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NUMERICAL SIMULATION OF ONE-DIMENSIONAL THREE-PHASE FLUID FLOW IN A STRATIFIED PETROLEUM RESERVOIR WITH NO CROSS-FLOW

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Approved by

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ABSTRACT

A numerical model for linear, three-phase fluid flow in a stratified petroleum reservoir has been developed for the purpose of studying water-flooding performance characteristics. Gravity forces, capillary pressure and cross-flow were neglected, but the fluids were considered compressible and evaluations of dissolved gas concentrations have been accounted for.

The model was used in evaluating the performance of a water flood project for a hypothetical reservoir which contained the three phases of oil, gas, and water. The results were then compared with those obtained using the Dykstra-Parsons technique. The model produces results which were found to be similar to those obtained by the Dykstra-Parsons method for the displacement of oil by water with no gas present, but poor agreement resulted when free gas saturations were formed. The results obtained using the numerical model are considered to be more realistic than those obtained by the Dykstra-Parsons method in depleted oil sands with a substantial gas saturation since the Dykstra-Parsons model was built on the assumption that only one phase flows at a point and water displace oil in a piston-like manner. When a free gas saturation is formed at the production well, the oil mobility is reduced substantially resulting in a lower oil recovery at water breakthrough than can be predicted by the Dykstra-Parsons technique.

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I. INTRODUCTION

Fluid injection methods have been used for many years for secondary and tertiary recovery and have contributed substantially to marked improvements in oil recovery. The number of such projects should increase in the future due to the rising costs of discovering and developing primary reserves.

To evaluate any proposed fluid injection program, it is first necessary to know how much oil can be recovered economically. This information must include accurate knowledge of oil rates, water-oil ratios, and gas-oil ratios with time or pressure declines. In order to best evaluate these parameters, it is desirable to simulate (by a complete mathematical description) the multi-phase fluid flow in the porous medium, and the complex interaction of natural forces and physical properties of a petroleum reservoir during depletion with various fluid injection programs.

The approach undertaken in this project, to simulate this multi-phase flow, was to sum the three continuity equations for the three mobil phases (oil, water and gas) in one dimension, and to force the sum of saturation derivatives to vanish, thus forming one partial differential equation in pressure, dependent upon the simultaneous multi-phase flow in the reservoir. The differential equation, with pressure as the dependent variable, was then replaced by a finite difference equation to expedite solution. Writing this equation about each discrete point in the system results in a tridiagonal matrix with coefficients which are both pressure and saturation dependent. This tridiagonal matrix is solved for pressure and yields pressure variances at the specific time level at the end of each time step. The method requires knowledge of rock and fluid properties, field geometry, well spacing, and known or proposed fluid injection rates.

The Dykstra-Parsons technique(1) was used as a reference to check the numerical model for cases in which the Dykstra-Parsons method is applicable and to test the Dykstra-Parsons technique for its applicability where a third phase, gas, appears and where the assumption of piston-like displacement of oil by water is not accurate.

The Dykstra-Parsons method employs a system of horizontal layers, not connected, except at the wells, in which permeability may vary vertically from layer to layer, with each layer being homogeneous. At any point in a layer only water or oil is flowing, but water is allowed to exist on one side of the front and oil on the other such that the velocity of fluids within a layer is dependent on the position of the front as well as the mobility ratio. No allowance is made for a third phase, gas, to exist in the system and all fluids are considered incompressible. Water-oil ratio is dependent on the fluid velocities in the individual layers and the number of layers which have experienced water breakthrough.

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(1) See References.

The Dykstra-Parsons technique was chosen as a reference method to indicate the validity of the mathematical model because it is believed that the Dykstra-Parsons method is very accurate within the limitations of the inherent assumptions.

With this introductory preface, one can say that the purpose of this research is two-fold:

1. To simulate a three-phase, one-dimensional petroleum reservoir, with water being injected at one boundary and oil, water and gas being produced at the other boundary. It can be readily noted that the approach is also useful for estimating vertical coverages.
2. To compare predicted performance of a hypothetical reservoir as evaluated by the numerical model with results obtained by the Dykstra-Parsons method.

The mathematical model and the Dykstra-Parsons method were programmed for an IBM 360 computer in Fortran IV language. The flow diagrams and programs are presented in Appendix D.

II. LITERATURE REVIEW

In 1856, the French engineer Henry Darcy(2) empirically formulated that the rate of flow of a single fluid through a porous medium is directly proportional to the potential gradient and the cross-sectional area normal to the direction of flow and inversely proportional to the viscosity of the fluid. This relationship is known today as Darcy's Law and has served as the cornerstone for describing fluid flow in porous media. Later it was shown that Darcy's Law is only valid for the viscous flow region usually defined for flow in porous media as that region where Reynolds number is equal to or less than one. Hubbert(3) showed that Darcy's Law can be deduced from the classical Navier-Stokes hydrodynamics equations. Other investigators found that Darcy's Law can be extended to fluids other than water insofar as Darcy's investigations were confined to water flow. The generalized form of Darcy's Law is written as,

$$V = - \frac{k}{\mu} \nabla \Phi^{*}$$

where

$\Phi = P - \rho g h$, h is taken positive downward.

In 1930, experimental studies were made to investigate the flow of immiscible fluids in porous media.(4) Experiments showed that the presence of a second phase reduces the conductance for both phases. The work of Wyckoff and Botset(5)

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*Terms defined in Nomenclature.

made it evident that the relative permeability to each phase of an immiscible two-phase system is a function of the phase saturation within the porous medium. Leverett(6) investigated the effect of fluid viscosity, pressure gradient and interfacial tension on the relative permeability and concluded that relative permeability is independent of fluid viscosity but is some function of pore size distribution, displacement pressure, pressure gradient and fluid saturation. In his work he introduced the concept of and defined the J-function, which relates capillary pressure to porosity and permeability. In 1941 Leverett and Lewis(7) reported the results of steady state flow tests on unconsolidated sands with three phases (oil, water and gas) present. From his work, the concept of three-phase relative permeabilities was established.

For a system containing two or more immiscible phases, a mathematical analysis consists of formulating differential equations obtained by combining Darcy's Law (with capillary pressure) and an equation of state for each phase with the Continuity Equation for each phase flowing. This results in a non-linear, second-order partial differential equation for each mobile phase. Due to the complexity of the resulting differential equation, a solution cannot be obtained by classical methods.

After the advent of high speed digital computing equipment, serious attempts at numerical solution were made. West, Garvin and Sheldon(8) studied horizontal, linear and radial systems produced under gas drive excluding capillary effects. Their numerical treatment employed a finite difference grid system and they used implicit methods for solving the resulting system of linear equations.

Hovanessian and Fayers(9) reported a numerical simulation for one-dimensional, inclined flow of two immiscible, incompressible fluids. They showed that the inclusion of capillary pressure and gravitational forces had a pronounced effect on both phase saturations and pressure distributions. Douglas et al.(10) studied a horizontal, one-dimensional system including capillary pressure and concluded that at high flow rates this model yields results similar to those obtained by the Buckley-Leverett method.(11)

Two dimensional techniques have also been attempted but these early studies indicated that such models were highly unstable. However, in 1958, Douglas(12) published a paper on the "Alternating Direction Implicit Procedure" (ADIP) which provided a stable method for the treatment of these two dimensional problems. In 1959, Douglas, Peaceman and Rachford(13) used this classic finite difference treatment (ADIP) to successfully simulate two-dimensional flow of immiscible fluids. They reported good agreement between predicted performance for displacement of oil by water and observed values obtained from laboratory experiments.

Thus, in the 1950's, digital computing equipment and techniques had been developed sufficiently to allow numerical solution of multi-phase, multi-dimensional flow problems. In 1955, Rapoport(14) combined the three dimensional partial differential equations describing immiscible, incompressible two-phase flow to obtain a single equation. He investigated the displacement of oil by water in terms of production history before and after water breakthrough.

In 1965, Fagin and Stewart(15) presented a two-dimensional, multi-phase reservoir simulator. This was a general flow model yielding accurate representations of pressure and multi-phase saturation changes with time.

In 1967, Coats, et al(16) presented the concept of vertical equilibrium in a horizontal model and adjusted a two-dimensional, two-phase model to account for saturation variation in the third dimension. Good agreement was reported when compared with three-dimensional calculations for a reservoir having definitive vertical communication.

In 1965, Quon, et al(17) reported the applicability of an "Alternating Direction Explicit Procedure" (ADEP) to a two-dimensional reservoir analysis where large time increments can be used to significantly reduce the time involved in solving such models. Their application of the ADEP technique was to simulate a gas reservoir and its flow characteristics and reported good agreement with similar results from an ADIP solution on the same model.

III. ASSUMPTIONS

The following assumptions are made to permit the desired solution;

1. The model simulates a reservoir consisting of horizontal layers of varying vertical thickness. The permeability and thickness are permitted to vary from layer to layer. Each layer consists of blocks of variable length and width adjacent to each other horizontally with the capability of varying permeability from block to block. It is assumed that there is an impermeable barrier between layers thus implying that there is no vertical cross-flow between layers.

2. Rock heterogeneity with respect to permeability (but not porosity) is assumed.

3. Capillarity, gravity and rock compressibility are considered negligible.

4. Water is injected in one end at constant rate, and oil, water and gas are produced at the other end at rates such that reservoir voidage equals water influx at reservoir conditions.

5. It is assumed that the simulation accurately accounts for three-phase flow, fluid compressibilities, and evolution of dissolved gas.

IV. THEORY

A. PARTIAL DIFFERENTIAL EQUATIONS

The basic concept employed in describing fluid flow in a porous media is based on the law of mass conservation and Darcy's Law.

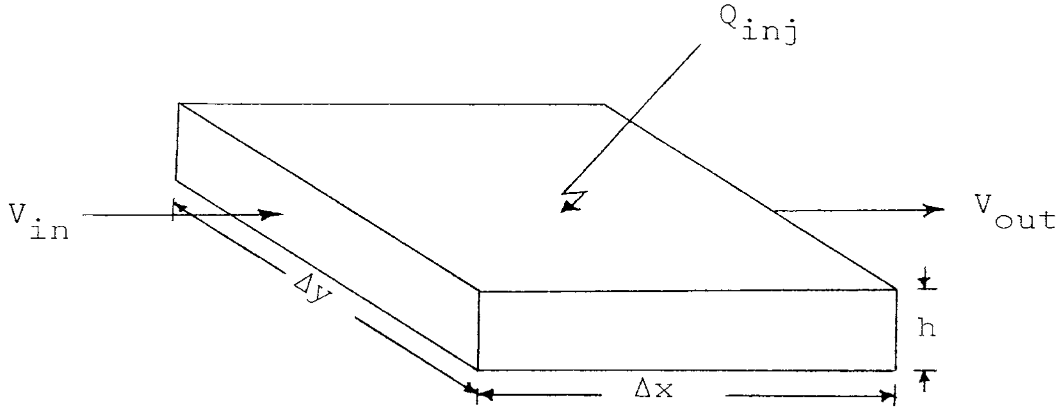


Fig. 1. A Typical Reservoir Elemental Block

If one considers three-phase fluid flow through an elemental block in the x-direction as shown in Figure (1), then one can apply the law of mass conservation on each phase making use of the Darcy's Law in the following manner:

Oil Phase:

a) Mass rate in - mass rate out = rate of mass accumulation (1)

b) Mass rate in = $\vec{Vo}_{in} A\rho_{os}/\beta_o + Q_{oinj}\rho_{os}$ (2)

where Q_{oinj} is negative if it is production instead of injection and,

$\vec{Vo}_{in}$ is the oil velocity entering the elemental block from the left hand side which is defined by Darcy's Law as:

$$\vec{V}o = -\frac{k_o}{\mu_o}\frac{\partial \Phi_o}{\partial x}$$

where;

Φ_o is the velocity potential defined as:

$$\Phi_o = P_o + \rho_o gh \quad .$$

c) The mass rate out can be expressed as;

Mass rate out = $\vec{V}o_{out}\ A\rho_{os}/\beta_o$

where $\vec{V}o_{out}$ is the velocity of oil leaving the elemental block through the right hand side and face of the elemental volume and is equal to

$$\vec{V}o_{in} + \frac{\partial \vec{V}_o}{\partial x}\Delta x. \tag{3}$$

The mass accumulation rate can be defined as;

d) Rate of mass accumulation = $\frac{\partial M_o}{\partial t}$ =

$$h\Delta x\Delta y\phi\rho_{os}\ \frac{\partial}{\partial t}\left(\frac{S_o}{\beta_o}\right) \tag{4}$$

where h, ϕ and ρ_{os} are constant factors.

Substituting Equations (2), (3) and (4) into Equation (1) and defining V as defined by Darcy's Law yields:

$$h\Delta x\Delta y\rho_{os}\ \frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o\beta_o}\ \frac{\partial\Phi_o}{\partial x}\right) + \rho_{os}Q_{oinj} = \phi h\Delta x\Delta y\rho_{os}\ \frac{\partial}{\partial t}\left(\frac{S_o}{\beta_o}\right) \ . \tag{5}$$

Dividing both sides of Equation (5) by $h\Delta x\Delta y\rho_{os}$ yields:

$$\frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o \beta_o}\frac{\partial \Phi_o}{\partial x}\right) + \frac{Q_{oinj}}{h\Delta x\Delta y} = \phi \frac{\partial}{\partial t}\left(\frac{S_o}{\beta_o}\right) . \qquad (6)$$

Expanding the time derivative (the right hand side of Equation (6)), and multiplying both sides by β_o yields:

$$\beta_o \frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o \beta_o}\frac{\partial \Phi_o}{\partial x}\right) + \beta_o \frac{Q_{oinj}}{h\Delta x\Delta y} = \phi \left[\frac{\partial S_o}{\partial t} - \frac{S_o}{\beta_o}\frac{\partial \beta_o}{\partial P_o}\frac{\partial P_o}{\partial t}\right] . \qquad (P\text{-}1)$$

Equation (P-1) is the partial differential equation describing the flow of oil in one dimension (x-direction) in a porous medium.

Water Phase:

Applying the logic similar to that employed in deriving the equation for the oil phase, a partial differential equation for the water phase flowing through an elemental block can be developed as follows:

$$\beta_w \frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w \beta_w}\frac{\partial \Phi_w}{\partial x}\right) + \frac{\beta_w Q_{winj}}{h\Delta x\Delta y} = \phi \left[\frac{\partial S_w}{\partial t} - \frac{S_w}{\beta_w}\frac{\partial \beta_w}{\partial P_w}\frac{\partial P_w}{\partial t}\right] \qquad (P\text{-}2)$$

where $\Phi_w = P_w + \rho_w gh$.

Gas Phase:

The derivation of a partial differential equation for the gas phase differs from that of oil and water phases only in that gas evolved from solution in oil must be accounted for. Therefore, the law of mass conservation may be written in the following manner:

Mass rate in + mass rate of gas evolution - mass rate out = rate of mass accumulation. (7)

Mathematically, Equation (7) is defined as

$$-h\Delta x\Delta y\ \rho_{gs}\ \frac{\partial}{\partial x}\left(\frac{\vec{V}g_{in}}{\beta_g}\right) + \rho_{gs}Q_{ginj} = -\frac{\partial M_{FG}}{\partial t} + \frac{\partial M_g}{\partial t} \qquad (8)$$

where $\frac{\partial M_{FG}}{\partial t}$ is the rate of change in the mass of gas evolved, and its derivation is based on the mass rate of gas evolution and is related to the oil phase present, the solution gas oil ratio (R_s), and their rate of change as follows:

$$\frac{\partial M_{FG}}{\partial t} = -h\Delta x\Delta y\ \rho_{gs}\ \phi\ \frac{\partial}{\partial t}\left(\frac{S_o}{\beta_o}\ R_s\right)\ . \qquad (9)$$

The minus sign in Equation (9) accounts for the fact that a decrease in the solution gas oil ratio is accompanied by an increase in free gas volumes. Also, in the same equation, R_s remains inside the derivative because it also varies with time being uniquely a function of pressure which is a function of time.

Substituting the Darcy's Law equivalent of the velocity potential and the results of Equation (9) into Equation (8) and then dividing both sides of the resulting equation by $h\Delta x\Delta y\ \rho_{gs}$, and multiplying through by β_g, yields:

$$\beta_g\ \frac{\partial}{\partial x}\left(\frac{k_g}{\mu_g\beta_g}\ \frac{\partial \Phi_g}{\partial x}\right) + \frac{\beta_g Q_{ginj}}{h\Delta x\Delta y} = \phi\beta_g\ \frac{\partial}{\partial t}\left(\frac{S_g}{\beta_g} + \frac{S_o}{\beta_o}\ R_s\right) \qquad (10)$$

where $\Phi_g = P_g + \rho_g gh$.

Expanding the time derivative of the right hand side of Equation (10) yields:

$$\phi\beta_g\left[\frac{\partial}{\partial t}\left(\frac{S_g}{\beta_g}+\frac{S_o}{\beta_o}R_s\right)\right] = \left[\frac{1}{\beta_g}\frac{\partial S_g}{\partial t} - \frac{S_g}{\beta_g^{2}}\frac{\partial \beta_g}{\partial P_g}\frac{\partial P_g}{\partial t} + \frac{S_o}{\beta_o}\frac{\partial R_s}{\partial P_g}\frac{\partial P_g}{\partial t} + R_s\frac{\partial}{\partial t}\left(\frac{S_o}{\beta_o}\right)\right]\phi\beta_g \ . \qquad (11)$$

Substituting the value of $\phi\frac{\partial}{\partial t}\left(\frac{S_o}{\beta_o}\right)$ as given by Equation (6) into Equation (11) yields:

$$\phi\frac{\partial S_g}{\partial t} + \left(\phi\beta_g\frac{S_o}{\beta_o}\frac{\partial R_s}{\partial P_g} - \phi\frac{S_g}{\beta_g}\frac{\partial \beta_g}{\partial P_g}\right)\frac{\partial P_g}{\partial t} + \beta_g R_s\frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o\beta_o}\frac{\partial \Phi_o}{\partial x}\right) + \beta_g R_s\frac{Q_{oinj}}{h\Delta x\Delta y} = \phi\beta_g\left[\frac{\partial}{\partial t}\left(\frac{S_g}{\beta_g}+\frac{S_o}{\beta_o}R_s\right)\right] \qquad (12)$$

Substituting the results shown in Equation (12) for the right hand side of Equation (10) and rearranging yields:

$$\beta_g\frac{\partial}{\partial x}\left(\frac{k_g}{\mu_g\beta_g}\frac{\partial \Phi_g}{\partial x}\right) + \phi\frac{S_g}{\beta_g}\frac{\partial \beta_g}{\partial P_g}\frac{\partial P_g}{\partial t} - \frac{\phi S_o\beta_g}{\beta_o}\frac{\partial R_s}{\partial P_g}\frac{\partial P_g}{\partial t} - \beta_g R_s\frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o\beta_o}\frac{\partial \Phi_o}{\partial x}\right) - \frac{\beta_g R_s Q_{oinj}}{h\Delta x\Delta y} + \frac{\beta_g Q_{ginj}}{h\Delta x\Delta y} = \phi\frac{\partial S_g}{\partial t} \ . \qquad (P\text{-}3)$$

There is no need to account for loss of oil due to the evolution of gas since this loss is totally accounted for by changes in the oil formation volume factor.

Since capillary pressure and gravity are considered negligible in this model, the following potential equivalences are plausible;

$$\Phi_o = \Phi_w = \Phi_g = P \ , \tag{13}$$

and individual phase pressures are identical, i.e.,

$$P_o = P_w = P_g = P \ .$$

Recognizing that the potential terms appearing in Equations (P-1), (P-2), and (P-3) are identical as defined by Equation (13), it may be noted that these three partial differential equations are simultaneous equations relating pressure and saturations (four dependent variables) to the independent variables, position and time. This system cannot be solved as there are three equations and four unknowns. This problem is readily overcome, however, by introducing another equation in the dependent variables. This equation can be based on the definition of saturations as follows:

$$S_o + S_w + S_g = 1.0 \ . \tag{14}$$

It may be further noted that the partial differential equations have been judiciously arranged so that they may be added to yield one equation in one dependent variable, notably pressure. Note that the three time derivatives of the saturations are isolated such that their only coefficient is a constant, porosity, and that taking the time derivative of Equation (14) yields:

$$\frac{\partial S_o}{\partial t} + \frac{\partial S_w}{\partial t} + \frac{\partial S_g}{\partial t} = 0 \ . \tag{15}$$

Summing Equations (P-1), (P-2), and (P-3), rearranging terms, and simplifying as discussed immediately above based on Equations (13), (14), and (15) yields:

$$\beta_g \frac{\partial}{\partial x}\left(\frac{k_g}{\mu_g \beta_g} \frac{\partial P}{\partial x}\right) + (\beta_o - \beta_g R_s) \frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o \beta_o} \frac{\partial P}{\partial x}\right) + \beta_w \frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w \beta_w} \frac{\partial P}{\partial x}\right)$$

$$= - \left[\frac{\beta_g Q_{ginj}}{h\Delta x\Delta y} + (\beta_o - \beta_g R_s) \frac{Q_{oinj}}{h\Delta x\Delta y} + \frac{\beta_w Q_{winj}}{h\Delta x\Delta y}\right]$$

$$- \phi \left[\frac{S_g}{\beta_g} \frac{\partial \beta_g}{\partial P} + \frac{S_o}{\beta_o}\left(\frac{\partial \beta_o}{\partial P} - \beta_g \frac{\partial R_s}{\partial P}\right) + \frac{S_w}{\beta_w} \frac{\partial \beta_w}{\partial P} \quad \frac{\partial P}{\partial t}\right] \qquad \text{(P-4)}$$

Equation (P-4) is a non-homogeneous, second-order non-linear, partial differential equation and no known technique for solving it by classical means exists. It is non-linear because the coefficients are in themselves pressure and saturation dependent. The equation is solvable, however, by finite difference approximation techniques.

B. FINITE-DIFFERENCE EQUATION

Equation (P-4) is in a form where a difference equation can be written, from which the value of the dependent variable (pressure) at time level t(n+1) can be calculated when the values of all parameters are known at the previous time level t(n). The terms of the left hand side of Equation (P-4) can be represented by a finite difference equation if the time-space plane is divided into discrete cells as shown in Figure (2) (for a single time), then expanded about some point (i) in the spatial grid pattern at some fixed time level.

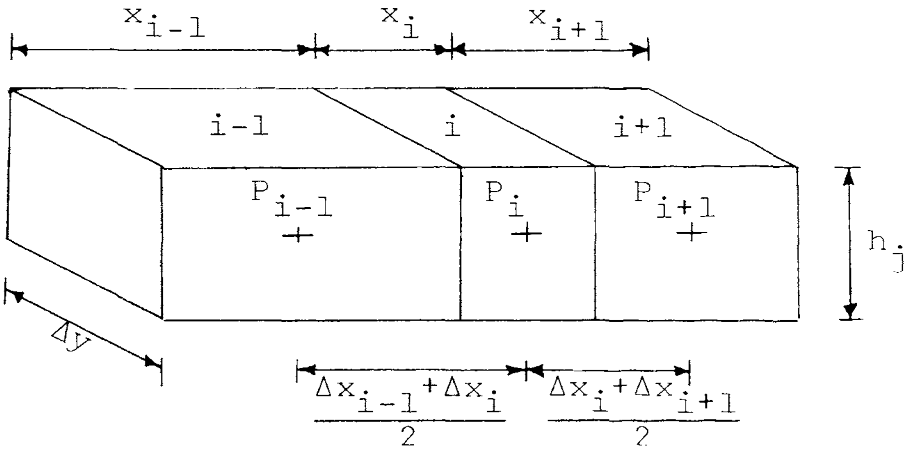


Fig. 2. Central Grid Point Illustration

The point at which all parameters are defined in each block is arbitrarily selected as its mid-point. Thus, the center is the focal point at which the pressure is assumed concentrated and this forms the basis for calculating pressure gradients.

Since there are three phases flowing, it is necessary to use relative permeabilities rather than absolute permeabilities. The former are defined by the following relations:

$$k_g = k\ k_{rg}\ ;$$

$$k_o = k\ k_{ro}\ ;$$

$$k_w = k\ k_{rw}\ .$$

Utilizing this fact the following equations may be written:

$$\beta_g \frac{\partial}{\partial x}\left(\frac{k\,k_{rg}}{\mu_g\beta_g}\frac{\partial P}{\partial x}\right) \simeq \frac{\beta_{gi}^{n+\frac{1}{2}}}{\Delta x_i}\left[\left\{\frac{k\,k_{rg}}{\mu_g\beta_g}\right\}_{i+\frac{1}{2}}^{n+\frac{1}{2}}\frac{(P_{i+1}-P_i)^{n+1}}{\left(\frac{\Delta x_{i+1}+\Delta x_i}{2}\right)} - \left\{\frac{k\,k_{rg}}{\mu_g\beta_g}\right\}_{i-\frac{1}{2}}^{n+\frac{1}{2}}\frac{(P_i-P_{i-1})^{n+1}}{\left(\frac{\Delta x_i+\Delta x_{i-1}}{2}\right)}\right], \tag{1-D}$$

and,

$$(\beta_o-\beta_g R_s)\frac{\partial}{\partial x}\left(\frac{k\,k_{ro}}{\mu\;\beta}\frac{\partial P}{\partial x}\right) \simeq \left[\frac{\beta_o-\beta_g R_s}{\Delta x_i}\right]^{n+\frac{1}{2}}\left[\left\{\frac{k\,k_{ro}}{\mu_o\beta_o}\right\}_{i+\frac{1}{2}}^{n+\frac{1}{2}}\frac{(P_{i+1}-P_i)^{n+1}}{\left(\frac{\Delta x_{i+1}+\Delta x_i}{2}\right)} - \left\{\frac{k\,k_{ro}}{\mu_o\beta_o}\right\}_{i-\frac{1}{2}}^{n+\frac{1}{2}}\frac{(P_i-P_{i-1})^{n+1}}{\left(\frac{\Delta x_i+\Delta x_{i-1}}{2}\right)}\right], \text{ and,} \tag{2-D}$$

$$\beta_w \frac{\partial}{\partial x}\left(\frac{k\,k_{rw}}{\mu_w\beta_w}\frac{\partial P}{\partial x}\right) = \frac{\beta_{wi}^{n+\frac{1}{2}}}{\Delta x_i}\left[\left\{\frac{k\,k_{rw}}{\mu_w\beta_w}\right\}_{i+\frac{1}{2}}^{n+\frac{1}{2}}\frac{(P_{i+1}-P_i)^{n+1}}{\left(\frac{\Delta x_{i+1}+\Delta x_i}{2}\right)} - \left\{\frac{k\,k_{rw}}{\mu_w\beta_w}\right\}_{i-\frac{1}{2}}^{n+\frac{1}{2}}\frac{(P_i-P_{i-1})^{n+1}}{\left(\frac{\Delta x_i+\Delta x_{i-1}}{2}\right)}\right]. \tag{3-D}$$

where the superscript $n+\frac{1}{2}$ represents the midpoint representation of the time level.

It is noted from Equations (1-D), (2-D), and (3-D), that a time average of the pressure and saturation dependent variables has been employed. Therefore, the pressure dependent variables should be evaluated at $\left(\frac{P^{n+1}+P^n}{2}\right)$ and the relative permeabilities should be evaluated at $\left(\frac{S^{n+1}+S^n}{2}\right)$ where the saturations have not been shown with subscript depending on the phase under evaluation. The parameters describing rock properties and model configuration are not time-dependent and can be isolated, forming a term, HKX, which may be defined as follows:

$$HKX_i = \frac{2k_{i+\frac{1}{2}}}{\Delta x_{i+1}+\Delta x_i} = \frac{2k_i k_{i+1}}{\Delta x_i k_{i+1}+\Delta x_{i+1} k_i} \qquad (16)$$

for the ith spatial point, and

$$HKX_{i-1} = \frac{2k_{i-\frac{1}{2}}}{\Delta x_{i-1}+\Delta x_i} = \frac{2k_{i-1} k_i}{\Delta x_{i-1} k_i+\Delta x_i k_{i-1}} \qquad (17)$$

for the i-1 spatial point.

$K_{i+\frac{1}{2}}$ and $K_{i-\frac{1}{2}}$ are series averaged values for rock permeability, evaluated at the interfaces bounding the ith block. Equations (16) and (17) are derived in Appendix .

At this point a difference equation for the left hand side of Equation (P-4) has been written. On the right hand side of Equation (P-4), the only term to be written as a difference is $\frac{\partial P}{\partial t}$ which may be expressed as follows;

$$\frac{\partial P}{\partial t} \simeq \frac{P_i^{n+1} - P_i^n}{\Delta t} . \qquad (4\text{-D})$$

Now substituting Equations (1-D), (2-D), (3-D), (16), (17) and (4-D) in Equation (P-4) yields:

$$\left\{\frac{\beta_{gi}}{\Delta x_i}\right\}^{n+\frac{1}{2}} \left[HKX_i \gamma_{gi+\frac{1}{2}}^{n+\frac{1}{2}} (P_{i+1} - P_i)^{n+1} - HKX_{i-1} \gamma_{gi-\frac{1}{2}}^{n+\frac{1}{2}} (P_i - P_{i-1})^{n+1} \right]$$

$$+ \left\{\frac{\beta_o - \beta_g R_s}{\Delta x_i}\right\}^{n+\frac{1}{2}} \left[HKX_i \gamma_{oi+\frac{1}{2}}^{n+\frac{1}{2}} (P_{i+1} - P_i)^{n+1} - HKX_{i-1} \gamma_{oi+\frac{1}{2}}^{n+\frac{1}{2}} (P_i - P_{i-1})^{n+1} \right]$$

$$+ \left\{\frac{\beta_{wi}}{\Delta x_i}\right\}^{n+\frac{1}{2}} \left[HKX_i \gamma_{wi+\frac{1}{2}}^{n+\frac{1}{2}} (P_{i+1} - P_i)^{n+1} - HKX_{i-1} \gamma_{wi+\frac{1}{2}}^{n+\frac{1}{2}} (P_i - P_{i-1})^{n+1} \right]$$

$$= - \frac{1}{h \Delta x_i \Delta y} \left[(\beta_g Q_{ginj})_i + (\beta_o - \beta_g R_s)_i (Q_{oinj})_i + (\beta_w Q_{winj})_i \right]^{n+\frac{1}{2}}$$

$$- \phi \left[\frac{S_{gi}}{\beta_{gi}} \beta_{gi}' + \frac{S_{oi}}{\beta_{oi}} (\beta_o' - \beta_g R_s')_i - C_w S_{wi} \right]^{n+\frac{1}{2}} \frac{P_i^{n+1} - P_i^n}{\Delta t} \qquad (5\text{-D})$$

where

$$\gamma_g = \frac{k_{rg}}{\mu_g \beta_g}, \quad \gamma_o = \frac{k_{ro}}{\mu_o \beta_o}, \quad \gamma_w = \frac{k_{rw}}{\mu_w \beta_w};$$

$$\beta_g' = \frac{\partial \beta_g}{\partial P}, \quad \beta_o' = \frac{\partial \beta_o}{\partial P}, \quad R_s' = \frac{\partial R_s}{\partial P}; \quad \text{and}$$

$$C_w = - \frac{1}{\beta_w} \frac{\partial \beta_w}{\partial P}.$$

Equation (5-D) is a complete solution of the partial differential Equation (P-4) in difference form. For simplicity, Equation (5-D) can be rearranged by collecting all terms associated with P_{i-1}^{n+1}, P_i^{n+1}, and P_{i+1}^{n+1}, and rewritten in the following form:

$$A_i P_{i-1}^{n+1} + B_i P_i^{n+1} + C_i P_{i+1}^{n+1} = D_i \qquad (6\text{-D})$$

where,

$$A_i = AOX1 + AWX1 + AGX1 \qquad (7\text{-D})$$

$$C_i = AOX2 + AWX2 + AGX2 \qquad (8\text{-D})$$

$$B_i = - A_i - C_i + TRM_i \qquad (9\text{-D})$$

$$D_i = QTERM_i + TRM_i \; P_i^n \qquad (10\text{-D})$$

$$AOX1 = \frac{HKX_{i-1}}{\Delta x_i} \left[(\beta_o - \beta_g R_s)_i \gamma_{oi-\frac{1}{2}} \right]^{n+\frac{1}{2}} \qquad (11\text{-D})$$

$$AWX1 = \frac{HKX_{i-1}}{\Delta x_i} \left[\beta_{wi} \gamma_{wi-\frac{1}{2}} \right]^{n+\frac{1}{2}} \qquad (12\text{-D})$$

$$AGX1 = \frac{HKX_{i-1}}{\Delta x_i} \left[\beta_{gi} \gamma_{gi-\frac{1}{2}} \right]^{n+\frac{1}{2}} \qquad (13\text{-D})$$

$$AOX2 = \frac{HKX_i}{\Delta x_i} \left[(\beta_o - \beta_g R_s)_i \gamma_{oi+\frac{1}{2}} \right]^{n+\frac{1}{2}} \qquad (14\text{-D})$$

$$AWX2 = \frac{HKX_i}{\Delta x_i} \left[\beta_{wi} \gamma_{wi+\frac{1}{2}} \right]^{n+\frac{1}{2}} \qquad (15\text{-D})$$

$$AGX2 = \frac{HKX_i}{\Delta x_i} \left[\beta_{gi} \gamma_{gi+\frac{1}{2}} \right]^{n+\frac{1}{2}} \qquad (16\text{-D})$$

$$TRM_i = \frac{\phi}{\Delta t} \left[\frac{S_{gi}}{\beta_{gi}} \beta'_{gi} + \frac{S_{oi}}{\beta_{oi}} (\beta'_o - \beta_g R'_s)_i - C_w S_{wi} \right]^{n+\frac{1}{2}} \qquad (17\text{-D})$$

$$\mathrm{QTERM}_i = -\frac{1}{h\Delta x_i \Delta y}\Big[(\beta_g Q_{ginj})_i + (\beta_o - \beta_g R_s)_i (Q_{oinj})_i$$

$$+ (\beta_w Q_{winj})_i\Big]^{n+\frac{1}{2}} \quad . \qquad (18\text{-D})$$

Equation (6-D), written about each point in the grid system, forms a set of simultaneous linear equations whose coefficient matrix is tridiagonal. The solution for a tridiagonal system is,

$$P_{m-1} = \alpha_{m-1}$$

$$P_i = \alpha_i - \frac{C_i P_{i-1}}{\lambda_i} \,, \qquad i = m-2, m-3, \ldots\ldots, 1$$

where α's and λ's are determined from the recursion formulae,

$$\lambda_1 = B_1, \; \alpha_1 = \frac{D_1}{\lambda_1}$$

$$\lambda_i = B_i - \frac{A_i C_{i-1}}{\lambda_{i-1}} \,, \qquad i = 2, 3, 4, \ldots\ldots, m-1$$

$$\alpha_i = \frac{D_i - A_i \alpha_{i-1}}{\lambda_i} \;. \qquad i = 2, 3, 4, \ldots\ldots m-1$$

It has been previously mentioned that the coefficients of Equation (6-D) are dependent on both pressure and saturation which are unknowns. Thus, the method involves a trial and error procedure. An estimate of future pressures

(P^{n+1}) and saturations (S^{n+1}) are made by linear extrapolation. Then the fluid properties are evaluated at $P^{n+\frac{1}{2}} = \frac{P^{n+1}+P^{n}}{2}$ while the rock properties (relative permeabilities) are evaluated at $S^{n+\frac{1}{2}} = \frac{S^{n+1}+S^{n}}{2}$. The unknown pressures are then calculated. Using these calculated pressures, an explicit determination for future saturations are made as described in the Material Balance Section. Both calculated pressures and saturations are then compared with the previously assumed values. If agreement is not within a prespecified tolerance, then an iteration is made. Otherwise, the calculations may proceed to the next time step. After the first time loop through, a linear extrapolation is made to predict pressures and saturations for the next time step. The extrapolation technique is shown in Appendix B.

C. STRATIFICATION

Since it is assumed that no cross-flow occurs between layers, then the performance of each layer is calculated independently, and the performance of the total reservoir is obtained by summing the results of each individual layer's performance. This method of treatment represents the approach used by both Stiles and Dykstra.

Injection and production (through wells) occurs through selected sources or sinks, respectively, in the mathematical simulation. The assignment of water injection into the left edge of each layer and the calculation of production of oil, water and gas from the right edge of each layer is the

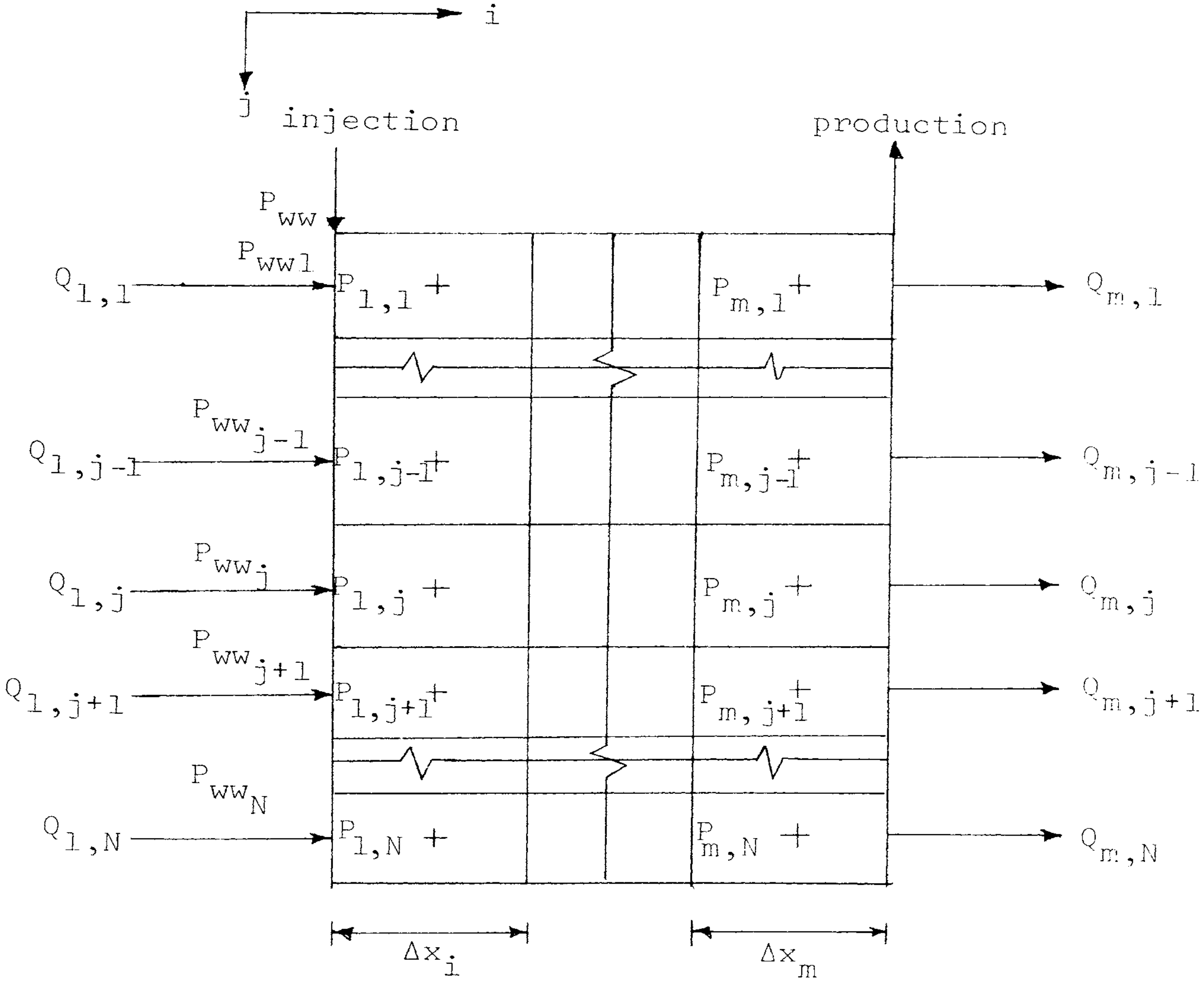


Fig. 3. Injection and Production Wells

convention assumed in this model and is treated as part of the boundary conditions. This is illustrated in Figure (3).

D. BOUNDARY CONDITIONS

The system simulated by this model is assumed to be closed at all boundaries except for those blocks containing wells, which for this model are those assumed to be at either end. No fluid is permitted to flow across these boundaries except that being produced or injected at the wells. The boundary barrier can be effected by either assigning zero permeability to an imaginary block outside the model or by assigning a pressure gradient of zero at these boundaries. The effect is identical. A schematic drawing of this model follows as Figure (4).

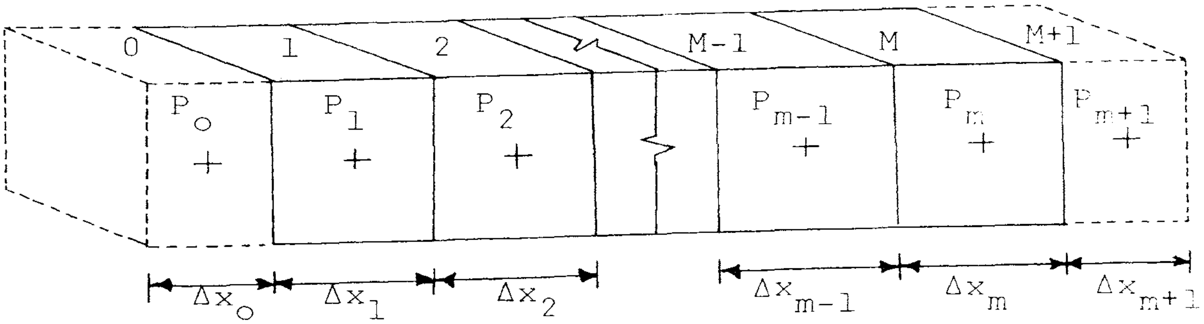


Fig. 4. Boundary Elements

Applying the latter assumption to Equation (6-D) and writing an equation about point i = 1, (see Figure (3)) with no flow across the left-hand boundary gives:

$$A_1 P_o^{n+1} + B_1 P_1^{n+1} + C_1 P_2^{n+1} = D_1 \ . \qquad (18)$$

The left hand boundary is closed by setting the pressure gradient to zero $\left[(\frac{\partial P}{\partial x})_{1-\frac{1}{2}} = 0 \right]$, which is equivalent to setting $P_o = P_1$.

Substituting this result into Equation (18) for i = 1 yields:

$$(A_1 + B_1) \ P_1^{n+1} + C_1 P_2^{n+1} = D_1$$

but, from Equation (9-D),

$$B_1 = - A_1 - C_1 + TRM_1$$

then the coefficient of P_1^{n+1} becomes

$$A_1 + B_1 = - C_1 + TRM_1 \ .$$

This result is equivalent to setting $A_1 = 0$, its value when the external block has a permeability of zero as shown below.

Equation (18) for the first block may be written as

$$B_1 P_1^{n+1} + C_1 P_2^{n+1} = D_1 \ . \qquad (19)$$

Closing the left hand boundary by assigning a zero permeability at the boundary ($k_o = 0$) gives the following result:

The coefficient A_1 in Equation (18) is defined as:

$$A_1 = \frac{HKX_o}{\Delta x_i} \left[(\beta_o - \beta_g R_s)_1 \gamma_{oi-\frac{1}{2}} + \beta_{wi} \gamma_{wi-\frac{1}{2}} + \beta_{gi} \gamma_{gi-\frac{1}{2}} \right] \qquad (20)$$

where,

$$HKX_o = \frac{2k_ik_o}{\Delta x_o k_i + \Delta x_i k_o} = \frac{2(k_i)\,(D)}{\Delta x_o k_i} = 0 \ .$$

Substituting this result into Equation (20) gives a value of

$$A_1 = 0 \ .$$

Equation (18) may, then, be written about the first point as shown by Equation (19), both methods giving identical results.

Second, using the same logic about point m (no flow across the right hand boundary), will result in $C_m = 0$, and an equation about point m, may be written as,

$$A_m P_{m-1}^{n+1} + B_m P_m^{n+1} = D_m \tag{21}$$

where

$$B_m = - A_m + TRM_m \ .$$

1. Injection End (left boundary).

The layer-wise distribution of water injection rate is obtained by first assuming a constant total injection rate in the "well", the left side of the model. A wellbore pressure at the top of the sand face (P_{ww}) is assigned and the wellbore pressure at the middle of each layer may be calculated as follows:

$$P_{ww_j} = P_{ww_{j-1}} + \rho_w \left(\frac{h_j + h_{j-1}}{2}\right) .$$

The horizontal pressure gradient entering element i,j is

$$\left(\frac{\partial P}{\partial x}\right)_{i,j} = \frac{2\left(P_{ww_{j-1}} + \frac{\rho_w}{2}(h_{j-1} + h_j) - P_{i,j}\right)}{\Delta x_i} . \qquad (22)$$

Replacing the gradient in Darcy's Law by the definition of the gradient as provided by Equation (22) yields:

$$(Q_{w1,j})_{inj} = \frac{2k_{1,j}k_{rw1,j}h_j\Delta y}{\beta_{w1,j}\mu_{w1,j}\Delta x_1}\left(P_{ww_{j-1}} + \frac{\rho_w}{2}(h_{j-1} + h_j) - P_{1,j}\right) \qquad (23)$$

The average pressure for elements 1,j (for all layers) may be defined as:

$$\bar{P} = \frac{\sum_{j=1}^{n} P_{1,j}h_j}{h_T} .$$

Then the average pressure gradient is:

$$\left(\frac{\partial \bar{P}}{\partial x}\right) = \frac{2\left(\left(P_{ww} + \frac{\rho_w}{2}h_T\right) - \bar{P}\right)}{\Delta x_1} . \qquad (24)$$

Applying the definition of gradient shown in Equation (24) to Darcy's Law and summing the capacity terms (kA) yields total flow rate to give the following relationship:

$$(Q_{Tw})_{inj} = \frac{2\sum_{j=1}^{n}(k_{1,j}k_{rw1,j}h_j)\Delta y}{\bar{\beta}_w\bar{\mu}_w\Delta x_1}\left(P_{ww} + \frac{\rho_w}{2}h_T - \sum_{j=1}^{n}\frac{P_{1,j}h_j}{h_T}\right) . \qquad (25)$$

The fraction of the total injection rate entering the jth layer is obtained by dividing Equation (23) by Equation (25) as follows:

$$\frac{(Q_{1,j})_{inj}}{(Q_{Tw})_{inj}} = \frac{k_{1,j}k_{rw1,j}h_j(P_{ww_{j-1}}+\frac{\rho_w}{2}(h_j+h_{j-1})-P_{1,j})}{(\sum_{j=1}^{n}(k_{1,j}k_{rw1,j}h_j))(P_{ww}+\frac{\rho_w}{2}h_T-\sum_{j=1}^{n}\frac{P_{1,j}h_j}{h_T})} \qquad (26)$$

and the actual injection rate into the 1,jth block is given as

$$(Q_{1,j})_{inj} = (Q_{Tw})_{inj}\,\frac{k_{1,j}k_{rw1,j}h_j(P_{ww_{j-1}}+\frac{\rho_w}{2}(h_j+h_{j-1})-P_{1,j})}{(\sum_{j=1}^{n}(k_{1,j}k_{rw1,j}h_j))(P_{ww}+\frac{\rho_w}{2}h_T-\sum_{j=1}^{n}\frac{P_{1,j}h_j}{h_T})} \qquad (27)$$

Equation (27) is used to calculate the water injection rate into first element of each layer at the injection end. It should be noted that for elements between the "injection end" and "production end" with no external flows, the QTERM is zero. The application of Equation (27) requires knowledge of the wellbore pressure at the top of the sand. A major difficulty arises from the fact that wellbore pressure is not constant when constant injection is specified. The most realistic approach to solving the problem is to assume a constant wellbore pressure rather than constant injection rate and calculate water injection rate into first element of each layer by using Equation (23). For simplicity, it is assumed that pressure gradient across the first element of each layer is constant; then, Equation (27) can be written as

$$(Q_{1,j})_{inj} \overset{\sim}{=} (Q_{Tw})_{inj} \frac{k_{1,j}k_{rw1,j}h_j}{\sum_{j=1}^{n} k_{1,j}k_{rw1,j}h_j} , \qquad (28)$$

since pressure gradient cancels out under this set of conditions.

2. Production End (right boundary).

The sum of oil, water and gas production rates evaluated at reservoir conditions at the production well from a particular layer is equal to the water injection rate (at reservoir conditions) into that layer at the injection well. The convention adopted in this model is that injection is positive and production is negative. Thus, the production-injection balance stated above may be written as follows:

$$(Q_{w_{inj}}\beta_w)_{1,j} = -(Q_{o_{prd}}\beta_o)_{m,j} - (Q_{w_{prd}}\beta_w)_{m,j} - (Q_{g_{prd}}\beta_g)_{m,j} . \qquad (29)$$

Darcy's Law is used to calculate water-oil ratio at reservoir conditions for each layer at the production well and may be written as follows including simplification:

$$\frac{(Q_{w_{prd}}\beta_w)_{m,j}}{(Q_{o_{prd}}\beta_o)_{m,j}} = \left(\frac{k_{rw}}{k_{ro}}\frac{\mu_o}{\mu_w}\right)_{m,j} ,$$

and

$$(Q_{w_{prd}}\beta_w)_{m,j} = (Q_{o_{prd}}\beta_o)_{m,j} \left(\frac{k_{rw}}{k_{ro}}\frac{\mu_o}{\mu_w}\right)_{m,j} . \qquad (30)$$

Writing an equation for gas-oil ratio in the same manner and solving for the gas production rate at reservoir conditions yields the following relation:

$$(Q_{g_{prd}}\beta_g)_{m,j} = (Q_{o_{prd}}\beta_o)_{m,j}\left(\frac{k_{rg}}{k_{ro}}\frac{\mu_o}{\mu_g}\right)_{m,j} . \qquad (31)$$

Substituting results shown by Equations (30) and (31) into the reservoir voidage balance given by Equation (29), and then solving for oil production rate at reservoir conditions, yields

$$(Q_{o_{prd}}\beta_o)_{m,j} = \frac{-(Q_{w_{inj}}\beta_w)_{1,j}}{1+\left\{\frac{k_{rw}}{k_{ro}}\frac{\mu_o}{\mu_w}+\frac{k_{rg}}{k_{ro}}\frac{\mu_o}{\mu_g}\right\}_{m,j}} . \qquad (32)$$

Having available reservoir oil production rates for each layer now allows values of the water and gas rates at reservoir conditions to be calculated by Equations (30) and (31), respectively, for each layer. Then all three rates are converted to stock tank conditions by dividing the reservoir rates calculated (which include volume factors) by the appropriate formation volume factor for each phase.

The total production rates for the well for each phase are now obtained by adding the respective rates from each layer. The water-oil ratio and gas-oil ratio from the well for the time period under consideration are computed as follows:

$$(Q_{To})_{m,prd} = \sum_{j=1}^{n} (Q_{om,j})_{prd} , \qquad (33)$$

$$(Q_{Tw})_{m,prd} = \sum_{j=1}^{n} (Q_{wj,j})_{prd} , \qquad (34)$$

$$(Q_{Tg})_{m,prd} = (\sum_{j=1}^{n} ((Q_{gm,j})_{prd}) + (Q_{om,j})_{prd}(\frac{R_{sm,j}^{n+1} + R_{sm,j}^{n}}{2}))) . \qquad (35)$$

$$WOR = (Q_{Tw})_{m,prd}/(Q_{To})_{m,prd} , \qquad (36)$$

and

$$GOR = (Q_{Tg})_{m,prd}/(Q_{To})_{m,prd} . \qquad (37)$$

E. MATERIAL BALANCE

Material balance calculations were made on each block in deriving the pressure equation and are also used to update saturations across each time step.

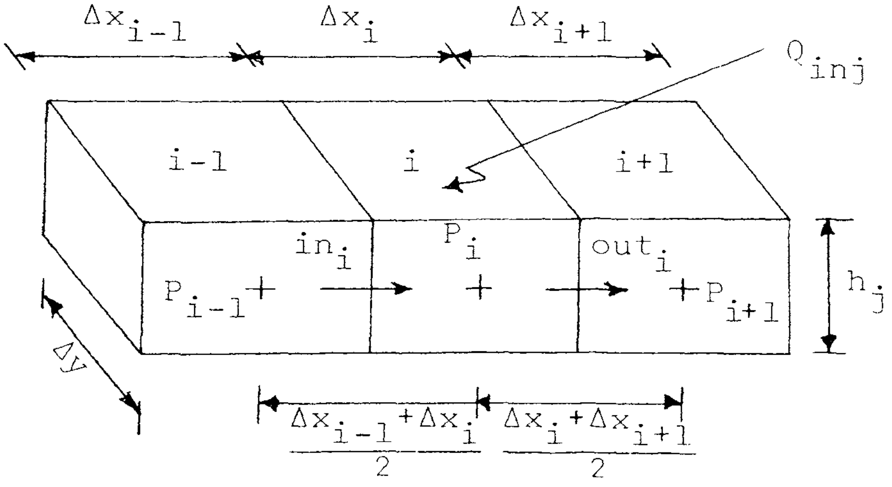


Fig. 5. Grid System for Material Balance

Referring to Figure (5), the saturation in block (i) at a new time level t(n+1) can be stated as the saturation at the beginning of the time step (old time level t(n)) plus the change in saturation during the time increment (Δt) (from n to n+1 time level), may be mathematically defined as,

$$S_i^{n+1} = S_i^n + \Delta S_i \ .$$

ΔS_i for a phase is the change in volume of that phase in block (i) during the time increment, divided by that block's pore volume.

However, in allowing for the compressibility of phases in this model, the accuracy of the calculation will be enhanced by making a mass balance rather than a volumetric balance and then converting the change in mass in the block to an equivalent change in saturation.

Thus, the following relation will be applicable for oil and water phases,

$$\Delta S_i = \frac{(\text{mass in})_i - (\text{mass out})_i + (\text{mass injected})_i}{\rho_{ri}(PV)_i} \ . \qquad (1\text{-M})$$

Darcy's Law is used at the (i-½) and (i+½) interfaces to evaluate the (mass in)$_i$ and (mass out)$_i$ terms, respectively, in Equation (1-M). The following combination of terms are applicable:

$$\Delta t \ \frac{\rho_s Q}{\beta} = - \frac{\rho_s k \ k_r A(\frac{\partial P}{\partial x})}{\beta \mu} \Delta t \ . \qquad (38)$$

Recognizing that the mass entering block (i) comes from block (i-1) as indicated in Figure (5), and taking the convention that flow is from left to right, it concluded that the following relations apply for the (mass in)$_i$ term:

$$\beta = \beta_{i-1},\ k = k_{i-\frac{1}{2}},\ \mu = \mu_{i-\frac{1}{2}},$$

$$(\partial P/\partial x)_{i-\frac{1}{2}} \stackrel{\sim}{=} (P_i - P_{i-1})/((\Delta x_i + \Delta x_{i-1})/2). \qquad (39)$$

and similarly for the (mass out)$_i$ term:

$$\beta = \beta_i,\ k = k_{i+\frac{1}{2}},\ \mu = \mu_{i+\frac{1}{2}},$$

$$(\partial P/\partial x)_{i+\frac{1}{2}} \stackrel{\sim}{=} (P_{i+1} - P_i)/((\Delta x_{i+1} + \Delta x_i)/2). \qquad (40)$$

Oil Phase:

Introducing the results shown immediately above into Equation (37) gives the following relations:

$$(\text{Mass in})_i = - \frac{\rho_{os} k_{i-\frac{1}{2}} \gamma_{oi-\frac{1}{2}} A (P_i - P_{i-1}) \Delta t}{\beta_{oi-1} \left\{ \frac{\Delta x_i + \Delta x_{i-1}}{2} \right\}} \qquad (2\text{-M})$$

$$(\text{Mass out})_i = - \frac{\rho_{os} k_{i+\frac{1}{2}} \beta_{oi+\frac{1}{2}} A (P_{i+1} - P_i) \Delta t}{\beta_{oi} \left\{ \frac{\Delta x_{i+1} + \Delta x_i}{2} \right\}} \qquad (3\text{-M})$$

$$(\text{Mass injected})_i = (Q_{oinj})_i\ \rho_{os} \Delta t \qquad (4\text{-M})$$

$$\rho_{or}\ (PV)_i = \Delta x_i A \frac{\rho_{os}}{\beta_{oi}} \phi\ . \qquad (5\text{-M})$$

The equality $\rho_{or} = \frac{\rho_{os}}{\beta_o}$ is assumed applicable and is shown in detail in Appendix B.

Substituting the results shown in Equations (2-M), (3-M), (4-M) and (5-M) into Equation (1-M) and introducing the terms HKX_i and HKX_{i-1} as defined by Equations (16) and (17), yields:

$$\Delta S_o = -\frac{\beta_{oi}\Delta t}{\Delta x_i \phi}\left[\frac{HKX_{i-1}\gamma_{oi-\frac{1}{2}}(P_i - P_{i-1})}{\beta_{oi-1}} - \frac{HKX_i\gamma_{oi+\frac{1}{2}}(P_{i+1}-P_i)}{\beta_{oi}} - \frac{(Q_{oinj})_i}{h\Delta y}\right]^{n+\frac{1}{2}} \qquad (6\text{-M})$$

where $A = h\Delta y$. It is well to note that all rock and fluids properties are evaluated at an average time, namely the $(n+\frac{1}{2})$ time level.

Water Phase:

Following a similar development as that shown for the oil phase, an equation for the change of water saturation during time increment Δt may be written as,

$$\Delta S_w = -\frac{\beta_{wi}\Delta t}{\Delta x_i}\left[\frac{HKX_{i-1}\gamma_{wi-\frac{1}{2}}(P_i - P_{i-1})}{\beta_{wi-1}} - \frac{HKX_i\gamma_{wi+\frac{1}{2}}(P_{i+1}-P_i)}{\beta_{wi}} - \frac{(Q_{winj})_i}{h\Delta y}\right]^{n+\frac{1}{2}} . \qquad (7\text{-M})$$

Gas Phase:

For the gas phase, an additional term should be incorporated into Equation (1-M) to account for the gas evolution as defined by Equation (9). With this additional term, the following equation describes the gas saturation incremental change:

$$\Delta S_{gi} = (\text{mass in})_i + (\text{mass of gas evolved})_i - (\text{mass out})_i + (\text{mass injected})_i / \rho_{gri}(PV)_i \ .$$

An explanation of the term accounting for solution gas was given previously and the term is listed as Equation (9). The present handling of this term is identical except that the ratio $(\frac{S_o}{\beta_o})$ will be factored out of the derivation as a constant at the average time over the increment. This is a realistic assumption since permitting $(\frac{S_o}{\beta_o})$ to vary may give rise to a situation in which all the gas in solution in the ΔS_o increment would be released during the time step. Thus, the "gas evolved term" in difference form will be,

$$\frac{\partial M_{FG}}{\partial t}\Delta t = -\Delta x_i A \phi \rho_{gs} \left(\frac{S_o}{\beta_o}\right)^{n+\frac{1}{2}} (R_s^{n+1} - R_s^{n})_i \ . \qquad (9\text{-M})$$

Applying logic like that used to derive ΔS_i for the oil and water phases and incorporating the gas evolution term, an equation may be written for the change in gas saturation in block (i) during time increment (Δt) as follows:

$$\Delta S_{gi} = -\frac{\beta_{gi}\Delta t}{\Delta x_i \phi}\left[\frac{HKX_{i-1}\gamma_{gi-\frac{1}{2}}(P_i - P_{i-1})}{\beta_{gi-1}} - HKX_i \gamma_{gi+\frac{1}{2}}(P_{i+1} - P_i) - \frac{(Q_{ginj})_i}{h\Delta y}\right]^{n+\frac{1}{2}} - \left\{\frac{S_o}{\beta_o}\right\}_i^{n+\frac{1}{2}} (R_s^{n+1} - R_s^{n})_i \beta_{gi}^{n+\frac{1}{2}} \ . \qquad (10\text{-M})$$

For simplicity, Equations (6-M), (7-M) and (10-M) may be written as follows:

$$\Delta S_{oi} = ZZ(AMOX1-AMOX2-QOTERM) \tag{11-M}$$

$$\Delta S_{wi} = ZZ(AMWX1-AMWX2-QWTERM) \tag{12-M}$$

$$\Delta S_{gi} = ZZ(AMGX1-AMGX2-QGTERM)-RSTERM \tag{13-M}$$

where,

$$ZZ = -\Delta t/\phi\Delta x_i$$

$$AMOX1 = HKX_{i-1}\left[\gamma_{oi-\frac{1}{2}}(P_i-P_{i-1})\left(\frac{\beta_{oi}}{\beta_{oi-1}}\right)\right]^{n+\frac{1}{2}}$$

$$AMWX1 = HKX_{i-1}\left[\gamma_{wi-\frac{1}{2}}(P_i-P_{i-1})\left(\frac{\beta_{wi}}{\beta_{wi-1}}\right)\right]^{n+\frac{1}{2}}$$

$$AMGX1 = HKX_{i-1}\left[\gamma_{gi-\frac{1}{2}}(P_i-P_{i-1})\left(\frac{\beta_{gi}}{\beta_{gi-1}}\right)\right]^{n+\frac{1}{2}}$$

$$AMOX2 = HKX_i\left[\gamma_{oi+\frac{1}{2}}(P_{i+1}-P_i)\right]^{n+\frac{1}{2}}$$

$$AMWX2 = HKX_i\left[\gamma_{wi+\frac{1}{2}}(P_{i+1}-P_i)\right]^{n+\frac{1}{2}}$$

$$AMGX2 = HKX_i\left[\gamma_{gi+\frac{1}{2}}(P_{i+1}-P_i)\right]^{n+\frac{1}{2}}$$

$$QOTERM = \frac{\beta_{oi}(Q_{winj})_i}{h\Delta y}$$

$$\text{QWTERM} = \frac{\beta_{wi}(Q_{winj})_i}{h\Delta y}$$

$$\text{QGTERM} = \frac{\beta_{gi}(Q_{ginj})_i}{h\Delta y}$$

$$\text{RSTERM} = \left(\frac{S_o}{\beta_o}\right)^{n+\frac{1}{2}} (R_s^{n+1} - R_s^n)\ \beta_{gi}\ .$$

The equations for updating oil, water and gas saturations are,

$$S_{oi}^{n+1} = S_{oi}^{n} + \Delta S_{oi}\ ;$$

$$S_{wi}^{n+1} = S_{wi}^{n} + \Delta S_{wi}\ ;$$

and,

$$S_{gi}^{n+1} = S_{gi}^{n} + \Delta S_{gi}\ .$$

V. RESULTS

Computational procedures for each of the analyses were developed and programmed in Fortran IV and the calculations were made by an IBM 360-50 computer. Flow diagrams and digital computer programs are presented in Appendix D.

Results of seventeen studies using four basic models are shown graphically by Figures 9, 10, 11, 12 and 13, and a summary of study runs with results are shown in Table I. Reservoir configuration and computational data are presented in Appendix C. Of the four basic models used in the study, two were numerical models, and two were computer-programmed versions of the Dykstra'-Parsons' technique.

Variable permeability in the vertical direction was simulated numerically by combining four linear models such that it simulates four vertical layers of different permeabilities. The first of these models included horizontal variations in permeability and is referred to as the "heterogeneous layered model". The results from a single run for an oil-water viscosity ratio of 1.96 using this model are presented in Figures 9, 10, and 11 as curve Number 4 in each, and as indicated on the graph in Figures 12 and 13.

The second of the numerical models used is layered in four vertical zones just as in the first model; however, each layer is homogeneous and the value of permeability for a layer was computed as a series average of the heterogeneous permeabilities in the first model. This second model is referred to as the "homogeneous layered model" and results of six runs obtained using this model are presented. The results for an oil-water viscosity ratio of 1.96 are shown for comparison purposes in Figures 9, 10 and 11, and referred to as curve Number 1. The results for oil-water viscosity ratios of 0.5, 1.96, 5.0, 10.0, 20.0 and 40.0 are shown in Figures 12 and 13.

The first Dykstra-Parsons model is layered in four vertical zones, and the value for permeability of a layer is the same as in the second version of the numerical model. The results of five runs are presented. The results for an oil-water viscosity ratio of 1.96 are shown in Figures 9, 10 and 11 and referred to as curve Number 2 and results using oil-water viscosity ratios of .5, 1.96, 5.0, 10.0 and 20.0 are shown in Figures 12 and 13.

The second Dykstra-Parsons model is layered in twenty-five vertical zones with an equal thickness of one foot in each of the layers. The permeability of each layer was read from Figure 8, which was obtained by plotting the series-averaged value for permeability of each of the four layers used previously as a function of cumulative thickness. Results of five runs using this model are presented. The results for an oil-water viscosity ratio of 1.96 are shown in Figures 9, 10 and 11, and referred to as curve Number 3 and results for oil-water viscosity ratios of 0.5, 1.96, 5.0, 10.0 and 20.0 are shown in Figures 12 and 13.

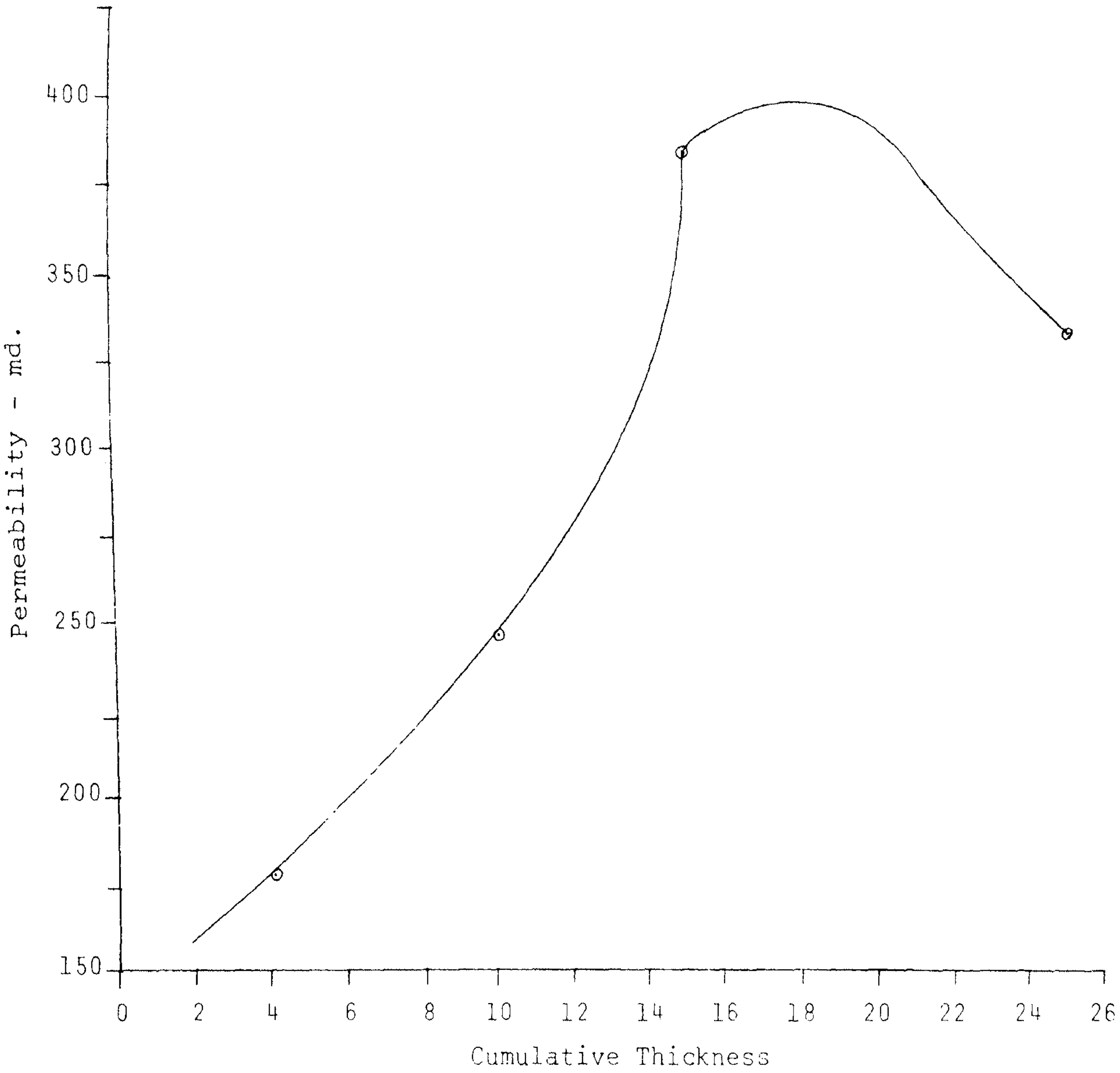


Fig. 8. Permeability Variations with Depth

TABLE I. Summary of Results

Model	% Recovery at WOR = 10 for Oil-Water Viscosity Ratio of					
	.5	1.96	5	10	20	40
Numerical (Heterogeneous Layered)	--	51.4	--	--	--	--
Numerical (Homogeneous Layered)	79.5	79	9.2	7.05	5.3	3.18
Dykstra-Parsons (25 Layers)	74.6	73.6	68.9	62.3	56	--
Dykstra-Parsons (4 Layers)	72.7	71.7	70.9	68.1	61.5	--

VI. DISCUSSION OF RESULTS

Figure (9) illustrates cumulative oil produced as a function of time for an oil-water viscosity ratio of 1.96 based on results from the four basic models described previously. Water breakthrough in all models occurred at approximately the same time (120-130 days). Curve 1 (numerical-homogeneous layers) and Curves 2 and 3 (Dykstra-Parsons, 4 layers and 25 layers, respectively) showed close agreement until water breakthrough occurs and slight deviation thereafter. Curve 2 was in closer agreement with Curve 1 than Curve 3 as expected, since the models of Curves 1 and 2 are in closer agreement, theoretically. However, final recovery was the same for both Dykstra-Parson models. Curve 4 (the numerical-heterogeneous layered model) showed a great reduction in recovery compared to Curves 1, 2 and 3. This was primarily due to the formation of free gas resulting from the heterogeneity in the fourth layer, which in turn caused the reservoir pressure to decline below the bubble point pressure. Although the other layers in this model were also heterogeneous, there was less variation in the permeabilities in these blocks prohibiting notable gas formation.

Figure (10), illustrates cumulative oil produced as a function of water-oil ratio. Curves 1 and 4 show that water-oil ratio changes gradually over most of the region except for regions with periodic rapid increases. This is explained by the fact that when any layer breaks through, the composite water-oil ratio increases significantly.

Thereafter, the water-oil ratio increases were gradual due to continually changing saturations until another layer breaks through resulting in another sudden increase. This face is not as pronounced in Curves 2 and 3 (Dykstra-Parsons), even though the results show some gradual changes in water-oil ratio. Because of the assumption of piston-like displacement in the Dykstra-Parsons method, only water is flowing in any layer after water breakthrough. The gradual change in water-oil ratio is due to changing velocities within the layers due to the front continually advancing in layers of different permeabilities. The changes are not due to saturation changes at the production face. In fact, Curves 2 and 3 are very nearly step-functions. It is also of interest to note that areal non-uniformity in permeability increased the life of the reservoir but resulted in less oil recovery and consistently higher water-oil ratios.

Figure (11) illustrates oil production rate as a function of time. Curve 1 shows fluctuations at the beginning of water injection which is due to the unsteady state nature of the model. A sudden decrease in oil production rate occurs when water breaks through in the most permeable layer. Curve 4 shows a sudden decrease in oil production rate until water breakthrough, and this is due to the formation of free gas.

Figure (12) illustrates the relationship between cumulative oil produced and cumulative water injected for various oil-water viscosity ratios. As expected, the recovery decreased as oil-water viscosity ratio increased for a particular model. This is also shown in Figure (13) by results from both numerical and Dykstra-Parson models. The deviation and reduction in oil recovery as compared by results from the numerical model from Dykstra-Parsons for oil-water viscosity ratios of 5.0, 20.0, and 40.0, results from the fact that the Dykstra-Parsons approach is not adapted for conditions where free gas is formed or is present as occurs in depleted oil sands. The model assumes only two-phase fluid flow (oil and water), and the method is not suitable for gas flow due to the assumption of piston-like displacement. It is also of interest to note that an increase in oil-water viscosity ratio results in a decline of reservoir pressure and an increase in the life of the reservoir by producing less oil at a higher water-oil ratio.

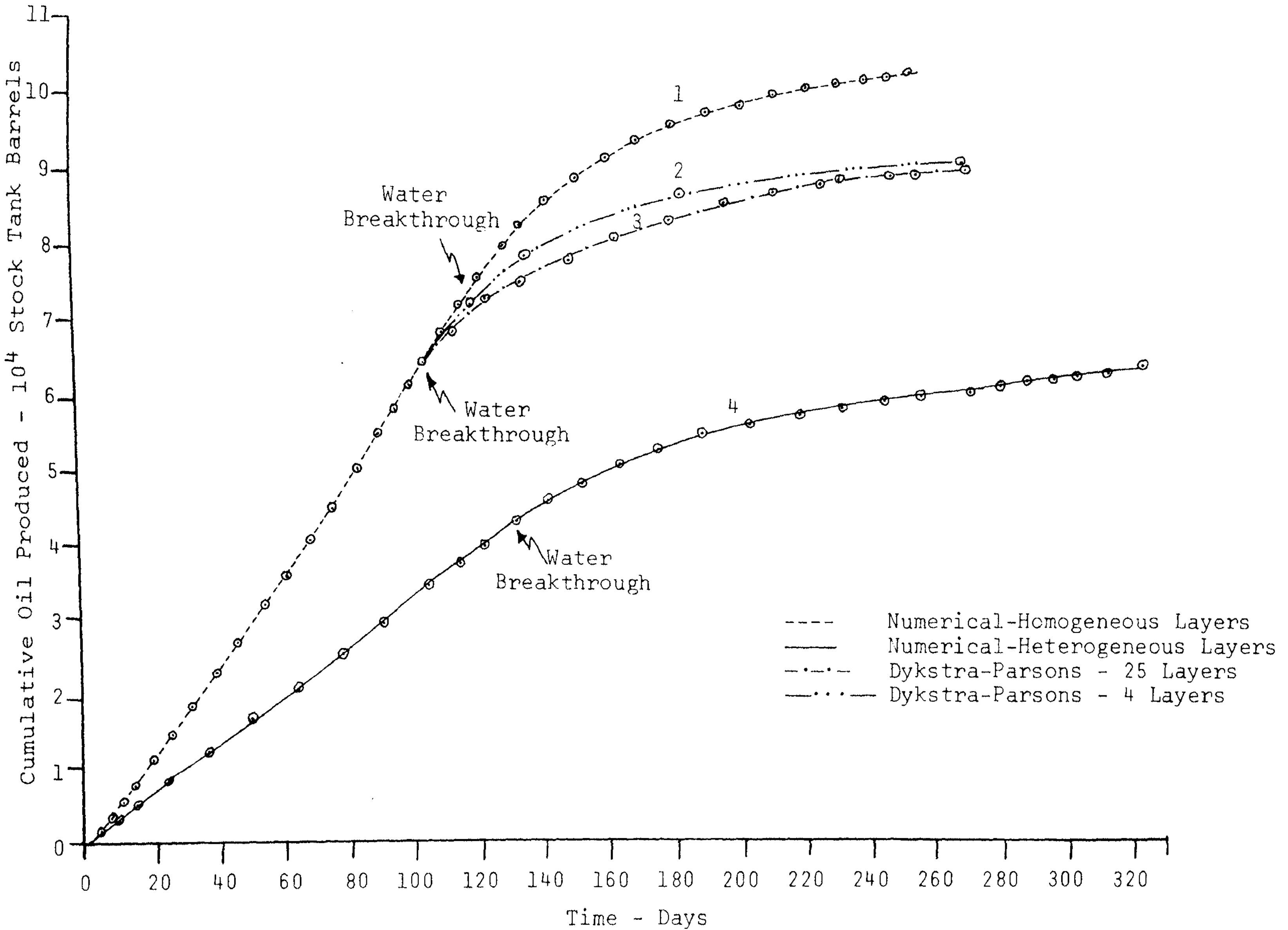


Fig. 9. Cumulative Oil Produced vs. Time for $\frac{\mu_o}{\mu_w}$ = 1.96

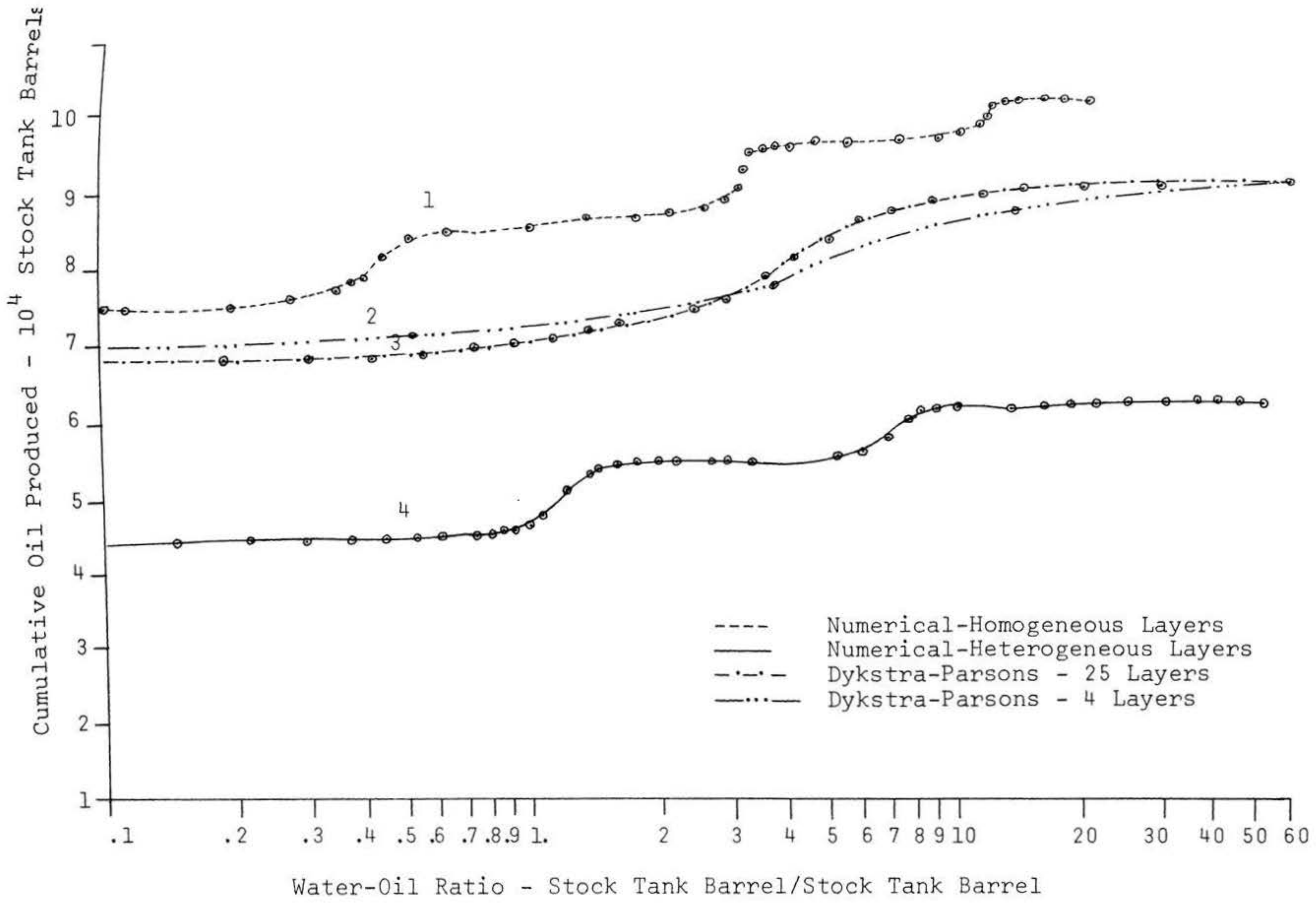


Fig. 10. Cumulative Oil Produced vs. Water-Oil Ratio for $\frac{\mu_o}{\mu_w} = 1.96$

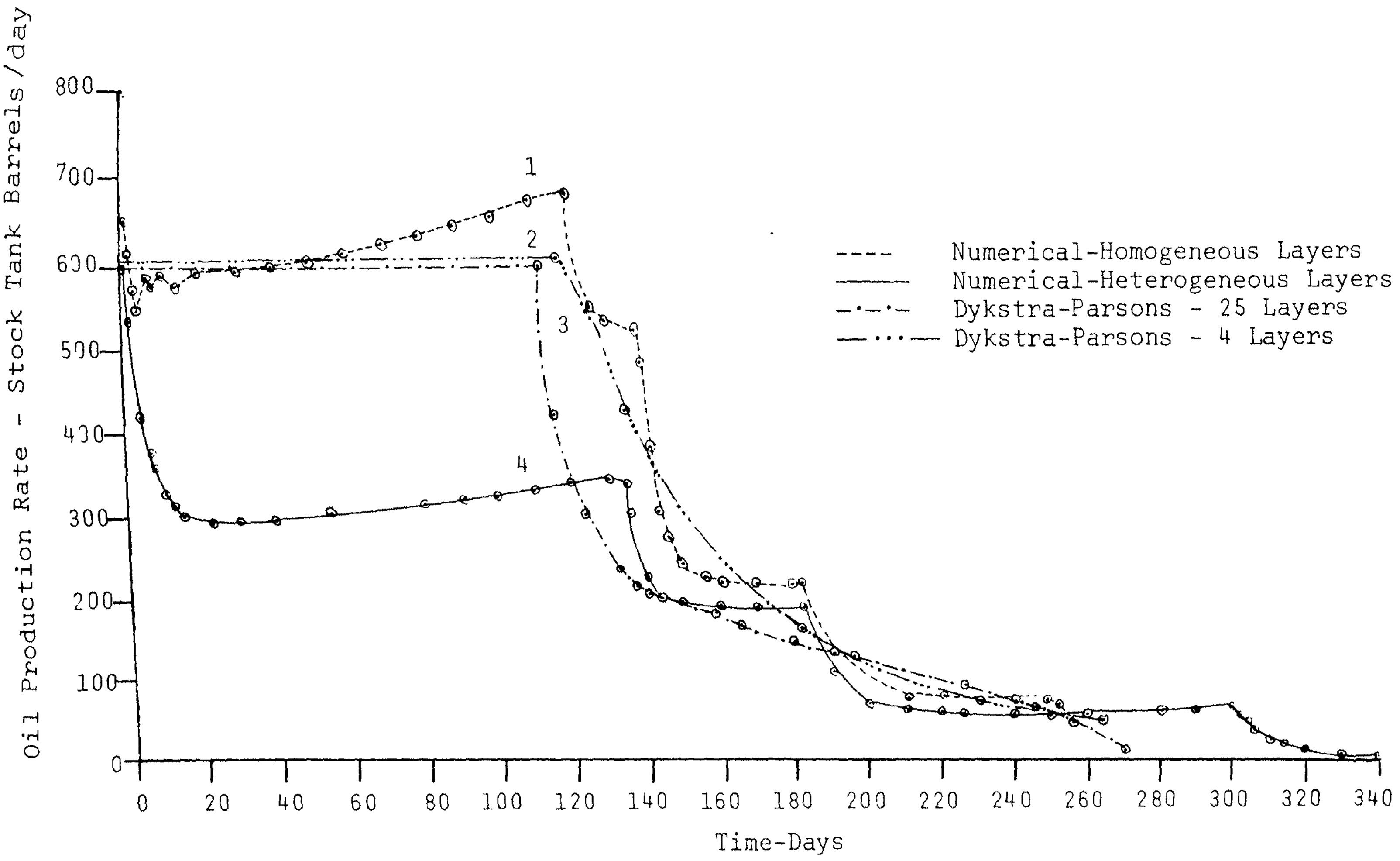


Fig. 11. Oil Production Rate vs. Time for $\frac{\mu_o}{\mu_w} = 1.96$

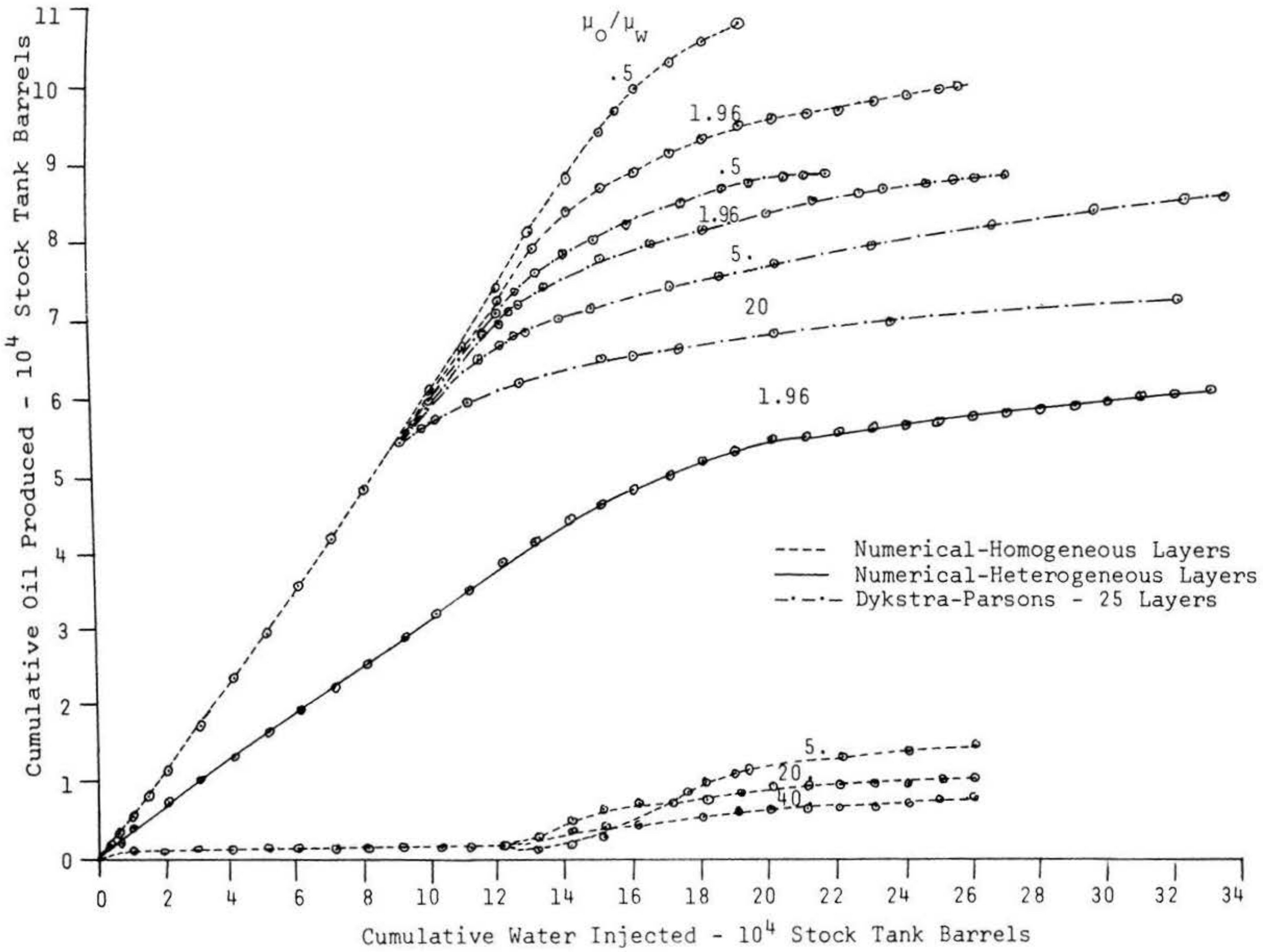


Fig. 12. Cumulative Oil Produced vs. Cumulative Water Injected for Various $\frac{\mu_o}{\mu_w}$ Ratios

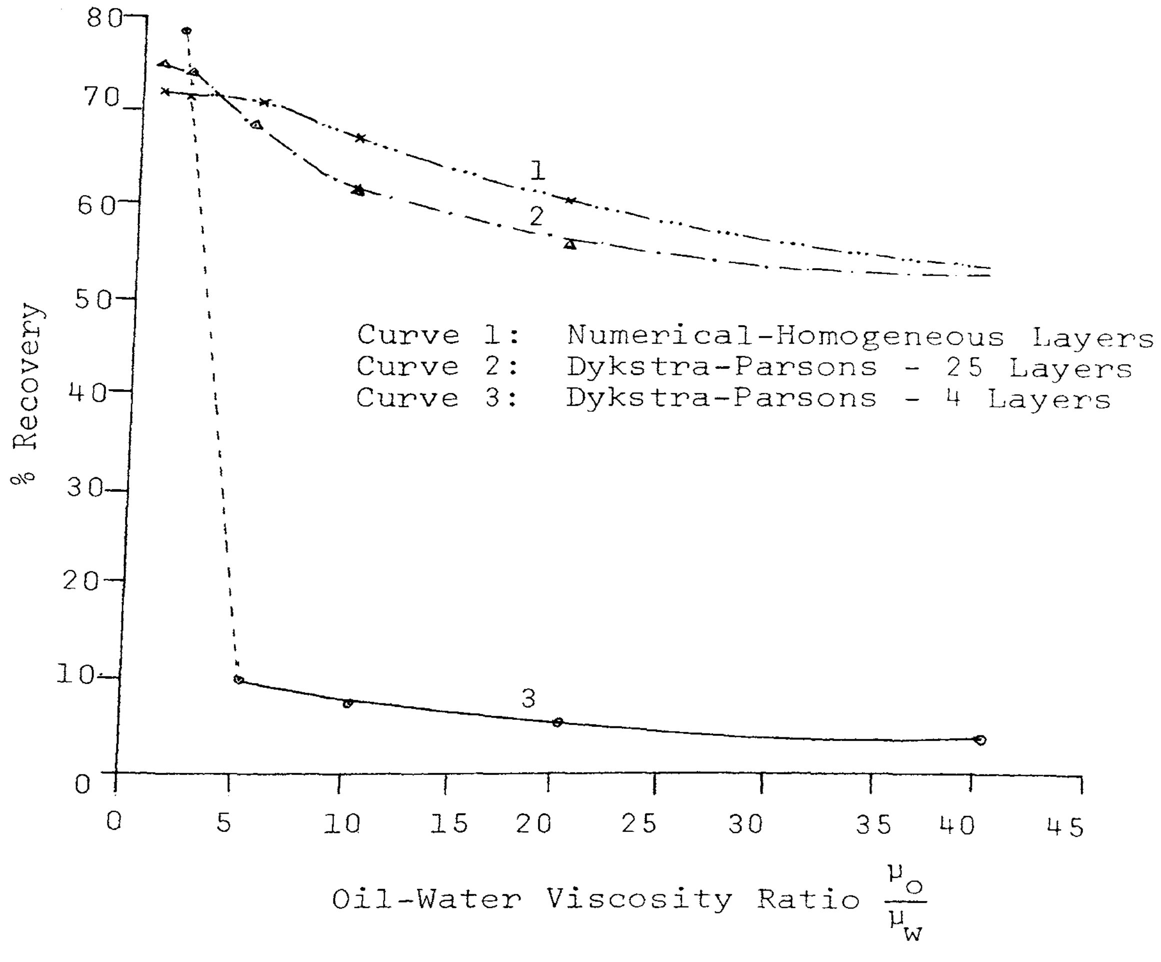


Fig. 13. Percent Recovery of Initial Oil in Place vs. Oil-Water Viscosity Ratio at WOR = 20:1

VII. CONCLUSIONS

1. Prediction of reservoir performance under a water drive by a three-phase numerical simulation model is shown to be feasible due to the favorable comparison of results with the Dykstra-Parsons method for those cases where the Dykstra-Parsons assumptions are satisfied.
2. The numerical method is valid for more general use than the Dykstra-Parsons method since the latter does not account for changes in fluid properties as a function of pressure, reservoir geometry, rock heterogeneity, and three-phase flow.
3. Results show that the prediction of performance of a water flood project by the Dykstra-Parsons method will not be valid when a mobile gas is present. The gas phase reduces oil mobility to such an extent that considerably less oil may be produced at water breakthrough than predicted by the Dykstra-Parsons technique.
4. Areal rock heterogeneity is not adequately accounted for by series-averaging individual permeabilities within a layer. Thus, a numerical simulator is needed for predictions where areal heterogeneity is obvious.
5. The Dykstra-Parsons calculations are faster than those by the numerical model and should be used in preference where conditions fit the assumptions of the Dykstra-Parsons method.

VIII. APPENDICES

APPENDIX A

1. NOMENCLATURE

A = cross sectional area normal to the flow direction, $A = \Delta yh$ - sq-cm

C = compressibility - atm^{-1}

g = acceleration of gravity - cm/sq-sec

h = layer thickness - cm

h_T = total reservoir thickness - cm

k = absolute permeability - Darcy

k_g, k_o, k_w = phase effective permeability

k_r = relative permeability

M = mass of accumulation - gm

m = number of elemental blocks in each layer

M_{FG} = mass of free gas evolved from solution gas for time period t_n to t_{n+1}

n = number of layers

P = pressure - tam

P_{md} = pressure at the interface between blocks - atm

PV = pore volume - cc

P_{ww} = pressure at top of the sand - atm

Q = production or injection rate - cc/sec

R_s = solution gas-oil ratio cc of gas/cc of oil

R_s' = $\frac{\partial R_s}{\partial P}$

S = saturation - fraction

SGC = equilibrium gas saturation - fraction

SOR = residual oil saturation - fraction

SWC = connate water saturation - fraction

t = time - sec

V = Darcy's velocity potential - cm/sec

v = volume - cc

Greek Letters and Derivatives:

β = formation volume factor - res cc/std cc

β' = $\frac{\partial \beta}{\partial P}$

γ = $\frac{k_r}{\mu \beta}$

ϕ = porosity - fraction

μ = viscosity - cp

Φ = pressure potential - atm

ρ = density - gm/cc

Δx = length of elemental block - cm

ΔP = pressure drop

ΔS_i = change in saturation to the ith elemental block from time t_n to t_{n+1} - fraction

Δt = time increment - sec

Δy = width of elemental block - cm

$\frac{\partial M}{\partial t}$ = mass rate of accumulation - gm/sec

$\frac{\partial M_{FG}}{\partial t}$ = mass rate of gas evolved from solution gas from time t_n to t_{n+1}

Subscripts:

b = bubble point

g = gas phase

i = indicating ith elemental block in x-direction

inj = injection

j = indicating jth layer in Z-direction

max = maximum

prd = production

s = standard conditions of pressure and temperature (1 atm and 60°F)

T = total

Superscript:

n = indicating nth time level

2. LIST OF COMPUTER SYMBOLS

FORTRAN SYMBOL	ALBEGRAIC SYMBOLS OR DEFINITION
AX	A
BGPRM	β_g'
BOPRM	β_o'
BTA	λ
BTAG	β_g
BTAO	β_o
BTAW	β_w
BTOBP	β_o at bubble point
BTWBP	β_w at bubble point
BX	B
COV	C = coverage
CUMOP	cumulative oil produced
CUMWIN	cumulative water injected
CX	C
DELT	Δt
DELTP	ΔN_p
DELX	Δx
DELY	Δy
DELZ	$Z_i - Z_{i-1}$
DX	D
GMA	α
GOR	gas-oil ratio

M	number of elemental blocks
N	number of layers
OILPR	cumulative oil production - N_P
P	pressure at n time level
PB	bubble point pressure
PERM	permeability - k
PERMAX	k_{max}
PHI	porosity - ϕ
PIN	initial pressure
PM	M
PN1	trial pressure for n+1 time level
PN2	pressure at n+1 time level
PRTOL	prespecified pressure tolerance check
QOR	oil production rate
RECOV	recovery = $\frac{\text{cumulative oil production}}{\text{initial oil in place}}$ x100
RKG	k_{rg}
RKO	k_{ro}
RKW	k_{rw}
RSPRM	R_s'
SATOL	prespecified saturation tolerance check
SGI	initial gas saturation
SOI	initial oil saturation
STOIP	initial oil in place
SUBROUTINE BETA	formation volume factors calculation
SUBROUTINE BETPRM	β_g', β_o', R_s' calculation

SUBROUTINE COFF	calculation of coefficients A, B, C and D
SUBROUTINE MBAL	material balance calculation
SUBROUTINE RPERM	relative permeabilities calculation
SUBROUTINE SOLGAS	solution gas-oil ratio calculation
SUBROUTINE TRIDAG	tri-diagonal solution
SUBROUTINE VISC	viscosity calculation
SUMH	total thickness
SUMKH	$\sum_{j=1}^{N} (kh)_j$
SUMQG	cumulative gas produced
SUMQO	cumulative oil produced
SUMQW	cumulative water produced
SUMQWI	$\sum_{j=1}^{N} QW_{inj,1,j}$
SW	saturation at n+1 time level
SWI	initial water saturation
SW1	saturation at n time level
SW2	trial saturation for n+1 time level
TEMP1	$(\beta_o-\beta_g R_s)_{i,j}^{n+\frac{1}{2}}$
TEMP2	$\beta_{w_{i,j}}^{n+\frac{1}{2}}$
TEMP3	$\beta_{g_{i,j}}^{n+\frac{1}{2}}$
TEMP4	$\Delta x_{i-1}/\Delta x_i$
TEMP5	$\beta_{o_{i,j}}^{n+\frac{1}{2}}$
TEMP6	$(\beta_o-\beta_g R_s)_{i-1,j}^{n+\frac{1}{2}}$
WI	cumulative water injected
WIR	water injection rate

WOR	water oil ratio
WP	cumulative water produced
XMUG	μ_g
XMUO	μ_o
XMUW	μ_w
XMUOBP	μ_o at bubble point
XMUWBP	μ_w at bubble point

APPENDIX B.

DERIVATION OF MISCELLANEOUS RELATIONSHIPS

1. Derivation of HKX

$K_{i+\frac{1}{2}}$ and $k_{i-\frac{1}{2}}$ are the series averaged values of permeability evaluated downstream and upstream, respectively, for the ith block. Each evaluation is based on the length and permeability of the two blocks bounding each respective interface horizontally.(18) The derivation of this parameter for the downstream interface is given below and is based on the geometric configuration shown in Figure (6).

Consider only that portion of the elemental block (i) and (i+1) between their mid points, an average permeability ($\bar{k}$) can be derived by Darcy's Law evaluated separately between the two blocks.

$$Q_i = \frac{k_i A_i \Delta P_i}{\mu \frac{\Delta x_i}{2}} \tag{1-A}$$

and

$$Q_{i+1} = \frac{k_{i+1} A_{i+1} \Delta P_{i+1}}{\mu \frac{\Delta x_{i+1}}{2}} \tag{2-A}$$

Note that pressures are evaluated at the mid point of each elemental block, therefore Δx's are divided by 2.

$$\Delta P_i = P_{md} - P_i$$

$$\Delta P_{i+1} = P_{i+1} - P_{md}$$

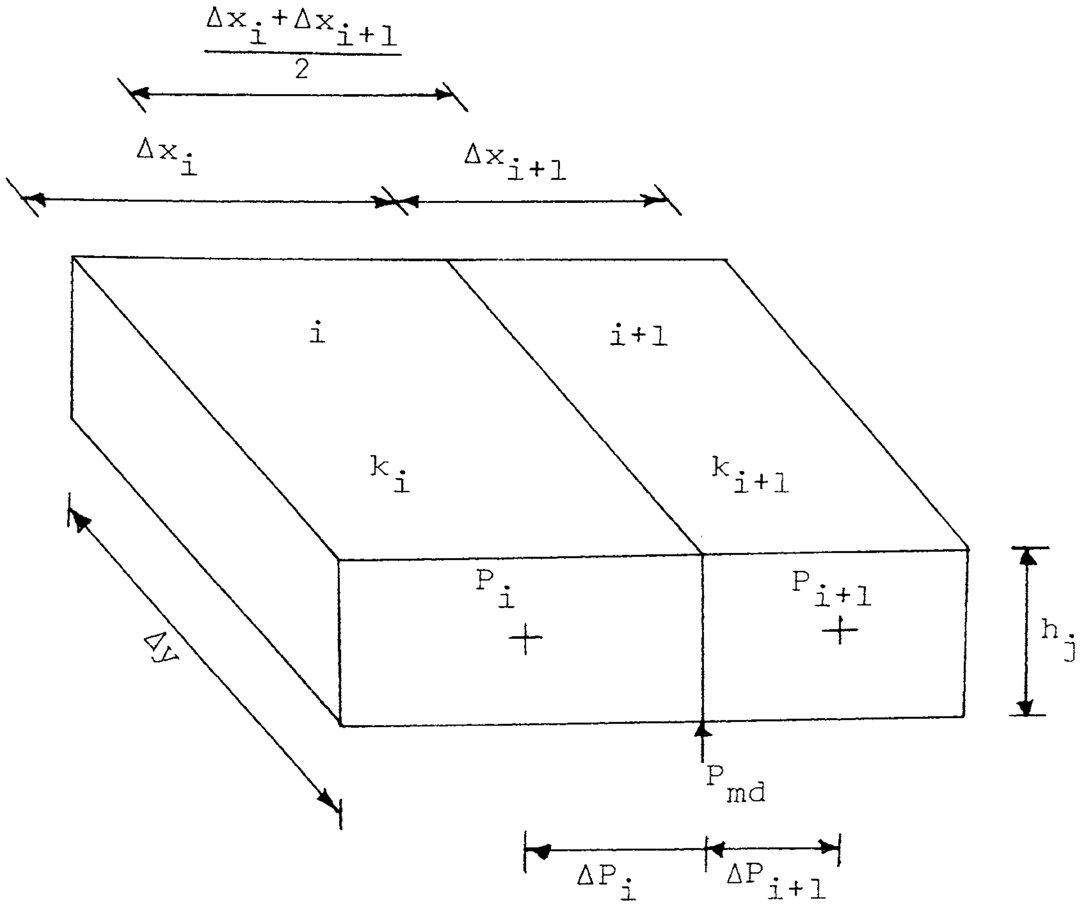


Fig. 6. Illustration of Series-Averaging Technique for Permeabilities

The total flow rate through both blocks in series is:

$$Q_T = \frac{\bar{k}A(P_{i+1}-P_i)}{\mu\frac{\Delta x_{i+1}+\Delta x_i}{2}} \quad , \ \bar{k} = k_{i+\frac{1}{2}} \tag{3-A}$$

Considering the flow rate constant through a series of elemental blocks yields:

$$Q_T = Q_i = Q_{i+1} \tag{4-A}$$

Further noting that,

$$\Delta P_i + \Delta P_{i+1} = (P_m - P_i) + (P_{i+1} - P_m) = P_{i+1} - P_i \tag{5-A}$$

Equations (1-A), (2-A) and (3-A) are solved for their respective ΔP's and these results are substituted into Equation (5-A) to yield:

$$\frac{Q\mu}{A}\left(\frac{\Delta x_{i+1}+\Delta x_i}{2\bar{k}}\right) = \frac{Q\mu}{A}\left(\frac{\Delta x_i}{2k_i} + \frac{\Delta x_{i+1}}{2k_{i+1}}\right)$$

or

$$\bar{k} = k_{i+\frac{1}{2}} = \frac{\Delta x_{i+1}+\Delta x_i}{\frac{\Delta x_i}{k_i} \quad \frac{\Delta x_{i+1}}{k_{i+1}}} = \frac{k_{i+1}k_i(\Delta x_{i+1}+\Delta x_i)}{\Delta x_i k_{i+1} + \Delta x_{i+1} k_i} \tag{6-A}$$

HKX_i is defined as,

$$HKX_i = \frac{2k_{i+\frac{1}{2}}}{\Delta x_i + \Delta x_{i+1}} \tag{7-A}$$

Substituting Equation (6-A) into Equation (7-A) yields:

$$HKX_i = \frac{2k_i k_{i+1}}{\Delta x_i k_{i+1} + \Delta x_{i+1} k_i} \tag{8-A}$$

and in the same manner:

$$HKX_{i-1} = \frac{2k_{i-1} k_i}{\Delta x_{i-1} k_i + \Delta x_i k_{i-1}} \tag{9-A}$$

2. Derivation of the Relationship $\rho_r = \frac{\rho_s}{\beta}$

$$\rho_r = \frac{\text{mass}}{V_r}, \quad \rho_s = \frac{\text{mass}}{V_s}$$

$$\frac{\rho_r}{\rho_s} = \frac{\text{mass}}{V_r} \times \frac{V_r}{\text{mass}} = \frac{V_s}{V_r} = \frac{1}{\beta}$$

Thus,

$$\rho_r = \frac{\rho_s}{\beta} \tag{10-A}$$

3. Linear Extrapolation

Assume a linear relationship between time and pressure Figure (7), then,

$$\frac{P^n - P^{n+1}}{t^{n+1} - t^n} = \frac{P^{n+1} - P^{n+2}}{t^{n+2} - t^{n+1}} \tag{11-A}$$

Letting

$$\Delta t_1 = t^{n+1} - t^n$$

and

$$\Delta t = t^{n+2} - t^{n+1}$$

Then solving Equation (11-A) for P^{n+2}, gives,

$$P^{n+2} = P^{n+1} + (P^{n+1} - P^{n}) \frac{\Delta t}{\Delta t_1} \tag{12-A}$$

In the same manner the extrapolated saturation is defined as:

$$S^{n+2} = S^{n+1} + (S^{n+1} - S^{n}) \frac{\Delta t}{\Delta t_1} \tag{13-A}$$

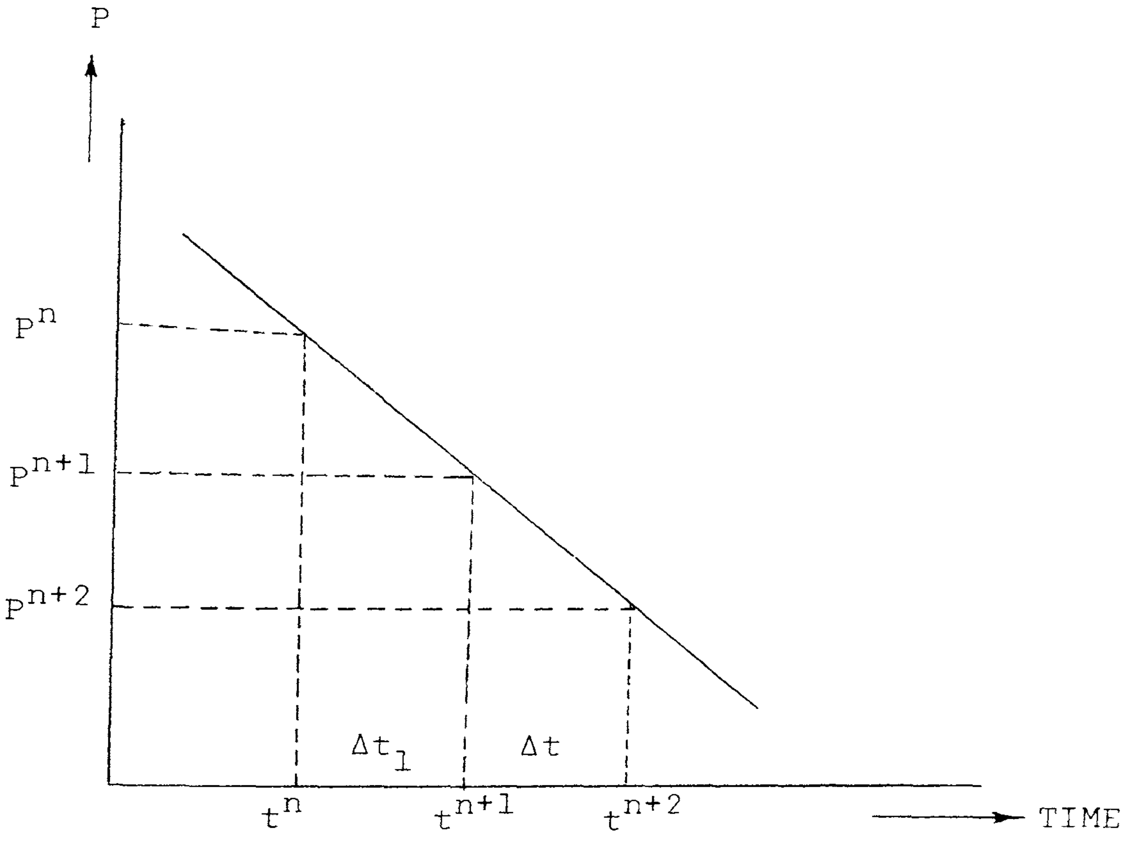


Fig. 7. Linear Extrapolation

APPENDIX C.

RESERVOIR DATA

A reservoir to be water flooded will be considered to have the following geometry, rock and fluid properties,*

Numerical Method (Heterogeneous Layered Case):

i	1	2	3	4	5	6	7	8	
j	Δx	Δx	Δx	Δx	Δx	Δx	Δx	Δx	
1	.15	.18	.17	.16	.2	.19	.19	.20	h = 4 ft.
2	.25	.23	.27	.22	.25	.27	.26	.25	h = 6 ft.
3	.35	.38	.37	.38	.40	.42	.39	.38	h = 5 ft.
4	.38	.36	.35	.32	.30	.38	.35	.30	h = 10 ft.
	100	150	200	250	250	200	150	100	ft.

x direction →

The number in each block in the above diagram is an absolute permeability value.

Total reservoir length = 1400 feet

Total reservoir thickness = 25 feet

Reservoir width Δy = 200 feet

Total water injection rate = 1000 barrels/day

---

*The data should be input in the computer program using field units. Conversion of field units to cgs units is made within the computer program.

Bubble point pressure P_b = 3487 psi

Oil compressibility CO = .003 atm^{-1}

Water compressibility CW = .0000045 atm^{-1}

Porosity ϕ = .20

Connate water saturation SWC = .2

Residual oil saturation SOR = .2

Initial oil saturation SOI = .8

Initial water saturation SWI = .2

Initial gas saturation SGI = 0.0

Equilibrium gas saturation SGC = .05

Oil formation volume factor at bubble point β_{ob} = 1.65

Water formation volume factor at bubble point β_{wb} = .95

Initial pressures in all the blocks of the first layer = 3487 psi

Initial pressures in the blocks of other layers are calculated by:

$$P_{i,j} = P_{i,j-1} + .433\left(\frac{h_j + h_{j-1}}{2}\right)$$

The fluid properties were suitably curve-fitted with the following empirical equations (using the least squares technique).

$$\left.\begin{aligned} \beta_o &= 1.09816 + 1.3487\text{x}10^{-4}\ P + 6.8235\text{x}10^{-9} P^2 \\ \frac{\partial \beta_o}{\partial P} &= 1.3487\text{x}10^{-4} + 2(6.82357)\text{x}10^{-9} P \end{aligned}\right\} \quad P < P_b$$

$$\left.\begin{aligned} \beta_o &= \beta_{ob}\ e^{-C_o(P-P_b)} \\ \frac{\partial \beta_o}{\partial P} &= -\ C_o \beta_o \end{aligned}\right\} \quad P > P_b$$

$$\beta_w = \beta_{wb}\ e^{-C_w(P-P_b)}$$

$$\beta_g = 1./(-19.937+9.126\text{x}10^{-2}\ P-2.1086\text{x}10^{-6}\ P^2)$$

$$\frac{\partial\beta_g}{\partial P} = -\beta_g{}^2\ (.09126-2(2.1086)P)$$

$$R_s = 44.2725+.266273P+7.7775\text{x}10^{-6}\ P^2 \qquad P \leq P_b$$

$$R_s = R_{s_b} \qquad P > P_b$$

$$\frac{\partial R_s}{\partial P} = .266273+2(7.7775\text{x}10^{-6}\ P) \qquad P < P_b$$

$$\frac{\partial R_s}{\partial P} = 0 \qquad P > P_b$$

$$\mu_o = 3.98117-1.992\text{x}10^{-2}P+8.0314\text{x}10^{-7}P^2-1.9869\text{x}10^{-10}P^3 \qquad P \leq P_b$$

$$\mu_o = \mu_{o_b} + .0001(P-P_b) \qquad P > P_b$$

$$\mu_w = \mu_{w_b} + .00001(P-P_b)$$

$$\mu_g = .002474+9.244\text{x}10^{-6}P-7.6556\text{x}10^{-10}P^2$$

<u>Relative Permeabilities</u>$^{(19)}$

$$k_{rg} = \frac{S_g{}^3(2-S_g-2SWC)}{(1-SWC)^4}, \quad S_w > SWC$$

$$k_{rg} = \frac{S_g{}^3(2-S_g-2SW)}{(1-SW)^4}, \quad S_w \leq SWC$$

$$k_{rw} = \left[\frac{S_w - SWC}{1 - SWC} \right]^4 , \quad S_w > SWC$$

$$k_{rw} = 0 \quad , \quad S_w \leq SWC$$

$$k_{ro} = \frac{(1 - S_g - SW)^3 (1 - S_g + SW - 2SWC)}{(1 - SWC)^4} , \quad S_w > SWC$$

$$k_{ro} = \left[\frac{1 - S_g - S_w}{1 - SWC} \right]^4 , \quad S_w \leq SWC$$

Numerical Method (Homogeneous Layered Case):

These data are the same as those for the heterogeneous case except that the layers are homogeneous. A series-averaged value of permeability was calculated for each heterogeneous layer and was assigned to that same layer in this model. These values are:

Layer	Permeability-Darcy
1	.18
2	.25
3	.38
4	.34

Dykstra-Parsons Method:

Initial gas saturation = 0.0

Initial oil saturation = .8

Initial water saturation = .2

Residual oil saturation = .2

Connate water saturation = .2

Reservoir length = 1400 feet

Reservoir thickness = 25 feet

Reservoir width = 200 feet

β_o = 1.65

μ_w = .75

μ_o = 1.47

μ_w = 1.0

k_{rw} at the residual oil saturation = .316

k_{ro} at the connate water saturation = .75

Layer's Thickness feet	Permeability md
4	180
6	250
5	380
10	340

These permeability variations were plotted versus cumulative thickness as in Figure (8) and rearranged for equal intervals of one foot as shown in Table II.

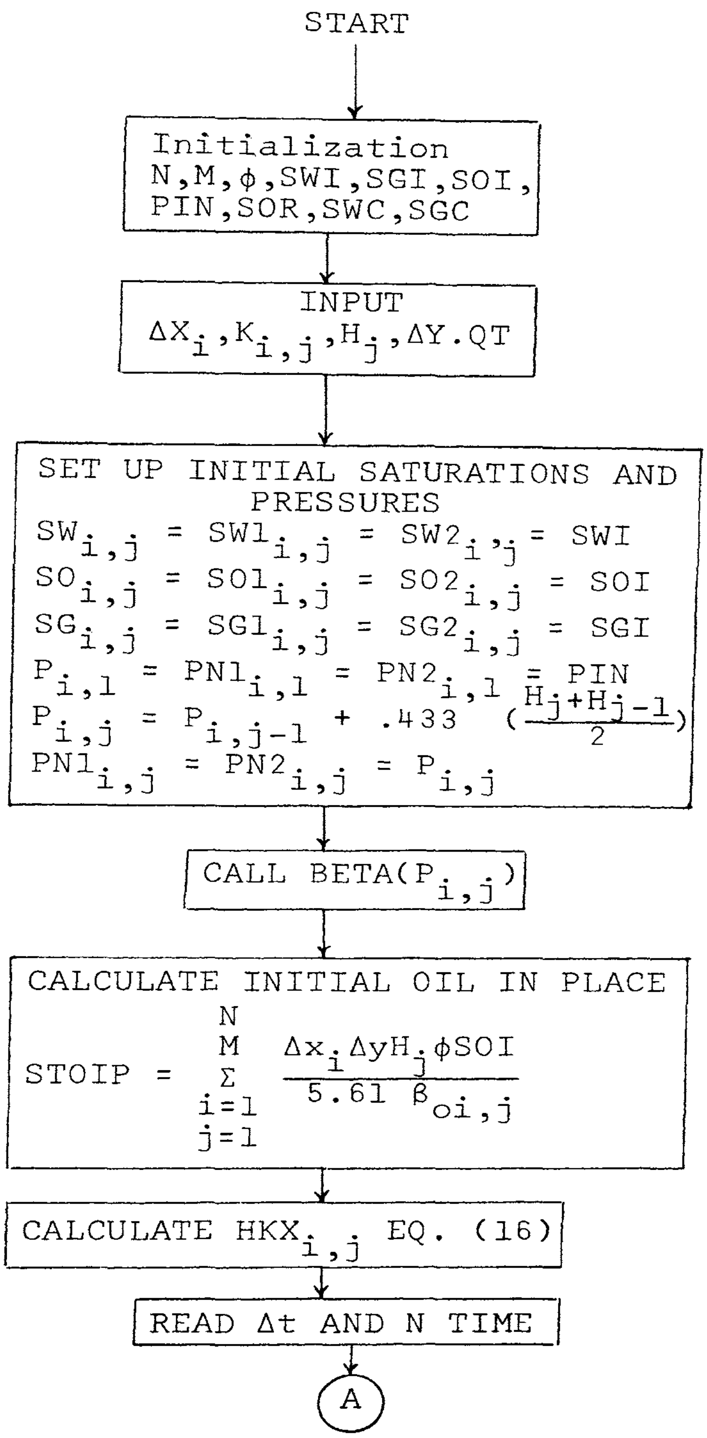
TABLE II. Rearranged Permeabilities

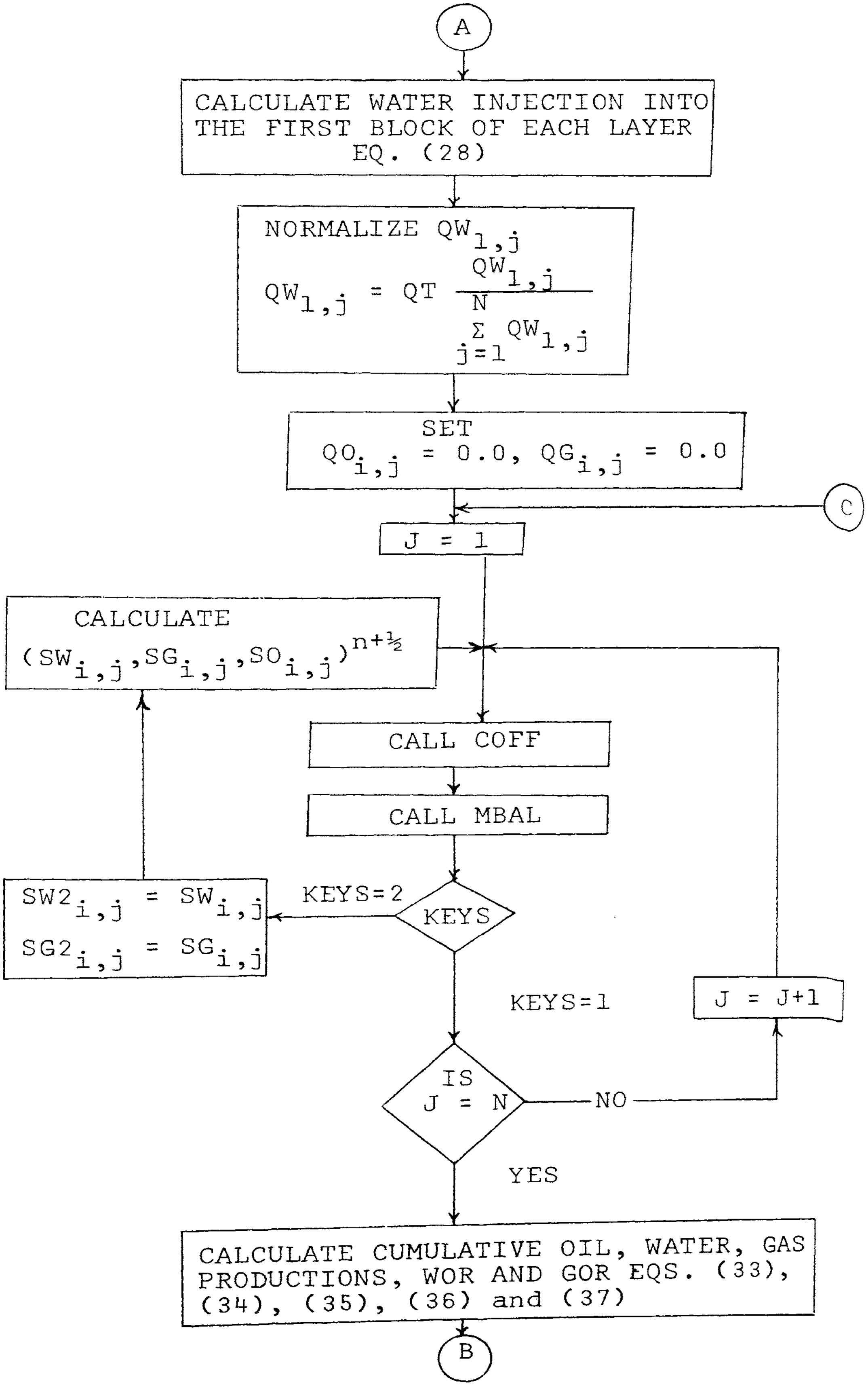
Cumulative Thickness feet	Permeability From Fig. 8 md.	Rearranged Permeability md.
1	170	398
2	172	398
3	175	397
4	180	395
5	183	388
6	189	385
7	200	380
8	212	376
9	230	365
10	250	358
11	272	350
12	298	340
13	327	327
14	358	298
15	380	272
16	395	250
17	398	230
18	398	212
19	397	200
20	388	189
21	385	183
22	376	180
23	365	175
24	350	172
25	340	170

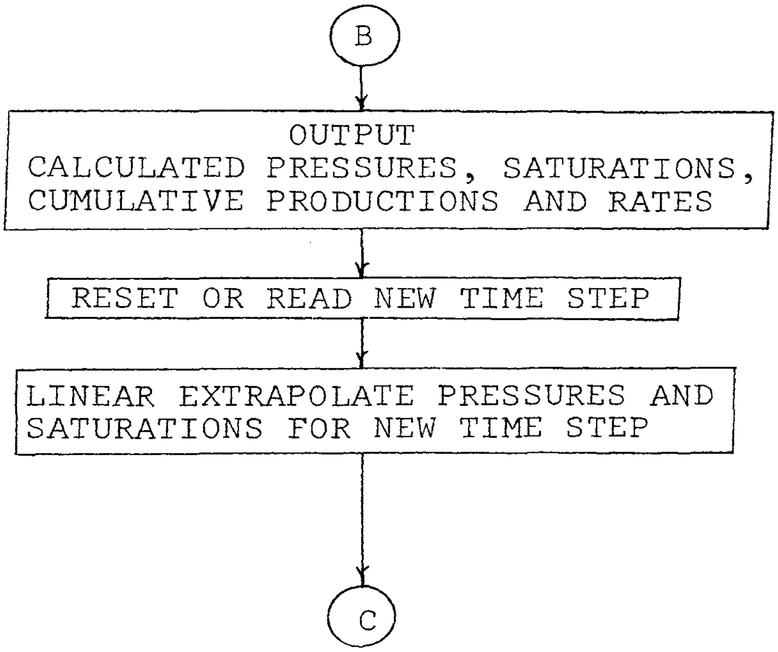
APPENDIX D

1. COMPUTER FLOW DIAGRAM FOR NUMERICAL MODEL

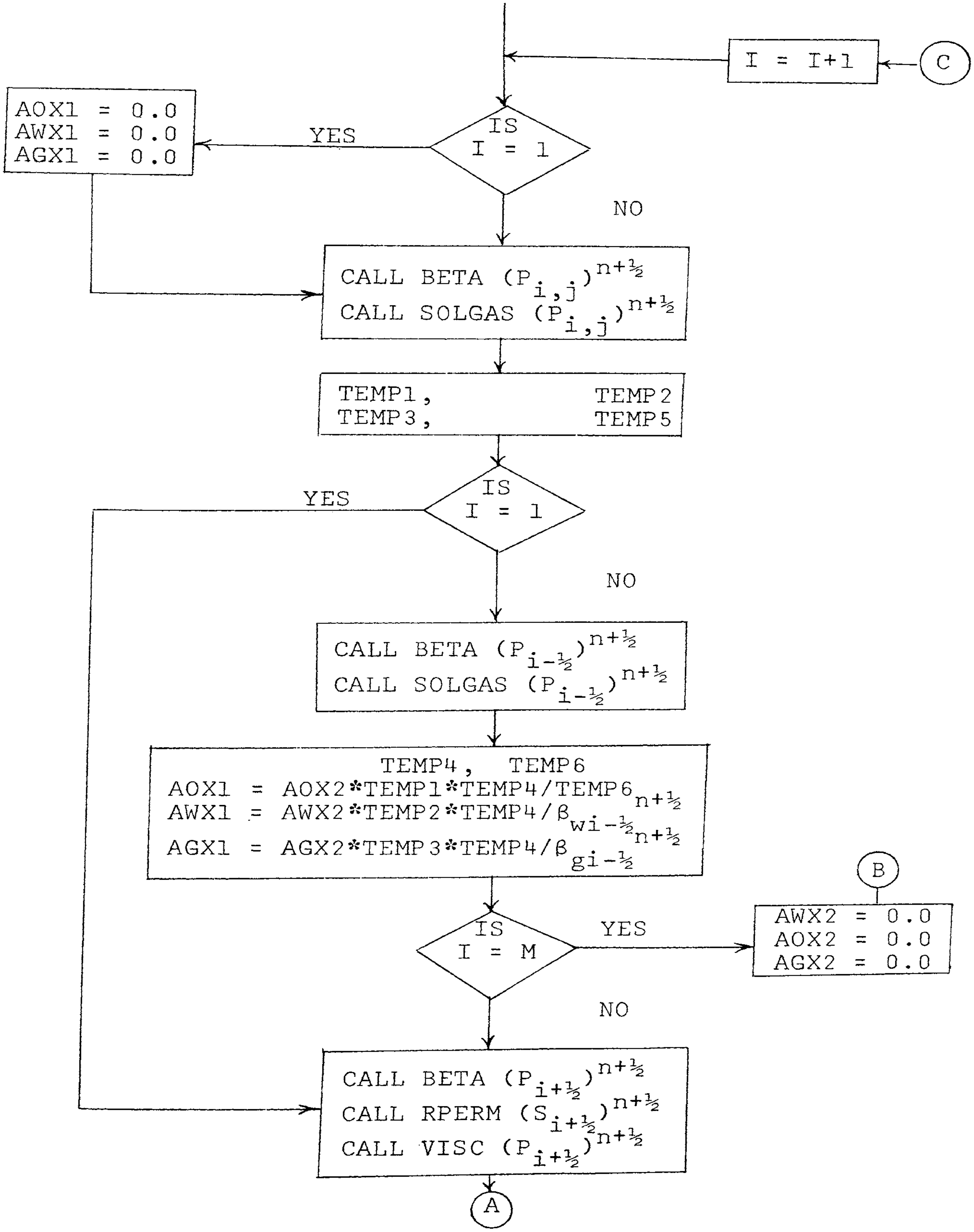
MAIN PROGRAM

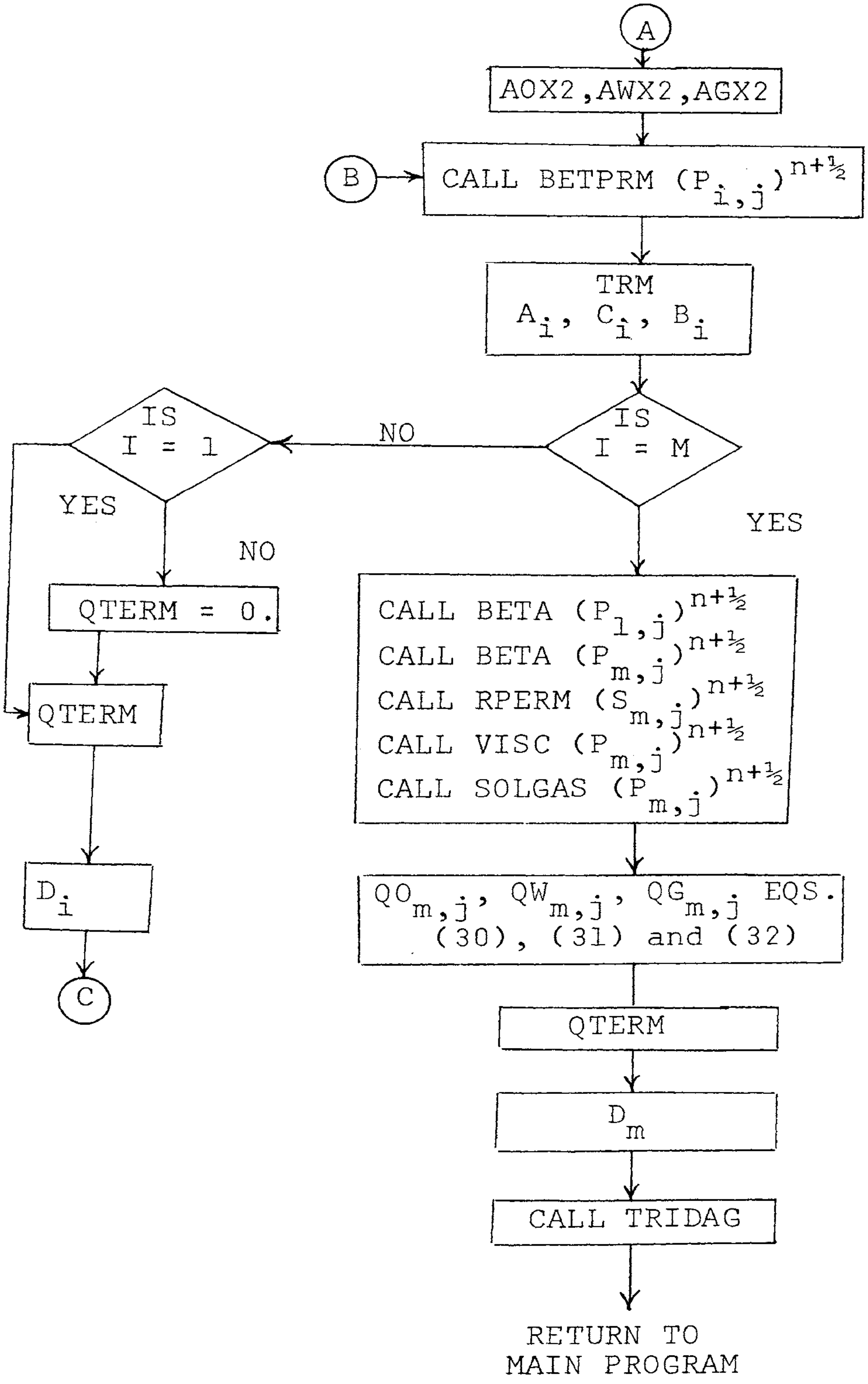



A
CALCULATE WATER INJECTION INTO THE FIRST BLOCK OF EACH LAYER EQ. (28)
NORMALIZE $QW_{1,j}$
$QW_{1,j} = QT \frac{QW_{1,j}}{\sum_{j=1}^{N} QW_{1,j}}$
SET
$QO_{i,j} = 0.0$, $QG_{i,j} = 0.0$
C
J = 1
CALCULATE
$(SW_{i,j}, SG_{i,j}, SO_{i,j})^{n+\frac{1}{2}}$
CALL COFF
CALL MBAL
KEYS=2
KEYS
$SW2_{i,j} = SW_{i,j}$
$SG2_{i,j} = SG_{i,j}$
KEYS=1
J = J+1
IS J = N
NO
YES
CALCULATE CUMULATIVE OIL, WATER, GAS PRODUCTIONS, WOR AND GOR EQS. (33), (34), (35), (36) and (37)
B

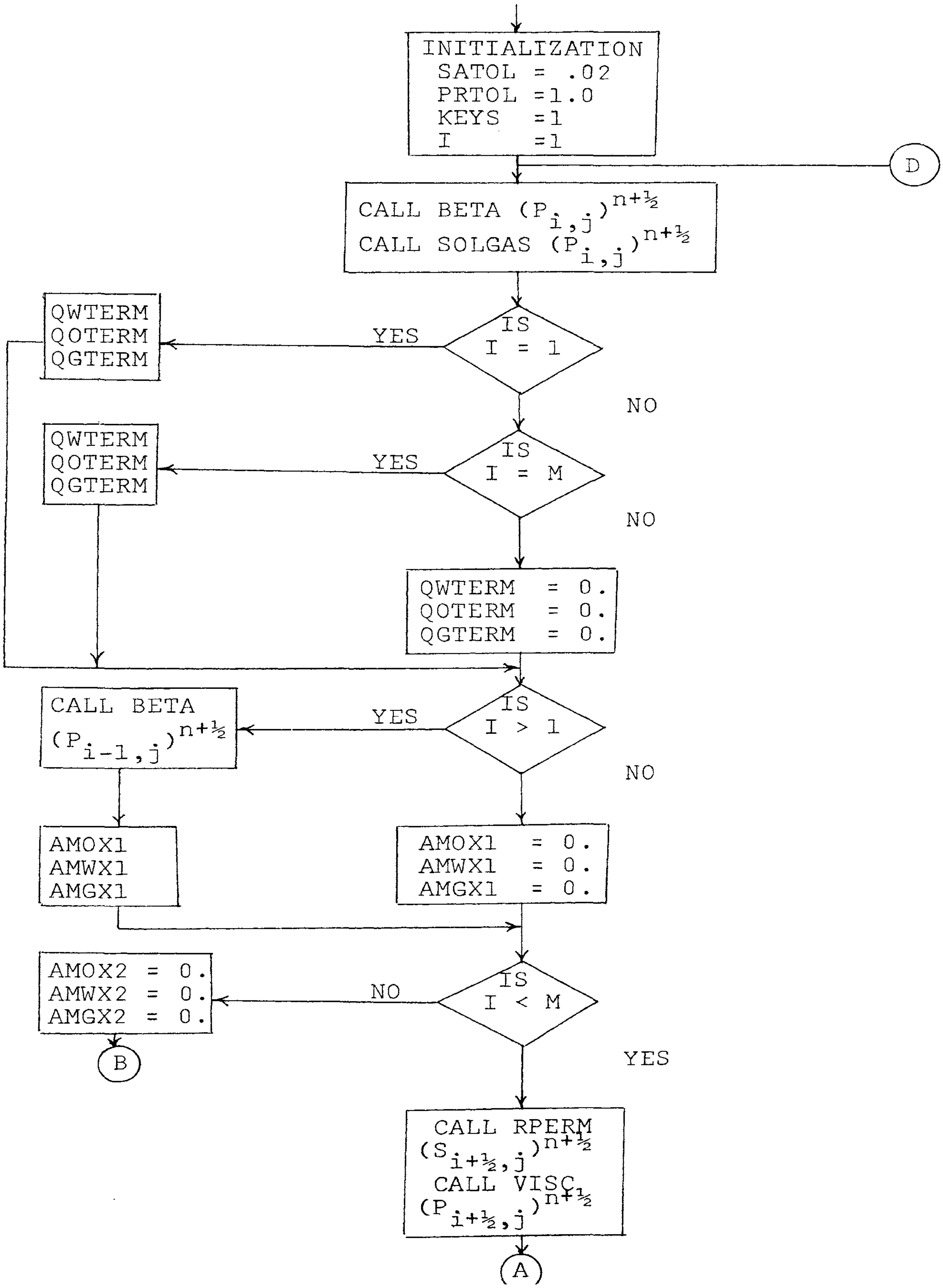

B
OUTPUT
CALCULATED PRESSURES, SATURATIONS,
CUMULATIVE PRODUCTIONS AND RATES
RESET OR READ NEW TIME STEP
LINEAR EXTRAPOLATE PRESSURES AND
SATURATIONS FOR NEW TIME STEP
C

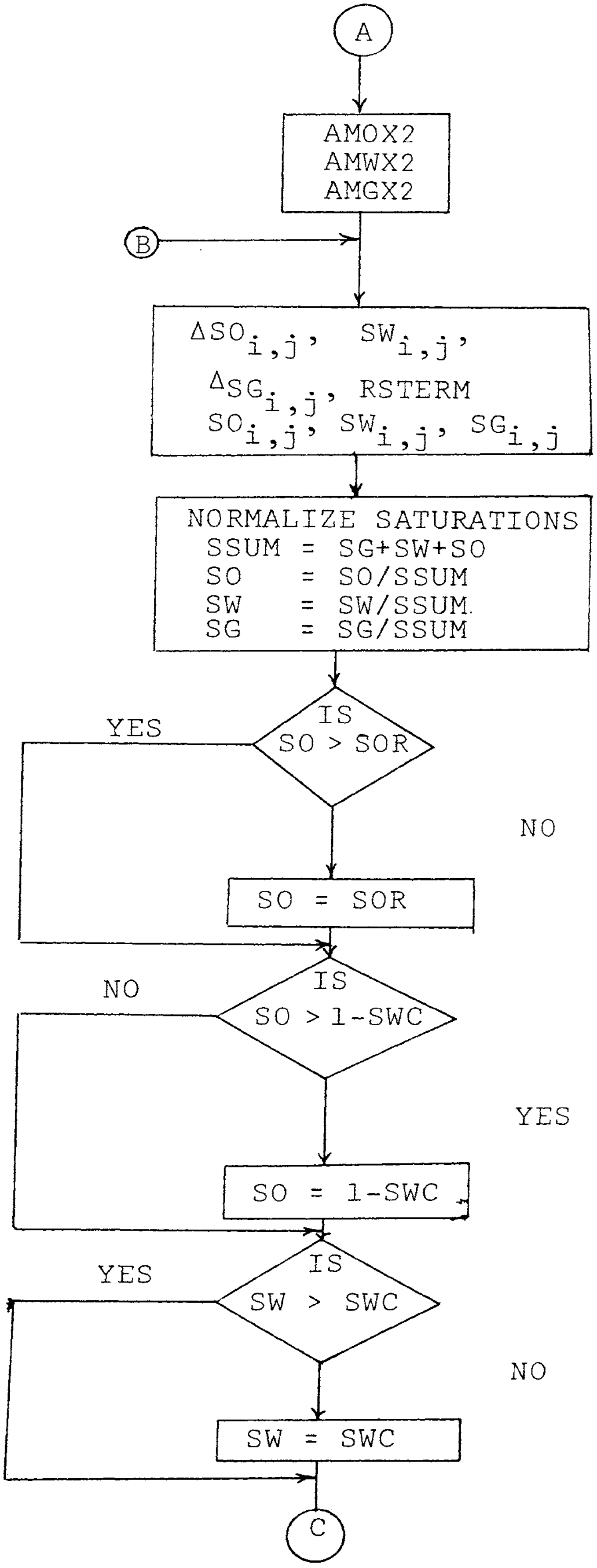
SUBROUTINE COFF

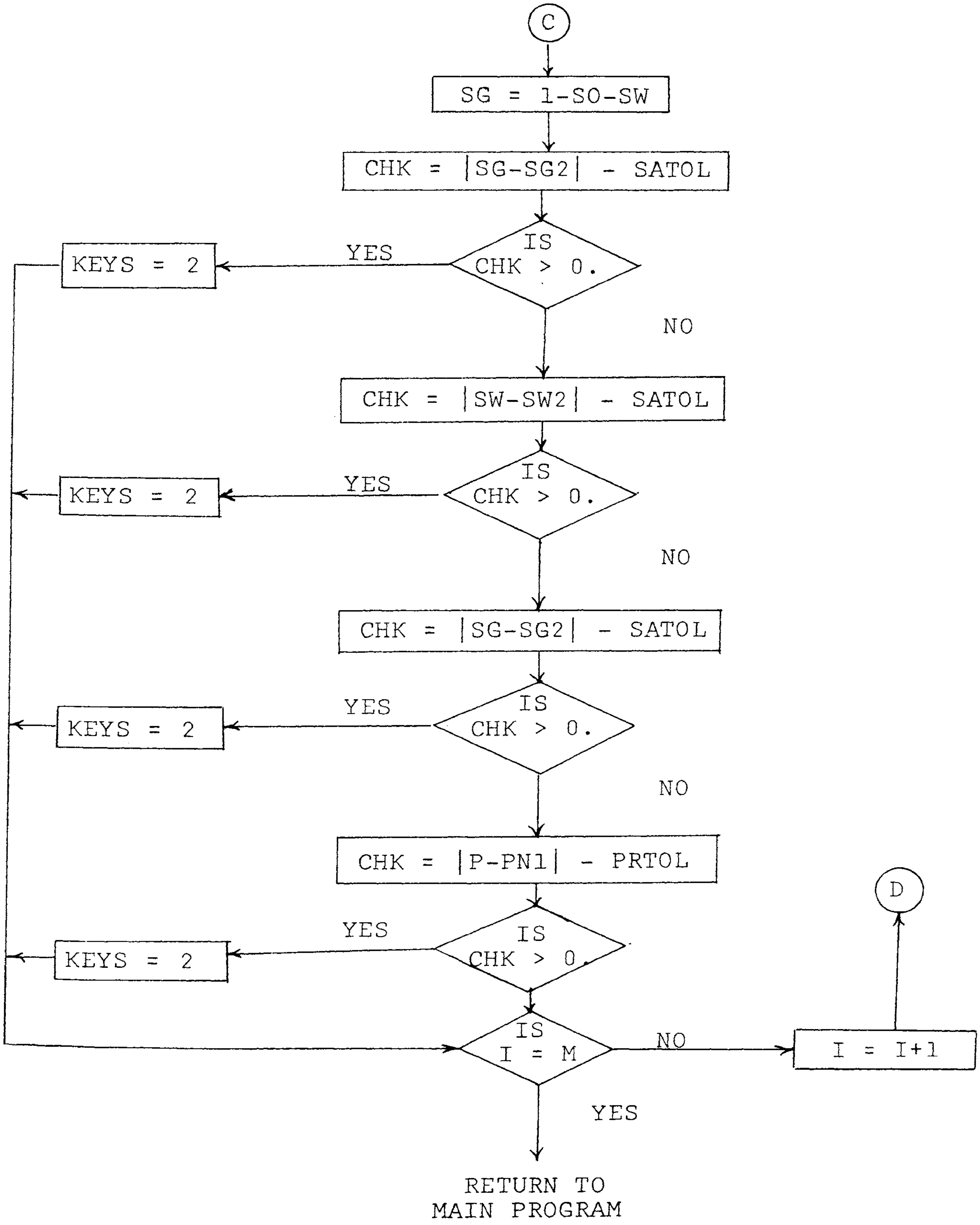



A
AOX2,AWX2,AGX2
B
CALL BETPRM $(P_{i,j})^{n+\frac{1}{2}}$
TRM
A_i, C_i, B_i
IS I = M
NO
YES
IS I = 1
YES
NO
QTERM = 0.
QTERM
D_i
C
CALL BETA $(P_{1,j})^{n+\frac{1}{2}}$
CALL BETA $(P_{m,j})^{n+\frac{1}{2}}$
CALL RPERM $(S_{m,j})^{n+\frac{1}{2}}$
CALL VISC $(P_{m,j})^{n+\frac{1}{2}}$
CALL SOLGAS $(P_{m,j})^{n+\frac{1}{2}}$
$QO_{m,j}$, $QW_{m,j}$, $QG_{m,j}$ EQS. (30), (31) and (32)
QTERM
D_m
CALL TRIDAG
RETURN TO MAIN PROGRAM

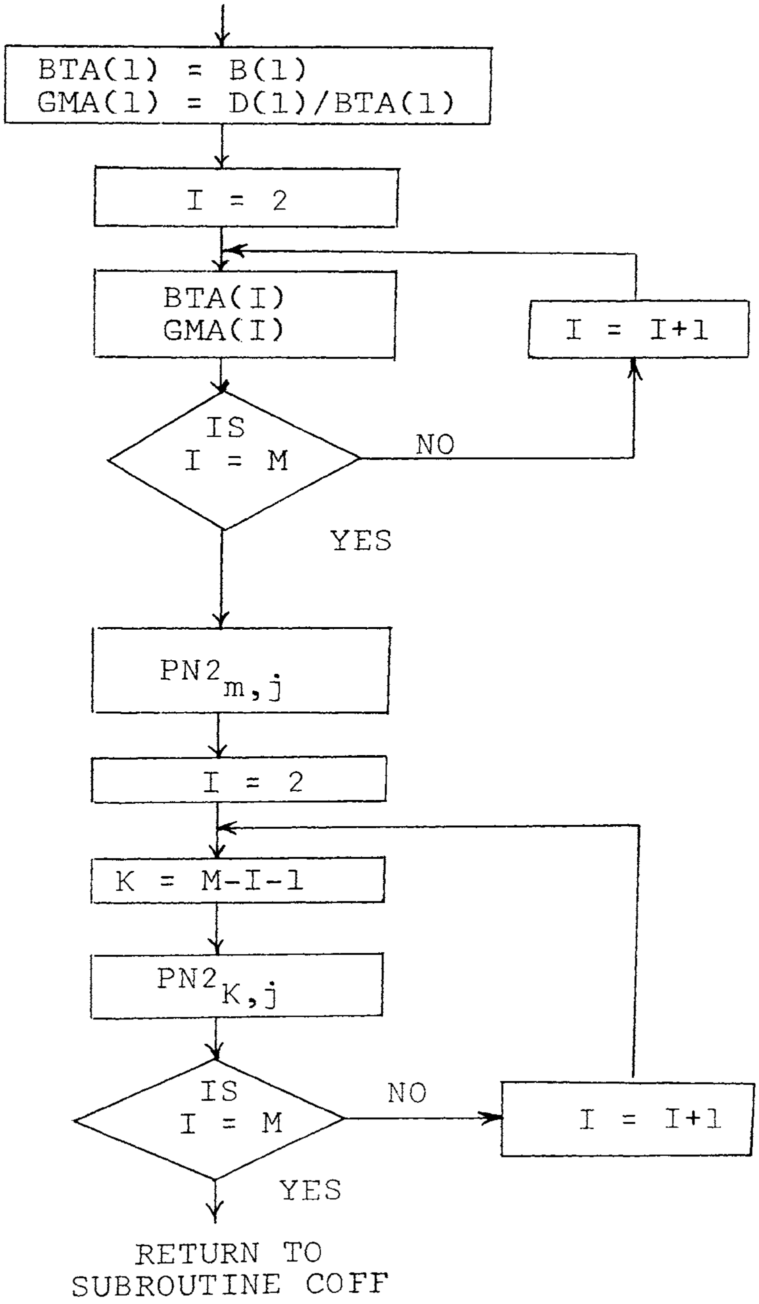
SUBROUTINE MBAL



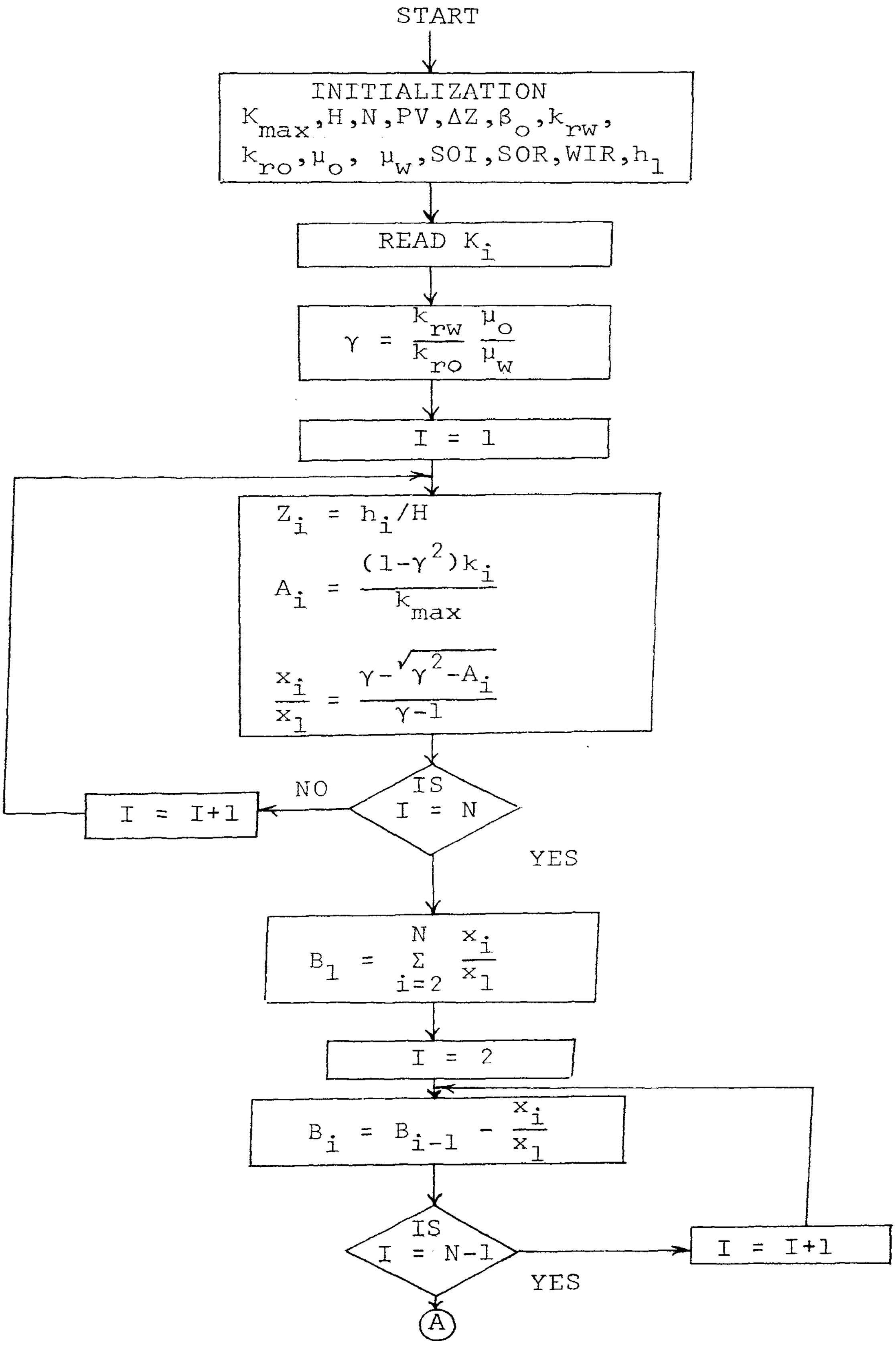

A
AMOX2
AMWX2
AMGX2
B
$\Delta SO_{i,j}$, $SW_{i,j}$,
$\Delta SG_{i,j}$, RSTERM
$SO_{i,j}$, $SW_{i,j}$, $SG_{i,j}$
NORMALIZE SATURATIONS
SSUM = SG+SW+SO
SO = SO/SSUM
SW = SW/SSUM
SG = SG/SSUM
IS SO > SOR
YES
NO
SO = SOR
IS SO > 1-SWC
NO
YES
SO = 1-SWC
IS SW > SWC
YES
NO
SW = SWC
C

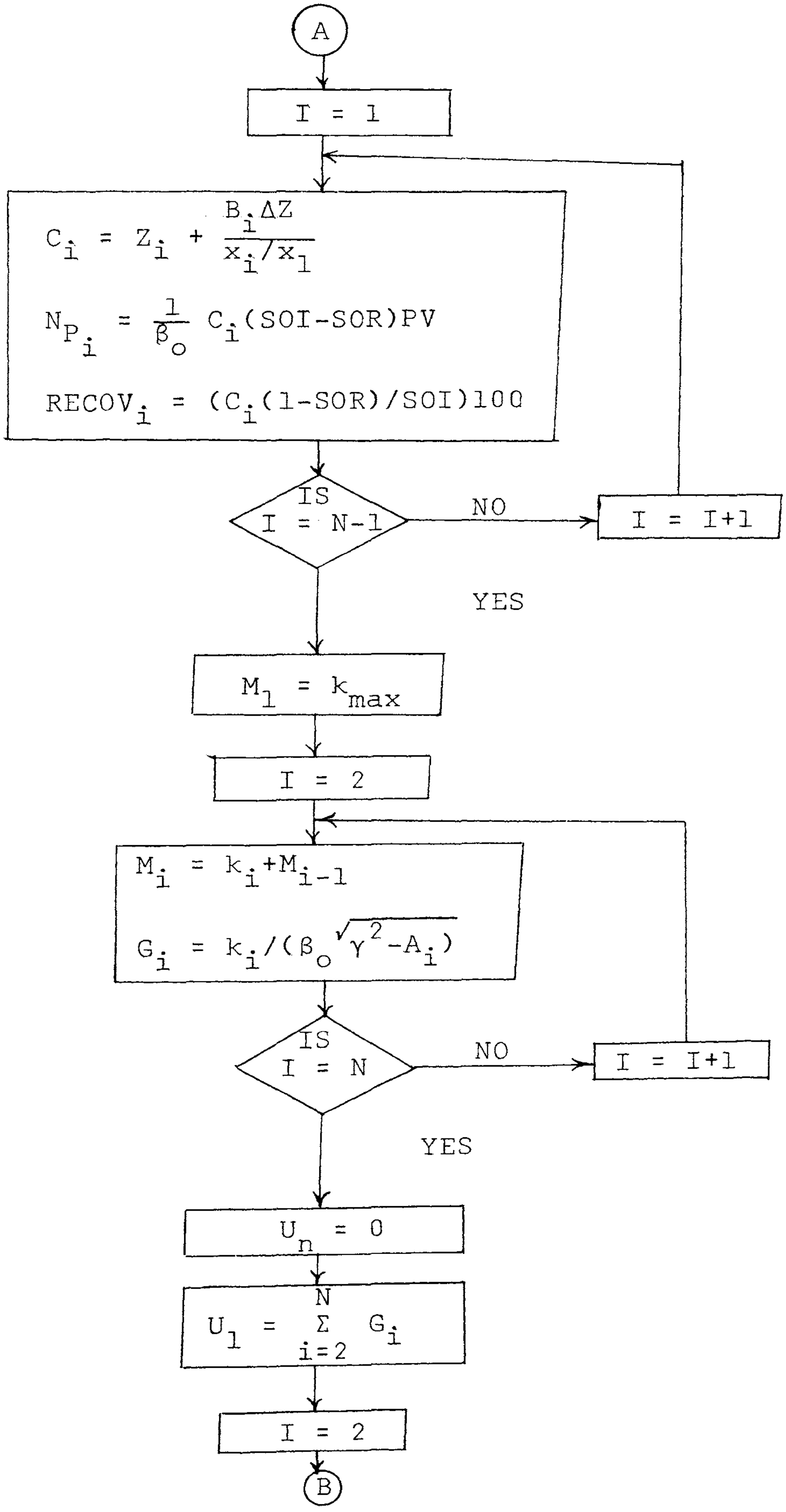

C
SG = 1-SO-SW
CHK = |SG-SG2| - SATOL
IS CHK > 0.
YES
KEYS = 2
NO
CHK = |SW-SW2| - SATOL
IS CHK > 0.
YES
KEYS = 2
NO
CHK = |SG-SG2| - SATOL
IS CHK > 0.
YES
KEYS = 2
NO
CHK = |P-PN1| - PRTOL
IS CHK > 0.
YES
KEYS = 2
D
IS I = M
NO
I = I+1
YES
RETURN TO MAIN PROGRAM

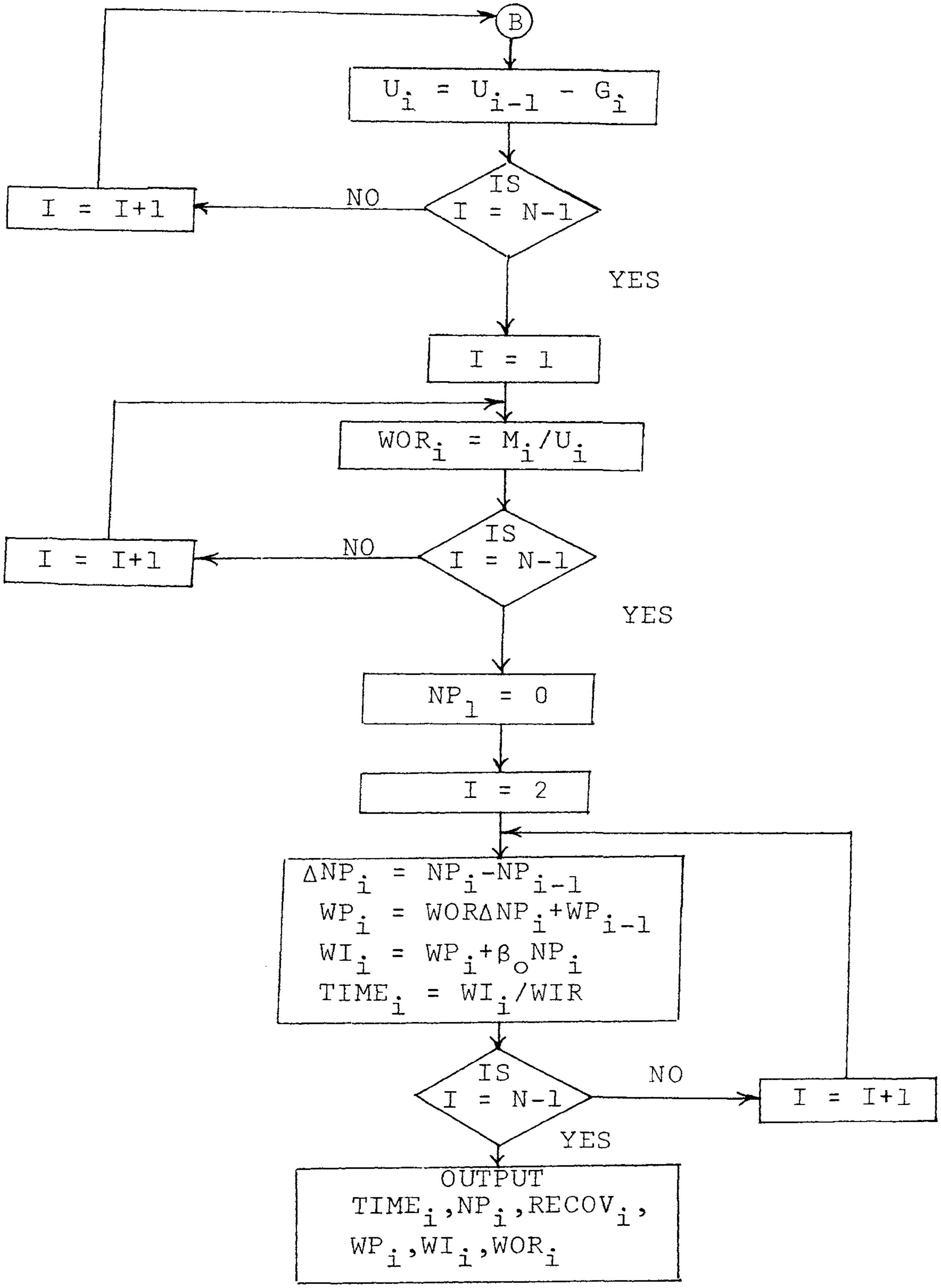
SUBROUTINE TRIDAG



2. COMPUTER FLOW DIAGRAM FOR DYKSTRA-PARSONS METHOD




A
I = 1
$C_i = Z_i + \frac{B_i \Delta Z}{x_i / x_1}$
$N_{P_i} = \frac{1}{\beta_o} C_i(SOI-SOR)PV$
$RECOV_i = (C_i(1-SOR)/SOI)100$
IS I = N-1
NO
I = I+1
YES
$M_1 = k_{max}$
I = 2
$M_i = k_i + M_{i-1}$
$G_i = k_i/(\beta_o \sqrt{\gamma^2 - A_i})$
IS I = N
NO
I = I+1
YES
$U_n = 0$
$U_1 = \sum_{i=2}^{N} G_i$
I = 2
B


B
$U_i = U_{i-1} - G_i$
IS $I = N-1$
NO
$I = I+1$
YES
$I = 1$
$WOR_i = M_i/U_i$
IS $I = N-1$
NO
$I = I+1$
YES
$NP_1 = 0$
$I = 2$
$\Delta NP_i = NP_i - NP_{i-1}$
$WP_i = WOR\Delta NP_i + WP_{i-1}$
$WI_i = WP_i + \beta_o NP_i$
$TIME_i = WI_i/WIR$
IS $I = N-1$
NO
$I = I+1$
YES
OUTPUT
$TIME_i, NP_i, RECOV_i,$
WP_i, WI_i, WOR_i

3. COMPUTER PROGRAM FOR NUMERICAL MODEL

```
FORTRAN IV G LEVEL 1, MOD 4          MAIN          DATE = 69290          18/06/54
              C     ONE DIMENSIONAL, THREE PHASE, STRATIFID, PETROLEUM RESERVOIR
              C     SIMULATOR    VARIABLE LENGTH(DELT X), VARIABLE LAYERS THICKNES
              C     (H), HETROGENOUS PERMEABILITY IN EACH LAYER, NEGLETING GRAVITY
              C     FORCES, CAPILLARY PRESSERS AND ROCK COPERSIBILITY, NO FLOW BETWEEN
              C     LAYERS,    INJECTING FROM ONE END AND PRODUCING FROM THE OTHER
              C     END
              C
              C                          ***** MAIN PROGRAM *****
              C
 0001                DIMENSION DELX(10),PERM(10,10),H(10),SO(10,10),SW(10,10),SG(10,10)
                    1,SO1(10,10),SO2(10,10),SW1(10,10),SW2(10,10),SG1(10,10),SG2(10,10)
                    1,P(10,10),PN1(10,10),PN2(10,10),HKX(10,10),QO(10,10),QW(10,10),
                    1QG(10,10),PN3(10,10)
 0002                N=4
 0003                M=8
 0004                TIME=0.0
 0005                ITCNT=0
 0006                CUMOP=0.0
 0007                CUMWP=0.0
 0008                SWC=.2
 0009                NN=N-1
 0010                PHI=.20
 0011                SWI=.2
 0012                SGI=0.0
 0013                SOI=.80
 0014                MM=M-1
 0015                KCOUNT=0
              C
 0016                READ(1,90)(DELX(I),I=1,M)
 0017                READ(1,100)((PERM(I,J),I=1,M),J=1,N)
 0018                READ(1,102)(H(I),I=1,N)
 0019                DELY=200.
 0020                PIN=3487.
              C
```

```
C     SET UP INITIAL SATURATIONS
C
      DO 98 I=1,M
      DO 98 J=1,N
      SW(I,J)=SWI
      SG(I,J)=SGI
      SO(I,J)=SOI
      SW1(I,J)=SW(I,J)
      SW2(I,J)=SW(I,J)
      SO1(I,J)=SO(I,J)
      SO2(I,J)=(I,J)
      SG1(I,J)=SG(I,J)
   98 SG2(I,J)=SG(I,J)
C     SW1,SG1,SO1 ARE SATURATIONS AT N TIME LEVEL (OLD TIME LEVEL)
C     SW2,SG2,SO2 ARE SATURATIONS AT N+1 TIME LEVEL AND K+1 ITERITIVE
C
C     SET UP INITIAL PRESSURES
C
      DO 97 I=1,M
      P(I,1)=PIN
      PN1(I,1)=P(I,1)
   97 PN2(I,1)=P(I,1)
      DO 96 J=2,N
      DO 96 I=1,M
      P(I,J)=P(I,J-1)+.433* (H(J)+H(J-1))/2.
      PN1(I,J)=P(I,J)
   96 PN2(I,J)=P(I,J)
C     P IS PRESSURE AT N TIME LEVEL (OLD TIME LEVEL)
C     PN1 IS PRESSURE AT N+1 TIME LEVEL AND K ITRITIVE
C     PN2 IS PRESSURE AT N+1 TIME LEVEL AND K+1 ITERITIVE
C
C     OUTPUT INITIAL CONDITIONS
      WRITE(3,90)(DELX(I),I=1,M)
      WRITE(3,102)(H(I),I=1,N)
      WRITE(3,100)((PERM(I,J),I=1,M),J=1,N)
      WRITE(3,121) TIME
      WRITE(3,241)
```

```
      WRITE(3,240)((PN2(I,J),I=1,M),J=1,N)
      WRITE (3,253)
      WRITE(3,251)((SO(I,J),I=1,M),J=1,N)
      WRITE(3,254)
      WRITE(3,251)((SW(I,J),I=1,M),J=1,N)
      WRITE(3,255)
      WRITE(3,251)((SG(I,J),I=1,M),J=1,N)
      DO 92 J=1,N
      DO 92 I=1,M
      P(I,J)=P(I,J)/14.7
      PN1(I,J)=PN1(I,J)/14.7
   92 PN2(I,J)=PN2(I,J)/14.7
C
      STOIP=0.0
      DO 91 J=1,N
      DO 91 I=1,M
      CALL BETA (P(I,J),BTAO,BTAW,BTAG)
   91 STOIP=STOIP+DELX(I)*DELY*H(J) *PHI*SOI/(5.61*BTAO)
      WRITE(3,256)STOIP
      DO 93 I=1,M
   93 DELX(I)=DELX(I)*30.48
      DO 94 J=1,N
   94 H(J)=H(J)*30.48
      DELY=DELY*30.48
C
C     CALCULATE HKX
C
      DO 1 J=1,N
      DO 2 I=1,MM
    2 HKX(I,J)=(2.*PERM(I,J)*PERM(I+1,J))/(DELX(I)*PERM(I+1,J)+DELX(I+1)
     1*PERM(I,J))
C
C     READ TIME STEP
C
      READ(1,108)DELT,NTIME
      TIME=TIME+DELT
      DELT=DELT*24.*3600.
```

```
      ITCNT=ITCNT+1
  897 CONTINUE
C
      QT=1000.*5.615*30.48*30.48*30.48/(24.*3600.)
      SUMKH=0.0
      SUMQWI=0.0
C     CALCULATE WATER INJECTION INTO THE FIRST BLOCK OF EACH LAYER
      PBIG=0.0
      DO 50 J=1,N
      IF(PBIG.LT.P(1,J)) PBIG=P(1,J)
   50 CONTINUE
      PBIG=PBIG+20.
      DO 899 J=1,N
  899 SUMKH=SUMKH+PERM(1,J)*H(J)*(PBIG-P(1,J))
      DO 898 J=1,N
      QW(1,J)=QT*PERM(1,J)*H(J)*(PBIG-P(1,J))/SUMKH
      SUMQWI=SUMQWI+QW(1,J)
      QO(1,J)=0.0
  898 QG(1,J)=0.0
      DO 901 J=1,N
      DO 901 I=2,MM
      QW(I,J)=0.0
      QO(I,J)=0.0
  901 QG(I,J)=0.0
      DO 892 J=1,N
  892 QW(1,J)=QT*(SW(1,J)/SUMQWI)
C
   53 CONTINUE
      J=1
      NCOUNT=0
   52 CONTINUE
      CALL COFF(SW1,SW2,SG1,SG2,HKX,P,PN1,PN2,M,MM,DELT,DELY,DELX,QO,1QW,QG,
     PHI,H,J,SO,SW,SG,PN3)
C
      CALL MBAL(PHI,DELX,DELY,HKX,QO,QW,QG,SO,SW,SG,PN2,P,DELT,H,SO1,1SO2,
     SW1,SW2,SG1,SG2,M,KEYS,J,PN1)
      NCOUNT=NCOUNT+1
```

```
      GO TO (55,59), KEYS
   59 CONTINUE
      IF(NCOUNT.GT.2) GO TO 55
      DO 60 I=1,M
      SW2(I,J)=SW(I,J)
      SW(I,J)=.5*(SW1(I,J)+SW2(I,J))
      SG2(I,J)=SG(I,J)
      SG(I,J)=.5*(SG1(I,J)+SG2(I,J))
      SO(I,J)=.5*(SO(I,J)+1.-SW1(I,J)-SG1(I,J))
   60 PN1(I,J)=PN2(I,J)
      GO TO 52
C
   55 IF(J.EQ.N) GO TO 56
      J=J 1
      NCOUNT=0
      GO TO 52
   56 CONTINUE
      WRITE(3,121)TIME
C
C     CALCULATE OIL,WATER AND GAS PRODUCTIONS,WOR,GOR AND OIL RECOVERY
C
      SUMQO=0.0
      SUMQW=0.0
      SUMQG=0.0
      DO 63 J=1,N
      SUMQO=SUMQO+QO(M,J)
      RQO=SUMQO*24.*3600./(30.43*30.48*30.48*5.615)
      SUMQW=SUMQW+QW(M,J)
      RQW=SUMQW*24.*3600./*30.48*30.48*30.48*5.615)
      CALL SOLGAS(P(M,J),RS1)
      CALL SOLGAS(PN2*M,J),RS2)
   63 SUMQG=SUMQG_QG(M,J)+QO(M,J)*(RS1+RS2)/2.
      IF(SUMQO.FQ.0.0) GO TO 58
      WOR=SUMQW/SUMQO
      GOR=5.615*SUMQG/SUMQO
      GO TO 54
   58 WOR=0.0
```

```
      GOR=0.0
   54 CONTINUE
      CUMOP=CUMOP+SUMQO*DELT/(30.48*30.48*30.48*5.615)
      CUMWP=CUMWP+SUMQW*DELT/(30.48*30.48*30.48*5.615)
      RECOV=100.*CUMOP/STOIP
      CUMWIN=QT*TIME*24.*3600./(30.48*30.48*30.48*5.615)
C
      WRITE(3,241)
      WRITE(3,240)((PN3(I,J),I=1,M),J=1,N)
      WRITE(3,253)
      WRITE(3,251)((SO(I,J),I=1,M),J=1,N)
      WRITE(3,254)
      WRITE(3,251)((SW(I,J),I=1,M),J=1,N)
      WRITE(3,255)
      WRITE(3,251)((SG(I,J),I=1,M),J=1,N)
      WRITE(3,257) RQO,RQW
      WRITE(3,252)CUMOP,CUMWP,RECOV,CUMWIN,WOR,GOR
      IE(WOR.GT.20.) GO TO 99
      KCOUNT=KCOUNT+1
C
C     RESET OR READ TIME STEP
C
      DELTN1=DELT
      IF(ITCNT.LT.NTIME) GO TO 57
      ITCNT=0
      READ(1,108,END=99) DELT,NTIME
      DELT=DELT*24.*3600.
   57 ITCNT=ITCNT+1
      TIME=TIME+DELT/(24.*3600.)
C
C     LINEAR INTERPOLATION FOR PRESSURES AND SATURATIONS FOR THE NEXT STEP TO
C     ACCELERATE CONVERGENCE
C
      DO 30 J=1,N
      DO 30 I=1,M
      SW2(I,J)=SW(I,J)+(SW(I,J)-SW1(I,J))/DELTN1*DELT
      SW1(I,J)=SW(I,J)
```

```
      SG2(I,J)=SG(I,J)+(SG(I,J)=SG1(I,J))/DELTN1*DELT
      SG1(I,J)=SG(I,J)
      SW(I,J)=.5*(SW1(I,J)+SW2(I,J))
      SG(I,J)=.5*(SG1(I,J)+SG2(I,J))
      SO(I,J)+1.=SW(I,J)-SG(I,J)
      PN1(I,J)=PN2(I,J)+(PN2(I,J)-P(I,J))/DELTN1*DELT
      P(I,J)=PN2(I,J)
   30 PN2(I,J)=PN1(I,J)
      IF(KCOUNT.GT.60) GO TO 500
      GO TO 53
  500 TIME=TIME-DELT/(24.*3600.)
      WRITE(2,502)((P(I,J),I=1,M),J=1,N)
      WRITE(2,502)((PN1(I,J),I=1,M),J=1,N)
      WRITE(2,502)((PN2(I,J),I=1,M),J=1,N)
      WRITE(2,503) TIME,DUMOP,CUMWP
      WRITE(2,504)((SW(I,J),I=1,M),J=1,N)
      WRITE(2,504)((SW1(I,J),I=1,M),J=1,N)
      WRITE(2,504)((SW2(I,J),I=1,M),J=1,N)
      WRITE(2,504)((SO(I,J),I=1,M),J-1,N)
      WRITE(2,504)((SO2(I,J),I=1,M),J=1,N)
      WRITE(2,504)((SG(I,J),I-1,M),J=1,N)
      WRITE(2,504)((SG1(I,J),I=1,M),J=1,N)
      WRITE(2,504)((SG2(I,J),I=1,M),J=1,N)
   99 STOP
   90 FORMAT(8F8.2)
  100 FORMAT(8F8.2)
  102 FORMAT(8F8.2)
  108 FORMAT(F10.5,14)
  121 FORMAT(/,40X,'*****TIME ELAPSED=',F16.7,'DAYS*****',//)
  240 FORMAT(20X,8F10.1)
  241 FORMAT(60X,'PRESSURE IN PSI',//)
  251 FORMAT(20X,8F10.4)
  253 FORMAT(//,60X,'OIL SATURATIONS',//)
  254 FORMAT(//,60X,'WATER SATURATIONS',//)
  255 FORMAT(//,60X,'GAS SATURATIONS',//)
  252 FORMAT(/,5X,'CUMULATIVE OIL PRODUCTION = ',F18.5,'STB',10X,'CUMLATIVE
     1WATER PRODUCTION = ',F18.5,'STB',/,15X,'PERCENT RECOVERY = 'F10.4,10X,
```

```
     1'CUMULATIVE WATER INJECTED = ',F18.5,'STB',/,5X,'WOR=',F10.4,10X,
     1'GOR = ',F10.4,//)
  256 FORMAT(/,10X,'ORIGINAL OIL IN PLACE',F16.2,'STB',//)
  257 FORMAT(/,5X,'OIL PRODUCTION RATE = ',F18.5,'STB/DAY',5X,'WATER
     1PRODUCTION RATE = ',F18.5,'STB/DAY')
  502 FORMAT(8F8.2)
  503 FORMAT(3F20.6)
  504 FORMAT(8F8.5)
      END
```

```
C                    *****CALCULATION OF COEFFICIENT*****
      SUBROUTINE COFF(SW1,SW2,SG1,SG2,HKX,P,PN1,PN2,M,MM,DELT,DELY,DELX,QO,
     1SQ,QG,PHI,H,J,SO,SW,SG,PN3)
      DIMENSION SW1(10,10),SW2(10,10),SG1(10,10),SG2(10,10),HKX(10,10),
     1P(10,10),PN1(10,10),PN2(10,10),DELX(10),QO(10,10),SQ(10,10),QG(10,120),
     1AX(10),BX(10),CX(10),DX(10),SO(10,10),SW(10,10),SG(10,10),H(10),PN3
     1(10,10)
      CW=.0000045
      SOR=.20
      SWC-.20
C
C     AT THIS POINT SW,SG,SO ARE TIME AVERAGED SATURATIONS BETWEEN N AND N+1
C     TIME LEVEL
C
      DO 50 I=1,M
      IF(I.EQ.1) GOTO 10
    2 PX=.5*(P(I,J)+PN2(I,J))
      CALL BETA (PX,BTAO,BTAW,BTAG)
      CALL SOLGAS(PX,RS)
      TEMP1=BTAO-BTAG*RS
      TEMP2=BTAW
      TEMP3=BTAG
      TEMP5=BTAO
      IF(I.EQ.1) GO TO 8
    3 PX=.5*(P(I-1,J)+PN2(I,J))
      CALL BETA(PX,BTAO,BTAW,BTAG)
      CALL SOLGAS(PX,RS)
      TEMP6=BTAO-BTAG*RS
      TEMP4=DELX(I-1)/DELX(I)
      AOX1=AOX2*TEMP1*TEMP4/TEMP6
      AWX1=AWX2*TEMP2*TEMP4/BTAW
      AGX1=AGX2*TEMP3*TEMP4/BTAG
C
      IF(I.EQ.M) GO TO 30
C
    8 PX=.25*(P(I,J)+P(I+1,J)+PN2(I,J)_PN2(I+1,J))
      SWA=.5*(SW(I,J)+SW(I+1,J))
```

```
      SGA=.5*(SG(I,J)+SG(I+1,J))
      SOA=.5*(SO(I,J)+SO(I+1,J))
      IF(SO(I,J).LE.SOR) SOA=0.0
      CALL BETA(PX,BTAO,BTAW,BTAG)
      CALL RPERM(SOA,SWA,SGA,RKO,RKW,RKG)
      CALL VISC (PX,XMUO,XMUW,XMUG)
      AOX2=HKX(I,J)/DELX(I)*TEMP1/BTAO*RKO/XMUO
      AWX2=HKX(I,J)/DELX(I)*TEMP2/BTAW*RKW/XMUW
      AGX2=HKX(I,J)/DELX(I)*TEMP3/BTAG*RKG/XMUG
      GO TO 11
C
   10 AOX1=0.0
      AWX1=0.0
      AGX1=0.0
      GO TO 2
C
   30 AOX2=0.0
      AWX2-0.0
      AGX2=0.0
C
   11 CONTINUE
      PX=(P(I,J)+PN1(I,J))*.5
      CALL BETPRM (PX,BOPRM,BGPRM,RSPRM)
      TRM=-SW(I,J)*CW+SG(I,J)*BGPRM/TEMP3+SO(I,J)*(BOPRM-RSPRM*TEMP3)/TEMP5
      TRM=PHI*TRM/DELT
      AX(I)=AOX1+AWX1+AGX1
      CX(I)=AOX2+AWX2+AGX2
      BX(I)=-AX(I)-CX(I)+TRM
C
      IF(I.EQ.M) GO TO 40
      IF(I.EQ.1) GO TO 12
C
      QTERM=0.0
      GO TO 13
   12 QTERM=QG(I,J)*TEMP3+QO(I,J)*TEMP1+QW(I,J)*TEMP2
      QTERM=-QTERM/(DELX(I)*DELY*H(J))
C
```

```
   13 DX(I)=QTERM+TRM*P(I,J)
      GO TO 50
C
   40 PX=.5*(P(1,J)+PN2(1,J))
      CALL BETA(PX,BTAO,BTAW,BTAG)
      TEMP2=BTAW
      PX=.5*(P(M,J)+PN2(M,J))
      SOA=SO(M,J)
      SWA=SW(M,J)
      SGA-SG(M,J)
      CALL BETA(PX,BTAO,BTAW,BTAG)
      CALL RPERM(SOA,SWA,SGA,RKO,RKW,RKG)
      CALL VISC(PX,XMUO,XMUW,XMUG)
      CALL SOLGAS(PX,RS)
C
      IF(RKO.LE.0.0) GO TO 41
      TEMP1=BTAO-BTAG*RS
      QO(M,J)=TEMP2*QW(1,j)/(1.+RKW/RKO*XMUO/XMUW+RKG/RKO*XMUO/XMUG)
      QO(M,J)=QO(M,J)/BTAO
      QW(M,J)=QO(M,J)*RKW*BTAO*XMUO/(RKO*BTAW*XMUW)
      QG(M,J)=QO(M,J)*RKG*BTAO*XMUO/(RKO*BTAG*XMUG)
      QTERM-QG(M,J)*BTAG+QO(M,J)*TEMP1+QW(M,J)*BTAW
      QTERM=QTERM/(DELX(M)*DELY*H(J))
C
      DX(M)=QTERM+TRM*P(M,J)
C
      GO TO 50
   41 QO(M,J)=0.0
      IF(RKG.LE.0.0) GO TO 42
      QG(M,J)=TEMP2*QW(1,J)/(1.+RKW/RKG*XMUG/XMUW)
      QG(M,J)=QG(M,J)/BTAG
      GO TO 43
   42 QG(M,J)=0.0
   43 QW(M,J)=(TEMP2*QW(1,J)-BTAG*QG(M,J))/BTAW
      QTERM=QG(M,J)*BTAG+QW(M,J)*BTAW
      QTERM-QTERM/(DELX(M)*DELY*H(J))
      DX(M)=QTERM+TRM*P(M,J)
```

```
C
   50 CONTINUE
      CALL TRIDAG(AX,BX,CX,DX,PN2,M,J,PN3)
      RETURN
      END
```

```
C
C                    *****MATERIAL BALANCE CALCULATION*****
      SUBROUTINE MBAL(PHI,DELX,DELY,HKX,QO,QW,QG,SO,SW,SG,PN2,P,DELT,H,SO1,
     1SO2,SW1,SW2,SG1,SG2,M,KEYS,J,PN1)
      DIMENSION SW1(10,10),SW2(10,10),SW(10,10),SG1(10,10),SG2(10,10),
     1SG(10,10),HKX(10,10),P(10,10),PN2(10,10),PN1(10,10),DELX(10),QO(10,10),
     1QW(10,10),QG(10,10),H(10),SO(10,10),SO1(10,10),SO2(10,10)
C
      SWC=.2
      SGC=.05
      SOR-.20
      SATOL=.02
      PRTOL=1.0
      KEYS=1
C
      DO 100 I=1,M
      ZZ=-DELT/(DELX(I)*PHI)
      PX=(P(I,J)+PN2(I,J))*.5
      CALL BETA (PX,BTOI,BTWI,BTGI)
      CALL SOLGAS(P(I,J),RSN)
      CALL SOLGAS(PN2(I,J),RSNP1)
C
C     CALCULATE QWTERM,QOTERM,QGTERM
C
      IF(I.EQ.1) GO TO 904
      IF(I.EQ.M) GO TO 905
C
      QWTERM=0.0
      QOTERM=0.0
      QGTERM=0.0
      GO TO 900
  904 QWTERM=QW(I,J)*BTWI/(H(J)*DELY)
      QOTERM=0.0
      QGTERM=0.0
      GO TO 900
  905 AWTERM=-QW(M,J)*BTWI/(H(J)*DELY)
      QOTERM=-QO(M,J)*BTOI/(H(J)*DELY)
```

```
      QGTERM=-QG(M,J)*BTGI/(H(J)*DELY)
  900 CONTINUE
C
      IF(I.GT.1) GO TO 5
C
      AOX1=0.0
      AWX1=0.0
      AGX1=0.0
C
      GO TO 7
C
    5 PX=(P(I-1,J)+PN2(I-1,J))*.5
      CALL BETA(PX,BTAO,BTAW,BTAG)
      AOX1=(BTOI/BTAO)*AOX2
      AWX1=(BTWI/BTAW)*AWX2
      AGX1=(BTGI/BTAG)*AGX2
C
    7 IF(I.LT.M) GO TO 8
C
      AOX2-0.0
      AWX2=0.0
      AGX2=0.0
      GO TO 17
C
    8 SWA=.5*(SW(I,J)+SW(I+1,J))
      SOA=.5*(SO(I,J)+SO(I+1,J))
      SGA=.5*(SG(I,J)+SG(I+1,J))
      IF(SO(I,J).LE.SOR) SOA=0.0
      CALL RPERM(SOA,SWA,SGA,RKO,RKW,RKG)
      PAV=.25*(P(I,J)+P(I+1,J)+PN2(I,J)+PN2(I+1,J))
      CALL VISC(PAV,XMJO,XMUW,XMUG)
      AOX2=HKX(I,J)*(RKO/XMUO)*(.5*(P(I+1,J)+PN2(I+1,J)-P(I,J)-PN2(I,J)))
      AGX2=HKX(I,J)*(RKG/XMUG)*(.5*(P(I+1,J)+PN2(I+1,J)-P(I,J)-PN2(I,J)))
      AWX2=HKX(I,J)*(RKW/XMUW)*(.5 (P(I+1,J)+PN2(I+1,J)-P(I,J)-PN2(I,J)))
C
   17 DELSW=ZZ*(AWX1-AWX2-AWTERM)
      DELSO=ZZ*(AOX1-AOX2-QOTERM)
```

```
      DELSG=ZZ*(AGX1-AGX2-QGTERM)
C
      RSTERM=SO(I,J)*(RSNP1-RSN)*(BTGI/BTOI)
      SG(I,J)=SG1(I,J)+DELSG-RSTERM
      SW(I,J)=SW1(I,J)+DELSW
      SO(I,J)=1.0-SW1(I,J)-SG1(I,J)+DELSO
C
C     SATURATION TOLERANCE CHECK
C
      SSUM=SG(I,J)+SW(I,J)+SO(I,J)
      SG(I,J)=SG(I,J)/SSUM
      SW(I,J)=SW(I,J)/SSUM
      SO(I,J)=SO(I,J)/SSUM
      IF(SO(I,J).GT.(1.-SWC)) SO(I,J)=1.-SWC
      IF(SO(I,J).GE.SOR) GO TO 20
      SO(I,J)=SOR
   20 IF(I.EQ.1) GO TO 21
      IF(SW(I,J).GT.SW(I-1,J)) SW(I,J)=SW(I-1,J)
   21 IF(SW(I,J).GE.SWC) GO TO 22
      SW(I,J)=SWC
   22 SG(I,J)=1.0=SO(I,J)-SW(I,J)
      CHK=ABS(SG(I,J)-SG2(I,J))-SATOL
      IF(CHK.GT.0.) KEYS=2
      CHK=ABS(SW(I,J)-SW2(I,J))-SATOL
      IF(CHK.GT.0.) KEYS=2
      CHK=ABS(SO(I,J)+SG2(I,J)+SW2(I,J)-1.)-SATOL
      IF(CHK.GT.0.) KEYS=2
      CHK=ABS(PN2(I,J)-PN1(I,J))-PRTOL
      IF(CHK.GT.0.) KEYS=2
      IF(SG(I,J).GT.0.0) GO TO 100
      SG(I,J)=0.0
      SW(I,J)=1.-SO(I,J)
C
  100 CONTINUE
      RETURN
      END
```

```
C                            *****TRI-DIAGONAL SOLUTION*****
      SUBROUTINE TRIDAG(AX,BX,CX,DX,PN2,M,J,PN3)
      DIMENSION AX(10),BX(10),CX(10),DX(10),PN2(10,10),BTA(10),GMA(10),PN3
     1(10,10)
      BTA(1)=BX(1)
      GMA(1)=DX(1)/BTA(1)
      DO 10 I=2,M
      BTA(I)=BX(I)-AX(I)*CX(I-1)/BTA(I-1)
   10 GMA(I)=(DX(I)-AX(I)*GMA(I-1))/BTA(I)
      PN2(M,J)=GMA(M)
      PN3(M,J)=GMA(M)*14.7
      DO 20 I=2,M
      K=M0I+1
      PN2(K,J)=GMA(K)-(CX(K)*PN2(K+1,J)/BTA(K))
   20 PN3(K,J)=PN2(K,J)*14.7
      RETURN
      END
```

```
      SUBROUTINE BETA(P,BTAO,BTAW,BTAG)
      AO=.1098168E01
      A1=.13487E-03
      A2=.682357E-08
      A3=-19.937
      A4=.9126E-1
      A5=-.21086E-5
      CW=.0000045
      CO=.0003
      PB=3487./14.7
      BTWBP=.95
      BTOBP=1.650
      IF(P.LT.PB) GO TO 4
      BTAO=BTOBP*(EXP(-CO*(P-PB)))
      GO TO 5
    4 BTAO=AO+A1*(14.7*P)+A2*(14.7*P)**2
    5 BTAG=A3+A4*14.7*P+A5*(14.7*P)**2
      BTAG=1./BTAG
      BTAW=BTWBP*(EXP(-CW*(P-PB)))
      RETURN
      END
```

```
      SUBROUTINE VISC(P,XMUO,XMUW,XMUG)
      A0=.397117E01
      A1=-.199219E-02
      A2=.80314E-06
      A3=-.1986899E-09
      A4=.21027E-13
      A5=.0024738
      A6=.9244E-5
      A7=-.76556E-9
      PB=3487./14.7
      CVISO=.0001
      CVISW=.00001
      XMUOBP=1.47
      XMUWBP=.75
      IF(P.LT.PB) GO TO 6
      XMUO=XMUOBP+CVISO*(P-PB)
      GO TO 7
    6 XMUO=A0+A1*14.7*P+A2*(14.7*P)**2+A3*(14.7*P)**3+A4*(14.7*P)**4
    7 XMUW=XMUWBP+CVISW*(P-PB)
      XMUG=A5+A6*14.7*P+A7*(14.7*P)**2
      RETURN
      END
```

```
      SUBROUTINE SOLGAS(PA,RS)
      PX=PA
      PB=3487./14.7
      IF(PX.GT.PB) PX=PB
      RS=44.27252+.266273*14.7*PX+.000007775651*(14.7*PX)**2
      RS=RS/5.615
      RETURN
      END
```

```
      SUBROUTINE RPERM(SOA,SWA,SGA,RKO,RKW,RKG)
      SGC=.05
      SWC=.20
      SOR=.20
      IF(SWA.GT.SWC) GO TO 10
      RKG=(SGA***3)*(2.-SGA-2.*SWA)/((1.-SWA)**4)
      RKW=0.0
      RKO=((1.-SGA-SWA)/(1.-SWC))**4
      GO TO 20
   10 CONTINUE
      RKG=(SGA**3)*(2.-SGA-2.*SWC)/((1.-SWC)**4
      RKW=((SWA-SWC)/(1.-SWC))**4
      RKO=((1.-SGA-SWA)**3)*(1.-SGA+SWA-2.*SWC)/((1.-SWC)**4)
C
C     ADJUST RKO FOR SOR AND RKG FOR SGC
   20 RKO=RKO*(SOA-SOR)/(1.-SOR)
      RKG=RKG*(SGA-SGC)/(1.-SGC)
C
      IF(RKO.LT.0.) RKO=0.0
      IF(SOA.LE.SOR) RKO=0.0
      IF(RKG.LT.0.) RKG=0.0
      IF(SGA.LE.SGC) RKG=0.0
      IF(RKW.LT.0.) RKW=0.0
      RKSUM=RKO+RKW+RKG
      IF(RKSUM.LE.1.) GO TO 6
      RKO=RKO/RKSUM
      RKW=RKW/RKSUM
      RKG=RKG/RKSUM
    6 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE BETPRM(PX,BOP,BGP,RSP)
      A1=.13487E-03
      A2=.682357E-08
      A3=-19.937
      A4=.09126
      A5=-.21086E-5
      A6=.266273
      A7=.7775651E-5
      BTOBP=1.65
      CO=.0003
      PB=3487./14.7
      IF(PX.LT.PB) GO TO 2
      BTAO=BTOBP*(EXP(-CO*(PX-PB)))
      BOP=-CO*BTAO
      RSP=0.0
      GO TO 3
    2 BOP=A1+2.*A2*PX*14.7
      RSP=A6+2.*A7*PX*14.7
      RSP=RSP/5.615
    3 BTAG=A3+A4*14.7*PX+A5*(14.7*PX)**2
      BTAG=1./BTAG
      BGP=-BTAG**2*(A4+2.*A5*PX*14.7)
      RETURN
      END
```

4. COMPUTER PROGRAM FOR DYKSTRA-PARSONS METHOD

```
/WAT 4     PT140710,TIME=01,PAGES=010   AKBAR ALI M         DEFAULT W JOB 75
/          CLASS=W,PRIORITY=13,READER=2
C              MODIFIED DYKSTRA-PARSONS METHOD
          DIMENSION PERM(50),H(50),Z(50),A(50),XRATIO(50),B(50),COV(50),OILPR(50),
         1PM(50),G(50),U(50),WOR(50),DELTP(50),WP(50),WI(50),TIME(50),RECOV(50) ,
         1QOR(50)
          TIME(1)=0.0
          PERMAX=399.
          SUMH=25.
          N=25
          NN=N-1
          PV=200.*25.*1400.*.2/5.615
          DELZ=.04
          BO=1.65
          RKW=.316
          RKO=.75
          XMUW=.75
          SOI=.8
          SOR=.2
          WIR=1000.
          H(1)=1.
          DO 4 I=2,N
        4 H(I)=1.+H(I-1)
          READ(1,100)(PERM(I),I=1,N)
        2 READ(1,400,END=50) XMUO
          VRATIO=XMUO/XMUW
          WRITE(3,410) VRATIO
          GAMA=RKW/RKO*XMUO/XMUW
          DO 10 I=1,N
          Z(I)=H(I)/SUMH
          A(I)=(1.-GAMA**2)*PERM(I)/PERMAX
       10 XRATIO(I)=(GAMA-(SQRT(GAMA**2+A(I))))/(GAMA-1.)
          BETA=0.0
          B(N)=0.0
          DO 11 I=2,N
```

```
   11 BETA=BETA+XRATIO(I)
      B(1)=BETA
      DO 12 I=2,NN
   12 B(I)=B(I-1)-XRATIO(I)
      DO 13 I=1,NN
      COV(I)=Z(I)+B(I)*DELZ/XRATIO(I)
      RECOV(I)=COV(I)*(1.-SOR/SOI)*100.
   13 OILPR(I)=COV(I)*(SOI-SOR)*PV/BO
      PM(1)=PERMAX
      DO 14 I=2,N
      PM(I)=PERM(I)+PM(I-1)
   14 G(I)=PERM(I)/(BO*(SQRT(GAMA**2+A(I))))
      SUMU=0.0
      U(N)=SUMU
      DO 15 I=2,N
   15 SUMU=SUMU+G(I)
      U(1)=SUMU
      DO 16 I=2,NN
   16 U(I)=U(I-1)-G(I)
      DO 17 I=1,NN
   17 WOR(I)=PM(I)/U(I)
      WRITE(3,200)
      WP(1)=0.0
      DO 18 I=2,NN
      DELTP(I)=OILPR(I)-OILPR(I-1)
      WP(I)=WOR(I)*DELTP(I)+WP(I-1)
      WI(I)=WP(I)+BO*OILPR(I)
      TIME(I)=WI(I)/WIR
      QOR(I)=DELTP(I)/(TIME(I)-TIME(I-1))
      WRITE(3,400)QOR(I)
      WRITE(3,300) TIME(I),OILPR(I),RECOV(I),WP(I),WI(I),WOR(I)
   18 CONTINUE
      GO TO 2
  100 FORMAT(7F10.3)
  200 FORMAT(//,8X,'TIME',8X,'CUMULATIVE OIL PROD',8X,'RECOVERY',7X,
     1'CUMULATIVE WATER PROD',5X,CUMULATIVE WATER INJ',15X,'WOR',//,
     18X,'DAYS',16X,'STB',19X,'%',21X,'STB',21X,'STB',21X,'BBL/STB',//)
```

```
  300 FORMAT(5X,F10.2,1X,F20.2,9X,F10.5,5X,F20.2,5X,F20.2,5X,F20.5,/)
  400 FORMAT(F10.5)
  410 FORMAT(/,5X,'OIL VISCO RATIO=',F10.3)
   50 STOP
      END
```

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