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Inclusion theorems for boundary value problems for delay differential equations

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INCLUSION THEOREMS FOR BOUNDARY VALUE PROBLEMS FOR DELAY DIFFERENTIAL EQUATIONS

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Abstract

In this thesis existence and uniqueness of solutions to certain second and third order boundary value problems for delay differential equations is established. Sequences of upper and lower solutions similar to those used by Kovač and Savčenko are defined by means of an integral operator, and conditions are given under which these sequences converge monotonically from above and below to the unique solution of the problem. Some numerical examples for the second order case are presented. Existence and uniqueness is also proved for the case where the delay is a function of the solution as well as the independent variable.

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Table of Contents

List of Tables

Page

I. Introduction

The study of delay differential equations has expanded rapidly in recent years due to numerous applications which have developed in engineering and the applied sciences. Some of the areas of application are automatic control theory [10], kinetics of biochemical reactions [1], and population growth [3].

Grimm and Schmitt [5], [6] have obtained results for boundary value problems for differential equations with deviating arguments in which the solution was contained in the region between upper and lower solutions satisfying certain differential inequalities. Similar results for equations of order 4k and 4k + 1, where k is a positive integer, were obtained by Kovač and Savcenkǒ [9] who also presented a method for iteratively improving the upper and lower solutions.

Analagous results for boundary value problems for ordinary differential equations have been obtained by Jackson and Schrader [7], Kovač [8], and Werner [12], who considered first order systems of ordinary differential equations.

This thesis presents a further application of differential inequalities to second and third order delay differential equations, and develops an iterative procedure which yields numerical estimates for the unique solution of the boundary value problem considered.

Some other numerical methods for differential equations with deviating arguments have been developed recently by Castleton and Grimm [2] for initial value problems, and by de Nevers and Schmitt [4] for boundary value problems.

II. Notation, Definitions, and Preliminaries

A delay differential equation (DDE) is a special type of differential equation with deviating arguments (DEDA). A DEDA is an equation of the form

$$y' = f(x,y(x),y(h(x,y)),y'(h(x,y))). \tag{2.1}$$

If f is independent of y' and if $h(x,y) \leqq x$, then the equation is of retarded type, or a DDE. For higher order equations a DEDA is a DDE if $h(x,y) \leqq x$ and f is independent of the highest order derivative.

The basic initial value problem for DDE's consists of determining a solution $y(x)$ of equation (2.1) for $x \geqq x_0$. Such that $y(x) = \phi(x)$ on the interval $E = (-\infty, x_0)$, where $\phi(x)$ is a given continuous function called the <u>initial function</u>. In case $h(x,y)$ is bounded below, the interval E may be finite.

The theory of initial value problems for DDE's where $h = x-\tau(x)$ has been well developed, and a number of results have also been obtained for other types of DEDA.

In this paper two point boundary value problems (BVP's) for the following second and third order scalar equations will be considered:

$$y''(x) = f(x,y(x),y(h(x,y(x)))), \quad 0 \leqq x \leqq a \tag{2.2}$$

and

$$y'''(x) = f(x,y(x),y(h(x,y(x)))),\ 0 \leqq x \leqq a. \qquad (2.3)$$

A solution to (2.2) or (2.3) is defined as a function $y(x)$ such that $y(x) = \phi(x)$ on E, $y(x) \varepsilon C^2$ or C^3 respectively on $(x_0,x_1]$, and $y(x)$ satisfies the boundary conditions given at x_0 and x_1. Existence and uniqueness of solutions will be shown, and in the case where h is independent of y, a method will be given in which the solution can be approximated by "upper" and "lower" solutions. A "lower solution" to the BVP for (2.2) is a function $Z(x)$ satisfying the given boundary conditions and satisfying the differential inequality:

$$Z''(x) - f(x,Z(x),Z(h(x,Z(x)))) \geqq 0.$$

An "upper solution" to the BVP for (2.2) is a function $V(x)$ satisfying the given BC's and satisfying the differential inequality:

$$V''(x) - f(x,V(x),V(h(x,V(x)))) \leqq 0.$$

The following results will be used repeatedly throughout this paper.

<u>Lemma 1</u>. Let $y(x) \varepsilon C^2$, $y(a) = y(b) = 0$ and $y''(x) \leqq 0$ on $[a,b]$. Then $y(x) \geqq 0$ on $[a,b]$.

<u>Proof</u>. Assume the contrary, i.e., $y(x) < 0$ for some $x \varepsilon (a,b)$. Since $y''(x) \leqq 0$, $y'(x)$ is non-increasing. But this means that if there exists $\xi \varepsilon (a,b)$ such that $y(\xi) \leqq 0$

then $y(x) < 0$ for all $x \geqq \xi$. In particular $y(b) < 0$, a contradiction.

A similar result will hold if the inequalities in Lemma 1 are reversed.

<u>Lemma 2</u>. Let $y(x) \varepsilon C^3$, $y'(a) = y'(b) = 0$, $y(a) = 0$, and $y'''(x) \leqq 0$ on $[a,b]$. Then $y(x) \geqq 0$ on $[a,b]$.

<u>Proof</u>. By Lemma 1 $y'(x) \geqq 0$ on $[a,b]$. Assume that $y(x) < 0$ on (a,b). But since the slope $(y'(x))$ is non-negative it is impossible for $y(0) = 0$, a contradiction.

As in Lemma 1, the inequalities in Lemma 2 can be reversed.

In the rest of this paper the interval $[0,1]$ will be denoted by I.

III. The Second Order Boundary Value Problem

A. Existence and Uniqueness. Consider the non-linear, second order BVP:

$$y''(x) = f(x,y(x),y(x-\tau(x))) \equiv f(x,y,y_\tau) \equiv f[y], \quad (3.1)$$

where $\tau(x) \geqq 0$, $\tau(x) \varepsilon C$ on I with boundary conditions

$$\begin{aligned} &y(x) = \phi(x) \text{ if } x \varepsilon E = \{x \mid x < 0\}, \\ &\phi(0) = 0, \; y(0) = y(1) = 0, \end{aligned} \quad (3.2)$$

where $\phi(x) \varepsilon C^2$ is a given initial function on E.

<u>Theorem 1</u>. Let f be continuous and have bounded derivatives, $\left|\frac{\partial f}{\partial y}\right| \leqq N$ and $\left|\frac{\partial f}{\partial y_\tau}\right| \leqq N$, on some compact region $\overline{D}_1$ containing

$$\overline{D} = \{(x,y,z) \mid 0 \leqq x \leqq 1, \; |y| \leqq \tfrac{B}{8}, \; |z| \leqq \tfrac{B}{8}\},$$

where

$$B = \sup_{\overline{D}_1} |f(x,y(x),y(x-\tau(x)))|.$$

Then if $N < 4$ in $\overline{D}$, there exists a unique continuous solution of the problem (3.1)-(3.2).

<u>Proof</u>. Let $Z_1(x)$ be a function such that $Z_1(x) = \phi(x)$ on E, $Z_1(x)$ is in the region $\overline{D}$, $Z_1(x) \varepsilon C^2$ on I, and $Z_1(0) = Z_1(1) = 0$. Also define $\alpha_1(x)$ on I by

$$Z_1''(x) - f[Z_1] = \alpha_1(x). \quad (3.3)$$

Now define the sequence of functions $\{Z_n(x)\}$ by the rule:

$$Z_{n+1}(x) = Z_n(x) - \sigma_n(x) \tag{3.4}$$

where

$$\sigma_n''(x) = \alpha_n(x) \text{ on } I \tag{3.5}$$

under the hypotheses

$$\begin{aligned} \sigma_n(x) &= 0 \text{ for } x \in E, \\ \sigma_n(0) &= \sigma_n(1) = 0 \end{aligned} \tag{3.6}$$

and

$$\alpha_n(x) = Z_n''(x) - f[Z_n] \text{ on } I. \tag{3.7}$$

From (3.5) and (3.6) it follows that

$$\sigma_n(x) = -\int_0^1 G(x,t)\alpha_n(t)dt \tag{3.8}$$

where

$$G(x,t) = \begin{cases} t(1-x), & 0 \leqq t \leqq x \\ x(1-t), & x \leqq t \leqq 1 \end{cases} .$$

It will now be shown that every function in the sequence $\{Z_n(x)\}$ is in the region $\overline{D}$. $Z_1(x)$ is in $\overline{D}$ by hypothesis. Assume $Z_n(x) \in \overline{D}$. Then

$$|Z_n(x)| \leqq \frac{B}{8} \quad \text{and} \quad |f[Z_n]| \leqq B.$$

Differentiate (3.4) twice to get

$$Z_{n+1}''(x) = Z_n''(x) - \sigma_n''(x),$$

but, from (3.7) and (3.5), this becomes

$$Z''_{n+1}(x) = \alpha_n(x) + f[Z_n] - \sigma''_n(x)$$

$$= f[Z_n] \; . \qquad (3.9)$$

This result combined with (3.8) gives

$$Z_{n+1}(x) = -\int_0^1 G(x,t)f[Z_n]\,dt \; . \qquad (3.10)$$

Since $G(x,t) \geqq 0$ on I and $\max_{x \in I} \int_0^1 G(x,t)dt = \frac{1}{8}$ (see Appendix I), it follows that

$$|Z_{n+1}(x)| \leqq \frac{B}{8} ,$$

which implies that $Z_{n+1}(x) \in \overline{D}$.

To show that $\lim_{n\to\infty} Z_n(x)$ exists and is a solution of (3.1)-(3.2) it is necessary to obtain bounds on $\alpha_n(x)$ and $\sigma_n(x)$. Use equations (3.4) and (3.7) to get

$$\alpha_{n+1}(x) = Z''_{n+1}(x) - f[Z_{n+1}]$$

$$= Z''_n(x) - \sigma''_n(x) - f[Z_{n+1}]$$

$$= \alpha_n(x) + f[Z_n] - \sigma''_n(x) - f[Z_{n+1}]$$

$$= f[Z_n] - f[Z_{n+1}] \; .$$

Now apply the mean value theorem to obtain

$$f[Z_n] - f[Z_{n+1}] = \frac{\partial \tilde{f}}{\partial y}[Z_n(x) - Z_{n+1}(x)] +$$

$$\frac{\partial \tilde{f}}{\partial y_\tau}[Z_n(x-\tau(x)) - Z_{n+1}(x-\tau(x))] \; .$$

In the above equation $\frac{\partial \tilde{f}}{\partial y}$ is the value of the partial derivative $\frac{\partial f}{\partial y}$ at some point in the region $\overline{D}$, and $\frac{\partial \tilde{f}}{\partial y_\tau}$ is the value of $\frac{\partial f}{\partial y_\tau}$ at some point in the region $\overline{D}$. Hence

$$\alpha_{n+1}(x) = \frac{\partial \tilde{f}}{\partial y} \sigma_n(x) + \frac{\partial \tilde{f}}{\partial y_\tau} \sigma_n(x-\tau(x)) \quad . \qquad (3.11)$$

Let $M = \max_{x \varepsilon I} |\alpha_1(x)|$. Then it follows from (3.8) with $n = 1$ that

$$|\sigma_1(x)| \leqq M(\tfrac{1}{8}) \quad .$$

Using this in equation (3.11) one obtains

$$|\alpha_2(x)| \leqq M(2N)(\tfrac{1}{8}) \quad .$$

Repeating the process for $n = 2$,

$$|\sigma_2(x)| \leqq M(2N)(\tfrac{1}{8})^2,$$

$$|\alpha_3(x)| \leqq M(2N)^2(\tfrac{1}{8})^2.$$

By induction one obtains the following inequalities:

$$\begin{aligned} |\sigma_n(x)| &\leqq M(2N)^{n-1}(\tfrac{1}{8})^n, \\ |\alpha_n(x)| &\leqq M(2N)^{n-1}(\tfrac{1}{8})^{n-1}. \end{aligned} \qquad (3.12)$$

Note that $\lim_{n\to\infty} Z_n(x)$ is equivalent to the sum of the series $Z_1(x) + [Z_2(x) - Z_1(x)] + [Z_3(x) - Z_2(x)] + \ldots$ and that $\lim_{n\to\infty} Z_n''(x)$ is equivalent to the sum of the series $Z_1''(x) + [Z_2''(x) - Z_1''(x)] + [Z_3''(x) - Z_2''(x)] + \ldots.$ Also,

$Z_{n+1}(x) - Z_n(x) = -\sigma_n(x)$ and $Z''_{n+1}(x) - Z''_n(x) = -\alpha_n(x)$.

Therefore, for $N < 4$, the series

$$Z_1(x) + [Z_2(x) - Z_1(x)] + \ldots + [Z_{n+1}(x) - Z_n(x)] + \ldots,$$

$$Z_1''(x) + [Z_2''(x) - Z_1''(x)] + \ldots + [Z''_{n+1}(x) - Z_n''(x)] + \ldots$$

converge absolutely and uniformly, their sums $\tilde{y}(x)$ and $\tilde{y}''(x)$ exist, are continuous and $\tilde{y}(x)$ satisfies the boundary conditions (3.2). To see that $\tilde{y}''(x)$ is the second derivative of $\tilde{y}(x)$, note that by uniform convergence the second series above can be integrated term by term twice to obtain the first series.

To show that $\tilde{y}(x)$ satisfies (3.1), rewrite equation (3.7) as follows:

$$Z_n(x) = -\int_0^1 G(x,t)(\alpha_n(t) + f[Z_n])dt. \qquad (3.13)$$

Taking the limit as $n \to \infty$ and using uniform convergence and continuity properties one obtains

$$\tilde{y}(x) = -\int_0^1 G(x,t) f[\tilde{y}]\,dt \ .$$

But this is equivalent to

$$\tilde{y}''(x) = f[\tilde{y}] \ ,$$

so $\tilde{y}(x)$ satisfies (3.1) and the sequence $\{Z_n(x)\}$ converges to a solution of (3.1)-(3.2).

To prove uniqueness assume that, in addition to $y(x)$, there is another function Y(x) satisfying (3.1)-(3.2) such that $F(x) = |y(x) - Y(x)|$ is not identically zero on I. Let $x = \xi$ be the point on I where F(x) takes on its maximum value, ($\max F(x) = \theta > 0$). Since y(x) and Y(x) satisfy (3.1) it is clear that

$$y''(x) - Y''(x) = f[y] - f[Y]. \tag{3.14}$$

Hence

$$y(x) - Y(x) = -\int_0^1 G(x,t)[f[y] - f[Y]]dt$$

and the following estimate will hold:

$$|y(x) - Y(x)| \leqq \int_0^1 G(x,t)[|\frac{\partial \tilde{f}}{\partial y}|(|y(t) - Y(t)|) + |\frac{\partial \tilde{f}}{\partial y_\tau}|(|y(t-\tau(t)) - Y(t-\tau(t))|]dt$$

$$< 2N\theta \max_{x\varepsilon I} \int_0^1 G(x,t)dt = 2N\theta(\tfrac{1}{8}) = \frac{N\theta}{4}.$$

So $F(x) < \frac{N\theta}{4}$ and in particular $F(\xi) = \theta < \frac{N\theta}{4}$ which gives $N > 4$. But this contradicts the hypothesis of the theorem that $N < 4$, so the solution is unique.

B. Inclusion Theorems. The next theorems give a method for approximating the solution of (3.1)-(3.2) by upper and lower solutions. Two cases will be considered.

1. Non-positive Derivatives. Assume that

$$\frac{\partial f}{\partial y} \leqq 0 \text{ and } \frac{\partial f}{\partial y_\tau} \leqq 0. \tag{3.15}$$

Assume there exist $Z_1(x)$ and $V_1(x)$ in $\overline{D}$ and in class C^2 which satisfy the BC's (3.2), such that

$$\begin{aligned} Z_1''(x) - f[Z_1] &= \alpha_1(x) \geqq 0 \text{ (lower solution)}, \\ V_1''(x) - f[V_1] &= \beta_1(x) \leqq 0 \text{ (upper solution)}. \end{aligned} \tag{3.16}$$

Construct the sequences of functions $\{Z_n(x)\}$ and $\{V_n(x)\}$ by the rules:

$$\begin{aligned} Z_{n+1}(x) &= Z_n(x) - \sigma_n(x), \\ V_{n+1}(x) &= V_n(x) - \omega_n(x), \end{aligned} \tag{3.17}$$

where

$$\sigma_n''(x) = \alpha_n(x), \quad \omega_n''(x) = \beta_n(x), \tag{3.18}$$

with boundary conditions

$$\sigma_n(x) = \omega_n(x) = 0 \text{ for } x \in E,$$
$$\sigma_n(0) = \sigma_n(1) = \omega_n(0) = \omega_n(1) = 0,$$

and

$$\begin{aligned} \alpha_n(x) &= Z_n''(x) - f[Z_n], \\ \beta_n(x) &= V_n''(x) - f[V_n]. \end{aligned} \tag{3.19}$$

<u>Theorem 2</u>. If for $x \in I$ (3.15) is satisfied and there exist functions $Z_1(x)$ and $V_1(x)$ satisfying (3.2) which are lower and upper solutions, respectively, as in (3.16), and if $y(x)$ is the solution of (3.1)-(3.2), then

$$Z_1(x) \leqq y(x) \leqq V_1(x). \tag{3.20}$$

Proof. The mean value theorem will again give

$$\begin{aligned}\alpha_{n+1}(x) &= \frac{\partial \tilde{f}}{\partial y}\,\sigma_n(x) + \frac{\partial \tilde{f}}{\partial y_\tau}\,\sigma_n(x-\tau(x))\\ \beta_{n+1}(x) &= \frac{\partial \tilde{f}}{\partial y}\,\omega_n(x) + \frac{\partial \tilde{f}}{\partial y_\tau}\,\omega_n(x-\tau(x)).\end{aligned} \tag{3.21}$$

It is clear from Lemma 1 and equation (3.18) that on I $\sigma_1(x) \leqq 0$ and $\omega_1(x) \geqq 0$, therefore

$$Z_2(x) \geqq Z_1(x) \text{ and } V_2(x) \leqq V_1(x).$$

Using (3.21) for n = 1,

$$\alpha_2(x) \geqq 0,\ \beta_2(x) \leqq 0$$

which gives

$$\sigma_2(x) \leqq 0,\ \omega_2(x) \geqq 0.$$

By induction, the following inequalities hold:

$$\alpha_n(x) \geqq 0,\ \sigma_n(x) \leqq 0, \tag{3.22}$$

$$\beta_n(x) \leqq 0,\ \omega_n(x) \geqq 0. \tag{3.23}$$

By Theorem 1, for $N < 4$, the sequences of functions $\{Z_n(x)\}$ and $\{V_n(x)\}$ converge to the solution of (3.1)-(3.2). Futhermore, by (3.22) and (3.23) it is clear that

$$Z_{n+1}(x) \geqq Z_n(x) \text{ and } V_{n+1}(x) \leqq V_n(x), \tag{3.24}$$

since $Z_{n+1}(x) = Z_n(x) - \sigma_n(x)$ and $V_{n+1}(x) = V_n(x) - \omega_n(x)$.

The conclusion of the theorem follows from the fact that $\{Z_n(x)\}$ is a monotonic non-decreasing sequence and $\{V_n(x)\}$ is a monotonic non-increasing sequence.

Corollary 1. If the hypotheses of Theorem 2 are satisfied and if the sequences $\{Z_n(x)\}$ and $\{V_n(x)\}$ are determined by (3.17), (3.18), and (3.19) then

$$Z_n(x) \leqq y(x) \leqq V_n(x). \tag{3.25}$$

Proof. It is evident from (3.19), (3.22), and (3.23) that $Z_n(x)$ is a lower solution and that $V_n(x)$ is an upper solution. Also, every $Z_n(x)$ and $V_n(x)$ satisfy (3.2), so the corollary follows from Theorem 2.

Corollary 2. If $V_1(x)$ ($Z_1(x)$) is an upper (lower) solution to (3.1)-(3.2) then the sequence $\{V_n(x)\}$ ($\{Z_n(x)\}$) defined as in Theorem 2 converges monotonically from above (below) to $y(x)$. In particular, if the function identically equal to zero on I is an upper (lower) solution to (3.1)-(3.2) then $y(x) \leqq 0$ ($\geqq 0$) on I.

Proof. The proof follows directly from Theorem 1 and the proof of Theorem 2.

2. Derivatives Bounded Above. Assume that there exist arbitrary constants M_0 and K_0 such that, in the region $\overline{D}$

$$M_0 \geqq \frac{\partial f}{\partial y}, \quad K_0 \geqq \frac{\partial f}{\partial y_\tau} \tag{3.26}$$

(Notice that if M_0, $K_0 \leqq 0$, this reduces to the case just considered.)

Suppose that in the region $\overline{D}$ there exist functions $Z_1(x)$ and $V_1(x)$ in class C^2 for $x \varepsilon I$ and which satisfy (3.2) such that

$$Z_1''(x) - f[Z_1] - A_1(x) = \alpha_1(x) \geqq 0,$$
$$V_1''(x) - f[V_1] + A_1(x) = \beta_1(x) \leqq 0, \tag{3.27}$$

where

$$A_1(x) = [-|M_0| - M_0][Z_1(x) - V_1(x)] + [-|K_0| - K_0][Z_1(x-\tau(x)) - V_1(x-\tau(x))].$$

Construct the sequences of functions $\{Z_n(x)\}$ and $\{V_n(x)\}$ by (3.17) and (3.18) where

$$\alpha_n(x) = Z_n''(x) - f[Z_n] - A_n(x),$$
$$\beta_n(x) = V_n''(x) - f[V_n] + A_n(x), \tag{3.28}$$

and

$$A_n(x) = [-|M_0| - M_0][Z_n(x) - V_n(x)] + [-|K_0| - K_0][Z_n(x-\tau(x)) - V_n(x-\tau(x))]. \tag{3.29}$$

<u>Theorem 3</u>. If there exist functions $Z_1(x)$ and $V_1(x)$ on I in class C^2 satisfying (3.2) and (3.27), and if the sequences $\{Z_n(x)\}$ and $\{V_n(x)\}$ are determined by (3.17), (3.18), and (3.28) then

$$Z_n(x) \leqq V_n(x),$$

$$Z_{n+1}(x) \geqq Z_n(x),$$
$$V_{n+1}(x) \leqq V_n(x).$$

<u>Proof</u>. Let $W_n(x) = Z_n(x) - V_n(x)$. Then from equation (3.27)

$$W_1''(x) = \alpha_1(x) - \beta_1(x) + f[Z_1] - f[V_1] + 2A_1(x).$$

Application of the mean value theorem to $f[Z_1] - f[V_1]$ yields

$$W_1''(x) = \alpha_1(x) - \beta_1(x) + \frac{\partial \tilde{f}}{\partial y} W_1(x) + \frac{\partial \tilde{f}}{\partial y_\tau} W_1(x-\tau(x)) + 2A_1(x),$$

which is equivalent to

$$W_1''(x) - [-2|M_0| - M_0 + \frac{\partial \tilde{f}}{\partial y} - M_0]\, W_1(x) - [-2|K_0| - K_0 + \frac{\partial \tilde{f}}{\partial y_\tau} - K_0]\, W_1(x-\tau(x)) + \beta_1(x) - \alpha_1(x) = 0. \quad (3.30)$$

But (3.30) is of the form

$$W_1''(x) - g(x, W_1(x), W_1(x-\tau(x))) = 0,$$

and Corollary 2 of Theorem 2 can be applied. $U(x) \equiv 0$ satisfies (3.2) and is an upper solution of (3.30), since

$$U''(x) - g[U] = \beta_1(x) - \alpha_1(x) \leqq 0.$$

Therefore by Corollary 2 of Theorem 2 $W_1(x) \leqq 0$ and $Z_1(x) \leqq V_1(x)$.

The rules (3.17), (3.18), and (3.28) for constructing

$\{Z_n(x)\}$ and $\{V_n(x)\}$ are equivalent to the following:

$$Z''_{n+1}(x) = f[Z_n] + A_n(x),$$
$$V''_{n+1}(x) = f[V_n] - A_n(x). \tag{3.31}$$

Since $W_1(x) \leqq 0$, for $n = 1$ (3.31) gives

$$W_2''(x) = [-2|M_0| - M_0 + \frac{\partial \tilde{f}}{\partial y} - M_0]\ W_1(x) + [-2|K_0| - K_0 + \frac{\partial \tilde{f}}{\partial y_\tau} - K_0]\ W_1(x-\tau(x)) \equiv \psi_1(x) \geqq 0,$$

it follows from Lemma 1 that $W_2(x) \leqq 0$ and $Z_2(x) \leqq V_2(x)$.

Since $\alpha_1(x) \geqq 0$ and $\beta_1(x) \leqq 0$, it also follows from Lemma 1 that $\sigma_1(x) \leqq 0$ and $\omega_1(x) \geqq 0$, which implies $Z_2(x) \geqq Z_1(x)$ and $V_2(x) \leqq V_1(x)$.

By applying (3.31) to (3.28) one gets

$$\alpha_{n+1}(x) = f[Z_n] - f[Z_{n+1}] + A_n(x) - A_{n+1}(x),$$

which is equivalent to

$$\alpha_{n+1}(x) = [-|M_0| + \frac{\partial \tilde{f}}{\partial y} - M_0]\ \sigma_n(x) + [-|K_0| + \frac{\partial \tilde{f}}{\partial y_\tau} - K_0]\ \sigma_n(x-\tau(x)) - [(-|M_0| - M_0)\ \omega_n(x) + (-|K_0| - K_0)\ \omega_n(x-\tau(x))]. \tag{3.32}$$

Similarly,

$$\beta_{n+1}(x) = [-|M_0| + \frac{\partial \tilde{f}}{\partial y} - M_0]\ \omega_n(x) + [-|K_0| + \frac{\partial \tilde{f}}{\partial y_\tau} - K_0]\ \omega_n(x-\tau(x)) - [(-|M_0| - M_0)\ \sigma_n(x)$$

$$+ (-|K_0| - K_0)\ \sigma_n(x-\tau(x))]. \tag{3.33}$$

For $n = 1$ these equations give

$$\alpha_2(x) \geqq 0 \text{ and } \beta_2(x) \leqq 0.$$

Therefore $\sigma_2(x) \leqq 0$ and $\omega_2(x) \geqq 0$, and so $Z_3(x) \geqq Z_2(x)$ and $V_3(x) \leqq V_2(x)$.

Continuing, by induction one obtains the desired inequalities using (3.32), (3.33), and

$$W''_{n+1}(x) = [-2|M_0| - M_0 + \frac{\partial \tilde{f}}{\partial y} - M_0]\ W_n(x) + [-2|K_0| - K_0 + \frac{\partial \tilde{f}}{\partial y_\tau} - K_0]\ W_n(x-\tau(x)) \equiv \psi_n(x). \tag{3.34}$$

<u>Theorem 4</u>. Let

$$P = \max\ (\sup_{\overline{D}_1} |[-|M_0| - M_0 + \frac{\partial f}{\partial y} - M_0]|,\ \sup_{\overline{D}_1} |[-|K_0| - K_0 + \frac{\partial f}{\partial y_\tau} - K_0]|),$$

and let $\{Z_n(x)\}$ and $\{V_n(x)\}$ be defined by (3.17), (3.18), and (3.28). If $P < 2$, then $\lim_{n\to\infty} Z_n(x)$ and $\lim_{n\to\infty} V_n(x)$ exist, $\lim_{n\to\infty} Z_n(x)$ is a lower solution to (3.1)-(3.2), and $\lim_{n\to\infty} V_n(x)$ is an upper solution to (3.1)-(3.2).

<u>Proof</u>. Let

$$M = \max\ (\sup_{x\varepsilon I} |\alpha_1(x)|,\ \sup_{x\varepsilon I} |\beta_1(x)|).$$

From (3.8) with $n = 1$ it follows that $|\sigma_1(x)| \leqq M(\frac{1}{8})$ and

$|\omega_1(x)| \leqq M(\frac{1}{8})$, and by using (3.32) and (3.33) one gets $|\alpha_2(x)| \leqq M(4P)(\frac{1}{8})$ and $|\beta_2(x)| \leqq M(4P)(\frac{1}{8})$.

By induction the following hold:

$$|\alpha_n(x)| \leqq M(4P)^{n-1}(\tfrac{1}{8})^{n-1},$$

$$|\beta_n(x)| \leqq M(4P)^{n-1}(\tfrac{1}{8})^{n-1},$$

$$|\sigma_n(x)| \leqq M(4P)^{n-1}(\tfrac{1}{8})^{n},$$

$$|\omega_n(x)| \leqq M(4P)^{n-1}(\tfrac{1}{8})^{n}.$$

Therefore for $P < 2$ the series

$$Z_1(x) + [Z_2(x) - Z_1(x)] + \ldots,$$

$$Z_1''(x) + [Z_2''(x) - Z_1''(x)] + \ldots,$$

$$V_1(x) + [V_2(x) - V_1(x)] + \ldots,$$

$$V_1''(x) + [V_2''(x) - V_1''(x)] + \ldots,$$

converge absolutely and uniformly to $\tilde{y}(x)$, $\tilde{y}''(x)$, $\tilde{z}(x)$, and $\tilde{z}''(x)$, respectively, as in Theorem 1. Again, Uniform convergence guarantees that $\tilde{y}''(x)$ and $\tilde{z}''(x)$ are the second derivatives of $\tilde{y}(x)$ and $\tilde{z}(x)$. Therefore $\lim_{n\to\infty} Z_n(x)$ and $\lim_{n\to\infty} V_n(x)$ exist.

From Theorem 3 $Z_n(x) - V_n(x) \leqq 0$ for all n. Hence from (3.29) $A_n(x) \geqq 0$ for all n and $\lim_{n\to\infty} A_n(x) \geqq 0$. Since

$$Z_{n+1}''(x) = f[Z_n] + A_n(x)$$

and

$$V''_{n+1}(x) = f[V_n] - A_n(x),$$

by taking the limit as $n\to\infty$ one obtains

$$\tilde{y}''(x) = f[\tilde{y}] + \lim_{n\to\infty} A_n(x)$$

and

$$\tilde{z}''(x) = f[\tilde{z}] - \lim_{n\to\infty} A_n(x).$$

But since $\lim_{n\to\infty} A_n(x) \geqq 0$ this is equivalent to

$$\tilde{y}''(x) - f[\tilde{y}] \geqq 0 \tag{3.35}$$

and

$$\tilde{z}''(x) - f[\tilde{z}] \leqq 0, \tag{3.36}$$

which are the definitions of lower and upper solutions.

<u>Theorem 5</u>. If the hypotheses of Theorem 4 are satisfied, and if, in addition, $\frac{\partial f}{\partial y_\tau} \leqq 0$, then

$$Z_n(x) \leqq y(x) \leqq V_n(x)$$

where y(x) is the solution of (3.1)-(3.2).

<u>Proof</u>. A result obtained by Grimm and Schmitt [6] will be used in the proof. By Theorem 4 $\tilde{y}(x)$ and $\tilde{z}(x)$ are lower and upper solutions, respectively, of (3.1)-(3.2), so Lemma 5 of Grimm and Schmitt gives

$$\tilde{y}(x) \leqq y(x) \leqq \tilde{z}(x).$$

Since $Z_n(x) \to \tilde{y}(x)$ monotonically from below and $V_n(x) \to \tilde{z}(x)$ monotonically from above, the desired result follows.

IV. The Third Order Boundary Value Problem

A. Existence and Uniqueness. Consider the third order BVP:

$$y'''(x) = f(x,y(x),y(x-\tau(x))) \equiv f(x,y,y_\tau) \equiv f[y], \quad (4.1)$$

where $\tau(x) \geqq 0$, $\tau(x) \in C$ on I with boundary conditions

$$\begin{aligned} &y(x) = \phi(x) \text{ for } x \in E = \{x \mid x < 0\}, \\ &\phi(0) = 0,\ \phi'(0) = 0,\ y(0) = 0,\ y'(0) = y'(1) = 0, \end{aligned} \quad (4.2)$$

where $\phi(x) \in C^3$ is a given initial function on E.

<u>Theorem 6</u>. Let f be continuous and have bounded derivatives, $\left|\frac{\partial f}{\partial y}\right| \leqq N$ and $\left|\frac{\partial f}{\partial y_\tau}\right| \leqq N$, on some compact region $\overline{D}_1$ containing

$$\overline{D}_2 = \{(x,y,z) \mid 0 \leqq x \leqq 1,\ |y| \leqq \tfrac{B}{12},\ |z| \leqq \tfrac{B}{12}\},$$

where

$$B = \sup_{\overline{D}_1} |f[y]|.$$

Then if $N < 6$ in $\overline{D}_2$, there exists a unique continuous solution of the problem (4.1)-(4.2).

<u>Proof</u>. Define the function $Z_1(x)$ such that $Z_1(x) = \phi(x)$ on E, $Z_1(x)$ is in the region $\overline{D}_2$, $Z_1(x) \in C^3$ on I, and $Z_1(0) = Z_1'(0) = Z_1'(1) = 0$. Also define $\alpha_1(x)$ on I by

$$\alpha_1(x) = Z_1'''(x) - f[Z_1]. \quad (4.3)$$

Construct the sequence of functions $\{Z_n(x)\}$ by the rule:

$$Z_{n+1}(x) = Z_n(x) - \sigma_n(x) \tag{4.4}$$

where

$$\sigma_n'''(x) = \alpha_n(x), \; x \in I, \tag{4.5}$$

with boundary conditions

$$\begin{aligned} &\sigma_n(x) = 0 \text{ if } x \in E, \\ &\sigma_n(0) = \sigma_n'(0) = \sigma_n'(1) = 0 \end{aligned} \tag{4.6}$$

and

$$\alpha_n(x) = Z_n'''(x) - f[Z_n]. \tag{4.7}$$

The solution to (4.5)-(4.6) is given by

$$\sigma_n(x) = -\int_0^1 \overline{G}(x,t)\alpha_n(t)dt \tag{4.8}$$

where

$$\overline{G}(x,t) = \begin{cases} \frac{t}{2}(2x - x^2 - t), & 0 \leqq t \leqq x \\ \frac{x}{2}(x - xt), & x \leqq t \leqq 1. \end{cases}$$

To show that $Z_n(x)$ is in $\overline{D}_2$ for all n note that if $Z_n(x)$ is in $\overline{D}_2$ then

$$|Z_n(x)| \leqq \frac{B}{12} \text{ and } |f[Z_n]| \leqq B.$$

By differentiating (4.4) three times one gets

$$Z'''_{n+1}(x) = f[Z_n]. \tag{4.9}$$

Hence

$$Z_{n+1}(x) = -\int_0^1 \overline{G}(x,t)f[Z_n]dt, \tag{4.10}$$

and since $\max_{x\varepsilon I} \int_0^1 \overline{G}(x,t)dt = \frac{1}{12}$ (see Appendix 1), it follows that $|Z_{n+1}(x)| \leqq \frac{B}{12}$, and $Z_{n+1}(x)$ is in $\overline{D}_2$.

As in Theorem 1 it can be shown that

$$\alpha_{n+1}(x) = \frac{\partial \tilde{f}}{\partial y}\sigma_n(x) + \frac{\partial \tilde{f}}{\partial y_\tau}\sigma_n(x-\tau(x)). \tag{4.11}$$

Let $M = \max_{x\varepsilon I} |\alpha_1(x)|$. Then by using (4.8) and (4.11) the following inequalities are obtained by induction:

$$|\alpha_n(x)| \leqq M(2N)^{n-1}\left(\frac{1}{12}\right)^{n-1},$$

$$|\sigma_n(x)| \leqq M(2N)^{n-1}\left(\frac{1}{12}\right)^{n}.$$

Therefore for $N < 6$ the series

$$Z_1(x) + [Z_2(x) - Z_1(x)] + \ldots,$$

$$Z_1'''(x) + [Z_2'''(x) - Z_1'''(x)] + \ldots$$

converge absolutely and uniformly, their sums exist, are continuous, and satisfy (4.2) as in Theorem 1. Call the sums $\overline{y}(x)$ and $\overline{y}'''(x)$ respectively. As before, uniform convergence guarantees that $\overline{y}'''(x)$ is the third derivative of $\overline{y}(x)$.

To show that $\overline{y}(x)$ is a solution of (4.1)-(4.2) rewrite (4.7) as follows:

$$Z_n(x) = -\int_0^1 \overline{G}(x,t)(\alpha_n(t) + f[Z_n])dt.$$

Now take the limit as $n \to \infty$ to get

$$\overline{y}(x) = -\int_0^1 \overline{G}(x,t)f[\overline{y}],$$

which implies that $\overline{y}'''(x) = f[\overline{y}]$, so $\overline{y}(x)$ is a solution of (4.1)-(4.2).

To prove uniqueness let $z(x)$ be another solution of (4.1)-(4.2) such that the function $F(x) = |y(x) - z(x)|$ is not identically zero on I. Let $x = \xi$ be the point on I where $F(x)$ achieves its maximum value, call it θ. (Note that $\theta > 0$.) Since $y(x)$ and $z(x)$ satisfy (4.1) it is clear that

$$y'''(x) - z'''(x) = f[y] - f[z].$$

Hence

$$y(x) - z(x) = -\int_0^1 \overline{G}(x,t)(f[y] - f[z])dt,$$

and the following estimate will hold:

$$|y(x) - z(x)| \leqq \int_0^1 \overline{G}(x,t)\Big[\Big|\frac{\partial \tilde{f}}{\partial y}\Big||y(t) - z(t)| + \Big|\frac{\partial \tilde{f}}{\partial y_\tau}\Big||y(x-\tau(x)) - z(x-\tau(x))|\Big]dt$$

$$< 2N\theta \max_{x \in I} \int_0^1 \overline{G}(x,t)dt = 2N\theta\left(\frac{1}{12}\right) = \frac{N\theta}{6}.$$

So $F(x) < \frac{N\theta}{6}$ on I and in particular for $x = \xi$, one gets $\theta < \frac{N\theta}{6}$ or $N > 6$. But this contradicts the hypothesis of the theorem, so $y(x)$ is the unique solution.

B. Inclusion Theorems. In this section results similar to those of Section III-B will be proved for the third order case, provided that the derivatives are non-positive. Assume that

$$\frac{\partial f}{\partial y} \leqq 0 \quad \text{and} \quad \frac{\partial f}{\partial y_\tau} \leqq 0, \tag{4.12}$$

and assume that there exist $Z_1(x)$ and $V_1(x)$ in class C^3 which lie in $\overline{D}_2$ and which satisfy the boundary conditions (4.2). Define $\alpha_1(x)$ and $\beta_1(x)$ such that

$$\begin{aligned} Z_1'''(x) - f[Z_1] &= \alpha_1(x) \geqq 0 \\ V_1'''(x) - f[V_1] &= \beta_1(x) \leqq 0. \end{aligned} \tag{4.13}$$

(Equation (4.13) can be taken as a definition of lower and upper solutions, respectively, for the third order BVP.) Construct the sequences of functions $\{Z_n(x)\}$ and $\{V_n(x)\}$ by the rules

$$\begin{aligned} Z_{n+1}(x) &= Z_n(x) - \sigma_n(x), \\ V_{n+1}(x) &= V_n(x) - \omega_n(x) \end{aligned} \tag{4.14}$$

where

$$\sigma_n'''(x) = \alpha_n(x), \ \omega_n'''(x) = \beta_n(x), \tag{4.15}$$

with boundary conditions

$$\sigma_n(x) = \omega_n(x) = 0 \text{ for } x \in E,$$

$$\sigma_n(0) = \sigma_n'(0) = \sigma_n'(1) = 0, \tag{4.16}$$

$$\omega_n(0) = \omega_n'(0) = \omega_n'(1) = 0,$$

and

$$\alpha_n(x) = Z_n'''(x) - f[Z_n],$$

$$\beta_n(x) = V_n'''(x) - f[V_n]. \tag{4.17}$$

<u>Theorem 7</u>. If for $x \in I$ (4.12) is satisfied and there exist functions $Z_1(x)$ and $V_1(x)$ satisfying (4.2) and (4.13), and if $y(x)$ is the solution of (4.1)-(4.2), then

$$Z_1(x) \leqq y(x) \leqq V_1(x).$$

<u>Proof</u>. Using (4.11), (4.15), and proceeding as in Theorem 2 one finds that

$$\alpha_n(x) \geqq 0, \quad \sigma_n(x) \leqq 0,$$

$$\beta_n(x) \leqq 0, \quad \omega_n(x) \geqq 0, \tag{4.18}$$

where $n = 1, 2, 3, \ldots$.

Hence, for $N < 6$ the sequences $\{|\alpha_n(x)|\}$, $\{|\beta_n(x)|\}$, $\{|\sigma_n(x)|\}$, and $\{|\omega_n(x)|\}$ converge to zero and therefore $\{Z_n(x)\}$ and $\{V_n(x)\}$ converge to the solution of (4.1)-(4.2).

It is clear from (4.18) and (4.14) that

$$Z_{n+1}(x) \geqq Z_n(x) \text{ and } V_{n+1}(x) \leqq V_n(x), \tag{4.19}$$

and so the proof follows as in Theorem 2.

Corollary 1. If the hypotheses of Theorem 7 are satisfied and if the sequence $\{Z_n(x)\}$ and $\{V_n(x)\}$ are determined by (4.14), (4.15), and (4.17) then

$$Z_n(x) \leqq y(x) \leqq V_n(x). \tag{4.20}$$

Proof. Same as Corollary 1, Theorem 2.

Corollary 2. If $V_1(x)$ $(Z_1(x))$ is an upper (lower) solution to (4.1)-(4.2) then the sequence $\{V_n(x)\}$ $(\{Z_n(x)\})$ defined as in Theorem 7 converges monotonically from above (below) to $y(x)$. In particular, if the function identically equal to zero on I is an upper (lower) solution to (4.1)-(4.2), then $y(x) \leqq 0$ $(\geqq 0)$ on I.

Proof. The proof follows from Theorem 6 and the proof of Theorem 7.

V. A Generalization

The results in Theorem 1 and Theorem 6 can be obtained for a more general situation, viz., when τ is a function of both x and y. However, the inclusion theorems will no longer necessarily hold when this is the case.

The second order BVP is now

$$y''(x) = f(x,y(x),y(x-\tau(x,y))) = f[y], \qquad (5.1)$$

$\tau(x,y) \geqq 0$ and continuous,

with boundary conditions

$$y(x) = \phi(x) \text{ for } x \in E = \{x \mid x < 0\}, \qquad (5.2)$$
$$\phi(0) = 0, \ y(0) = y(1) = 0,$$

where $\phi(x) \in C^2$ is a given initial function on E.

Theorem 8. Let f be continuous and have bounded derivatives, $\left|\frac{\partial f}{\partial y}\right| \leqq N$ and $\left|\frac{\partial f}{\partial y_\tau}\right| \leqq N$, on some compact region $\overline{D}_1$ containing

$$\overline{D} = \{(x,y,z) \mid 0 \leqq x \leqq 1, \ |y| \leqq \tfrac{B}{8}, \ |z| \leqq \tfrac{B}{8}\},$$

where

$$B = \sup_{\overline{D}_1} |f[y]|.$$

Also let $\left|\frac{\partial \tau}{\partial y}\right| \leqq \frac{m}{B}$ where m is any positive integer.

Then if $N < \frac{4}{m+1}$ in $\overline{D}$, there exists a unique continuous solution of (5.1)-(5.2).

<u>Proof</u>. The proof is identical to the proof of Theorem 1 up to the derivation of (3.11). At this point the mean value theorem is applied as follows:

$$\begin{aligned}
\alpha_{n+1}(x) &= f[Z_n] - f[Z_{n+1}] \\
&= \frac{\partial \tilde{f}}{\partial y}\left(Z_n(x) - Z_{n+1}(x)\right) + \frac{\partial \tilde{f}}{\partial y_\tau}\Big(Z_n(x-\tau(x,Z_n)) - \\
&\quad Z_{n+1}(x-\tau(x,Z_{n+1}))\Big) \\
&= \frac{\partial \tilde{f}}{\partial y}\sigma_n(x) + \frac{\partial \tilde{f}}{\partial y_\tau}\Big(Z_n(x-\tau(x,Z_{n+1})) - \\
&\quad Z_{n+1}(x-\tau(x,Z_{n+1})) - Z_n(x-\tau(x,Z_{n+1})) + \\
&\quad Z_n(x-\tau(x,Z_n))\Big) \\
&= \frac{\partial \tilde{f}}{\partial y}\sigma_n(x) + \frac{\partial \tilde{f}}{\partial y_\tau}\Big(\sigma_n(x-\tau(x,Z_{n+1})) + \\
&\quad Z_n'(\xi)(\tau(x,Z_{n+1}) - \tau(x,Z_n))\Big) \\
&= \frac{\partial \tilde{f}}{\partial y}\sigma_n(x) + \frac{\partial \tilde{f}}{\partial y_\tau}\Big(\sigma_n(x-\tau(x,Z_{n+1})) + \\
&\quad Z_n'(\xi)\frac{\partial \tilde{\tau}}{\partial y}(-\sigma_n(x))\Big) \\
&= \left(\frac{\partial \tilde{f}}{\partial y} - \left(\frac{\partial \tilde{f}}{\partial y_\tau}\right)(Z_n'(\xi))\left(\frac{\partial \tilde{\tau}}{\partial y}\right)\right)\sigma_n(x) + \\
&\quad \frac{\partial \tilde{f}}{\partial y_\tau}\sigma_n(x-\tau(x,Z_{n+1})). \qquad (5.3)
\end{aligned}$$

Bounds need to be determined for $Z_n'(\xi)$ in order to be able to use this equation.

From (3.9), $Z_n''(x) = f[Z_{n-1}] \leqq B$. One finds by integration that

$$|Z_n'(x)| = |Z_n'(0) + \int_0^x f[Z_{n+1}(s)]ds|$$

$$\leqq |Z_n'(0)| + \int_0^x Bds$$

$$\leqq |Z_n'(0)| + Bx$$

$$\leqq |Z_n'(0)| + B \text{ on } I.$$

Assume that there exists $\delta > 0$ such that $Z_n'(0) > B + \delta$. Then

$$Z_n'(x) > B + \delta - \int_0^1 Bds,$$

$$Z_n'(x) > \delta \text{ for all } x \in I.$$

Hence

$$Z_n(x) > Z_n(0) + \delta x = \delta x \text{ for all } x \in (0,1].$$

But this means $Z_n(1) > 0$, a contradiction. So $Z_n'(x) \leqq 2B$ for all n.

Now assume that there exists $\delta > 0$ such that $Z_n'(0) < -B - \delta$. Then

$$Z_n'(x) < -B - \delta + \int_0^1 Bds,$$

$$Z_n'(x) < -\delta \text{ for all } x \in I.$$

Hence

$$Z_n(x) < -\delta x \text{ for all } x \in (0,1].$$

But now $Z_n(1) < 0$, a contradiction. So $Z_n'(x) \geqq -2B$ for all n.

Therefore $|Z_n'(x)| \leqq 2B$.

Since $\left|\frac{\partial \tau}{\partial y}\right| \leqq \frac{m}{B}$ by hypothesis, where m is any positive integer, the following inequality will hold:

$$\alpha_{n+1}(x) \leqq (2m + 1)N\ \sigma_n(x) + N\ \sigma_n(x-\tau(x,Z_{n+1})). \qquad (5.4)$$

Let $\max_{x \varepsilon I} |\alpha_1(x)| = M$. Then by (3.8) $|\sigma_1(x)| \leqq M(\frac{1}{8})$.

Using (3.8) and (5.4) one obtains the following by induction:

$$|\alpha_n(x)| \leqq M(2Nm + 2N)^{n-1}(\tfrac{1}{8})^{n-1}$$
$$|\sigma_n(x)| \leqq M(2Nm + 2N)^{n-1}(\tfrac{1}{8})^{n}. \qquad (5.5)$$

So for $N < \frac{4}{m+1}$ the series

$$Z_1(x) + [Z_2(x) - Z_1(x)] + \ldots,$$

$$Z_1''(x) + [Z_2''(x) - Z_1''(x)] + \ldots,$$

converge absolutely and uniformly to $y(x)$ and $y''(x)$. The remainder of the proof is analogous to the proof of Theorem 1.

VI. A Numerical Example

The equation

$$y''(x) = -y(x) - y(x - \tfrac{1}{2}),$$

with boundary conditions

$$y(x) = -x, \quad x \in [-\tfrac{1}{2}, 0],$$

$$y(0) = y(1) = 0,$$

together with the functions

$$Z_1(x) = \begin{cases} -x, & x \in [-\tfrac{1}{2}, 0] \\ 0, & x \in I \end{cases}$$

and

$$V_1(x) = \begin{cases} -x, & x \in [-\tfrac{1}{2}, 0] \\ x - x^2, & x \in I \end{cases}$$

satisfy the conditions of Theorem 2 and its corollaries. The iteration scheme is defined by

$$Z_{n+1}(x) = -\int_0^1 G(x,t) f[Z_n] dt$$

$$V_{n+1}(x) = -\int_0^1 G(x,t) f[V_n] dt.$$

The results obtained using the IBM 360 model 50 digital computer at the University of Missouri-Rolla are given in tables I and II. As an example of the notation used in the tables, 0.4573D-03 means 0.4573×10^{-3}.

Table I

x	$Z_5(x)$	$V_5(x)$	$V_5(x) - Z_5(x)$
0.000	0.0000D 00	0.0000D 00	0.0000D 00
0.125	0.2257D-02	0.2257D-02	0.1400D-09
0.250	0.2777D-02	0.2777D-02	0.2732D-08
0.375	0.1780D-02	0.1780D-02	0.9102D-08
0.500	0.0000D 00	0.1312D-07	0.1312D-07
0.625	0.3270D-04	0.3271D-04	0.1734D-07
0.750	0.3224D-04	0.3225D-04	0.1545D-07
0.875	0.1203D-04	0.1204D-04	0.8129D-08
1.000	0.0000D 00	0.0000D 00	0.0000D 00

Table II

x	$Z_{20}(x)$	$V_{20}(x)$	$V_{20}(x) - Z_{20}(x)$
0.000	0.0000D 00	0.0000D 00	0.0000D 00
0.125	0.2257D-02	0.2257D-02	0.4337D-18
0.250	0.2777D-02	0.2777D-02	0.0000D 00
0.375	0.1780D-02	0.1780D-02	0.0000D 00
0.500	0.0000D 00	0.6585D-35	0.6585D-35
0.625	0.3270D-04	0.3270D-04	0.3388D-20
0.750	0.3224D-04	0.3224D-04	0.0000D 00
0.875	0.1203D-04	0.1203D-04	0.0000D 00
1.000	0.0000D 00	0.0000D 00	0.0000D 00

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Vita

Leon Morris Hall, Jr. was born on July 31, 1946 in Springfield, Missouri. In 1950 his family moved to Sedalia, Missouri where he received his primary and secondary education. He entered the University of Missouri-Rolla in the fall of 1964. He also attended the University of Missouri-Columbia in the summer and fall of 1967 as part of the cooperative mathematics-education program. In January 1969 he received a Bachelor of Science Degree in Education from the University of Missouri-Columbia, and in June 1969 he received a Bachelor of Science Degree in Applied Mathematics from the University of Missouri-Rolla.

He taught mathematics at Normandy Senior High School in St. Louis, Missouri for the 1969-1970 school year, and has held teaching assistantships at the University of Missouri-Rolla for the 1968-1969 and 1970-1971 school years.

On December 16, 1967 he was married to Pennye Kaye Nichols of Sedalia.

Appendix I

Bounds for the Integrals of the Green's Functions

A. Green's Function for the BVP of Section III. Let $G(x,t)$ be defined by

$$G(x,t) = \begin{cases} t(1-x), & 0 \leqq t \leqq x \\ x(1-t), & x \leqq t \leqq 1. \end{cases}$$

It is easily verified by integration that

$$\int_0^1 G(x,t)dt = \frac{1}{2}(x - x^2).$$

Differentiating the above function, one finds that its maximum for x in I occurs at $x = \frac{1}{2}$. Hence

$$\max_{x\varepsilon I} \int_0^1 G(x,t)dt = \frac{1}{2}(\frac{1}{2} - \frac{1}{4}) = \frac{1}{8}.$$

B. Green's Function for the BVP of Section IV. Let $\overline{G}(x,t)$ be defined by

$$\overline{G}(x,t) = \begin{cases} \frac{t}{2}(2x - x^2 - t), & 0 \leqq t \leqq x \\ \frac{x}{2}(x - xt), & x \leqq t \leqq 1. \end{cases}$$

Again, one finds by integration that

$$\int_0^1 \overline{G}(x,t)dt = \frac{x^2}{4} - \frac{x^3}{6},$$

and in this case the maximum occurs at $x = 1$. Hence

$$\max_{x\varepsilon I} \int_0^1 \overline{G}(x,t)dt = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$