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A digital computer program for studying elasto-plastic structural behavior due to cyclic loading

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A DIGITAL COMPUTER PROGRAM FOR STUDYING

ELASTOPLASTIC STRUCTURAL BEHAVIOR DUE TO CYCLIC LOADING

by

RAMESHCHANDRA CHANDULAL HAZARIWALA, 1942 –

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ABSTRACT

A computer program has been developed to study the behavior of plane stress structures under cyclic loading. Such phenomena as thermal ratcheting, alternate plasticity, shake-down and the Bauschinger effect may be considered. The incremental theory of plasticity has been used. The program deals with realistic conditions such as nonlinear strain hardening, nonlinear temperature distribution and occurrence of both compressive and tensile plastic flow. The concept of an average material property has been used.

Thermal ratcheting of a beam subjected to a constant bending moment and a temperature cycle, has been studied in detail. The analysis shows analytically that the rate of plastic strain growth reduces with an increase in the number of loading cycles. Applications of the computer program have been discussed.

Further, the thermal ratcheting of a two bar model has been discussed considering the simplifying assumptions of linear strain hardening and the absence of compressive plastic flow.

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LIST OF SYMBOLS

σ	Stress, psi.
ε	Strain, inch/inch.
ε_T	Total mechanical strain in uniaxial case, inch/inch.
ε_e	Elastic component of total mechanical strain ε_T, inch/inch.
ε_p	Plastic component of total mechanical strain ε_T, inch/inch.
E	Modulus of elasticity, psi.
n	Shape parameter for the strain hardening characteristics of a stress-strain curve.
$\sigma_{.7}$, $\sigma_{.85}$, etc.	Stresses determined from the intersection of the stress-strain curve by the lines of slopes 0.7E and 0.85E, respectively, drawn from the origin, psi.
σ_e	Equivalent stress, psi.
ε_{et}	Equivalent total mechanical strain, inch/inch.
$\Delta\varepsilon_p$	Increment of plastic strain in uniaxial case, inch/inch.
ε_p	Summation of increments of plastic strain in uniaxial case, inch/inch.

σ_{xx}, σ_{zz}, etc.	Stress in 'x' direction on an 'x' face, etc., psi.
ε_{xx}, ε_{zz}, etc.	Strain in 'x' direction on an 'x' face, etc., inch/inch.
μ	Poisson's ratio.
α	Coefficient of thermal expansion, in./in.°F.
T	Temperature above the reference temperature, °F.
$\Delta\varepsilon_{xx}^{p}$, $\Delta\varepsilon_{zz}^{p}$, etc.	Increment of plastic strain in 'x' cirection on an 'x' face, etc., inch/inch.
ε_{xx}^{p}, ε_{zz}^{p}, etc.	Summation of plastic strain increments in 'x' direction on an 'x' face, etc., inch/inch.
G	Modulus of rigidity, psi.
$\Delta\varepsilon_{ep}$	Equivalent plastic strain increment, inch/inch.
ε_{xx}', ε_{zz}', etc.	Modified total strain in an 'x' direction on an 'x' face, etc., inch/inch.
ε_{et}'	Equivalent modified total strain, inch/inch.
$\Delta\sigma_{e}$	Increment of an equivalent stress σ_{e}, psi.
h	Half of the total height of the section, inch.

b	Half of the total width of the section, inch.
T_O	Reference temperature, °F.
σ_O	Yield stress at the reference temperature T_O, psi.
S_1, S_2, etc.	σ_{xx}/σ_O, σ_{zz}/σ_O, etc.
E_O	Modulus of elasticity at the reference temperature T_O, psi.
ε_O	Yield strain at the reference temperature T_O, σ_O/E_O, inch/inch.
H	Dimensionless quantity, E/E_O.
τ	Dimensionless quantity, $\alpha T/\varepsilon_O$.
η_x, η_z	Dimensionless quantities indicating X/b and Z/h respectively.
e_{xx}, e_{zz}, etc.	Dimensionless strain in 'x' direction on an 'x' face, $\varepsilon_{xx}/\varepsilon_O$, etc.
Δe_{xx}^p, Δe_{zz}^p, etc.	Dimensionless increment of plastic strain in an 'x' direction on an 'x' face, etc.
e_{xx}^p, e_{zz}^p, etc.	Summation of Δe_{xx}^p, etc.
P	Axial load, psi.
M	Bending moment, lb. inch.
t	Time, hr.
K	Thermal conductivity, btu/hr.ft.°F.

k	Specific conductivity, ft/hr.
C	Specific heat, btu/lb.°F.
ρ	Density, lbs/ft^3.
$\sigma_{act,i}$	Actual stress at the end of the current load increment, psi.
A, B	Areas of the cross sections of the bars A and B, in^2.
σ_m	Mean stress, psi.
$\sigma_{B\frac{1}{2}}$, $\sigma_{A\frac{1}{2}}$, etc.	Stress in the bar B at the end of half of the temperature cycle, etc., psi.
σ_y	Yield stress, psi.
n_1	$\sigma_{A\frac{1}{2}}/\sigma_y$.
n_2	$\sigma_{B\frac{1}{2}}/\sigma_y$.
ε_g	Plastic strain growth, inch/inch.
$\varepsilon_{A\frac{1}{2}}$, $\varepsilon_{B\frac{1}{2}}$, etc.	Strain in the bar A at the end of half of the temperature cycle, etc., inch/inch.
E_s	Tangent modulus of elasticity, psi.
θ	$\tan^{-1} E$.
ϕ	$\tan^{-1} E_s$.
ε_{TK}	Total mechanical strain at the end of the K th cycle, inch/inch.
ε_{gK}	Total plastic strain growth at the end of the k th cycle, inch/inch.

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I. INTRODUCTION

The presence of time dependent temperature gradients in structures normally produces complex thermal stress distributions. These stresses combine with the stress due to external mechanical loads. Thus, the design of structures such as those for high speed flight, turbine blades, etc., where large thermal stresses may occur, requires a knowledge of the structural behavior under the combined effects of mechanical and thermal loads. To define the allowable loads and safety factors for design, several types of problems must be studied.

Parkes[1]* listed the following types of structural behavior as a result of cyclic thermal loading.

Elastic	The stresses always lie in the elastic range.
Shake-down	The stresses are inelastic in the first cycle of the loading, but thereafter acquire a prestress which ensures that they remain elastic during the following cycles.
Alternate Plasticity	There are both positive and negative plastic flow in each cycle of zero net value. A stable hysteresis loop is developed.

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*Numbers in parenthesis refer to the bibliography.

Incremental Collapse or Thermal Ratcheting	There is plastic flow in each cycle of non-zero net value, and therefore, growth of deformation. This growth can result in failure of the structure.
Sudden Collapse	This will occur if the sum of the product of the ultimate stress which has been attained and the cross-sectional area fails to balance the applied load.
Ablation	The member becomes hot enough that it may melt and suffer loss of material.

Further, when cyclic loading is involved, failure may occur by (1) the achievement of a specified amount of distortion, (2) the appearance of a first crack, (3) crack penetration of a specified amount, or (4) fracture of the cross-section. In some cases of cyclic loading, the condition that finally develops is one in which stress and strain alternate about a mean component. For thermal ratcheting, this mean component increases progressively and monotonically even though the magnitude of external loading does not increase in successive cycles. In such cases, the ultimate termination of the useful life of a structure may be due to excessive deformation.

In Appendix A the author has derived the equations for progressive thermal distortion (thermal ratcheting) for a simple two bar model. Even with the simplifying

assumptions of linear strain hardening and the absence of

plastic flow in the reverse direction, the equations are

complex.

The problem is further complicated by the occurrence

of nonlinear strain hardening, tensile and compressive

plastic flow and nonlinear temperature distribution. In

addition, thermal shock which is the result of the tran-

sient nature of many thermal problems may occur. Such pro-

perties as specific heat and thermal conductivity, which

do not enter directly into consideration for thermal

stresses under known conditions of temperature, become

important in thermal shock applications because these

properties determine the temperature, the temperature

gradients, and the rate of change of the gradients.[2]

Consideration of these factors complicates the problem

even more.

The object of this work has been to develop a com-

puter program to study the structural behavior and the

pattern of thermal stresses and strains during cyclic

thermal loading of two dimensional elastoplastic problems.

The mechanical load which may be present may or may not be

constant. The stress-strain curve has been assumed to have

a nonlinear strain hardening characteristic. Nonlinear

transient temperature distributions and hence the possi-

bility of occurrence of thermal shock have been con-

sidered. It is noteworthy that in order to take into

account the change in stress-strain characteristics with
an increase in temperature, the concept of an average
property has been used. Also, account has been taken of
the fact that a certain portion of the cross-section may
undergo strain hardening, while other portions may remain
elastic. Hence, the stress-strain states will be different
at different locations in the same cross-section. The
method of successive approximations and the incremental
theory of plasticity have been used.

II. REVIEW OF LITERATURE

Under repeated thermal loading, it has been noted
that four types of stress-strain systems may be set up,
namely, permanent elasticity, shake-down, alternate plas-
ticity and thermal ratcheting. E.W. Parkes, in 1954, in-
vestigated the behavior of simplified aircraft wings under
repeated thermal stress.[3] He considered an I section for
his analysis. He assumed a linearly elastic perfectly
plastic stress-strain curve and the absence of temperature
gradients in the web and flanges. These assumptions, of
course, minimize the algebraic work and tend to simplify
the analysis. In cases involving a strain hardening
material and the presence of temperature gradients, Parkes'
results may be far from reality.

Further, the same author, in 1956,[4] derived the ex-
pressions for incremental collapse of the I section taking
into account the variation of yield stress. In the previous
paper[3] the impression was given that incremental collapse
was a rather rare phenomenon which could only occur when
the initial stresses due to applied mechanical loads were
extremely high. He subsequently suggested that incre-
mental collapse might be of much wider concern. The wing
was assumed to carry a constant bending moment due to
flight loads and to be initially at a zero stress-strain
state at a reference temperature taken as zero. It was

then subjected to a prescribed cycle. Parkes, however, did not consider the temperature history and the external load variation. Both assumptions, that is, elastic perfectly plastic material and the absence of temperature gradients, were considered in the discussion.

In another paper[1] in 1958, Parkes discussed a design philosophy for repeated thermal loading. He described conditions and zones using a curve of load versus temperature to describe the various behavior patterns of a two bar model. A conclusion of his investigation was that "Factors applied to stress or load are not satisfactory criteria of safety under repeated thermal loading, and it is possible that a statistical approach to safety, based on the expected life history of the structure, will have to be used."

In 1964, K.B. Ayers[5] investigated the behavior patterns of I beams carrying pure bending moments and subjected to repeated thermal cycles. These fall into a number of patterns which are basically similar to those obtained by Parkes. Here, also, the basic assumption of a linearly elastic perfectly plastic stress-strain curve was used. The product $E\alpha$ was assumed to remain constant and the rate of change of temperature was not considered.

In 1959, D.R. Miller[6] investigated the thermal stress ratchet mechanism in pressure vessels. He considered a three bar model assuming linear strain hardening

and also assuming no compressive plastic deformation
during the temperature cycle. He derived the equations
for thermal ratcheting through a graphical construction.

In 1959, G.H. Sprague and P.C. Huang[7] considered
the fact that structures are subject to nonlinear stress
patterns due to the presence of temperature gradients.
The significance of such nonlinear stress systems, when
combined with external loading, on the inelastic behavior
and buckling characteristic of structures was presented.
The effect of residual stresses, resulting from inelastic
behavior under nonlinear stress systems on subsequent
structural behavior was also considered. They remarked,

> "In general, any of the four conditions as
> described by Parkes and discussed previously can
> occur at random, dependent on the specific load-
> temperature conditions. (.....) While inelastic
> behavior appreciably minimizes the problems of
> thermal stress for single applications of high
> load level conditions, a detailed analysis
> of the structural behavior under subsequent
> loading must be made for design conditions in-
> volving a repeated load-temperature spectrum.
> Thus, in missile design, where relatively
> few applications of high level loading are en-
> countered, a greater degree of plasticity can
> be considered than would be permissable in the
> design of an aircraft with its long life."

In 1966, S.S. Manson[2] in his text, Thermal Stress
and Low Cycle Fatigue, limited his discussion primarily
to fatigue since, in his words, "..... the mechanism of
progressive distortion being as yet only poorly analyzed."
Though he admits that in dealing with ductile materials

it is to be expected that failure will be due to distortion and/or fatigue rather than to fracture upon the application of a single cycle (see page 307 of reference 2).

As a result of the remarks of Parkes, G.H. Sprague and P.C. Huang on the randomness of occurrence of these four important behaviors, it was decided to devise a computer program which would be capable of taking into account a stress-strain curve with nonlinear strain hardening characteristics, the occurrence of plastic flow in both directions, variation of external loads and the variation of material properties. The incremental method of plasticity was selected because it enables one to trace the history of externally applied loads and temperature variations.

III. UNIAXIAL AND MULTIAXIAL STRESS-STRAIN CURVE
FOR STRAIN HARDENING MATERIALS

A. Stress-Strain Curve for Strain Hardening Materials

A conventional stress-strain curve is shown in Fig. 1. The linear part of the curve extends to point A, which is called the proportional limit. It is in this range that the linear theory of elasticity is valid. Upon further increase of the load, the strain no longer increases linearly with the stress, but the material still remains elastic, that is, upon removal of the load, the specimen returns to its original length. This condition will prevail until a point B, called the elastic limit or yield point, is reached. In most materials there is very little difference between the proportional limit A and the elastic limit B. In this work, points A and B have been assumed to coincide. Beyond the elastic limit, permanent plastic deformation takes place. As the load is increased beyond the elastic limit, the strain increases at a greater rate than the stress. However, the specimen will not deform further unless the load is increased. This condition is called work hardening or strain hardening. The stress required for further plastic flow is called flow stress. Finally, a point C is reached where the load is a maximum. Beyond this point, called the ultimate or tensile strength, a complicated triaxial state of stress

exists in ductile materials.

If at any point between the elastic limit B and the maximum load point C the load is removed, unloading will take place along a line parallel to the elastic line, OB (see Fig. 1). Only a portion of the strain is thus recovered. The total mechanical strain is therefore considered as being made up of two parts, ε_e, the elastic component, and ε_p, the plastic component. That is,

$$\varepsilon_T = \varepsilon_e + \varepsilon_p \tag{1}$$

In terms of stress, the total mechanical strain may be approximated as,

$$\varepsilon_T = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)n \tag{2}$$

where n is a shape parameter of the strain hardening portion of the stress-strain curve and B is a constant. This expression is only intended to provide a convenient method of representing the stress-strain curve for later numerical work. The determination of these two quantities is explained in the discussion that follows.

Ramburg and Osgood[8] suggested a formula for describing the stress-strain curve in terms of Young's modulus E, a secant yield stress $\sigma_{.7}$ ($\sigma_{.7}$ is taken as the stress determined from the intersection of the stress-strain curve by a line of slope 0.7E drawn from the origin (see

Fig. 2)), and a parameter n which describes the shape of a stress-strain curve in the yield region. In order to determine n, another stress $\sigma_{.85}$ is used. The value of $\sigma_{.85}$ is determined from the intersection of the curve by a line of slope 0.85E drawn through the origin. The suggested relation is

$$\frac{E\varepsilon_T}{\sigma_{.7}} = \frac{\sigma}{\sigma_{.7}} + \left(\frac{\sigma}{\sigma_{.7}}\right)^n \tag{3}$$

The equation for the shape parameter n is

$$n = 1 + \frac{\log_e (17/7)}{\log_e (\sigma_{.7}/\sigma_{.85})} \tag{4}$$

Dividing both sides of Eq. (3) by $E/\sigma_{.7}$ gives

$$\varepsilon_T = \frac{\sigma}{E} + \left(\frac{\sigma}{\sigma_{.7}^{\frac{n-1}{n}} \, E^{\frac{1}{n}}}\right)^n \tag{5}$$

Equating Eqs. (2) and (5) gives

$$B = \sigma_{.7}^{\frac{n-1}{n}} \, E^{\frac{1}{n}} \tag{6}$$

From Eqs. (4) and (6), it can be concluded that a knowledge of $\sigma_{.7}$, $\sigma_{.85}$, and E is necessary if the Ramburg-Osgood equation is to be used to define the stress-strain curve.

B. <u>Behavior of Materials Under Uniaxial Stress</u>

Figure 3 shows the stress-strain curve as the line OABC. If the material is continuously loaded until the stress at B is reached and is then unloaded, unloading proceeds along BDE. If the stress is reduced to D and then again increased, the path will be DBC and the material acts as a new material having a yield point B and a stress-strain curve EDBC. Also it is assumed, for this discussion, that the material has the same stress-strain curve in compression and tension. Thus GFEBC represents a new stress-strain curve after the material has once been subjected to tensile or compressive stresses which bring the stress state to point D where D can be any point on the elastic line BF (see page 111 of reference 2).

Thus, for a uniaxially loaded material that has once been subjected to a plastic flow ε_p, the strain at any subsequent stress σ , whether loaded further (for example, σ_c) or unloaded (for example, σ_D) is

$$\varepsilon_T = \frac{\sigma_B}{E} + \varepsilon_p + \Delta\varepsilon_{pl}$$

where $\Delta\varepsilon_{pl}$ is any new plastic flow. In case of unloading from B, $\Delta\varepsilon_{pl}$ is obviously zero. Whether new plastic flow takes place depends on the final stress and a consideration of the new stress-strain curve CBDEFG. If the stress reaches the value σ_c , the increment in plastic strain is

$\Delta\varepsilon_{p1}$. If the stress reaches σ_D, there is no further plastic strain even though σ_D may be high enough to have caused plastic strain in the initial condition of the material prior to the plastic strain ε_p. If the stress goes to G, the strain becomes

$$\varepsilon_T = \frac{\sigma_G}{E} + \varepsilon_p - \Delta\varepsilon_{p2}$$

C. Behavior of Materials Under Multiaxial Stress

When multiaxial stresses are present, the criterion for further plastic flow is whether the equivalent stress exceeds a specified value. If for example, plastic flow has already occurred in a body and the equivalent stress is σ_B , as shown in Fig. 3, the material at that point must be regarded as having the relation between "equivalent stress, σ_e" and "equivalent total mechanical strain, ε_{et}" given by the curve FEBC. If the stress components are changed so that the equivalent stress is reduced, no further plastic flow takes place and the material unloads along the line BDE. For such loading, elastic strains are determined by Hooke's law. If, however, the stress components are changed so that the equivalent stress is increased to σ_c , further plastic flow takes place. Since the stress-strain curve is EDBC, the increased equivalent plastic strain is $\Delta\varepsilon_{ep1}$. The relation between uniaxial stress-strain curve and the multiaxial stress-strain curve has been discussed in article V.

IV.   THE CONCEPT OF AN AVERAGE STRESS-STRAIN CURVE

The stress-strain characteristics of materials change
with changes in temperature. These changes depend upon
the sensitivity of Young's modulus E, and the stresses
$\sigma_{0.7}$ and $\sigma_{.85}$, to the changes in temperature. In
reality, the stress-strain curve would be different at
different locations in a structure and would vary with the
temperature variations during the cycle. Further, the
rate of change of the mechanical properties increases with
an increase in temperature (Fig. 4). It is proposed that
as an adequate compromise a single stress-strain curve at
each location be chosen such that it represents the
average properties for the temperature range experienced
by the station. Hence, if each location experiences a
different maximum temperature, each location will have a
different stress-strain curve. Therefore, with prior
knowledge of maximum temperature, temperature range
and with curves such as shown in Fig. 4, the average values
of E, $\sigma_{.7}$ and $\sigma_{.85}$ can be calculated in order to define the
stress-strain curve as

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n$$

for each station.

V. GENERAL RELATIONS FOR PLANE ELASTOPLASTIC PROBLEMS

A. The Equilibrium, Compatibility and Stress-Strain Relations for the Plane Stress Problems

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} = 0 \tag{7}$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = 0 \tag{8}$$

and

$$\varepsilon_{xx} = \frac{1}{E} \{ (1 + \mu) \sigma_{xx} - \mu (\sigma_{xx} + \sigma_{zz}) \} + \alpha T + \varepsilon_{xx}^{p} + \Delta \varepsilon_{xx}^{p}$$

$$\varepsilon_{zz} = \frac{1}{E} \{ (1 + \mu) \sigma_{zz} - \mu (\sigma_{xx} + \sigma_{zz}) \} + \alpha T + \varepsilon_{zz}^{p} + \Delta \varepsilon_{zz}^{p}$$

$$\varepsilon_{yy} = \frac{1}{E} \{ - \mu (\sigma_{xx} + \sigma_{zz}) \} + \alpha T + \varepsilon_{yy}^{p} + \Delta \varepsilon_{yy}^{p}$$

$$\varepsilon_{xz} = \frac{\sigma_{xz}}{2G} + \varepsilon_{xz}^{p} + \Delta \varepsilon_{xz}^{p} \tag{9}$$

where,

σ_{xx} is the stress in 'X' direction on an 'X' face,

ε_{xx} is the strain in 'X' direction on an 'X' face,

μ is the Poisson's ratio,

α is the coefficient of thermal expansion,

T is the temperature above the reference temperature,

$\Delta\varepsilon^p_{xx}$ is the unknown plastic strain increment occurring during the current increment of loading,

$\varepsilon^p_{xx} = \sum\limits_{k=1}^{i-1} \Delta\varepsilon^p_{xx,k}$ is the plastic strain accumulated during the first i - 1 increments of loading (i is the current increment of load).

and

G is the shear modulus.

The Prandtl - Reuss relations are

$$\Delta\varepsilon^p_{xx} = \frac{\Delta\varepsilon_{ep}}{2\sigma_e} (2\sigma_{xx} - \sigma_{zz})$$

$$\Delta\varepsilon^p_{zz} = \frac{\Delta\varepsilon_{ep}}{2\sigma_e} (2\sigma_{zz} - \sigma_{xx})$$

$$\Delta\varepsilon^p_{xz} = \frac{3\Delta\varepsilon_{ep}}{2\sigma_e} \sigma_{xz} \qquad (10\text{-}a)$$

and assuming volume constancy, we have

$$\Delta\varepsilon^p_{yy} = -\Delta\varepsilon^p_{xx} - \Delta\varepsilon^p_{zz} \qquad (10\text{-}b)$$

The corresponding plastic strain-total strain relations are

$$\Delta\varepsilon_{xx}^{p} = \frac{\Delta\varepsilon_{ep}}{3\varepsilon_{et}'} (2\varepsilon_{xx}' - \varepsilon_{yy}' - \varepsilon_{zz}')$$

$$\Delta\varepsilon_{zz}^{p} = \frac{\Delta\varepsilon_{ep}}{3\varepsilon_{et}'} (2\varepsilon_{zz}' - \varepsilon_{yy}' - \varepsilon_{xx}')$$

$$\Delta\varepsilon_{xz}^{p} = \frac{\Delta\varepsilon_{ep}}{\varepsilon_{et}'} \varepsilon_{xz}' \qquad (11)$$

where the primed quantities are the modified total strains

$$\varepsilon_{xx}' = \varepsilon_{xx} - \varepsilon_{xx}^{p}$$

$$\varepsilon_{yy}' = \varepsilon_{yy} - \varepsilon_{yy}^{p}$$

$$\varepsilon_{zz}' = \varepsilon_{zz} - \varepsilon_{zz}^{p} \qquad etc.,$$

and

$$\Delta\varepsilon_{ep} = \frac{2}{\sqrt{3}} [(\Delta\varepsilon_{xx}^{p})^2 + (\Delta\varepsilon_{zz}^{p})^2 + \Delta\varepsilon_{xx}^{p}\Delta\varepsilon_{zz}^{p} + (\Delta\varepsilon_{xz}^{p\,2})]^{\frac{1}{2}}$$

$$(12)$$

$$\varepsilon_{et}' = \frac{\sqrt{2}}{3} [(\varepsilon_{xx}' - \varepsilon_{zz}')^2 + (\varepsilon_{zz}' - \varepsilon_{yy}')^2 + (\varepsilon_{yy}' - \varepsilon_{xx}')^2 + 6(\varepsilon_{xz}')^2]^{\frac{1}{2}}$$

$$\sigma_{e} = (\sigma_{xx}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{zz} + 3\sigma_{xz}^2)^{\frac{1}{2}} \qquad (13)$$

where $\Delta\varepsilon_{ep}$ denotes the equivalent plastic strain increment,

ε'_{et} denotes the equivalent modified total strain, and σ_e is the equivalent stress.

B. Relationship Between Equivalent Plastic Strain
Increment $\Delta\varepsilon_{ep}$, Equivalent Stress σ_e and Equivalent
Modified Total Strain ε'_{et}

The relation between ε'_{et}, $\Delta\varepsilon_{ep}$ and σ_e is given by

$$\varepsilon'_{et} = \frac{2}{3}(1 + \mu)\frac{\sigma_e}{E} + \Delta\varepsilon_{ep} \qquad (14)$$

Referring to the uniaxial stress-strain curve (see Fig. 5), let $\Delta\sigma_e$ be the increment in stress to which corresponds a plastic strain increment $\Delta\varepsilon_{ep}$. Let σ_e be the stress at the end of the current load increment. Then ε_{et} is the sum of the plastic strain increments plus the total elastic strain multiplied by $\frac{2}{3}(1 + \mu)$. Solving Eq. (14) for $\Delta\varepsilon_{ep}$ results in

$$\Delta\varepsilon_{ep} = \varepsilon'_{et} - \frac{2}{3}(1 + \mu)\frac{\sigma_e}{E} \qquad (15)$$

The equivalent stress σ_e can now be eliminated from Eqs. (14) and (15). Let the stress preceding the current load increment be $\sigma_{e,i-1}$, that is,

$$\sigma_{e,i} = \sigma_{e,i-1} + \Delta\sigma_e$$

Then expanding $\sigma_{e,i}$ in a Taylor series about $\sigma_{e,i-1}$ gives

$$\sigma_{e,i} = \sigma_{e,i-1} + \left(\frac{d\sigma_e}{d\varepsilon_{ep}}\right)_{i-1} \Delta\varepsilon_{ep} \qquad (16)$$

The higher order terms in $\Delta\varepsilon_{ep}$ have been neglected. Substituting for $\sigma_{e,i}$ into Eq. (15) and solving for $\Delta\varepsilon_{ep}$ gives

$$\Delta\varepsilon_{ep} = \frac{\varepsilon'_{et} - \frac{2}{3}[(1 + \mu)/E]\,\sigma_{e,i-1}}{1 + \frac{2}{3}[(1 + \mu)/E]\left(\dfrac{d\sigma_e}{d\varepsilon_{ep}}\right)_{i-1}} \qquad (17)$$

This equation is useful in determining the magnitude of the increment of plastic strain. For linear strain hardening, Eq. (17) is exact (see page 127 of ref. 10).

VI. GENERAL EQUATIONS FOR SYMMETRICAL BEAMS

Consider a beam of depth '2h' and width '2b' as shown in Fig. 6.

The stress-strain relations are given by equation (9). In order to give Eq. (9) a non-dimensional form[10] let

$$S_1 = \frac{\sigma_{xx}}{\sigma_o} \quad , \quad S_2 = \frac{\sigma_{zz}}{\sigma_o} \quad , \quad S_3 = \frac{\sigma_{xz}}{\sigma_o} \tag{18}$$

$$S = S_1 + S_2 \tag{19}$$

$$H = \frac{E}{E_o} \quad , \quad \tau = \frac{\alpha T}{\varepsilon_o} \quad , \quad \eta_z = \frac{Z}{h} \quad , \quad \eta_x = \frac{X}{b} \tag{20}$$

$$e_{xx} = \frac{\varepsilon_{xx}}{\varepsilon_o} \quad , \quad e^p_{xx} = \frac{\varepsilon^p_{xx}}{\varepsilon_o} \quad , \quad \Delta e^p_{xx} = \frac{\Delta e^p_{xx}}{\varepsilon_o} \quad ,$$

$$e_{zz} = \frac{\varepsilon_{zz}}{\varepsilon_o} \quad , \quad e^p_{zz} = \frac{\varepsilon^p_{zz}}{\varepsilon_o} \quad , \quad \Delta e^p_{zz} = \frac{\Delta \varepsilon^p_{zz}}{\varepsilon_o} \quad ,$$

$$e_{xz} = \frac{\varepsilon_{xz}}{\varepsilon_o} \quad , \quad e^p_{xz} = \frac{\varepsilon^p_{xz}}{\varepsilon_o} \quad , \quad \Delta e^p_{xz} = \frac{\Delta \varepsilon^p_{xz}}{\varepsilon_o} \tag{21}$$

where σ_o is the yield stress at a reference temperature, T_o, $\varepsilon_o = \sigma_o/E_o$ is the yield strain at the reference temperature, and E_o is the modulus of elasticity at the reference temperature.

Equation (9) can be written as,

$$e_{xx} = (1 + \mu) \frac{S_1}{H} - \mu \frac{S}{H} + \tau + e_{xx}^p + \Delta e_{xx}^p \qquad (22)$$

$$e_{yy} = - \mu \frac{S}{H} + \tau + e_{yy}^p + \Delta e_{yy}^p \qquad (23)$$

and

$$e_{zz} = (1 + \mu) \frac{S_2}{H} - \mu \frac{S}{H} + \tau + e_{zz}^p + \Delta e_{zz}^p \qquad (24)$$

Also,

$$\varepsilon_{xz} = \frac{\sigma_{xz}}{2G} + \varepsilon_{xz}^p + \Delta \varepsilon_{xz}^p \qquad (25)$$

where

$$G = \frac{E}{2(1 + \mu)} \qquad (26)$$

Substituting Eq. (26) into Eq. (25) and then writing the result in non-dimensional form yields

$$e_{xz} = (1 + \mu) \frac{S_3}{H} + e_{xz}^p + \Delta e_{xz}^p \qquad (27)$$

The compatibility equation for plane stress is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}$$

In non-dimensional form, the compatibility equation can be written as

$$\frac{\partial^2 e_{xx}}{\partial \eta_z^2} + \frac{\partial^2 e_{zz}}{\partial \eta_x^2} = 2 \frac{\partial^2 e_{xz}}{\partial \eta_x \partial \eta_z} \tag{28}$$

Now substituting Eqs. (22), (23), and (27) into Eq. (28) yields

$$\frac{\partial^2}{\partial \eta_z^2} (1 + \mu)(\frac{S_1}{H}) - \mu \frac{\partial^2}{\partial \eta_z^2} (\frac{S}{H}) + \frac{\partial^2 \tau}{\partial \eta_z^2} + \frac{\partial^2 e_{xx}^p}{\partial \eta_z^2}$$

$$+ \frac{\partial^2 \Delta e_{xx}^p}{\partial \eta_z^2} + \frac{\partial^2}{\partial \eta_x^2} [(1 + \mu) \frac{S_2}{H} - \mu\frac{S}{H} + \tau + e_{zz}^p$$

$$+ \Delta e_{zz}^p]$$

$$= 2 \left(\frac{\partial^2}{\partial \eta_x \partial \eta_x} (1 + \mu)(\frac{S_3}{H}) + \frac{\partial^2 e_{xz}^p}{\partial \eta_x \partial \eta_z} + \frac{\partial^2 \Delta e_{xz}^p}{\partial \eta_x \partial \eta_z} \right) \tag{29}$$

The equilibrium equations are

$$\frac{\partial \sigma_{xz}}{\partial z} = - \frac{\partial \sigma_{xx}}{\partial x} \tag{30}$$

$$\frac{\partial \sigma_{xz}}{\partial x} = - \frac{\partial \sigma_{zz}}{\partial z} \tag{31}$$

In non-dimensional form, Eqs. (30) and (31) can be re-written as

$$\frac{\partial}{\partial \eta_z} (\frac{S_3}{H}) = - \frac{\partial}{\partial \eta_x} (\frac{S_1}{H}) \tag{32}$$

$$\frac{\partial}{\partial \eta_x} (\frac{S_3}{H}) = - \frac{\partial}{\partial \eta_z} (\frac{S_2}{H}) \tag{33}$$

respectively. Differentiating Eq. (32) with respect to η_x and Eq. (33) with respect to η_z, and then adding, results in

$$2 \frac{\partial^2}{\partial \eta_x \partial \eta_z} (\frac{S_3}{H}) = - \frac{\partial^2}{\partial^2 \eta_x} (\frac{S_1}{H}) - \frac{\partial^2}{\partial^2 \eta_z} (\frac{S_2}{H}) \tag{34}$$

Substitution of Eq. (34) into Eq. (29) gives

$$(1 + \mu) \ [\nabla^2 \frac{(S_1 + S_2)}{H}] - \mu \ \nabla^2 (\frac{S}{H}) + \nabla^2 (\tau)$$

$$= - \frac{\partial^2}{\partial \eta_z{}^2} (e_{xx}^p + \Delta e_{xx}^p) - \frac{\partial^2}{\partial \eta_x{}^2} (e_{zz}^p + \Delta e_{zz}^p)$$

$$+ 2 \frac{\partial^2 (e_{xz}^p + \Delta e_{xz}^p)}{\partial \eta_x \partial \eta_z} \tag{35}$$

where $\nabla^2 = (\dfrac{\partial^2}{\partial \eta_x^2} + \dfrac{\partial^2}{\partial \eta_z^2})$

Simplifying Eq. (35) yields

$$\nabla^2 \frac{(S_1 + S_2)}{H} - \nabla^2 \tau = - \frac{\partial^2}{\partial \eta_z^2} (e_{xx}^p + e_{xx}^p) - \frac{\partial^2}{\partial \eta_x^2}$$

$$(e_{zz}^p + \Delta e_{zz}^p) + \frac{2 \, \partial^2}{\partial \eta_x \, \partial \eta_z} (e_{xz}^p + \Delta e_{xz}^p) \qquad (36)$$

Eq. (36) is the general equation governing plane elasto-plastic problems.

Assume now that the quantities 2b and 2h are small compared to the length L and that the temperature varies in the 'Z' direction only, that is, $T = T(Z)$. Because of the thinness of the beam, the plane stress assumption that $\sigma_{yy} = \sigma_{yz} = \sigma_{xy} = 0$ is made (see page 279 of ref. 12). The resulting two dimensional problem can be reduced to a one dimensional problem for the example discussed in Appendix B by making the assumption that

$$\sigma_{zz} = \sigma_{xz} = 0$$

and

$$\sigma_{xx} = \sigma_{xx}(Z) \qquad (37)$$

Now Eq. (36) can be written as,

$$\nabla^2 \left(\frac{S_1}{H}\right) + \nabla^2 \tau = \frac{\partial^2}{\partial \eta_z^2} (e_{xx}^p + \Delta e_{xx}^p) \tag{38}$$

Note that from the above assumptions, stress varies only in the 'Z' direction and hence Eq. (38) becomes

$$\frac{\partial^2}{\partial \eta_z^2} \left(\frac{S_1}{H} + \tau + e_{xx}^p + \Delta e_{xx}^p\right) = 0 \tag{39}$$

Integrating Eq. (39) twice gives

$$\frac{S_1}{H} + \tau + e_{xx}^p + \Delta e_{xx}^p = C_1 \eta_z + C_2 \tag{40}$$

or

$$e_{xx} = C_1 \eta_z + C_2 \tag{41}$$

From Eqs. (40) and (41)

$$S_1 = H (e_{xx} - \tau - e_{xx}^p - \Delta e_{xx}^p) \tag{42}$$

is obtained. The constants C_1 and C_2 can be evaluated from the boundary conditions

$$\int_{-h_1}^{+h_1} \sigma_{xx} dA = p \tag{43}$$

and

$$\int_{-h_1}^{+h_1} \sigma_{xx} Z \, dA = M \tag{44}$$

In nondimensional form, Eqs. (43) and (44) can be written as

$$\int_{-1}^{1} S_1 d\eta_z = \frac{P}{h\sigma_o} = P^* \tag{45}$$

and

$$\int_{-1}^{1} S_1 \eta_z d\eta_z = \frac{M}{h^2 \sigma_o \, 2b} = M^* \tag{46}$$

respectively.

Substituting for S_1 into Eqs. (45) and (46) yields

$$\int_{-1}^{1} H \, (C_1 \eta_z + C_2) \, d\eta_z - \int_{-1}^{1} (\tau + e_{xx}^p + \Delta e_{xx}^p) \, d\eta_z = P^* \tag{47}$$

and

$$\int_{-1}^{1} H \, (C_1 \eta_z + C_2) \eta_z d\eta_z - \int_{-1}^{1} (\tau + e_{xx}^p + \Delta e_{xx}^p) \eta_z d\eta_z = M^* \tag{48}$$

(10)

Solving Eqs. (47) and (48), for constants C_1 and C_2 yields

$$C_1 = B_1(M* + D) - B_2 (P* + F) \qquad (49)$$

and

$$C_2 = B_3(P* + F) - B_2 (M* + D) \qquad (50)$$

where

$$B_1 = \frac{\displaystyle\int_{-1}^{1} H d\eta_z}{\displaystyle\int_{-1}^{1} H d\eta_z \int_{-1}^{1} H\eta_z^2 \, d\eta_z - \left(\int_{-1}^{1} H\eta_z d\eta_z\right)^2}$$

$$B_2 = \frac{\displaystyle\int_{-1}^{1} H\eta_z d\eta_z}{\displaystyle\int_{-1}^{1} H d\eta_z \int_{-1}^{1} H\eta_z^2 \, d\eta_z - \left(\int_{-1}^{1} H\eta_z \, d\eta_z\right)^2}$$

and

$$B_3 = \frac{\displaystyle\int_{-1}^{1} H\eta_z^2 \, d\eta_z}{\displaystyle\int_{-1}^{1} H\eta_z^2 \, d\eta_z \int_{-1}^{1} H d\eta_z - \left(\int_{-1}^{1} H\eta_z d\eta_z\right)^2}$$

Equation (41) can now be written as

$$e_{xx} = (B_1 \eta_z - B_2)(M^* + D) + (B_3 - B_2 \eta_z)(P^* + F) \qquad (51)$$

where

$$D = \int_{-1}^{1} H(\tau + e_{xx}^p + \Delta e_{xx}^p) \eta_z d\eta_z$$

and

$$F = \int_{-1}^{1} H(\tau + e_{xx}^p + \Delta e_{xx}^p) d\eta_z$$

Equations (23) and (24) can also be written as

$$e_{yy} = -\mu(C_1 \eta_z + C_2) + (1 + \mu)\tau + (\mu - 1)(e_{xx}^p + \Delta e_{xx}^p)$$

$$- (e_{zz}^p + \Delta e_{zz}^p) \qquad (52)$$

and

$$e_{zz} = -\mu(C_1 \eta_z + C_2) + (1 + \mu)\tau + (e_{zz}^p + \Delta e_{zz}^p)$$

$$+ \mu(e_{xx}^p + \Delta e_{xx}^p) \qquad (53)$$

Note that the relation $\Delta e_{yy}^p = -\Delta e_{xx}^p - \Delta e_{zz}^p$ has been used in deriving Eq. (52).

From Eq. (40)

$$\sigma_{xx} = \sigma_o \quad H \ (C_1 \eta_z + C_2 - \tau - e_{xx}^p - \Delta e_{xx}^p) \qquad (54)$$

is obtained.

The strains ε_{xx}, ε_{yy}, ε_{zz} and stress σ_{xx} can now be calculated by Eqs. (51), (52), (53) and (54), respectively.

VII. THEORIES OF PLASTICITY

A. Total Deformation Theory of Plasticity

This theory establishes a relation between the stresses and the total strains so that if the stresses are known, the strains can be directly calculated. The path by which a given stress distribution is reached, presumably, does not influence the strains. This cannot be generally correct. However, it is a useful concept in certain cases. If the deformation theory is used, all the previous equations with Δ's and primes removed are valid.[10]

Consider now the equation

$$\varepsilon_{ep} = \varepsilon_{et} - 2/3 \; (1 + \mu) \frac{\sigma_e}{E} \tag{55}$$

The relation between ε_{et} and σ_e is contained in the uniaxial stress-strain curve. For any selected value of ε_{ep}, the value of σ_e can be determined. Thus, Eq. (55) represents a direct relationship between ε_{et} and ε_{ep} which can be determined from the stress-strain curve (see Fig. 7). This curve can then be used in place of the original stress-strain curve. It can be seen that a plot of ε_{et} versus ε_{ep} will have a slope of approximately unity. A small error in ε_{et} will, therefore, produce the same order-of-magnitude error in ε_{ep} without any magnification.

B.   The Incremental Theory of Plasticity

In this theory, each step consists of obtaining the increment in the plastic strain as the material passes from one stress state to the next stress state. Consider the equation

$$\varepsilon_{xx} = \frac{1}{E} \{\sigma_{xx} - \mu(\sigma_{xx} + \sigma_{zz})\} + \alpha T + \sum \Delta\varepsilon_{xx}^{p} + \Delta\varepsilon_{xx}^{p} \qquad (56)$$

If the summation term is transposed to the left hand side of the equation, it will be seen that this equation becomes

$$\varepsilon_{xx}' = \frac{1}{E} \{\sigma_{xx} - \mu(\sigma_{yy} + \sigma_{zz})\} + \alpha T + \Delta\varepsilon_{xx}^{p} \qquad (57)$$

where $\varepsilon_{xx}' = \varepsilon_{xx} - \sum \Delta\varepsilon_{xx}^{p}$.

The expression $\varepsilon_{xx} - \sum \Delta\varepsilon_{xx}^{p}$ is the total strain component in the X direction due to the thermal and mechanical stresses. It takes into account the fact that the free length of the element has been changed by the plastic flow (see page 113 of reference 2). Denoting this term as the "modified total strain component" it is evident that an equivalent modified total strain can be computed from the components in a manner analagous to that of Eq. (13). This strain will be related to the increment in the plastic strain by Eq. (11). This observation makes it

possible to perform any one computation in the same
manner as in a deformation theory computation.

The solutions of problems by the incremental theory
is desirable whenever unloading is possible and whenever
a nonproportional loading condition exists. It consists
thus, of several individual computations each representing
a stage actually encountered by the body during its
mechanical or thermal load history.

VIII. DETERMINATION OF THE MAGNITUDE OF THE

PLASTIC STRAIN INCREMENT $\Delta\varepsilon_{ep}$

During the first increment of loading, there are two possibilities regarding the state of stress which may exist at the end of computation, (1) the stress may lie in the elastic range, and (2) the stress may exceed the elastic limit.

In the first case, the equivalent total mechanical strain will not exceed the limiting equivalent mechanical strain. This suggests that a condition during the computation should be imposed which will set $\Delta\varepsilon_{ep}$ and, hence, $\Delta\varepsilon_{xx}^p$, $\Delta\varepsilon_{zz}^p$, etc., to zero. The elastic solution may thus be obtained.

In the second case, there are two possibilities, (a) the stress may lie near the yield or limit stress (A_1') or (b) the stress may be far from the yield point (A_1'').

These two cases have been illustrated in Fig. (9-a). Consider Eq. (17), that is,

$$\Delta\varepsilon_{ep} = \frac{\varepsilon_{et}' - 2/3[(1 + \mu)/E]\, \sigma_{e,i-1}}{1 + 2/3[(1 + \mu)/E]\, (\dfrac{d\sigma_e}{d\varepsilon_{ep}})_{i-1}}$$

Obviously, the stress $\sigma_{e,i-1}$ and the slope $(\dfrac{d\sigma_e}{d\varepsilon_{ep}})_{i-1}$

should be taken as the stress and slope of the stress-strain curve at the current yield point. When the loading is such that the actual stress state is at A_1', then the plastic strain increment may be satisfactory. Eq. (17) shows that the strain hardening characteristic is linear with the slope $(\frac{d\sigma_e}{d\varepsilon_{ep}})_{i-1}$. If the actual stress state is to be at A_1'', then the plastic strain increment will be inaccurate. The alternative in such a case will be to use the expression

$$\varepsilon_{ep} = (\frac{\sigma_e}{B})^n$$

and to calculate corresponding plastic strains for selected stress values. The least square method can then be used to find a straight line relationship between the stress σ_e and the plastic strain ε_{ep}. The slope of this line can be substituted into Eq. (17) in place of the term $(\frac{d\sigma e}{d\varepsilon_{ep}})_{i-1}$. Equation (17) in a sense limits the size of the load increment. For better results, it is necessary that smaller load increments be chosen in the area of the yield point.

Now consider any other increment of loading. At the end of the current load increment, the stress state may be at point B (see Fig. 9-b) if unloading is experienced, or it may be at C if the load is increased. Because of strain hardening, the new stress-strain curve during the current increment of loading is considered to be O'AC.

The equivalent limit stress and equivalent limit strain
are $\sigma_{A,i-1}$ and $\sigma_{A,i-1}/E'$, where $E' = E/\frac{2}{3}(1 + \mu)$. The
subscript 'i' indicates the current increment of load.
The stress $\sigma_{A,i-1}$ is known. When the equivalent modified
total strain ε'_{et} does not exceed $\sigma_{A,i-1}/E'$, unloading or
loading in the elastic region occurs and the stress state
reaches a point B somewhere on the line O'A. Observation
of this condition, therefore, automatically ensures that
unloading has been done elastically. Further, if the
equivalent modified total strain ε'_{et} is greater than the
current equivalent limit strain, Eq. (17) can be used to
determine the magnitude of the associated plastic strain
increment. Point C, which indicates the state of stress
at the end of the current load increment in case of fur-
ther loading, should not lie far from the current stress
limit (i.e., a large difference between the stress due to
the current load increment and the stress in the plastic re-
gion due to next increment of load , should not be per-
mitted). Also, it should be noted that if the magnitude
of the stress at the end of the previous load increment is
less than that of the limit stress, the value of the
actual stress should be considered to be equal to the
limit stress, in order to compute the plastic strain in-
crement, $\Delta\varepsilon_{ep}$.

IX. ONE DIMENSIONAL TRANSIENT HEAT CONDUCTION

Most of the one dimensional problems can be treated in a manner similar to the flat plate problem (see page 276 of ref. 2). In this problem, the plate is assumed to have its faces and ends insulated so that the temperature varies in the Z direction.

Consider the model shown in Fig. 10. The conduction equation is

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \frac{\partial T}{\partial t} \tag{58}$$

where

$k = \dfrac{K}{\rho C}$ = specific conductivity, ft./hr.

K = thermal conductivity, btu/hr. ft. °F.

C = specific heat, btu/lb.°F.

ρ = density, lbs./ft.^{3}.

Let

$$\phi = T - T_f$$

where T_f denotes the fluid temperature. The solution of Eq. (58) is[11]

$$\phi = \sum_{n=1,3,5}^{\alpha} \frac{4(T_o - T_f)}{n\pi} e^{\frac{-n^2\pi^2 kt}{L^2}} \sin \frac{n\pi z}{L} \tag{59}$$

where T_o is the initial temperature of the plate.

X. EXPLANATION OF PROCEDURE AND FLOW LIST

The beam section shown in Fig. 10 was divided into a number of stations. An odd number was preferred in order to carry out the numerical integration (see Fig. 11). The stress-strain curve was determined for each station from the maximum temperature and the temperature range as explained previously. Numerical integration may be used or the equations for average properties may be derived by referring to curves such as shown in Fig. 3 (flow list: steps 2 and 3).

The proportional limit stress and strain were determined from the arbitrary condition that at the proportional limit the value of plastic strain ε_p is approximately 0.0001. The equation

$$\sigma = B \, \varepsilon_{ep}^{\frac{1}{n}}$$

was used to determine the value of limit stress at a plastic strain ε_{ep} of 0.0001 (flow list: step 4).

Before beginning the application of mechanical and/or thermal load, the summation terms $\sum \varepsilon_{xx}^p$, $\sum \varepsilon_{zz}^p$, $\sum \varepsilon_{yy}^p$, etc., were set to zero. Then the loading cycle was started. To calculate the stresses and strains for any particular increment of loading, the iterative procedure was then started. The plastic strain increments $\Delta \varepsilon_{xx}^p$, $\Delta \varepsilon_{zz}^p$, etc., were set to zero for the first iteration. The total

strains ε_{xx} , ε_{yy} , ε_{zz} etc., were then calculated in order to determine the equivalent total strain. The integrals appearing in the equations for ε_{xx} , ε_{yy} , ε_{zz} (Eqs. (51), (52), (53), respectively) were evaluated by a numerical method. It should be noted that though Simpson's rule was used, any of the Newton-Cote formulae can be used (flow list steps 5 to 7-b).

The equivalent modified total strain ε'_{et} was then calculated by using Eq. (13). If ε'_{et} is less than the equivalent limit strain, then the stress and strains are in the elastic range. But if the equivalent modified total strain ε_{et} exceeds the equivalent limit strain, then the stress and strains due to the current load increment must be in the plastic range and hence $\Delta\varepsilon_{ep}$ should be calculated as explained earlier. Step 7-g solves for the plastic strain components, $\Delta\varepsilon_{xx}^{p}$, $\Delta\varepsilon_{zz}^{p}$, and $\Delta\varepsilon_{yy}^{p}$ by using the set of Eq. (11). Thus, the plastic strain increments $\Delta\varepsilon_{xx}^{p}$, $\Delta\varepsilon_{zz}^{p}$ and $\Delta\varepsilon_{yy}^{p}$ were calculated at each station. At this stage, a check for convergence was carried out as shown by step 7-i. When convergence was not achieved, the values of the plastic strain increments $\Delta\varepsilon_{xx}^{p}$, $\Delta\varepsilon_{zz}^{p}$, and $\Delta\varepsilon_{yy}^{p}$, obtained during the current iteration were substituted into the integrals appearing in the equations for total strains. The procedure described by steps 7-b to 7-g was repeated until convergence was obtained.

The stresses were then computed by using Eq. (54) (flow list: step 8).

The next step was to determine a new limit stress and
and a limit strain before considering the next loading
increment. Setting of the limit stress and strain has been
discussed in detail in the following article.

Flow List:

    (1)   Data:

        (a)   Geometry of a section.

        (b)   Values of 'η_z' following the selection of a
suitable number of stations (preferrably an
odd number).

        (c)   Maximum temperature and temperature range
for each station.

        (d)   Thermo-physical properties such as specific
heat, thermal conductivity and coefficient
of thermal expansion.

        (e)   Mechanical properties such as Poisson's
ratio, modulus of elasticity, stresses
$\sigma_{0.85}$, $\sigma_{0.87}$ suitable for describing a
stress-strain curve.

        (f)   Mechanical loads.

        (g)   Thermal loads.

        (h)   Physical property: density

    (2)   Calculate average mechanical properties such as
E, $\sigma_{.7}$ and $\sigma_{.85}$ which are used to compute an
average stress-strain curve.

(3) Compute the average stress-strain curve mentioned in step 2.

(4) Calculate the limit stress and strain at each station for $\varepsilon_{ep} \simeq 0.0001$ using Eq. (2).

(5) Set summation terms for plastic strain increment to zero, that is, $\sum \Delta \varepsilon_{xx}^p = \sum \Delta \varepsilon_{zz}^p = \sum \Delta \varepsilon_{yy}^p = 0$.

(6) Apply mechanical and thermal loads. Thermal loads are calculated by varying the time intervals in Eq. (59).

(7) Start the iteration procedure:

(a) For the first iteration, set the plastic strain increments $\Delta \varepsilon_{xx}^p$, $\Delta \varepsilon_{zz}^p$, and $\Delta \varepsilon_{yy}^p$ to zero.

(b) Evaluate integrals appearing in Eqs. (51), (52) and (53) by Simpson's rule.

(c) Compute total strains ε_{xx}, ε_{zz}, and ε_{yy} using Eqs. (51), (52) and (53).

(d) Calculate modified total strains and the equivalent modified total strain using Eqs. (11) and (13).

(e) If equivalent modified total strain is less than or equal to equivalent limit strain, then the plastic strain increment is zero. If not, go to (7-f).

(f) Compute the equivalent plastic strain increment $\Delta \varepsilon_{ep}$ using Eq. (17).

(g) Compute the components of plastic strain $\Delta\varepsilon_{ep}$, i.e., $\Delta\varepsilon_{xx}^{p}$, $\Delta\varepsilon_{zz}^{p}$ and $\Delta\varepsilon_{yy}^{p}$ using Eq. (11).

(h) Calculate these components at all stations.

(i) Check for convergence at all stations.

Conditions:
$$\frac{\left|\Delta\varepsilon_{xx,i,j}^{p} - \Delta\varepsilon_{xx,i,j-1}^{p}\right|}{\Delta\varepsilon_{xx,i,j}^{p}} \leq .001,$$

etc.

(j) If convergence is not achieved at one or more stations, substitute the values of $\Delta\varepsilon_{xx}^{p}$, $\Delta\varepsilon_{zz}^{p}$ and $\Delta\varepsilon_{yy}^{p}$ in (7-b) and repeat the procedure. If convergence is achieved at all stations, go to step 8.

(8) Calculate the stresses using Eq. (54).

(9) Calculate the equivalent stress σ_{e} and the summation of the plastic strain increments.

(10) Determine the equivalent limit stress and strain for the next increment of load.*

(11) Go to step 6.

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*See section XI.

XI. DETERMINATION OF THE LIMIT STRESS AND
LIMIT STRAIN FOR THE NEXT INCREMENT OF LOADING

When cyclic loading is encountered, it becomes necessary to consider the Bauschinger effect. According to this effect, plastic flow in one direction (say tension) reduces the stress at which yielding will occur in the opposite direction (compression). The stress-strain path which may be, presumably, traveled due to this effect has been shown by a broken line in Fig. 12.[13]

Before considering the Bauschinger effect, let us examine the effect of the usual assumption of isotropic strain hardening. Assume that during an early stage of the loading cycle, the stress-strain state has reached the point A as shown in Fig. 12. Because of the assumption under consideration, during unloading or reversed loading, the limit stress will be considered to be equal to σ_C , where σ_C is equal to σ_A. The stress-strain state, therefore, will follow the path ABCD and hence plastic flow ε_{ep2} which could have occurred, will not be computed. The stress-strain state which would have been at C' would be calculated as if it were at some point on the line AC. Further, assume that the stress-strain state is at D', somewhere on the path CD, due to reversed loading. Reversed loading is then discontinued and once again loading in the initial direction is resumed. In this case, the

limit stress will be considered to be σ_F where σ_F is

equal in magnitude to σ_D, . It can be seen that this

assumption will lead to incorrect results. To be closer

to reality, it is, therefore, necessary to reset the limit

stress and strain as per the Bauschinger effect. However,

in order to reduce the error introduced due to the assump-

tion of isotropic strain hardening, the limit stress can

be calculated from $\sigma = B \, \varepsilon_{p3}^{\frac{1}{n}}$, where ε_{p3} is the total

plastic strain at the end of a loading cycle (see Fig. 13).

Due to the interaction of the mechanical load and

different temperature gradients at different stations in a

structure, it becomes quite difficult to predict when

plastic flow in the direction opposite to that of previous

plastic flow takes place. Thus, setting of the limit

stress and limit strain according to the Bauschinger effect

becomes difficult. The following discussion assumes

simple mathematical models for the Bauschinger effect and

describes the method of determination of limit stress and

strain accordingly.

Assume now that during cyclic loading the stress-

strain state follows the path shown in Fig. 14. The

stresses $\sigma_{y,t}$ and $\sigma_{y,c}$ are the initial yield stresses in

tension and compression, respectively. It is assumed that

during the cyclic loading plastic flow occurs and a new

yield stress σ_A is obtained due to strain hardening. The

yield stress $\sigma_{y,c}$ is assumed here to have a value equal to the stress $\sigma_{y,t}$.[13] Also assume, for simplification, that the ratio of the stress σ_A to the stress $\sigma_{y,t}$ remains constant during the cyclic loading. The last assumption has been made in order to simplify the discussion that follows. If a mathematical model for the Bauschinger effect which relates the number of changes in the direction of plastic flow and the magnitude of plastic flow associated with each change were available, it could be easily employed. From Fig. (15-a) it can be seen that during a typical loading cycle, the path traveled is $OYAB_i$. The point B_i (tensile stress) indicates the actual stress-strain state at the end of the current load increment. The point A indicates the limit stress-strain state for the current load increment. The point C indicates the limit stress-strain state for reverse plastic flow.

It can now be expected that at the end of the next load increment, there may exist three possible states of stress and strain, namely,

(1) The state may be on the elastic line AC (for example, point B'_{i+1}), if unloading occurs or in case of reverse loading which does not cause plastic flow in the opposite direction.

(2) The state may be on the strain hardening portion AH (e.g., point B''_{i+1}), if further loading is continued and plastic flow occurs in the same direction.

(3) The state may be on the strain hardening portion CJ (e.g., point $B_{i+1}^{''\,'}$), if the loading condition is such that plastic flow may occur in the opposite direction.

Assume now that the load increment size is small enough so that if the stress at the end of the current load increment is in one direction (tension), the stress that will be produced at the end of the next load increment will not exceed the yield limit C in the opposite direction. The purpose of this assumption will become clear from the discussion that follows. This assumption can be utilized by dividing the loading cycle into a number of load increments. The size of the loading increments must be kept small from other considerations as well. For a particular case such as shown in Fig. (15-a), the third possibility will be ruled out due to the small increment assumption. The actual stress-strain state can, therefore, be somewhere on the path CAH. It is now clear that the limit stress-strain state is at A for the next load increment.

In Fig. (15-b), the point B_i corresponds to the compressive stress σ_{B_i} . From the previous discussion and in view of the assumptions made, the new limit stress-strain state can be determined to be at C ($\sigma_c = \sigma_{y,t}$ for the first loading cycle). In other words, the limit stress state should be selected such that it has the same direction as that of the actual stress state at the end of the current

load increment (for example, if the actual stress is posi-
tive, the limit stress should also be positive). Assume
that the actual stress is σ_{B_i}. It is compressive and
very near to zero. The stress σ_C will be selected as a
new limit stress. Also, unloading or loading may occur
during the next load increment. If loading occurs and
the state of the actual stress is as shown by the point
B_{i+1}', the stress indicated by point B_{i+1}' should not ex-
ceed the stress of σ_F (the magnitude of stress σ_F is
equal to the magnitude of stress σ_C). If care is not taken
and loading occurs, then yielding will be shown to occur
at σ_F, which is obviously incorrect because in case of
loading, the limit stress should be σ_A. Therefore, the
size of the load increment should be kept small enough so
that the difference in actual stress due to two consecu-
tive load increments does not exceed the stress σ_C (in
general, the lower of the two possible limit stresses).

Consider now Fig. (15-c). It is shown in this figure
that reverse plastic flow has occurred. It can also be
seen that the two possible stress limits will be σ_K and
σ_L. The magnitude of σ_L is given by

$$\sigma_L = B \ \varepsilon_{ep}^{\frac{1}{n}} \tag{60}$$

Further, the same figure shows five limit stress-
strain states for a typical loading cycle. These states
are as follows:

(1) State Y (at the beginning of the cycle).

(2) State A (plastic flow in one direction has occurred).

(3) State C (unloading or loading in the reverse direction has occurred but not sufficient to cause plastic flow in the reverse direction).

(4) State K (plastic flow in the reverse direction has occurred).

(5) State L (loading or unloading along the elastic line LK).

It should, however, be noted that if plastic flow in the reverse direction does not take place, the point K will coincide with the point C and the point L will coincide with the point A.

From the above discussion, it should now be noted that knowledge of four conditions is required in order to choose the proper limit stress-strain state. These conditions are as follows:

(1) The magnitude and direction of the limit stress for the current load increment.

(2) The magnitude and direction of the actual stress at the end of the current load increment.

(3) The direction of the first plastic flow which occurs during the cycle under consideration.

(4) Occurrence of plastic flow in the reverse direction.

The direction of the first plastic flow is useful in the determination of the occurrence of plastic flow in the opposite direction. In Fig. (16-a), assume that the limit stress during the computation for a particular load increment is σ_A. The actual stress is σ_{B_1}. During the next and subsequent load increments the actual stress changes. As long as the actual stress state remains positive and on the elastic line O'A, the limit stress state remains at A. But when the actual stress state becomes negative, the limit stress state will have to be set at C, that is, the yield point in the opposite direction. This limit is obtained by dividing σ_A by the constant established during the first cycle (see Fig. 14: $K = \dfrac{\sigma_{l,t}}{\sigma_{y,c}}$). It should be noted that if a more appropriate mathematical model for the Bauschinger effect were available, the relation between these two limit stresses would be known. Also, if the model incorporates the effect of the number of stress strain cycles (hysteresis loops), then the constant used above may vary with the number of cycles. Now consider a case wherein the actual stress state does not go beyond point C but reaches a point C_1 on the elastic line O'C. Then it returns to a point D_1 on the positive side during the next load increment. It will be seen that as soon as the actual stress-state becomes positive, a need arises to establish the limit stress-state at A. In the absence of knowledge of the direction of plastic flow,

the only known factors are

(1) The magnitude and direction of the yield stress for the current load increment and,

(2) The magnitude and direction of the actual stress at the end of the current load increment.

Reconsider now Fig. (16-a). Assume that during the current load increment, the limit stress was σ_c. This is because the actual stress-state was at C_1 at the end of the previous load increment. It is now observed that at the end of the current load increment the actual stress-state is at σ_A. In the absence of the knowledge of the direction of prior plastic flow, the situation would be as described below:

(1) The yield stress for the current load increment is negative.

(2) The actual stress at the end of the current load increment is positive.

Consider now Fig. (16-b). Two possible situations that may arise have been described by the points C_1D_1 and C_2D_2 on either side of the stress axis. Fig. (16-c) shows the typical paths that are possible in passing through these points during the first cycle. When a loading path passes through the points D_1 and C_1, tensile plastic flow occurs, whereas in the case of the loading path passing through the points C_2 and D_2, compressive plastic flow is observed. In the first case, it is necessary to multiply

the limit stress σ_c by the constant K, whereas in the second case, division is necessary. The decision regarding multiplication or division can be made only when the direction of plastic flow is known.

Now reconsider Fig. (15-c). When the actual stress-strain state proceeds along the elastic line KL, the possible limit stress-strain states are at L and K. The limit stresses σ_L and σK can be related by a factor (e.g., $K_f = \sigma_L/\sigma_K$). It should be noted that as long as the value of the actual stress lies along the elastic line LK, the value of K_f will remain constant. It may, however, change during subsequent cycles. The direction of plastic flow again helps to decide whether multiplication or division is necessary. Illustration of the procedure has been given below. The particular case shown in Fig. (15-c) has been considered.

S_i is the actual stress state at the end of the current load increment.

σ_K is the limit stress for the current load increment since B_i was the actual stress at the end of the previous load increment.

σ_L is another possible limit stress. This stress is obtained by the equation,

$$K_f = \frac{\sigma_L}{\sigma_K}$$

Since stress σ_{S_i} is positive, the positive limit

stress σ_L is to be selected. The limit stress σ_L can
be obtained by either

$$\sigma_L = B \quad \varepsilon_{ep}{}^{\frac{1}{n}}$$

or

$$\sigma_L = \sigma_K \quad K_f$$

During the computation, before starting a particular
cycle of load, the number indicating the plastic flow for
that cycle must be set to zero. In the program, plastic
flow is indicated as shown below:

(1) No plastic flow   :   0

(2) Compressive plastic flow :   1

(3) Tensile plastic flow   :   2

These help to indicate how the loading path is being
traced for that particular loading cycle. However, it is
necessary that the direction of the plastic flow which
occurs for the first time during the previous loading
history be retained in the memory of the computer. In
order to avoid confusion in the discussion which follows,
this plastic flow is termed the 'very first plastic flow'.
The reason for retaining the direction of the very first
plastic flow has been explained in Fig. 17.

Assume that at a certain station in the section,
compressive stress occurs in the early stage of the loading

cycle (see Fig. (17-a)). However, this stress does not reach the yield point. Due to further load increments the stress-strain state travels along the path $OYAC_1$. The point C_1 indicates the actual stress-strain state at the end of the first loading cycle. Similarly, point C_2 indicates the actual stress-strain state at the end of the second loading cycle. The very first plastic flow, in this particular case, is tensile. It is seen that first plastic flow during the second cycle is compressive. Assume now that σ_L is the limit stress for the current load increment. Due to the loading conditions of the current load increment, unloading or reverse loading occurs and the actual stress becomes σ_{B_i} . The stress σ_{B_i} is compressive, and hence, the new limit stress should also be compressive. The limit stress σ_K , therefore, should be chosen as a new limit stress. This can be obtained by the equation,

$$\sigma_K = \sigma_L/K_f \tag{61}$$

Suppose the direction of the very first plastic flow is not considered. Then one may depend only on the knowledge of the first plastic flow which takes place during the cycle. Considering Fig. (17-b), and following

the earlier discussion, the stress σ_K will be obtained by
the equation,

$$\sigma_K = \sigma_L K \tag{62}$$

which is incorrect for this case.

Further, consider Fig. (17-c). The directions of the
very first plastic flow and the first plastic flow for the
second cycle are identical. The limit stress σ_K is
obtained by the equation,

$$\sigma_K = \sigma_L / K \tag{63}$$

Compare now Eqs. (61) and (63). It can be seen that
a decision regarding multiplication or division solely
depends upon the direction of the very first plastic flow.
It should be noted that in the absence of reverse plastic
flow, the constant K and K_f will have the same value.
Note that Fig. (17-c) represents a typical case where the
very first plastic flow and the first plastic flow during
a particular loading cycle have the same direction.

If no plastic flow is indicated during the particular
cycle, it is necessary to consider the direction of the
plastic flow in the previous cycle.

The preliminary computation procedure has been illus-
trated in the flow diagram shown in Fig. 18. This method
of resetting the stress limit gives freedom of selection

of the stress limit according to available data. Hence,
it does not confine one to the limitation of the assumption
that the material has the same stress-strain curve in
both directions and that alteration of the stress-strain
curve is the same in compression and tension. In fact,
mathematical models of the Bauschinger effect could be
studied.

XII. DISCUSSION AND CONCLUSIONS

A computer program has been developed (see Appendix B) which has the potential to investigate the following phenomena.

A. The Influence of the Bauschinger Effect on the Cyclic Life of the Material

Because the Bauschinger effect is more the rule than the exception in the behavior of engineering alloys, and has the tendency of changing the yield stress with each change in the direction of the plastic flow, it has a very important bearing on the cyclic stress-strain patterns. As the yield point lowers at one station in the section, more load is shared by the portions which have not yielded as well as by those which have yielded but still possess higher yield points than the yield point of the station under consideration. This may cause gradual yielding at more and more stations as the number of cycles increases. It is, therefore, possible that a situation may develop when the entire section may yield causing a sudden collapse by failing to balance the applied load. It should, however, be noted that the life of the structure may be terminated because of excessive deformation or the appearance of cracks before a condition of sudden collapse is reached. It should be of interest to predict analytically when the condition of collapse is reached, that

is , to investigate the mechanical and thermal loads and the number of loading cycles required to cause collapse in typical structures.

B. The Phenomena of Thermal Ratcheting and Alternate Plasticity

In case of alternate plasticity, as defined by Parkes[1] there is both positive and negative plastic flow in each cycle of net zero value. This can be true when linearly elastic perfectly plastic material is considered. If strain-hardening characteristics are considered, it is not possible to show steady behavior of alternate plasticity immediately after the formation of the first hysteresis loop due to cyclic loading. The reason for this is that at any station in the section, the magnitude of tensile and compressive yield stresses should become smaller with each change in the direction of plastic flow and this, in turn, affects the stress-strain pattern not only at that particular station, but at other stations also. It should be noted that during an early stage of cyclic loading, the rate of change of the magnitude of the yield stresses due to the Bauschinger effect may be high. This rate would finally become negligible resulting in stationary yield points for the cycles that follow. One may, therefore, expect alternate plasticity to occur at a station when the yield stresses become steady subject to the condition that the stress-strain patterns at other

stations do not affect the stress-strain pattern at the station under consideration.

For thermal ratcheting, Ayer[5] observed that "The stress pattern repeats itself after the first cycle and a constant increment of strain is added on each cycle." He considered the linearly elastic perfectly plastic material. Considering the linear strain hardening characteristic, the author has proven analytically that an asymptotic stress-strain condition may exist after a certain number of loading cycles and further strain growth may not occur (see Appendix A). This phenomenon has been observed experimentally (see page 128 of ref. 2). Thus, it is possible that different behavior may be observed at the same station with an increase in the number of loading cycles. Further, different stations in the section may exhibit different patterns of behavior at the same time and these patterns are likely to change. Thermal ratcheting may start at one or more stations and then progressively extend to the neighboring stations.

For design, it is important to investigate the loading conditions which may cause thermal ratcheting or other structural behavior in the section and how rapidly the detrimental effect of thermal ratcheting may spread to other stations. The behavior patterns should also be examined when the applied mechanical load is also varying. For example, the bending moment on the wing of

a bomber aircraft will vary with the variation in the aerodynamic loads. Variation in the mechanical loads may occur with the release of the bombs and consumption of fuel during flight. All of these ideas could be studied with the program which has been developed.

The plots shown in Fig. 19 illustrate structural behavior for cyclic thermal loading at four stations in a rectangular section of aluminum alloy 2024-T3 (see Appendix B). Thermal ratcheting is observed at all four stations. At the end stations (21) and (11), both tensile and compressive plastic flow are observed, whereas at neighboring stations, plastic flow in the opposite direction is not observed. An increase in stress and strain is observed for thermal ratcheting. It should be noted that the rate of increase in stress and strain reduces with an increase in the number of cycles as predicted in Appendix A.

Further, Fig. 20 illustrates the stress distribution along the Z axis of the section under consideration at several temperature conditions as described in the figure. Figure 21 has been reproduced from reference 3 to compare the pattern of the stress distribution while passing from one temperature to another. It can be seen that the stress distribution shown in Fig. 20 has the same tendency as that exhibited in Fig. 21.

The average stress-strain curve which the material

(aluminum 2024-T3) is supposed to follow has been shown
by the path OAB in Fig. 22. The curve OYC is the path
plotted during the computation procedure. It can be seen
that the path OYC does not coincide with the path OAB and
hence a cumulative error is observed. Possible reasons
for this error are:

(1) The size of the load increments in the region
of the yield point may be important. Eq. (17) is exact
for linear strain hardening. Convergence takes place on a
straight line of slope $(\frac{d\sigma_e}{d\varepsilon_{ep}})$. A large increment, there-
fore, will cause a large error (see article IX).

(2) The number of terms retained in the Taylor
series used in the development of Eq. (17) can also have
an effect.

XIII. APPENDICES

APPENDIX   A

Thermal Ratcheting of a Two Bar Model
Under Steady External Load

The model shown in Fig. 23 is a two bar assembly having areas A and B, respectively. The ends of the bars are attached to rigid plates, P_1 and P_2. Bar B is assumed to be heated and cooled in succession. When heated, the bar tends to expand, but the expansion is partially restrained by bar A through attachments P_1 and P_2. These plates are rigid enough so that the net lengths of both bars are always equal (see page 184 of ref. 2).

In addition to thermal loading resulting from nonuniform heating, there is present the external load P, which must be supported by the bars A and B acting jointly. The fraction of the load carried by each bar shifts, however, during heating and cooling, as a result of plastic flow.

Loading Cycle: Initially, it is considered that the model is at uniform temperature and the load P is uniformly distributed over the entire area A and B. Assume that the temperature of the outer bar is uniformly increased to a temperature T, while bar A is maintained at the reference temperature. Then the temperature of B again returns to the reference value.

This temperature cycle is assumed to be repeated while the external mechanical load P remains constant.

Analysis: For simplification of the analysis, the following assumptions are made.

(1) Plastic flow in the reverse direction does not occur.

(2) Strain hardening is linear.

When the model is at a uniform temperature initially, and the external load P is applied, the average stress is

$$\sigma_m = \frac{P}{(A + B)} \tag{1-A}$$

When the temperature of the bar B is uniformly increased to a temperature T while the bar A is maintained at the reference temperature, the attempted expansion of bar B will transfer some of the load to A. If the expansion αT is small, the action may be completely in the elastic range, but if αT is large enough, the condition indicated by the points $A_{\frac{1}{2}}$ and $B_{\frac{1}{2}}$ may develop at the end of the first half cycle of heating (see Fig. 24). There is plastic flow in the bar A, while the bar B has unloaded elastically to the point $B_{\frac{1}{2}}$. The strain at $A_{\frac{1}{2}}$ is greater than the strain at $B_{\frac{1}{2}}$ by an amount αT. At the end of the half cycle, for equilibrium,

$$P = B\sigma_{B_{\frac{1}{2}}} + A \sigma_{A_{\frac{1}{2}}} \tag{2-A}$$

Therefore, with the help of Eq. (1-A), we obtain,

$$\sigma_{B\frac{1}{2}} = (1 + \frac{A}{B}) \sigma_m - \frac{A}{B} n_1 \sigma_y \qquad (3\text{-}A)$$

where

$$n_1 = \sigma_{A\frac{1}{2}}/\sigma_y$$

When the temperature of bar B is returned to the reference value, bar A, which has been stretched, unloads along the line $A_{\frac{1}{2}}A_1$ parallel to the elastic line and bar B assumes a greater portion of the load, straining along the line $B_{\frac{1}{2}}NB_1$. At the end of the first cycle, therefore, from the condition of equilibrium, we obtain

$$\sigma_{A_1} = (1 + \frac{B}{A}) \sigma_m - \frac{B}{A} n_2 \sigma_y \qquad (4\text{-}A)$$

where

$$n_2 = \sigma_{B_1}/\sigma_y$$

The overall length of the model is permanently increased by plastic strain ε_g.

From Fig. 24,

$$\varepsilon_g = JK$$

$$= LM - LJ - KM$$

$$= \alpha T - \frac{\sigma_{B_1} - \sigma_{B_{\frac{1}{2}}}}{E} - \frac{\sigma_{A_{\frac{1}{2}}} - \sigma_{A_1}}{E}$$

$$= \alpha T - \frac{n_2 \sigma_y - \sigma_{B_{\frac{1}{2}}}}{E} - \frac{n_1 \sigma_y - \sigma_{A_1}}{E} \qquad (5\text{-}A)$$

is obtained.

Substituting Eqs. (3-A) and (4-A) into Eq. (5-A) yields

$$\epsilon_g = \alpha T + \left(2 + \frac{A}{B} + \frac{B}{A}\right) \frac{\sigma_m}{E} - \left(1 + \frac{A}{B}\right) \frac{n_1 \sigma_y}{E}$$

$$- \left(1 + \frac{B}{A}\right) \frac{n_2 \sigma_y}{E} \qquad (6\text{-}A)$$

Eq. (6-A) can be rewritten as

$$\epsilon_g = \alpha T + K_1 \sigma_m - K_2 n_1 - K_3 n_2 \qquad (7\text{-}A)$$

where K_1, K_2 and K_3 are constants.

A relation can be established between n_1 and n_2. Consider the triangle CHK in Fig. 25. From this triangle we obtain

$$HK = \frac{1}{\tan \theta} \left[\sigma_{A_{\frac{1}{2}}} - \left\{\sigma_m \left(1 + \frac{B}{A}\right) - \frac{B}{A} \sigma_{B_1}\right\}\right] \qquad (8\text{-}A)$$

Also from triangle CPJ we obtain

$$HK = \cot \phi \left(\sigma_{A_{\frac{1}{2}}} - \sigma_{B_1}\right) \qquad (9\text{-}A)$$

Equating Eqs. (8-A) and (9-A) gives

$$(\sigma_{A_{\frac{1}{2}}} - \sigma_{B_1}) \tan \theta \cdot \cot \phi = \sigma_{A_{\frac{1}{2}}} - \sigma_m(1 + \frac{B}{A}) + \frac{B}{A} \sigma_{B_1}$$

(10-A)

Dividing both sides of Eq. (10-A) by σ_y and substituting E and E_s for $\tan \theta$ and $\tan \phi$, respectively, yields,

$$n_2 = \frac{\sigma_m}{\sigma_y} \frac{(1 + \frac{B}{A})}{(\frac{E}{E_s} + \frac{B}{A})} + n_1 \frac{(\frac{E}{E_s} - 1)}{(\frac{E}{E_s} + \frac{B}{A})}$$

(11-A)

Eq. (11-A) can be written as

$$n_2 = K_4 n_1 + K_5 \frac{\sigma_m}{\sigma_y}$$

(12-A)

substituting Eq. (12-A) into Eq. (7-A) yields,

$$\varepsilon_g = \alpha T + \sigma_m(K_1 - \frac{K_3 K_5}{\sigma_y}) - n_1(K_2 + K_3 K_4)$$

(13-A)

Also from Fig. 24 we obtain

$$\varepsilon_{B_{\frac{1}{2}}} = \frac{\sigma_m}{E} - \frac{\sigma_m - \sigma_{B_{\frac{1}{2}}}}{E}$$

(14-A)

$$\varepsilon_{A_{\frac{1}{2}}} = \frac{\sigma_{A_{\frac{1}{2}}} - \sigma_y}{E} + \frac{\sigma_y}{E}$$

(15-A)

$$\varepsilon_{A_{\frac{1}{2}}} = \varepsilon_{B_{\frac{1}{2}}} + \alpha T$$

(16-A)

Substituting for $\varepsilon_{A\frac{1}{2}}$ and $\varepsilon_{B\frac{1}{2}}$ into Eq. (16-A) and then solving for n_1 yields

$$n_1 = \frac{(1 + \frac{A}{B})\sigma_y}{(\frac{1}{E_s} + \frac{A}{BE})} \cdot \frac{\sigma_m}{E} + \frac{\sigma_y}{(\frac{1}{E_s} + \frac{A}{BE})} \alpha T$$

$$+ \frac{(\frac{1}{E_s} - \frac{1}{E})\sigma_y^2}{(\frac{1}{E_s} + \frac{A}{BE})} \tag{17-A}$$

Eq. (17-A) can be rewritten as

$$n_1 = K_6 \sigma_m + K_7 \alpha T + K_8 \tag{18-A}$$

where K_6, K_7 and K_8 are constants.

Now substituting Eq. (18-A) into Eq. (17-A) yields

$$\varepsilon_g = (1 - K_2 K_7 - K_3 K_4 K_7)\alpha T + (K_1 - \frac{K_3 K_5}{\sigma_y} - K_6 K_2$$

$$- K_3 K_4 K_6)\sigma_m - (K_2 K_8 + K_3 K_4 K_8) \tag{19-A}$$

Rewriting Eq. (19-A) yields

$$\varepsilon_g = K_9 \sigma_m + K_{10} \alpha T - K_{11} \tag{20-A}$$

where K_9, K_{10} and K_{11} are constants.

It should be noted that 'ε_g' is the plastic strain growth which may occur at the end of the first cycle. Also, it can be seen that the factors to choose for design are T, σ_m , the ratio A/B and the plastic strain growth ε_g .

Consider now the state of stress and strain at the end of 1½ and 2 cycles (see Fig. 26). At the end of 1½ cycles, the stress in bar B is

$$\sigma_{B_{3/2}} = (1 + \frac{A}{B})\sigma_m - \frac{A}{B}\sigma_{A_{3/2}} \tag{21-A}$$

Also, the strains in bars B and A are given by

$$\varepsilon_{B_{3/2}} = \varepsilon_1 - \frac{\sigma_{B_1} - \sigma_{B_{3/2}}}{E} \tag{22-A}$$

and

$$\varepsilon_{A_{3/2}} = \frac{\sigma_y}{E} + \frac{\sigma_{A_{3/2}} - \sigma_y}{E_s} , \tag{23-A}$$

respectively. And

$$\varepsilon_{A_{3/2}} = \varepsilon_{B_{3/2}} + \alpha T \tag{24-A}$$

Therefore,

$$\varepsilon_{A_{3/2}} = \varepsilon_1 - \frac{\sigma_{B_1} - \sigma_{B_{3/2}}}{E} + \alpha T \qquad (25\text{-}A)$$

Substituting Eqs. (21-A) and (23-A) into Eq. (25-A) yields

$$\sigma_{A_{3/2}} \left(\frac{1}{E_s} + \frac{A}{B}\right) = \left(\frac{1}{E_s} - \frac{1}{E}\right)\sigma_{B_1} + \left(1 + \frac{A}{B}\right)\sigma_m + \alpha T \qquad (26\text{-}A)$$

Further, at the end of the second cycle, the stress and strain are given by

$$\sigma_{A_2} = \sigma_m \left(1 + \frac{B}{A}\right) - \frac{B}{A} \cdot \sigma_{B_2} \qquad (27\text{-}A)$$

and

$$\varepsilon_2 = \frac{\sigma_{A_{3/2}} - \sigma_{A_2}}{E} \qquad (28\text{-}A)$$

Also,

$$\varepsilon_2 = \frac{\sigma_y}{E} + \left(\frac{\sigma_{B_2} - \sigma_y}{E_s}\right) \qquad (29\text{-}A)$$

Equating Eqs. (28-A) and (29-A) and then substituting for σ_{A_2} from Eq. (27-A) yields

$$\sigma_{B_2} = \{\frac{(1 + \frac{B}{A})}{\frac{B}{A} - \frac{E}{E_s}}\} \ \sigma_m - \{\frac{1}{(\frac{B}{A} - \frac{E}{E_s})}\} \ \sigma_{A_{3/2}}$$

$$+ \{\frac{(1 - \frac{E}{E_s})}{(\frac{B}{A} - \frac{E}{E_s})}\} \ \sigma_y \qquad (30-A)$$

Substituting the value of $\sigma_{A_{3/2}}$ obtained by Eq. (26-A) into Eq. (30-A) gives

$$\sigma_{B_2} = \{\frac{(1 + \frac{B}{A})}{(\frac{B}{A} - \frac{E}{E_s})} - \frac{(1 + \frac{A}{B})}{(\frac{1}{E_s} + \frac{A}{B})(\frac{B}{A} - \frac{E}{E_s})}\} \ \sigma_m$$

$$-\{\frac{1}{\frac{1}{E_s} + \frac{A}{B}}\} \alpha T - \{\frac{(\frac{1}{E_s} - \frac{1}{E})}{(\frac{1}{E_s} + \frac{A}{B})(\frac{B}{A} - \frac{E}{E_s})}\} \ \sigma_{B_1}$$

$$+ \{\frac{(1 - \frac{E}{E_s})}{(\frac{B}{A} - \frac{E}{E_s})}\} \ \sigma_y$$

$$= K_{12}\sigma_m - K_{13}\sigma_{B_1} - K_{14} \ \alpha T + K_{15} \qquad (31-A)$$

where K_{12} , K_{13} , K_{14} , and K_{15} are constants.

Assume now that a temporary decision has been made about mean stress σ_m and αT (i.e., the external load and temperature range T have been selected). Consider, therefore, that σ_m and αT are constants. Eq. (31-A) can

now be rewritten as

$$\sigma_{B_2} = K_{16} - K_{13} \sigma_{B_1} \tag{32-A}$$

From the repetitive nature of the loading and following the same arguments, Eq. (32-A) can be generalized as

$$\sigma_{B_K} = \sum_{n=2}^{n=K} (-1)^{n-2} K_{13}^{n-2} K_{16} + (-1)^{K-1} K_{13}^{K-1} \sigma_{B_1}$$

$$\tag{33-A}$$

where σ_{B_K} is the stress in bar B at the end of the K_{th} cycle. Note that $K>1$. The total strain at the end of the K_{th} cycle is obtained from

$$\epsilon_{KT} = \frac{\sigma_y}{E} + \frac{\sigma_{B_K} - \sigma_y}{E} \tag{34-A}$$

The strain growth is the plastic component of the total strain. This plastic component is given by

$$\epsilon_{gK} = \sigma_y \left(\frac{1}{E} - \frac{1}{E_s} \right) + \sigma_{B_K} \left(\frac{1}{E_s} - \frac{1}{E} \right)$$

$$= K_{17} + K_{18} \sigma_{B_K} \tag{35-A}$$

where K_{17} and K_{18} are constants.

Substituting Eq. (33-A) into Eq. (35-A) yields

$$\varepsilon_{gK} = K_{17} + K_{18} [\sum_{n=2}^{n=K} (-1)^{n-2} K_{13}^{n-2} K_{16}$$

$$+ (-1)^{K-1} K_{13}^{K-1} \sigma_{B_1}] \qquad (36\text{-}A)$$

Also, from Eq. (12-A), solving for σ_{B_1} gives

$$\sigma_{B_1} = K_5 \sigma_m + K_4 \sigma_{A_{\frac{1}{2}}} \qquad (37\text{-}A)$$

Substituting Eq. (37-A) into Eq. (36-A) yields

$$\varepsilon_{gK} = K_{17} + \sum_{n=2}^{n=K} (-1)^{n-2} K_{13}^{n-2} K_{19}$$

$$+ (-1)^{K-1} K_{13}^{K-1} K_{20} \sigma_{A_{\frac{1}{2}}} + (-1)^{K-1} K_{13}^{K-1} K_{21} \qquad (38\text{-}A)$$

where

$$K_{19} = K_{16} K_{18}$$

$$K_{20} = K_4 K_{18}$$

and

$$K_{21} = K_5 K_{18} \sigma_m$$

Expanding K_{19}, K_{20} and K_{21} yields

$$K_{19} = \sigma_m \frac{(\frac{1}{E_s} - \frac{1}{E})}{(\frac{B}{A} - \frac{E}{E_s})} \left\{ (1 + \frac{B}{A}) - \frac{(1 + \frac{A}{B})}{(\frac{1}{E_s} + \frac{A}{B})} \right\}$$

$$+ (\frac{1}{E_s} - \frac{1}{E}) \left\{ \sigma_y \frac{(1 - \frac{E}{E_s})}{(\frac{B}{A} - \frac{E}{E_s})} - \frac{\alpha T}{(\frac{1}{E_s} + \frac{A}{B})} \right\}$$

$$K_{20} = (\frac{E}{E_s} - 1)(\frac{1}{E_s} - \frac{1}{E}) / (\frac{E}{E_s} + \frac{B}{A})$$

and

$$K_{21} = \frac{(1 + \frac{B}{A})(\frac{1}{Es} - \frac{1}{E})}{(\frac{E}{E_s} + \frac{B}{A})} \sigma_m$$

Expansion of K_{13} gives

$$K_{13} = \frac{(\frac{1}{E_s} - \frac{1}{E})}{(\frac{1}{E_s} + \frac{A}{B})(\frac{B}{A} - \frac{E}{E_s})}$$

The exponential nature of Eq. (38-A) indicates that the total growth of plastic strain increases as the number of cycles increases. However, the rate of growth of plastic strain decreases. After a certain number of cycles,

this rate may become negligible and an asymptotic stress condition may be achieved.

Substituting $\sigma_m = 0$ into Eq. (20-A) gives

$$\varepsilon_g = K_{10}\alpha T - K_{11} \tag{39-A}$$

When $\varepsilon_g = 0$, Eq. (39-A) yields

$$\alpha T = \frac{K_{11}}{K_{10}}$$

$$\frac{\sigma_y^{\,3}}{E} \cdot \frac{\left(\dfrac{1}{E_s} - \dfrac{1}{E}\right)}{\left(\dfrac{1}{E_s} + \dfrac{A}{BE}\right)} \left\{ \frac{\left(1 + \dfrac{A}{B}\right) + \left(1 + \dfrac{B}{A}\right)\left(\dfrac{E}{E_s} - 1\right)}{\left(\dfrac{E}{E_s} + \dfrac{B}{A}\right)} \right\} \\ = \frac{}{1 - \dfrac{\sigma_y^{\,2}}{E\left(\dfrac{1}{E_s} + \dfrac{A}{BE}\right)} \left\{ \dfrac{\left(1 + \dfrac{A}{B}\right)\left(1 + \dfrac{B}{A}\right)\left(\dfrac{E}{E_s} - 1\right)}{\left(\dfrac{E}{E_s} + \dfrac{B}{A}\right)} \right\}}$$

$$\tag{40-A}$$

Eq. (40-A) shows that when the free thermal expansion αT of bar B exceeds the value expressed by the right hand side of the equation, strain growth may be observed even in the absence of external load.

APPENDIX   B

Statement of an Example Problem and the Program

   1.   <u>Statement of an example problem</u>.   Study the
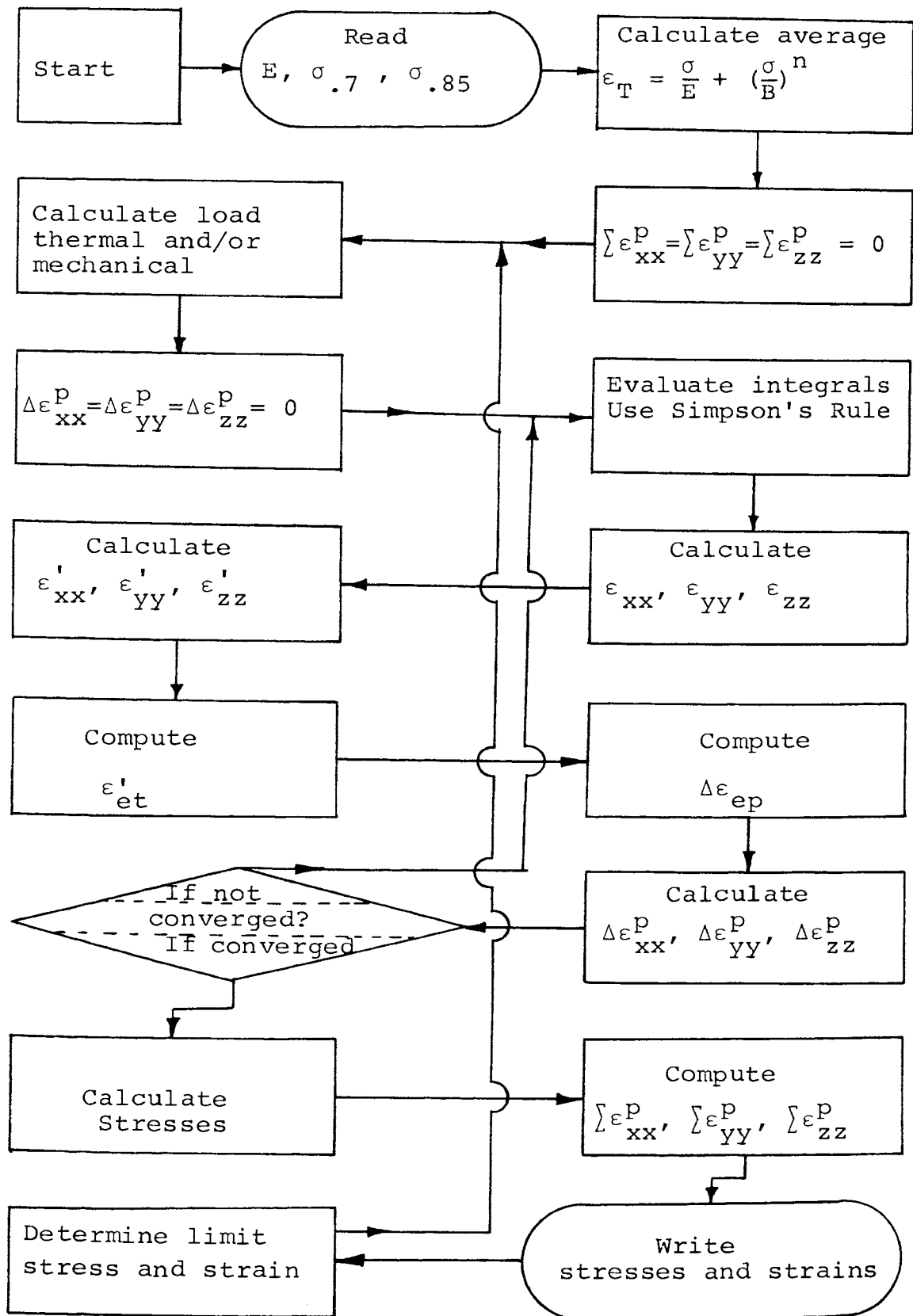structural behavior of a rectangular section for the fol-
lowing conditions:

(a)	Bending moment	15000.0 lb. in.
(b)	Axial load	0.0 lbs.
(c)	Maximum temperature	375.0 °F
(d)	Minimum temperature	75.0 °F
(e)	Cross section	
	(1) Height	4.0 in.
	(2) Width	0.23 in.

The structure undergoes cyclic thermal loading. The
material of the structure is Aluminum alloy 2024-T3. The
mechanical and thermo-physical properties have been listed
in Table I.

   In the computer program, the equations to calculate
average properties cover the range of 75-700 F. The mini-
mum temperature has been assumed to remain constant in this
analysis.

   The cross section has been divided into 21 stations as
shown in Fig. 11.

   Program Language: Fortran IV

   Computer   :   IBM-360-50

   Plotter   :   Calcomp-750

Start

Read
E, $\sigma_{.7}$, $\sigma_{.85}$

Calculate average
$\varepsilon_T = \dfrac{\sigma}{E} + \left(\dfrac{\sigma}{B}\right)^n$

Calculate load
thermal and/or
mechanical

$\sum \varepsilon_{xx}^p = \sum \varepsilon_{yy}^p = \sum \varepsilon_{zz}^p = 0$

$\Delta\varepsilon_{xx}^p = \Delta\varepsilon_{yy}^p = \Delta\varepsilon_{zz}^p = 0$

Evaluate integrals
Use Simpson's Rule

Calculate
ε_{xx}', ε_{yy}', ε_{zz}'

Calculate
ε_{xx}, ε_{yy}, ε_{zz}

Compute
ε_{et}'

Compute
$\Delta\varepsilon_{ep}$

If not
converged?
If converged

Calculate
$\Delta\varepsilon_{xx}^p$, $\Delta\varepsilon_{yy}^p$, $\Delta\varepsilon_{zz}^p$

Calculate
Stresses

Compute
$\sum \varepsilon_{xx}^p$, $\sum \varepsilon_{yy}^p$, $\sum \varepsilon_{zz}^p$

Determine limit
stress and strain

Write
stresses and strains

Flow Chart

2. <u>The program</u>. The list of symbols used in the program:

A	Time, seconds.
ACRV(I)	Shape parameter of a stress-strain curve at 'I'th station.
ALPHA	Coefficient of thermal expansion, in./(in.) (F).
AM	Poisson's ratio.
AMT	Bending moment, lb/inch.
ANITA(I)	Ratio h/Z at 'I'th station.
B1, B2 & B3	Constants appearing in equations for ϵ_{xx}, ϵ_{yy}, and ϵ_{zz}.
BCRV(I)	Constant B in equation $(\sigma/B)^n$ at 'I'th station.
BP	Axial load, lbs.
C	Specific heat, btu/(lb.)(F).
C1 and C2	Constants appearing in equations for ϵ_{xx}, ϵ_{yy} and ϵ_{zz}.
CICLE	Temperature cycle.
CK	Thermal conductivity, btu/(hr.)(ft.)(F)).
CONCON	Condition for convergence.
CRTEX	Modified strain in 'X' direction, in./in.
CRTEZ	Modified strain in 'Z' direction, in./in.
DISC(I)	Distance of an 'I'th station from one end of the section, in.

DIST(I)	Distance from the reference point in the section, in.
DLTEXP(I,J)	Dimensionless increment of plastic strain in 'X' direction at 'I'th station and 'J'th iteration.
DLTEZP(I,J)	Dimensionless increment of plastic strain in 'Z' direction at 'I'th station and 'J'th iteration.
DMTEXP(I,J)	Increment of plastic strain in 'X' direction at 'I'th station and 'J'th iteration in./in.
EC(I)	Modulus of elasticity at 'I'th station.
ECR	Modulus of elasticity at reference temperature, lb./in.^{2}
EEP(I)	Equivalent plastic strain increment.
EQLIM(I)	Equivalent limit strain at 'I'th station.
EQSGMA(I)	Equivalent limit stress at 'I'th station.
EXM(I)	Mechanical strain in 'X' direction, in./in.
EZM(I)	Mechanical strain in 'Z' direction, in./in.
PFLOW1(M,I)	First plastic flow during 'M'th cycle.
PFLOW2(M,I)	Plastic flow in the direction opposite to that of the first plastic flow at 'I'th station during 'M'th cycle.
PLSTRN(I)	Plastic strain at 'I'th station.
PREFLO(I)	Previous plastic flow at 'I'th station.
PRESTS(I)	Initial yield stress, psi.

SECFL(I)   Plastic flow having the direction opposite
                    to that of the very first plastic flow.

SECLIM(I)   Limit Stress having the direction opposite
                    to that of the stress which causes very
                    first plastic flow.

SDTEXP(I,2)   Summation of plastic strain increment in
                    'X' direction (dimensionless).

SUMEXP(I,2)   Summation of plastic strain increment in
                    'X' direction, in./in.

SIGMAR   Yield stress at the reference temperature.

SLOPE(I)   Slope of stress-plastic strain curve at
                    limit stress.

TREF   Reference temperature.

TLOW   Lower temperature.

TMAX   Maximum temperature.

THKW   Width of section.

TM   Time, hr.

TNCRIS   Increase in temperature.

TDCRIS   Decrease in temperature.

```
         C        R. C. HAZARIWALA ME490, JOB ME140767G INCREMENTAL COLLAPSE
0001             DIMENSION DLTEXP(21,50 ),DLTEZP(21,50 ),SDTEXP(21,2),SDTEZP(21,21,
                1C1(50 ),C2(50 ),SIGMAX(21),EX(21),EY(21),EZ(21),EP(21),EET(21),
                1EEP(21),TO(21),B1(21),B2(21),B3(21),ANITA(21),GL(21),TM(21),T(21),
                1TOW(21),X(21),H(21),F(21),FN(21),HA(21),AINGRL(21),BINGRL(21)
0002             DIMENSION TMAXC(21),EC(21),FOET(21),FOS(21),STRESS(45),CCRV(21),
                1ACRV(21),BCRV(21),SLOPE(21)
0003             DIMENSION STRES(21),PLSTRN(21),DIST(21),EQSGMA(21),EQLIM(21),
                1SUMEXP(21,2),SUMEZP(21,2),DMTEXP(21,50 ),SIGMAC(21),DISC(21)
0004             DIMENSION PRESTS(21),PREFO(21),STRAIN(45)
0005             DIMENSION XSIG21(450),XSIG20(450),XSIG11(450),XSIG10(450),
                1EXM(21),EXM21(450),EXM20(450),EXM11(450),EXM10(450)
0006             DIMENSION CONST(21),SETSTS(21),SECFL(21),FSTLIM(21)
0007             DIMENSION SECLIM(21),CONST2(21),FIRSTE(21),PREFLO(21),
                1PFLOW1(10,21),PFLOW2(10,21),FLOWNO(21),FPLOW1(21),FPLOW2(21)
0008             CALL PENPOS('HAZARIWALA RAMESHCHA',20,1)
0009             AMT=15000.0
0010             BP=0.0
0011             TREF=75.0
0012             TMAX=375.0
0013             TLOW=75.0
0014             TDCRIS=15.0
0015             TNCRIS=TDCRIS
0016             TINCR=(TMAX-TLOW)/TNCRIS
0017             TINCRT=TINCR+1
0018             AM=0.3125
0019             READ(1,201)(ANITA(J), J=1,21)
0020             READ(1,201)(GL(J), J=1,21)
0021             READ(1,202)(H(K),K=1,5)
0022             DO 500 I=1,21
0023        500  TMAXC(I)=TMAX
0024             DO 501 I=1,1
0025             IF(TMAXC(I)-300.0)502,502,503
0026        502  EC(I)=10833125.0-887.50*(TMAXC(I)+75.0)
0027             FOS(I)=40100.0-7.325*(TMAXC(I)+75.0)
0028             FOET(I)=36834.0-5.560*(TMAXC(I)+75.0)
0029             STRESS(1)=33000.0
0030             GO TO 504
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```
0031        503 IF(TMAXC(I)-500.0)504,504,505
0032        504 EC(I)=(2365500000.0+10585000.0*(TMAXC(I)-300.0)-4750.0*
            1(TMAXC(I)**2.0-90000.0))/(TMAXC(I)-75.0)
0033            FOS(I)=(8612000.0+52050.0*(TMAXC(I)-300.0)-27.25*
            1(TMAXC(I)**2.0-90000.0))/(TMAXC(I)-75.0)
0034            FOET(I)=(7830000.0+48550.0*(TMAXC(I)-300.0)-26.75*
            1(TMAXC(I)**2.0-90000.0))/(TMAXC(I)-75.0)
0035            STRESS(1)=28000.0
0036            GO TO 506
0037        505 EC(I)=(3733500000.0+13400000.0*(TMAXC(I)-500.0)-5000.00*
            1(TMAXC(I)**2.0-250000.0))/(TMAXC(I)-75.0)
0038            FOS(I)=(14662000.+71300.0* (TMAXC(I)-500.0)-46.5*
            1(TMAXC(I)**2.0-250000.0))/(TMAXC(I)-75.0)
0039            FOET(I)=(12260000.+66050.0*(TMAXC(I)-500.0)-43.25*
            1(TMAXC(I)**2.0-250000.0))/(TMAXC(I)-75.0)
0040            STRESS(1)=22000.0
0041        506 CCRV(I)=(FOS(I)/FOET(I))
0042            ACRV(I)=1.0+((ALOG(2.4285))/(ALOG(CCRV(I))))
0043            BCRV(I)=(FOS(I)**((ACRV(I)-1.0)/ACRV(I)))*(EC(I)**(1./ACRV(I)))
0044            STRES(I)=STRESS(1)
0045        514 PLSTRN(I)=(STRES(I)/BCRV(I))**ACRV(I)
0046            IF(PLSTRN(I)-0.00010)512,512,513
0047        512 STRES(I)=STRES(I)+350.0
0048            GO TO 514
0049        513 IF(PLSTRN(I)-0.00011)515,515,516
0050        516 STRES(I)=STRES(I)-200.0
0051            GO TO 514
0052        515 ELLIM=(STRES(I)/EC(I))
0053            EQLIM(I)=((2.0*(1.0+AM))/3.0)*ELLIM
0054        501 CONTINUE
0055            WRITE(3,520)EC(1),FOS(1),FOET(1),STRES(1),PLSTRN(1)
0056            DO 525 I=2,21
0057            EC(I)=EC(1)
0058            FOS(I)=FOS(1)
```

```
0059          FOET(I)=FOET(1)
0060          STRES(I)=STRES(1)
0061          EQLIM(I)=EQLIM(1)
0062          PLSTRN(I)=PLSTRN(1)
0063          ACRV(I)=ACRV(1)
0064          BCRV(I)=BCRV(1)
0065      525 CONTINUE
       C      ***  IN THIS PARTICULAR EXAMPLE, TMAXC(I) HAS BEEN ASSUMED TO BE
       C      CONSTANT AT ALL POINTS. ALL PROPERTIES, THEREFORE, WILL BE SAME
       C      AT ALL POINTS. DO LOOPS DEFINED BY STATEMENT NUMBERS 525 AND 501
       C      HAVE BEEN INTRODUCED TO ELIMINATE REPEATITION OF COMPUTATIONS. ***
0066          SIGMAR=27500.0
0067          ECR=10300000.0
0068          STRNYR=SIGMAR/ECR
0069          ALPHA=0.000014
0070          P=172.8
0071          C=0.2375
0072          CK=75.0
0073          PK=(CK)/(P*C)
0074          TL=0.33333
0075          THKW=0.23
0076          HTCW=2.0
0077      155 DO 163 I=1,21
0078          SDTEXP(I,2)=0.0
0079          SDTEZP(I,2)=0.0
0080          PFLOW1(1,I)=0.0
0081          PFLOW2(1,I)=0.0
0082          FLOWNO(I)=0.0
0083          FIRSTF(I)=0.0
0084          CONST2(I)=1.0
0085          PREFLO(I)=0.0
0086          SECFL(I)=0.0
0087          CONST(I)=1.0
0088          PRESTS(I)=STRES(I)
0089          EQSGMA(I)=PRESTS(I)
0090          EP(I)=PLSTRN(I)
0091          FSTLIM(I)=PRESTS(I)
0092          SECLIM(I)=PRESTS(I)
0093      163 SETSTS(I)=PRESTS(I)
0094          STRSIN=400.0
0095          N=1
0096     2502 STRESS(N)=PRESTS(1)+STRSIN-4000.0
0097          STRAIN(N)=(STRESS(N)/EC(1))+(STRESS(N)/BCRV(1))**ACRV(1)
0098          IF(STRESS(N)-37500.0)2500,2500,2501
```

```
0099        2500  N=N+1
0100              STRSIN=STRSIN+400.0
0101              GO TO 2502
0102        2501  CALL NEWPLT(9.0,5.0,18.0)
0103              CALL  ORIGIN(0.0,0.0)
0104              CALL  XSCALE(-0.0150,0.015,15.0)
0105              CALL  YSCALE(-50000.0,50000.0,8.0)
0106              CALL  XAXIS(0.0025)
0107              CALL  YAXIS(5000.0)
0108              CALL  XYPLT(STRAIN,STRESS,N,1,-1)
0109              CALL  ENDPLT
0110              NC=4
0111              KZ=4
0112              N=1
0113              DO  484 M=2,NC
0114              DO  2400 I=1,21
0115              FPLOW1(I)=0.0
0116              FPLOW2(I)=0.0
0117              PFLOW1(M,I)=0.0
0118        2400  PFLOW2(M,I)=0.0
0119              CICLE=0.25
0120              DO 550 I=1,21
0121         550  T(I)=TLOW
0122              GO TO 622
0123         551  T(11)=T(11)+TNCRIS
0124              T(21)=T(11)
0125              IF(T(11)-TMAX)622,622,560
0126         560  CICLE=0.50
0127              A=0.2
0128         614  DO  11 J=1,11
0129              DO  12 K=1,4
0130              TF=TMAX
0131              TO(J)=TLOW
0132              TM(J)=(A/3600.0)
0133              B=(3.1428*H(K)*(GL(J)))/(TL)
0134              AN=(4.0*(TO(J)-TF))/(H(K)*3.1428)
0135              EXPT=((1)/(EXP(((H(K))**2*(3.1428)**2*(PK)*(TM(J)))/((TL)**2))))
0136          12  X(K)=B*AN*EXPT
0137              SUM=X(1)+X(2)+X(3)+X(4)
0138              T(J)=SUM+TF
0139              IF(T(J)-TLOW)41,41,11
0140          41  T(J)=75.0
0141          11  CONTINUE
0142              DO 178 J=12,21
```

```
0143        178 T(J)=T(J-10)
0144            GO TO 622
0145        553 IF(A-2.2)611,612,612
0146        611 A=A+0.25
0147            GO TO 614
0148        612 IF(A-24.2)617,618,618
0149        617 A=A+1.0
0150            GO TO 614
0151        618 IF(A-44.2)619,620,620
0152        619 A=A+2.0
0153            GO TO 614
0154        620 IF(A-74.2)2619,2620,621
0155       2619 A=A+5.0
0156            GO TO 614
0157       2620 A=100.0
0158            DO 31 J=1,21
0159            T(J)=TF
0160         31 CONTINUE
0161            GO TO 622
0162        621 CICLE=0.75
0163        555 T(11)=T(11)-TOCRIS
0164            T(21)=T(11)
0165            IF(T(11)-TLOW)561,622,622
0166        561 CICLE=1.0
0167            A=0.2
0168        400 DO 461 J=1,11
0169            DO 401 K=1,4
0170            TO(J)=TMAX
0171            TF=TLOW
0172            TM(J)=(A/3600.0)
0173            B=(3.1428*H(K)*(GL(J)))/(TL)
0174            AN=(4.0*(TO(J)-TF))/(H(K)*3.1428)
0175            EXPT=((1)/(EXP(((H(K))**2*(3.1428)**2*(PK)*(TM(J)))/((TL)**2))))
0176        401 X(K)=B*AN*EXPT
0177            SUM=X(1)+X(2)+X(3)+X(4)
0178            T(J)=SUM+TF
0179            IF(T(J)-TO(J))402,403,403
0180        403 T(J)=TO(J)
0181            GO TO 461
0182        402 IF(T(J)-TLOW)462,462,461
0183        462 T(J)=TLOW
0184        461 CONTINUE
0185            DO 405 J=12,21
```

```
0186        405 T(J)=T(J-10)
0187            GO TO 622
0188        556 IF(A-2.2)429,430,430
0189        429 A=A+0.25
0190            GO TO 400
0191        430 IF(A-24.2)431,432,432
0192        431 A=A+1.0
0193            GO TO 400
0194        432 IF(A-44.2)433,434,434
0195        433 A=A+2.0
0196            GO TO 400
0197        434 IF(A-74.2)2433,2434,436
0198       2433 A=A+5.0
0199            GO TO 400
0200       2434 A=100.0
0201            DO 435 J=1,21
0202        435 T(J)=TF
0203            GO TO 622
0204        436 WRITE(3,457)
0205            GO TO 483
        C       DOUBLE INTEGRATION BY SIMSON'S RULE IS REQUIRED FOR TWO
        C       DIMENSIONAL PROBLEMS.
0206        622 DO 179 J=1,21
0207            TOW(J)=(ALPHA*(T(J)-TLOW))/STRNYR
0208            B1(J)=1.50
0209            B2(J)=0.0
0210            B3(J)=0.5
0211        179 CONTINUE
0212            STARMW=AMT/(SIGMAR*HTCW**2.0*THKW)
0213            STARPW=BP/(SIGMAR*HTCW*THKW)
0214       1001 DO 175 I=1,21
0215            DO 176 J=1,2
0216            DLTEXP(I,J)=0.0
0217            DMTEXP(I,J)=0.0
0218        176 DLTEZP(I,J)=0.0
0219        175 CONTINUE
0220         32 DO 16 J=2,50
0221       1114 DO 1 I=1,21
0222            HA(I)=EC(I)/ECR
0223            F(I)=HA(I)*(TOW(I)+SDTEXP(I,2)+DLTEXP(I,J))
0224          1 FN(I)=F(I)*ANITA(I)
0225            FINGRL=(0.1/3.0)*(F(21)+F(11)
           1+4.0*(F(20)+F(10)+F(18)+F(8)+F(16)+F(6)+F(14)+F(4)+F(12)+F(2))+
           12.0*(F(19)+F(9)+F(17)+F(7)+F(15)+F(5)+F(13)+F(3)+F(1)))
0226            FINGRL=(0.1/3.0)*(FN(21)+FN(11)+4.0*(FN(20)+FN(10)+FN(8)+
           1FN(18)+FN(16)+FN(6)+FN(14)+FN(4)+FN(12)+FN(2))+
           12.*(FN(19)+FN(9)+FN(17)+FN(7)+FN(15)+FN(5)+FN(13)+FN(3)+FN(1)))
```

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```
0227            DO 2 I=1,21
0228            AINGRL(I)=FINGRL
0229            BINGRL(I)=TINGRL
0230            C1(J)=B1(I)*STARMW+B1(I)*BINGRL(I)-B2(I)*STARPW-B2(I)*AINGRL(I)
0231            C2(J)=B3(I)*AINGRL(I)-B2(I)*BINGRL(I)+B3(I)*STARPW-B2(I)*STARMW
0232            EX(I)=C1(J)*ANITA(I)+C2(J)
0233            EZ(I)=-AM*(C1(J)*ANITA(I)+C2(J))+(1.0+AM)*TOW(I)+(SDTEZP(I,2)+
               1DLTEZP(I,J))+AM*(SDTEXP(I,2)+DLTEZP(I,J))
0234            EY(I)=-AM*(C1(J)*ANITA(I)+C2(J))+(1.0+AM)*TOW(I)+
               1(AM-1.0)*(SDTEXP(I,2)+DLTEXP(I,J))-(SDTEZP(I,2)+DLTEZP(I,J))
0235            EX(I)=EX(I)*STRNYR
0236            EZ(I)=EZ(I)*STRNYR
0237            EY(I)=EY(I)*STRNYR
0238            EXM(I)=EX(I)-TOW(I)*STRNYR
0239            CRTEX=EX(I)-SDTEXP(I,2)*STRNYR
0240            CRTEZ=EZ(I)-SDTEZP(I,2)*STRNYR
0241            CRTEY=EY(I)+(SDTEXP(I,2)+SDTEZP(I,2))*STRNYR
0242            DIFXY=ABS(CRTEX-CRTEY)
0243            DIFYZ=ABS(CRTEY-CRTEZ)
0244            DIFZX=ABS(CRTEZ-CRTEX)
0245            EET(I)=((2.0**0.5)/3.0)*(DIFXY**2.0+DIFYZ**2.0+DIFZX**2.0)**0.5
0246            IF(ABS(EET(I))-EQLIM(I))165,165,144
0247        165 EEP(I)=0.0
0248            K=J+1
0249            DLTEXP(I,K)=0.0
0250            DLTEZP(I,K)=0.0
0251            GO TO 2
0252        144 IF(J-2)1440,1440,147
0253       1440 IF(EQSGMA(I)-STRES(I))146,146,147
0254        146 EQSGMA(I)=STRES(I)
0255            EP(I)=(STRES(I)/BCRV(I))**ACRV(I)
0256            SLOPE(I)=(BCRV(I)*(1.0/ACRV(I)))/((ABS(EP(I)))**((ACRV(I)-1.0)
               1/ACRV(I)))
0257        853 RAM=0.66667*((1.0+AM)/EC(I))
0258        147 EEP(I)=(EET(I)-RAM*(EQSGMA(I)))/(1.0+RAM*SLOPE(I))
0259        140 K=J+1
0260            DLTEZP(I,K)=0.3333*(ABS(EEP(I)/EET(I)))*(2.0*CRTEZ-CRTEY-CRTEX)
               1/STRNYR
0261            DLTEXP(I,K)=0.3333*(ABS(EEP(I)/EET(I)))*(2.0*CRTEX-CRTEY-CRTEZ)
               1/STRNYR
```

```
0262            2 CONTINUE
0263          522 DO 3 I=1,21
0264              IF(ABS(DLTEXP(I,K))-0.0)141,142,141
0265          142 GO TO 3
0266          141 DMTEXP(I,K)=DLTEXP(I,K)*STRNYR
0267              DMTEXP(I,J)=DLTEXP(I,J)*STRNYR
0268              CONCON=ABS(DMTEXP(I,K)-DMTEXP(I,J))
0269              IF(ABS(DMTEXP(I,K))-0.001)242,242,243
0270          242 IF(CONCON-0.000001)3,3,24
0271          243 IF(ABS(DMTEXP(I,K))-0.01)244,244,245
0272          244 IF(CONCON-0.00001)3,3,24
0273          245 CONCON=ABS((DLTEXP(I,K)-DLTEXP(I,J))/DLTEXP(I,K))
0274              IF(CONCON-0.0010)3,3,24
0275            3 CONTINUE
0276              DO 4 I=1,21
0277              DIST(I)=GL(I)*12.0
0278          510 SIGMAX(I)=SIGMAR*HA(I)*(C1(J)*ANITA(I)+C2(J)-TOW(I)-SDTEXP(I,2)
0279             1-DLTEXP(I,K))
0280              SDTEXP(I,2)=SDTEXP(I,2)+DLTEXP(I,K)
0281              SDTEZP(I,2)=SDTEZP(I,2)+DLTEZP(I,K)
0282              SUMEXP(I,2)=SDTEXP(I,2)*STRNYR
0283              SUMEZP(I,2)=SDTEZP(I,2)*STRNYR
0284              EP(I)=(2.0/(3.0**0.5))*(ABS((ABS(SUMEXP(I,2)))**2.0+
0285             1(ABS(SUMEZP(I,2)))**2.0+(SUMEXP(I,2)*(SUMEZP(I,2))))**0.5
0286            4 CONTINUE
0287              DO 2001 I=1,21
0288              EQSGMA(I)=ABS(SIGMAX(I))
0289              IF(EQSGMA(I)-STRES(I))2001,2001,2002
0290         2002 SETSTS(I)=SIGMAX(I)
0291              IF(SIGMAX(I) -0.0)2003,2003,2004
0292         2003 IF(PFLOW1(M,I)-1.0)2005,2005,2006
0293         2005 PFLOW1(M,I)=1.0
0294              PREFLO(I)=1.0
0295              IF(FPLOW1(I)-1)8001,8002,8001
0296         8001 FLOWNO(I)=FLOWNO(I)+1
0297         8002 FPLOW1(I)=PFLOW1(M,I)
0298              GO TO 2200
0299         2006 PFLOW2(M,I)=1.0
0300              IF(FPLOW2(I)-1)8003,8004,8003
0301         8003 FLOWNO(I)=FLOWNO(I)+1
0302         8004 FPLOW2(I)=PFLOW2(M,I)
0303              GO TO 2200
                  2004 IF(PFLOW1(M,I)-1.0)2007,2008,2007
                  2007 PFLOW1(M,I)=2.0
```

```
0304              PREFLO(I)=2.0
0305              IF(FPLOW1(I)-2)8005,8005,8006
0306         8005 FLOWNO(I)=FLOWNO(I)+1
0307         8006 FPLOW1(I)=PFLOW1(M,I)
0308              GO TO 2200
0309         2008 PFLOW2(M,I)=2.0
0310              IF(FPLOW2(I)-2)8007,8007,8008
0311         8007 FLOWNO(I)=FLOWNO(I)+1
0312         8008 FPLOW2(I)=PFLOW2(M,I)
0313         2200 IF(FIRSTF(I) -1.0)2201,2099,2099
0314         2201 IF(PFLOW1(M,I)-1.0)2203,2204,2205
0315         2203 FIRSTF(I)=0.0
0316              GO TO 2099
0317         2204 FIRSTF(I)=1.0
0318              SECFL(I)=2.0
0319              GO TO 2099
0320         2205 FIRSTF(I)=2.0
0321              SECFL(I)=1.0
0322         2099 IF(FIRSTF(I)-1.0)2001,2209,2210
0323         2209 IF(SIGMAX(I)-0.)2401,2401,2212
0324         2212 SECLIM(I)=SIGMAX(I)
0325              GO TO 2001
0326         2401 FSTLIM(I)=SIGMAX(I)
0327              GO TO 2001
0328         2210 IF(SIGMAX(I)-0.0)2214,2214,2402
0329         2402 FSTLIM(I)=SIGMAX(I)
0330              GO TO 2001
0331         2214 SECLIM(I)=SIGMAX(I)
0332         2001 CONTINUE
0333              WRITE(3,5111)SIGMAX(21),SETSTS(21),FSTLIM(21),SECLIM(21),STRES(21)
                 1,CONST(21),CONST2(21)
0334              WRITE(3,5111)SIGMAX(11),SETSTS(11),FSTLIM(11),SECLIM(11),STRES(11)
                 1,CONST(11),CONST2(11)
0335         5111 FORMAT(/,5X,F10.2,5X,F10.2,5X,F10.2,5X,F10.2,5X,F9.2,5X,F5.2,5X,
                 1F10.4)
0336              DO 2010 I=1,21
0337              IF(SETSTS(I)-0.0)2011,2011,2012
0338         2011 IF(SIGMAX(I)-0.0)2013,2013,2014
0339         2014 IF(PFLOW2(M,I)-1.0)2015,2278,2279
0340         2278 IF(SECFL(I)-1.0)2178,2178,2041
0341         2279 IF(SECFL(I)-1.0)2042,2042,2179
0342         2015 IF(PFLOW1(M,I)-1.0)2071,2141,2142
```

```
0343      2141  IF(FIRSTF(I)-1.0)2041,2041,2178
0344      2142  IF(FIRSTF(I)-1.0)2179,2179,2042
0345      2041  IF(CONST(I)-1.0)2050,2050,2051
0346      2050  CONST(I)=ABS(FSTLIM(I))/PRESTS(I)
0347            GO TO 2051
0348      2071  IF(PFLOW2(M-1,I)-1.0)2077,2278,2279
0349      2077  IF(PFLOW1(M-1,I)-1.0)2242,2141,2142
0350      2242  IF(PREFLO(I)-1.0)2141,2141,2142
0351      2012  IF(SIGMAX(I)-0.0)2017,2017,2013
0352      2017  IF(PFLOW2(M,I)-1.0)2019,2379,2378
0353      2379  IF(SECFL(I)-1.0)2179,2179,2042
0354      2378  IF(SECFL(I)-1.)2041,2041,2178
0355      2019  IF(PFLOW1(M,I)-1.0)2072,2342,2345
0356      2342  IF(FIRSTF(I)-1.0)2042,2042,2179
0357      2345  IF(FIRSTF(I)-1.0)2178,2178,2045
0358      2045  IF(CONST(I)-1.0)2052,2052,2051
0359      2052  CONST(I)=ABS(FSTLIM(I))/PRESTS(I)
0360            GO TO 2051
0361      2072  IF(PFLOW2(M-1,I)-1.0)2080,2379,2378
0362      2080  IF(PFLOW1(M-1,I)-1.0)2245,2342,2345
0363      2245  IF(PREFLO(I)-1.0)2342,2342,2345
0364      2013  STRES(I)=ABS(SETSTS(I))
0365            GO TO 2010
0366      2024  STRES(I)=PRESTS(I)
0367            GO TO 2010
0368      2178  IF(ABS(SUMEXP(I,2))-PLSTRN(I))2024,2024,3025
0369      3025  FSTLIM(I)=(BCRV(I)*(ABS(SUMEXP(I,2)))**(1.0/ACRV(I)))
0370            CONST2(I)=ABS(FSTLIM(I)/SECLIM(I))
0371            STRES(I)=ABS(SECLIM(I))*CONST2(I)
0372            GO TO 2010
0373      2179  IF(ABS(SUMEXP(I,2))-PLSTRN(I))2024,2024,2025
0374      2025  FSTLIM(I)=(BCRV(I)*(ABS(SUMEXP(I,2)))**(1.0/ACRV(I)))
0375            CONST2(I)=ABS(FSTLIM(I)/SECLIM(I))
0376            STRES(I)=ABS(FSTLIM(I))/CONST2(I)
0377            GO TO 2010
0378      2051  STRES(I)=(ABS(FSTLIM(I)))/CONST(I)
0379            SECLIM(I)=-FSTLIM(I)/CONST(I)
0380            GO TO 2010
0381      2042  STRES(I)=ABS(SECLIM(I))*CONST(I)
0382      2010  EQLIM(I)=0.6667*(1.0+AM)*(STRES(I)/EC(I))
0383            XSIG21(N)=SIGMAX(21)
0384            XSIG20(N)=SIGMAX(20)
```

```
0385      XSIG11(N)=SIGMAX(11)
0386      XSIG10(N)=SIGMAX(10)
0387      EXM21(N)=EXM(21)
0388      EXM20(N)=EXM(20)
0389      EXM11(N)=EXM(11)
0390      EXM10(N)=EXM(10)
0391      IF(N-1)700,701,702
0392   702 IF(N-21)700,701,703
0393   703 IF(N-69)700,701,704
0394   704 IF(N-89)700,701,705
0395   705 IF(N-137)700,701,700
0396   701 DO 706 I=1,11
0397      DISC(I)=DIST(12-I)
0398   706 SIGMAC(I)=SIGMAX(12-I)
0399      DO 707 I=12,21
0400      SIGMAC(I)=SIGMAX(I)
0401   707 DISC(I)=DIST(I)
0402      CALL NEWPLT(0.0,5.0,10.0)
0403      CALL ORIGIN(0.0,0.0)
0404      CALL XSCALE(0.0,4.0,8.0)
0405      CALL YSCALE(-70000.0,70000.0,10.0)
0406      CALL XAXIS(0.25)
0407      CALL YAXIS(5000.0)
0408      CALL XYPLT(DISC,SIGMAC,21,1,-1)
0409      IF(N-137)700,708,700
0410   708 CALL ENDPLT
0411   700 N=N+1
0412      IF(CICLE-0.25)551,551,552
0413   552 IF(CICLE-0.50)553,553,554
0414   554 IF(CICLE-0.75)555,555,556
0415    24 IF(J-49)16,25,25
0416    16 CONTINUE
0417   483 IF(M-K7)2061,2061,2061
0418  2061 CALL NEWPLT(9.0,5.0,18.0)
0419      CALL ORIGIN(0.0,0.0)
0420      CALL XSCALE(-0.0150,0.015,15.0)
0421      CALL YSCALE(-50000.0,50000.0,8.0)
0422      CALL XAXIS(0.0025)
0423      CALL YAXIS(5000.0)
0424      CALL XYPLT(EXM11,XSIG11,N-1,1,-1)
0425      CALL ENDPLT
0426      CALL NEWPLT(9.0,5.0,18.0)
0427      CALL ORIGIN(0.0,0.0)
0428      CALL XSCALE(-0.0150,0.015,15.0)
0429      CALL YSCALE(-50000.0,50000.0,8.0)
0430      CALL XAXIS(0.0025)
```

```
0431            CALL YAXIS(5000.0)
0432            CALL XYPLT(EXM10,XSIG10,N-1,1,-1)
0433            CALL ENDPLT
0434            CALL NEWPLT(9.0,5.0,18.0)
0435            CALL ORIGIN(0.0,0.0)
0436            CALL XSCALE(-0.0150,0.015,15.0)
0437            CALL YSCALE(-50000.0,50000.0,8.0)
0438            CALL XAXIS(0.0025)
0439            CALL YAXIS(5000.0)
0440            CALL XYPLT(EXM20,XSIG20,N-1,1,-1)
0441            CALL ENDPLT
0442            CALL NEWPLT(9.0,5.0,18.0)
0443            CALL ORIGIN(0.0,0.0)
0444            CALL XSCALE(-0.0150,0.015,15.0)
0445            CALL YSCALE(-50000.0,50000.0,8.0)
0446            CALL XAXIS(0.0025)
0447            CALL YAXIS(5000.0)
0448            CALL XYPLT(EXM21,XSIG21,N-1,1,-1)
0449            CALL ENDPLT
0450       2060 DO 2021 I=1,21
0451            IF(PFLOW2(M,I)-1.0)2075,2021,2021
0452       2075 CONST2(I)=CONST(I)
0453       2021 CONTINUE
           C    NEW LOADING CYCLE IS TO BE RESUMED.
0454            IN=N-1
0455            WRITE(3,465)IN
0456            IF(M-KZ)484,2046,484
0457       2046 N=1
0458            KZ=KZ+3
0459        484 CONTINUE
0460            GO TO 231
0461         25 WRITE(3,230)
0462        201 FORMAT(7F10.6/7F10.6/7F10.6)
0463        202 FORMAT(5F10.5)
0464        203 FORMAT(7F10.2/7F10.2/7F10.2/7F10.2/7F10.2/7F10.2/7F10.2/7F10.2)
0465        222 FORMAT(//,6X,'TIME',5X,'DISTANCE',3X,'TEMP.',5X,'ANITA',
                15X,'TOTAL   EX',7X,'TOTAL  EY',7X,'TOTAL   EZ',8X,'SIGMAX',5X,
                1'SUMPLX',/)
0466        226 FORMAT(4X,F6.2,4X,F7.4,4X,F6.2,4X,F5.2,4X,F12.8,4X,F12.8,
                14X,F12.8,4X,F10.2,4X,F10.7)
0467        520 FORMAT(6X,F14.2,5X,F10.2,5X,F10.2,5X,F10.2,5X,F12.7)
0468        521 FORMAT(6X,F12.7,5X,F12.7,5X,F12.7)
0469        624 FORMAT(/,15X,'****  COOLING  STARTS   ****',/)
0470        457 FORMAT(/,15X,'****  CYCLE    ENDS      ****',/)
0471        230 FORMAT(15X,'****  DOES NOT CONVERGE IN 49 ITERATIONS  **')
0472        465 FORMAT(/,15X,'***   NO. OF INCREMENTS  :',2X,I3,/)
0473        231 CALL LSTPLT
0474            CALL EXIT
0475            END
```

TABLE I

LIST OF PROPERTIES OF ALUMINUM 2024-T3

Temp. °F.	Time Hr.	E psi x 10^6	$\sigma_{0.7}$ KSl	$\sigma_{0.85}$ KSl	n	μ
Room	2	10.7	39.0	36.0	11.5	0.3125
300	2	10.3	35.7	33.5	15.0	-
500	2	8.4	24.8	22.8	10.9	-
700	2	6.4	6.2	5.5	8.2	-

Material: Aluminum Alloy 2024-T3; sheet and plate, heat treated, thickness $\leq$ 0.25
inch.

ρ   0.10 lb./in.^{3}

C   0.23 BTU/(lb.)(F)

K   75.0 BTU/[(hr)(Ft) (F)]

α   0.14 in./in./(F)

XIV. FIGURES
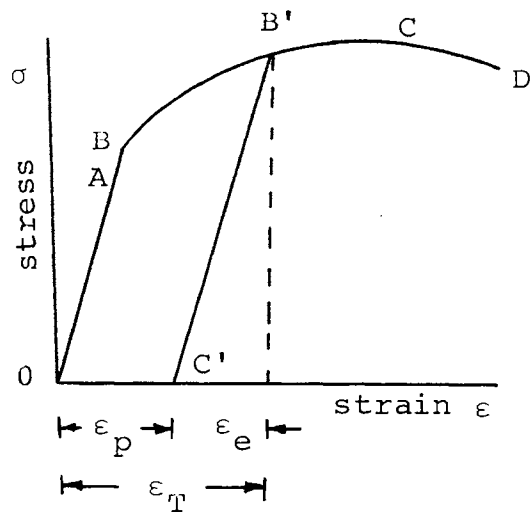
Figure 1. Conventional stress-strain curve.
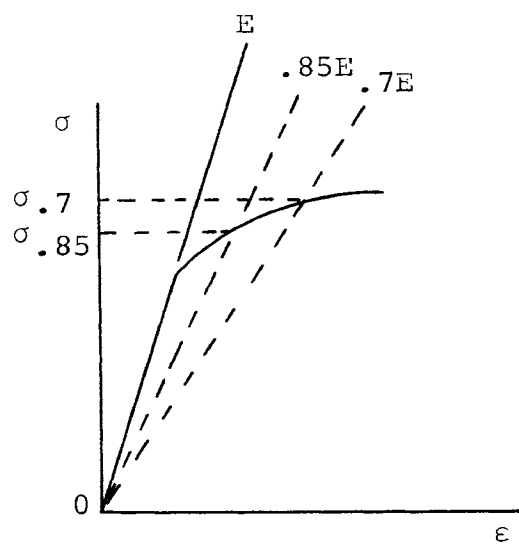
Figure 2. Stress-strain curve described by
three parameters.
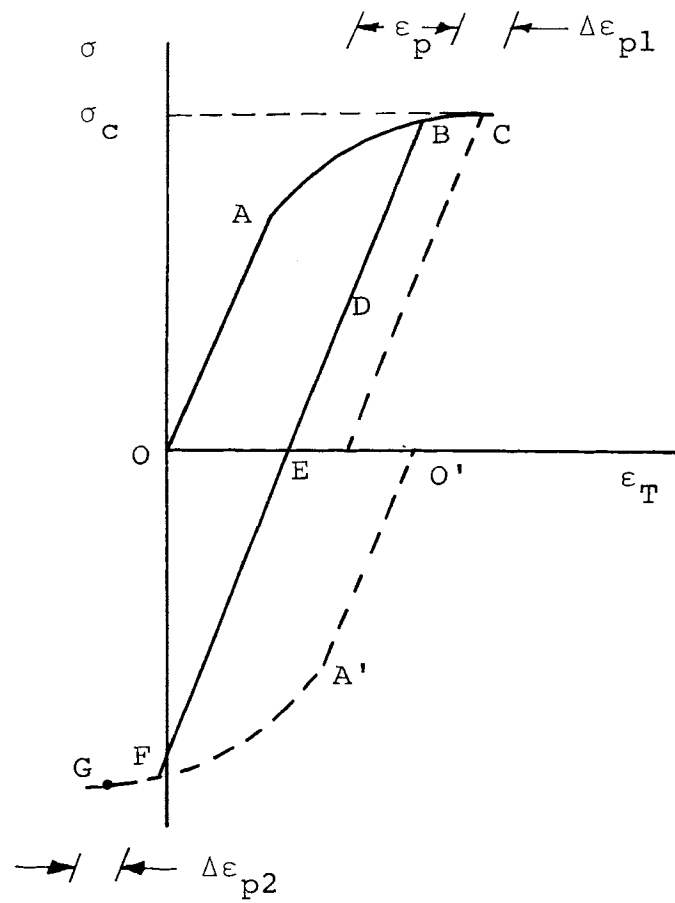
Figure 3. Behavior of materials under uniaxial stress.
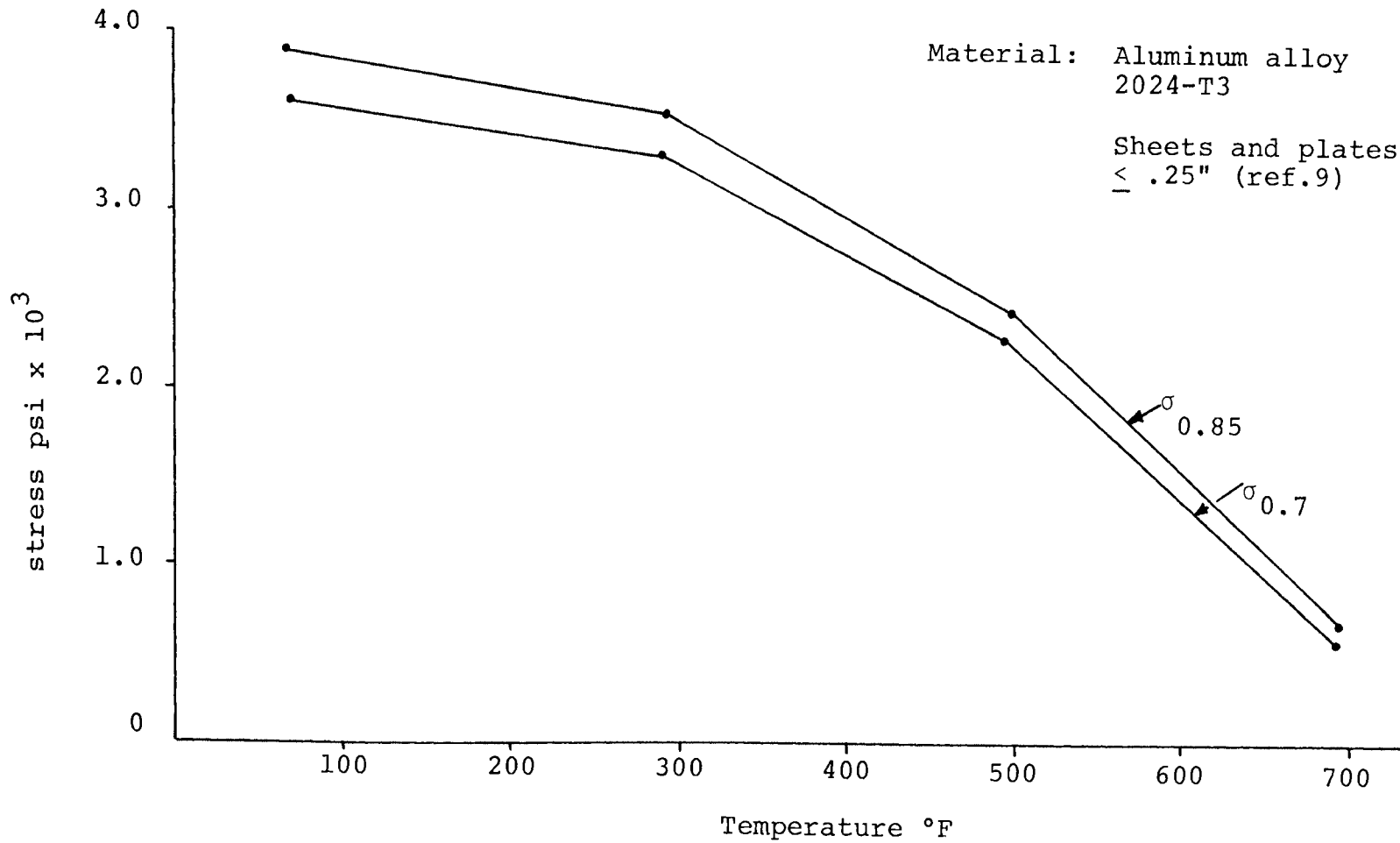
Figure 4. Variation of mechanical properties with changes in temperature.
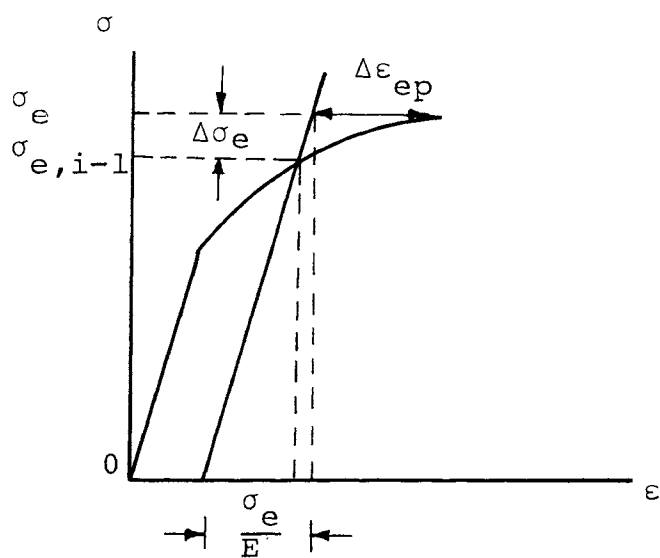
Figure 5. Relation between ε'_{et} , σ_e , and $\Delta\varepsilon_{ep}$.
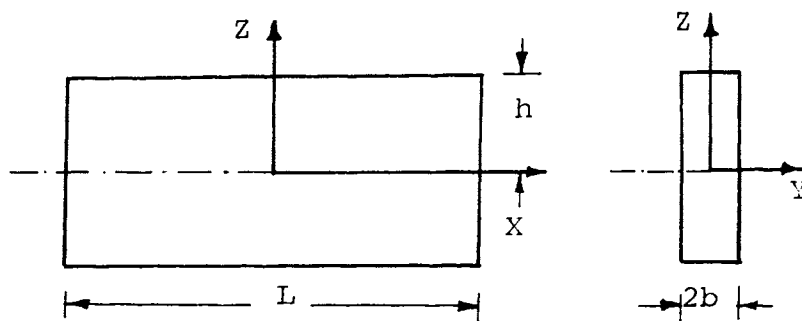
Figure 6. Beam with rectangular section.
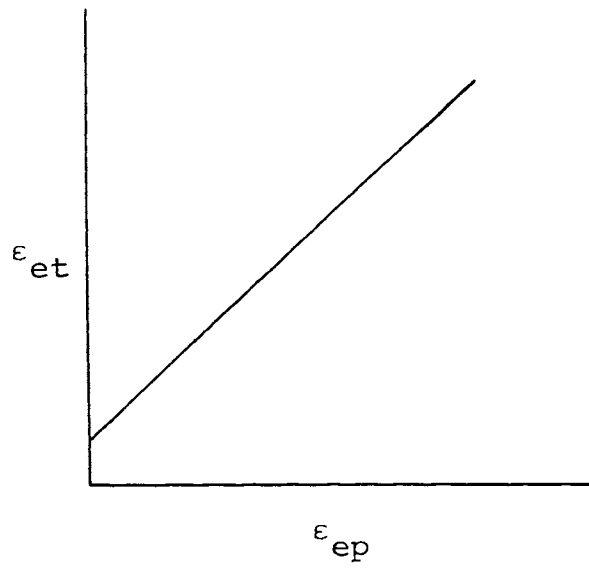
Figure 7. ε_{et} versus ε_{ep} curve.
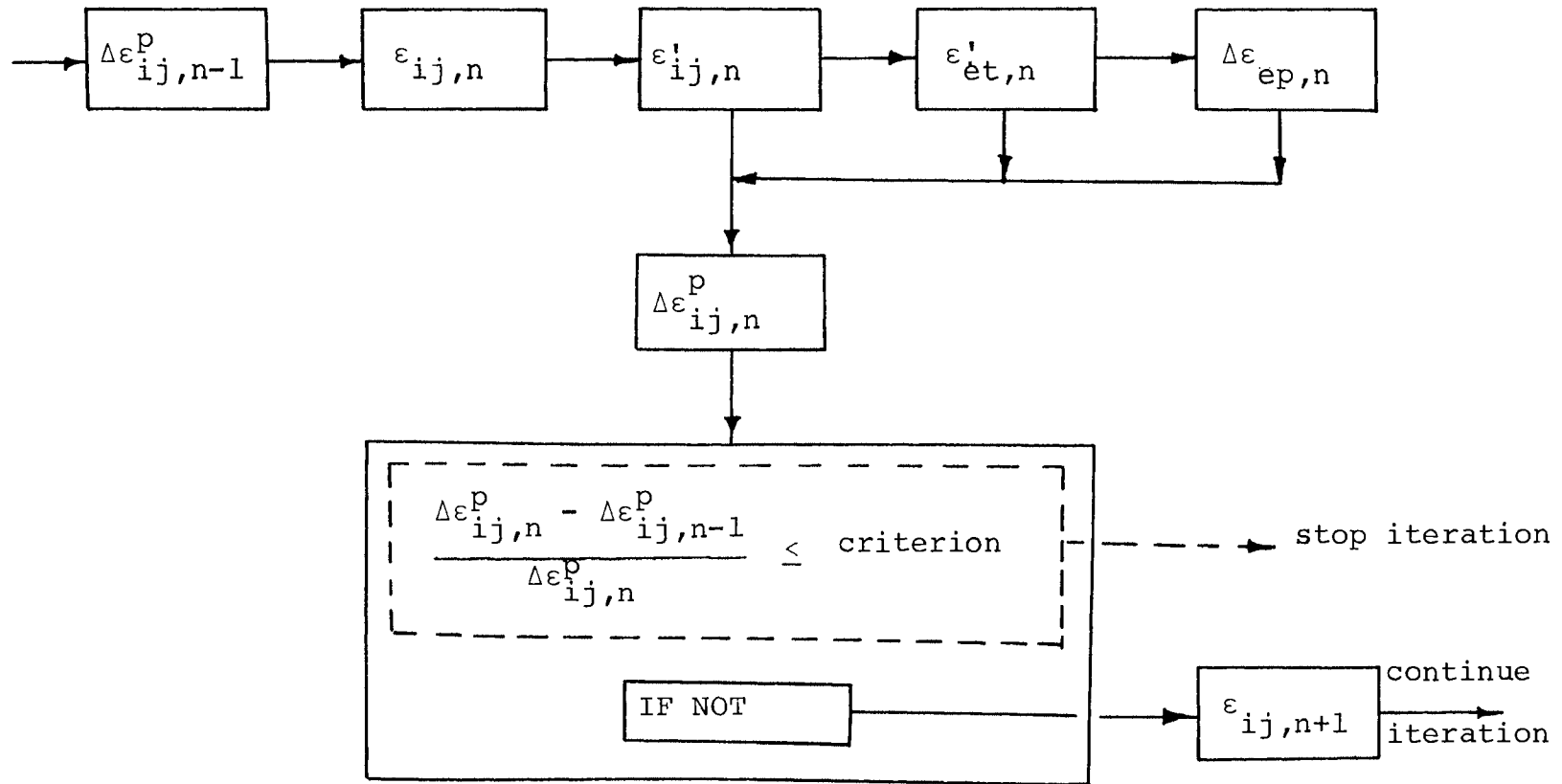
Figure 8. Flow diagram for a rapidly convergent successive approximation scheme.
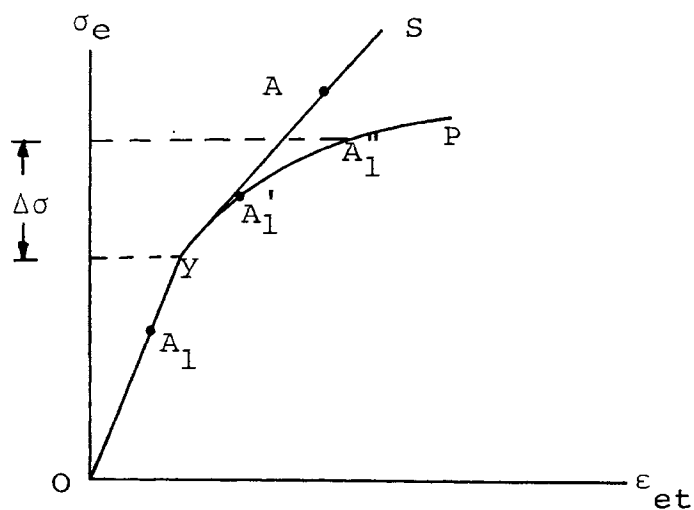
(a) Effect of size of load increment
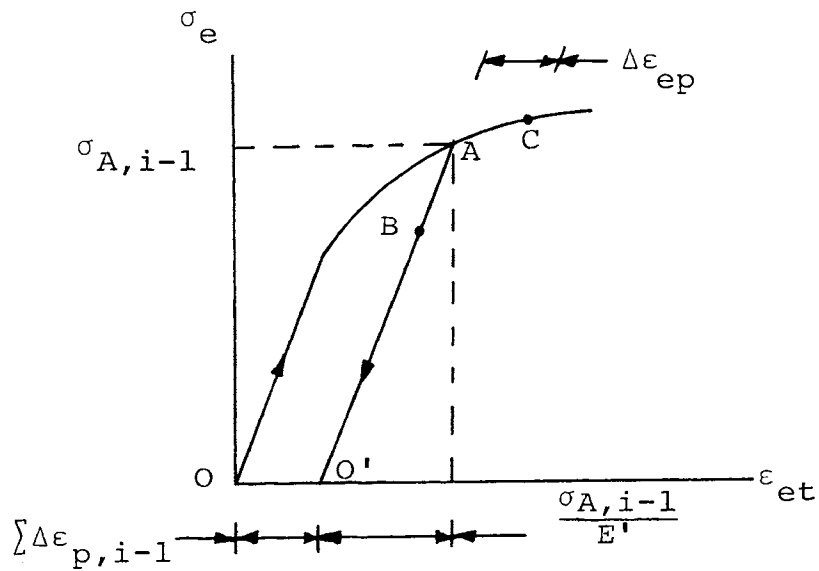
(b) Limit stress $\sigma_{A,i-1}$ for the next load increment

Figure 9. Determination of magnitude of plastic strain increment $\Delta\varepsilon_{ep}$.
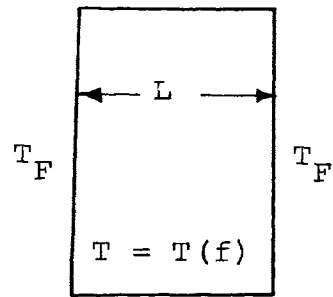
Figure 10. Model for one dimensional transient
temperature distribution (faces and
ends are insulated).
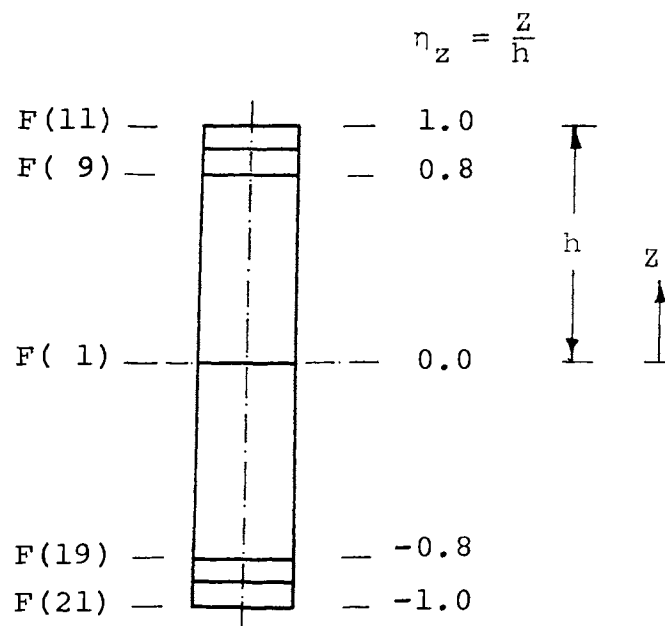
Figure 11. Model of the rectangular section with
21 stations.
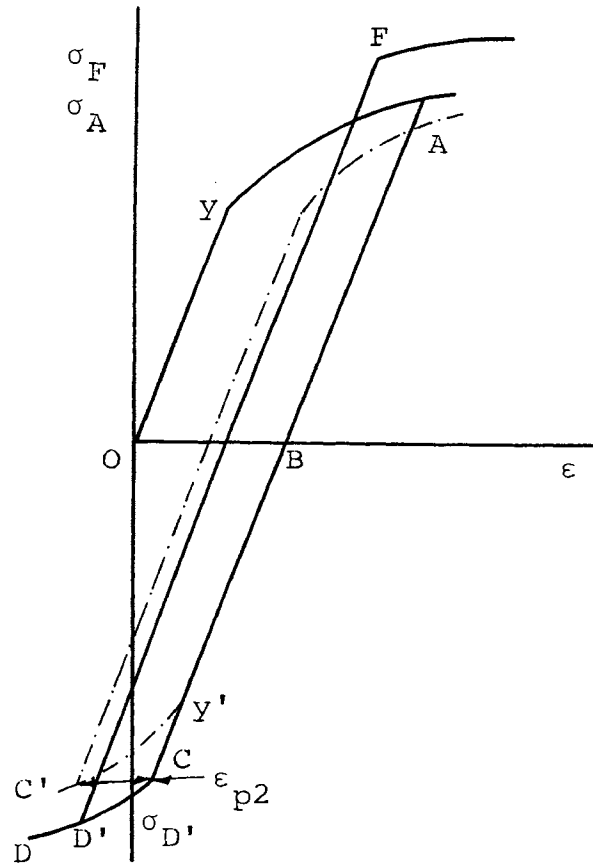
—·—·—·— path which may, presumably be traveled if
the Bauschinger effect is considered

Figure 12. Effect of isotropic strain hardening.
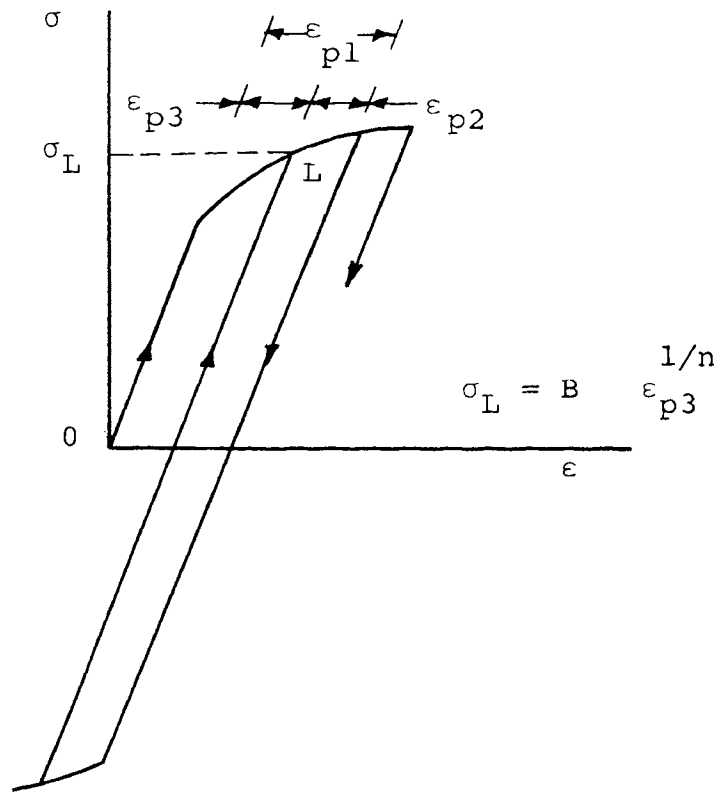
Figure 13. Limit stress for the next loading cycle;
$\sigma_L = B \; \varepsilon_{p3}^{1/n}$.

$$\frac{\sigma_{1,t}}{\sigma_{1,c}} \ = \ K \ = \ \text{constant}$$

$$\frac{\sigma_{2,t}}{\sigma_{2,c}} \ = \ K$$

$$\sigma_B \ = \ B \ \varepsilon_p^{1/n}$$

Figure 14. The simplified stress-strain curve for cyclic
           loading.
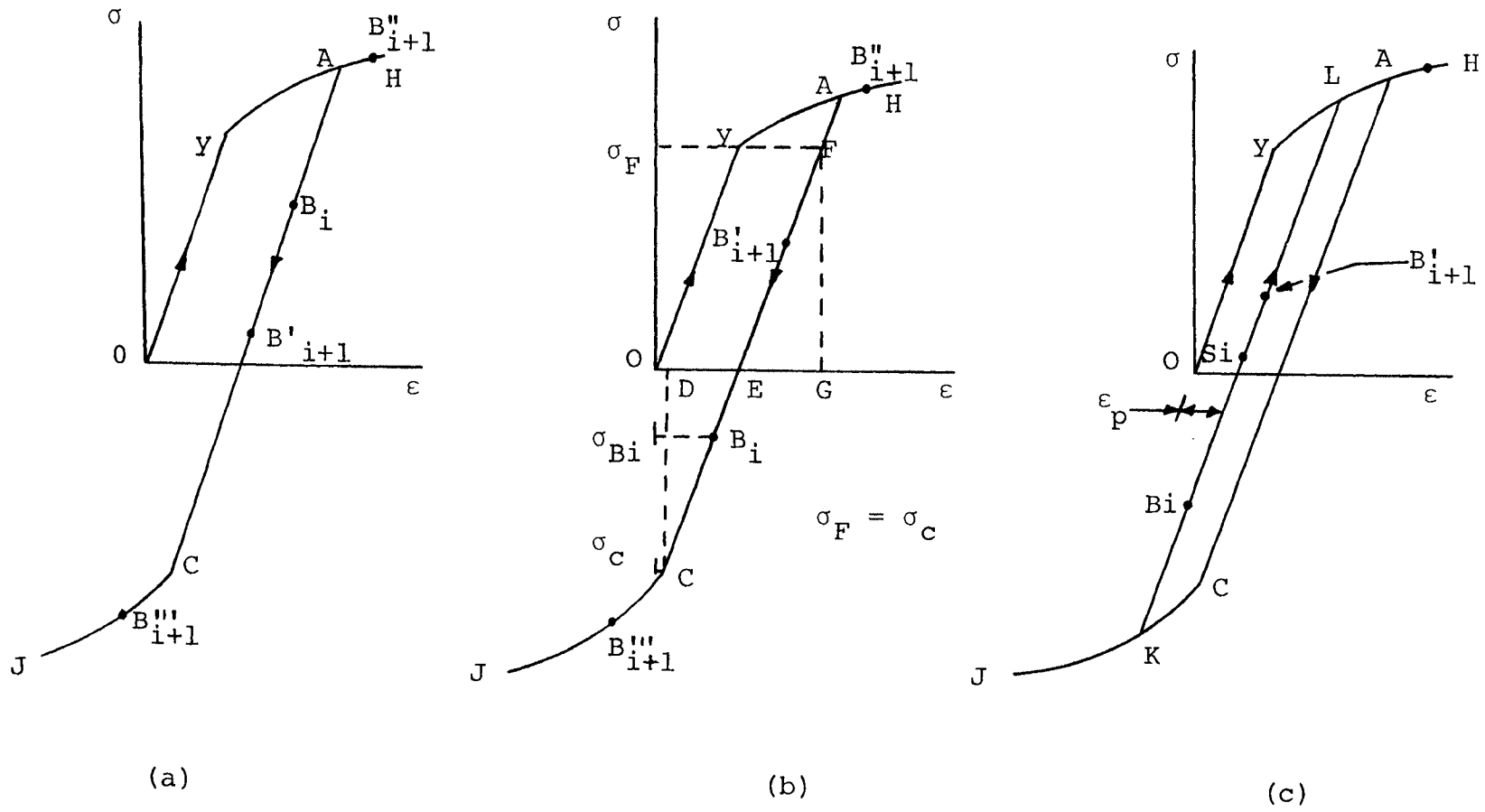
Figure 15. The size of the load increment and possible limit stresses.
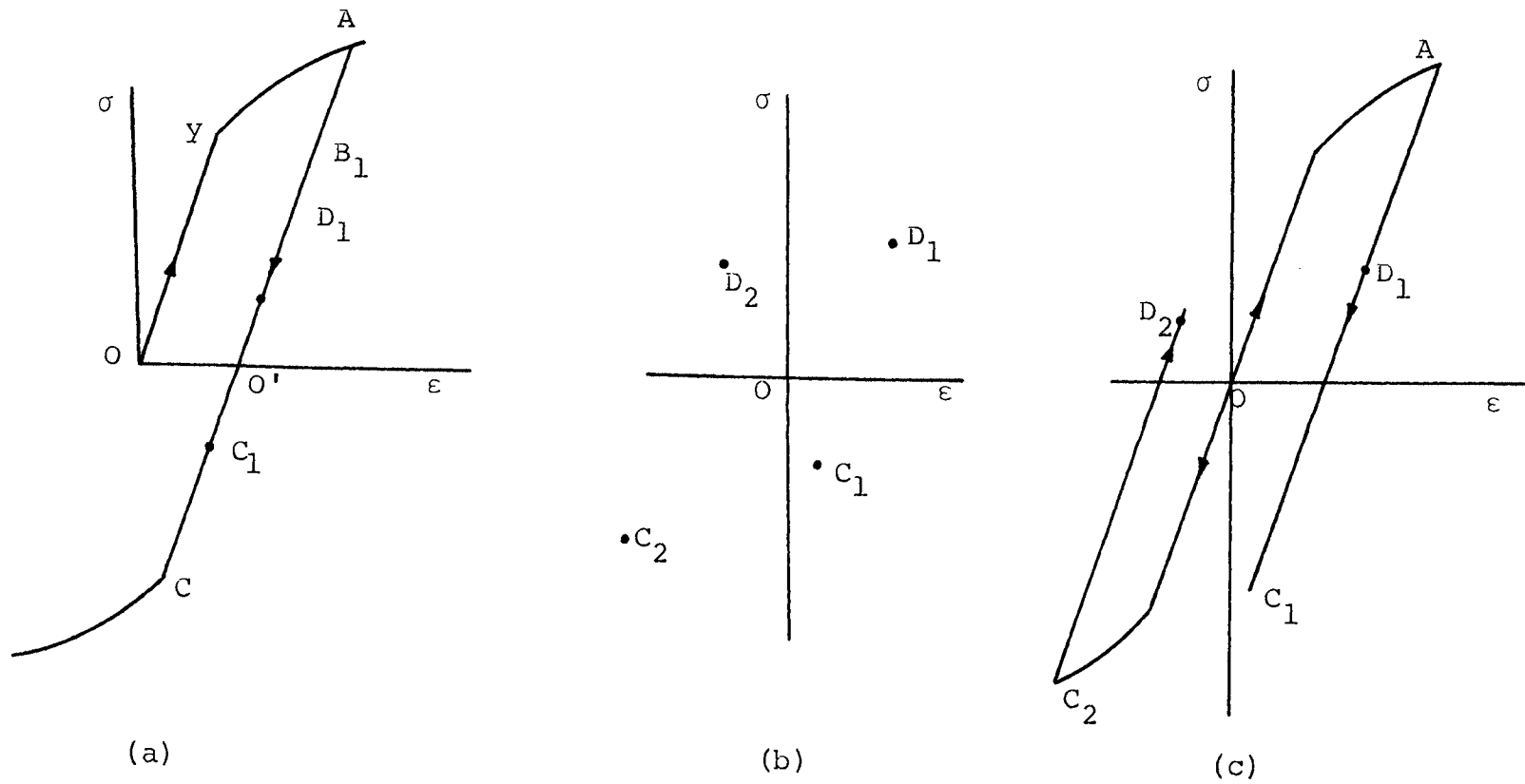
Figure 16. Importance of direction of plastic flow.
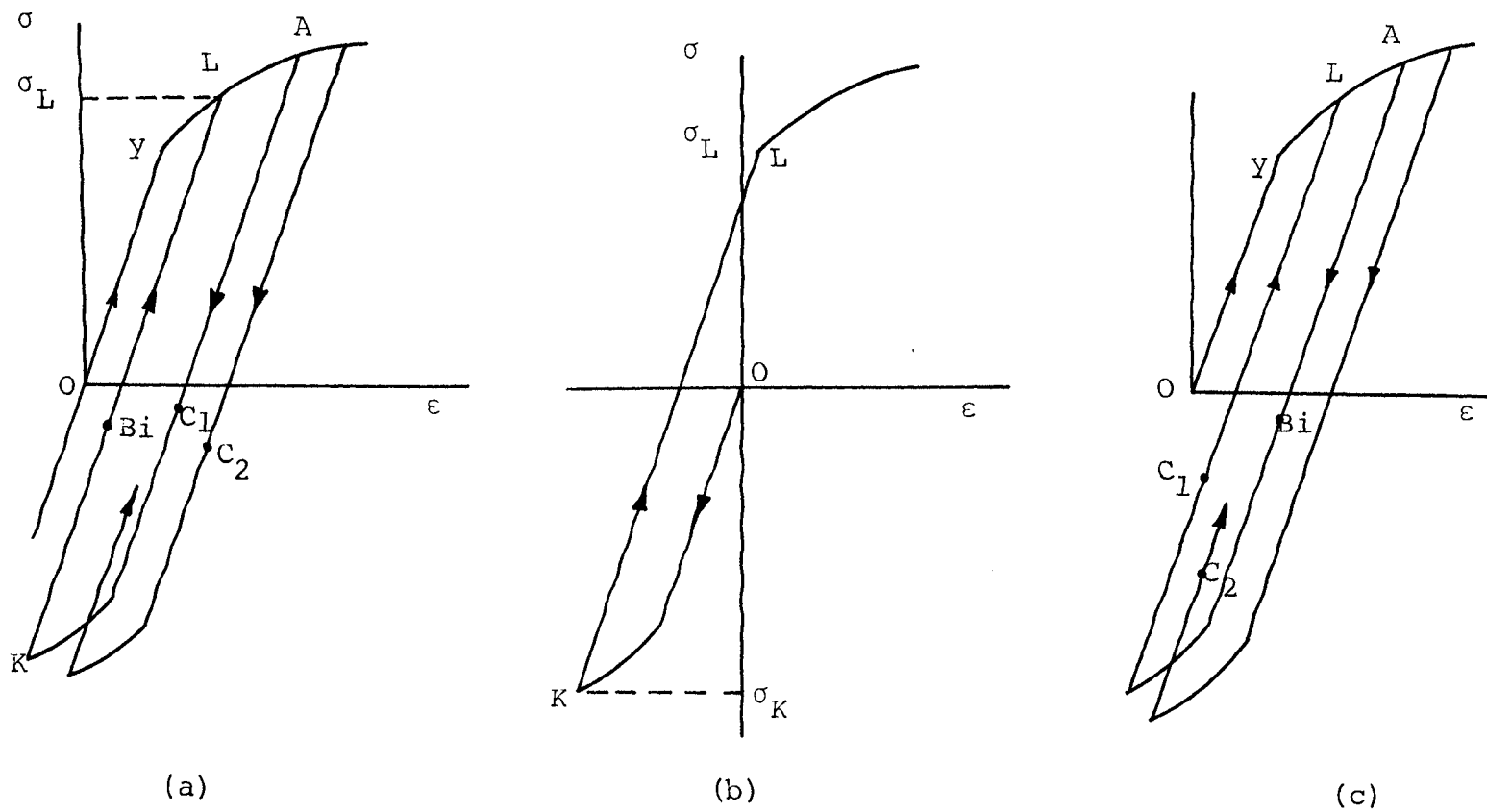
Figure 17. Determination of a limit stress for the next load increment.

Figure 18. Preliminary flow diagram
to determine a limit stress
for the next increment of
load.
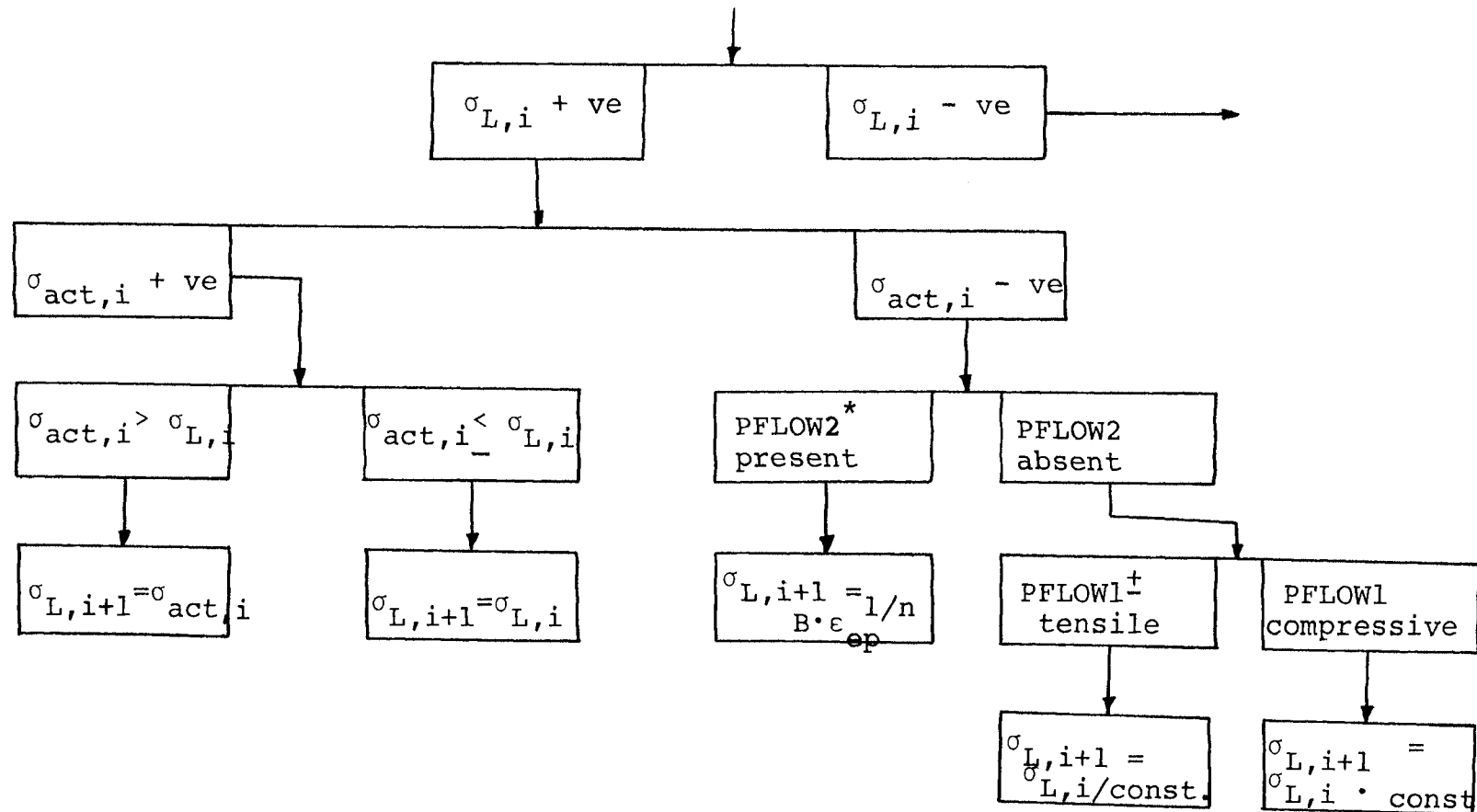
Figure 18

* plastic flow in reversed direction

$\pm$ First plastic flow

$\mp$ Only to explain the basic idea. Division or multiplication by K_f has not been included. Also, effect of change in the direction of the first plastic flow has not been included. See Article XI for futher details.
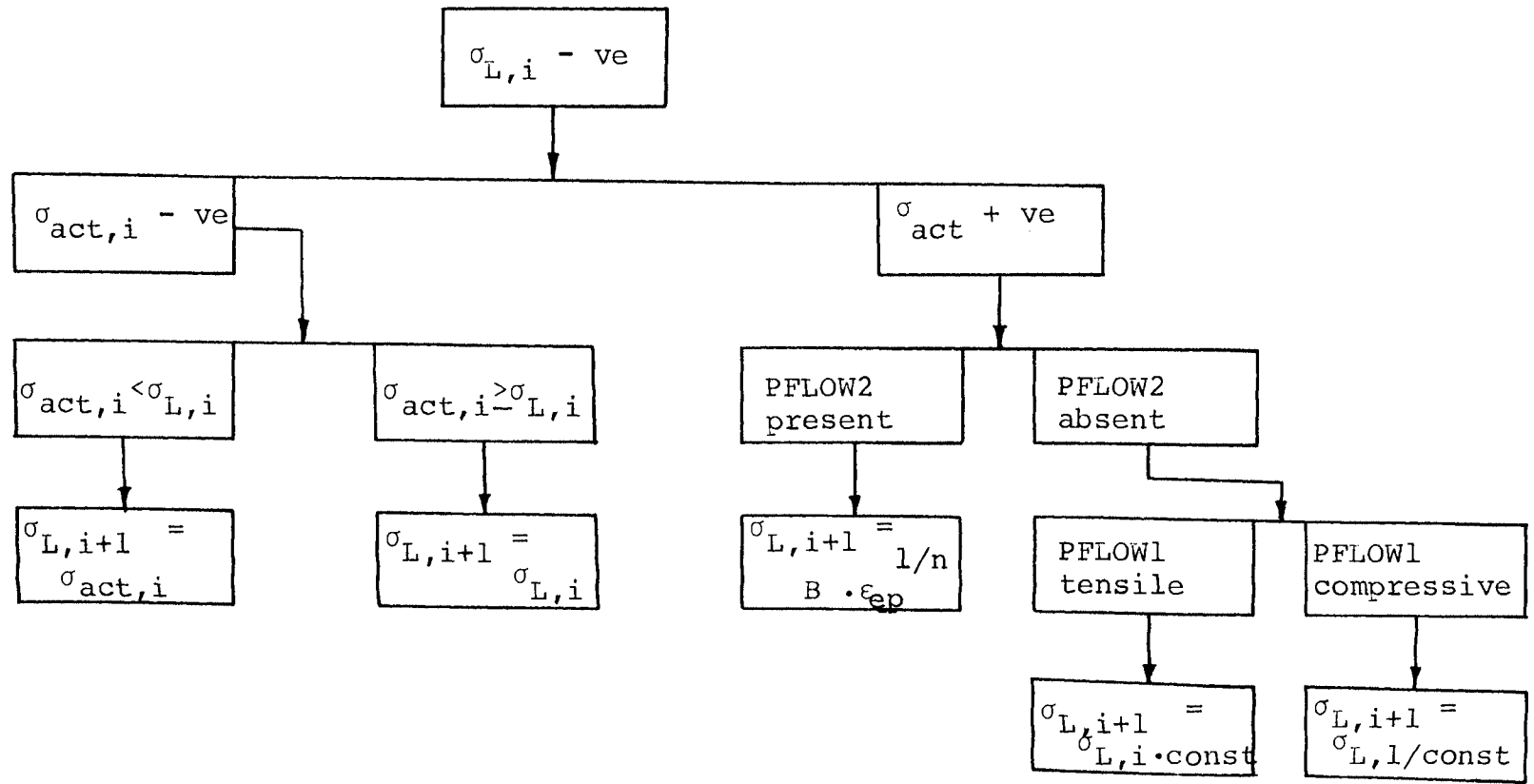
Figure 18

Figure 19. Structural behavior at different stations

of the section due to cyclic thermal

load.

Total cycles: 3
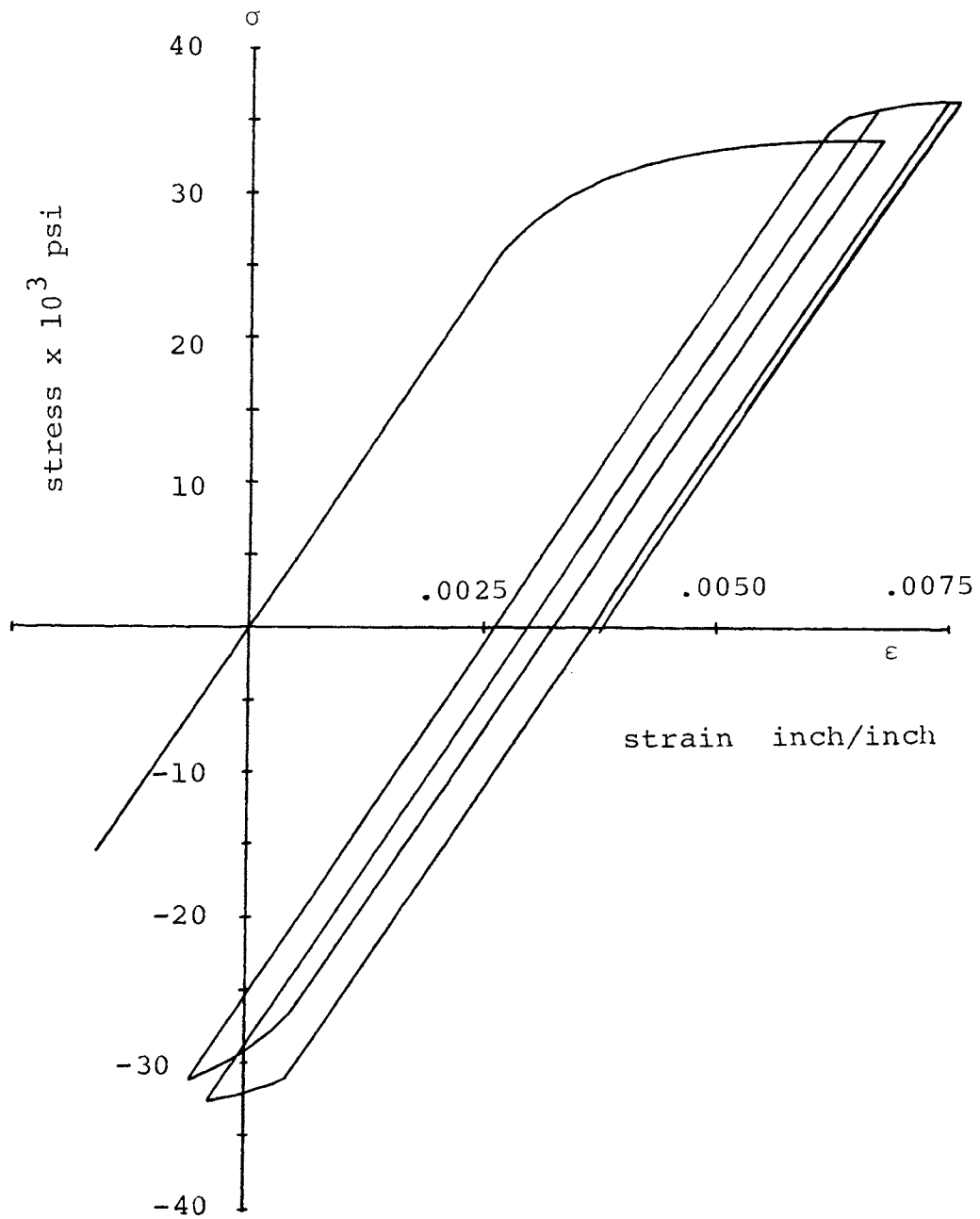
Figure 19

(a) Structural behavior at station (11).
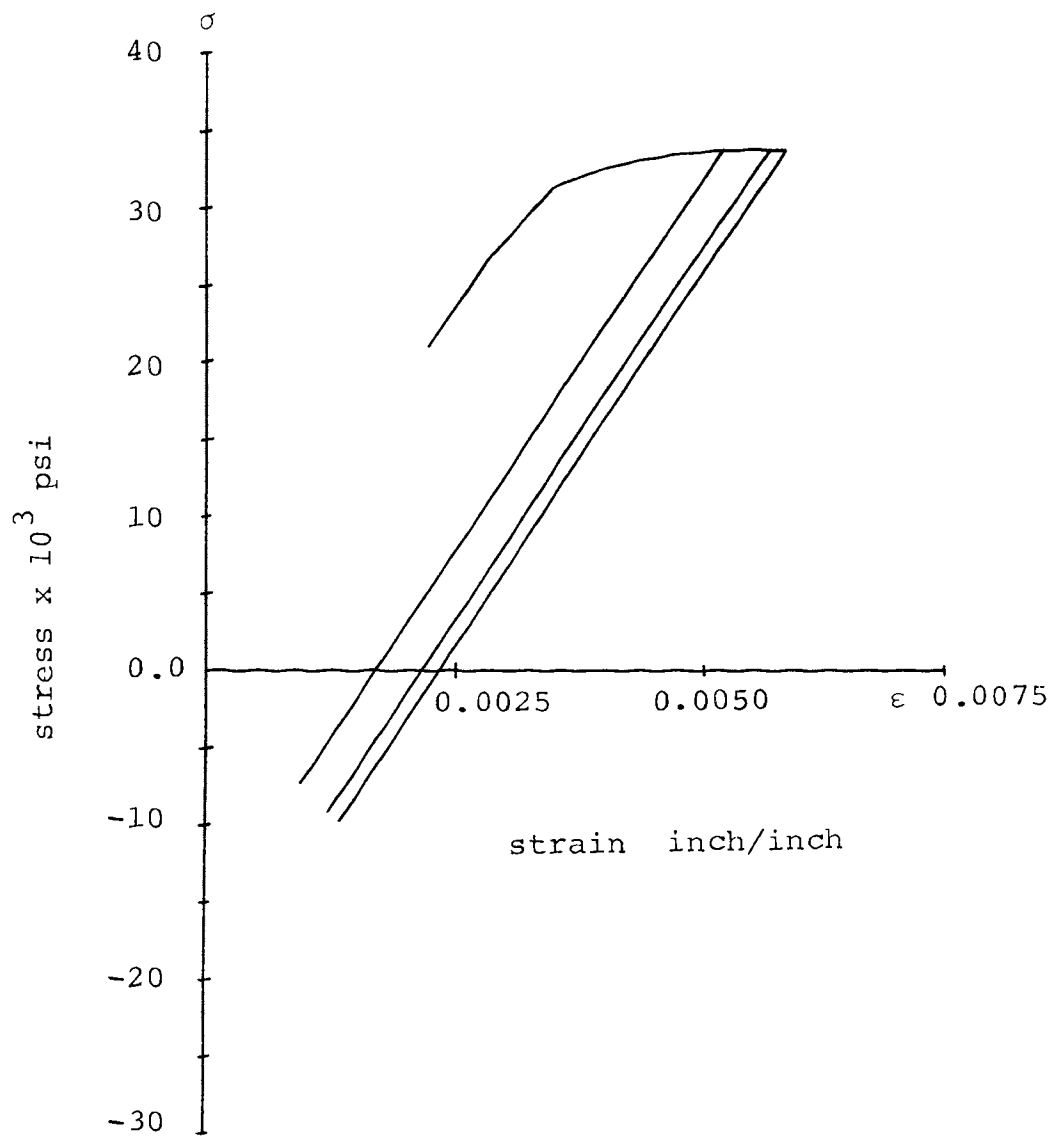
Figure 19

(b) Structural behavior at station (10).
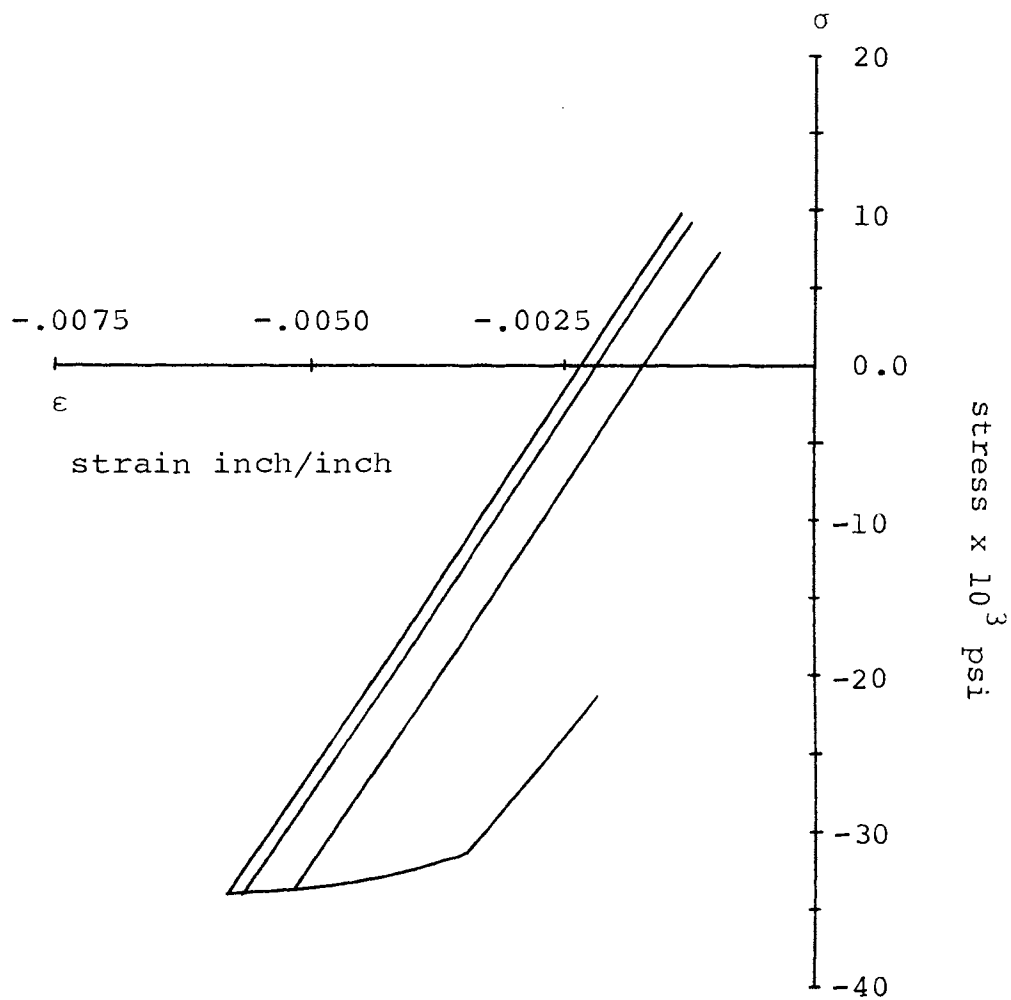
Figure 19

(c) Structural behavior at station (20).
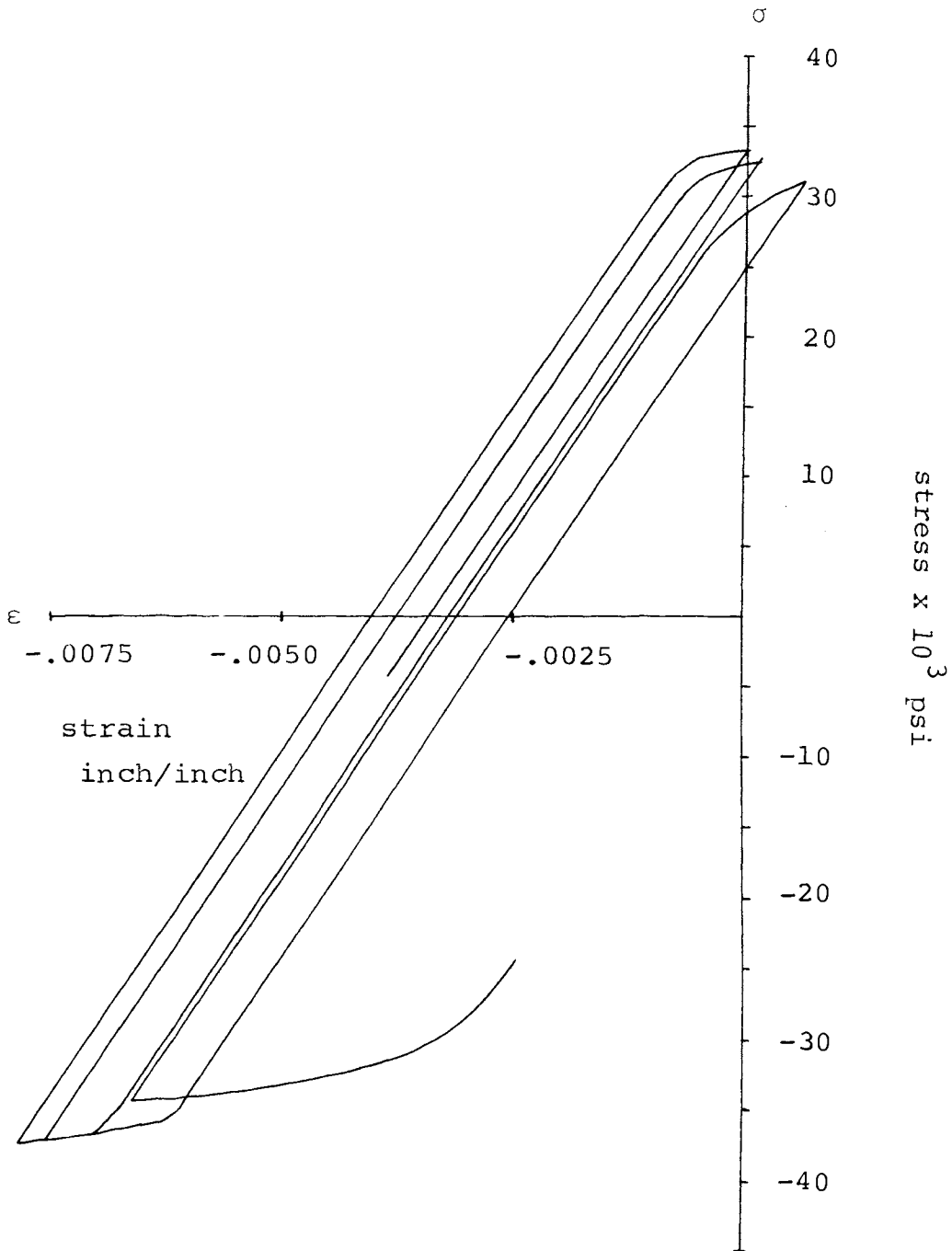
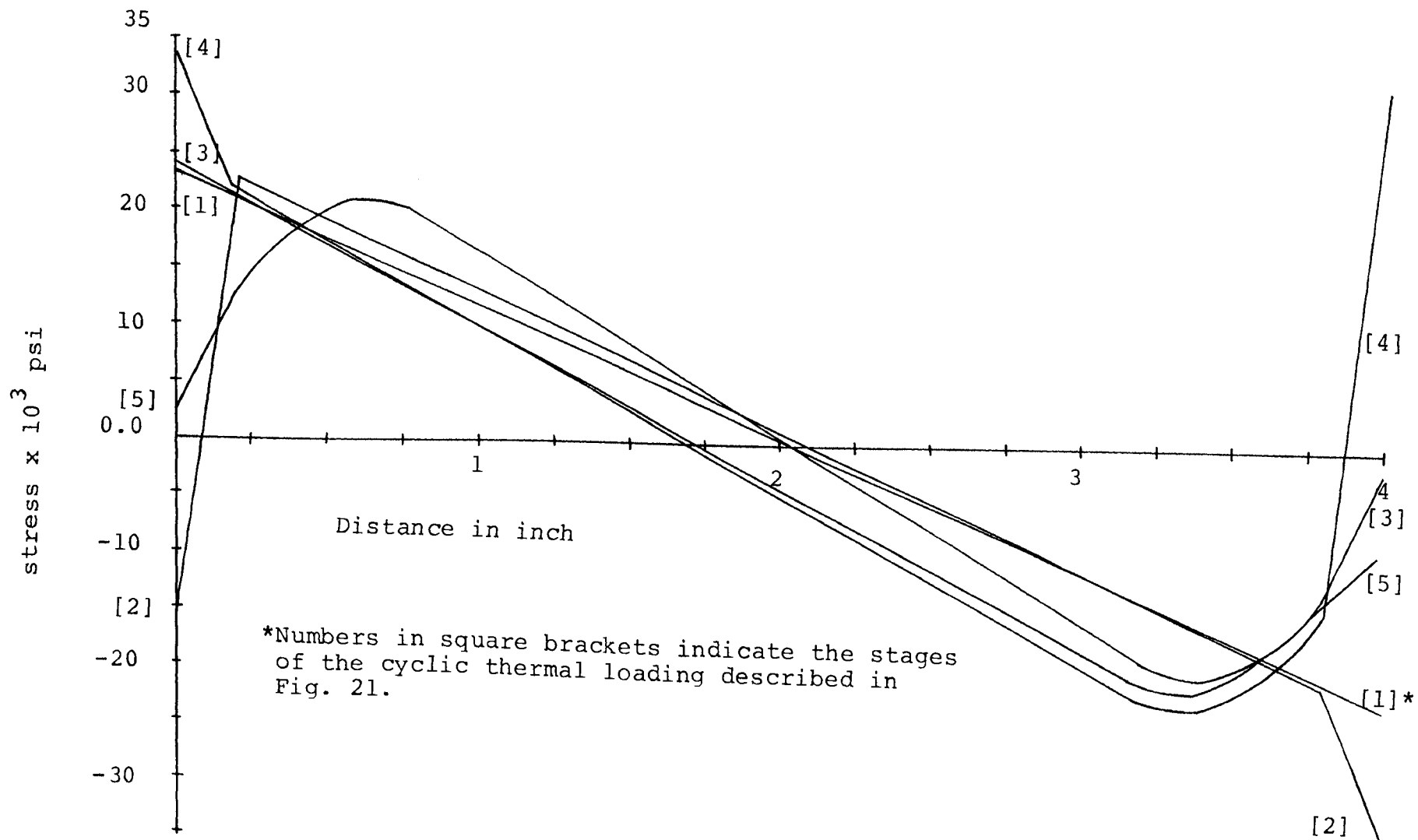Figure 19

(d) Structural behavior at station (21).

Figure 20. Stress distribution in a rectangular section due to cyclic thermal loading.

*Numbers in square brackets indicate the stages of the cyclic thermal loading described in Fig. 21.

Figure 21. Stresses in an I section due to
cyclic thermal load.

A_s :   Area of the flange of an I section

A_w :   Area of the web of an I section

σ_y :   Yield stress

$$\sigma_o = \sigma_y \frac{(A_s - \frac{A_w}{2})}{\frac{(A_s + A_w)}{6}}$$
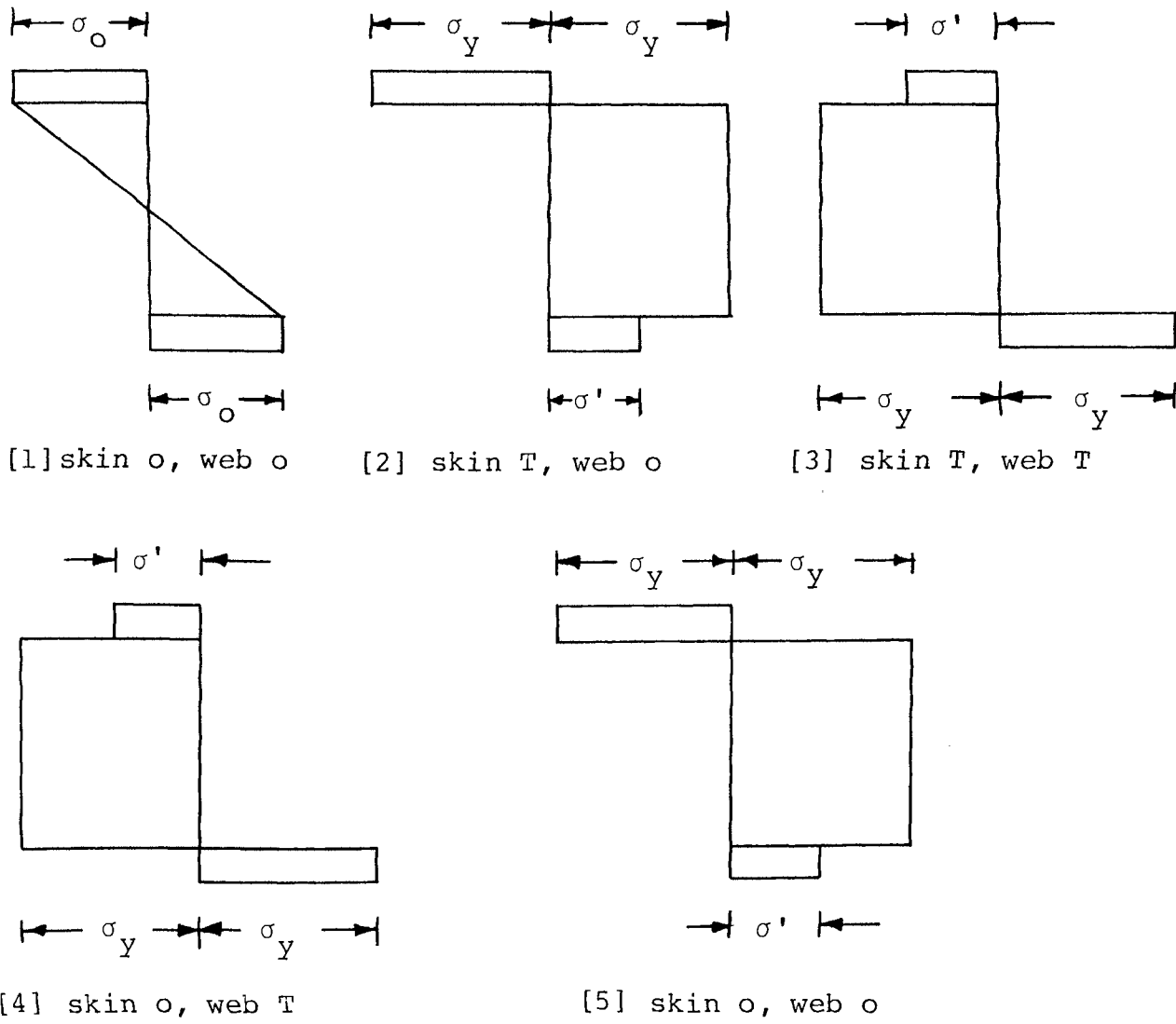
$$\sigma' = \sigma_y \frac{(A_s - A_w)}{A_s}$$

$$\sigma_o$$

$$\sigma_o$$

[1] skin o, web o

$$\sigma_y \quad \sigma_y$$

$$\sigma'$$

[2] skin T, web o

$$\sigma'$$

$$\sigma_y \quad \sigma_y$$

[3] skin T, web T

$$\sigma'$$

$$\sigma_y \quad \sigma_y$$

[4] skin o, web T

$$\sigma_y \quad \sigma_y$$
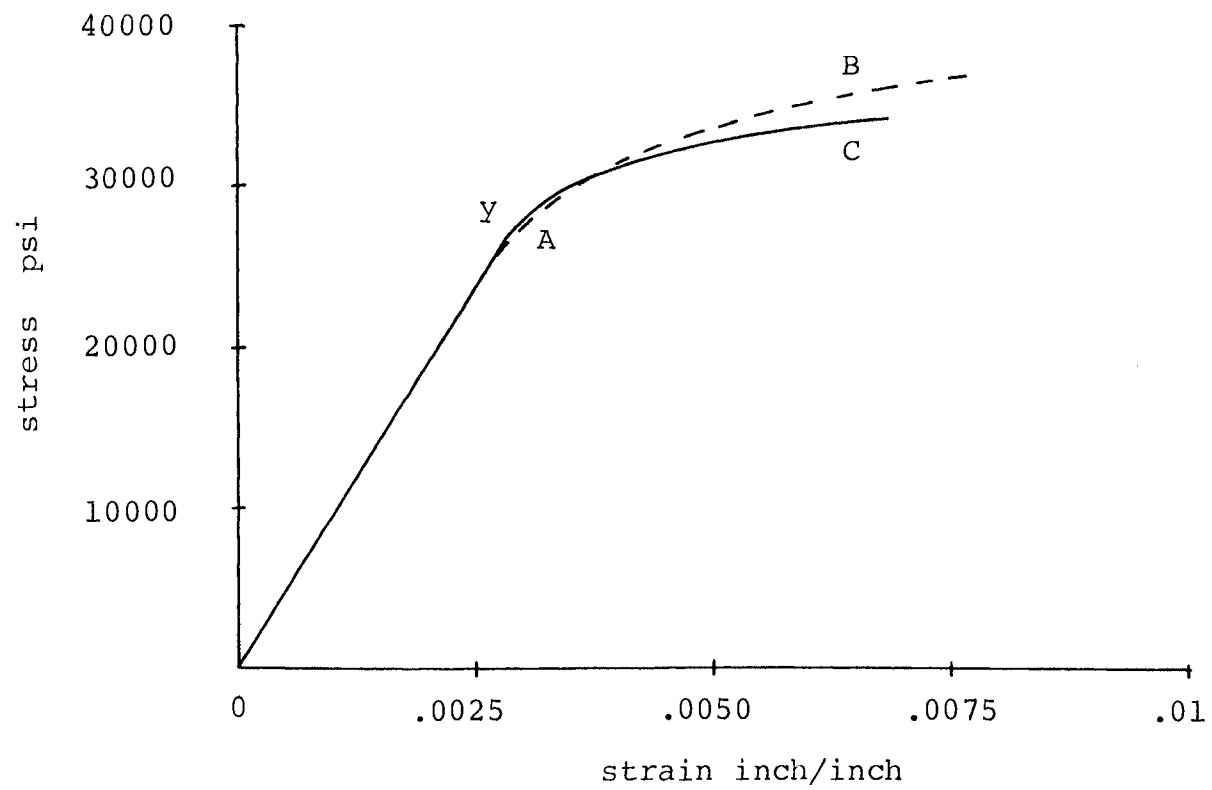
$$\sigma'$$

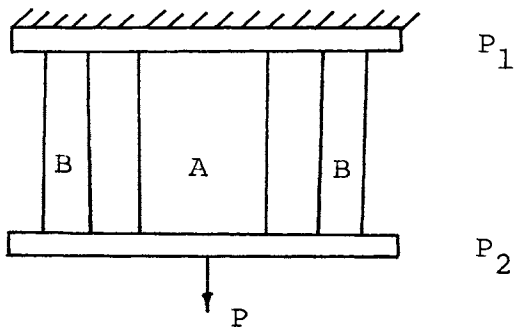[5] skin o, web o

Figure 21

Figure 22. Error in the computation procedure.

Figure 23.   Two bar model.

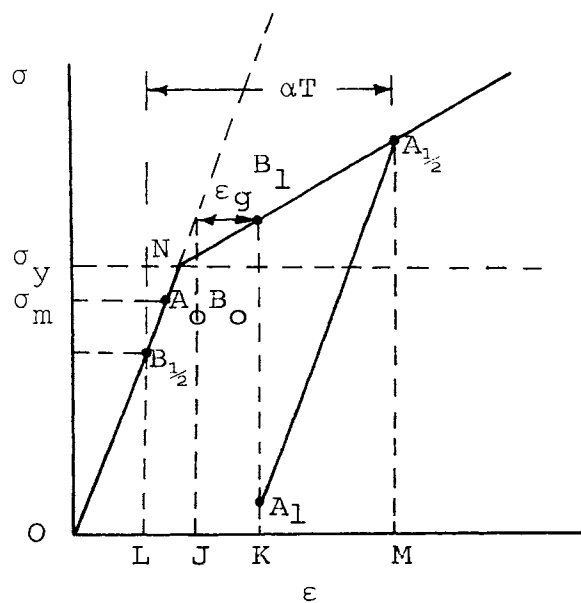Figure 24.   State of stress and strain in bars
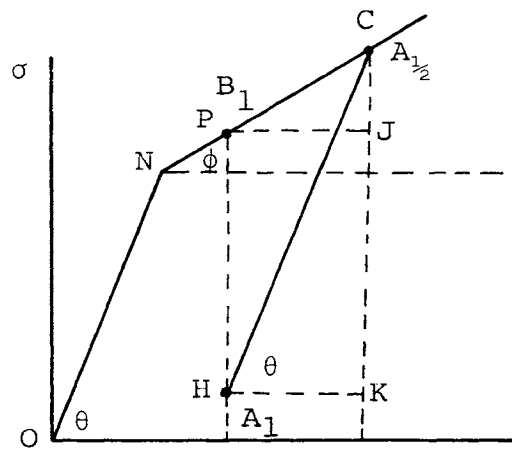A and B at the end of the first
cycle.

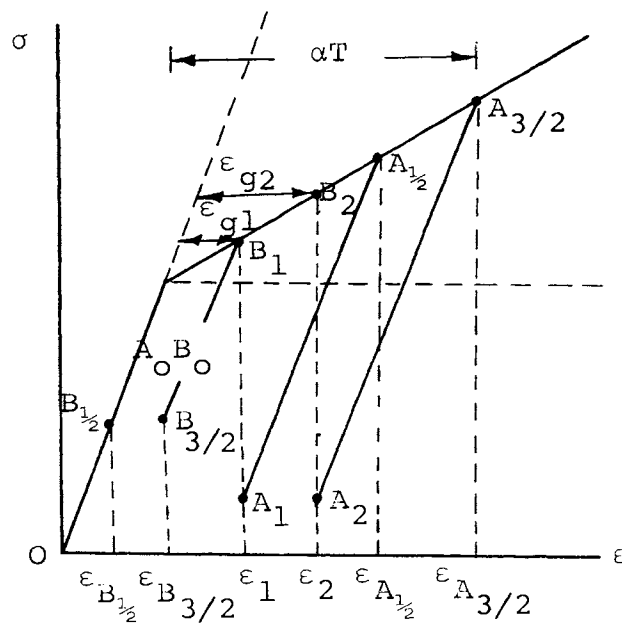Figure 25. Relation between stresses $\sigma_{A_{\frac{1}{2}}}$ and σ_{B_1} .

Figure 26. Growth of plastic strain in a two bar model.

XV . BIBLIOGRAPHY

(1) Parkes, E.W., "A Design Philosophy for Repeated
      Thermal Loading," Agardograph No. 213 (1958),
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XVI. VITA

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