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EVALUATION OF METHODS FOR ANALYSIS OF
MULTI-DEGREE-OF-FREEDOM SYSTEMS
WITH DAMPING

BY

BRIJ. R. MOHTA, 1942

A

THESIS

submitted to the faculty of

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Approved by

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ABSTRACT

A general review of various methods for studying the behavior of linear lumped parameter systems with viscous damping is presented.

Five methods are discussed. These are: (1) Normal Mode Technique (2) Holzer's Method (3) Impedance Method (4) Graphical Technique (5) A Method for Reducing Degrees-of-freedom.

For solution of vibration problems by the Normal Mode Technique, the systems are classified as (1) classically damped or (2) non-classically damped. It is shown that the classically damped systems are relatively easy to solve. For non-classically damped systems, the method proposed by K. A. Foss has been employed. This method is quite complex, but does provide an exact solution in most cases. In Holzer's Method, equations for both undamped and damped systems are derived. A sample table is presented which is employed to solve these equations. Systems having dampers between masses as well as between the masses and ground have been discussed. Also branched systems have been treated. In the Impedance Method, the four-pole parameters of a mass, spring and damper are derived and the formulas for solving tandem and parallel connections are presented. In the Graphical Technique, procedures for arranging the equations of motion in a form suitable for graphical solution are outlined. Application of this method to branched systems is discussed. In the Method for Reducing Degrees-of-Freedom, two problems are presented to illustrate the use of this method. The results obtained have been compared with exact solutions.

Advantages and disadvantages of each of these methods are discussed on a comparative basis. A sample problem is solved by all of these methods and the results are compared. Suggestions for further work are made.

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LIST OF SYMBOLS

(Listed in order of appearance)

- \dot{X}_i = displacement coordinate associated with i^{th} mass,
 $\dot{\dot{X}}_i$ = velocity of i^{th} coordinate,
 \ddot{X}_i = acceleration of i^{th} coordinate,
 C_{ij} = coefficient of viscous damping between i^{th} and j^{th} mass
 K_{ij} = spring constant between i^{th} and j^{th} mass
 F_i = force on the i^{th} mass
 η_i = displacement of i^{th} normal coordinate
 $\dot{\eta}_i$ = velocity of i^{th} normal coordinate
 $\ddot{\eta}_i$ = acceleration of i^{th} normal coordinate
 T = kinetic energy
 U = potential energy
 V = dissipation function
 P = virtual work
 W = circular frequency
 M_i = mass of i^{th} element
 $[Q]$ = modal matrix
 $[N] = [\sqrt{M}]$
 $[I]$ = identity matrix
 a_{ne} = a constant
 J_i = mass moment of inertia of i^{th} disk
 θ_i = angular displacement of i^{th} disk
 $\dot{\theta}_i$ = angular velocity of i^{th} disk
 $\ddot{\theta}_i$ = angular acceleration of i^{th} disk
 $\{\Phi_i\}$ = a column vector of amplitudes in $2N$ space

$A_{11}, A_{12}, A_{21}, A_{22}$ = four pole parameters

w_0 = selected value of w within frequency range of interest

Td_j = displacement transmissibility of j^{th} element

Tf_j = force transmissibility of j^{th} element

w_{oj} = individual natural frequency of j^{th} spring-mass system

b_j = ratio of viscous damping to the critical damping of j^{th} spring-mass-damper system.

$$\lambda_i = -\frac{C_i}{2M_i} \pm \frac{\sqrt{C_i^2 - 4K_iM_i}}{2M_i}$$

h_i = coefficient of viscous damping between i^{th} mass and ground

A, B, C, = as defined in context

β = phase angle

I. INTRODUCTION

It should be remarked at the outset that this is a report of the tutorial type. By this, it is meant that many of the methods to be presented have appeared somewhere in the literature. However, engineers with only basic knowledge of the subject of vibrations often have difficulty in understanding and applying these ideas, many of which are presented in relatively obscure forms. The purpose here is to collect these ideas together and to describe, evaluate, and compare them in such a way that the method best suited to solution of a particular problem can be determined and applied.

Many engineering vibration problems can be treated by the theory of one-degree-of-freedom systems. More complex systems may possess several degrees of freedom. The standard technique to solve such systems, if the degrees of freedom are not more than three, is to obtain the equations of motion by Newton's law of motion, by the method of influence coefficients, or by Lagrange's equations. Then the differential equations of motion are solved by assuming an appropriate solution. Solving of differential equations of motion becomes increasingly more laborious as the number of degrees of freedom increases. While the solution is straight forward for an undamped multi-degree-of-freedom system, it becomes much more complex for a damped system.

The forces F_1 arising due to damping associated with the co-ordinates x_1, x_2, \dots, x_n will have the form

$$\begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{Bmatrix} \text{damp} = - \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{1n} \\ C_{21} & C_{22} & & & & C_{2n} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ C_{n1} & \cdot & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{Bmatrix}$$

For a less accurate solution, the damping forces associated with the normal coordinates can be written as:

$$\begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{Bmatrix} \text{damp} = - \begin{bmatrix} C_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & C_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & C_3 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & C_n \end{bmatrix} \begin{Bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\eta}_n \end{Bmatrix}$$

Note that the off-diagonal, or coupling terms have been assumed negligibly small. However, for many engineering applications such as aircraft design, this method fails to provide an accurate solution. For this reason several different techniques have been suggested for solving a damped multi-mass system.

The objective of this thesis is to analyze, simplify, and compare these techniques in a manner useful to practicing engineers. To facilitate this, a sample problem will be solved by each of these methods. This will also be of value in illustrating the use of the mathematical theory presented.

II. REVIEW OF LITERATURE

Generally textbooks^{(1,2,3,4)*} very thoroughly cover the analysis of undamped multimass systems, with practically no mention of damping in such systems. The only exception to the above statement is reference (2) where the author has derived the equations of motion for a damped n-degree-of-freedom system and has also presented an approximate solution for such a system.

Because of the wide applications of damped multimass systems, several technical papers with different solutions to the problem, have appeared recently. K. A. Foss⁽¹¹⁾, T. K. Caughey⁽¹²⁾, and M. E. J. O'Kelly⁽¹³⁾ have developed the Normal Mode Technique which is widely used in solving the equations for damped multimass systems. However, this method has limitations which will be pointed out later.

In a modification of the Normal Mode Technique, S. E. Staffeld⁽⁷⁾ has suggested a method by which the number of degrees of freedom can be reduced in mathematical models of damped linear dynamic systems.

W. T. Thomson⁽¹⁾, E. H. Eddy⁽⁸⁾, and J. P. Den Hartog^(9,10) describe and demonstrate the use of Holzer's method, which is one of the most important tools in solving damped torsional multidisk systems.

C. T. Molloy⁽⁵⁾ has suggested the use of four-pole parameters for solving vibration problems. The graphical technique of reference (6)

* Numbers in parentheses refer to the list of references at the end of the thesis.

is also quite useful in determining the frequency response of linear mechanical systems.

III. ANALYSIS OF DAMPED MULTIMASS SYSTEMS

Several methods are available for analyzing a damped multi-degree-of-freedom system. Some of these will be presented here in the following order:

1. Normal Mode Technique
2. Holzer's Method
3. Impedance Method
4. A Graphical Technique
5. Method for Reducing Degrees-of-freedom

Where the development of mathematical theory is complicated, undamped systems will be considered first and then damping will be introduced in such systems. In other methods which are easier to follow, undamped systems are considered as a special case of the general problem with damping. Should it be desired to apply these methods to an undamped system, the damping term is simply set equal to zero.

IV. NORMAL MODE TECHNIQUE

The equations of motion of a system can be derived by a number of different methods. These are derived here by using Lagrange's Equations⁽¹⁴⁾, the energy method most frequently encountered in engineering analysis.

To use Lagrange's equation, it is necessary to define:

- (1) Generalized Coordinates: A set of independent coordinates used to completely describe the motion of a system.
- (2) Holonomic System: A system such that the number of degrees of freedom equals the number of coordinates required to completely describe it.
- (3) Non-Holonomic System: For such a system, the number of degrees of freedom is less than the required number of coordinates required to completely describe the motion of the system. In such systems coordinates cannot be eliminated by using the constraint equations. Therefore, systems containing non-holonomic constraints always require more coordinates for their description than there are degrees of freedom. Such systems may occur in Dynamics of Particles⁽¹⁴⁾ but are rarely encountered in vibration analysis.

Lagrange's equations for a holonomic system with n degrees of freedom can be expressed as

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_1} - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} + \frac{\partial V}{\partial x_1} = \frac{\partial P}{\partial x_1} \quad (1)$$

Here V is a dissipation function which accounts for the losses due to viscous damping. P is virtual work. The concept of virtual work is explained⁽¹⁴⁾ as follows:

Suppose that the forces F_1, F_2, \dots, F_N are applied at the corresponding coordinates in the direction of the increasing coordinate in each case. Now imagine that, at a given instant, the system is given arbitrary small displacements $\delta x_1, \delta x_2, \dots, \delta x_N$ of the corresponding coordinates. The work done by the applied forces is

$$\delta P = \sum_{j=1}^{j=N} F_N \delta x_N$$

and is known as the virtual work. The small displacements are called virtual displacements because they are imaginary in the sense that they are assumed to occur without the passage of time, the applied forces remaining constant.

Derivation of Equations of Motion of Multi-Degree of Freedom System

Consider a system of n discrete masses m_i coupled together through springs and dashpots as shown in figure 1. Let x_i denote the n generalized coordinates to specify the motion of the system. Define $x_i = 0$ for all i when the system is in stable equilibrium. For such a system,

$$T = \frac{1}{2} \sum_{i=1}^n m_i \dot{x}_i^2 \quad (2)$$

$$U = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n K_{ij} x_i x_j \quad (3)$$

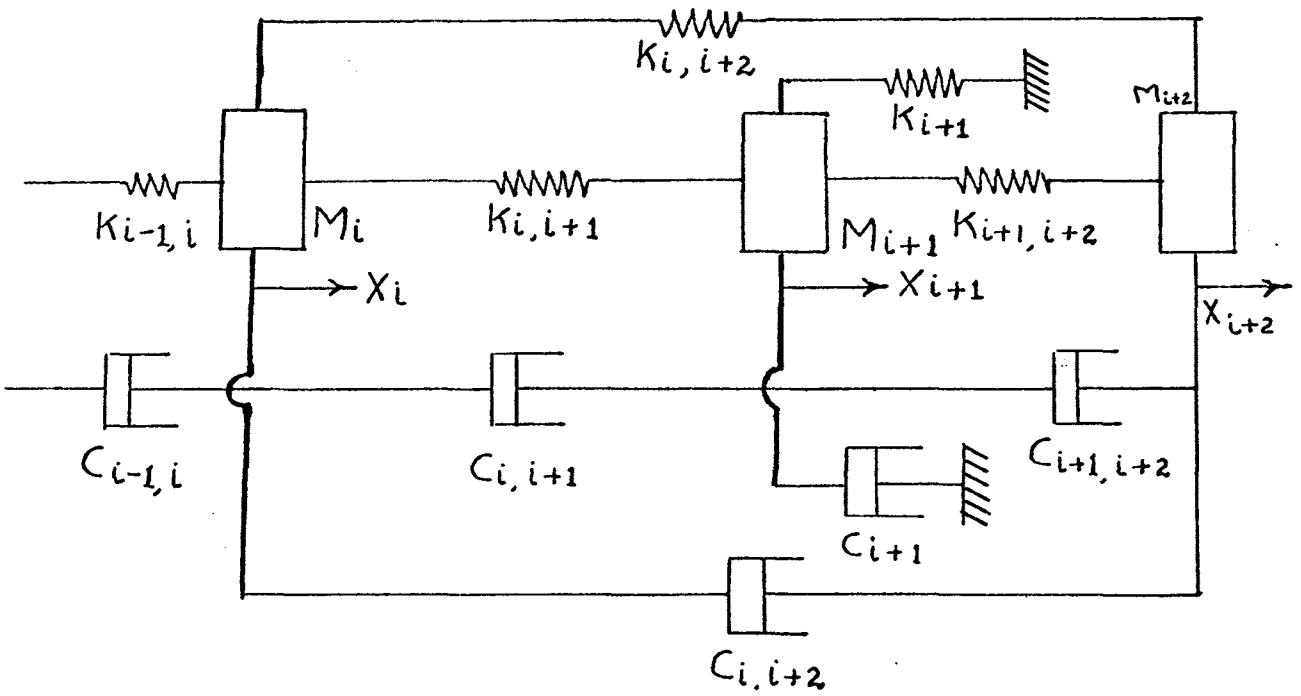


Figure 1

Mass, Spring, and Damper System

$$V = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n C_{ij} \dot{x}_i \dot{x}_j \quad P = \sum_{i=1}^n F_i \delta x_i \quad (4)$$

From these equations,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_i} \right) = \sum_{i=1}^n m_i \ddot{x}_i \quad \frac{\partial T}{\partial x_i} = 0 \quad \frac{\delta P}{\delta x_i} = \sum_{i=1}^n F_i$$

It can be shown by use of the reciprocity theorem* that

$$K_{ij} = K_{ji} \quad C_{ij} = C_{ji}$$

Using the above relationships,

$$\frac{\partial U}{\partial x_i} = \sum_{j=1}^n K_{ij} x_j^{**}$$

* for proof, see Appendix A.

** The notation is explained as follows:

Let $i = j = 2$

$$\begin{aligned} U &= \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 K_{ij} x_i x_j \\ &= \frac{1}{2} K_{11} x_1^2 + \frac{1}{2} K_{12} x_1 x_2 + \frac{1}{2} K_{21} x_2 x_1 + \frac{1}{2} K_{22} x_2^2 \end{aligned}$$

Since $K_{12} = K_{21}$

$$U = \frac{1}{2} K_{11} x_1^2 + K_{12} x_1 x_2 + \frac{1}{2} K_{22} x_2^2$$

$$\frac{\partial U}{\partial x_1} = K_{11} x_1 + K_{12} x_2, \quad \frac{\partial U}{\partial x_2} = K_{22} x_2 + K_{12} x_1$$

$$\begin{aligned} \frac{\partial U}{\partial x_i} &= K_{11} x_1 + K_{22} x_2 + K_{12} x_1 + K_{12} x_2 \\ &= \sum_{j=1}^2 K_{ij} x_j \quad \text{for } i = 1, 2 \end{aligned}$$

Similarly $\frac{\partial V}{\partial \dot{x}_i} = \sum_{j=1}^n C_{ij} \dot{x}_j$ for $i = 1, 2, \dots, n$.

$$\frac{\partial V}{\partial \dot{x}_i} = \sum_{j=1}^n C_{ij} \dot{x}_j$$

Substituting the above expressions in equation (1), the equations of motion for the system may be written

$$m_i \ddot{x}_i + \sum_{j=1}^n K_{ij} x_j + \sum_{j=1}^n C_{ij} \dot{x}_j = F_i^*, \quad i = 1, 2, \dots, n \quad (5)$$

The above equation can be represented in matrix form as

$$[M] \{\ddot{x}\} + [C] \{\dot{x}\} + [K] \{x\} = \{F\} \quad (6)$$

where

$$\text{Mass Matrix } [M] = \begin{bmatrix} m_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & M_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & M_n \end{bmatrix} = [M_i]$$

$$\text{Spring Matrix } [K] = \begin{bmatrix} K_{11} & K_{12} & \cdot & \cdot & \cdot & \cdot & K_{1n} \\ K_{21} & K_{22} & \cdot & \cdot & \cdot & \cdot & K_{2n} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ K_{n1} & \cdot & \cdot & \cdot & \cdot & \cdot & K_{nn} \end{bmatrix} = [K_{ij}]$$

* Here F_i is sinusoidal in nature. For discussion of solution with other forms of F_i forcing function, refer to section (IX).

Damping Matrix

$$[c] = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ c_{n1} & \cdot & \cdot & \cdot & \cdot & c_{nn} \end{bmatrix} = [c_{ij}]$$

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{Bmatrix} \quad \text{a column vector of order } Nx1$$

and

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ F_n \end{Bmatrix} \quad \text{a column vector of order } Nx1$$

For a physically realizable system, $[M]$, $[K]$, and $[C]$ are symmetric matrices. This follows from the fact that the system obeys the reciprocity theorem. Also $[M]$ is a diagonal matrix if the motion of each mass is described by a different absolute coordinate.

A review of the properties of matrices useful for vibrational analysis is given in Appendix (B).

Undamped Systems:

For undamped systems, equation (6) reduces to

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\} \quad (7)$$

To solve this set of equations by classical methods it is necessary to first solve the homogeneous equation

$$[M]\{\ddot{x}\} + [K]\{x\} = 0 \quad (8)$$

This equation is also known as the equation of free vibrations of the undamped system.

The equation (8) can be solved by assuming a solution of the form

$$\{x\} = \{q\} e^{i\omega t} \quad (9)$$

where $\{q\}$ is a column vector of order $N \times 1$, the elements of which are independent of time. On substituting equation (9) into equation (8),

$$(-\omega^2 [M] + [K])\{q\} e^{i\omega t} = 0$$

or

$$(-\omega^2 [M] + [K])\{q\} = 0 \quad (10)$$

For non-trivial solutions⁽¹⁷⁾ of equation (10),

$$\|[-\omega^2 [M] + [K] \| = 0 \quad (11)$$

Equation (11), known as the characteristic equation, is a polynomial of degree n in ω^2 when the above determinant of order n is expanded. Since both $[M]$ and $[K]$ are symmetric and positive definite, the roots of this equation are all real and positive⁽¹⁵⁾. Neglecting for the present the case of repeated roots, there exist n distinct values of ω^2 which satisfy equation (11). For each distinct ω_1^2 there exists a

vector $\{q^i\}$ which satisfies the following equation:

$$\left[-w_i^2[M] + [K]\right] \{q^i\} = 0$$

The vectors $\{q^i\}$ form a linearly independent set⁽¹⁷⁾.

Since $[M]$ and $[K]$ are symmetric and positive definite, there exists a transformation⁽¹⁸⁾ $[Q]$ such that

$$\begin{aligned} [Q]^T [M] [Q] &= [\bar{M}] \quad \text{a diagonal matrix} \\ [Q]^T [K] [Q] &= [\bar{K}] \quad \text{a diagonal matrix} \end{aligned}$$

where Q is known as the Modal Matrix, the columns of which are the eigenvectors of the system.

The particular solution is obtained by letting

$$\{x\} = [Q]\{n(t)\}. \quad (12)$$

On substituting equation (12) into equation (7)

$$[M][Q]\{\ddot{n}(t)\} + [K][Q]\{n(t)\} = \{F(t)\} \quad (13)$$

Premultiplying equation (13) by $[Q]^T$,

$$[Q]^T [M][Q]\{\ddot{n}(t)\} + [Q]^T [K][Q]\{n(t)\} = [Q]^T \{F(t)\}$$

or

$$[\bar{M}]\{\ddot{n}(t)\} + [\bar{K}]\{n(t)\} = \{G(t)\} \quad (14)$$

where

$$[Q]^T \{F(t)\} = \{G(t)\}$$

and

$$\{G(t)\} = \left\{ \begin{array}{c} g_1(t) \\ g_2(t) \\ \cdot \\ \cdot \\ \cdot \\ g_n(t) \end{array} \right\}$$

Equation (14) is a system of uncoupled equations of type

$$\bar{M}_{ii} \ddot{\eta}_i(t) + \bar{K}_{ii} \eta_i(t) = \{g_i(t)\}, \quad (15)$$

where $\frac{\bar{K}_{ii}}{\bar{M}_{ii}} = \omega_i^2$. The complete solution of equations (15) is

$$\eta_i = A \cos \omega_i t + B \sin \omega_i t + \frac{P_i}{K_{ii} - M_{ii} \omega^2} \cos \omega t$$

where A and B are constants of integration to be determined by the initial conditions and g_i is of the form $P_i \cos \omega t$. Having found η_i , x_i is obtained from equation (12). Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{bmatrix} q^1 & q^2 & \cdot & \cdot & q^n \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \eta_n \end{pmatrix}$$

where the column vectors q^1, q^2, \dots, q^n are the eigenvectors of the system.

From the above analysis it is evident that any undamped forced system can be solved by the Normal Mode Technique provided the roots of the frequency equation are distinct. This method has been presented in great detail in reference (2) and the author has solved several representative problems.

For the case of repeated roots, see Appendix (B) where the possibility of a solution with repeated roots is discussed.

Damped Systems:

For the purpose of analyzing damped systems, we shall divide such systems into two categories:

(a) Classically Damped Systems are those systems in which the matrix $[C]$ is also diagonalized by the same transformation which uncouples the corresponding undamped system.

(b) Non-classically Damped Systems are those where $[C]$ cannot be diagonalized by the transformation which uncouples the corresponding undamped system.

Classically Damped Systems:

For a linear damped system, the equations of motion are

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \quad (16)$$

where $[C]$ is symmetric and non-negative definite.

As shown in Appendix B, there exists a transformation $[Q]$ such that

$$[Q]^T [M][Q] = [\bar{M}] \quad \text{is a diagonal matrix}$$

$$[Q]^T [K][Q] = [\bar{K}] \quad \text{is a diagonal matrix}$$

Now if $[C]$ is such that

$$[Q]^T [C][Q] = [\bar{C}] \quad \text{is a diagonal matrix}$$

then it is possible to completely uncouple the above equations of motion.

Let $\{x\} = [Q]\{n(t)\} \quad (17)$

Substituting equation (17) into equation (16)

$$[M][Q]\{\ddot{n}(t)\} + [C][Q]\{\dot{n}(t)\} + [K][Q]\{n(t)\} = \{F(t)\}$$

Premultiplying the above equation by $[Q]^T$,

$$[Q]^T[\bar{M}][Q]\{\ddot{n}(t)\} + [Q]^T[C][Q]\{\dot{n}(t)\} + [Q]^T[K][Q]\{n(t)\} = [Q]^T\{F(t)\}$$

$$[\bar{M}]\{\ddot{n}(t)\} + [\bar{C}]\{\dot{n}(t)\} + [\bar{K}]\{n(t)\} = \{g_i(t)\}$$

This is a set of uncoupled equations of type

$$\bar{M}_{ii}\ddot{n}_i + \bar{C}_{ii}\dot{n}_i + \bar{K}_{ii}n_i = g_i(t)$$

The complete solution is of the form

$$n_i(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{P_i \cos(\omega t - \beta)}{\sqrt{(K_{ii} - M_i \omega^2)^2 + (C_{ii} \omega)^2}}$$

where $g_i = P_i \cos \omega t$

C_1 and C_2 are constants of integration to be determined by initial conditions.

Thus, in a classically damped system, it is always possible to obtain a solution of type

$$\{x\} = [Q]\{n\}$$

where $[Q]$ simultaneously diagonalizes $[M]$, $[K]$, and $[C]$.

The above analysis is possible only if $[C]$ is such that the transformation which uncouples the undamped system will also uncouple the damped system. Dr. T. K. Caughey⁽¹²⁾ derived sufficient conditions for $[C]$ such that the equations can be uncoupled as above. However, the necessary conditions for uncoupling the damping matrix have not yet been developed.

Caughey shows that a sufficient though not necessary condition that a system possesses classical normal modes is that

$$[N]^{-1}[C][N]^{-1} = \sum_{n=0}^{\infty} \sum_{\ell=1}^{n-1} a_{n\ell} \left[[N]^{-1}[K][N]^{-1} \right]^{\frac{\ell}{n}}$$

The above expression is arrived at by making use of the transformation $\{X\} = [N]\{L\}$ and reducing the equations of motion to the following form

$$[I]\{\ddot{L}\} + [A]\{\dot{L}\} + [B]\{L\} = 0$$

where

$$[A] = [N]^{-1}[C][N]$$

and

$$[B] = [N]^{-1}[K][N]$$

It is then proved that if diagonal matrices $[a]$ and $[b]$, obtained from $[A]$ and $[B]$ respectively by some other transformation, can be expressed as $[a] = [b]^{\ell}$ where ℓ is an integer then $[A] = [B]^{\ell}$. Thus, if $[A] = [B]^{\ell}$, it can be concluded that $[a] = [b]^{\ell}$. By a more extensive analysis on similar lines, Caughey's sufficient condition can be derived. Caughey's condition is equivalent to stating that if $[C]$ can be expressed as a linear combination of $[M]$ and $[K]$, then the system is classically damped.

Non-classically Damped Systems:

If the system possesses non-classical damping, the methods of solution presented above are not applicable. K. A. Foss⁽¹¹⁾ has developed a method for solving some of these systems. In this method the original system is transformed in $2N$ space in which the equations of motion of the system can be uncoupled.

Method of K. A. Foss

We define a pair of $2N \times 1$ column vectors

$$\{Z\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} \{0\} \\ \{F(t)\} \end{Bmatrix}$$

and the following set of matrices of order $2N \times 2N$

$$[R] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}$$

$$[P] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}$$

where $\{x\}$, $\{\dot{x}\}$, $\{0\}$ and $F(t)$ are column vectors of order $N \times 1$ associated with the equations of motion. With the above definitions the equations of motion of the original system can now be reduced to

$$[R]\{\dot{Z}\} + [P]\{Z\} = \{F(t)\} \quad (18)$$

The above equation, after performing the indicated matrix operations, results in two matrix equations, one of which is the original equation of motion and other is an identity. Thus

$$\begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{Bmatrix} + \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{f(t)\} \end{Bmatrix}$$

or $[M]\{\ddot{x}\} - [M]\{\dot{x}\} = \{0\}$

and $[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\}$

To obtain a solution of equation (18) we first solve the homogeneous equation

$$[R]\{\dot{Z}\} + [P]\{Z\} = \{0\} \quad (19)$$

Assume
$$\{Z\} = \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} = e^{\alpha t} \begin{Bmatrix} \Phi \\ \phi \end{Bmatrix} = e^{\alpha t} \begin{Bmatrix} \alpha \phi \\ \phi \end{Bmatrix} \quad (20)$$

on substituting equation (20) in equation (19) and rearranging

$$[\alpha[R] + [P]]\{\Phi\} = \{0\} \quad (21)$$

Premultiplying by $[P]^{-1}$ and dividing through by α , the above equation becomes

$$[[P]^{-1}[R] + \frac{1}{\alpha} [I]]\{\Phi\} = \{0\} \quad (22)$$

It is easily shown that

$$[P]^{-1} = \begin{bmatrix} -[M]^{-1} & [0] \\ [0] & [K]^{-1} \end{bmatrix}$$

and therefore it will always exist. Thus,

$$[P]^{-1}[R] = \begin{bmatrix} [0] & -[I] \\ [K]^{-1}[M] & [K]^{-1}[C] \end{bmatrix} = [U]$$

Equation (22) now becomes

$$[[u] + \frac{1}{\alpha} [I]]\{\Phi\} = \{0\} \quad (23)$$

Equation (23) is the usual form of an eigenvalue problem. $[u]$ will be symmetric only if

$$[K]^{-1}[M] = -[I] \text{ and}$$

$$[K]^{-1}[C] \text{ is a symmetric matrix}$$

If $[u]$ is symmetric, then $2N$ independent eigenvectors will always exist (as shown in Appendix B) and Foss's method will give a solution. However, in general $[u]$ is unsymmetric and $2N$ linearly independent eigenvectors will always exist only if there are $2N$ distinct roots of the frequency equation in $2N$ space

$$\|[[u] + \frac{1}{\alpha} [I]]\| = 0 \quad (24)$$

For each distinct root $\frac{1}{\alpha_1}$ of the above equation there exists an independent

vector $\{\Phi^i\}$. As shown in Appendix B, a root of multiplicity K may or may not have K associated linearly independent eigenvectors. The eigenvalues of the system are obtained by solving equation (24). Equation (23) then gives the eigenvectors $\{\Phi^i\}$ in $2N$ space. Having found $\{\Phi^i\}$, $\{x_i\}$ can be obtained from equation (20).

Forced Vibrations

Nonhomogeneous solution of the equations of motion in $2N$ space is obtained by letting

$$\{Z\} = [\Omega]\{\xi\}$$

where $[\Omega]$ is a matrix of order $2N \times 2N$, the columns of which are $\{\Phi\}$ and $\{\xi\}$ is a column vector of order $2N \times 1$.

Using the orthogonality condition in $2N$ space, equation (18) reduces to

$$R_i \dot{\xi}_i - \alpha_i R_i \xi_i = \left\{ \begin{matrix} \alpha_i \phi_i \\ \phi_i \end{matrix} \right\}^T \{F(t)\} \quad (25)$$

where R_i is a scalar and is defined as

$$R_i = \left\{ \begin{matrix} \alpha_i \phi_i \\ \phi_i \end{matrix} \right\}^T [R] \left\{ \begin{matrix} \alpha_i \phi_i \\ \phi_i \end{matrix} \right\}$$

The solution of equation (25) is

$$\xi_i = \frac{1}{R_i} \int_0^t e^{\alpha_i(t-\tau)} \left\{ \begin{matrix} \alpha_i \phi_i \\ \phi_i \end{matrix} \right\}^T \{F(\tau)\} d\tau + A_i e^{\alpha_i t}$$

where A_i is an arbitrary constant depending upon initial conditions.

Having found ξ_i , $\{Z\}$ is obtained from $\{Z\} = [\Omega]\{\xi\}$.

Since $\{Z\} = \left\{ \begin{matrix} \{\dot{x}\} \\ \{x\} \end{matrix} \right\}$, $\{x\}$ can be obtained by expanding $\{Z\}$.

The detailed derivations for forced vibrations are given in references (11) and (13).

Limitations of Foss's Method: For the application of Foss's method, it is assumed that

$$\{Z\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} = e^{\alpha t} \{\bar{\phi}\} = e^{\alpha t} \begin{Bmatrix} \alpha\phi \\ \phi \end{Bmatrix} \quad (26)$$

Since $\{x\} = e^{\alpha t} \{\phi\}$

$$\{\dot{x}\} = e^{\alpha t} \{\alpha\phi\}, \text{ and} \quad (27)$$

$$\{\dot{Z}\} = \alpha e^{\alpha t} \begin{Bmatrix} \alpha\phi \\ \phi \end{Bmatrix} \quad (28)$$

It is shown below that for critically damped systems, equation (27) does not apply.

Consider a critically damped system such that the i th uncoupled equation

$$\bar{M}_{ii} \ddot{q}_i + \bar{C}_{ii} \dot{q}_i + \bar{K}_{ii} q_i = 0$$

has the solution of type

$$(A_i + B_i t) e^{\alpha_i t}$$

where

$$\alpha_i = \frac{-\bar{C}_{ii}}{2\bar{M}_{ii}}, \text{ and}$$

A_i and B_i are constants.

Here

$$\{x\} = (A_i + B_i t) e^{\alpha_i t} \{\phi_i\}$$

and

$$\{\dot{x}\} = (A_i + B_i t) e^{\alpha_i t} \{\alpha_i \phi_i\} + B_i e^{\alpha_i t} \{\phi_i\}$$

therefore in this case,

$$\{Z\} = (A_i + B_i t)e^{\alpha_i t} \begin{Bmatrix} \{\alpha_i \phi_i\} \\ \{\phi_i\} \end{Bmatrix} + B_i e^{\alpha_i t} \begin{Bmatrix} \{\phi_i\} \\ \{0\} \end{Bmatrix} \quad (29)$$

The time derivative of the above equation cannot be expressed in the form of equation (28) and therefore Foss's method does not give a solution for critically damped systems. Summarizing, Foss's method does not work for the case

- (1) where eigenvalues are repeated and the eigenvectors do not form a linearly independent set,
- (2) when the system is critically damped at one or more of its modes of free vibration.

V. HOLZER'S METHOD

The Holzer method is a tabular method for the analysis of multi-mass lumped-parameter systems. It is applicable for the study of free and forced vibrations, systems with or without damping, systems with any boundary conditions, and systems with angular or rectilinear motion.

The Holzer method is a trial-and-error method. It can be used to find natural frequencies, and each frequency can be determined independently of the others. In addition to the natural frequencies, this method also gives the amplitude ratios of the masses and the nodes in a system at its principal modes of vibration.

Undamped System:

For an undamped system, it can be assumed that no external torque is required to maintain a conservative system vibrating at its principal modes. Consider the system shown in figure 2. The equations of motion are

$$J_1 \ddot{\theta}_1 + K_{12}(\theta_1 - \theta_2) = 0 \quad (30)$$

$$J_2 \ddot{\theta}_2 - K_{12}(\theta_1 - \theta_2) + K_{23}(\theta_2 - \theta_3) = 0 \quad (31)$$

$$J_3 \ddot{\theta}_3 - K_{23}(\theta_2 - \theta_3) = 0 \quad (32)$$

Summing the three equations gives

$$J_1 \ddot{\theta}_1 + J_2 \ddot{\theta}_2 + J_3 \ddot{\theta}_3 = 0 \quad (33)$$

Hence the sum of the inertia torques must equal zero at all times.

The motions of all disks are simple harmonic. Letting

$$\theta_1 = A \cos \omega t$$

$$\theta_2 = B \cos \omega t$$

$$\theta_3 = C \cos \omega t$$

and substituting for θ_1 and θ_2 in equation (30) above,

$$-J_1 A \omega^2 + K_{12}(A - B) = 0$$

or

$$B = A \left(1 - \frac{J_1 \omega^2}{K_{12}} \right)$$

If A is taken as unity, B can be calculated.

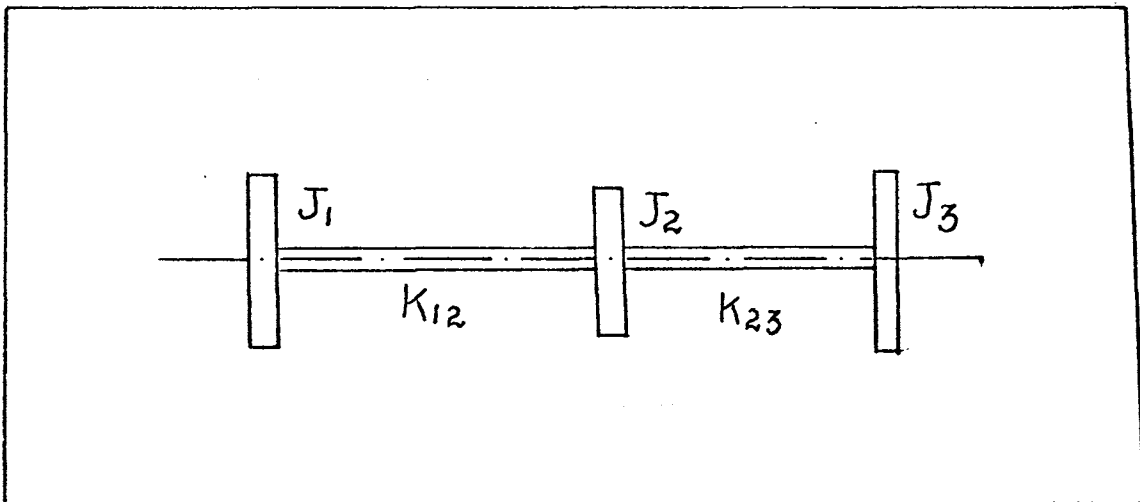


Figure 2

A Three-Disk Torsional System

Adding equations (30) and (31),

$$(-J_2 B \omega^2 - J_1 A \omega^2) + K_{23}(B - C) = 0$$

or

$$C = B - \frac{J_2 B \omega^2 + J_1 A \omega^2}{K_{23}}$$

Knowing B, C can be calculated. Generalizing the above procedure,

$$\theta_n = \theta_{n-1} - \frac{\sum_{i=1}^{i=n-1} J_i \theta_i \omega^2}{K_{n,n-1}} \quad (34)$$

Equations (33) and (34) provide the basis for the Holzer table which is presented below:

Trial Frequency ω							
Station Number	Mass Moment of Inertia J	$J\omega^2$	Amplitude θ	Inertia Torque $J\omega^2\theta$	$\sum J\omega^2\theta$	Spring constant $K_{i,i-1}$	$\frac{\sum J\omega^2\theta}{K_{i,i-1}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)

Holzer Table

A similar table can be constructed for rectilinear motion if J is replaced by M and θ by X.

The physical meaning of the various columns in this table is as follows:

Column 2: is the inertia torque of each disk for an amplitude of 1 radian at the trial frequency selected;

Column 3: is the angular amplitude θ of each disk;

Column 4: is the inertia torque of each disk at the amplitude θ ;

Column 5: gives the value of the shaft torque beyond the disk in question;

Column 6: shows the torsional spring constants;

Column 7: gives the relative angle of twist between the disk in question and the next disk.

The computations of the natural frequencies of an undamped system by Holzer's Method are straight forward. Assuming a trial value for a natural frequency, the process is carried out by letting the angular displacement of the first disk arbitrarily be one radian. The angular displacement of each disk of the system is found by repeatedly applying equation (34) in a sequential fashion. If the algebraic sum of the inertia torques is zero, that is, if equation (33) is satisfied, then the assumed frequency is a natural frequency. If equation (33) is not satisfied, a new value of w is assumed and the process is repeated. The magnitude and sign of remainder torque are a measure of how far the trial frequency is removed from a natural frequency.

Selection of Trial Frequencies:

The difficulties often encountered with the Holzer's Method are (1) in estimating the initial trial value for w ; and (2) in selecting a second trial value for w if the initial trial value does not satisfy equation (33).

The initial value of the trial frequency is usually obtained by reducing the multidisk system to an approximate two or three disk system. The approximate frequency to be assumed may then be found by using frequency equations presented in Normal Mode Technique. A positive remainder torque for the first natural frequency indicates that the system has surplus inertia torque. Therefore to balance the system, a higher trial value is selected. The reverse is true for a negative remainder torque. Consequently the trial value is decreased. For torsional vibrations, the mode of the vibration is usually taken to be the same as the number of nodes^{*}, which means that the first natural frequency has one node, the second two, and so on. Each time a node is passed, the sign in the amplitude column changes. The number of sign changes in the amplitude column indicates the number of natural frequencies which lie below the trial value. If there are many sign changes in the amplitude column for the first trial, a much lower value of w should be selected for the next trial.

To obtain the higher natural frequencies, it should be noted that the inertia torque on the last disk changes sign as the frequency is changed from below to above a natural frequency. A plot of remainder torque as a function of trial frequency should be made to aid in the estimation of trial frequencies (see sample problem). The points where the curve intersects the frequency-axis should be used as trial eigenvalues.

* A node is a point along the shaft with a zero deflection.

To meet all possible boundary conditions, an appropriate Holzer table has to be constructed. For example, if the left-hand end is built-in, then θ_1 is required to be zero and an arbitrary value for inertia torque on the first disk is selected. If the left-hand is free then inertia torque is taken zero and θ_1 has some arbitrary value. Similarly if right-hand is a free end, the torque at this end is zero and if it is built-in, then the amplitude has to be zero.

Once the eigenvalues have been found, the amplitude column of the Holzer table directly gives the eigenvectors for each of these eigenvalues.

Forced Vibrations:

If a torque $T_0 \cos \omega t$ is applied to one of the disks, forced vibrations will be produced. Consider the system of figure 2 except that in this case a pulsating torque $T_0 \cos \omega t$ is acting on the first disk. It is required to obtain the angular displacements of all disks which result due to applied torque of frequency ω . The equations of motion are

$$J_1 \ddot{\theta}_1 + K_{12}(\theta_1 - \theta_2) = T_0 \cos \omega t \quad (35)$$

$$J_2 \ddot{\theta}_2 - K_{12}(\theta_1 - \theta_2) + K_{23}(\theta_2 - \theta_3) = 0 \quad (36)$$

$$J_3 \ddot{\theta}_3 - K_{23}(\theta_2 - \theta_3) = 0 \quad (37)$$

As before, let

$$\begin{aligned} \theta_1 &= A \cos \omega t \\ \theta_2 &= B \cos \omega t \\ \theta_3 &= C \cos \omega t \end{aligned} \quad (38)$$

Substituting for θ_1 , θ_2 , and θ_3 in above equations of motion and adding the resulting equations,

$$w^2(J_1A + J_2B + J_3C) = -T_0 \quad (39)$$

From equations (35) and (38),

$$B = A - \frac{J_1Aw^2}{K_{12}} - \frac{T_0}{K_{12}}$$

The Holzer table is completed as before and the remainder torque is equated to $-T_0$ as indicated by equation (39). This gives A in terms of T_0 . Knowing A, B and C can be calculated.

In the above example, the shaft considered is free-free and the remainder torque is equated to $-T_0$. However, this may not always be true. For example, if the last mass is attached to a rigid support, then the remainder amplitude is equated to zero. Similarly other boundary conditions have to be taken care of while solving forced vibrations problems by this method.

Branched Systems:

For use of this method in studying branched system behavior, separate tables have to be made for each branch of the system. The application of Holzer's Method to the branched systems can be best illustrated by the following example: Consider the branched system shown in figure 3. A Holzer table for branch 1 is made. Branch 1 is arbitrarily taken as the primary branch with a unit angular deflection at the left-hand end. Proceed through branch 1 as previously outlined to obtain the deflection

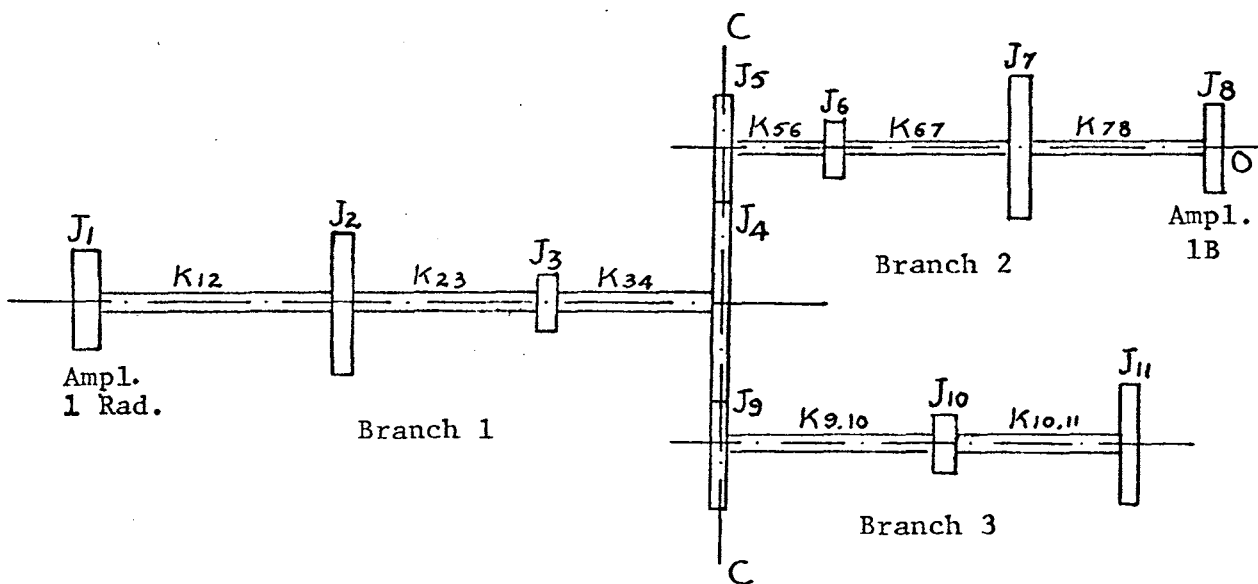


Figure 3

A Geared Torsional System

at section C-C. The calculation for branch 2 is started from the opposite end at point O. Assuming an arbitrary deflection of magnitude B at this end where B is a constant, proceed towards section C-C. Since the deflection at section C-C is already known, the two deflections are equated to solve for B. The inertia torques of branch 1 and branch 2 are added. Calculations are then carried for branch 3 starting from section C-C with inertia torque equal to the sum of inertia torques of branch 1 and branch 2. If the trial frequency is a natural frequency of the entire system, it will meet the boundary conditions at the end of branch 3. If not, then the whole procedure is repeated for a new trial frequency.

Use of this method requires that all branches of the system must have a one-to-one gear ratio. This is accomplished by multiplying the stiffness and mass moment of inertia of each geared shaft by n^{2*} , where n is the speed ratio of the geared shaft to the reference shaft. This

* If the oscillatory amplitude of reference shaft is θ_1 , the oscillatory amplitude of geared shaft will be $\theta_2 = n\theta_1$. The oscillatory kinetic energy of disk 2 is then

$$T = \frac{1}{2} J_2 \dot{\theta}_2^2 = \frac{1}{2} n^2 J_2 \dot{\theta}_1^2$$

and the equivalent inertia of disk 2 referred to shaft 1 is $n^2 J_2$.

To determine the equivalent stiffness of geared shaft, note that if θ_1 is the angle of twist of reference shaft, then the geared shaft turns through an angle of $n\theta_1$. The potential energy of the geared shaft

is illustrated in figure 4 which is equivalent of figure 3 with gears reduced to common speeds.

(continued)

corresponding to twist of n_1 is

$$U = \frac{1}{2} K_2 (n\theta_1)^2 = \frac{1}{2} (n^2 K_2) \theta_1^2$$

The geared shaft referred to shaft 1 must therefore have a stiffness of $n^2 K_2$.

The energy dissipated due to viscous damping in the geared shaft is

$$V = \frac{1}{2} C_2 \dot{\theta}_2^2$$

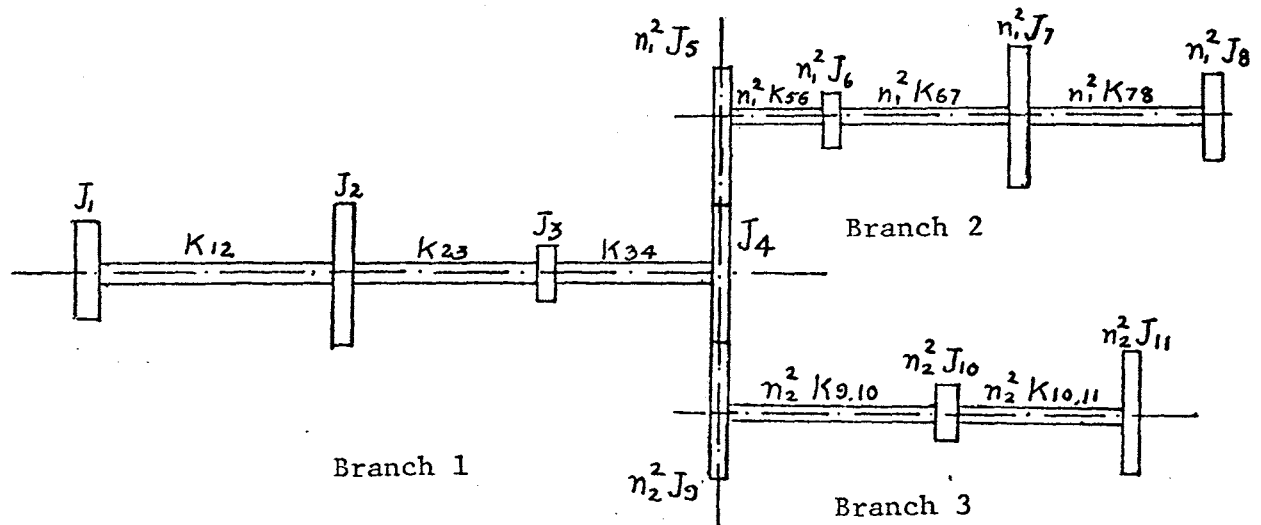
Since

$$\theta_2 = n\theta_1$$

$$V = \frac{1}{2} (C_2 n^2) \dot{\theta}_1^2$$

The damping coefficient in the geared shaft referred to shaft 1 is therefore multiplied by n^2 .

Though the damping has not yet been introduced in the system, for convenience, it has been discussed here.



$$n_1 = \frac{\text{Speed of branch 2}}{\text{Speed of branch 1}}$$

$$n_2 = \frac{\text{Speed of branch 3}}{\text{Speed of branch 1}}$$

Figure 4

Equivalent System With One-to-One Gearing

Damped Systems:

In an undamped system all disks move either in phase or out of phase by 180 degrees with the disturbing torque and no energy is dissipated. If damping is introduced into the system, the motion of each disk will in general be out of phase by an angle other than 180 degrees. Therefore the computational work is done in a mathematically complex plane and computations are subject to the rules governing complex numbers.

The basic Holzer table equations for a general case may be derived with the aid of figure 5. This includes viscous damping between the disks as well as between the disks and ground. It is necessary to distinguish the viscous damping between the disks, which is designated as C_{ij} , from the viscous damping between the disks and the fixed member, which is designated as h_i .

Considering each disk as a free body, the following equations of motion are obtained:

$$J_1 \ddot{\theta}_1 + h_1 \dot{\theta}_1 + K_{12}(\theta_1 - \theta_2) + C_{12}(\dot{\theta}_1 - \dot{\theta}_2) = 0 \quad (40)$$

$$J_2 \ddot{\theta}_2 + h_2 \dot{\theta}_2 - K_{12}(\theta_1 - \theta_2) - C_{12}(\dot{\theta}_1 - \dot{\theta}_2) + K_{23}(\theta_2 - \theta_3) + C_{23}(\dot{\theta}_2 - \dot{\theta}_3) = 0 \quad (41)$$

$$J_3 \ddot{\theta}_3 + h_3 \dot{\theta}_3 - K_{23}(\theta_2 - \theta_3) - C_{23}(\dot{\theta}_2 - \dot{\theta}_3) = 0 \quad (42)$$

Since damping is present, complex numbers are used. Let

$$\theta_i = A_i e^{j\omega t}$$

where $j = \sqrt{-1}$

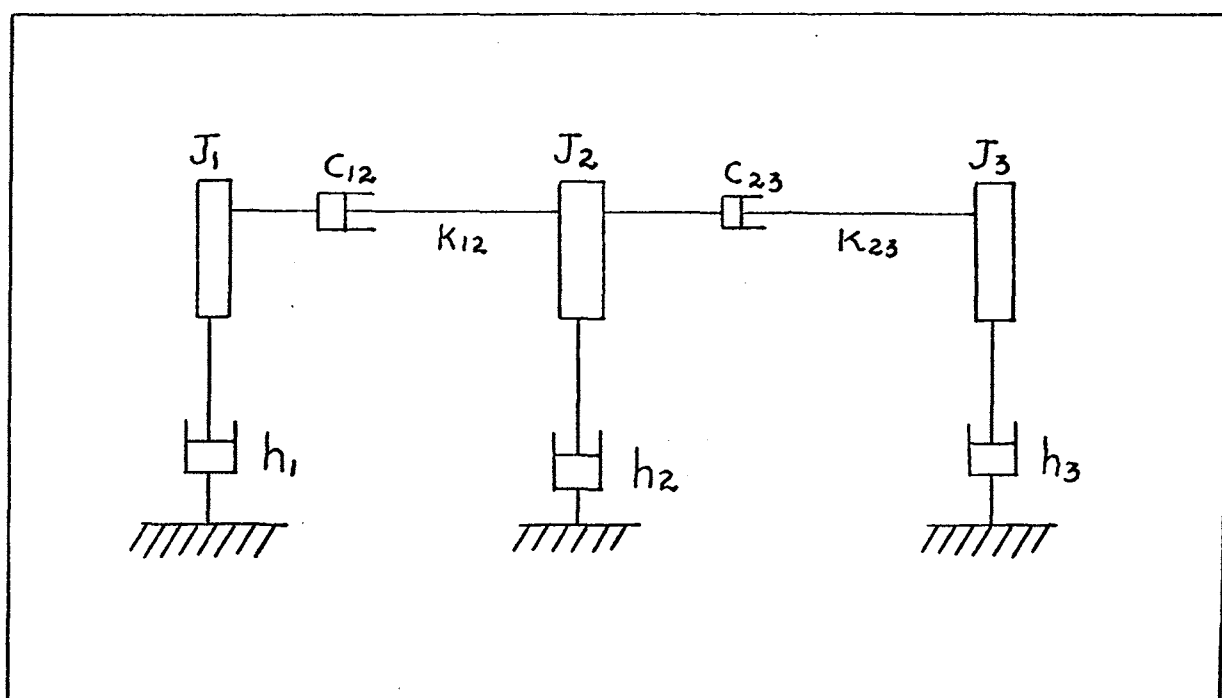


Figure 5

A Damped Torsional Multidisk System

Substituting for θ_i and its derivatives into equations (40), (41), and (42) gives the following simultaneous equations:

$$-J_1 \omega^2 A_1 + j\omega h_1 A_1 - (K_{12} + j\omega C_{12})(A_2 - A_1) = 0 \quad (43)$$

$$\begin{aligned} -J_2 \omega^2 A_2 + j\omega h_2 A_2 + (K_{12} + j\omega C_{12})(A_2 - A_1) \\ - (K_{23} + j\omega C_{23})(A_3 - A_2) = 0 \end{aligned} \quad (44)$$

$$-J_3 \omega^2 A_3 + j\omega h_3 A_3 + (K_{23} + j\omega C_{23})(A_3 - A_2) = 0 \quad (45)$$

Equation (43) may be written in the form

$$A_2 - A_1 = \frac{-J_1 \omega^2 A_1 + j\omega h_1 A_1}{K_{12} + j\omega C_{12}}$$

or

$$A_2 = A_1 - \frac{(J_1 \omega^2 - j\omega h_1) A_1}{K_{12} + j\omega C_{12}} \quad (46)$$

Now if an amplitude of A_1 for disk 1 is assumed, equation (46) may be solved for A_2 .

Similarly from equation (44),

$$A_3 = A_2 - \frac{A_2 (J_2 \omega^2 - j\omega h_2) - (K_{12} + j\omega C_{12})(A_2 - A_1)}{K_{23} + j\omega C_{23}} \quad (47)$$

But from equation (43),

$$(K_{12} + j\omega C_{12})(A_2 - A_1) = -J_1 \omega^2 A_1 + j\omega h_1 A_1$$

Substituting this expression in equation (47) gives

$$A_3 = A_2 - \frac{A_2 (J_2 \omega^2 - j\omega h_2) + (J_1 \omega^2 - j\omega h_1) A_1}{K_{23} + j\omega C_{23}} \quad (48)$$

which may be solved for A_3 by substituting in the calculated value of A_2 from equation (46).

Equation (48) may be written in general form for the motion of disk i ,

$$A_i = A_{i-1} - \frac{\sum_{Z=1}^{Z=i-1} A_Z (J_Z \omega^2 - jwh_Z)}{K_{i,i-1} + jwC_{i,i-1}} \quad (49)$$

Equation (49) is a general form of the Holzer equation for damped systems which may be solved in tabular form for an assumed value of A_1 . Equation (49) is similar to the corresponding equation (34) for undamped systems. Comparing the two equations, $J\omega^2$ term of equation (34) is replaced by $(J\omega^2 - jwh)$ in equation (49) and $K_{i,i-1}$ in this case becomes $(K_{i,i-1} + jwC_{i,i-1})$. Accordingly, column 2 and column 6 of Holzer's table presented previously have to be modified for damped systems.

Since vibrational energy is continuously dissipated in the form of heat in a system having damping, any free vibration of such a system is a transient one. The resonant frequency for a damped system has to be defined in a different manner because the residual torque never passes through zero⁽⁸⁾. However, the remainder torque when plotted against various trial frequencies has a minimum where the real part of the torque passes through zero. This frequency can be designated as the damped natural frequency⁽²²⁾. Whereas an undamped system vibrates without any external torque at its natural frequencies, a damped system requires an input to sustain such vibrations. However, this input should be minimum when the system passes through its natural frequencies.

This explains why the minimum point corresponds to one of the natural frequencies of the damped system.

It should be mentioned that the above system had free-free boundary conditions and therefore the remainder torque gave an indication of natural frequencies. For other boundary conditions such as fixed-fixed, the remainder amplitude has to be used in estimating a natural frequency.

The applications of Holzer's Method to damped systems with external input and/or with branches is similar to that of undamped systems.

VI. IMPEDANCE METHOD

Use of the analogy between linear passive electrical networks and a mechanical system can simplify the study of a complicated mechanical system with either lumped or distributed parameters. The impedance method⁽⁵⁾ makes use of this fact and describes the steady state system behavior.

Consider a mechanical system shown in figure 6. The elastic system can be any combination of lumped, linear, mechanical elements such as masses, springs, and dampers. It can also be combinations of linear, distributed parameter systems, such as beams, plates. etc.

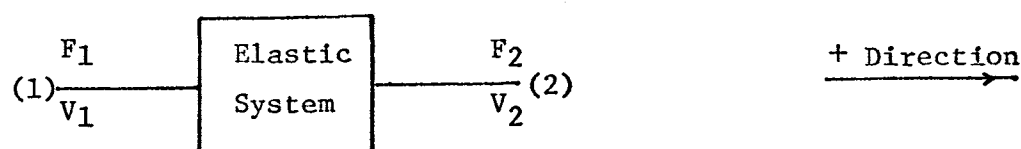


Figure 6

Four-pole Parameter Notation

The elastic system must have two identifiable connection points (1) and (2) which are called the input and output points respectively. At the input point there exists an input force F_1 and a velocity V_1 . The input force and velocity are produced by connection of point (1) to that portion of the complete mechanical system which precedes it. At the output point (2) there exists a force F_2 and a velocity V_2 which result from the application of F_1 .

and V_1 at point (1) and the reaction of the portion of the passive mechanical system between point (1) and point (2).

The performance of any component of the system can be described by the following equations:

$$F_1 = A_{11} F_2 + A_{12} V_2 \quad (50)$$

$$V_1 = A_{21} F_2 + A_{22} V_2 \quad (51)$$

The four coefficients A_{11} , A_{12} , A_{21} , and A_{22} in the above equations are called four-pole parameters. The four-pole parameters for a mass, spring, and a viscous damper are obtained below.

Four-pole Parameter for a Mass: Since the mass is regarded as a rigid body,

$$V_1 = V_2 \quad (52)$$

and by Newton's law,

$$F_1 - F_2 = m \frac{dV_1}{dt} = m \frac{dV_2}{dt} \quad (53)$$

For a sinusoidal time variation motion, let

$$\begin{aligned} F_1 &= F_A e^{i\omega t} \\ F_2 &= F_B e^{i\omega t} \\ V_1 &= V_A e^{i\omega t} \\ V_2 &= V_B e^{i\omega t} \end{aligned} \quad (54)$$

where F_A , F_B , etc., may be complex functions of mass, stiffness, resistance, and frequency, but are time independent. Note that no phase angles are necessary in the $e^{i\omega t}$ term since the F_A , etc., terms are

allowed to be complex. The relations between velocity, displacement, and acceleration can be expressed as follows:

$$\begin{aligned} V &= V_A e^{i\omega t} \\ X &= \int V_A e^{i\omega t} dt = \frac{V_A}{i\omega} e^{i\omega t} = \frac{V}{i\omega} \\ \ddot{X} &= \frac{d}{dt}(V) = i\omega V_A e^{i\omega t} = i\omega V \end{aligned}$$

Displacements at each input and output point are to be measured from the respective equilibrium positions of these points. The equilibrium position is defined as the position occupied by the point when no sinusoidal excitation is applied to the point.

Equations (52) and (53) become

$$F_1 = F_2 + i\omega m V_2 \quad (55)$$

$$V_1 = 0 \cdot F_2 + V_2 \quad (56)$$

Comparing equations (55) and (56) with equations (50) and (51), the four-pole parameters for a mass are $A_{11} = 1$, $A_{12} = i\omega m$, $A_{21} = 0$, and $A_{22} = 1$.

Four-pole Parameter for a Massless Spring: The force applied at the input point of the spring is the same as the force which the spring delivers at its output point:

$$F_1 = F_2 \quad (57)$$

Also,

$$F_2 = K(x_1 - x_2) = K\left(\frac{V_1}{i\omega} - \frac{V_2}{i\omega}\right) \quad (58)$$

The equations (57) and (58) can be written as

$$F_1 = F_2 + 0 \cdot V_2$$

$$V_1 = \frac{i\omega}{K} F_2 + V_2$$

Hence the four-pole parameters for a spring are $A_{11} = 1$, $A_{12} = 0$, $A_{21} = \frac{i\omega}{K}$, and $A_{22} = 1$.

Four-pole Parameters for a Viscous Damper: For a damper,

$$\text{and } F_1 = F_2 \quad (59)$$

$$F_2 = C(V_1 - V_2) \quad (60)$$

where C is the coefficient of viscous damping. Rearranging equations (59) and (60),

$$F_1 = F_2 + 0 \cdot V_2$$

$$V_1 = \frac{1}{C} F_2 + V_2$$

and the four-pole parameters are $A_{11} = 1$, $A_{12} = 0$, $A_{21} = \frac{1}{C}$, and $A_{22} = 1$.

Connection of Four-pole Networks: There are two ways by which two four-pole networks can be connected to each other;

(A) Tandem Connection

(B) Parallel Connection

(A) Tandem Connection: Two four-pole networks are said to be in tandem connection when the output from the first is precisely the input to the second. The analysis of this type of connection is efficiently handled by matrix techniques. The structure resulting from a tandem connection of elements is simply another four-pole network. Equations (50) and (51) can be written in the matrix form thus

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} \times \begin{bmatrix} F_2 \\ V_2 \end{bmatrix} \quad (61)$$

where the subscripts signify that the parameters belong to the four-pole network numbered (1).

If the output of four-pole network (1) is the input of four-pole network (2), then

$$\begin{bmatrix} F_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix} \times \begin{bmatrix} F_3 \\ V_3 \end{bmatrix} \quad (62)$$

replacing $\begin{bmatrix} F_2 \\ V_2 \end{bmatrix}$ in equation (61) by its value from (62),

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} A_{11}^1 & A_{21}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} \times \begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix} \times \begin{bmatrix} F_3 \\ V_3 \end{bmatrix} \quad (63)$$

generalizing the above process for n networks in tandem,

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} \\ V_{n+1} \end{bmatrix}$$

where $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is obtained by multiplying parameter matrices of all the four-pole networks.

Note that each matrix in equation (63) is associated with a single four-pole network and that any changes made in that network affect only its own matrix. This is a considerable convenience in certain problems.

(B) Parallel Connection: When two four-pole networks are connected so that

- (1) all their input and output junctions move with the same velocity;
 (2) the input force to the composite four-pole network is the sum of the input forces of the individual networks;
 (3) the output force from the composite four-pole network is the sum of the output forces of the individual networks;
- then the networks are said to be connected in parallel. A parallel connection is shown below:

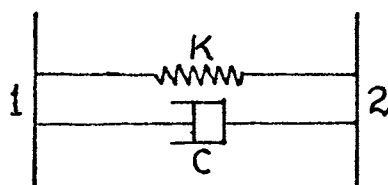


Figure 7

Two Four-pole Elements Connected in Parallel

The four-pole parameters for the composite system of n four-pole networks are given by the following formulas*:

$$A_{11} = \frac{A}{B}, \quad A_{12} = \frac{AC}{B} - B, \quad A_{21} = \frac{1}{B}, \quad \text{and} \quad A_{22} = \frac{C}{B}$$

where

$$A = \sum_{\ell=1}^{\ell=n} \left(\frac{A_{11}^{\ell}}{A_{21}^{\ell}} \right)$$

$$B = \sum_{\ell=1}^{\ell=n} \left(\frac{1}{A_{21}^{\ell}} \right)$$

$$C = \sum_{\ell=1}^{\ell=n} \left(\frac{A_{22}^{\ell}}{A_{21}^{\ell}} \right)$$

With the use of above equations, the four-pole parameters of the combination spring and damper system shown in figure 7 can be found as follows:

* for discussion of proof, refer to Appendix C.

$$A = \left(\frac{A_{11}}{A_{21}}\right)^1 + \left(\frac{A_{11}}{A_{21}}\right)^2$$

$$= \frac{K}{iw} + C$$

$$B = \left(\frac{1}{A_{21}}\right)^1 + \left(\frac{1}{A_{21}}\right)^2$$

$$= \frac{K}{iw} + C$$

$$C = \left(\frac{A_{22}}{A_{21}}\right)^1 + \left(\frac{A_{22}}{A_{21}}\right)^2$$

$$= \frac{K}{iw} + C$$

which gives,

$$A_{11} = \frac{A}{B} = 1$$

$$A_{12} = \frac{AC}{B} - B = 0$$

$$A_{21} = \frac{1}{B} = \frac{1}{\frac{K}{iw} + C}$$

$$A_{22} = \frac{C}{B} = 1$$

Having obtained the four-pole parameters of the composite system between points (1) and (2), the composite system is reduced to a single element having the above four-pole parameters. All parallel connections in the system are reduced to single elements in this fashion. These elements are now used in tandem connections and the system can be simplified by applying equation (63).

VII. GRAPHICAL TECHNIQUE

This technique⁽⁶⁾ for determining the frequency response of damped linear mechanical systems can be used to obtain solutions either by graphical means or by digital computer. The technique involves determining the displacement and force transmissibilities of each element of a multimass system as an individual single-degree-of-freedom system. The equations of motion are then written in terms of these individual transmissibilities and desired response calculated by either graphical or computer methods.

Derivation of Equations: Consider the system shown in figure 8 where masses M_j are connected in series by springs and viscous dampers. The springs and dampers are described by the spring constants K_j and the damping constants C_j . The masses have linear motion only. The harmonic excitation f_u is applied to a particular mass M_u . The equations of motion for this system are

$$M_o \ddot{X}_o = C_1 (\dot{X}_1 - \dot{X}_o) + K_1 (X_1 - X_o) \quad (64)$$

$$M_j \ddot{X}_j = C_{j+1} (\dot{X}_{j+1} - \dot{X}_j) + K_{j+1} (X_{j+1} - X_j) - C_j (\dot{X}_j - \dot{X}_{j-1}) - K_j (X_j - X_{j-1}) + \begin{cases} f_u & @ j = u \\ 0 & @ j \neq u \end{cases} \quad (65)$$

$$M_n \ddot{X}_n = -C_n (\dot{X}_n - \dot{X}_{n-1}) - K_n (X_n - X_{n-1}) \quad (66)$$

Since the excitation is assumed to be harmonic*, $-w^2 X_j$ and iwC_j can be substituted for \ddot{X}_j and \dot{X}_j respectively.

* for other forcing functions, see section IX.

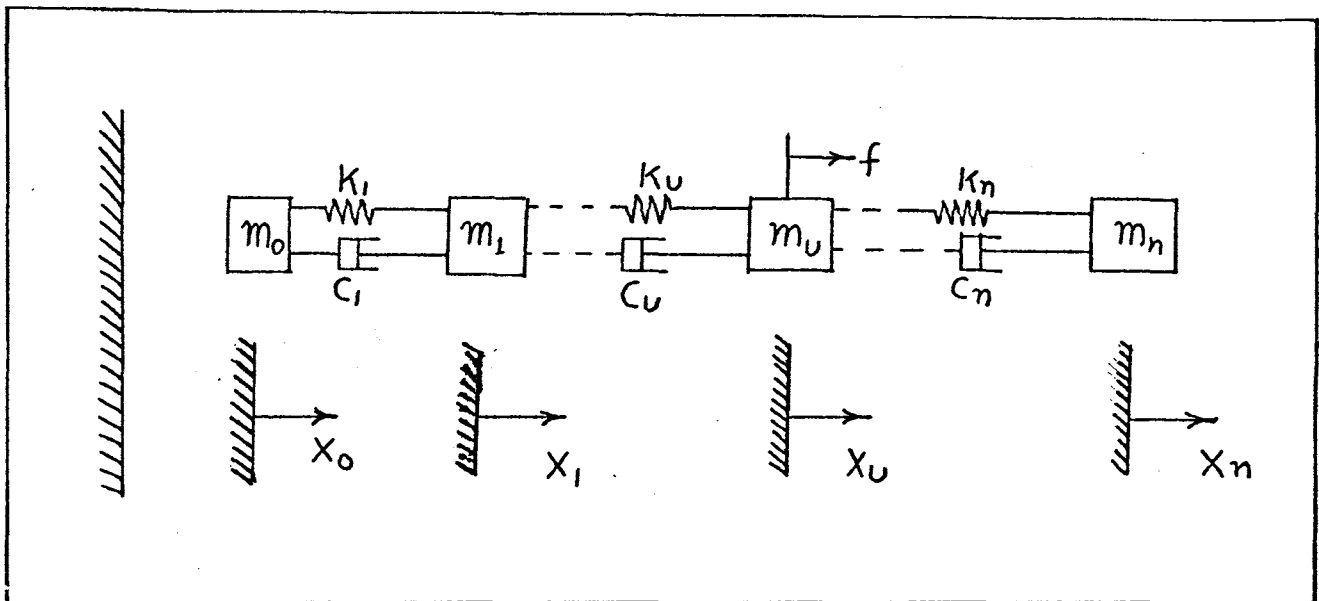


Figure 8

Damped Multi-degree-of-freedom System

The above equations yield the following subsidiary equations:

$$(-M_0 \omega^2 + iC_1 \omega + K_1) X_0 - (iC_1 \omega + K_1) X_1 = 0 \quad (67)$$

$$\begin{aligned} -(iC_j \omega + K_j) X_{j-1} + (-M_j \omega^2 + iC_j \omega + K_j + iC_{j+1} \omega + K_{j+1}) X_j \\ - (iC_{j+1} \omega + K_{j+1}) X_{j+1} = \begin{cases} f_u @ j=u \\ 0 @ j \neq u \end{cases} \end{aligned} \quad (68)$$

$$(-M_n \omega^2 + iC_n \omega + K_n) X_n - (iC_n \omega + K_n) X_{n-1} = 0 \quad (69)$$

Considering each element as a single-degree-of-freedom system, define

$$T_{d_j} = \frac{i\omega C_j + K_j}{-M_j \omega^2 + i\omega C_j + K_j} = \frac{1 + i2b_j \frac{\omega}{\omega_{oj}}}{1 - \left(\frac{\omega}{\omega_{oj}}\right)^2 + i2b_j \frac{\omega}{\omega_{oj}}} \quad (70)$$

$$T_{f_j} = \frac{-M_j \omega^2}{-M_j \omega^2 + i\omega C_j + K_j} = \frac{-\left(\frac{\omega}{\omega_{oj}}\right)^2}{1 - \left(\frac{\omega}{\omega_{oj}}\right)^2 + i2b_j \frac{\omega}{\omega_{oj}}} \quad (71)$$

where

$$\omega_{oj} = \left(\frac{K_j}{M_j}\right)^{\frac{1}{2}}$$

$$b_j = \frac{C_j}{2(M_j K_j)^{\frac{1}{2}}}$$

T_{d_j} is defined as the displacement transmissibility of the individual element while T_{f_j} is defined as the corresponding force transmissibility. It should be noted that in most of the textbooks^(1,19), displacement and force transmissibilities have identical expressions. However, these have been defined here such that

$$T_{f_j} = 1 - T_{d_j} \quad (72)$$

Defining Tf_j as above helps in expressing the equations of motion in terms of these two quantities.

From equations (70) and (71),

$$ic_j w + K_j = -M_j w^2 \left(\frac{Td_j}{Tf_j} \right) \quad (73)$$

$$-M_j w^2 + iC_j w + K_j = \frac{-M_j w^2}{Tf_j} \quad (74)$$

Substituting these relationships into equations (67), (68), and (69)

$$(-M_0 w^2 - M_1 w^2 \frac{Td_1}{Tf_1}) X_0 + (M_1 w^2 \frac{Td_1}{Tf_1}) X_1 = 0 \quad (75)$$

$$(M_j w^2 \frac{Td_j}{Tf_j}) X_{j-1} + \left(\frac{-M_j w^2}{Tf_j} - M_{j+1} w^2 \frac{Td_{j+1}}{Tf_{j+1}} \right) X_j \quad (76)$$

$$-(-M_{j+1} w^2 \frac{Td_{j+1}}{Tf_{j+1}}) X_{j+1} = \begin{cases} f_u & j=u \\ 0 & j \neq u \end{cases}$$

$$-(-M_n w^2 \frac{Td_n}{Tf_n}) X_{n-1} + \left(\frac{-M_n w^2}{Tf_n} \right) X_n = 0 \quad (77)$$

Dividing equations (75), (76), and (77) through by $-w^2$, the following equations are obtained:

$$(M_0 + M_1 \frac{Td_1}{Tf_1}) X_0 - (M_1 \frac{Td_1}{Tf_1}) X_1 = 0 \quad (78)$$

$$\begin{aligned} & -(M_j \frac{Td_j}{Tf_j}) X_{j-1} + \left(\frac{M_j}{Tf_j} + M_{j+1} \frac{Td_{j+1}}{Tf_{j+1}} \right) X_j - (M_{j+1} \frac{Td_{j+1}}{Tf_{j+1}}) X_{j+1} \\ & = \begin{cases} -\frac{f_u}{w^2} & j = u \\ 0 & j \neq u \end{cases} \quad (79) \end{aligned}$$

$$-(M_n \frac{Td_n}{Tf_n}) X_{n-1} + (\frac{M_n}{Tf_n}) X_n = 0 \quad (80)$$

The above equations are rearranged to obtain a form that can be readily adapted to a graphical solution. From equation (78),

$$\frac{X_0}{X_1} = \frac{M_1 \frac{Td_1}{Tf_1}}{M_0 + M_1 \frac{Td_1}{Tf_1}} \quad (81)$$

or

$$\frac{X_0}{X_1} = \left[1 + \frac{M_0}{M_1} \frac{Tf_1}{Td_1} \right]^{-1}$$

From equation (79), three equations are obtained corresponding to

$$\begin{aligned} j &< u \\ j &= u \\ \text{and } j &> u \end{aligned}$$

For $j < u$: Let $j = 1$. From equation (79),

$$-(M_1 \frac{Td_1}{Tf_1}) X_0 + (\frac{M_1}{Tf_1} + M_2 \frac{Td_2}{Tf_2}) X_1 - (M_2 \frac{Td_2}{Tf_2}) X_2 = 0 \quad (82)$$

substituting for X_0 from equation (81),

$$-(M_1 \frac{Td_1}{Tf_1}) \frac{(M_1 \frac{Td_1}{Tf_1}) X_1}{(M_0 + M_1 \frac{Td_1}{Tf_1})} + (\frac{M_1}{Tf_1} + M_2 \frac{Td_2}{Tf_2}) X_1 - (M_2 \frac{Td_2}{Tf_2}) X_2 = 0 \quad (83)$$

Equation (83) gives

$$\frac{X_1}{X_2} = \frac{M_2 \frac{Td_2}{Tf_2}}{\frac{M_1}{Tf_1} + M_2 \frac{Td_2}{Tf_2} - \frac{(M_1 \frac{Td_1}{Tf_1})^2}{M_0 + M_1 \frac{Td_1}{Tf_1}}}$$

rearranging,

$$\frac{X_1}{X_2} = \left[1 + \frac{M_1}{M_2} \frac{Tf_2}{Tf_1} \frac{1}{Td_2} \left(1 - \frac{M_1 Td_1^2 / Tf_1}{M_0 + M_1 \frac{Td_1}{Tf_1}} \right) \right]^{-1} \quad (84)$$

substituting $\frac{X_0}{X_1}$ for $\frac{M_1 \frac{Td_1}{Tf_1}}{M_0 + M_1 \frac{Td_1}{Tf_1}}$, the above equation reduces to

$$\frac{X_1}{X_2} = \left[1 + \frac{M_1}{M_2} \frac{Tf_2}{Tf_1} \frac{1}{Td_2} \left(1 - Td_1 \frac{X_0}{X_1} \right) \right]^{-1} \quad (85)$$

Generalizing the above derivation,

$$\frac{X_{j-1}}{X_j} = \left[1 + \frac{M_{j-1}}{M_j} \frac{Tf_j}{Td_j} \frac{1}{Tf_{j-1}} \left(1 - Td_{j-1} \frac{X_{j-2}}{X_{j-1}} \right) \right]^{-1} \quad (86)$$

For $j = u$: Equation (79) becomes

$$-(M_u \frac{Td_u}{Tf_u}) X_{u-1} + (\frac{M_u}{Tf_u} + M_{u+1} \frac{Td_{u+1}}{Tf_{u+1}}) X_u - (M_{u+1} \frac{Td_{u+1}}{Tf_{u+1}}) X_{u+1} = -\frac{f_u}{w^2} \quad (87)$$

Dividing through by $(X_u M_u)$ and rearranging, equation (87) yields

$$-\frac{M_u X_u w^2}{f_u} = \left[-\frac{Td_u}{Tf_u} \frac{X_{u-1}}{X_u} + \frac{1}{Tf_u} + \frac{M_{u+1}}{M_u} \frac{Td_{u+1}}{Tf_{u+1}} - \frac{M_{u+1}}{M_u} \frac{Td_{u+1}}{Tf_{u+1}} \frac{X_{u+1}}{X_u} \right]^{-1} \quad (88)$$

For $j > u$: Write equation (79) as

$$(M_{j+1} \frac{Td_{j+1}}{Tf_{j+1}}) X_j - (\frac{M_{j+1}}{Tf_{j+1}} + M_{j+2} \frac{Td_{j+2}}{Tf_{j+2}}) X_{j+1} + (M_{j+2} \frac{Td_{j+2}}{Tf_{j+2}}) X_{j+2} = 0 \quad (89)$$

Dividing through by X_{j+1} and rearranging,

$$\frac{X_j}{X_{j+1}} = \left[\frac{\left(\frac{M_{j+1}}{Tf_{j+1}} + M_{j+2} \frac{Td_{j+2}}{Tf_{j+2}} \right) - \left(M_{j+2} \frac{Td_{j+2}}{Tf_{j+2}} \right) \frac{X_{j+2}}{X_{j+1}}}{M_{j+1} \frac{Td_{j+1}}{Tf_{j+1}}} \right] \quad (90)$$

The above equation is now written in the following form:

$$\frac{X_{j+1}}{X_j} = Td_{j+1} \left[1 + \frac{M_{j+2}}{M_{j+1}} \frac{Td_{j+2}}{Tf_{j+2}} Tf_{j+1} \left(1 - \frac{X_{j+2}}{X_{j+1}} \right) \right]^{-1} \quad (91)$$

Lastly, from equation (80),

$$\frac{X_n}{X_{n-1}} = Td_n \quad (92)$$

Equations (81), (86), (88), (91), and (92) are used for determining the system response. The amplitude ratio between the inertia force of mass j and the external force applied at location u is obtainable by the relation

$$\left| \frac{M_j w^2 X_j}{f_u} \right| = \frac{M_j}{M_u} \left| \frac{M_u w^2 X_u}{f_u} \right| \left| \frac{X_j}{X_u} \right| \quad (93)$$

Similarly, the amplitude ratio between any two displacements is obtainable by the relation

$$\left| \frac{X_j}{X_{j-2}} \right| = \left| \frac{X_j}{X_{j-1}} \right| \left| \frac{X_{j-1}}{X_{j-2}} \right| \quad (94)$$

If the external forces are applied at more than one location, then there will be an equation similar to equation (88) for each external force.

The contribution due to each force can be determined by repeatedly using equation (93).

To apply the general solution to specific models, several simplifications can be made. If the input is applied to an end mass, begin the analysis at the opposite end and progress toward the input. If the force is applied at each end, the analysis is carried out by taking into account one force at a time and beginning the analysis at the opposite end of this force. The net response is obtained by principle of superposition.

For the case where the first mass is attached to a rigid foundation, M_0 is assumed to be infinite. Then X_0 vanishes and the ratio $\frac{X_1}{X_2}$ is available as a function of the parameters describing the first and second elements.

Branched Systems: For the branched system, the transmissibility of the nodal mass is simply a function of the sum of the branch transmissibilities - the bracketed term in equations (81), (86), (88), (91), and (92) then has the format $1 + \text{Branch 1} + \text{Branch 2} + \dots$. To illustrate the setting up of the equations, consider the system shown in figure 9. Let $n = 3$, which is the number of masses in the longest chain, and let $j = 0$. Then from equation (92),

$$\frac{X_3}{X_2} = Td_3 \quad (95)$$

$$\frac{X_4}{X_1} = Td_4 \quad (96)$$

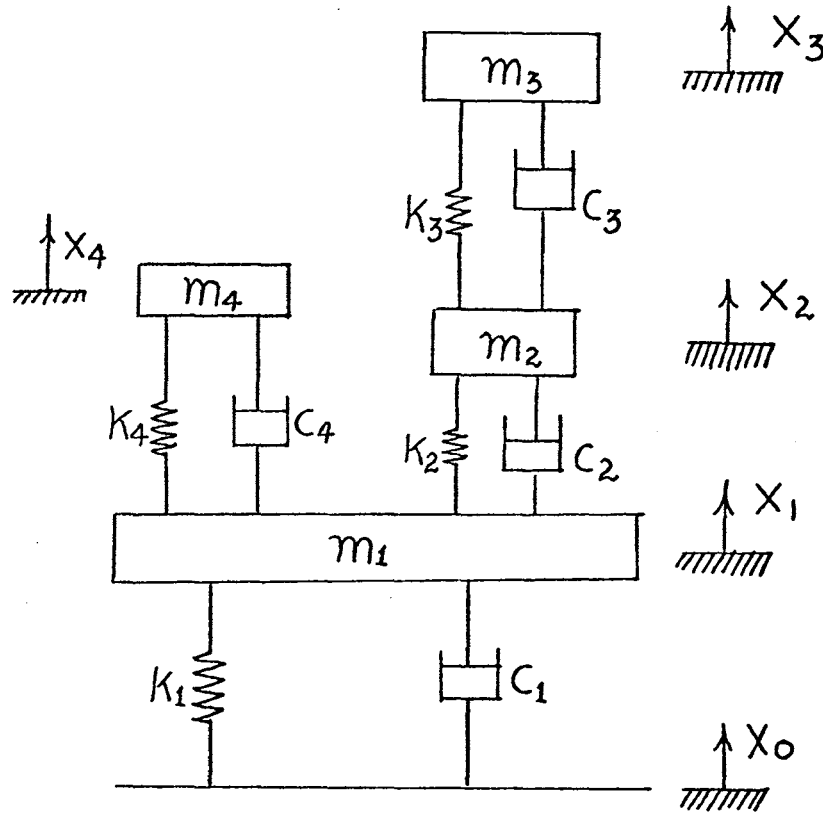


Figure 9

Damped Four-degree-of-freedom System Excited
by Foundation Motion

From equation (91),

$$\begin{aligned}\frac{X_2}{X_1} &= Td_2 \left[1 + \frac{M_3}{M_2} \frac{Td_3}{Tf_3} Tf_2 (1 - Td_3) \right]^{-1} \\ &= Td_2 \left[1 + \frac{M_3}{M_2} Td_3 Tf_2 \right]^{-1}\end{aligned}\quad (97)$$

$$\begin{aligned}\frac{X_1}{X_o} &= Td_1 \left[1 + \frac{M_2}{M_1} \frac{Td_2}{Tf_2} Tf_1 \left(1 - \frac{X_2}{X_1}\right) + \frac{M_4}{M_1} \frac{Td_4}{Tf_4} Tf_1 \left(1 - \frac{X_4}{X_1}\right) \right]^{-1} \\ &= Td_1 \left[1 + \frac{M_2}{M_1} \frac{Td_2}{Tf_2} Tf_1 \left(1 - \frac{X_2}{X_1}\right) + \frac{M_4}{M_1} \frac{Td_4}{Tf_4} Tf_1 (1 - Td_4) \right]^{-1}\end{aligned}$$

noting $Tf_4 = 1 - Td_4$, the above equation has the final form

$$\frac{X_1}{X_o} = Td_1 \left[1 + \frac{M_2}{M_1} \frac{Td_2}{Tf_2} Tf_1 \left(1 - \frac{X_2}{X_1}\right) + \frac{M_4}{M_1} Td_4 Tf_1 \right]^{-1}\quad (98)$$

Using equations (97) and (98) gives $\frac{X_2}{X_o}$,

$$\frac{X_2}{X_o} = \frac{X_2}{X_1} \cdot \frac{X_1}{X_o}\quad (99)$$

similarly from equations (95) and (99),

$$\frac{X_3}{X_o} = \frac{X_3}{X_2} \cdot \frac{X_2}{X_o}\quad (100)$$

and from equations (96) and (98),

$$\frac{X_4}{X_o} = \frac{X_4}{X_1} \cdot \frac{X_1}{X_o}\quad (101)$$

Equations (98) through (101) give the ratio between the amplitude at any location and the input X_o .

Note that the rotational motion has been neglected and the masses are assumed to have translational motion only. (For solving a system having both rotational and translational motions, see section IX).

Graphical Method for Solution: The graphical solution of equations (81), (86), (88), (91), and (92) involves the plotting of the individual transmissibilities Td_j and Tf_j . Magnitude and phase angle plots of these equations are discussed in texts on basic servomechanism theory⁽²⁰⁾. Magnitudes in decibels are conventionally plotted against frequency ratio $\frac{w}{w_{0j}}$ on a semilogarithmic scale. Multiplication is then equivalent to graphical addition. Some of Td and Tf plots⁽⁶⁾ of both magnitude and phase versus frequency ratio are shown in figures 10 and 11. To proceed with the graphical method, first determine the equations of motion and evaluate all the known parameters such as masses, damping ratios, and the undamped natural frequencies of the individual spring-mass-damper elements. The working curves are based on "uncoupled" natural frequencies and damping ratios. The appropriate curves of Td and Tf versus $\frac{w}{w_{0j}}$ that are required by the specific problem are then traced onto a sheet of semilog paper. The curves representing those terms that are to be multiplied together are graphically added. If the solution requires the evaluation of a reciprocal of a function, revolve the magnitude curve of the function about the 0-decibel line and reverse the original phase relationship. Mass-ratio terms may be combined with Tf and Td curves by first determining the decibel value of the mass ratio and then shifting the associated Tf and Td magnitude curve up or down by this value. The accuracy of the graphical technique is dependent upon the quality of the working curves.

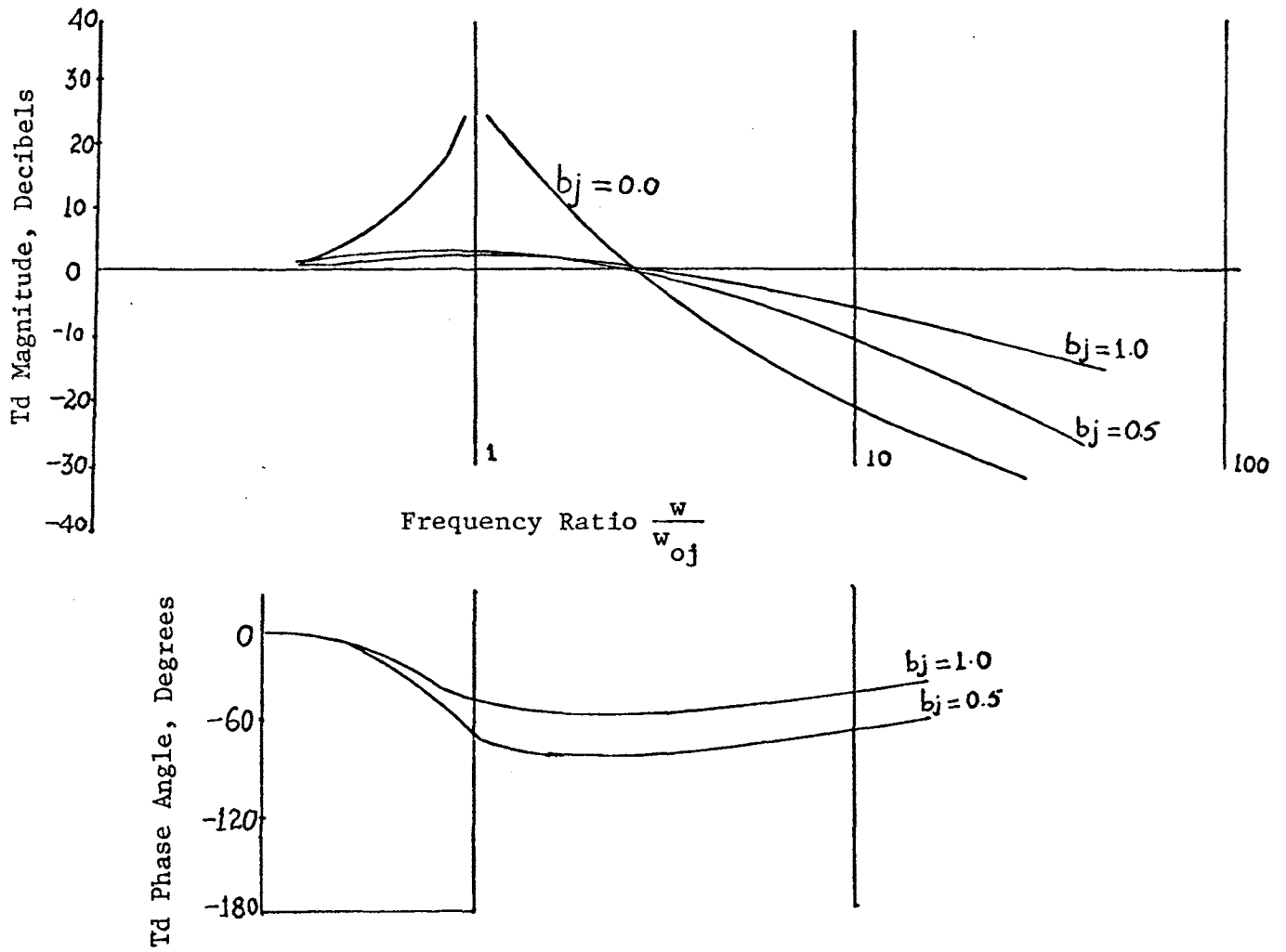


Figure 10

Td Magnitude and Phase-angle Characteristics

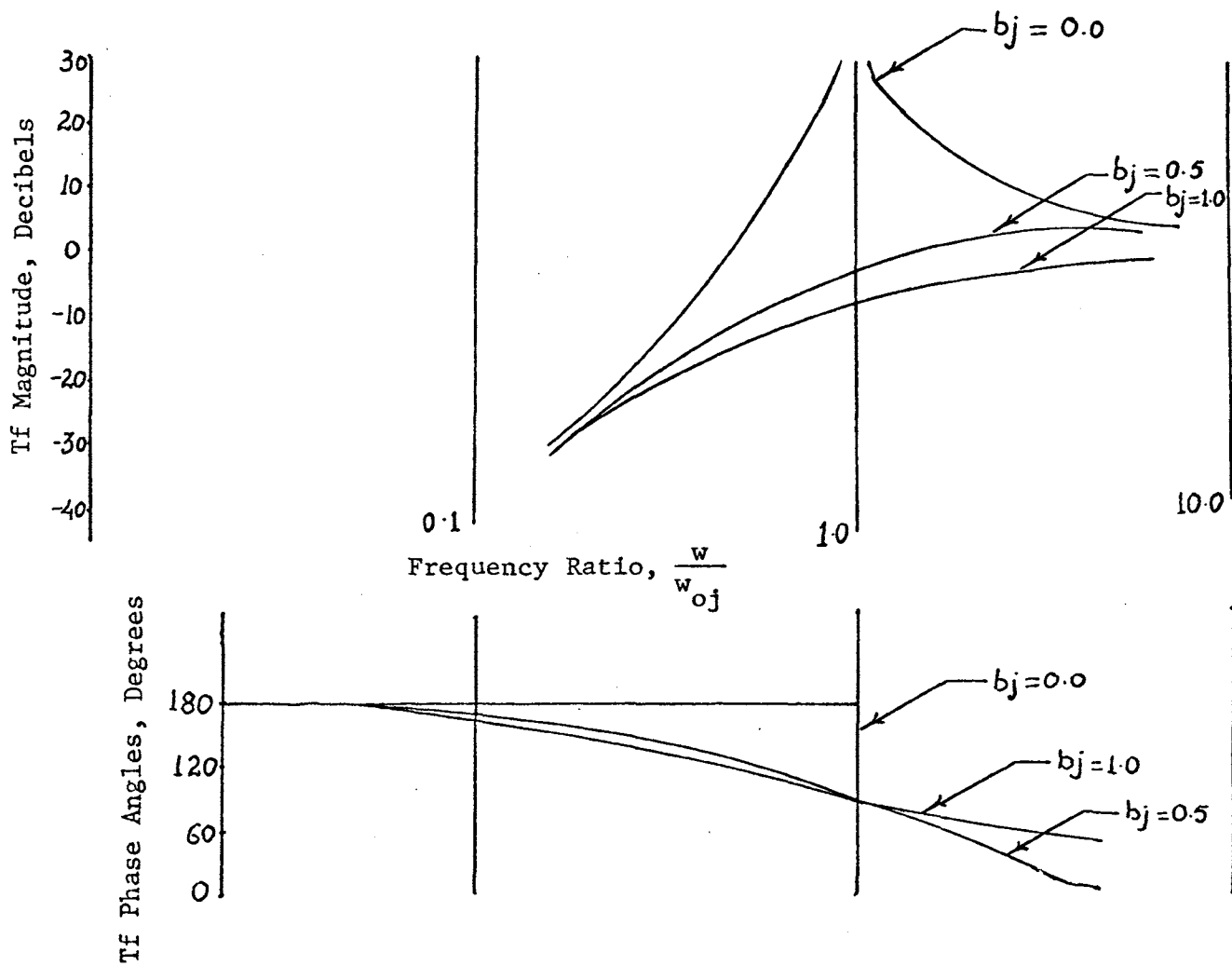


Figure 11

Tf Magnitude and Phase-angle Characteristics

Another form of graphical solution consists of vector diagrams where various vectors can be added or subtracted. Before drawing vector diagrams, all multiplications and divisions are carried out by algebraic methods and the final expression is reduced to a form which contains additions and subtractions of vectors only.

To illustrate this procedure, consider equation (85):

$$\frac{X_1}{X_2} = \frac{1}{1 + \frac{M_1}{M_2} \frac{Tf_2}{Tf_1} \frac{1}{Td_2} - \frac{M_1}{M_2} \frac{Tf_2}{Tf_1} \frac{Td_1}{Td_2} \frac{X_o}{X_1}} \quad (85)$$

It is desired to obtain the magnitude and the phase angle of $\frac{X_1}{X_2}$. For this, first evaluate the products $\frac{M_1}{M_2} \frac{Tf_2}{Tf_1} \frac{1}{Td_2}$ and $\frac{M_1}{M_2} \frac{Tf_2}{Tf_1} \frac{Td_1}{Td_2} \frac{X_o}{X_1}$. The two vectors thus obtained are then added in a vector diagram. Add vectorially 1 to the resultant vector. This gives the denominator of the right-hand side of the above equation. The expression for $\frac{X_1}{X_2}$ is reduced to the form

$$\frac{X_1}{X_2} = \frac{1}{A \angle \beta^\circ}$$

where A is the magnitude of the denominator and β is the phase angle.

$\frac{X_1}{X_2}$ now become

$$\frac{X_1}{X_2} = (A)^{-1} \angle \beta^\circ$$

which is the desired result.

Either of these graphical methods may be employed depending upon the individual's preference for solving a particular problem. There appears to be no specific advantage of one method over the other.

With a relatively small amount of labor and considerable saving of time, one can solve multimass transmissibility problems by the graphical technique without solving mathematically the equations of motion.

VIII. METHOD FOR REDUCING DEGREES-OF-FREEDOM

When the degrees of freedom in a mathematical model are large, solution by conventional methods becomes increasingly complex. In many practical problems, engineers are generally interested in a frequency range far less broad than the full range of resonant frequencies. This method makes use of this fact and reduces a large system of equations to a much smaller system reproducing approximately the same response as the original in the limited frequency range of interest.

S. E. Staffeld⁽⁷⁾ has presented a mathematical theory for reducing the degrees of freedom of a system and then solving this reduced system. His approach is similar to the Normal Mode Technique presented earlier, but the procedures appear unnecessarily complicated. The method presented below makes use of Staffeld's basic ideas but the solution is obtained by conventional methods.

A broad outline of this method is presented first and then a few problems will be solved to illustrate the use and accuracy of results obtained from it. For a given frequency range, the steady state system response can be obtained with satisfactory accuracy by performing the following steps:

- (1) Determine the natural frequencies of individual spring-mass combinations.
- (2) Eliminate those masses from spring-mass combinations whose natural frequencies are significantly higher than the frequency range of interest.

- (3) Solve the reduced system for its steady state response by any of the conventional methods.

Step (2) is reasonable because: (1) the lowest system eigenvalue is less than the lowest frequency of any individual spring-mass combination, (2) the eigenvalues of the system which are greater than the frequency range of interest are due to those masses whose individual spring-mass combination frequencies are higher than the frequency range of interest. Therefore, elimination of these masses from the system has little effect on system response within the frequency range of interest. The elimination of the masses whose individual spring-mass frequencies are closer to the operating frequency depends upon individual judgement. With experience, it should be possible to eliminate the masses whose omission from the system does not produce a large error in the system response.

Elimination of some of the masses automatically reduces the degrees of freedom. However, the effect of these masses in the frequency range of interest is included by altering the spring constants associated with the original spring-mass combinations. These are altered to account for the inertia force produced by the missing masses. The original spring constant K is replaced by $K - mw_0^2$, where w_0 is the usual (or typical) operating frequency. The system modified as above will have exactly the same behavior as the original system at w_0 and will generally be a good approximation to it in the neighborhood of w_0 .

If the mass to be eliminated happens to be the outermost mass, then the spring attached to this mass is assumed to have infinite stiffness*. This permits coupling of this mass with the one which precedes it.

If viscous damping is present, it is assumed that the normal modes not in the frequency range of interest may be represented by modified springs without mass and damper. The resulting reduced damped system equations are then solved by one of the methods presented previously, for the remaining eigenvalues and eigenvectors.

The potential savings in work arises from two sources:

- (1) All of the eigenvalues and eigenvectors need not be found.
- (2) Fewer forced vibration equations are required and the uncoupling procedure is thereby simplified.

Illustrative Examples: As a first example, consider the system shown in figure 12 with

$$M_1 = M_2 = M_3 = 1 \text{ lb-sec}^2/\text{in.}$$

$$K_1 = 100 \text{ lbs/in.}, K_2 = 4 \text{ lbs/in.}, K_3 = 1 \text{ lb/in.}$$

$$\text{Operating frequency range} = 0 \text{ to } 3 \text{ rad/sec.}$$

It is required to find natural frequencies in the operating frequency range and the corresponding amplitudes of the masses.

* See illustrative example on page 67.

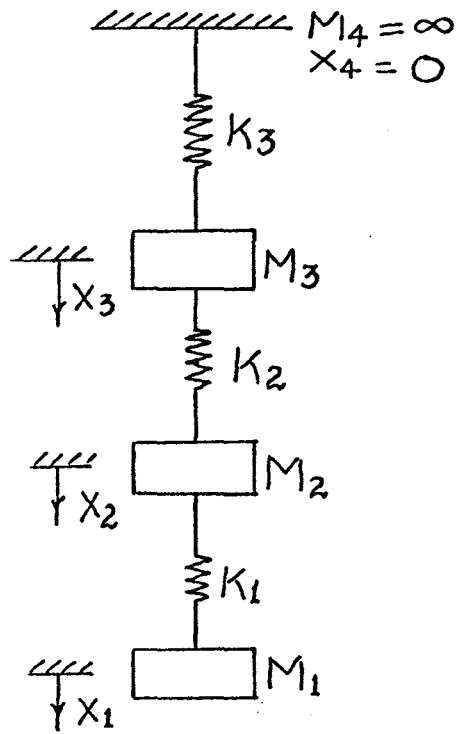


Figure 12

A Spring-Mass System

Solution: The actual solution is presented first. For this, any of the conventional methods can be used; the tabular method of Holzer has been employed here. The boundary condition for this problem is $X_4 = 0$ at all times. Holzer tables for the three natural frequencies are given below:

Holzer Table for First Natural Frequency

Table I: $w^2 = 0.30$

Mass No	M	Mw^2	X_i	Mw^2X_i	$\sum Mw^2X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2X_i}{K_{i,i+1}}$
1	1	.3	1.0	.3	.3	100	.003
2	1	.3	.997	.297	.597	4	.149
3	1	.3	.848	.254	.851	1	.851
4	∞		-.003				

The amplitude of the first mass is arbitrarily taken as one unit. For $w^2 = .3$, $X_4 = -.003 \approx 0$. This shows that the trial frequency is natural frequency of the system and since only one sign change in the amplitude column occurs, there are no natural frequencies below this*. Therefore, $w_1 = \sqrt{.3} = 0.548$ rad/sec.

* For a detailed explanation, refer to Holzer's Method on page 27.

Holzer Table for Second Natural Frequency

Table II: $w^2 = 6.64$

Mass No	M	Mw ²	X _i	Mw ² X _i	∑Mw ² X _i	K _{i,i+1}	$\frac{\sum Mw^2 X_i}{K_{i,i+1}}$
1	1	6.64	1.0	6.64	6.64	100	.0664
2	1	6.64	.9336	6.195	12.835	4	3.209
3	1	6.64	-2.2754	15.11	-2.275	1	-2.275
4	∞						-.0004

$$w_2 = \sqrt{6.64} = 2.582 \text{ rad/sec}$$

A similar table gives the third natural frequency $w_3 = 14.28 \text{ rad/sec}$.

Reduced System: Individual natural frequencies are

$$w_1 = \sqrt{\frac{K_1}{M_1}} = \sqrt{\frac{100}{1}} = 10 \text{ rad/sec}$$

$$w_2 = \sqrt{\frac{K_2}{M_2}} = \sqrt{\frac{4}{1}} = 2 \text{ rad/sec}$$

$$w_3 = \sqrt{\frac{K_3}{M_3}} = \sqrt{\frac{1}{1}} = 1 \text{ rad/sec}$$

Since the operating frequency range is 0 to 3 rad/sec, w_1 lies outside the frequency range of interest. Therefore mass M_1 can be eliminated.

Since M_1 is outermost mass, dropping M_1 results in omission of K_1 also.

To take care of such situations, K_1 is assumed to have infinite stiffness.

This enables M_1 to be added to M_2 and the reduced system is represented

as

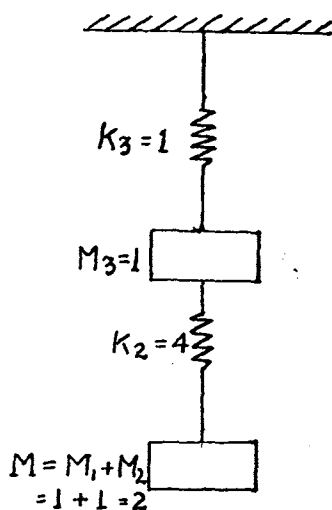


Figure 13. Reduced Spring-Mass System

In other words, it is assumed that the amplitude of M_1 is the same as that of M_2 . This assumption is supported by the actual displacements of M_1 and M_2 as observed from Holzer tables for first and second natural frequencies.

Holzer tables for the reduced system are as follows:

Table I: $w^2 = 0.3$

Mass No	M	Mw^2	X_i	$Mw^2 X_i$	$\sum Mw^2 X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2 X_i}{K_{i,i+1}}$
1	2	.6	1	.6	.6	4	.15
2	1	.3	.85	.255	.855	1	.855
3	∞						-.005

$$w_1 \approx \sqrt{.3} = .548 \text{ rad/sec}$$

Table II: $w^2 = 6.7$

Mass No	M	Mw^2	X_i	$Mw^2 X_i$	$\sum Mw^2 X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2 X_i}{K_{i,i+1}}$
1	2	13.4	1.0	13.4	13.4	4	3.35
2	1	6.7	-2.35	-15.75	-2.35	1	-2.35
3	∞		0				

$$w_2 = \sqrt{6.7} = 2.588 \text{ rad/sec}$$

A table comparing the actual and reduced system solutions is presented below. For this table, X_1 and X_2 represent vectors consisting of amplitudes of only those masses whose individual eigenvalues lie within frequency range of interest. These are obtained from the amplitude columns associated with the first and second natural frequencies.

Comparison of Actual and Reduced Systems

	w_1	w_2	X_1	X_2
Actual solution	.548	2.582	$\begin{Bmatrix} 1.0 \\ .85 \end{Bmatrix}^*$	$\begin{Bmatrix} 1.0 \\ -2.4 \end{Bmatrix}^*$
Reduced System Solution	.548	2.588	$\begin{Bmatrix} 1 \\ .85 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ -2.35 \end{Bmatrix}$

It can be seen from the above table that eigenvalues in the frequency range of interest obtained by this method compare closely with the actual eigenvalues. Also the displacements given by this method do not vary significantly from actual displacements of the original system.

* Note that the amplitude of the mass M_2 has been scaled to 1.

As a second example, consider the system shown below, where the mass to be eliminated happens to be other than the outermost mass.

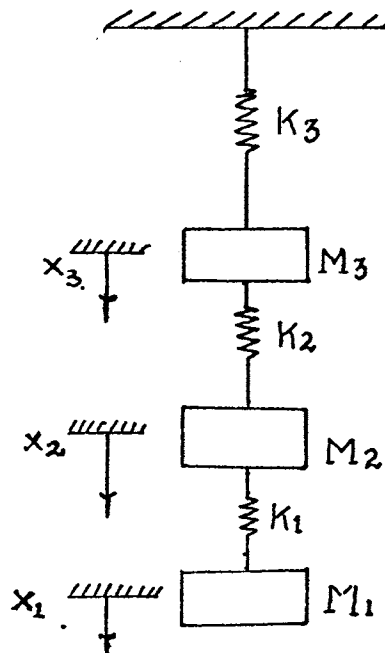


Figure 14. A Spring-Mass System

Given data: $M_1 = M_2 = M_3 = 1 \text{ lb-sec}^2/\text{in}$

$K_1 = 1 \text{ lb/in.}, K_2 = 5 \text{ lb/in.}, K_3 = 10 \text{ lb/in.}$

Usual operating frequency = 2 rad/sec.

It is required to find eigenvalues and the corresponding eigenvectors in the neighborhood of the operating frequency.

Actual Solution: (Holzer's Method)

Table I: $w^2 = 0.71$

Mass No	M	Mw^2	X_i	Mw^2X_i	$\sum Mw^2X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2X_i}{K_{i,i+1}}$
1	1	.71	1.0	.71	.71	1	.71
2	1	.71	.29	.206	.916	5	.183
3	1	.71	.107	.076	.992	10	.0992
4	∞		.008				

$$w_1 \approx \sqrt{.71} = .8425 \text{ rad/sec}$$

Table II: $w^2 = 4.05$

Mass No	M	Mw^2	X_i	Mw^2X_i	$\sum Mw^2X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2X_i}{K_{i,i+1}}$
1	1	4.05	1.0	4.05	4.05	1	4.05
2	1	4.05	-3.05	-12.35	-8.3	5	-1.66
3	1	4.05	-1.39	-5.64	-13.94	10	-1.394
4	∞		+0.004				

$$w_2 \approx 2.013 \text{ rad/sec}$$

And the third natural frequency w_3 is found to be 4.15 rad/sec.

Reduced System: The individual spring-mass combination frequencies are

$$w_1 = 1.0 \text{ rad/sec}$$

$$w_2 = 2.27 \text{ rad/sec}$$

$$w_3 = 3.17 \text{ rad/sec}$$

which indicates that M_3 can be eliminated. The effect of elimination of M_3 is included by modifying K_3 as follows:

$$\text{modified stiffness } K = K_3 - M_3 w_o^2$$

where w_o is used usual operating frequency.

$$\begin{aligned} K &= 10 - (1)(4) \\ &= 6 \text{ lb/in.} \end{aligned}$$

The Holzer tables for Reduced System are given below:

Table I: $w^2 = .675$

Mass No	M	Mw^2	X_i	$Mw^2 X_i$	$\sum Mw^2 X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2 X_i}{K_{i,i+1}}$
1	1	.675	1	.675	.675	1	.675
2	1	.675	.325	.219	.894	2.73	.327
3	∞		-.002				

$$w_1 \approx .822 \text{ rad/sec.}$$

Table II: $w^2 = 4.05$

Mass No	M	Mw^2	X_i	$Mw^2 X_i$	$\sum Mw^2 X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2 X_i}{K_{i,i+1}}$
1	1	4.05	1.0	4.05	4.05	1	4.05
2	1	4.05	-3.05	-12.35	-8.3	2.73	-3.04
3	∞		-.01				

$$w_2 \approx 2.013 \text{ rad/sec.}$$

The reduced system for the second illustrative example is

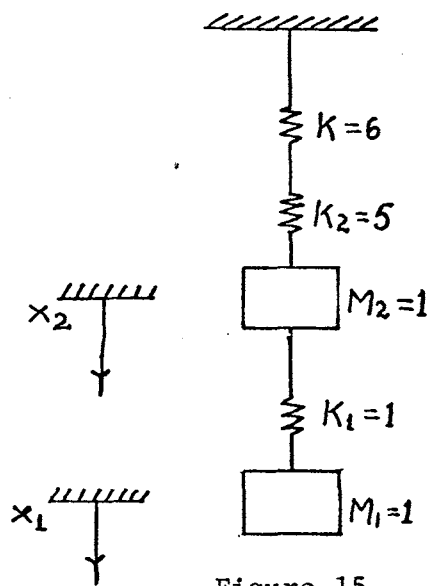


Figure 15

Reduced Spring-Mass System

which is equivalent to

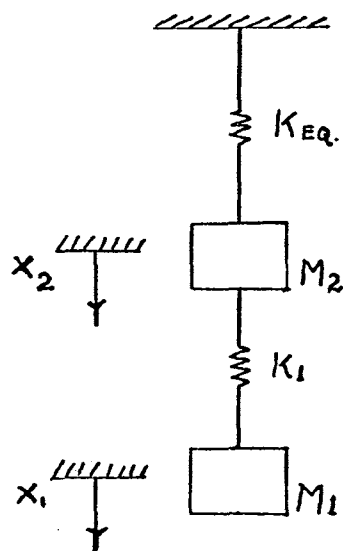


Figure 16

Equivalent Reduced System

$$K_{eq} = \frac{K_2 K}{K_2 + K} = \frac{5 \times 6}{5 + 6} = 2.73 \text{ lb/in.}$$

A comparison of results obtained by actual and reduced system solutions is presented below:

Comparison of Actual and Reduced Systems

	w_1	w_2	X_1	X_2
Actual solution	.8425	2.013	$\begin{Bmatrix} 1 \\ .29 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ -3.05 \end{Bmatrix}$
Reduced system solution	.822	2.013	$\begin{Bmatrix} 1 \\ .325 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ -3.05 \end{Bmatrix}$

It can be seen from above two examples that approximating a system can reduce the degrees of freedom and at the same time give results which compare well with the actual solution. With a little experience, this method can save a considerable amount of labor when a large number of degrees of freedom is involved.

IX. SUMMARY AND CONCLUSIONS

In this work various methods for the analysis of linear damped multi-mass systems have been studied. Naturally, a question arises as to the suitability of these methods for solution of a specific problem. Any comparison between these methods should be based on the following points:

- (1) The amount of labor and time required to formulate a mathematical model and to obtain the natural frequencies and system response.
- (2) The approximations involved and consequently the accuracy of results.
- (3) Application to systems such as (a) those with masses and dampers connected to a reference frame; and (b) those having both linear and angular motions.
- (4) Ease of application when used (a) manually; or (b) with a digital or analog computer.

The advantages and disadvantages of each of the methods presented earlier will now be discussed on the basis of the above points.

Normal Mode Technique:

- (1) Even for systems with many degrees of freedom the derivation of the equations of motion is straightforward and does not present any problem. However, when the degrees of freedom are large, required operations involving the mass, stiffness, and damping matrices can be laborious. If the system is classically damped, the natural frequencies

and the system response can be obtained without great difficulty. However, non-classically damped systems are extremely difficult to solve even for as few as three degrees of freedom, as is illustrated by the sample problem in Appendix D. Another major disadvantage of this method lies in the fact that to ascertain the nature of the system (whether classically or non-classically damped), M, K, and C matrices have to be obtained. This requires the derivation of the equations of motion as a first step. If it is discovered that the system is non-classically damped, any of the other methods may be more desirable to employ.

(2) The results obtained by this method will always be exact^{*}, since there are no approximations involved.

(3) This method provides the solution to any kind of system^{*}. If there are combined linear and angular motions, the degrees of freedom will be more than the number of masses and consequently there will be additional equations of motion. However, a complete solution is mathematically possible, with two exceptions^{**}: (1) when there are less than $2N$ independent eigenvectors in the case of repeated roots of the frequency equation of non-classically damped systems; (2) when one or more of the free vibrational modes is critically damped.

* These comments apply to only linear systems.

** Note that these conditions rarely occur in practice.

Of the five methods discussed, this is the only method which provides the system transient response. Though not the topic of this thesis, this is an advantage of the Normal Mode Technique over the other methods.

(4) For classically damped systems having more than two or three degrees of freedom, obtaining the solution by the Normal Mode Technique can be time-consuming if manually done. Therefore, use of a digital computer is recommended for systems having more than 3 degrees of freedom. For non-classically damped systems, use of a computer is essential.

Holzer's Method:

(1) This method does not make use of the frequency equation. The eigenvalues and system response are obtained by trial and error without deriving the equations of motion. This is one of the principal advantages of this method. Another significant advantage is that the higher eigenvalues can be obtained as easily and with as much accuracy as the fundamental eigenvalue. The eigenvectors are directly obtained from the amplitude column of the Holzer table without additional effort. Also, Holzer's Method is ideal for obtaining steady state system response to external harmonic inputs. Only one table needs to be constructed for the impressed frequency; no trial and error procedure is involved.

The disadvantage of this method is that a trial and error method is required for determining the eigenvalues; this can be a difficult task if the system has more than a few masses or if damping is present.

(2) Since a trial and error procedure is involved, some accuracy may have to be sacrificed unless a digital computer is employed. However, use of a computer should permit any practical degree of accuracy to be attained in a reasonable time.

(3) Holzer's Method can be employed to solve systems with dampers connected either between two masses or between a mass and the reference frame, as shown in section V. However, systems having coupled rotational and translational motions have not been treated by this method. In this case the mass has a combination of inertia force and inertia torque which makes the problem quite complex and beyond the scope of this work.

Another application of Holzer's Method is in the study of transverse vibrations of beams. When a beam is replaced by lumped masses connected by massless beam sections, a modification of Holzer's Method can be used to compute progressively the deflection, slope, moment, and shear from one station to the next. This method has been developed by N. O. Myklestad and M. A. Prohl^(1,23).

(4) When the degrees of freedom are four or less, computations can be performed manually without great difficulty. However, for large systems, use of a computer can result in substantial savings of time and labor.

Impedance Method:

(1) The formulation of the equations which describe system performance by the Impedance Method is relatively simple. It does not require derivation of the equations of motion. At the same time, neither the

natural frequencies nor the mode shapes can be obtained directly from this method, which may be a major disadvantage. A second disadvantage of the Impedance Method is that it requires solution of numerous simultaneous equations to obtain the desired relationship between input and output quantities. This can be a laborious task. However, compared to the Normal Mode Technique, this method is much simpler to use for steady-state forced vibrations.

(2) There are no approximations involved and therefore this method is exact for harmonic inputs.

(3) The method is applicable to a very large class of systems. The only practical exception is a system with coupled linear and angular motions. In this case it may be difficult to determine and use the four-pole parameters for the various system elements.

(4) Most of the computations for this method are relatively simple to perform and do not require the use of a digital computer. However, some experience in combining the simultaneous system performance equations can result in considerable savings of time.

Graphical Technique:

(1) Derivation of the equations of motion and expressing these in a form which is suitable for plotting comprises the major part of the work for this method. The eigenvalues cannot be found directly by this technique. However, some estimation of natural frequencies can be obtained by determining system responses for various harmonic input frequencies. The peaks in the plot of the output magnitude versus fre-

quency closely correspond to the natural frequencies of the system. The system response is relatively easy to obtain once the equations are expressed in proper format.

(2) The results obtained by this method are not, in general, as accurate as those obtained from other methods. The accuracy depends to a large extent upon the quality of the working curves.

(3) This technique is ideally suited for systems with masses connected by dampers and springs and having no other elements. Dampers connecting masses to the reference frame can be handled by this method but complicated equations often result. Therefore, Holzer's Method is recommended in this case. The Graphical Method cannot be applied directly to problems having coupled linear and angular motions, because the composite transmissibilities are not defined in this case.

(4) Since the solution is obtained by graphical means, all of the work has to be done manually. However, each stage of the graphical solution can be computerized for improved accuracy.

Reduced System:

Once the original system is reduced to a new system of fewer degrees of freedom, it can be solved by any one of the conventional methods. While the procedure for reducing the degrees of freedom is a simple one, considerable experience may be required to eliminate only those masses and dampers whose presence has little effect in the frequency range of interest.

As is obvious, this is an approximate method. It can be useful, however, when the degrees of freedom of a system are large. Finding the responses of such systems may tax the capacity of a digital computer. Reducing the degrees of freedom of such systems often helps in handling these systems mathematically, without significant loss of accuracy.

For undamped systems, results obtained by this method agree closely with the actual solutions. The accuracy of this method when applied to damped systems will depend upon the accuracy of the conventional method employed to solve the reduced system.

It should be mentioned here that the inputs considered for obtaining the steady state response are sinusoidal in nature. To obtain the response for other types of inputs, several methods⁽²⁾ are available such as Fourier Series Expansion, Laplace Transformation, etc.

It is thus concluded that all of the methods presented in this thesis can produce accurate results if an appropriate method is selected. This summary is intended to serve as a guide for selection of the method best suited to the solution of a particular problem. .

X. SUGGESTIONS FOR FURTHER WORK

There is little need to stress the desirability of attempting to evaluate further some of the methods presented in this thesis. A few suggestions for further work are:

- (1) A systematic procedure for solving systems with coupled angular and linear motions by Holzer's Method, the Impedance Method, and the Graphical Technique needs to be developed.
- (2) Several classes of systems need to be analyzed by the Graphical Technique and the Reduced System Method to ascertain the applicability as well as the accuracy of these methods.
- (3) Further work and experience in solving systems by reducing the degrees of freedom should permit development of a standard procedure for eliminating only those masses which have little effect on system behavior in the frequency range of interest.
- (4) An approximate solution by the Normal Mode Technique can be obtained by ignoring off-diagonal terms of the damping matrix, but the results may be inaccurate. A method should be developed to include, in an optimum manner, the effect of the off-diagonal terms by altering the diagonal elements of the damping matrix.

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APPENDICES

APPENDIX A

Reciprocity Theorem

Maxwell's reciprocity theorem will be proved⁽²¹⁾ here for influence coefficients. An influence coefficient δ_{ij} is defined as the static deflection of the system at station i owing to a unit force applied at station j of the system. Thus, by definition, δ_{ij} is reciprocal of K_{ij} . To prove the theorem, consider the simply supported beam of figure 17 in which two vertical forces F_1 and F_2 are applied at stations 1 and 2. For this illustration, the influence coefficients are δ_{11} , δ_{12} , δ_{21} , and δ_{22} . The deflection at station 1 owing to the force F_2 applied at station 2 is $F_2 \delta_{12}$.

Assume that the procedure of loading is separated into two steps. First, F_1 is applied to station 1 and then the force F_2 is applied to station 2. When F_1 is applied alone, the potential energy in the beam by virtue of its deformation, is equal to $\frac{1}{2} F_1^2 \delta_{11}$. Now, when F_2 is applied, the additional deflection at station 1 owing to the force F_2 is $F_2 \delta_{12}$. The work done by F_1 corresponding to this deflection is $F_1 (F_2 \delta_{12})$. Hence the total potential energy in the system is

$$U = \frac{1}{2} F_1^2 \delta_{11} + F_1 (F_2 \delta_{12}) + \frac{1}{2} F_2^2 \delta_{22}$$

The last two terms of this equation represent the additional potential energy which is due to the application of F_2 .

Second, F_2 is applied to station 2 and then the force F_1 is applied to station 1. The potential energy of the system is

$$U = \frac{1}{2} F_2^2 \delta_{22} + F_2 (F_1 \delta_{21}) + \frac{1}{2} F_1^2 \delta_{11}$$

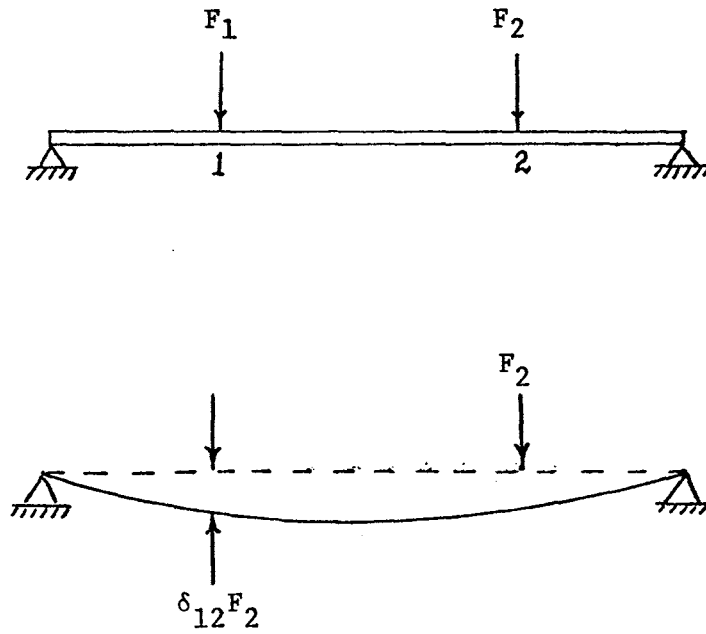


Figure 17

A Simply Supported Beam With Two Loads

The last two terms of this equation are due to the application of F_1 .

Since the final states of the system are identical for the two methods of loading, by the law of conservation of energy, the potential energies expressed in the two cases are the same. By equating the two expressions for potential energy, it is deduced that $\delta_{12} = \delta_{21}$. Generalizing the above proof for several loads, we have

$$\delta_{ij} = \delta_{ji}$$

Since δ_{ij} is the reciprocal of K_{ij} , it follows that $K_{ij} = K_{ji}$.

On the same lines, it can be proved that

$$C_{ij} = C_{ji}$$

The above relationships hold for any linear system.

APPENDIX B

Review of Matrix Operations

There are many properties and relationships that pertain to the eigenvalues and eigenvectors of a square matrix⁽¹⁸⁾, and those that are useful in vibration problems will be mentioned here.

(1) The eigenvalues of a real symmetric matrix are always real and consequently the eigenvectors of such a matrix are also real. This is true because in this case all the coefficients in the characteristic equation are real⁽¹⁶⁾.

(2) All eigenvectors associated with distinct eigenvalues will be linearly independent⁽¹⁷⁾. Furthermore, the eigenvectors are frequently linearly independent even when there are repeated eigenvalues. For a symmetric matrix of order n , n linearly independent eigenvectors will always exist even though the eigenvalues may be repeated.

(3) All of the eigenvectors of a symmetric matrix are orthogonal to one another. To prove this, consider two different eigenvalues E_i and E_j of a symmetric matrix A . These eigenvalues satisfy the equations

$$AX_i = E_i X_i \quad (1)$$

$$AX_j = E_j X_j \quad (2)$$

in which X_i and X_j are the eigenvectors associated with E_i and E_j respectively. Premultiplying equation (1) by X_j^T (transpose of the column vector X_j) and equation (2) by X_i^T ,

$$X_j^T A X_i = E_i X_j^T X_i \quad (3)$$

$$X_i^T A X_j = E_j X_i^T X_j \quad (4)$$

Taking the transposes of equations (3) and (4) yields two more equations

$$X_i^T A X_j = E_i X_i^T X_j \quad (5)$$

$$X_j^T A X_i = E_j X_j^T X_i \quad (6)$$

The left-hand sides of equations (3) and (6) are the same, and similarly for equations (4) and (5), the right-hand sides of these equations may be equated. Thus,

$$E_i X_j^T X_i = E_j X_j^T X_i$$

$$E_i X_i^T X_j = E_j X_i^T X_j$$

Rearranging these equations gives

$$(E_i - E_j) X_j^T X_i = (E_i - E_j) X_i^T X_j = 0 \quad (7)$$

Because E_i and E_j are assumed to be different eigenvalues, it follows from equation (7) that

$$X_j^T X_i = X_i^T X_j = 0$$

which shows that the eigenvectors X_i and X_j are orthogonal vectors.

By a more involved analysis⁽¹⁸⁾, the same results can be confirmed for eigenvectors that are associated with repeated eigenvalues. Thus, the important conclusion is obtained that all of the eigenvectors of a symmetric matrix are orthogonal to one another.

The above orthogonality condition applies when the matrix product of $[M]^{-1} [K]$ is symmetric. For the case when $[M]^{-1} [K]$ is not symmetric, the following orthogonality conditions for the normal modes exist:

For the i th normal mode,

$$[K] \{q_i\} = w_i^2 [M] \{q_i\} \quad (8)$$

Similarly, for the j th normal mode,

$$[K] \{q_j\} = w_j^2 [M] \{q_j\} \quad (9)$$

Premultiplying equations (8) and (9) by the transposes of the j^{th} and i^{th} modal columns respectively, we can write

$$\{q_j\}^T [K] \{q_i\} = w_i^2 \{q_j\}^T [M] \{q_i\} \quad (10)$$

$$\{q_i\}^T [K] \{q_j\} = w_j^2 \{q_i\}^T [M] \{q_j\} \quad (11)$$

taking the transpose of both sides of equation (10) and noting that both $[M]$ and $[K]$ are symmetric matrices, we obtain

$$\{q_i\}^T [K] \{q_j\} = w_i^2 \{q_i\}^T [M] \{q_j\} \quad (12)$$

Subtracting equation (12) from equation (11),

$$0 = (w_j^2 - w_i^2) \{q_i\}^T [M] \{q_j\} \quad (13)$$

From above equation, it can be concluded that for distinct eigenvalues,

$$\{q_i\}^T [M] \{q_j\} = 0 \quad (14)$$

Similarly, it can be shown that for $w_i \neq w_j$,

$$\{q_i\}^T [K] \{q_j\} = 0 \quad (15)$$

Equations (14) and (15) are the orthogonality conditions for the normal modes. If there are repeated eigenvalues, each of the associated normal modes will be orthogonal to any normal mode associated with a different eigenvalue⁽²⁾.

(4) Any square matrix of order $n \times n$ can be diagonalized⁽¹⁶⁾ (transformation of a matrix to a diagonal matrix) provided that the matrix has n linearly independent eigenvectors.

The following example will exhibit some of the above properties:

$$\text{Let } [B] = \begin{bmatrix} 78 & -60 & 15 \\ 150 & -117 & 30 \\ 200 & -160 & 43 \end{bmatrix}$$

The eigenvalues of this matrix are

$$E_1 = 3$$

$$E_2 = 3$$

$$E_3 = -2$$

Because two of the eigenvalues are the same and the matrix is unsymmetric, it is not clear at this point whether or not $[B]$ can be diagonalized. However, for the repeated eigenvalue $E = 3$, the solution of the resulting homogeneous equations has two linearly independent solutions:

$$X_1 = B_1 \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} \qquad X_2 = B_2 \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$$

That the above solutions are linearly independent can be verified from the fact that the vector X_1 cannot be expressed as a linear combination of vector X_2 . Since the vectors are linearly independent, the matrix can be diagonalized. The third eigenvalue gives an eigenvector equal to

$$X_3 = B_3 \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Letting $B_1 = B_2 = B_3 = 1$, the modal matrix Q , the columns of which are vectors X_1 , X_2 , and X_3 , become

$$[Q] = \begin{bmatrix} 4 & -1 & 3 \\ 5 & 0 & 6 \\ 0 & 5 & 8 \end{bmatrix}$$

$$[Q]^{-1} = \frac{1}{5} \begin{bmatrix} 30 & -23 & 6 \\ 40 & -32 & 9 \\ -25 & 20 & -5 \end{bmatrix}$$

and

$$[Q]^{-1} [B] [Q] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = [D]$$

where $[D]$ is a diagonal matrix the elements of which are eigenvalues of the original matrix.

From the above example, it is clear that the eigenvectors associated with repeated eigenvalues of an unsymmetric matrix are not necessarily orthogonal to one another (as can be observed from vectors X_1 and X_3 or X_1 and X_2).

Some definitions pertaining to a symmetric matrix which have been used in Section (IV), will be mentioned here⁽¹⁵⁾.

A symmetric matrix is said to be Positive Definite if all of its eigenvalues are positive.

If all eigenvalues are positive and at least one is zero, the matrix is Positive Semidefinite.

A matrix that may be either Positive Semidefinite or Positive Definite is called Nonnegative Definite.

APPENDIX C

Parallel Four-pole Networks

To derive the formulas for a parallel four-pole network, we have

$$F_1 = A_{11} F_2 + A_{12} V_2 \quad (1)$$

$$V_1 = A_{21} F_2 + A_{22} V_2 \quad (2)$$

Solving the above equations simultaneously for V_2 gives,

$$V_2 = \frac{V_1 A_{11} - A_{21} F_1}{A_{11} A_{22} - A_{12} A_{21}} \quad (3)$$

From equations (1) and (3),

$$F_2 = \frac{F_1}{A_{11}} - \left(\frac{A_{11} V_1 - F_1 A_{21}}{A_{11} A_{22} - A_{21} A_{12}} \right) \frac{A_{12}}{A_{11}} \quad (4)$$

It should be observed that for individual four-pole parameters of mass, spring, and damper, the following relationship holds⁽⁵⁾:

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = 1 \quad (5)$$

The physical significance of this relationship is that these elements obey the reciprocity principle. However, the above relationship is not necessarily true for all systems which follow the reciprocity principle.

Equations (1) through (5) are written for the spring and damper individually. The total force carried by the spring and the damper is then equated to the force at the input or at the output point. The desired relationships can then be obtained after solving some simultaneous equations.

APPENDIX DSample Problem:

Often practicing engineers experience considerable difficulty in deciding upon a method for solution of vibration problems. Appropriate selection of a method can result in great savings of time and labor without sacrifice of accuracy. A problem is presented below to illustrate this point and also to demonstrate the use of the mathematical theory presented. For this purpose, the system shown in figure 18 is selected. This system is an approximation of a vehicle driven on a rough road. It is assumed that (1) the vehicle is constrained to three degrees of freedom in the vertical direction and the rotational motion of any of the masses does not occur; and (2) the tires do not leave the road surface*.

In figure 18, M_1 represents the mass of passengers in the vehicle; M_2 is the sprung mass of the vehicle and M_3 is the unsprung mass of the tire assembly. The values of the springs and the dampers between the masses are reasonable approximations of the physical system. The vehicle forward speed is 20 mph, and the road surface varies sinusoidally with a period of 10.5 ft. and an amplitude X_0 of 2.0 in..

It is desired to obtain the undamped natural frequencies and the mode shapes, as well as the response of the system to the harmonic excitation, by the methods presented earlier. The results will then be compared.

*This assumption permits a continuous displacement input to the system.

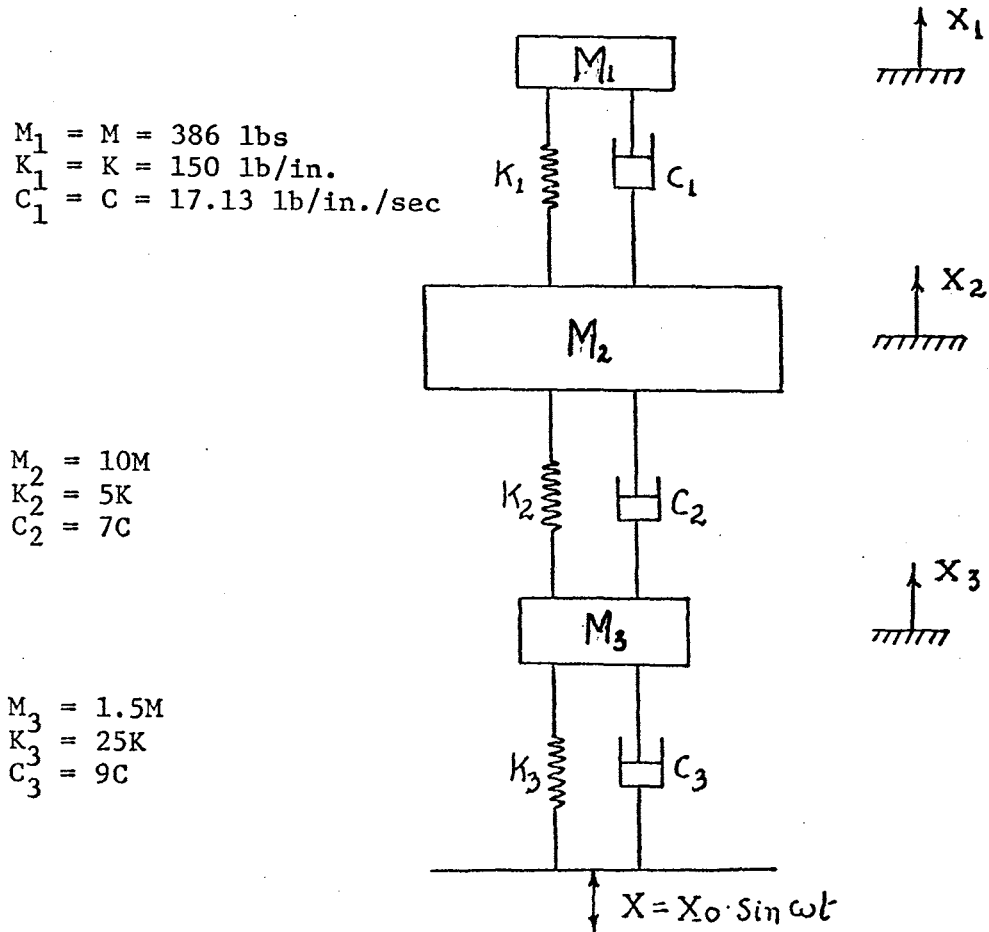


Figure 18

A Damped Three Degree-of-freedom System
 Excited by Foundation Motion

Solution:

$$\text{Speed} = 20 \text{ mph} = 28.8 \text{ fps.}$$

The excitation frequency is

$$\omega = \frac{28.8}{10.5} = 2.74 \text{ cps} = 17.2 \text{ rad/sec}$$

Normal Mode Technique:

Applying Newton's law of motion, the equations of motion of the system are obtained as

$$M_1 \ddot{X}_1 + C_1 (\dot{X}_1 - \dot{X}_2) + K_1 (X_1 - X_2) = 0 \quad (1)$$

$$M_2 \ddot{X}_2 + C_2 (\dot{X}_2 - \dot{X}_3) + K_2 (X_2 - X_3) - C_1 (\dot{X}_1 - \dot{X}_2) - K_1 (X_1 - X_2) = 0 \quad (2)$$

$$M_3 \ddot{X}_3 + C_3 \dot{X}_3 + K_3 X_3 - K_2 (X_2 - X_3) - C_2 (\dot{X}_2 - \dot{X}_3) = F_0 \quad (3)$$

where F_0 is a complex number and is given by

$$F_0 = K_3 X_0 \sin \omega t + C_3 \omega X_0 \cos \omega t \quad (4)$$

The equations of motion in the matrix form can be expressed as

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{X}_3 \end{Bmatrix} + \begin{bmatrix} C_1 & -C_1 & 0 \\ -C_1 & C_1 + C_2 & -C_2 \\ 0 & -C_2 & C_2 + C_3 \end{bmatrix} \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{Bmatrix} + \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K_2 + K_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_0 \end{Bmatrix} \quad (5)$$

Substituting the values of masses, damping coefficients and spring constants in equation (5),

$$\begin{bmatrix} M & 0 & 0 \\ 0 & 10M & 0 \\ 0 & 0 & 1.5M \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{X}_3 \end{Bmatrix} + \begin{bmatrix} C & -C & 0 \\ -C & 8C & -7C \\ 0 & -7C & 16C \end{bmatrix} \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{Bmatrix} + \begin{bmatrix} K & -K & 0 \\ -K & 6K & -5K \\ 0 & -5K & 30K \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_0 \end{Bmatrix} \quad (6)$$

Undamped Free Vibrations: For an undamped free system, equation (6) reduces to

$$\begin{bmatrix} M & 0 & 0 \\ 0 & 10M & 0 \\ 0 & 0 & 1.5M \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{X}_3 \end{Bmatrix} + \begin{bmatrix} K & -K & 0 \\ -K & 6K & -5K \\ 0 & -5K & 30K \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (7)$$

Substitution of the trial solution

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \sin \omega t \quad (8)$$

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into the equations of motion leads to

$$\begin{bmatrix} K-M\omega^2 & -K & 0 \\ -K & 6K-10M\omega^2 & -5K \\ 0 & -5K & 30K-1.5M\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (9)$$

Setting the determinant of the coefficient matrix equal to zero results in the characteristic equation

$$w^6 - 21.6 \frac{K}{M} w^4 + 31.0 \frac{K^2}{M^2} w^2 - 8.34 \frac{K^3}{M^3} = 0 \quad (10)$$

which yields the three eigenvalues

$$\begin{aligned} w_1^2 &= 0.363 \frac{K}{M} \\ w_2^2 &= 1.17 \frac{K}{M} \\ w_3^2 &= 20.1 \frac{K}{M} \end{aligned} \quad (11)$$

Substituting for K and M in equation (11),

$$w_1 = 7.38 \text{ rad/sec}$$

$$w_2 = 13.25 \text{ rad/sec}$$

$$w_3 = 54.9 \text{ rad/sec}$$

To obtain the eigenvectors, substitute the first eigenvalue in equation (9),

$$\begin{bmatrix} .637K & -K & 0 \\ -K & 2.37K & -5K \\ 0 & -5K & 29.45K \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the homogeneous equations to be solved are

$$.637 A_1 - A_2 = 0$$

$$-A_1 + 2.37 A_2 - 5 A_3 = 0$$

$$-5 A_2 + 29.45 A_3 = 0$$

Setting arbitrarily the amplitude of the first mass to unity, the mode shape at the first natural frequency is

$$X_1 = \begin{Bmatrix} 1.0 \\ .637 \\ .108 \end{Bmatrix}$$

Repeating the above procedure for the remaining eigenvalues, the corresponding mode shapes are

$$X_2 = \begin{Bmatrix} 1.0 \\ -.17 \\ -.032 \end{Bmatrix}, \quad X_3 = \begin{Bmatrix} 1.0 \\ -19.1 \\ 760 \end{Bmatrix}$$

The modal matrix whose columns are the vectors X_1 , X_2 , and X_3 , becomes

$$[Q] = \begin{bmatrix} 1.0 & 1.0 & 1 \\ .637 & -.17 & -19.1 \\ .108 & -.032 & 760 \end{bmatrix}$$

Forced Vibrations With Damping: To obtain the response to the harmonic excitation, we first determine whether the system is classically or non-classically damped. Thus,

$$[Q]^T [M] [Q] = \begin{bmatrix} 5.067M & 0 & 0 \\ 0 & 1.29M & 0 \\ 0 & 0 & 8.7 \times 10^5 M \end{bmatrix} = [\bar{M}]$$

$$[Q]^T [K] [Q] = \begin{bmatrix} 1.82K & 0 & 0 \\ 0 & 1.45K & 0 \\ 0 & 0 & 1.74 \times 10^7 \end{bmatrix} = [\bar{K}]$$

$$[Q]^T [C] [Q] = c \begin{bmatrix} 2.193 & -.72 & -2130 \\ -.118 & 1.263 & 555 \\ -2140 & 575 & 9.45 \times 10^6 \end{bmatrix}$$

It is observed from the above matrices that the transformation which diagonalizes the M and K matrices does not diagonalize the damping

matrix and therefore the system is non-classically damped. This requires the use of Foss's method to obtain the system response.

As already defined in section IV,

$$[u] = \begin{bmatrix} [0] & [-I] \\ [K]^{-1} [M] & [K]^{-1} [C] \end{bmatrix}$$

Now

$$[K]^{-1} [M] = \frac{M}{25K} \begin{bmatrix} 31 & 60 & 1.5 \\ 6 & 60 & 1.5 \\ 1 & 10 & 1.5 \end{bmatrix}$$

and

$$[K]^{-1} [C] = \frac{C}{25K} \begin{bmatrix} 25 & 10 & -26 \\ 0 & 35 & -26 \\ 0 & 0 & 9 \end{bmatrix}$$

Foss's method requires the solution of the characteristic equation in $2N$ space,

$$\| [u + \frac{I}{\alpha}] \| = 0$$

Substituting for u , the characteristic equation becomes,

$$\begin{vmatrix} \frac{1}{\alpha} & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 & 0 & -1 \\ \frac{31M}{25K} & \frac{60M}{25K} & \frac{1.5M}{25K} & \frac{C}{K} + \frac{1}{\alpha} & \frac{10C}{25K} & \frac{-26C}{25K} \\ \frac{6M}{25K} & \frac{60M}{25K} & \frac{1.5M}{25K} & 0 & \frac{35C}{25K} + \frac{1}{\alpha} & \frac{-26C}{25K} \\ \frac{M}{25K} & \frac{10M}{25K} & \frac{1.5M}{25K} & 0 & 0 & \frac{9C}{25K} + \frac{1}{\alpha} \end{vmatrix} = 0$$

Solving the above determinant of order 6×6 is a long and tedious task.

This is one of the practical limitations of Foss's method even though

mathematically the solution is obtainable. Only a portion of the solution method is presented here, because the calculations are impractical without use of a digital computer. Refer to section IV for a summary of the remainder of the solution.

To obtain an approximate solution by the Normal Mode Technique, the off-diagonal terms of the damping matrix can be ignored⁽²⁾.

Consider the following equation which has been derived in section IV:

$$[\bar{M}]\{\ddot{n}\} + [\bar{C}]\{\dot{n}\} + [\bar{K}]\{n\} = [Q]^T\{F\}$$

Here $[\bar{C}]$ becomes,

$$[\bar{C}] = \begin{bmatrix} 2.193C & 0 & 0 \\ 0 & 1.263C & 0 \\ 0 & 0 & 9.45 \times 10^6 C \end{bmatrix}$$

and

$$[Q]^T [F] = \begin{Bmatrix} .108F_o \\ -.032F_o \\ 760F_o \end{Bmatrix}$$

The uncoupled equations of motion in the matrix form are

$$\begin{bmatrix} 5.067M & 0 & 0 \\ 0 & 1.29M & 0 \\ 0 & 0 & 8.7 \times 10^5 M \end{bmatrix} \begin{Bmatrix} \ddot{n}_1 \\ \ddot{n}_2 \\ \ddot{n}_3 \end{Bmatrix} + \begin{bmatrix} 2.193C & 0 & 0 \\ 0 & 1.263C & 0 \\ 0 & 0 & 9.45 \times 10^6 C \end{bmatrix} \begin{Bmatrix} \dot{n}_1 \\ \dot{n}_2 \\ \dot{n}_3 \end{Bmatrix} + \begin{bmatrix} 1.82K & 0 & 0 \\ 0 & 1.45K & 0 \\ 0 & 0 & 1.74 \times 10^7 K \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} .108F_o \\ -.032F_o \\ 760F_o \end{Bmatrix} \quad (12)$$

The steady state response is obtained by the equation

$$\eta_i = \frac{F_i}{[(K_{ii} - M_i \omega^2) + (C_i \omega)^2]^{1/2}} \quad (13)$$

and the phase angle ϕ_i of η_i with respect to F_i is given by

$$\phi_i = \tan^{-1} \frac{C_i \omega}{K_{ii} - M_i \omega^2} \quad (14)$$

which is a lag angle.

Using equations (13) and (14), the following results are obtained

$$\begin{aligned} \eta_1 &= 0.721, \quad \phi_1 = -28^\circ \\ \eta_2 &= -.702, \quad \phi_2 = -66.5^\circ \\ \eta_3 &= .00193, \quad \phi_3 = 49.5^\circ \end{aligned}$$

Using $\{X_i\} = [Q]\{\eta_i\}$, the system response to the harmonic excitation is,

$$\begin{aligned} X_1 &= .021 \angle 17^\circ \\ X_2 &= .5764 \angle -31.5^\circ \\ X_3 &= 1.564 \angle 84.5^\circ \end{aligned}$$

The phase angle of X_i is with respect to X_0 (at 0°). The vector F_0 given by equation (4) leads X_0 by an angle of 35° . Consequently this angle is added to ϕ_i (which is with respect to F_0) to obtain the phase angle of X_i with respect to X_0 .

The damped natural frequency ω_{di} can be calculated by

$$\omega_{di} = \sqrt{1 - b_i^2} \omega_{ni}$$

where w_{ni} is the corresponding undamped natural frequency and b_i is the damping ratio, given by

$$b_i = \frac{C_i}{2\sqrt{K_i M_i}}$$

The damped natural frequencies are found to be

$$w_{d1} = 6.36 \text{ rad/sec}$$

$$w_{d2} = 10.0 \text{ rad/sec}$$

For w_{d3} , b_3 is greater than one and therefore, the system becomes overdamped. Hence, vibratory motion will not exist in this case.

The system response obtained by approximating the damped system will be compared later with the exact response obtained by Holzer's Method or by the Impedance Method.

Holzer's Method:

Undamped Free Vibrations: With no foundation motion ($X=0$), the end support has zero displacement. This is the boundary condition for construction of Holzer table. We shall use the values of the natural frequencies obtained by the Normal Mode Technique as the trial values for the Holzer tables.

Holzer Table for First Natural Frequency

$$w^2 = 0.363 \frac{K}{M}$$

Mass No	M	Mw ²	X _i	Mw ² X _i	∑Mw ² X _i	K _{i,i+1}	$\frac{\sum Mw^2 X_i}{K_{i,i+1}}$
1	M	.363K	1.0	.363K	.363K	K	.363
2	10M	3.63K	0.637	2.31K	2.673K	5K	.534
3	1.5M	.5445K	0.103	0.056K	2.73K	25K	.108
							-.005

Since the remainder $X_i = -.005 \approx 0$,

$$w_1^2 = 0.363 \frac{K}{M}$$

or

$$w_1 = 7.38 \text{ rad/sec}$$

Holzer Table for Second Natural Frequency

Starting with a trial value of $w^2 = 1.17 \frac{K}{M}$ for the second natural frequency, several trials were made and the corresponding remainder displacements found. These are tabulated below:

w ²	Remainder Amplitude
1.17 $\frac{K}{M}$	0.027
1.16 $\frac{K}{M}$	0.0071
1.15 $\frac{K}{M}$	-0.01

A graph of the remainder displacement versus w^2 is shown on the next page. The points lie on a straight line which crosses w^2 axis at

$$w^2 = 1.155 \frac{K}{M}$$

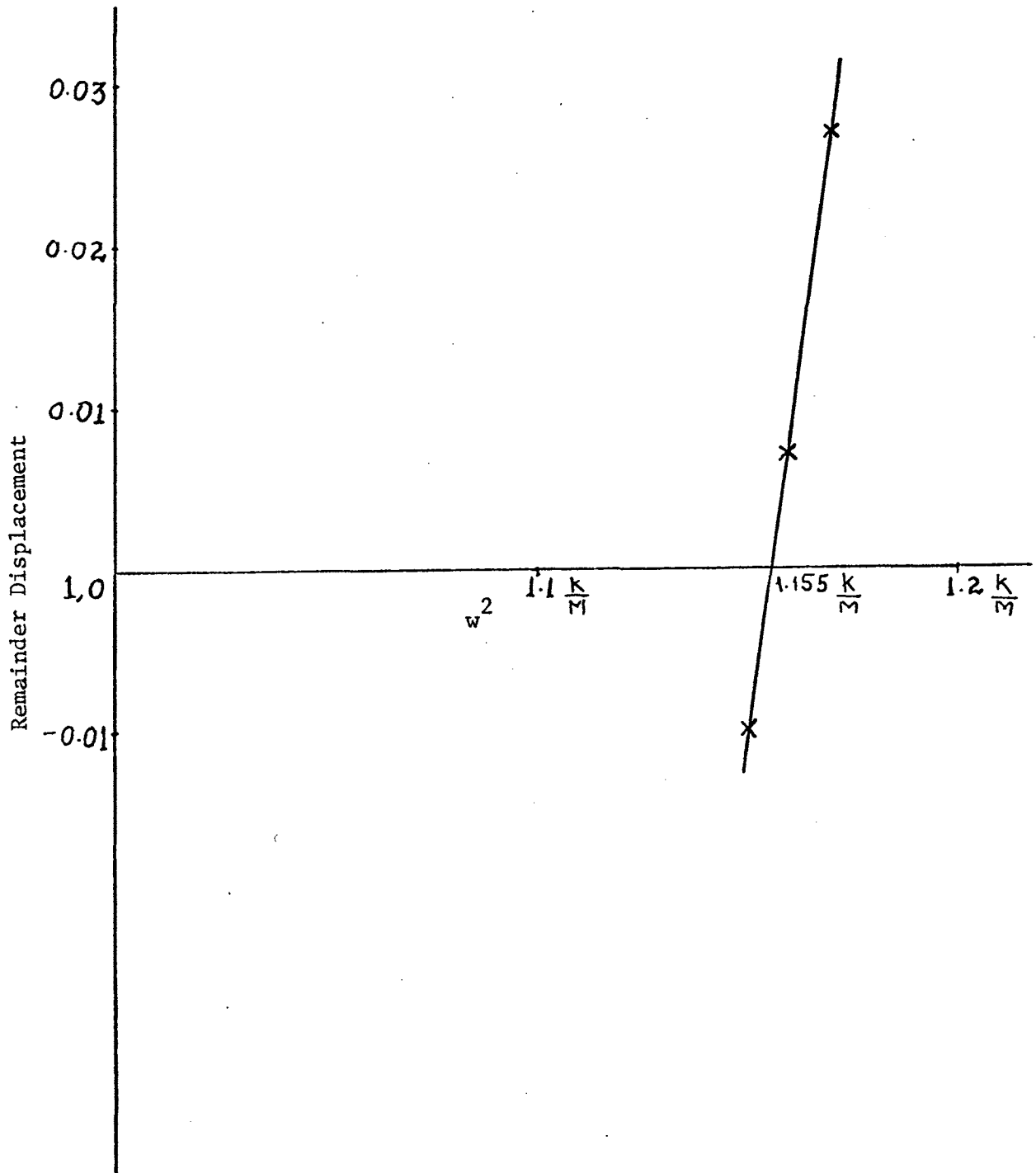


Figure 19

Remainder Displacement From Holzer's Method

Therefore, the next trial value is selected as $w^2 = 1.155 \frac{K}{M}$.

$$w^2 = 1.155 \frac{K}{M}$$

Mass No	M	Mw^2	X_i	$Mw^2 X_i$	$\sum Mw^2 X_i$	$K_{i,i+1}$	$\frac{\sum Mw^2 X_i}{K_{i,i+1}}$
1	M	1.155K	1.0	1.155K	1.155K	K	1.155
2	10M	11.55K	-.155	-1.79K	-.635K	5K	-.127
3	1.5M	1.73K	-.028	-.0485	-.6835	25K	-.0274
							-.0006

The magnitude of the remainder displacement is very close to zero and therefore,

$$w_2^2 = 1.155 \frac{K}{M}$$

or

$$w_2 = 13.16 \text{ rad/sec}$$

Similarly the third natural frequency is

$$w_3 = 54.9 \text{ rad/sec}$$

The mode shapes are obtained directly from the amplitude column of the Holzer tables. These are

$$X_1 = \begin{Bmatrix} 1.0 \\ 0.637 \\ 0.103 \end{Bmatrix}, \quad X_2 = \begin{Bmatrix} 1.0 \\ -.155 \\ -.028 \end{Bmatrix}, \quad X_3 = \begin{Bmatrix} 1.0 \\ -19.1 \\ 741.0 \end{Bmatrix}$$

Damped Forced Vibrations:

Using Holzer's Method, to obtain the system response to an input, only one table at the impressed frequency needs to be constructed. The Holzer table corresponding to the impressed frequency $w = 17.2 \text{ rad/sec}$

is shown on the next page. For this table, the displacement of the first mass is assumed to be X_1 . The displacements of the remaining masses are obtained in terms of X_1 . The remainder displacement (at the end support) is then equated to X_0 which is 2.0 in. in this case.

From the Holzer table, the displacement at the end support is

$$(-1.393 + 1.i)X_1 = 2 \quad (15)$$

Because of the damping in the system, X_1 is a complex number. Let

$$X_1 = a + ib$$

Substituting for X_1 in equation (15) and rearranging,

$$(-1.393a - b) + i(a - 1.393b) = 2 \quad (16)$$

Equating the real and the imaginary parts of the above equation provides the following two equations:

$$-1.393a - b = 2 \quad (17)$$

$$a - 1.393b = 0 \quad (18)$$

which gives

$$a = -.95$$

$$b = -.681$$

Therefore,

$$X_1 = 1.17 \angle 215.6^\circ$$

Since the displacements of the remaining masses* are in terms of X_1 , these can be easily calculated. Thus,

* This is obtained from the displacement column of the Holzer table.

$$\underline{\omega^2 = 294}$$

Mass Number (1)	M (2)	$M\omega^2$ (3)	X_j (4)	$M\omega^2 X_j$ (5)	$\sum M\omega^2 X_j$ (6)	$K_{j,j+1}$ $+i\omega C_{j,j+1}$ (7)	Column 6 Column 7 (8)
1	1	294	X_1	$294X_1$	$294X_1$	150+i295	$.405X_1$ $-(.794X_1)i$
2	10	2940	$.595X_1$ $+(.794X_1)i$	$1750X_1$ $+(2330X_1)i$	$2044X_1$ $+(2330X_1)i$	750+i2060	$1.315X_1$ $-(.51X_1)i$
3	1.5	441	$-.72X_1$ $+(1.30X_1)i$ $-1.393X_1$ $+(X_1)i$	$-318X_1$ $+(570X_1)i$	$1726X_1$ $+(2900X_1)i$	3750+i2660	$.673X_1$ $+(.3X_1)i$

Holzer Table for an Impressed Frequency of 17.2 rad/sec

$$X_2 = 1.16 \angle -91.2^\circ$$

and

$$X_3 = 1.74 \angle -25.5^\circ$$

The phase angles of X_1 are relative to the input X_0 .

Since there is no approximation involved, the response obtained above is exact.

For purposes of illustration, a Holzer table to obtain a damped natural frequency is presented on the next page. Using w_{d2} obtained from the first method, as a trial value, the table is constructed as explained in section V. For a trial value to be a damped natural frequency, the real part of the remainder displacement should be zero*. For $w = 10.0$ rad/sec, the real part of the remainder displacement is $-.003$ which is close to zero and therefore $w = 10$ rad/sec is one of the damped natural frequencies of the system.

Impedance Method:

In order to apply this method, the various elements of the system are connected between the points which have been numbered as shown in figure 20. Thus across points (1) and (2) are a spring and a damper in parallel and across points (2) and (3) is a mass, and so on. The velocity V_1 at point (1) is $w x$ and the force F_1 at this point is unknown. The force F_7 at point (7) is zero as it is a free end. With these boundary conditions and knowing the four-pole parameters which relate any two points,

* For detailed explanation, refer to section V.

$$\underline{\omega^2 = 100.0}$$

Mass Number	M	M ω^2	X _j	M ω^2 X _j	\sum M ω^2 X _j	K _{j,j+1} +i ω c _{m,j+1}	<u>Column 6</u> Column 7 (8)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	
1	1.0	100	1.0+0.0i	100+0i	100+0i	150+171.3i	.289-.321i
2	10.0	1000	.711+.321i	711+321i	811+321i	750+1200i	.429-.368i
3	1.5	150	.22+.689i	33+103i	844+424i	3750+1545i	.223-.0176i
							-.003+.707i

A Holzer Table for a Damped Natural Frequency

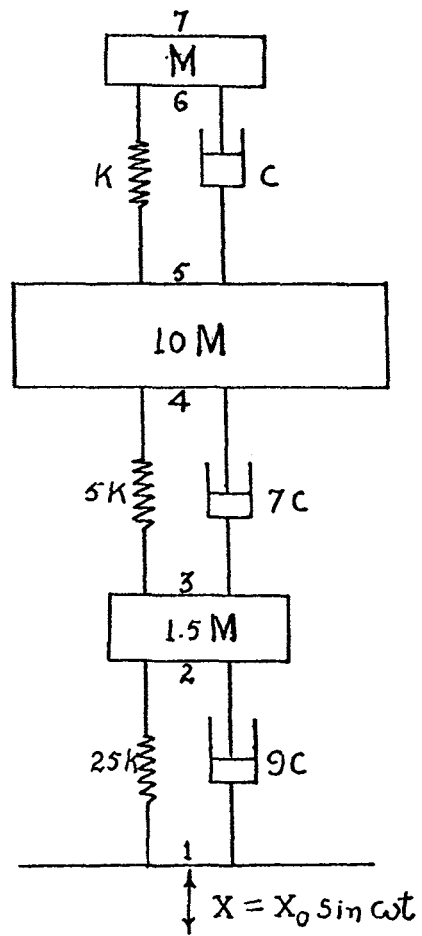


Figure 20

A Damped Three Degree-of-freedom System
With Harmonic Excitation

the steady state system response can be determined. Using the theory developed in section VI, the following relationships are obtained:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .00215+i(.00304) & 1 \end{bmatrix}^* \begin{bmatrix} F_2 \\ V_2 \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} F_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 25.8i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_3 \\ V_3 \end{bmatrix} \quad (20)$$

$$\begin{bmatrix} F_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .00736+i(.00268) & 1 \end{bmatrix} \begin{bmatrix} F_4 \\ V_4 \end{bmatrix} \quad (21)$$

$$\begin{bmatrix} F_4 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & 172i & F_5 \\ 0 & 1 & V_5 \end{bmatrix} \quad (22)$$

$$\begin{bmatrix} F_5 \\ V_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .0464+(.0236)i & 1 \end{bmatrix} \begin{bmatrix} F_6 \\ V_6 \end{bmatrix} \quad (23)$$

$$\begin{bmatrix} F_6 \\ V_6 \end{bmatrix} = \begin{bmatrix} 1 & 17.2i & 0 \\ 0 & 1 & V_7 \end{bmatrix} \quad (24)$$

Applying equation (63) derived in section VI, between points (1) and (7),

* This is matrix of four-pole parameters of spring K_3 and damper C_3 which are connected in parallel. For formulas of parallel connections, refer to section VI.

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} -5+8.04i & -170+100i \\ -.0327+.076i & -1.4+1.0i \end{bmatrix} \begin{bmatrix} 0 \\ V_7 \end{bmatrix} \quad (25)$$

Similarly for points (2) and (7),

$$\begin{bmatrix} F_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} -5+8.04i & -170+100i \\ .00246+.074i & -.73+1.31i \end{bmatrix} \begin{bmatrix} 0 \\ V_7 \end{bmatrix} \quad (26)$$

From equations (25) and (26),

$$V_1 = (-1.4 + 1.0i)V_7 \quad (27)$$

and

$$V_7 = \frac{V_2}{-.73 + 1.31i} \quad (28)$$

Substituting for V_7 from equation (28) into equation (27) and simplifying,

$$\begin{aligned} V_1 &= 1.15 V_2 \angle 25.5^\circ \\ \text{or } V_2 &= .87 V_1 \angle -25.5^\circ \\ \text{or } X_2 &= .87 X_1 \angle -25.5^\circ \end{aligned} \quad (29)$$

$$\text{Now } X_1 = X_0 = 2$$

Therefore,

$$X_2 = 1.74 \angle -25.5^\circ$$

which is the displacement vector of M_3 .

From equations (19) and (20),

$$F_1 = F_2 = F_3 + 25.8i V_3 \quad (30)$$

From equation (25),

$$F_1 = (-170 + 100i)V_7 \quad (31)$$

Substituting for V_7 from equation (28) into equation (31) and simplifying,

$$F_1 = (113 + 67i)V_2 \quad (32)$$

Substitute the above value of F_1 into equation (30), which gives

$$F_3 = (113 + 41.2i)V_2 \quad (33)$$

From equations (20) and (21),

$$V_2 = V_3 \quad (34)$$

$$F_3 = F_4 \quad (35)$$

$$V_3 = (.00736 + .00268i)F_4 + V_4 \quad (36)$$

Solving equations (33) through (36) simultaneously,

$$V_4 = (.277 - .606i)V_2 \quad (37)$$

or

$$X_4 = (.277 - .606i)X_2 \quad (38)$$

Substituting for X_2 in the above equation gives X_4 ,

$$X_4 = 1.16 \angle -91^\circ$$

which is the displacement vector of M_2 .

From equations (24) and (28),

$$V_6 = V_7 = \frac{V_2}{-.73 + 1.31i}$$

or

$$X_6 = \frac{X_2}{-.73 + 1.31i}$$

Substituting for X_2 in the above equation,

$$X_6 = 1.163 \angle 215.5^\circ$$

which is the displacement vector of M_1 .

Expressing the final results in terms of the original system coordinates,

$$x_1 = 1.163 \angle 215.5^\circ$$

$$x_2 = 1.16 \angle -91^\circ$$

$$x_3 = 1.74 \angle -25.5^\circ$$

These results are in good agreement with those obtained by Holzer's Method.

Graphical Technique:

For obtaining the steady state system response by this method, define the coordinates of the system as shown in figure 21. This permits the direct application of equation (91) derived in section VII.

The individual spring-mass combination frequencies are

$$w_{01} = 50.0 \text{ rad/sec}$$

$$w_{02} = 8.66 \text{ rad/sec}$$

$$w_{03} = 12.25 \text{ rad/sec}$$

and

$$\frac{w}{w_{01}} = .345, \quad \frac{w}{w_{02}} = 1.985, \quad \frac{w}{w_{03}} = 1.4$$

where

$$w = 17.2 \text{ rad/sec}$$

The ratio of the damping coefficient and the critical damping for individual M, K, and C is

$$b_1 = 1.0, \quad b_2 = 0.692, \quad b_3 = 0.7$$

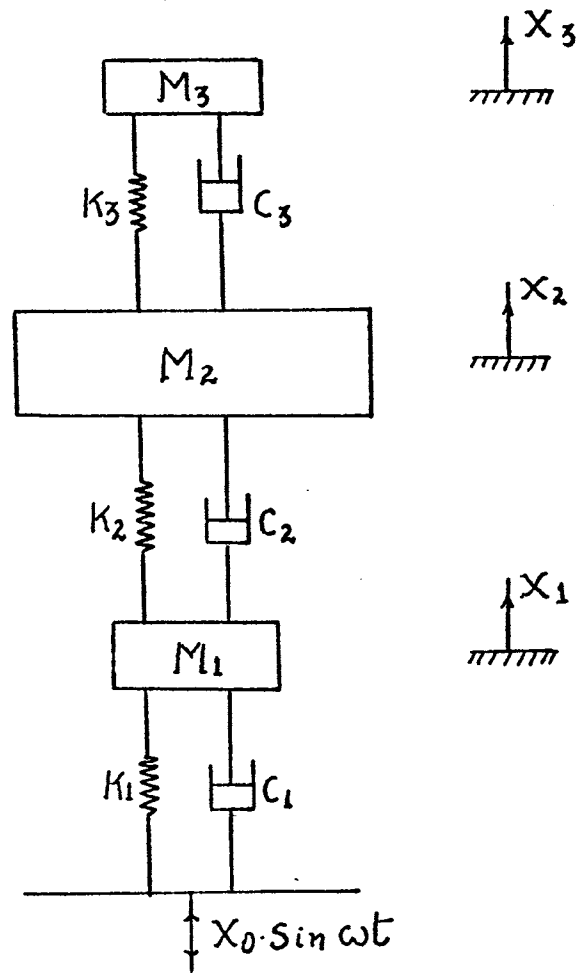


Figure 21

A Damped Three Degree-of-freedom
System With Harmonic Excitation

Corresponding to $\frac{w}{w_{oj}}$ and b_j , the value of Td_j is read from the displacement transmissibility curve^(1,2). Tf_j is then obtained by the relation

$$Tf_j = 1 - Td_j$$

The values of Td_j and Tf_j are presented below in a polar notation:

$$\begin{aligned} Td_1 &= 1.1 \angle 0^\circ, & Tf_1 &= 0.1 \angle 180^\circ \\ Td_2 &= 0.72 \angle -60^\circ, & Tf_2 &= 0.885 \angle 42.4^\circ \\ Td_3 &= 1.0 \angle -54^\circ, & Tf_3 &= 0.918 \angle 62^\circ \end{aligned}$$

Using equation (91) of section VII, the amplitude ratios of the various coordinates can be expressed in terms of Td_j and Tf_j by the following equations:

$$\frac{X_3}{X_2} = Td_3 \quad (39)$$

$$\begin{aligned} \frac{X_2}{X_1} &= Td_2 \left[1 + \frac{M_3}{M_2} \frac{Td_3}{Tf_3} Tf_2 \left(1 - \frac{X_3}{X_2} \right) \right]^{-1} \\ &= Td_2 \left[1 + \frac{M_3}{M_2} Td_3 Tf_2 \right]^{-1} \end{aligned} \quad (40)$$

$$\frac{X_1}{X_o} = Td_1 \left[1 + \frac{M_2}{M_1} \frac{Td_2}{Tf_2} Tf_1 \left(1 - \frac{X_2}{X_1} \right) \right]^{-1} \quad (41)$$

Equation (39) gives

$$\frac{X_3}{X_2} = 1.0 \angle -54^\circ \quad (42)$$

Equations (40) and (41) can be solved graphically as explained in section VII. The vector diagram for solving equation (40) is shown

on the next page. Thus,

$$\begin{aligned}\frac{X_2}{X_1} &= \frac{.72 \angle -60^\circ}{1 + \frac{1}{10} (1 \angle -54^\circ) (.885 \angle 42.4^\circ)} \\ &= \frac{.72 \angle -60^\circ}{1 + .0885 \angle -11.6^\circ} \\ &= \frac{.72 \angle -60^\circ}{1.086 \angle -2^\circ}\end{aligned}$$

The value of $1.086 \angle -2^\circ$ is obtained from the vector diagram. Thus

$$\frac{X_2}{X_1} = 0.663 \angle -58^\circ \quad (43)$$

Substituting the numerical values in equation (41) and simplifying,

$$\frac{X_1}{X_0} = 1.125 \angle -27.6^\circ \quad (44)$$

Expressing the amplitude ratios in terms of the original system coordinates,

$$\frac{X_3}{X_0} = 1.125 \angle -27.6^\circ$$

$$\frac{X_2}{X_3} = .663 \angle -58^\circ$$

$$\frac{X_1}{X_2} = 1.0 \angle -54^\circ$$

which gives

$$X_3 = 2.25 \angle -27.6^\circ$$

$$X_2 = 1.49 \angle -85.6^\circ$$

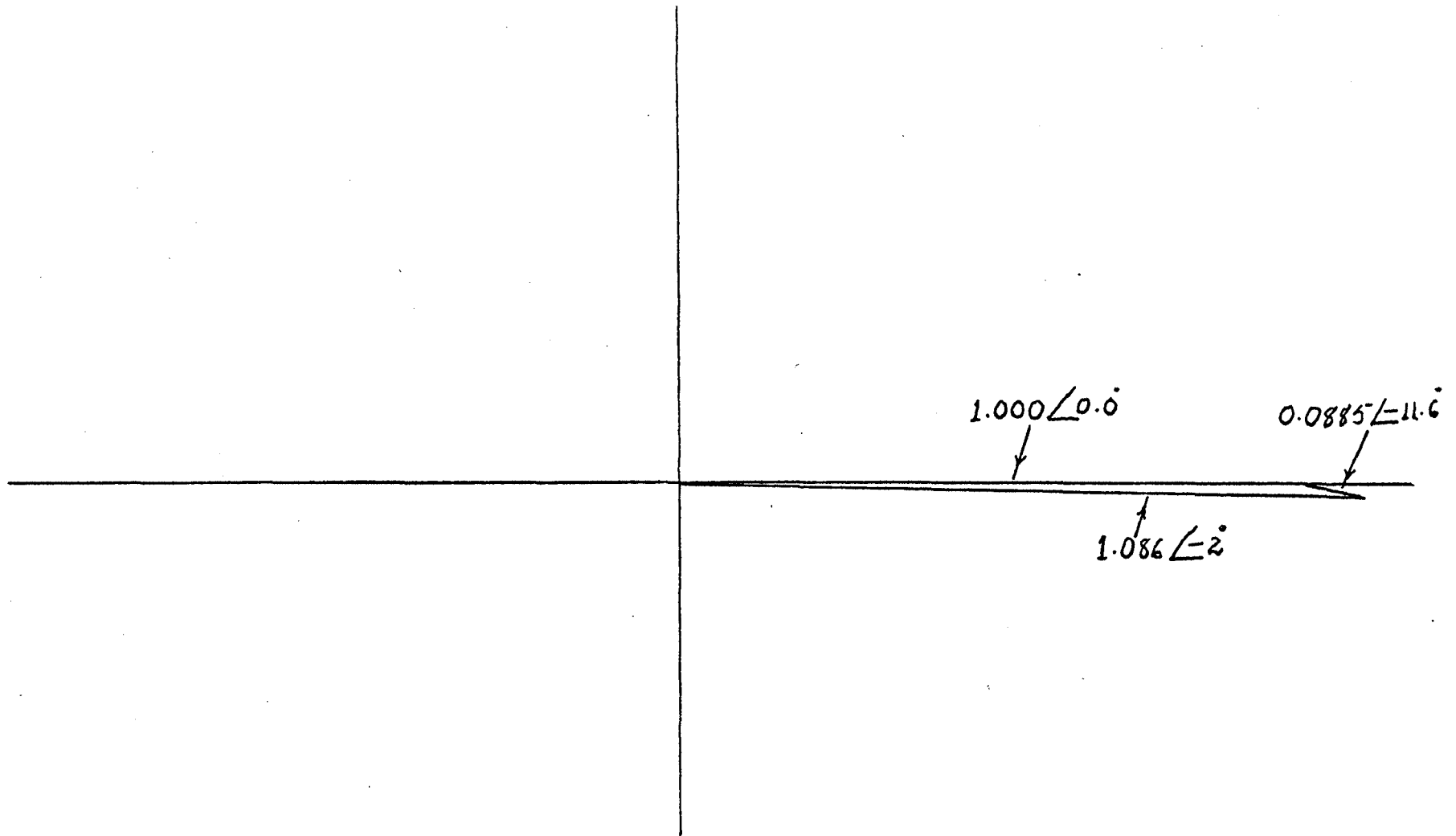


Figure 22

Vector Diagram for the Graphical Technique

$$X_1 = 1.49 \angle -139.6^\circ$$

or

$$X_1 = 1.49 \angle 220.4^\circ$$

The above displacements do not agree with those obtained by Holzer's Method or by the Impedance Method. The discrepancy lies in the ratio $\frac{X_3}{X_0}$. From Holzer's Method,

$$\frac{X_3}{X_0} = 0.87 \angle -25.5^\circ$$

whereas by this method,

$$\frac{X_3}{X_0} = 1.125 \angle -27.6^\circ$$

The calculations for $\frac{X_3}{X_0}$ were checked and rechecked. However, extensive investigations failed to provide the desired ratio. The source of this deviation could not be traced.

The remaining amplitude ratios obtained by the Graphical Technique are reasonably close to those obtained by Holzer's Method.

Reduced System:

The impressed frequency w_0 is 17.2 rad/sec and the individual spring-mass combination frequencies are

$$w_{n1} = 12.25, w_{n2} = 8.66, w_{n3} = 50.0$$

According to the theory presented in section VIII, M_3 and C_3 should be eliminated. The reduced system is shown in figure 23.

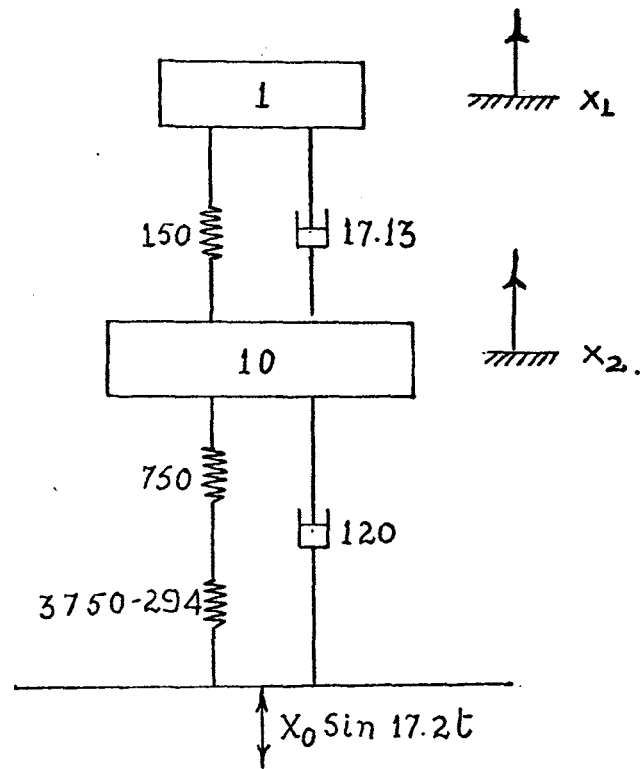


Figure 23

Initial Reduction of the Original System

The above figure is equivalent to the system shown below,

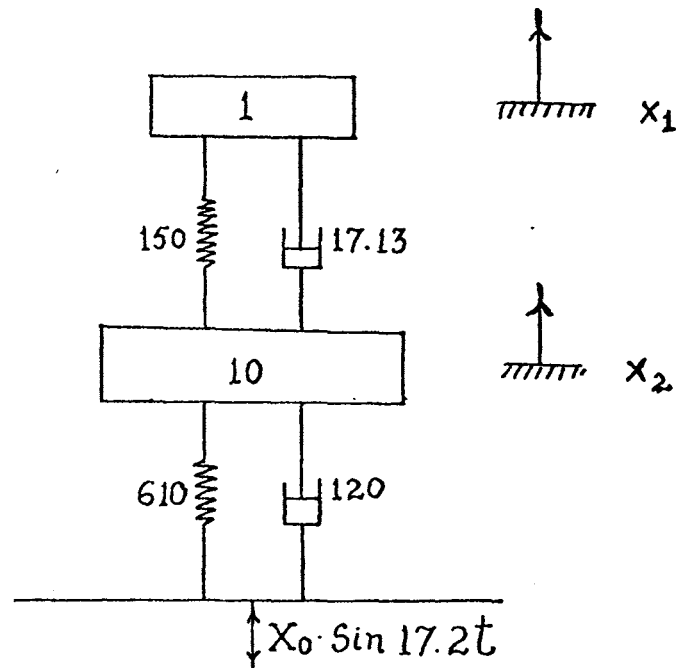


Figure 24

Reduced Two-Degrees-of-freedom System

Undamped Free Vibrations: Using Holzer's Method, the following undamped natural frequencies and the corresponding mode shapes are obtained:

$$w_1 = 7.25 \text{ rad/sec}, \quad X_1 = \begin{Bmatrix} 1.0 \\ .65 \end{Bmatrix}$$

$$w_2 = 13.16 \text{ rad/sec}, \quad X_2 = \begin{Bmatrix} 1.0 \\ -.155 \end{Bmatrix}$$

These results are close to the exact solution obtained by Holzer's Method.

Damped Forced Vibrations: To obtain the response of the reduced system to the harmonic excitation, a Holzer's table similar to the one presented for the exact solution is constructed. This table appears on the next page.

Equating the remainder displacement to X_0 gives two simultaneous equations. Thus,

$$-(.715)(a + ib) + i1.394(a + ib) = 2$$

where

$$X_1 = a + ib$$

As before, the real and the imaginary parts of the above equation are equated which gives

$$a = -.583, \quad b = -1.13$$

Therefore,

$$X_1 = 1.27 \angle 242.8^\circ$$

Substituting for X_1 in the expression for X_2 (obtained from the displacement column of the Holzer's table),

$$\underline{w^2 = 294.0}$$

Mass Number (1)	M (2)	Mw ² (3)	X _j (4)	$\sum Mw^2 X_j$ (5)	Mw ² X _j (6)	K _{j,j+1} +iwC _{j,j+1} (7)	<u>Column 6</u> <u>Column 7</u> (8)
1	1	294	X ₁	294X ₁	294X ₁	150+i295	.405X ₁ -(.794X ₁)i
2	10	2940	595X ₁ +(.794X ₁)i -.715X ₁ +(1.394X ₁)i	1750X ₁ +(2330X ₁)i	2044X ₁ +(2330X ₁)i	610+i2060	1.31X ₁ -(.6X ₁)i

Holzer Table for Impressed Frequency of 17.2 rad/sec

$$x_2 = 1.266 \angle -64^\circ$$

The magnitudes of the displacements of the masses are reasonably close to the exact system response. However, the phase angles of the displacement vectors are in considerable error. An interesting point is that the ratio for $\frac{x_1}{x_2}$ obtained by the actual solution as well as by reducing the system is same.

It should be pointed out that approximating the system by ignoring the off-diagonal terms of the damping matrix can result in serious error. This is the case for this problem as can be observed by comparing the exact solution and the approximate solution. For a discussion on comparison of various methods, the reader is referred to section IX. The final results obtained by various methods are presented in a tabular form on the next page.

Method	Natural Frequency	Mode Shapes (Undamped)	Amplitude Ratios (Forced)
Normal Mode	7.38	$\begin{bmatrix} 1.0 & .637 & .108 \end{bmatrix}$.782 $\angle 84.5^\circ$
	13.25	$\begin{bmatrix} 1.0 & -.17 & -.032 \end{bmatrix}$.368 $\angle -116^\circ$
	54.9	$\begin{bmatrix} 1.0 & -19.1 & 760 \end{bmatrix}$.036 $\angle 38.5^\circ$
			**
Holzer	7.38	$\begin{bmatrix} 1.0 & .637 & .103 \end{bmatrix}$.87 $\angle -25.5^\circ$
	13.16	$\begin{bmatrix} 1.0 & -.155 & -.028 \end{bmatrix}$.666 $\angle -65.7^\circ$
	54.9	$\begin{bmatrix} 1.0 & -19.1 & 741 \end{bmatrix}$	1.06 $\angle -53.2^\circ$
Impedance	*	*	.87 $\angle -25.5^\circ$
			.666 $\angle -65.5^\circ$
			1.02 $\angle -53.5^\circ$
Graphical	*	*	1.125 $\angle -27.6^\circ$
			.663 $\angle -58^\circ$
			1.0 $\angle -54^\circ$
Reduced System	7.25	$\begin{bmatrix} 1.0 & .65 \end{bmatrix}$	
	13.16	$\begin{bmatrix} 1.0 & -.155 \end{bmatrix}$	1.0 $\angle -53.2^\circ$

Comparison of Results Obtained by Various Methods

**These ratios are for the approximate system obtained by ignoring the off-diagonal terms of the damping matrix.

*Values cannot be determined.

For the above table, the natural frequencies and the mode shapes are listed in the serial order. Amplitude ratios are in the following order:

$$\frac{x_3}{x_0}$$

$$\frac{x_2}{x_3}$$

$$\frac{x_1}{x_2}$$

VITA

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Immediately thereafter he was employed by Voltas Limited, Bombay, in the Earthmoving and Mining Division. He attained the position of Senior Engineer - Sales before leaving Voltas in December 1966.

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