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AN ANALYSIS OF DIRAC DELTA GENERATED SPHERICAL WAVES IN AN INFINITE MAXWELL MEDIUM

by

William Francis Breig, 1940

A Thesis
Submitted to the Faculty of

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ABSTRACT

It is an accepted fact that the simple Maxwell and Voigt models do not usually represent the behavior of real materials. In order to make the results of a model more realisitic, other combinations of springs and dashpots must be considered. To understand the more complicated models, it is desirable to have a knowledge of the Maxwell model since this element usually occurs either in series or in parallel in the advanced models.

This investigation reports solutions of the spherical wave equation in both the elastic and viscoelastic media. Laplace transform techniques are used to obtain the parameters: stress, velocity, and acceleration for the Maxwell solid and velocity, acceleration, displacement, stress, and strain for the elastic solid. The delta pressure pulse was chosen because of its simple transform (unity) and because the solution for any other pressure pulse can be obtained by convolution. Simpson integration was performed to obtain the numerical data.

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LIST OF SYMBOLS

$u(r,t)$	displacement in the radial direction
r	radial direction
ρ	density of the medium
λ	Lame's constant (elastic)
μ	elastic shear modulus
$\phi(r,t)$	displacement potential ($u = \frac{\partial \phi}{\partial r}$)
c	dilatational wave velocity ($c = (\frac{\lambda + 2\mu}{\rho})^{1/2}$)
$\sigma(r,t)$	normal radial stress
r_o	cavity radius
P_o	constant
R	$(r-r_o)/c$
E	elastic spring constant (Young's modulus)
η	coefficient of viscosity
ω_o	E/η
γ	Poisson's ratio (constant)
$I_o(z)$	zero order modified Bessel function of first kind
$I_1(z)$	first order modified Bessel function of first kind
$\delta(t)$	Dirac delta function
$H(t)$	unit step function
$v(r,t)$	particle velocity
$\varepsilon(r,t)$	strain
$a(r,t)$	acceleration

$\doteqdot$	Van der Pol's symbol
s	Laplace transform variable
t	time
$\bar{}$	transformed expresssion for the quantity over which the bar appears

CHAPTER I

INTRODUCTION

The problem of stress-wave propagation from a spherical source has been investigated extensively in the last three decades by such authors as Sharpe (1), Lockett (2), and Blake (3). The early solutions were, in general, either valid only for an elastic medium, or were asymptotic solutions which made contributions only for large times after the pressure pulse was applied. In 1964, Lee (4) extended the spherical analysis in the viscoelastic medium by reporting closed form solutions to the propagation of spherical waves from an internal source and also to the transient response of a viscoelastic medium to a pressure pulse. His work exhibited the damping effect of the medium and the influence at relatively short time intervals and distances. Berry (5) solved for displacement resulting from spherical waves in a Maxwell medium without recourse to a potential function. He assumed a Dirac delta pressure pulse. His method of solution was a Laplace transform technique which involved the same inversion formula (equation 3-1) as used in this paper to find stress, acceleration and velocity. No comparison between results are available.

The simplicity and ease of obtaining the transform solution should not lull the reader into a false feeling that the solution is readily tractable. In general, for even the simplest problems of viscoelasticity, transformed solutions arise for which inverse functions are at present not tabulated in even the most advanced tables. It is possible, however, at least in theory, to apply the inversion integral and by suitably choosing the contour, perform an integration.

This barrier of not being able to find the inverse function can be partly overcome by resorting to numerical techniques. Doetsch (6) discusses a method which involves the selection of a certain parameter. At present, no way has been found to optimize it. As a result, selecting an arbitrary value of the parameter may yield a valid expression, but one which has no immediate value due to the fact that more accuracy will be needed for convergence of the solution than can be carried on conventional computers.

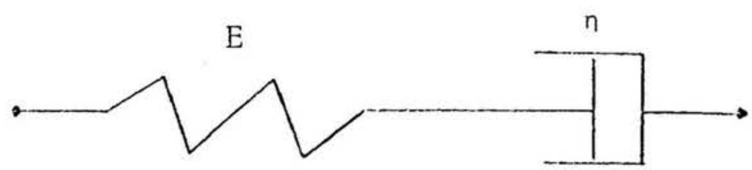
Bellman (7) reduced the limits of the Laplace integral to the finite interval (0,1) and used Legendre polynominals in an attempt to approximate the unknown function. The procedure involved solving a system of equations. As the number of equations increased, it was found that the solution was not stable.

CHAPTER II

FUNDAMENTAL CONSIDERATIONS AND THE MAXWELL MODEL

MAXWELL MODEL

The basic elements in any linear viscoelastic model are elastic springs and viscous dashpots. Different combinations of these elements arranged in such a manner as to produce no degenerate effects (e.g., two springs in series can be replaced by an equivalent spring) constitute the basic models. The proposed Maxwell model consists of an elastic and viscous element in series and will be referred to as a Maxwell element. Schematically, the Maxwell element is represented as



where the nomenclature is

E = modulus of elasticity of the spring

η = viscosity of the dashpot

The stress (σ) across both elements is the same, and the strain (ε) is the sum of the strains of the two elements. The stress-strain relationship of the Maxwell element can be written as

$$E \frac{\partial \varepsilon}{\partial t} = (\frac{E}{\eta} + \frac{\partial}{\partial t})\sigma \tag{2-1}$$

METHOD OF SOLUTION

Problems in linear viscoelasticity can be reduced to solving the same problem by elastic analysis and applying the correspondence principal (Bland -8) to obtain the viscoelastic solution. This procedure is tailored to fit into the field of transform calculus. This paper will make use of the one-sided Laplace transformation defined for an arbitrary piece-wise continuous, exponential order function f(t) as follows

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt = \overline{f}(s) \tag{2-2}$$

Using van der Pol's notation, a symbol $\doteqdot$ which connects the transform plane with the time plane, an operation in the transformed plane can be directly related to the corresponding operation in the time plane. This is best seen by replacing the variable in the transform plane by some constant times the variable (Doetsch - 9). In van der Pol's notation, this would appear as

$$\frac{1}{a} f(\frac{t}{a}) \doteqdot \overline{f}(as) \qquad (a>0) \tag{2-3}$$

UNIT STEP FUNCTION

The unit step function is a discontinuous function which first appears to have been introduced into the literature by Heaviside, although it was known at an earlier time by Cauchy. The value of the unit step function at the point of discontinuity varies depending upon the author. Since the point of discontinuity is irrelevant to the calculation of the Laplace integral, it may be convenient for the function to have the value zero, one, or one-half at this point. Throughout this paper, the unit step function will be defined as

$$H(t-t_o) = \begin{cases} 0 & (t<t_o) \\ \frac{1}{2} & (t=t_o) \\ 1 & (t>t_o) \end{cases} \tag{2-4}$$

The Laplace transform of the unit step function is found to be

$$L\left\{H(t-t_o)\right\} = \int_0^\infty H(t-t_o)e^{-st}dt = \int_{t_o}^\infty e^{-st}dt = \frac{e^{-st_o}}{s} \tag{2-5}$$

DELTA FUNCTION

The Dirac delta function is defined as the derivative of the unit step function. It is apparent that this derivative will be zero everywhere except at the point of discontinuity at which point the derivative is not defined. Approximating the unit step function by a continuous function, it is found that the derivative has an infinite value at the point of discontinuity. (The assumption here is that the limit and derivative operation may be interchanged). It follows that

$$\delta(t-t_o) = \begin{cases} 0 & t \neq t_o \\ \infty & t = t_o \end{cases} \qquad (2\text{-}6)$$

with the additional property

$$\int_{-\infty}^{\infty} \delta(t-t_o)\,dt = 1 \qquad (2\text{-}7)$$

The Laplace transform of the delta function is

$$L\left\{\delta(t-t_o)\right\} = \int_{0}^{\infty} \delta(t-t_o)e^{-st}dt = e^{-st_o}$$

CHAPTER III

THE ANALYSIS OF STRESS, VELOCITY, AND ACCELERATION OF THE SPHERICAL MAXWELL MODEL WITH DELTA PRESSURE PULSE

In solving for the acceleration, velocity, and stress of the Maxwell spherical wave with delta pressure pulse, use is made of the correspondence principle which relates the elastic solution to the viscoelastic solution. This method involves replacing the elastic parameters in the transform plane by their corresponding viscoelastic parameters. In general, difficulty is experienced in trying to invert the viscoelastic solution into the physical plane. The method of series has been successfully used by Clark, Rupert and Jamison (10). Convolution integrals have also been used extensively. It has been found that a particular inversion formula used by Berry (5) is applicable to the form of the Maxwell spherical equation. This formula is

$$e^{-\frac{\alpha t}{2}}\left[f(t) + \frac{\alpha}{2}\int_{0}^{t} f(\sqrt{t^2-\beta^2})\, I_1(\tfrac{\alpha}{2}\beta)\, d\beta\right] \doteqdot \bar{f}(\sqrt{s(s+\alpha)}) \qquad (3\text{-}1)$$

where α is a real non-negative constant, and β is a dummy variable of integration.

In order to find the viscoelastic transform of the displacement potential, substitute equations (B-4), (B-7) and (B-10) into the elastic transform equation (A-21) to arrive at

$$r\phi(r,t) \doteqdot \frac{-P_o r_o}{3\,\frac{\mu s}{s+\omega_o}} \left(\frac{c^2 s}{s+\omega_o}\right) \frac{\exp\left[-(r-r_o)s/\sqrt{\frac{c^2 s}{s+\omega_o}}\right]}{\left[s^2 + \frac{4s}{3r_o}\sqrt{\frac{c^2 s}{s+\omega_o}} + \frac{4c^2}{3r_o^2}\,\frac{s}{(s+\omega_o)}\right]} \tag{3-2}$$

Equation (3-2) can be put in the form

$$r\phi(r,t) \doteqdot \frac{-P_o r_o c^2}{3\mu}\,\frac{s(s+\omega_o)}{s^2}\,\frac{\exp\left(-R\sqrt{s(s+\omega_o)}\right)}{\left[s(s+\omega_o) + c_1\sqrt{s(s+\omega_o)} + c_o\right]} \tag{3-3}$$

where

$$R = \frac{(r-r_o)}{c}$$

$$c_o = \frac{4c^2}{3r_o^2}$$

$$c_1 = \frac{4c}{3r_o}$$

ACCELERATION

Multiplying the transform by s^2 has the effect of taking the second derivative in the time plane (Doetsch -11). The result is a function of $\sqrt{s(s+\omega_o)}$:

$$r\frac{\partial \phi}{\partial t^2}(r,t) \doteqdot \frac{-P_o r_o c^2}{3\mu} f(\sqrt{s(s+\omega_o)}) \tag{3-4}$$

where

$$f(\sqrt{s(s+\omega_o)}) = \frac{(\sqrt{s(s+\omega_o)})^2 \exp(-R\sqrt{s(s+\omega_o)})}{\left[(\sqrt{s(s+\omega_o)})^2 + c_1\sqrt{s(s+\omega_o)} + c_o\right]} \tag{3-5}$$

From equation (3-5)

$$f(t) \doteqdot \frac{s^2 e^{-Rs}}{s^2 + c_1 s + c_o} = e^{-Rs}\left[1 - \frac{c_1 s + c_o}{s^2 + c_1 s + c_o}\right] \tag{3-6}$$

Equation (3-6) can be inverted (Erdelyi -12) to give

$$f(t) = \delta(t-R) - H(t-R)e^{-\frac{c_1}{2}(t-R)}\left[c_1 \cos D(t-R) + A \sin D(t-R)\right] \tag{3-7}$$

where

$$D = \sqrt{c_o - \frac{c_1^2}{4}}$$

$$A = \frac{c_o - \frac{c_1^2}{2}}{D}$$

By the use of equation (3-1) with $\alpha = \omega_o$

$$r\frac{\partial^2\phi}{\partial t^2}(r,t) = \frac{-P_o r_o c^2}{3\mu} e^{-\frac{\omega_o t}{2}} \left[\delta(t-R) - \left\{ e^{-\frac{c_1}{2}(t-R)} \right.\right.$$

$$\left. \times \left[c_1 \cos D(t-R) + A \sin D(t-R) \right] \right\} H(t-R)$$

$$+ \frac{\omega_o}{2} \int_0^t \left\{ \delta(\sqrt{t^2-\beta^2}-R) - e^{-\frac{c_1}{2}(\sqrt{t^2-\beta^2}-R)} H(\sqrt{t^2-\beta^2}-R) \right.$$

$$\left.\left. \times \left[c_1 \cos D(\sqrt{t^2-\beta^2}-R) + A \sin D(\sqrt{t^2-\beta^2}-R) \right] \right\} I_1(\frac{\omega_o \beta}{2}) \, d\beta \right] \qquad (3\text{-}8)$$

From appendix (D), $\delta(\sqrt{t^2-\beta^2}-R)$ and $H(\sqrt{t^2-\beta^2}-R)$ can be linearized and the sifting property of the delta function can be applied to the integral in equation (3-8) to obtain $(t>R)$

$$r\frac{\partial^2\phi}{\partial t^2}(r,t) = \frac{-P_o r_o c^2}{3\mu} e^{-\frac{\omega_o t}{2}} \left\{ \frac{\frac{\omega_o}{2} R I_1(\frac{\omega_o}{2}\sqrt{t^2-R^2})}{\sqrt{t^2-R^2}} \right.$$

$$- e^{-\frac{c_1}{2}(t-R)} \left[c_1 \cos D(t-R) + A \sin D(t-R) \right]$$

$$- \frac{\omega_o}{2} \int_R^t e^{-\frac{c_1}{2}(z-R)} \left[c_1 \cos D(z-R) \right.$$

$$\left.\left. + A \sin D(z-R) \right] \frac{z I_1(\frac{\omega_o}{2}\sqrt{t^2-z^2})}{\sqrt{t^2-z^2}} dz \right\} \qquad (3\text{-}9)$$

The acceleration can be obtained from equation (3-9) by noting that

$$\frac{\partial}{\partial r}\left[\frac{\partial^2}{\partial t^2}\phi(r,t)\right] = \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial r}\phi(r,t)\right) = \frac{\partial^2}{\partial t^2}u(r,t) = a(r,t) \qquad (3\text{-}10)$$

The application of $\frac{\partial}{\partial r}$ to equation (3-9) by Leibniz's rule (Churchill -13) yields upon collecting terms $(t>R)$

$$a(r,t) = \frac{-P_o r_o c^2}{3\mu r^2} H(t-R)\, e^{-\frac{\omega_o t}{2}} \Bigg\{ e^{-\frac{c_1}{2}} \Big[(A-rm)\sin D(t-R) + (c_1-rN)\cos D(t-R)\Big] + \frac{\omega_o}{2}\int_R^t e^{-\frac{c_1}{2}(z-R)} \Big[(c_1-rN)\cos D(z-R) + (A-rm)\sin D(z-R)\Big] \frac{z I_1\left(\frac{\omega_o}{2}\sqrt{t^2-z^2}\right)}{\sqrt{t^2-z^2}}\,dz + \frac{\omega_o I_1\left(\frac{\omega_o}{2}\sqrt{t^2-R^2}\right)}{\sqrt{t^2-R^2}} \times \left[\frac{R^2 r}{c(t^2-R^2)} + \frac{r}{2c}(1+c_1 R) - \frac{R}{2}\right] - \frac{\frac{\omega_o^2 r R^2}{2} I_o\left(\frac{\omega_o}{2}\sqrt{t^2-R^2}\right)}{4c(t^2-R^2)} \Bigg\} \qquad (3\text{-}11)$$

where

$$M = \frac{Ac_1}{2c} + \frac{Dc_1}{c}$$

and

$$N = \frac{c_1^2}{2c} - \frac{AD}{c}$$

VELOCITY

From the definition of displacement potential ϕ, equation (A-2), it follows that

$$u(r,t) = \frac{\partial \phi}{\partial r}(r,t) \doteqdot \frac{P_o r_o c^2}{3\mu} \frac{(s+\omega_o)e^{-R\sqrt{s(s+\omega_o)}}}{s\left[s(s+\omega_o) + c_1\sqrt{s(s+\omega_o)} + c_o\right]}$$

$$\times\left[\frac{1}{r^2} + \frac{\sqrt{s(s+\omega_o)}}{cr}\right] \qquad (3\text{-}12)$$

Using the s multiplying property (Doetsch -11) results in

$$v(r,t) \doteqdot \frac{P_o r_o c^2 (s+\omega_o)e^{-R\sqrt{s(s+\omega_o)}}}{3\mu\left[s(s+\omega_o) + c_1\sqrt{s(s+\omega_o)} + c_o\right]}\left[\frac{1}{r^2} + \frac{\sqrt{s(s+\omega_o)}}{cr}\right] \qquad (3\text{-}13)$$

Using the same procedure as in the acceleration, equation (3-13) can be written as

$$v(r,t) \doteqdot \frac{P_o r_o c^2}{3\mu}\left[s+\omega_o\right] f(\sqrt{s(s+\omega_o)}) = \frac{P_o r_o c^2}{3\mu}\left[sf(\sqrt{s(s+\omega_o)})\right.$$

$$\left. + \omega_o f(\sqrt{s(s+\omega_o)})\right] \qquad (3\text{-}14)$$

where

$$f(\sqrt{s(s+\omega_o})) = \frac{\exp(-R\sqrt{s(s+\omega_o)})}{\left[(\sqrt{s(s+\omega_o)})^2 + c_1\ s(s+\omega_o) + c_o\right]} \times \left[\frac{1}{r^2} + \frac{\sqrt{s(s+\omega_o)}}{cr}\right] \qquad (3\text{-}15)$$

From equation (3-1), and the properties of the delta function of appendix (D), equation (3-15) becomes

$$f(\sqrt{s(s+\omega_o)}) \doteqdot \frac{e^{-\frac{\omega_o t}{2}}}{cr^3} \left\{ H(t-R)\left[r^2 \cos M(t-R) + P \sin M(t-r)\right] e^{-N(t-R)} + \frac{\omega_o}{2} \int_R^t e^{-N(z-R)} \left[r^2 \cos M(z-R) + P \sin M(z-R)\right] \times \frac{z I_1(\frac{\omega_o}{2}\sqrt{t^2-z^2})}{\sqrt{t^2-z^2}}\, dz \right\} \qquad (3\text{-}16)$$

where

$$M = \sqrt{c_o - \frac{c_1^2}{4}}$$

$$N = \frac{c_1}{2}$$

$$P = \frac{cr - Nr^2}{M}$$

Noting

$$L\left\{ F'(t)\ H(t-R) \right\} = \int_0^\infty H(t-R)\ F'(t)e^{-st}\,dt$$

$$= \int_R^\infty F'(t)e^{-st}\,dt = e^{-st}\ F(t)\ \Big|_R^\infty$$

$$+ s\int_0^\infty H(t-R)\ F(t)e^{-st}\,dt = -e^{-Rs}F(R)$$

$$+ sL\left\{ H(t-R)\ F(t) \right\} \tag{3-17}$$

it follows that

$$L\left\{ F'(t)\ H(t-R) + F(R)\ \delta(t-R) \right\} = sL\left\{ F(t)H(t-R) \right\} \tag{3-18}$$

The $F(R)$ term in equation (3-18) must be interpreted as $F(R^+)$ and not the functional value since the introduction of the term $H(t-R)$ makes the product a discontinuous function.

From equation (3-16)

$$F(R^+) = \frac{e^{-\frac{\omega_o R}{2}}}{cr}$$

Thus, taking $\frac{\partial}{\partial t}$ of equation (3-16) results in

$$\frac{\partial f}{\partial t}(r,t) + \frac{e^{-\frac{\omega_o R}{2}}\,\delta(t-R)}{cr} \doteq sf(\sqrt{s(s+\omega_o)}) \tag{3-19}$$

In applying Leibnitz's rule to equation (3-16), the method of appendix (C) must be used. The partial derivative is (t>R)

$$\begin{aligned}\frac{\partial f(r,t)}{\partial t} = \frac{1}{cr^3}\Bigg\{ & H(t-R)\exp(-\xi t+NR)\Big[(MP-\xi r^2)\cos M(t-R) \\ & -(Mr^2+\xi P)\sin M(t-R)\Big] - \frac{\omega_o^2}{4}\exp\left(-\frac{\omega_o t}{2}\right)\int_R^t \exp(-N(z-R)) \\ & \times\Big[r^2\cos M(z-R) + P\sin M(z-R)\Big]\frac{zI_1\left(\frac{\omega_o}{2}\sqrt{t^2-z^2}\right)}{\sqrt{t^2-z^2}}\,dz \\ & + \frac{\omega_o}{2}\exp\left(-\frac{\omega_o t}{2}\right)\int_R^t \exp(-N(z-R))\Big[r^2\cos M(z-R) \\ & + P\sin M(z-R)\Big]\, z\left[\frac{\omega_o t I_o\left(\frac{\omega_o}{2}\sqrt{t^2-z^2}\right)}{(t^2-z^2)} - \frac{2tI_1\left(\frac{\omega_o}{2}\sqrt{t^2-z^2}\right)}{(t^2-z^2)^{3/2}}\right]dz \\ & + \frac{\omega_o^2}{8}\,t\exp\left(-\frac{\omega_o t}{2} - N(t-R)\right)\Big[r^2\cos M(t-R) + P\sin M(t-R)\Big]\Bigg\}\end{aligned} \tag{3-20}$$

where

$$\xi = N + \frac{\omega_o}{2}$$

$$P = \frac{cr - Nr^2}{M}$$

Substituting equations (3-16), (3-18), and (3-20) into equation (3-14), and collecting terms, it follows that $(t>R)$

$$v(r,t) = \frac{cr_oP_o}{3\mu r^2}\Bigg[\Bigg\{\Big[\exp\ (NR-\xi t)\Big]\Big[(MP-\xi r^2 + \omega_o r^2)\cos M(t-r)$$

$$+ (P\omega - Mr^2-\xi P)\sin M(t-R)\Big] + \frac{\omega_o{}^2}{8}\, t\Big[\exp(NR-\xi t)\Big]\Big[r^2 \cos M(t-R)$$

$$+ P \sin M(t-R)\Big]\Bigg\} H(t-R) + \frac{\omega_o}{2}\exp(-\frac{\omega_o}{2})\int_R^t \exp(-N(z-R))\Big[r^2\cos M(z-R)$$

$$+ P\sin M(z-R)\Big]\frac{z}{\sqrt{t^2-z^2}}\Bigg[\frac{\omega_o t I_o(\frac{\omega_o}{2}\sqrt{t^2-z^2})}{2\sqrt{t^2-z^2}}$$

$$+ (\frac{\omega_o}{2} - \frac{2t}{(t^2-z^2)})\, I_1\, (\frac{\omega_o}{2}\sqrt{t^2-z^2})\Bigg]\, dz\Bigg] \qquad (3\text{-}21)$$

STRESS

From equation (A-7), the normal radial elastic stress is

$$\sigma(r,t) = 3\mu\frac{\partial u(r,t)}{\partial r} + \frac{2\mu u(r,t)}{r} \tag{3-22}$$

Taking the Laplace transform of equation (3-22), and using equations (B-7) and (A-2), yields

$$\sigma(r,t) \doteq \left(\frac{\mu s}{s+\omega_o}\right)\left[3\frac{\partial^2\bar{\phi}}{\partial r^2} + \frac{2}{r}\frac{\partial\bar{\phi}}{\partial r}\right] \tag{3-23}$$

Inserting the first and second derivatives of equation (3-3) into equation (3-23) gives the transformed stress as

$$\frac{\sigma(r,t)}{-P_o r_o c^2} \doteq \frac{\exp\left(-R\sqrt{s(s+\omega_o)}\right)}{\left[s(s+\omega_o) + c_1\sqrt{s(s+\omega_o)} + c_o\right]}\left\{\frac{4}{3r^3} + \right.$$

$$\left. + \frac{4}{3r^2c}\sqrt{s(s+\omega_o)} + \frac{s(s+\omega_o)}{rc^2}\right\} \tag{3-24}$$

After some calculations, it follows that $(t>R)$

$$
\begin{aligned}
-\frac{\sigma(r,t)}{P_o r_o c^2} &= \exp\left(-\frac{\omega_o t}{2}\right) \\
&\times\Bigg[\Bigg\{\exp\left(-J(t-r)\right)\Big[M\cos V(t-R) + Q\sin V(t-R)\Big] \\
&+ \frac{B\omega_o}{4}\,\frac{R I_1\left(\frac{\omega_o}{2}\sqrt{t^2-R^2}\right)}{\sqrt{t^2-R^2}}\Bigg\} H(t-R) \\
&+ \frac{\omega_o}{2}\int_R^t \exp(-J(z-R))\Big[M\cos V(z-R) \\
&+ Q\sin V(z-R)\Big]\frac{z I_1\left(\frac{\omega_o}{2}\sqrt{t^2-z^2}\right)}{\sqrt{t^2-z^2}}\,dz\Bigg]
\end{aligned}
\tag{3-25}
$$

where

$$M = \frac{4}{3r^2c} - \frac{c_1}{rc^2}$$

$$P = \frac{4}{3r^3} - \frac{c_o}{rc^2}$$

$$J = \frac{c_1}{2}$$

$$Q = \frac{P-JM}{V}$$

$$V^2 = \frac{8c^2}{9r^2}$$

NUMERICAL COMPUTATIONS

The non-elementary integrands of the normalized parameters stress, velocity, and acceleration necessitated the use of numerical methods in obtaining the data for the graphs of Figures 1, 2, and 3. The computations were performed on an IBM 360 computer. Simpson integration was used with accuracy of 10^{-3}. This tolerance sufficed for the purpose of obtaining graphical data and reducing the tremendous machine time required to integrate the above parameters.

The constants r_o, ω_o, and c were assigned the values

$$r_o = 50 \text{ ft}$$

$$\omega_o = 600$$

$$c = 20{,}000 \text{ ft/sec}$$

The radial distance variable was incremented in multiples of twenty-five starting with an initial value of seventy-five feet. After each radial increment, the time parameter was varied over the range of the arrival time to thirty milliseconds in increments of one millisecond.

DISCUSSION OF RESULTS

In all of the viscoelastic solutions obtained in this paper, the values of the parameters were infinite (contained delta functions) at the time of arrival of the wave. The elastic solutions did not contain the delta function due to the initial conditions on the transform of the elastic wave equation. The delta functions and their derivatives (non-zero only at the arrival time) were omitted from the solutions since only time greater than the arrival time was considered.

The infinite value of the parameters at the arrival time of the wave can best be understood by considering the behavior of the Maxwell model when it is instantaneously subject to an infinite stress. The viscous element acts as a rigid member since it requires a finite time to respond. As a result, the elastic element alone is responsible for the initial response (infinite).

The data plotted in Figures 1, 2, and 3 in the vicinity of the arrival time do not represent the actual limiting values, but values which are obtained from the next integral value of time after the arrival time. The limiting values can be obtained from either the elastic or viscoelastic solutions.

The particle velocity (Fig. 2) takes on a positive value immediately after the arrival of the wave front, decreases rapidly to zero, changes direction and oscillates about the zero state to eventually approach zero velocity from the positive side.

The particle acceleration, by nature of its definition $\frac{\partial v}{\partial t}$, is the slope of the velocity curve. This relationship between velocity and acceleration is illustrated in Fig. 2 and 3 by the fact that the acceleration is zero at maximum and minimum points of the velocity.

The stress (Figure 1) decays rapidly for short periods of time and approaches the zero state without oscillation.

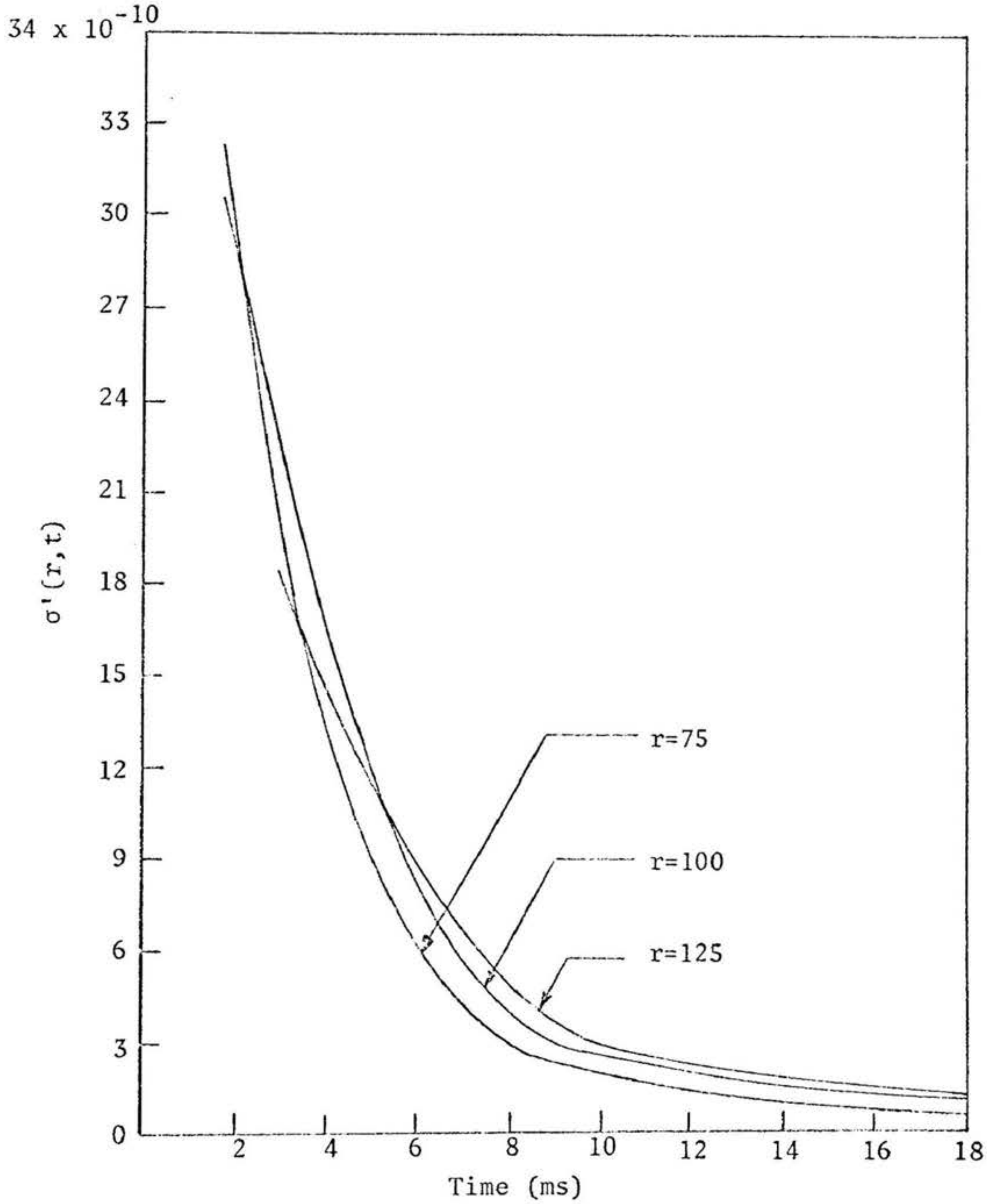


Figure 1. Normalized stress $\sigma'(r,t) = \sigma(r,t)/p_o r_o c^2$ for $\delta(t)$ forcing function for Maxwell spherical wave.

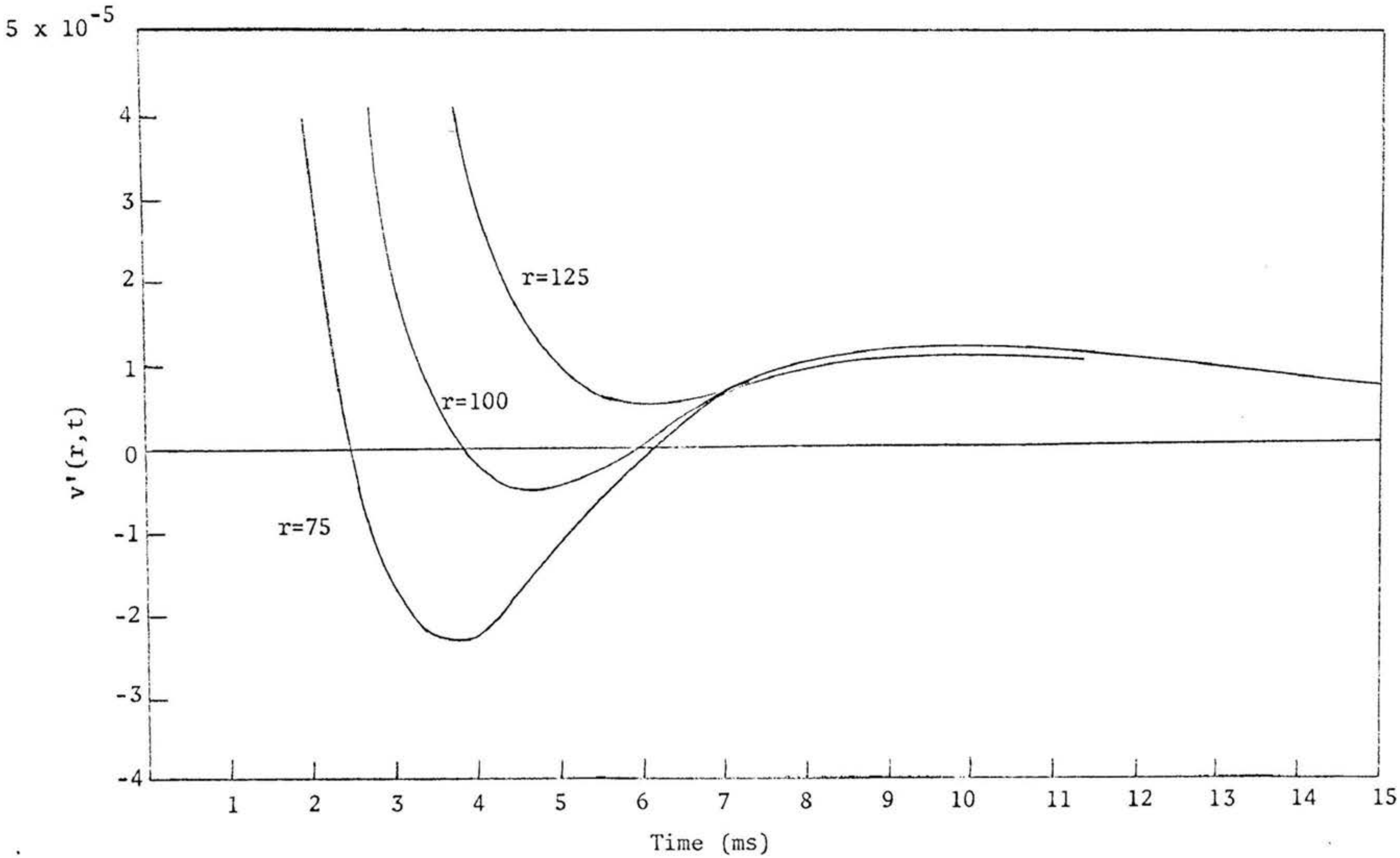


Figure 2. Normalized particle velocity $v'(r,t) = (12 / r_o p_o c^2)\ v(r,t)$ for $\delta(t)$ forcing function for Maxwell spherical wave.

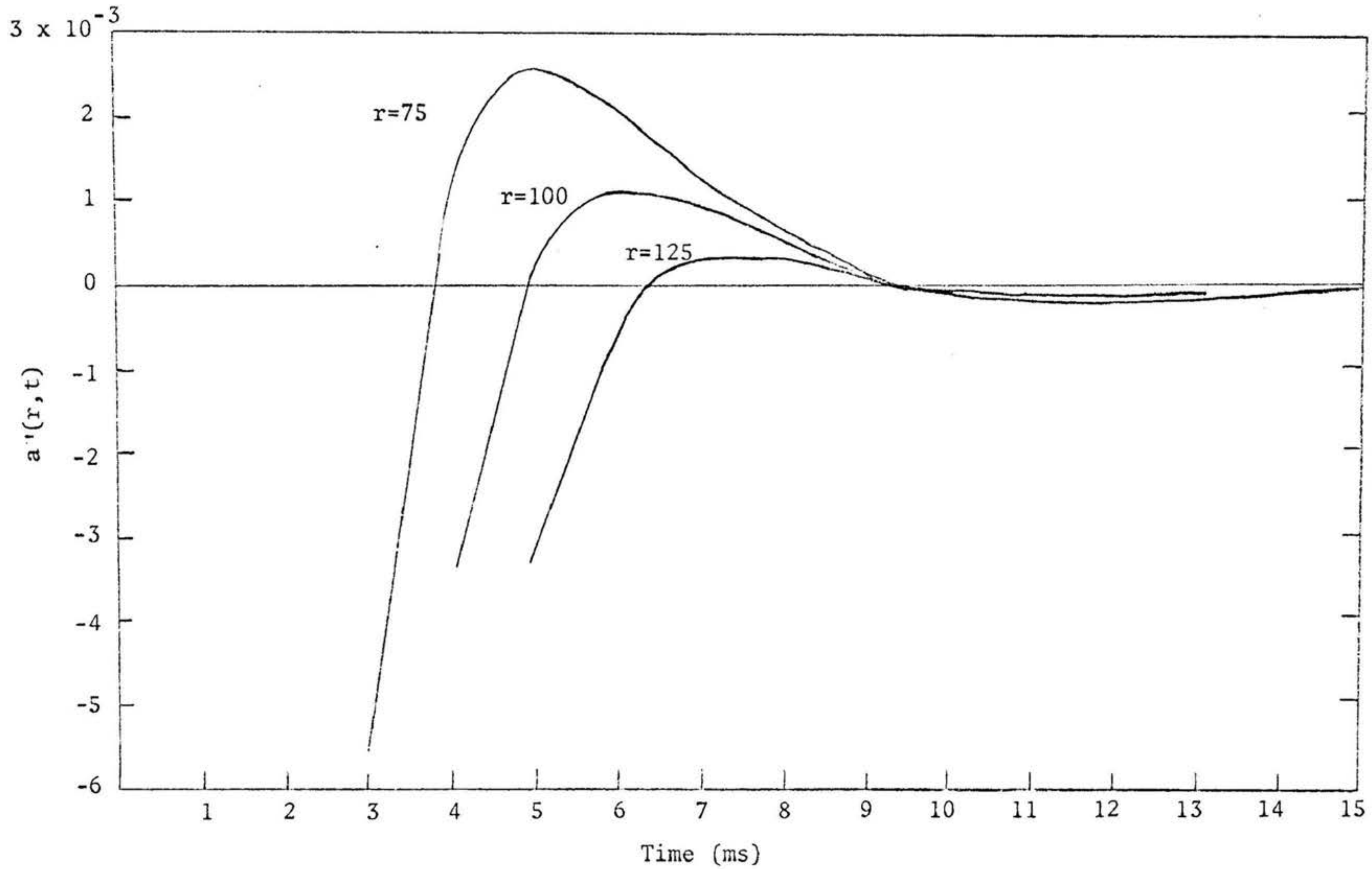


Figure 3. Normalized acceleration $a'(r,t) = (3\mu/r_o p_o c^2)\ a(r,t)$ for $\delta(t)$ forcing function for Maxwell spherical wave.

CHAPTER IV

SUMMARY AND CONCLUSIONS

The purpose of this paper was to obtain and analyze solutions in closed form for the parameters stress, velocity, and acceleration of the Maxwell solid. As has been indicated several times, the Maxwell model does not sufficiently represent real behavior. It appears that a combination of at least three or more basic elements must be used since the Maxwell element represents stress relaxation and the Voigt model alone represents retarded deformation.

The delta forcing function does not represent a realistic pressure function which would closely approximate observed real wave phenomena due to the instantaneous rise and fall time. It was chosen because of its simple transform and integral properties. The solution for any other pressure function can be obtained from the delta solution by convolution. The operation of convolution introduces an integration which, in general, must be performed by numerical methods since non-elementary functions are usually involved.

As is expected, the addition of more elements into the model greatly increased the complexity of the function whose inverse is desired. It may be that the inversion formula (eq. 3-1) used in the Maxwell spherical model can be applied to more complicated models, say, the Burger model. In general, the viscoelastic transform involves square roots which hinder the inversion. Since the solution for only the simplest models can be found from tabulated inverse transform pairs, other techniques should be at the disposal of the investigator. A knowledge of contour integration and complex function theory can be used to obtain a solution from the inversion integral. Also, expanding a function in a power series and inverting term by term has been used with success.

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APPENDIX A

SOLUTIONS OF THE SPHERICAL WAVE EQUATION IN AN ELASTIC MEDIUM WITH A DIRAC DELTA PRESSURE PULSE

From Bland (14), the basic equation for a spherical elastic wave with normal impact on the boundary of a spherical cavity in an infinite medium is found to be

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}u = \frac{\rho}{\lambda+2\mu}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{A-1}$$

The displacement potential ϕ is defined as

$$u = \frac{\partial \phi}{\partial r} \tag{A-2}$$

Equation (A-1) has the form

$$\frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r} + \frac{2}{r}u\right] = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{A-3}$$

From equation (A-2)

$$\frac{\partial u}{\partial r} = \frac{\partial^2 \phi}{\partial r^2} \tag{A-4}$$

and, from equation (A-3)

$$\frac{\partial^2}{\partial r^2}\left[\frac{\partial u}{\partial r} + \frac{2}{r}u\right] = \frac{1}{c^2}\frac{\partial}{\partial r}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{\partial u}{\partial r}\right] \tag{A-5}$$

It follows that the spherical wave expressed in terms of the displacement potential becomes

$$\frac{\partial^2 (r\phi)}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2}(r\phi) \tag{A-6}$$

From Bland (14) the normal radial stress σ is

$$\sigma = (\lambda+2\mu) \frac{\partial u}{\partial r} + 2\lambda \frac{u}{r} \tag{A-7}$$

The imposed boundary conditions are

$$(\lambda+2\mu) \frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial r}\right) + \frac{2\lambda}{r} \frac{\partial \phi}{\partial r} = +\sigma = -P_o \delta(t) \qquad \begin{matrix} r=r_o \\ t>0 \end{matrix} \tag{A-8}$$

$$\frac{\partial}{\partial t}(r\phi) = 0 \qquad \begin{matrix} r>r_o \\ t=0 \end{matrix} \tag{A-9}$$

$$r\phi = 0 \qquad \begin{matrix} r>r_o \\ t=0 \end{matrix} \tag{A-10}$$

$$\lim_{r\to\infty} (r\phi) = 0 \tag{A-11}$$

Assuming that the system is initially in an unpreturbed state allows for the following initial conditions

$$u(r,0) = 0 \qquad r>r_o \qquad \text{(A-12)}$$

$$v(r,0) = 0 \qquad \text{(A-13)}$$

$$\varepsilon(r,0) = 0 \qquad \text{(A-14)}$$

$$\sigma(r,0) = 0 \qquad \text{(A-15)}$$

Taking the Laplace Transform of equation (A-6) yields

$$\frac{d^2}{dr^2}(r\overline{\phi}) = \frac{1}{c^2}\left[s^2(r\overline{\phi}) - s(r\phi)_{t=0} - \left(\frac{\partial(r\phi)}{\partial t}\right)_{t=0}\right] \qquad \text{(A-16)}$$

Applying equations (A-9) and (A-10) to equation (A-16) produces the second order ordinary differential equation

$$\frac{d^2}{dr^2}(r\overline{\phi}) = \frac{s^2}{c^2}(r\,\bar{\phi}) \qquad \text{(A-17)}$$

which has a solution of the form

$$r\overline{\phi} = A(s)\, e^{-\frac{s}{c}r} + B(s)\, e^{-\frac{s}{c}r} \qquad \text{(A-18)}$$

The application of the Laplace Transform to equations (A- 8) and (A-11) requires that B(s) = 0 and

$$(\lambda+2\mu)\,\frac{d^2\overline{\phi}}{dr^2} + \frac{2\lambda}{r}\,\frac{d\overline{\phi}}{dr} = -P_{\dot{o}} \qquad r=r_o \qquad \text{(A-19)}$$

When equation (A-18), with B=0, is substituted into equation (A-19), with $\lambda=\mu$, the unknown function A(s) is found to be

$$A(s) = \frac{-P_o r_o c^2}{3\mu} \frac{e^{\frac{r_o s}{c}}}{s^2 + \frac{4cs}{3r_o} + \frac{4c^2}{3r_o^2}} \tag{A-20}$$

Inserting equation (A-20) into equation (A-18) results in the transformed elastic spherical wave equation for the displacement potential with delta pulse. Equation (A-18) then becomes

$$r\bar{\phi} = \frac{-P_o r_o c^2}{3\mu} \frac{e^{-\left(\frac{r-r_o}{c}\right) s}}{s^2 + \frac{4cs}{3r_o} + \frac{4c^2}{3r_o^2}} \tag{A-21}$$

Solving equation (A-21) for $\bar{\phi}$ and finding $\frac{\partial\bar{\phi}}{\partial r}$ results in

$$\frac{\partial\bar{\phi}}{\partial r} = \bar{u}(r,s) = \frac{P_o c^2 r_o}{3\mu} \frac{Bs + D}{s^2 + c_1 s + c_o} \quad e^{-Rs} \tag{A-22}$$

where

$$R = \frac{r-r_o}{c}$$

$$c_o = \frac{4c^2}{3r_o^2}$$

$$c_1 = \frac{4c}{3r_o}$$

$$B = \frac{1}{cr}$$

$$D = \frac{1}{r^2}$$

$$M^2 = c_o - \frac{c_1{}^2}{4}$$

It can be shown that equation (A-22) has the inverse

$$u(r,t) = \frac{P_o r_o c^2}{3\mu} e^{-\frac{c_1(t-R)}{2}} \left[B \cos M(t-R) + \left(\frac{D-\frac{Bc_1}{2}}{M}\right) \sin M(t-R) \right] H(t-R) \tag{A-23}$$

From equation (A-23) it follows that for t>R

$$v(r,t) = \frac{P_o r_o c^2}{3\mu} e^{-\frac{c_1(t-R)}{2}} \left[(D-Bc_1) \cos M(t-R) - \left(MB + \frac{c_1 D}{2M} - \frac{Bc_1{}^2}{4M}\right) \sin M(t-R) \right] H(t-R) \tag{A-24}$$

$$a(r,t) = \frac{P_o r_o c^2}{3\mu} e^{-\frac{c_1(t-R)}{2}} \Bigg[\Bigg\{ \frac{c_1}{2}\left(MB + \frac{c_1 D}{2M} - \frac{Bc_1^2}{4M}\right)$$

$$- M(D-Bc_1) \Bigg\} \sin M(t-R) - \Bigg\{ M\left(BM + \frac{Dc_1}{2M} - \frac{Bc_1^2}{4M}\right)$$

$$+ \frac{c_1}{2}(D-Bc_1) \Bigg\} \cos M(t-R) \Bigg] H(t-R) \qquad \text{(A-25)}$$

$$\varepsilon(r,t) = \frac{P_o r_o c^2}{3\mu} e^{-\frac{c_1(t-R)}{2}} \Bigg[\Bigg\{ \frac{M}{c^2 r} - \frac{2}{Mr^3} + \frac{c_1}{Mcr^2} - \frac{c_1^2}{4c^2 Mr} \Bigg\} \sin M(t-R)$$

$$+ \Bigg\{ \frac{c_1}{rc^2} - \frac{2}{cr^2} \Bigg\} \cos M(t-R) \Bigg] H(t-R) \qquad \text{(A-26)}$$

$$\sigma(r,t) = P_o r_o c^2 e^{-\frac{c_1(t-R)}{2}} \Bigg[\Bigg\{ \frac{c_1}{rc^2} - \frac{2}{cr^2} + \frac{2B}{3r} \Bigg\} \cos M(t-R)$$

$$+ \Bigg\{ \frac{M}{c^2 r} - \frac{2}{Mr^3} + \frac{c_1}{Mcr^2} - \frac{c_1^2}{4c^2 Mr} + \frac{2D}{3rM} - \frac{Bc_1}{3rM} \Bigg\} \sin M(t-R) \Bigg] H(t-R) \qquad \text{(A-27)}$$

APPENDIX B

TRANSFORMED VISCOELASTIC OPERATORS

The Maxwell model has the force-displacement (stress-strain) relationship

$$\left(E\frac{\partial}{\partial t}\right)\varepsilon = \left(\omega_o + \frac{\partial}{\partial t}\right)\sigma \qquad \text{(B-1)}$$

where

$$\omega_o = \frac{E}{\eta} \qquad \text{(B-2)}$$

The result of applying the Laplace transform to equation (B-1) is (σ and ε initially zero)

$$\bar{\sigma} = \frac{Es}{s+\omega_o}\bar{\varepsilon} \qquad \text{(B-3)}$$

From which the transformed viscoelastic operator corresponding to Young's modulus is

$$E \longrightarrow \frac{Es}{s+\omega_o} \qquad \text{(B-4)}$$

To obtain the corresponding viscoelastic operator for the elastic parameter μ, consider the relationship between μ and E as found in Timoshenko (15)

$$\mu = \frac{E}{2(1+\gamma)} \qquad \text{(B-5)}$$

Applying equation (B-4) to equation (B-5) results in

$$\mu \longrightarrow \left(\frac{Es}{s+\omega_o}\right)\left(\frac{1}{2(1+\gamma)}\right) = \frac{E}{2(1+\gamma)} \frac{s}{s+\omega_o} \tag{B-6}$$

Substituting equation (B-5) into (B-6) gives the transformed viscoelastic operator

$$\mu \longrightarrow \frac{\mu s}{s+\omega_o} \tag{B-7}$$

The remaining viscoelastic operator is found from the definition of the dilatational wave velocity

$$c = \left[\frac{\lambda+2\mu}{\rho}\right]^{\frac{1}{2}} \tag{B-8}$$

setting $\lambda=\mu$ and using equation (B-7) results in

$$c \longrightarrow \left[\frac{3}{\rho}\left(\frac{\mu s}{s+\omega_o}\right)\right]^{\frac{1}{2}} = \left[\left(\frac{3\mu}{\rho}\right)\left(\frac{s}{s+\omega_o}\right)\right]^{\frac{1}{2}} \tag{B-9}$$

A combination of equations (B-8) and (B-9) gives the transformed viscoelastic operator

$$c \longrightarrow \left[\frac{c^2 s}{s+\omega_o}\right]^{\frac{1}{2}} \tag{B-10}$$

APPENDIX C

THE EVALUATION OF A LIMIT IN THE MAXWELL SPHERICAL ANALYSIS

In the course of applying Leibniz's rule for differentrating integrals, it may be that an indeterminate form is obtained when the integral is evaluated at the upper or lower limits and multiplied by the derivative of that upper or lower limit respectively. This is indeed what happens in integrals of the form

$$F(t) = \int_R^t g(z,R) \frac{I_1(b\sqrt{t^2-z^2})\,dz}{\sqrt{t^2-z^2}} \qquad \text{(C-1)}$$

Assuming that g(z,R) and its derivative are finite at z=t, the limit can be evaluated by an application of L'Hospital's rule

$$L = \lim_{z \to t} \left[\frac{g(z,R) I_1(b\sqrt{t^2-z^2})}{\sqrt{t^2-z^2}} \right] =$$

$$\lim_{z \to t} \left[\frac{g(z,R)\frac{d}{dz} I_1(b\sqrt{t^2-z^2}) + I_1(b\sqrt{t^2-z^2})\frac{d}{dz}g(z,R)}{-z/\sqrt{t^2-z^2}} \right]$$

but,

$$\lim_{z \to t} \left[\frac{\sqrt{t^2-z^2}}{z} I_1(b\sqrt{t^2-z^2})\frac{dg}{dz}(z,R) \right] = 0 \qquad \text{(C-2)}$$

from which equation (C-2) becomes

$$L = - \lim_{z \to t} \left[\frac{g(z,R)}{z} \sqrt{t^2 - z^2} \frac{d}{dz} I_1 (b\sqrt{t^2 - z^2}) \right] \tag{C-3}$$

The derivative of the Bessel function can be obtained by making a substitution

$$v = b\sqrt{t^2 - z^2} \tag{C-4}$$

From tables of mathematical functions (16), it can be shown that

$$\frac{dI_\gamma (v)}{dv} = I_{\gamma - 1}(v) - \frac{\gamma}{v} I_\gamma (v) \tag{C-5}$$

With $\gamma = 1$ and $v = b\sqrt{t^2 - z^2}$, it follows that

$$\frac{dI_1 (v)}{dz} = \frac{dI_1 (v)}{dv} \left(\frac{dv}{dz}\right) = \frac{-bz}{\sqrt{t^2 - z^2}} \left[I_o(v) - \frac{1}{v} I_1(v) \right]$$

$$= \frac{-bz}{\sqrt{t^2 - z^2}} \left[I_o (b\sqrt{t^2 - z^2}) - \frac{1}{b\sqrt{t^2 - z^2}} I_1 (b\sqrt{t^2 - z^2}) \right] \tag{C-6}$$

Thus, equation (C-3) becomes

$$L = - \lim_{z \to t} \left[(-bg(z,R)) \left\{ I_o (b\sqrt{t^2 - z^2}) - \frac{I_1 (b\sqrt{t^2 - z^2})}{b\sqrt{t^2 - z^2}} \right\} \right] \tag{C-7}$$

However,

$$\lim_{z \to t} bg(z,R)I_o(b\sqrt{t^2-z^2}) = bg(t,R) \tag{C-8}$$

and

$$\lim_{z \to t} \left\{ \left[-bg(z,R)\right] \left[\frac{-I_1(b\sqrt{t^2-z^2})}{b\sqrt{t^2-z^2}}\right] \right\} = L \tag{C-9}$$

hence, equation (C-3) can be written as

$$L = bg(t,R) - L \tag{C-10}$$

Solving equation (C-10) for L yields

$$L = \lim_{z \to t} \left[\frac{g(z,R)I_1(b\sqrt{t^2-z^2}}{\sqrt{t^2-z^2}}\right] = \frac{b}{2}g(t,R) \tag{C-11}$$

APPENDIX D

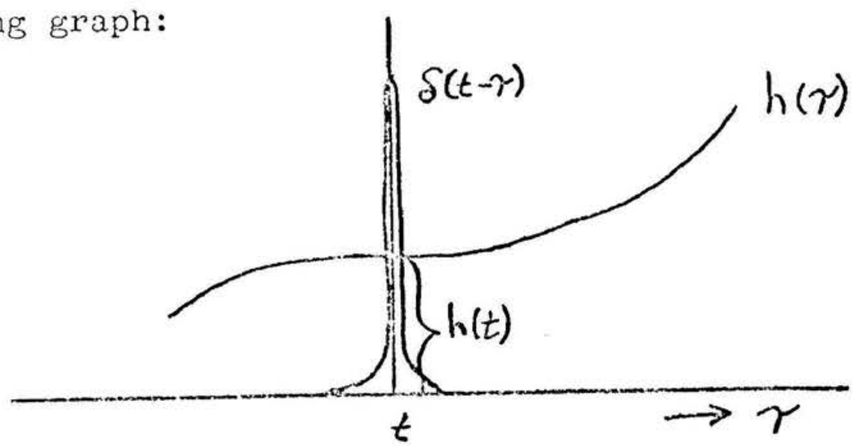
AN APPLICATION OF THE SIFTING PROPERTY OF THE DIRAC DELTA FUNCTION WITH A NON-LINEAR ARGUMENT

The usefulness of the Dirac delta function in operational mathematics depends upon its integrating property

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = h(t) \qquad \text{(D-1)}$$

Symbolically, the delta function acts as if it were a sieve; after multiplying an arbitrary function $h(t-\tau)$ by $\delta(\tau)$, and then integrating over the real t-axis, the value of $h(t)$ is selected at $\tau = 0$. This property will be referred to as the sifting property of the delta function.

The sifting property can be illustrated by the following graph:



Noting that $h(\tau)$ is approximately constant (equal to $h(t)$) in the region where the delta function is large, the integral then is equal to h(t) times the area below the $\delta(t-\tau)$ curve. This agrees with equation (D-1).

In order to apply the sifting property, the argument of the delta function must be of the form of equation (D-1).

Considering integrals of the form

$$F(t,R) = \int_0^t \delta(\sqrt{t^2-\beta^2}-R)\Big[A\cos(\sqrt{t^2-\beta^2}-R) +$$

$$B\sin(\sqrt{t^2-\beta^2}-R)\Big] I_1(a\beta)\,d\beta \tag{D-2}$$

and making the substitution

$$z = \sqrt{t^2-\beta^2} \tag{D-3}$$

equation (D-2) becomes

$$F(t,R) = -\int_t^0 \delta(z-R)\Big[A\cos(z-R) +$$

$$B\sin(z-R)\Big]\frac{zI_1(a\sqrt{t^2-z^2})}{\sqrt{t^2-z^2}}dz \tag{D-4}$$

Interchanging limits and applying the sifting property yields

$$F(t,R) = \frac{ARI_1(a\sqrt{t^2-R^2})}{\sqrt{t^2-R^2}} \qquad t>R \tag{D-5}$$

It is seen that the limits of integration of equation (D-1) which defines the sifting property are infinite while those of equation (D-2) are finite ($t<\infty$). Both equations are consistant since the infinite limits may be replaced by finite limits provided the critical point (here $z = R$ in equation (D-4)) lies interior to the interval $(0,t)$. The nature of R assures that this condition is met.

If the critical point is on the end of the interval, then the sifting property (Equation D-1) will have to be modified as follows

$$\int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)\delta(t-\tau)d\tau + \int_{t}^{\infty} h(\tau)\delta(t-\tau)d\tau = h(t) \qquad \text{(D-6)}$$

If equation (D-6) is to have the value $h(t)$, then it must follow that

$$\int_{-\infty}^{t} h(\tau)\delta(t-\tau)d\tau = \tfrac{1}{2}h(t) = \int_{t}^{\infty} h(\tau)\delta(t-\tau)d\tau \qquad \text{(D-7)}$$

VITA

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