bmatrix} \tag{27}$$

where

$$P_{mn} = \int_L I_n(s') \; \pi(s_m, s')ds' \tag{28}$$

$$Q_m = \cos ks_m \tag{29}$$

$$V_m = -j\left(\frac{V}{60}\right) \sin ks_m \tag{30}$$

evidently depend on the choice of expansion set and the location of match points. For the integration of the right hand side of equation (28), Weddle's rule (see Appendix B) is used.

Finally by inserting C_n obtained from equation (27) into equation (25), the current distribution for the problem can be obtained.

The input impedance Z_{in} is given by

$$Z_{in} = \frac{V}{C_1}. \tag{31}$$

In what follows, we will consider the case with rectangular pulses and piecewise sinusoids as the expansion sets for the currents.

1. RECTANGULAR PULSE EXPANSION FOR THE CURRENT

Pulse functions are defined in such a way that each term of the current expansion is zero except in a specific subsection $\Delta s'$ as depicted in Figure 4. For example, the pulse function $P_n(s')$ is unity over the increment, $\Delta s' = s'_{n+1} - s'_n$, and is given by

$$P_n(s') = U(s'-s'_n) - U(s'-s'_{n+1}) \tag{32}$$

where U(s') is the familiar unit step function,

$$U(s') = \begin{cases} 1 \quad , & s'>0 \\ 0 \quad , & s'<0 \end{cases} \tag{33}$$

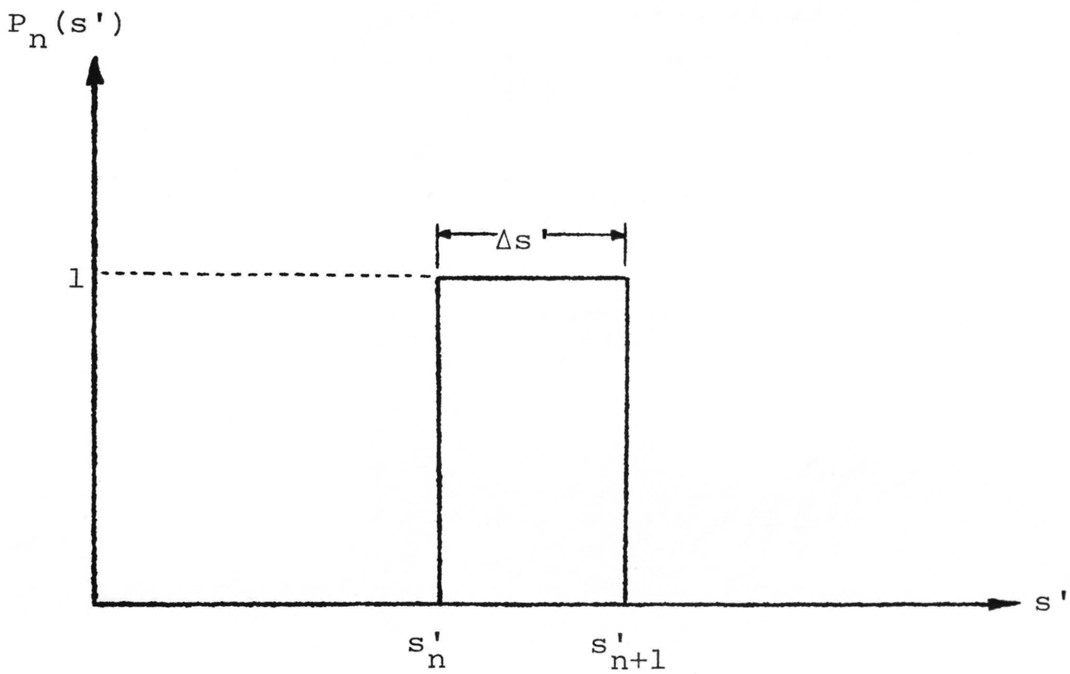

Figure 4. Representative Subsection of a Pulse Function Expansion

Figure 5 illustrates the form of the pulse function expansion for the current along the antenna from the delta-gap to the end of the wire. Thus each C_n specifies the value of a constant current over the interval. We then have

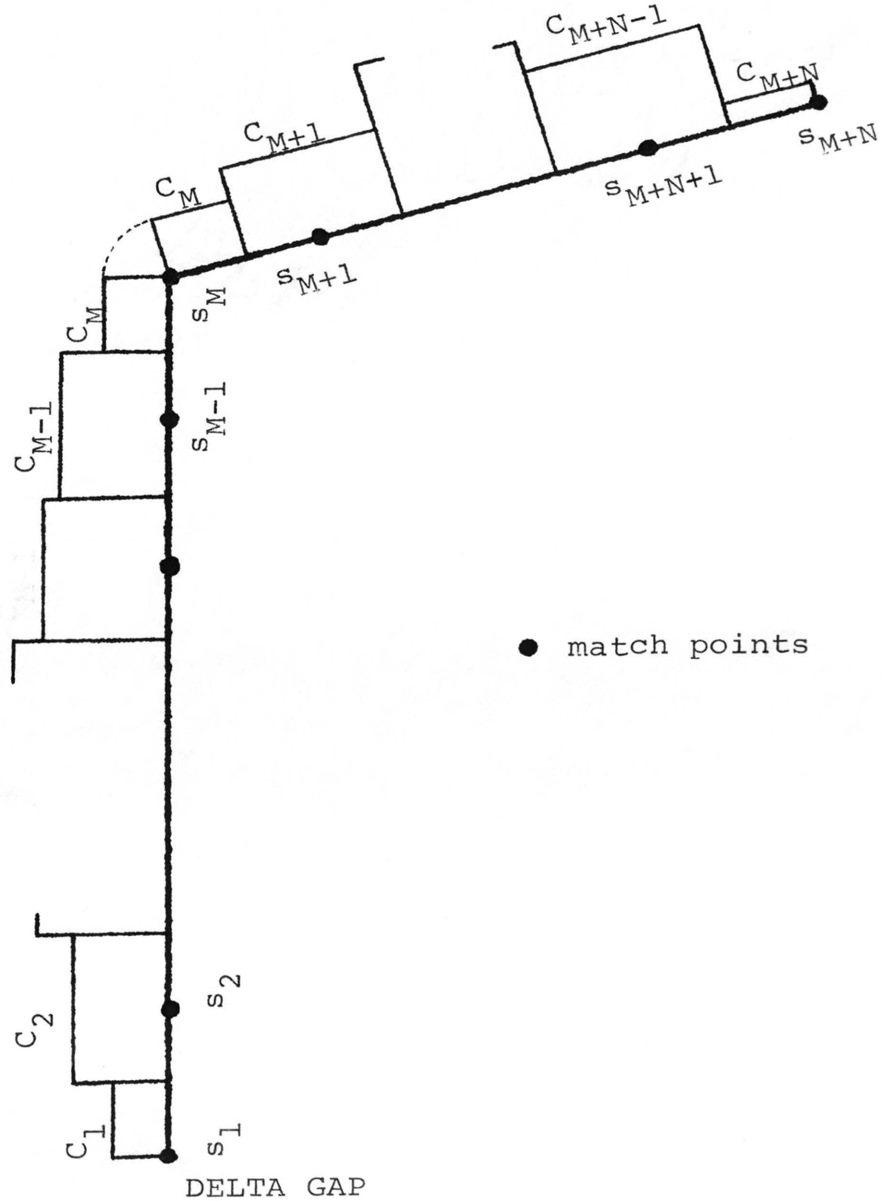

Figure 5. Pulse Function Expansion for the Current Distribution

$$I(s') = \sum_{n=1}^{M+N} C_n[U(s'-s_n') - U(s'-s_{n+1}')]. \qquad (34)$$

By inserting equation (34) into equation (25) and choosing match points as shown in Figure 5, we have the M+N simultaneous linear equations for the M+N unknowns. In this case, the specific values of matrix elements P_{mn} are of the form

$$P_{mn} = \int_{s_n'}^{s_{n+1}'} \pi(s_m, s')ds'. \qquad (35)$$

2. <u>PIECEWISE SINUSOIDAL EXPANSION FOR THE CURRENT</u>

Generally it is understood that the piecewise sinusoidal expansion as shown in Figure 6 is more similar to the exact shape of the current distribution and hence will require fewer terms to accurately calculate the current than the pulse function expansion as shown in Figure 4 [7].

Therefore, by assuming the current distribution as depicted in Figure 7, equation (25) can be written as

$$I(s') = \frac{C_1}{\Delta} \sin k(s_2'-s') \; U(s_2'-s')$$

$$+ \frac{1}{\Delta} \sum_{n=2}^{M+N-1} C_n\{\sin k(s'-s_{n-1}')[U(s'-s_{n-1}')-U(s'-s_n')]$$

$$+ \sin k\,(s_{n+1}'-s')[U(s'-s_n')-U(s'-s_{n+1}')]\}$$

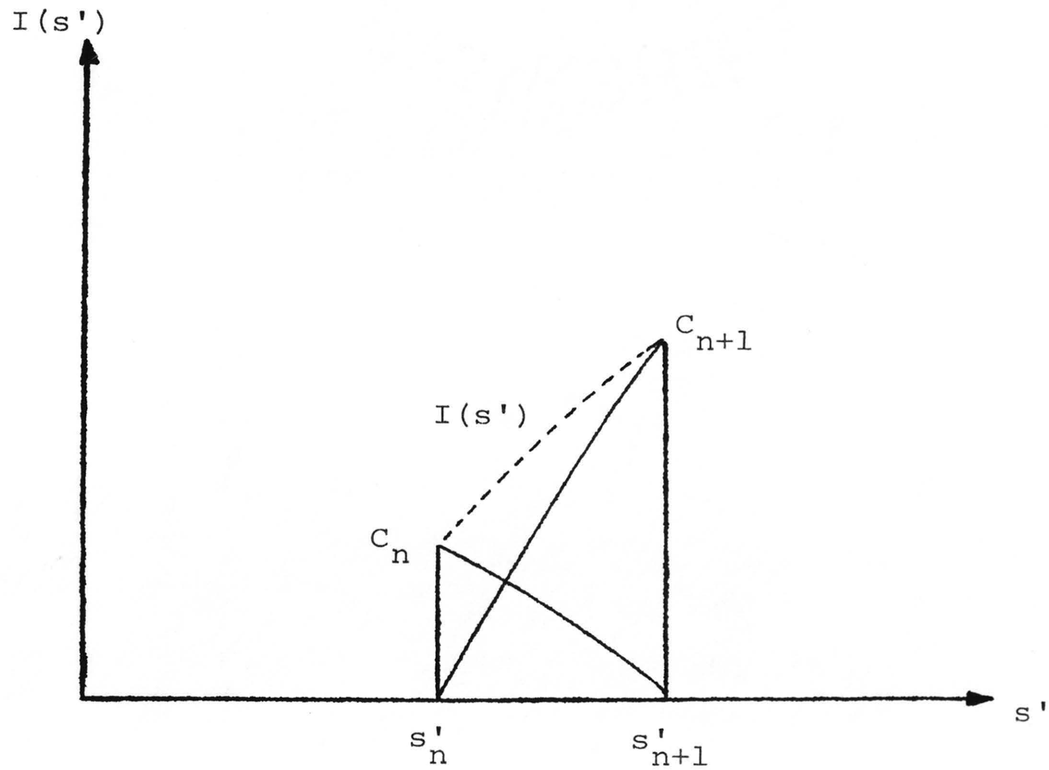

Figure 6. Representative Subsection of a Piecewise Sinusoidal Current Expansion

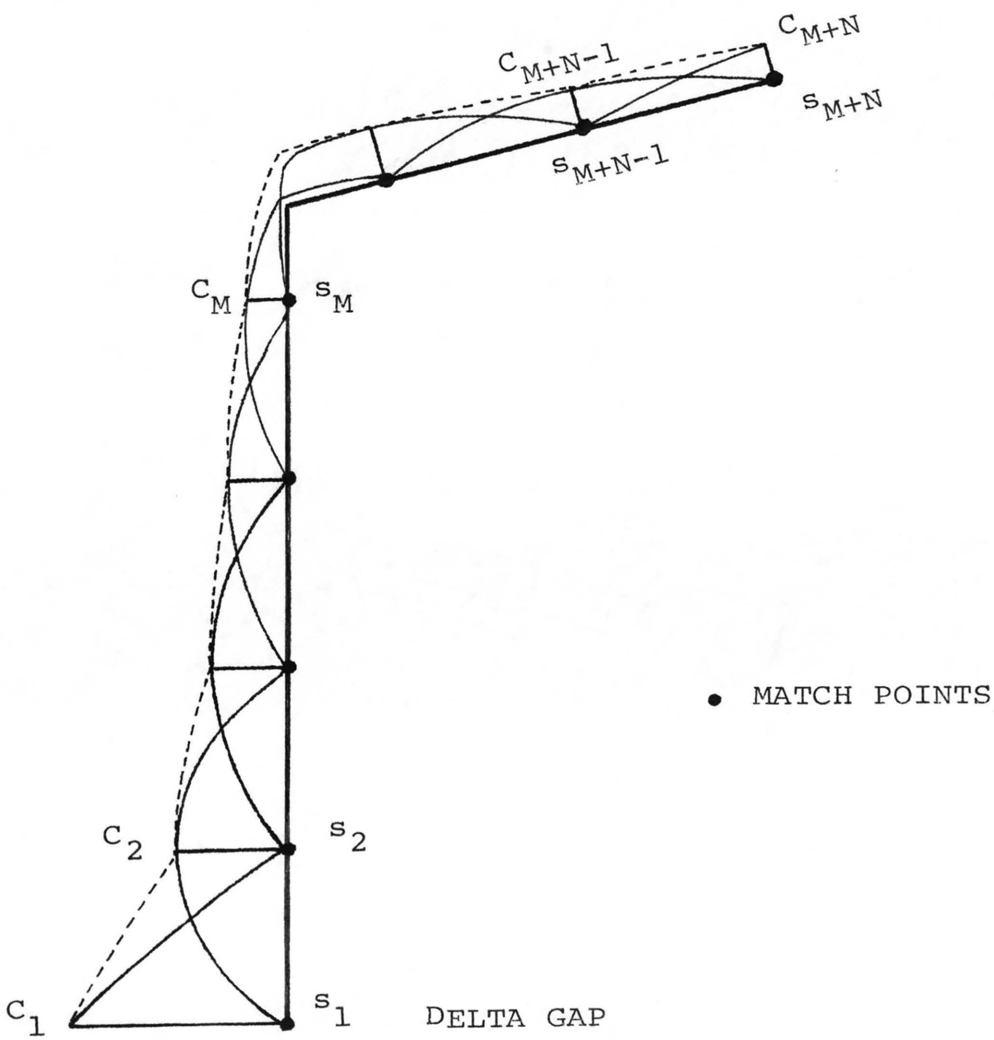

Figure 7. Piecewise Sinusoidal Expansion for the Current Distribution

$$+ \frac{C_{M+N}}{\Delta} \sin k(s'-s'_{M+N+1}) \; U(s'-s'_{M+N+1}) \tag{36}$$

where

$$\Delta = \sin k\Delta s'.$$

Using a procedure similar to that used for the pulse expansion case, a set of simultaneous linear equations is obtained which has the same form as the equation (27). However, the implicit matrix elements P_{mn} will be different. In this case, they are given by

$$P_{m1} = \frac{1}{\Delta} \int_{s'_1}^{s'_2} \pi(s_m, s') \sin k(s'_2 - s')ds' \tag{37}$$

$$P_{mn} = \frac{1}{\Delta} \int_{s'_{n-1}}^{s'_n} \pi(s_m, s') \sin k(s' - s'_{n-1})ds'$$

$$+ \frac{1}{\Delta} \int_{s'_n}^{s'_{n+1}} \pi(s_m, s') \sin k(s'_{n+1} - s')ds' \tag{38}$$

where

$$m = 1,2,3 \; . \; . \; . \; M+N \qquad\qquad n = 2,3,4 \; . \; . \; . \; M+N-1.$$

III. RESULTS AND CONCLUSIONS

The necessary calculations were programmed in FORTRAN IV (single precision) and carried out on IBM 360/50 digital computer at the University of Missouri - Rolla. The results are given in Figures 8 through 11. In all cases, the total length of the dipole considered is $\frac{\lambda}{2}$ or the monopole length $\frac{\lambda}{4}$.

Figure 8 shows the current distribution for the case of bend angle zero. This special case corresponds to the fundamental case of straight dipole and results are in close agreement with the known values. In obtaining these results the total number of subsections used was 20 for piecewise sinusoids and rectangular pulses, the corresponding computation time being approximately 3 minutes and 4 minutes, respectively. The purpose of investigating this special case was to make a specific comparison between the two expansion functions and the results indicate that the piecewise sinusoids give a considerably faster convergence than the rectangular pulses.

The effect of changing the proportions between the vertical element and the bent element was investigated for the case of $\theta = \frac{\pi}{2}$ (the inverted-L antenna) and it is illustrated in terms of input impedance in order to compare with another data obtained by Simpson [6] in Figure 9. Simpson's datas are based on a formulation which has two separate sets of integral equations on the vertical and the horizontal elements which are to satisfy the boundary condition at the bend point. It is seen that while the imaginary part of the current is a reasonably good agreement between Simpson's results and our's, the real parts are considerably different except two special cases corresponding to the vertical dipole and the horizontal dipole. However, it can be seen that the input impedance increases as the horizontal portion of the wire decreases.

The current distribution for the inverted-L wire is plotted in Figure 10 for several different combinations of vertical and horizontal elements. A sharp spike may be noted in the neighborhood of bend point which is understandable on the basis of strong mutual coupling between currents in this region.

Figure 11 shows the effect on current distribution as θ is changed from zero to several different values. It can be seen that the increase in bend angles tend to amplify the overall current including the sharp spike in the bend region. Computer programs are given in Appendix D.

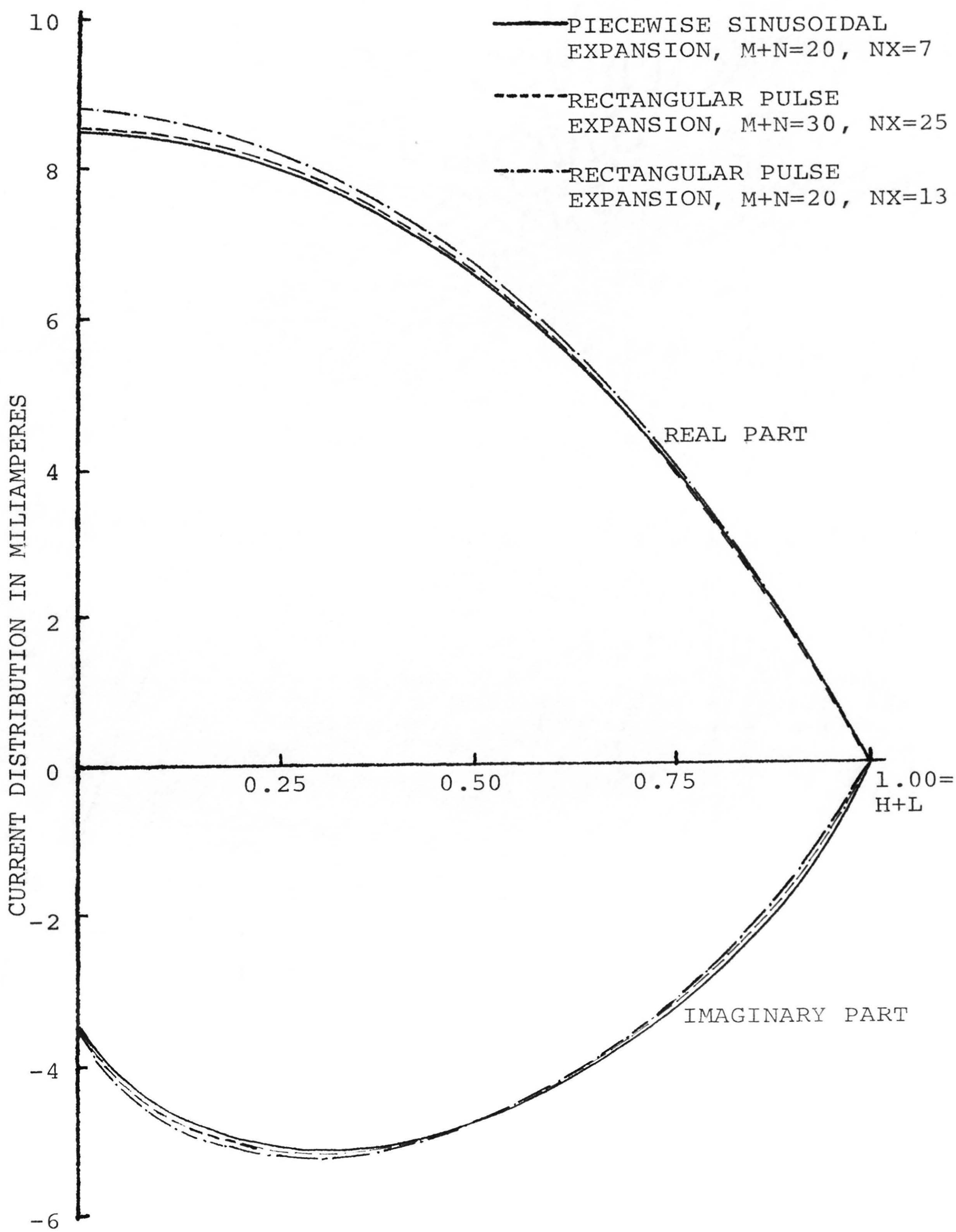

Figure 8. Current Distribution of a Straight Wire.
$H+L= \frac{\lambda}{4}$, $A= \frac{\lambda}{240}$ and $\theta=0.0$. NX=number of Weddle subsections

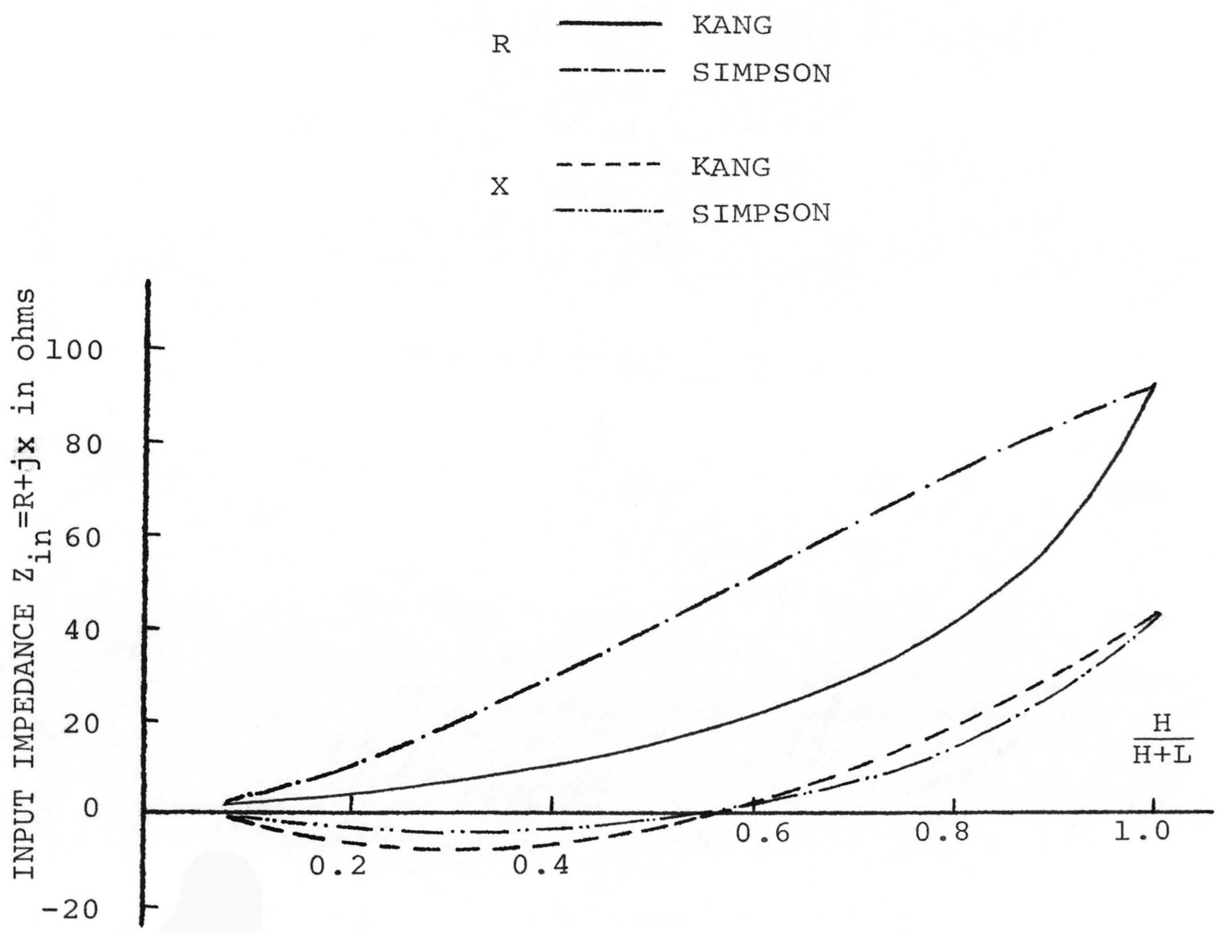

Figure 9. Input Impedance of an Inverted-L Antenna for Various Ratios of Vertical Portion Over the Total Length $\frac{\lambda}{4}$ ($A = \frac{1}{250}\lambda$, M+N=20, NX=7)

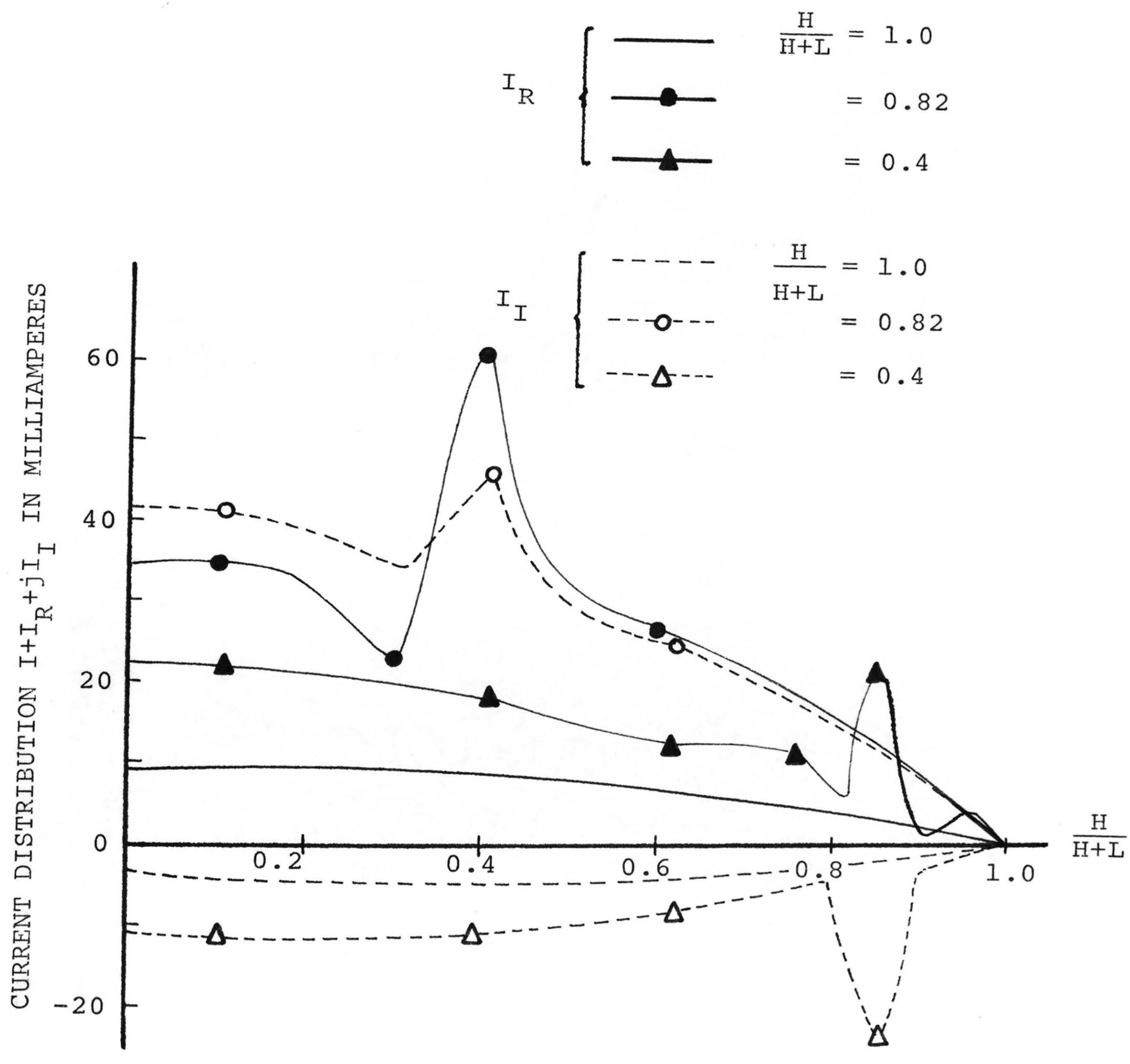

Figure 10. Current Distribution of an Inverted-L Wire for Various Ratios of the Vertical Portion Over the Total Length $\frac{\lambda}{4}$. ($A = \frac{\lambda}{250}$, N=20, NX=7)

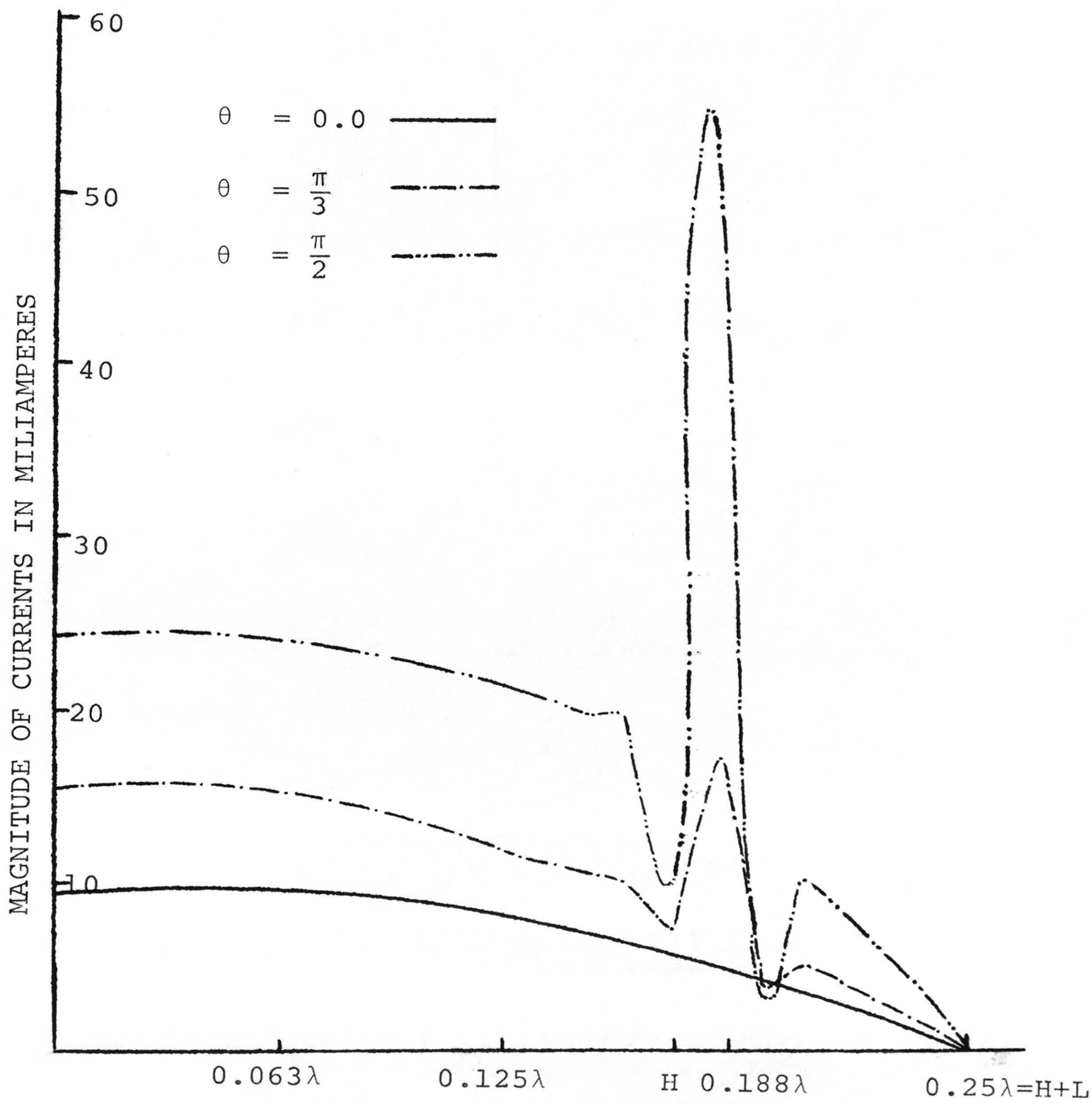

Figure 11. Current Distribution of Bent Wire Antennas With Various Bend-Angles ($H=0.177\lambda$, $N=20$, $Nx=7$)

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VITA

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APPENDIX A

THE METHOD OF MOMENTS [8]

The method of moments is a procedure to reduce an operator equation to a system of linear simultaneous equations which can be solved by matrix inversion and matrix multiplication.

Let's consider the inhomogeneous equation written in symbolic notations

$$L(I) = V \tag{A-1}$$

where L is a linear operator, V is known, and I is to be determined. Let I be expanded in a series of functions $I_1, I_2, I_3, \ldots I_N$ in the domain L, as

$$I = \sum_{n}^{N} C_n I_n \tag{A-2}$$

where the C_n are unknown constants, and we shall call the I_n's expansion functions or basis functions. For approximate solutions, (A-2) is usually a finite summation.

Substituting (A-2) into (A-1) and using linearity of L, we have

$$\sum_{n}^{N} C_n L(I_n) = V. \tag{A-3}$$

It is assumed that a suitable inner product $\langle I,V\rangle$ has been determined for the problem. Now define a set of test functions or weighting functions $w_1, w_2, w_3, \ldots, w_N$ in the range of L, and the inner product of (A-3) with w_m. This results in

$$\sum_{n}^{N} C_n \langle w_m, LI_n\rangle = \langle w_m, V\rangle \tag{A-4}$$

for m = 1, 2, 3, . . ., N. This set of equations can be written in matrix form as

$$[1_{mn}] \quad [C_n] \quad = \quad [V_m] \tag{A-5}$$

where

$$[1_{mn}] = \begin{bmatrix} \langle w_1, LI_1\rangle & \langle w_1, LI_2\rangle & \cdots \\ & & \\ \langle w_2, LI_1\rangle & \langle w_2, LI_2\rangle & \cdots \end{bmatrix} \tag{A-6}$$

$$[C_n] = \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ C_N \end{bmatrix} \tag{A-7}$$

$$[V_m] = \begin{bmatrix} \langle w_1, V\rangle \\ \langle w_2, V\rangle \\ \cdot \\ \langle w_N, V\rangle \end{bmatrix} . \tag{A-8}$$

If matrix $[l_{mn}]$ is nonsingular, its inverse $[l_{mn}]^{-1}$ exists. The $[C_n]$ are then given by

$$[C_n] = [l_{mn}]^{-1} [V_m]. \qquad \text{(A-9)}$$

The solution for I is obtained by inserting C_n's into the equation (A-2).

This solution by the method of moments may be exact or approximate depending on the choice of expansion functions I_n and the test functions w_m. If I_n's are subdomain functions, the method is called the method of subsections. The particular choice $w_m = \delta_m$, Dirac delta functions, is known as the point matching or the method of collocation.

APPENDIX B

WEDDLE'S RULE OF INTEGRATION

Weddle's rule is a numerical integration method which uses a 6th-order Newton forward-form interpolating polynomial.

Let us consider the evaluation of the following definite integral

$$I = \int_{x_o}^{x_n} ydx, \qquad y = f(x) \tag{B-1}$$

from a set of numerical values of the integrand, (x_i, y_i) for $i = 0, 1, 2, \ldots, n$, where the functional form of y is unknown. We first determine an n-th degree polynomial of the form

$$P_n(x) = C_o + C_1(x-x_o) + C_2(x-x_o)(x-x_1) + \ldots$$
$$\ldots + C_n(x-x_o)(x-x_1) \ldots (x-x_{n-1}) \tag{B-2}$$

which approximates $y = f(x)$ over the interval (x_o, x_n) and which coincides with the function at the n+1 evenly spaced match points x_i, i.e., the polynomial $P_n(x)$ satisfies the contraints equations

$$P_n(x_i) = y_i \quad (i = 0,1,2, \ldots, n). \tag{B-3}$$

Substituting equation (B-3) and $h = (x_j - x_i)/(j-i)$ into equation (B-2), we obtain the coefficients C_k of the n-th degree Newton's forward-form interpolating polynomial. Thus

$$P_n(x) = y_o + \frac{\Delta y_o}{h}(x-x_o) + \frac{\Delta^2 y_o}{2h}(x-x_o)(x-x_1) + \ldots$$

$$\ldots + \frac{\Delta^n y_o}{n!h^n}(x-x_o)(x-x_1)\ldots(x-x_{n-1}). \quad \text{(B-4)}$$

Let $u = (x-x_o)/h$ and $du = dx/h$. Then, inserting equation (B-4) into (B-1), we find

$$\int_{x_o}^{x_o+nh} y\,dx = h\int_o^m (y_o + u\Delta y_o + \frac{u(u-1)}{2!}\Delta^2 y_o + \ldots$$

$$\ldots + \frac{u(u-1)(u-2)\ldots(u-5)}{5!}\Delta^6 y_o + \ldots)du \quad \text{(B-5)}$$

which is after integration

$$\int_{x_o}^{x_o+nh} y\,dx = h[ny_o + \frac{n^2}{2}\Delta y_o + (\frac{n^3}{3} - \frac{n^2}{2})\frac{n^2 y_o}{2!} +$$

$$\ldots + (\frac{n^7}{7} + \ldots)\frac{\Delta^6 y_o}{6!} + \ldots]. \quad \text{(B-6)}$$

Putting n = 6 and neglecting all higher order terms beyond sixth, we get

$$\int_{x_o}^{x_o+6h} y\,dx = h[6y_o + 18\Delta y_o + 27\Delta^2 y_o + 24\Delta^3 y_o$$

$$+ \frac{123}{10}\Delta^4 y_o + \frac{33}{10}\Delta^5 y_o + \frac{41}{141}\Delta^6 y_o]. \qquad \text{(B-7)}$$

Substituting the relation [9]

$$\Delta^k y_o = y_k - \binom{k}{1} y_{k-1} + \binom{k}{2} y_{k-2} - \cdots + (-1)^k y_o$$

into (B-7), we have

$$\int_{x_o}^{x_o+6h} y\,dx = h[41y_o + 216\,y_1 + 27y_2 + 272\,y_3 + 27\,y_4$$

$$+ 216\,y_5 + 41\,y_6]/140. \qquad \text{(B-8)}$$

For the next set of six intervals from x_6 to x_{12}, where n is now a multiple of six, we similarly obtain

$$\int_{x_o+6h}^{x_o+12h} y\, dx = h[41\ y_6 + 216\ y_n + 27\ y_8 + 272\ y_9 + 27\ y_{10}$$

$$+ 216\ y_{11} + 41\ y_{12}]/140. \qquad \text{(B-9)}$$

Adding all such expressions as equation (B-8) and (B-9) over the interval (x_o, x_n), where n is now a multiple of six, we get

$$\int_{x_o}^{x_o+nh} y\, dx = \frac{h}{140}\ [41\ y_o + 216\ y_1 + \ldots + 216\ y_5 + 82\ y_6$$

$$+ 216\ y_7 + \ldots + 272\ y_{n-3} + 27\ y_{n-2} + 216\ y_{n-1}$$

$$+ 41\ y_n] = \frac{k}{140} \sum_{n=0}^{n} k_n y_n,$$

where k_n = 41, 216, 27, 272, 27, 216, 84, 216, 27, . . . etc. Weddle's rule described above is essentially Newton-Cotes 6th order quadrature formula [9].

APPENDIX C

DERIVATION OF AN EXPRESSION FOR THE KERNEL FUNCTION $\pi(s,s')$

In this section we will derive the specific expressions for the Kernel function π as defined in equation [19].

Defining the distances between observation points and source points s' as shown in Figure 12, we have

$$R_{11} = (a^2+(s'-\xi)^2)^{\frac{1}{2}}$$

$$R_{12} = (a^2+(s'+\xi)^2)^{\frac{1}{2}}$$

$$R_{13} = ((H-\xi)^2+(s'-H)^2+2(H-\xi)(s'-H)\cos\theta+a^2)^{\frac{1}{2}}$$

$$R_{14} = ((H+\xi)^2+(s'-H)^2+2(H+\xi)(s'-H)\cos\theta+a^2)^{\frac{1}{2}}$$

$$R_{34} = ((2(H+(\xi-H)\cos\theta))^2+(s'-\xi)^2+4(H+(\xi-H)\cos\theta)(s'-\xi)$$
$$\cos\theta+a^2)^{\frac{1}{2}}$$

$$R_{31} = ((\xi-H)^2+(H-s')^2+2(\xi-H)(h-s')\cos\theta+a^2)^{\frac{1}{2}}$$

$$R_{32} = ((\xi-H)^2+(H+s')^2+2(\xi-H)(H+s')\cos\theta+a^2)^{\frac{1}{2}} \quad .$$

We have the corresponding Green's function for each case,

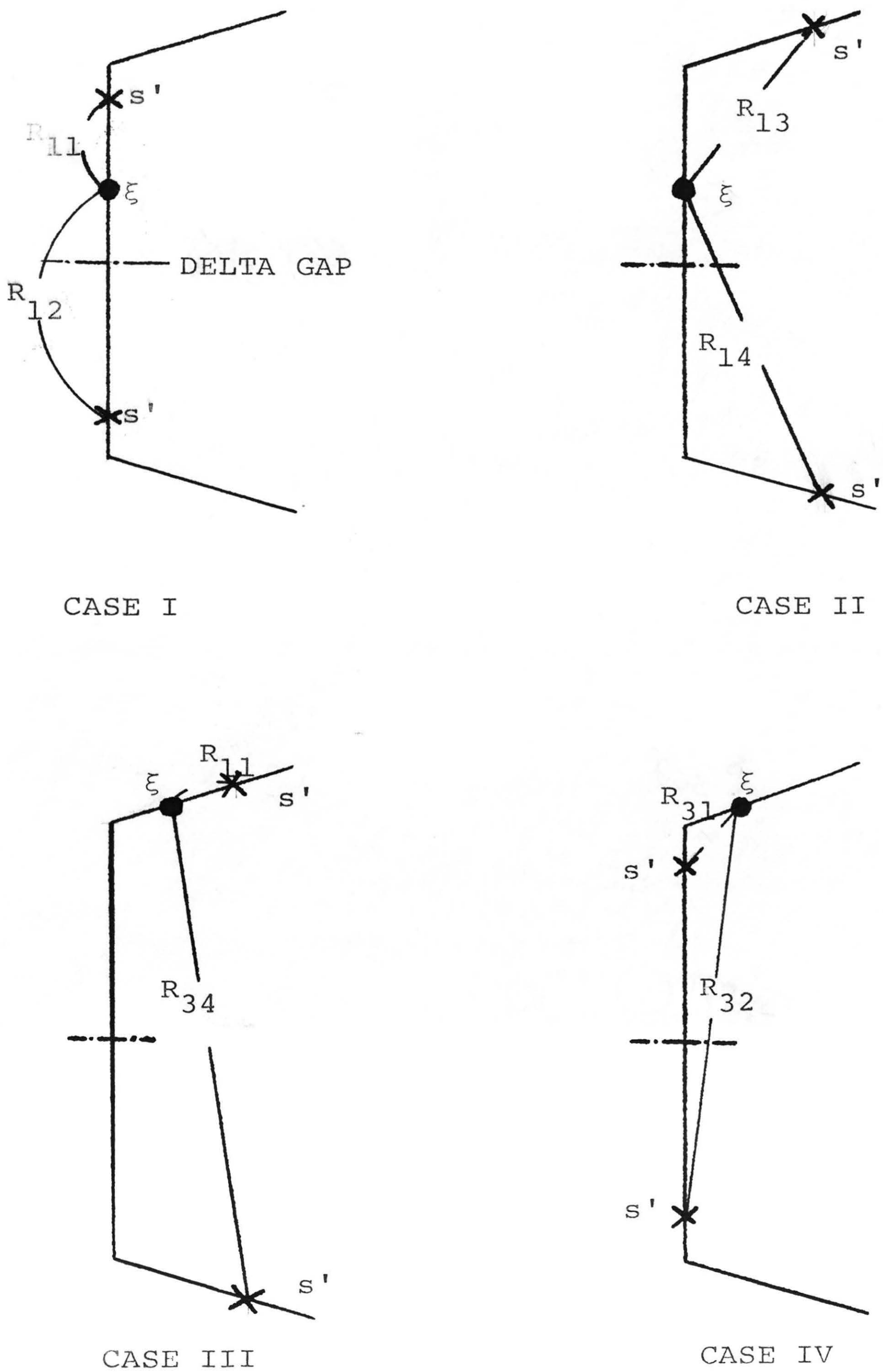

Figure 12. Geometry of Dipole and Relevant Symbols

$$G_{11} = \exp(-jkR_{11})/R_{11}$$

$$G_{12} = \exp(-jkR_{12})/R_{12}$$

$$G_{13} = \exp(-jkR_{13})/R_{13}$$

$$\vdots$$

Differentiating the above Green's function with respect to s' and ξ gives

$$\partial G_{11}/\partial s' = G_{11}(-1-jkR_{11})(s'-\xi)/R_{11}^2$$

$$\partial G_{11}/\partial \xi = -\partial G_{11}/\partial s'$$

$$\partial G_{12}/\partial s' = -G_{12}(-1-jkR_{12})(s'+\xi)/R_{12}^2$$

$$\partial G_{12}/\partial \xi = -\partial G_{12}/\partial s'$$

$$\partial G_{13}/\partial s' = G_{13}(-1-jkR_{13})[(s'-H)+(H-\xi)\cos\theta]/R_{13}^2$$

$$\partial G_{13}/\partial \xi = G_{13}(-1-jkR_{13})[(\xi-H)+(H-s')\cos\theta]/R_{13}^2$$

$$\partial G_{14}/\partial s' = -G_{14}(-1-jkR_{14})[(s'-H)+(H+\xi)\cos\theta]/R_{14}^2$$

$$\partial G_{14}/\partial \xi = G_{14}(-1-jkR_{14})[(\xi+H)+(s'-H)\cos\theta]/R_{14}^2$$

$$\partial G_{34}/\partial s' = -G_{34}(-1-jkR_{34})[(s'-\xi)+2(H+(\xi-H)\cos\theta)\cos\theta]/R_{34}^2$$

$$\partial G_{34}/\partial \xi = G_{34}(-1-jkR_{34})[2(H+(\xi-H)\cos\theta)\cos\theta+(\xi-s')(1-2\cos^2\theta)]/R_{34}^2$$

$$\partial G_{31}/\partial s' = -G_{31}(-1-jkR_{31})[(H-s')+(\xi-H)\cos\theta]/R_{31}^2$$

$$\partial G_{32}/\partial s' = -G_{32}(-1-jkR_{32})[s'+H)+(\xi-H)\cos\theta]R_{32}^2$$

$$\partial G_{32}/\partial \xi = G_{32}(-1-jkR_{32})[(\xi-H)+(H+s')\cos\theta]/R_{32}^2$$

The above equations are now substituted into equation (19) and the scalar products contained in equation (19) are replaced by their appropriate forms for each case defined in Figure 12. The resulting Kernel functions $\pi(s,s')$ are given by

CASE I

$$\pi(s,s') = G_{11}(s,s') + G_{12}(s,s')$$

CASE II

$$\pi(s,s') = (G_{13}(s,s') + G_{14}(s,s'))\cos\theta - \int_0^s ((\partial G_{13}/\partial s' + \partial G_{14}/\partial s') +$$

$$(G_{13}/\partial \xi + \partial G_{14}/\partial \xi)\cos\theta)\cos k(s-\xi)d\xi$$

CASE III

$$\pi(s,s')=G_{11}(s,s')+G_{34}(s,s')\cos 2\theta-\int_{O}^{H}(\partial G_{13}/\partial s'+\partial G_{14}/\partial s')+$$

$$(\partial G_{13}/\partial \xi+\partial G_{14}/\partial \xi)\cos\,\theta)\cos\,k(s-\xi)d\xi-$$

$$\int_{H}^{s}(\partial G_{34}/\partial s'+\partial G_{34}/\partial \xi\,\cos 2\theta)\cos\,k(s-\xi)d\xi$$

CASE IV

$$\pi(s,s')=(G_{31}(s,s')+G_{32}(s,s'))\cos\theta-\int_{H}^{s}((\partial G_{31}/\partial s'+\partial G_{32}/\partial s')+$$

$$(\partial G_{31}/\partial \xi+\partial G_{32}/\partial \xi)\cos\,\theta)\cos\,k(s-\xi)d\xi.$$

APPENDIX D

COMPUTER PROGRAM

The following are computer programs that were used for the calculation of current distribution of bent thin wire antennas. The function subprograms CABL, CABU, CKCD, CFG and CKAE for obtaining matrix elements P_{mn} are included. Subroutine subprogram CWEDF is for the integration by using Weddle's rule and subroutine subprogram CMIN1 is for the complex matrix inversion by using the Gauss-Jordan elimination method, which are not included. All programs are written in single precision IBM360/50 digital computer FORTRAN IV language. Typical execution time for each run is less than five minutes for M+N=20 and NX=7, and core requested is about 120k.

```
      C          BENTED DIPOLE PROBLEM
      C          SLICE GENERATOR EXCITATION & PIECE WISE SINUSOIDAL EXPANSION
      C          DIMENSION STATEMENT
01               IMPLICIT REAL*4(A-B,E-H),COMPLEX*8(C,O-Z)
02               COMPLEX*8 S(30,30),DQM(30),DMF(30),CN(30),CMPLX,ZINPUT,CANS,DETERM
                *,P(30,181),X(30)
03               REAL*4 BS(181),BSM(30),ACN(30),SIN,COS,SQRT
04               COMMON FK,ASQ,FFI,AMAT,AS,ALPHA,H,BNL,BNU
05               EXTERNAL CABU,CKCD,CKAE,CKFG,CABL
      C          INITIAL CONSTANTS
06               MMAX=14
07               NMAX=6
08               H=(2.*MMAX-1.)/(8.*(MMAX+NMAX-1.))
09               ALPHA=1.5708
10               A=1./250.
11               MP1=MMAX+1
12               MP2=MMAX+2
13               MN=MMAX+NMAX
14               MNN1=MN-1
15               KMP1=MP1*6-5
16               KMNP=MN*6+1
17               KMNN=MNN1*6-5
18               KMN=MN*6-5
19               HDEL=2.*H/(2.*MMAX-1.)
20               HDELH=HDEL/2.
21               BDEL=HDEL/6.
22               ASQ=A*A
23               FI=31416
24               ETA=120.*FI
25               FK=2.*FI
26               FCQ=(4.*FI)/ETA
27               EMF=1.0
28               FCV=(FCQ*EMF)/2.
      C
      C          COMPUTATION OF BSN
```

```
      DO 11 K=1,KMNP
      KM1=K-1
      BS(K)=KM1*BDEL
   11 CONTINUE
C     COMPUTATION OF DQM,DMFM
      DO 13 M=1,MN
      BSM(M)=M*HDEL-HDEL
   13 CONTINUE
      DO 14 M=1,MN
      FQ=FCQ*COS(FK*BSM(M))
      DQM(M)=CMPLX(0.0,FQ)
      FV=-FCV*SIN(FK*SBM(M))
      DMF(M)=CMPLX(0.0,FV)
   14 CONTINUE
C     COMPUTATION OF SMN ANDPMK
      DO 15 M=1,MMAX
      AMAT=BSM(M)
      DO 15 N=1,MMAX
      BNL=(N-1)*HDEL
      BNU=N*HDEL
      CALL CWEDF(CABU,BNL,BNU, 7,CANS)
      S(M,N)=CANS/SIN(FK*HDEL)
      ANP=N-2
      IF(ANP) 15,50,50
   50 BNU=BSM(N)
      BNL=BSM(N-1)
      CALL CWEDF(CABL,BNL,BNU,7,CANS)
      S(M,N)=S(M,N)+CANS/SIN(FK*HDEL)
   15 CONTINUE
      DO 16 M=1,MMAX
      AMAT=BSM(M)
      DO 16 N=MP1,MNN1
```

```
      KL=6*N-11
      KM=6*N-5
      KU=6*N+1
      BNL=BSM(N-1)
      BNU=BSM(N+1)
      DO 61 K=KL,KU
      AS=BS(K)
      ANL=0.0
      ANU=AMAT
      CALL CWEDF(CKCD,ANL,ANU, 7, CANS)
      HC=SQRT((AMAT-H)**2+(AS-H)**2+2.*(H-AMAT)*(AS-H)*COS(ALPHA)+ASQ)
      HD=SQRT((AMAT+H)**2+(AS-H)**2+2.*(H+AMAT)*(AS-H)*COS(ALPHA)+ASQ)
      CGC=CMPLX(COS(FK*HC),-SIN(FK*HC))/HC
      CGD=CMPLX(COS(FK*HD),-SIN(FK*HD))/HD
      P(M,K)=-CANS+(CGC+CGD)*COS(ALPHA)
      IF(K -KM) 62,63,63
62    P(M,K)=P(M,K)*SIN(FK*(AS-BNL))
      GO TO 61
63    P(M,K)=P(M,K)*SIN(FK*(BNU-AS))
61    CONTINUE
      S(M,N)=(41.*P(M,6*N-11)+216.*P(M,6*N-10)+27.*P(M,6*N-9)+272.*
     2P(M,6*N-8)+27.*P(M,6*N-7)+216.*P(M,6*N-6)+82.*P(M,6*N-5)+216.*
     3P(M,6*N-4)+27.*P(M,6*N-3)+272.*P(M,6*N-2)+27.*P(M,6*N-1)+216.*P
     4(M,6*N)+41.*P(M,6*N+1))*BDEL/(140.*SIN(FK*HDEL))
16    CONTINUE
      DO 17 M=MP1,MN
      AMAT=BSM(M)
      DO 17 N=1,MP1
      KU=6*N+1
      BNU=BSM(N+1)
      IE(N.EQ.1) GO TO 53
```

```
      BNL=BSM(N-1)
      KL=6*N-11
      KM=6*N-5
      GO TO 54
   53 KL=6*N-5
      KM=KL
      BNL=BSM(N)
   54 DO 71 K=KL,KU
      AS=BS(K)
      ANL=H
      ANU=AMAT
      CALL CWEDF(CKFG,ANL,ANU, 7,CANS)
      HF=SQRT((AMAT-H)**2+(H-AS)**2-2.*(H-AMAT)*(H-AS)*COS(ALPHA)+ASQ)
      HG=SQRT((AMAT-H)**2+(H+AS)**2-2.*(H-AMAT)*(H+AS)*COS(ALPHA)+ASQ)
      CGF=CMPLX(COS(FK*HF),-SIN(FK*HF))/HF
      CGG=CMPLX(COS(FK*HG),-SIN(FK*HG))/HG
      P(M,K)=      -CANS+(CGF+CGG)*COS(ALPHA)
      IF(K-KM) 55,57,57
   55 P(M,K)=P(M,K)*SIN(FK*(AS-BNL))
      GO TO 71
   57 P(M,K)=P(M,K)*SIN(FK*(BNU-AS))
   71 CONTINUE
      IF(N.EQ.1) GO TO 58
      S(M,N)=(41.*P(M,6*N-11)+216.*P(M,6*N-10)+27.*P(M,6*N-9)+272.*
     2P(M,6*N-8)+27.*P(M,6*N-7)+216.*P(M,6*N-6)+82.*P(M,6*N-5)+216.*
     3P(M,6*N-4)+27.*P(M,6*N-3)+272.*P(M,6*N-2)+27.*P(M,6*N-1)+216.*P
     4(M,6*N)+41.*P(M,6*N+1))*BDEL/(140.*SIN(FK*HDEL))
      GO TO 17
   58 S(M,N)=(41.*P(M,6*N-5)+216.*P(M,6*N-4)+27.*P(M,6*N-3)+272.*P(M.
     26*N-2)+27.*P(M,6*N-1)+216.*P(M,6*N)+41.*P(M,6*N+1))*BDEL/(140.*
     3SIN(FK*HDEL))
   17 CONTINUE
      DO 19 M=MP1,MN
      AMAT=BSM(M)
```

```
      DO 19 N=MP1,MNN1
      KL=6*N-11
      KM=6*N-5
      KU=6*N+1
      BNL=BSM(N-1)
      BNU=BSM(N+1)
      DO 91 K=KL,KU
      IF((N.EQ.MP1).AND.(K.LE.KM)) GO TO 91
      AS=BS(K)
      ANL=0.0
      ANU=H
      CALL CWEDF(CKCD,ANL,ANU, 7,CANS
      P(M,K)=-CANS
      ANL=H
      ANU=AMAT
      CALL CWEDF(CKAE,ANL,ANU, 7,CANS)
      HA=SQRT(ASQ+(AS-AMAT)**2)
      HE=SQRT((2.*(H+(AS-H)*COS(ALPHA)))**2+(AS-AMAT)**2-2.*(2.*(H+(AS
     2-H)*COS(ALPHA)))*(AS-AMAT)*COS(ALPHA)+ASQ)
      CGA=CMPLX(COS(FK*HA),-SIN(FK*HA))/HA
      CGE=CMPLX(COS(FK*HE),-SIN(FK*HE))/HE
      P(M,K)=P(M,K)-CANS+CGA+CGE*COS(2.*ALPHA)
      IF(K-KM)  95,97,97
   95 P(M,K)=P(M,K)*SIN(FK*(AS-BNL))
      GO TO 91
   97 P(M,K)=P(M,K)*SIN(FK*(BNU-AS))
   91 CONTINUE
      S(M,N)=(41.*P(M,6*N-11)+216.*P(M,6*N-10)+27.*P(M,6*N-9)+272.*
     2P(M,6*N-8)+27.*P(M,6*N-7)+216.*P(M,6*N-6)+82.*P(M,6*N-5)+216.*
     3P(M,6*N-4)+27.*P(M,6*N-3)+272.*P(M,6*N-2)+27.*P(M,6*N-1)+216.*P
     4(M,6*N)+41.*P(M,6*N+1))*BDEL/(140.*SIN(FK*HDEL))
   19 CONTINUE
      DO 21 M=1,MN
      S(M,MN)=DQM(M)
```

```
   21     CONTINUE
          WRITE(3,105)((M,N,S(M,N),N=1,MN),M=1,MN)
  105     FORMAT(5X,2I2,5X,2E15.7)
C         SIMULTANEOUS EQUATIONS SOLUTION
          CALL CMIN1(MN,S,30,DETERM)
          DO 22 M=1,MN
          CN(M)=0.0
          DO 600 N=1,MN
  600     CN(M)=CN(M)+DMF(N)*S(M,N)
          ACN(M)=CABS(CN(M))
   22     WRITE(3,151) M,ACN(M),CN(M)
  151     FORMAT(2X,I2,5X,'ACN=',E16.7,5X,'CN=',2E16.7)
          AINPUT=EMF/CN(1)
          WRITE(3,152) ZINPUT
  152     FORMAT(2X,'ZINPUT=',2E16.7)
          STOP
          END
```

```
          COMPLEX FUNCTION CABL(BE)
          IMPLICIT COMPLEX*8(C)
          COMPLEX*8 CMPLX
          REAL*4 SQRT,COS,SIN
          COMMON FK,ASQ,FFI,AMAT,AS,ALPHA,H,BNL,BNU
          GP=SQRT(ASQ+(AMAT-BE)**2)
          GI=SQRT(ASQ+(AMAT+BE)**2)
          FKR=COS(FK*GP)/GP+COS(FK*GI)/GI
          FKI=-SIN(FK*GP)/GP-SIN(FK*GI)/GI
          CABL=CMPLX(FKR,FKI)*SIN(FK*(BE-BNL))
          RETURN
          END
```

```
      COMPLEX FUNCTION CABU(BE)
      IMPLICIT COMPLEX*8(C)
      COMPLEX*8 CMPLX
      REAL*4 SQRT,COS,SIN
      COMMON FK,ASQ,FFI,AMAT,AS,ALPHA,H,BNL,BNU
      GP=SQRT(ASQ+(AMAT-BE)**2)
      GI=SQRT(ASQ+(AMAT+BE)**2)
      FKR=COS(FK*GP)/GP+COS(FK*GI)/GI
      FKI=-SIN(FK*GP)/GP-SIN(FK*GI)/GI
      CABU=CMPLX(EKR,FKI)*SIN(FK*(BNU-BE))
      RETURN
      END
```

```
C     FUNCTION SUBPROGRAM FOR CKCD
      COMPLEX FUNCTION CKCD(BE)
      IMPLICIT COMPLEX*8(C)
      COMPLEX*8 CMPLX
      REAL*4 SQRT,COS,SIN
      COMMON FK,ASQ,FFI,AMAT,AS,ALPHA,H,BNL,BNU
      HC=SQRT((BE-H)**2+(AS-H)**2+2.*(H-BE)*(AS-H)*COS(ALPHA)+ASQ)
      HD=SQRT((BE+H)**2+(AS-H)**2+2.*(H+BE)*(AS-H)*COS(ALPHA)+ASQ)
      CGC=CMPLX(COS(FK*HC),-SIN(FK*HC))/HC
      CGD=CMPLX(COS(FK*HD),-SIN(FK*HD))/HD
      CKCD=(CGC*CMPLX(-1./HC**2,-FK/HC)*(AS-H)+CGD*CMPLX(-1./HD**2,-FK/
     2HD)*(H-AS))*SIN(ALPHA)**2*COS(FK*(AMAT-BE))
      RETURN
      END
```

```
      COMPLEX FUNCTION CKFG(BE)
      IMPLICIT COMPLEX*8(C)
      COMPLEX*8 CMPLX
      REAL*4 SQRT,COS,SIN
      COMMON FK,ASQ,FFI,AMAT,AS,ALPHA,H,BNL,BNU
      HF=SQRT((BE-H)**]+(H-AS)**2+2.*(BE-H)*(H-AS)*COS(ALPHA)+ASQ)
      HG=SQRT((BE-H)**]+(H+AS)**2+2.*(BE-H)*(H+AS)*COS(ALPHA)+ASQ)
      CGF=CMPLX(COS(FK*HF),-SIN(FK*HE))/HE
      CGG=CMPLX(COS(FK*HG),=SIN(FK*HG))/HG
      CKEG=(CGF*CMPLX(-1./HF**2,-FK/HE)*(AS-H)+CGG*CMPLX(-1./HG**2,-FK
     2/HG)*(-AS-H))*SIN(ALPHA)**2*COS(FK*(AMAT-BE))
      RETURN
      END
```

```
      COMPLEX FUNCTION CKAE(BE)
      IMPLICIT COMPLEX*8(C)
      COMPLEX*8 CMPLX
      REAL*4 SQRT,COS,SIN
      COMMON FK,ASQ,FFI,AMAT,AS,ALPHA,H,BNL,BNU
      HA=SQRT(ASQ+(AS-BE)**2)
      HE=SQRT((2.*(H+(AS-H)*COS(ALPHA)))**2+(AS-BE)**2-2.*(2.*(H+(AS
     2-H)*COS(ALPHA)))*(AS-BE)*COS(ALPHA)+ASQ)
      CGA=CMPLX(COS(FK*HA),-SIN(FK*HA))/HA
      CGE=CMPLX(COS(FK*HE),-SIN(FK*HE))/HE
      CKAE=(CGE*CMPLX(-1./HE**2,-FK/HE)*((AS-BE)*(-1.-COS(2.*
     2ALPHA)+2.*COS(ALPHA)**2*COS(2.*ALPHA)))+2.*(H+(BE-H)*COS(ALPHA))
     3*COS(ALPHA)*(-1.+COS(2.*ALPHA)))*COS(FK*(AMAT-BE))
      RETURN
      END
```