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Performance comparison of receivers designed to combat intersymbol interference due to specular multipath

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PERFORMANCE COMPARISON OF RECEIVERS DESIGNED TO COMBAT INTERSYMBOL INTERFERENCE DUE TO SPECULAR MULTIPATH

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ABSTRACT

Bit error probabilities for biphase transmission have been computed for a known two-component multipath channel with a variety of sub-optimum receivers. Performance curves are presented for each receiver, and the relative performance is summarized. Degradation due to imperfect channel knowledge is calculated, and the feasibility of countering this interference with these specially designed receivers is discussed.

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Dr. David Cunningham and Dr. Rodger Ziemer for their helpful comments and discussions, and to Continental Oil Company for providing the financial support for this project.

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I. INTRODUCTION AND LITERATURE REVIEW

Multipath has long been known to be the source of the severe selective fading encountered in digital microwave communications systems, and much effort has been devoted to minimizing the effect of this fading as indicated by the extensive bibliography of Lindsey [1]. Another aspect of multipath interference, which has been less extensively considered, is intersymbol interference (ISI). This aspect is becoming a more significant problem with the advent of very high data rate systems [2], particularly in offshore and underwater communications systems [3]. In this thesis, a variety of receivers specially designed to combat two-component multipath ISI are considered. Their performance is compared to the performance of the standard integrate-and-dump (I/D) receiver.

In particular, the following receiver structures are examined:

1) An I/D receiver which begins integrating at time τ, the multipath delay, and stops integrating at the end of the bit period;
2) An I/D receiver which begins integrating at time τ, and stops at time $T + \tau$, where T is the bit period;
3) An I/D receiver which changes its decision threshold depending upon the previous decision. This last structure was analyzed at baseband by Aein and Hancock [4], and its performance will be indicative of the optimum receiver of Gonsalves [5], since the switched threshold receiver is a truncated version of the optimum.

These receivers are analyzed first for bit error probabilities assuming the channel is completely known. Receivers 1) and 3), along with the standard I/D receiver, are then analyzed and compared for the case in which the delay and phase of the reflected component are not exactly known, but are estimated, and are consequently subject to error.

Although multipath intersymbol interference is a special case of many papers which treat generalized types of ISI (see bibliography of Valerdi and Simpson [6]), few of these papers have addressed themselves specifically to practical solutions to the multipath problem. Gonsalves [5] has found the optimum (maximum likelihood) receiver for the 2-component specular reflected path case in which the channel is completely known, and Aein and Hancock [4] have investigated the performance of two sub-optimum receivers: a variable threshold correlator and an optimum memoryless receiver.

At this point, the contrast between fading and ISI should be noted. Fading refers to the destructive interference occurring when the multipath component is out of phase with the direct component, causing an effective cut in the transmitted power. ISI, on the other hand, refers to the interference introduced by the overlap of the previous symbol of the multipath component with the current symbol of the direct component. Thus, the source of the performance degradation is slightly different in the two cases: with fading, the degradation is a result of the ambient background noise, while with ISI the degradation stems directly from the multipath component and would, in fact, be a function of the character sequence transmitted.

Section IIA will examine the relative importance of these two types of interference for the standard I/D receiver, and will examine performance of the specially designed receivers in the ideal case of a perfectly known channel. Section IIB will consider the more realistic case in which the channel is unknown and must be estimated. Section III summarizes the conclusions which can be drawn from this work.

II. RESULTS AND DISCUSSION

A. Performance Comparison Assuming Known Channel

The received signal is assumed to be composed of a direct binary PSK component, a specular reflected component of relative amplitude α (which suffers a delay, τ and phase change θ) and an additional white Gaussian noise component:

$$S_T(t) = S_s(t) + S_r(t) + n(t) \;, \qquad (1)$$

$$\text{where } S_s(t) = \sqrt{2E/T} \sin\left[\omega_0 t + k_i \frac{\pi}{2}\right] \;, \qquad (2)$$

$$\text{with } k_i = \pm 1 \text{ and } (i-1)T \leq t \leq iT \;,$$

$$S_r(t) = \alpha\sqrt{2E/T} \sin\left[\omega_0 (t-\tau) - \theta + k_i \frac{\pi}{2}\right] \qquad (3)$$

$$\text{with } 0 \leq \alpha \leq 1 \;,\; (i-1)T \leq t-\tau \leq iT$$

and $n(t)$ is white Gaussian noise with two sided spectral density $N/2$.

It is assumed that $k_i = 1$ and $k_i = -1$ are equally probable and that k_i is independent of $k_{j\neq i}$. Note that E is the bit energy in the direct component, and T is the bit period.

For all receivers perfectly coherent detection is assumed; that is, the receivers are perfectly synchronized with respect to carrier reference phase and bit period. The receivers are thus of the form shown in Figure 1.

The standard I/D receiver is considered first in order to indicate when intersymbol interference is a problem in multipath channels. It

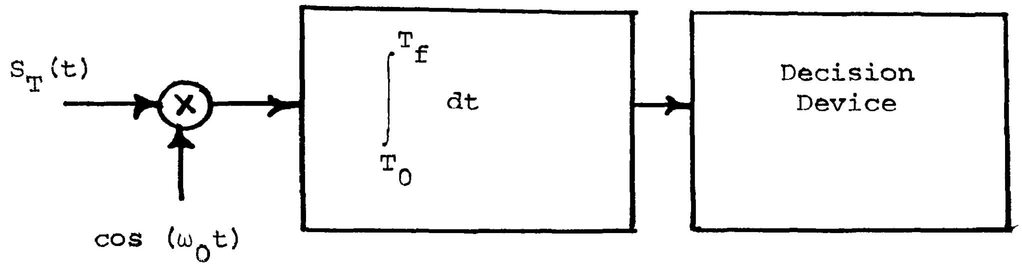

Figure 1. Coherent Receiver Structure

is shown in Appendix A that the bit error probability for this receiver is given by

$$P_E = \frac{1}{4}[\text{erfc}\ \{\sqrt{\frac{E}{N}}(1 + f - 2f\ \frac{\tau}{T})\} + \text{erfc}\ \{\sqrt{\frac{E}{N}}(1 + f)\}]\ , \quad (4)$$

where $f = \alpha \cos\ (\omega_0\tau + \theta)$. (5)

Using this expression, error probabilities have been calculated for various values of $\frac{E}{N}$, f, and τ. Note that the parameter f is a measure of the fading experienced by the channel. Negative f represents fading, and the closer f is to -1, the more severe the fading.

Figures 2 and 3 indicate the results of the calculations. Figures 2a and 2b are simply plots of error probability as a function of the signal-to-noise ratio (SNR), $\frac{E}{N}$, for $\frac{\tau}{T}$ constant and for various values of f. The intuitively obvious results are immediately apparent: the performance is degraded as f becomes more negative and $\frac{\tau}{T}$ becomes larger. Using this type of plot, Figure 3 was compiled, and it demonstrates these results more explicitly. The degradation plotted in Figure 3 is defined as the increase in the SNR necessary to result in

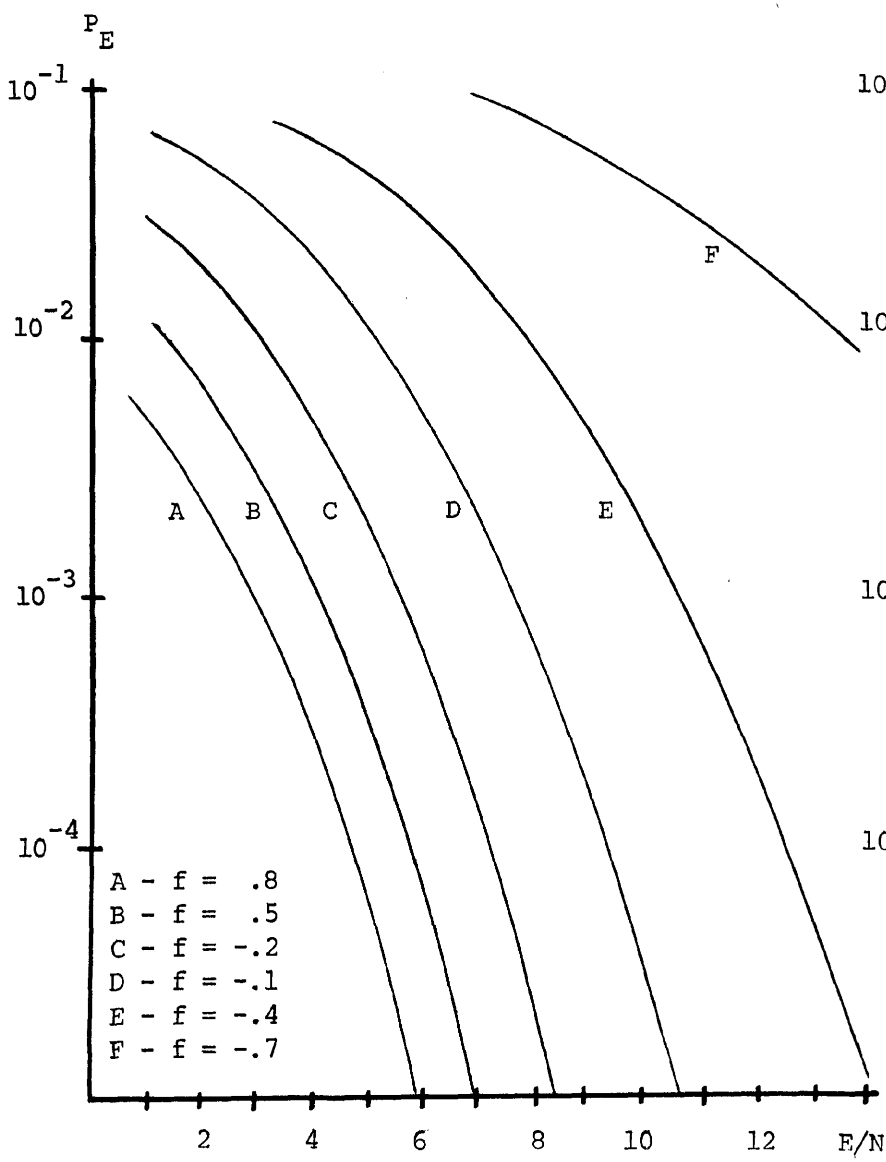

Figure 2a. Bit Error Probability versus E/N For Integrate-and-Dump Receiver: τ = .2T

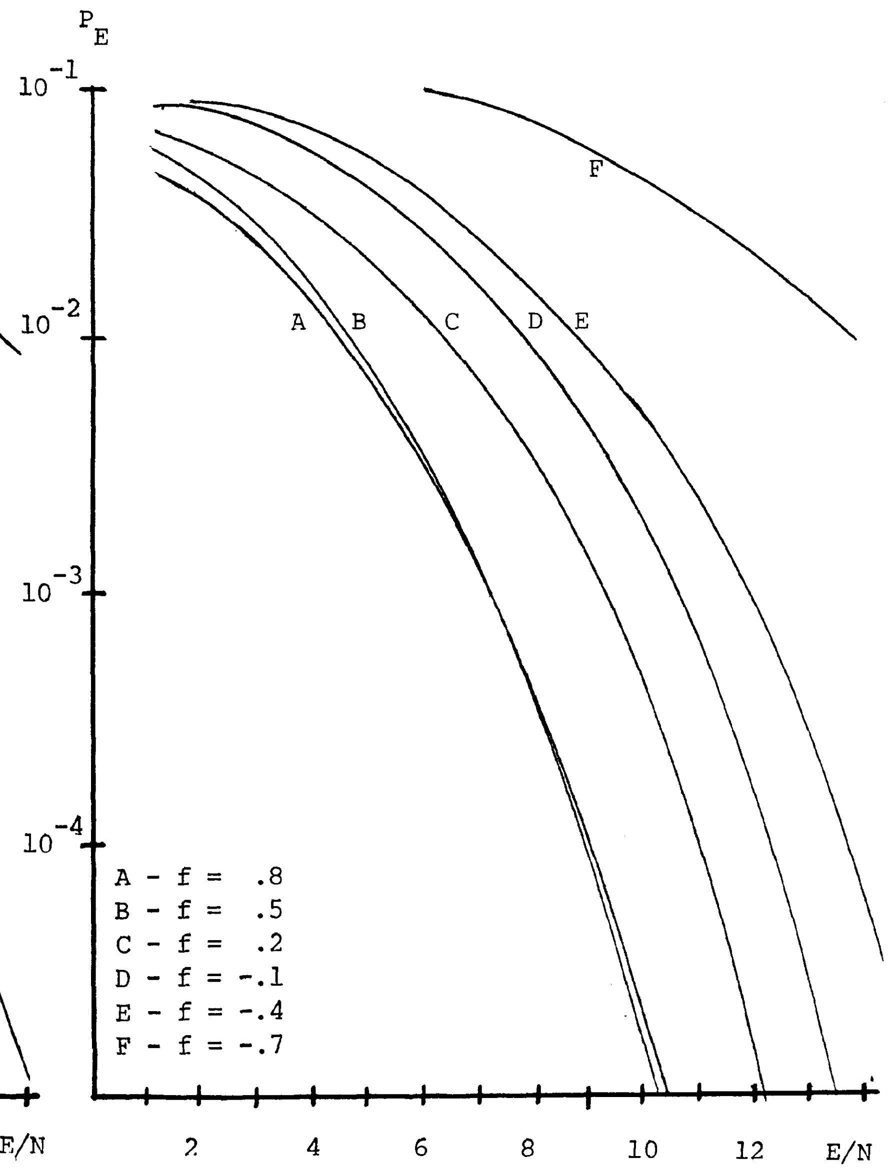

Figure 2b. Bit Error Probability versus E/N For Integrate-and-Dump Receiver: τ = .8T

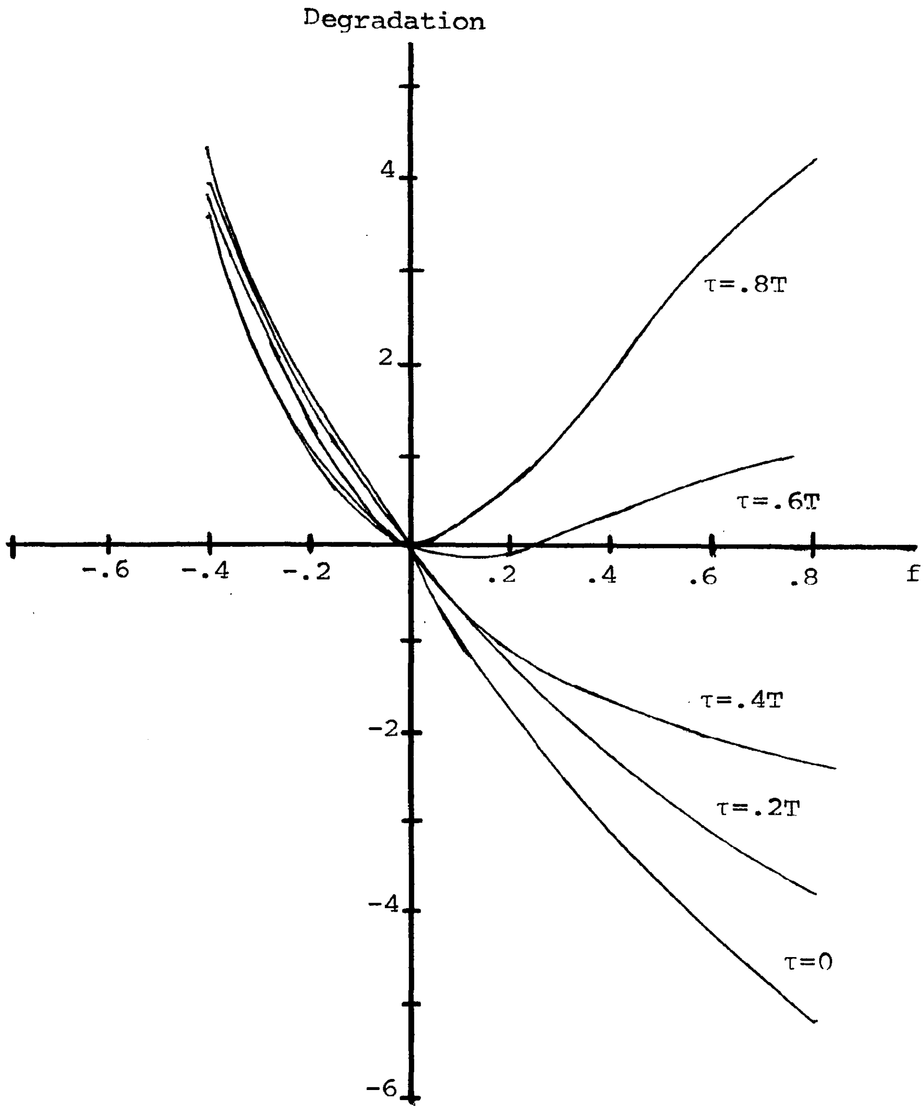

Figure 3. Degradation versus f For Integrate and Dump Receiver

an error probability of 10^{-4} compared to the SNR required for an error probability of 10^{-4} when $f = 0$; i.e., no reflected component is present. From this figure it is clear that when $f < 0$, intersymbol interference is negligible because variations in $\frac{\tau}{T}$ have no significant effect on the error probability -- the effects of fading dominate. On the other hand, when $f > 0$, significant variations in P_E as $\frac{\tau}{T}$ changes indicate that intersymbol interference is playing a significant role in performance degradation. Specifically, when $\frac{\tau}{T}$ is greater than about .5, performance begins to deteriorate. With the high data rate systems now being developed (100 MBS and higher [2]) this represents a path differential of 1.5 m or less which could easily occur over typical microwave transmission distances.

In designing receivers to combat this type of ISI, the simplest approach, both conceptually and practically, is simply to begin integrating at time τ, so that no previous bit information is included in the decision. Of course, the direct signal component is also lost in the interval $0 - \tau$, so that one would expect that this technique would be valuable only when intersymbol interference is severe; i.e., $f \cong 1$. In order to verify this intuition, analytically, one may calculate the error probability for the receiver shown in Figure 4. The error

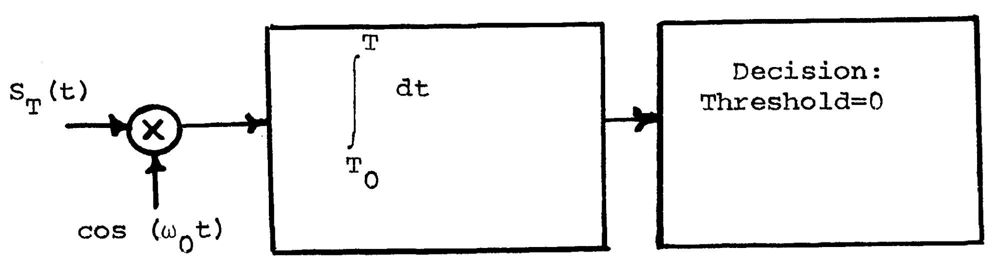

Figure 4. Coherent Detector With Arbitrary Start Time

probability may then be differentiated with respect to T_0. When $\frac{\partial P_E}{\partial T_0} < 0$, performance is improved by shortening the integration period. It is shown in Appendix B that there are indeed conditions for which $\frac{\partial P_E}{\partial T_0} < 0$, and that $f \cong 1$ is among these conditions.

The error probability for the receiver which begins integrating at τ has been evaluated numerically as a function of τ, and results have been tabulated graphically in the same format as the results for the standard I/D receiver. These results are presented in Figures 5 and 6. Comparison of receiver performance is presented at the end of this section.

A logical extension of this last receiver is one which integrates over the period τ to $T + \tau$. Bit error probabilities for this receiver have been calculated analytically in Appendix 1, and some numerical results are presented in Figure 7.

Both the above receivers are clearly sub-optimum. The optimum receiver would recognize that information about both adjacent bits is necessary to make an optimum decision concerning any given bit. Gonsalves [5] recognized this Markov structure of the problem in formulating the optimum receiver. Aein and Hancock [4] partially realized this structure in their switched threshold correlator, but since they were considering practically realizable receivers they concerned themselves only with using knowledge of the previous bit. In Appendix C their receiver is rederived as a logical extension of the optimum receiver which uses only current bit information, and the error probability is determined in terms of parameters f and $\frac{\tau}{T}$. Note the energy convention adopted in this paper: E is the bit energy of the

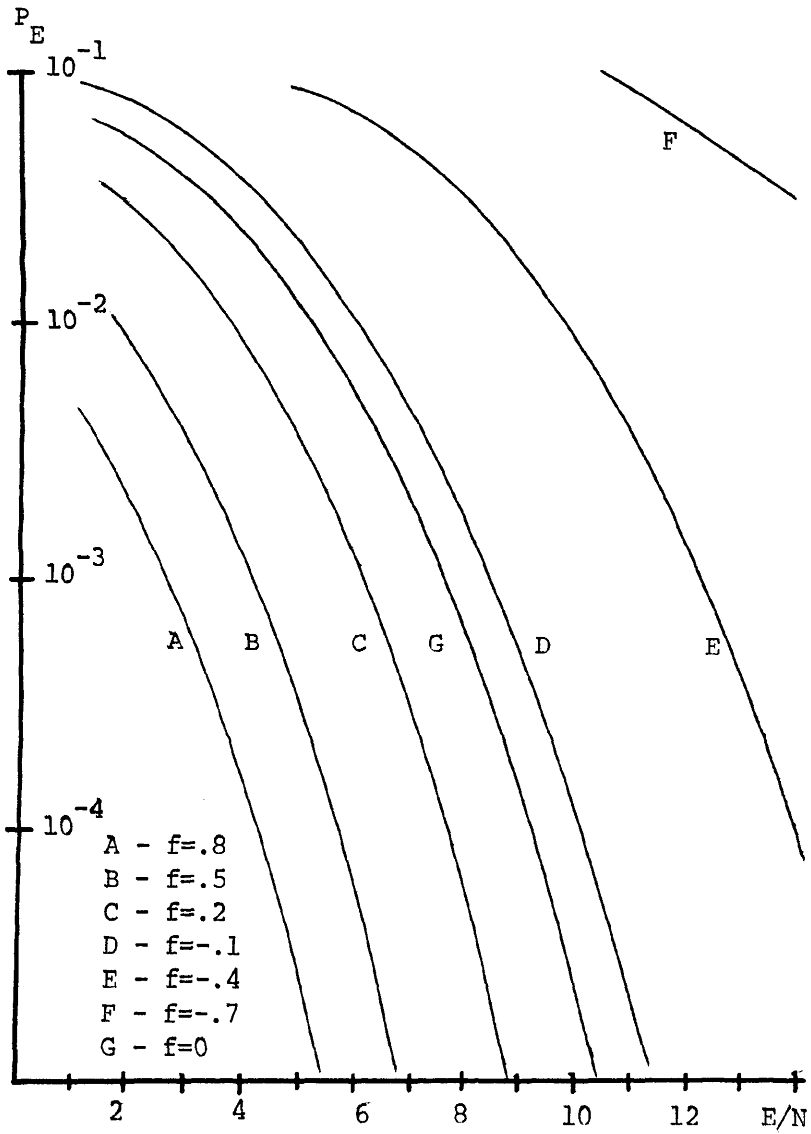

Figure 5a. Bit Error Probability versus E/N For Delayed Start Receiver: τ = .2T

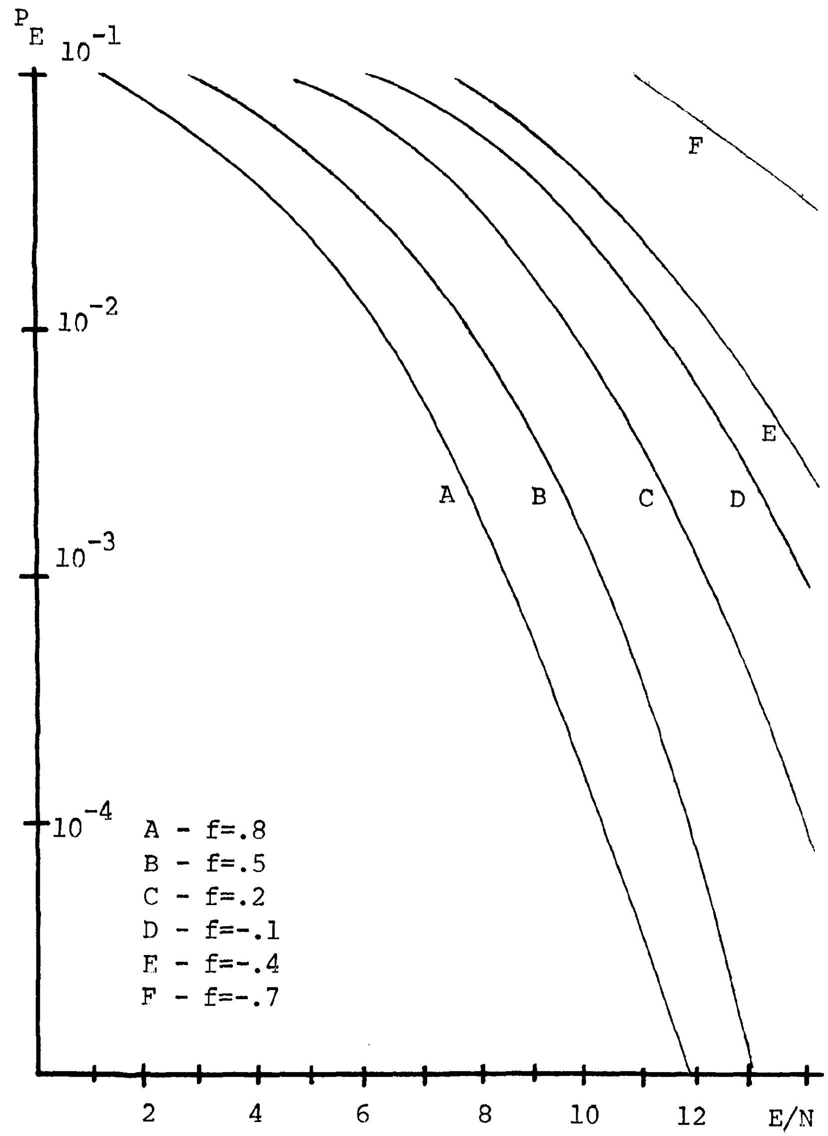

Figure 5b. Bit Error Probability versus E/N For Delayed Start Receiver: τ = .8T

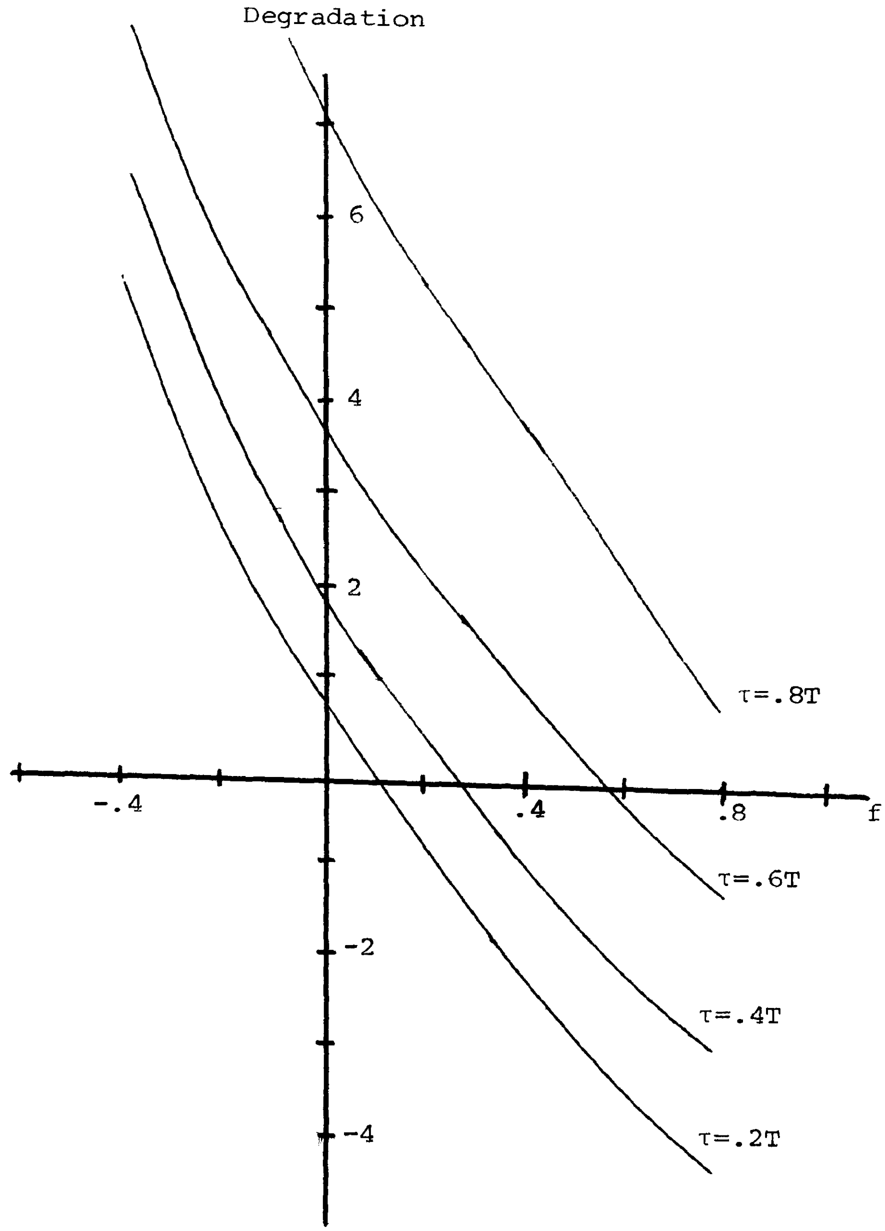

Figure 6. Degradation versus f For Delayed Start Receiver

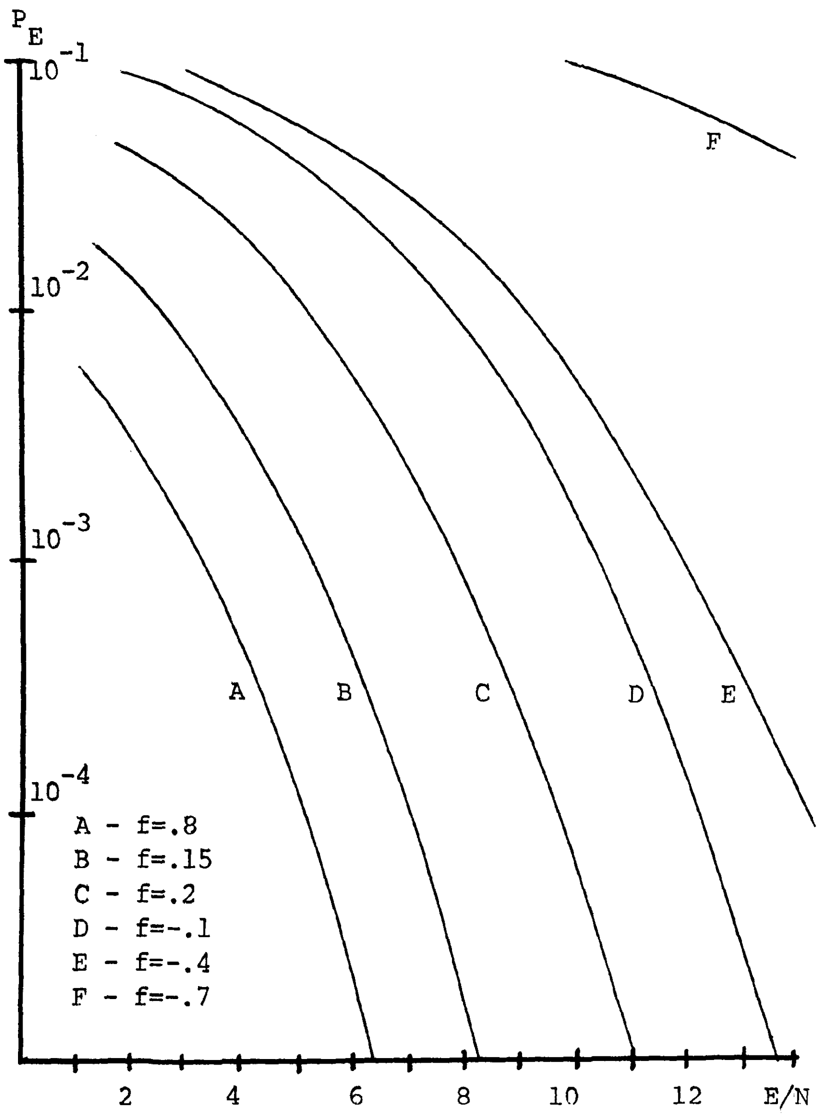

Figure 7a. Bit Error Probability versus E/N For Delayed Start and Stop Receiver: τ = .2T

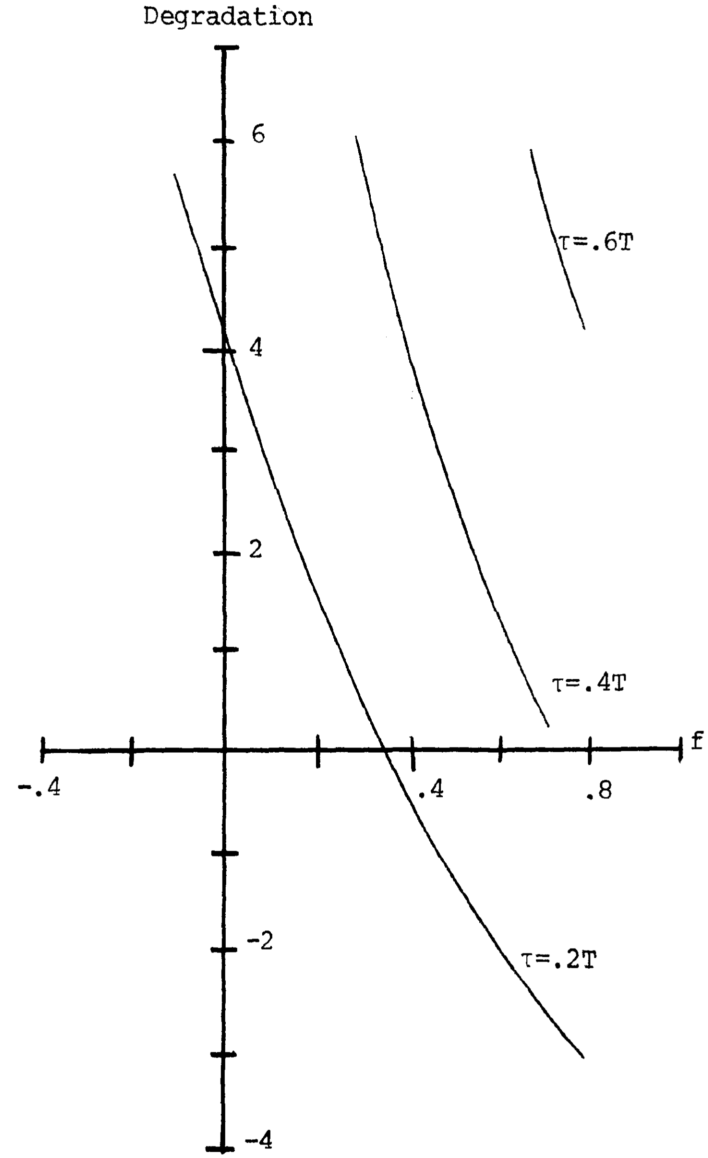

Figure 7b. Degradation versus f For Delayed Start and Stop Receiver

direct component of the received signal. The computed error probabilities are tabulated graphically in Figures 8 and 9.

The receivers are compared in Figure 10. The following points should be noted:

1) The delayed start and stop receiver performed considerably worse than the standard I/D receiver in every case. This intuitively obvious result is of significance because it indicates that bit synchronization, which could be lost due to the interference, must somehow be carefully controlled.

2) The delayed start receiver allows an improved performance when f is greater than about .6, but degrades the performance when f is smaller.

3) The switched threshold receiver not only eliminates the multipath ISI, but also improves performance under fading conditions for large τ.

B. Performance Comparison Assuming Unknown Channel

From the results of the previous section, it is clear that performance can be improved in certain cases of specular multipath interference if a delayed start or switched threshold receiver is used. However, both these receivers require knowledge of the channel: the delayed start receiver requires τ, the delay of the reflected component, and the switched threshold receiver requires both τ and f. These parameters will not usually be known a priori and consequently will need to be estimated with some device. This estimate will, of course,

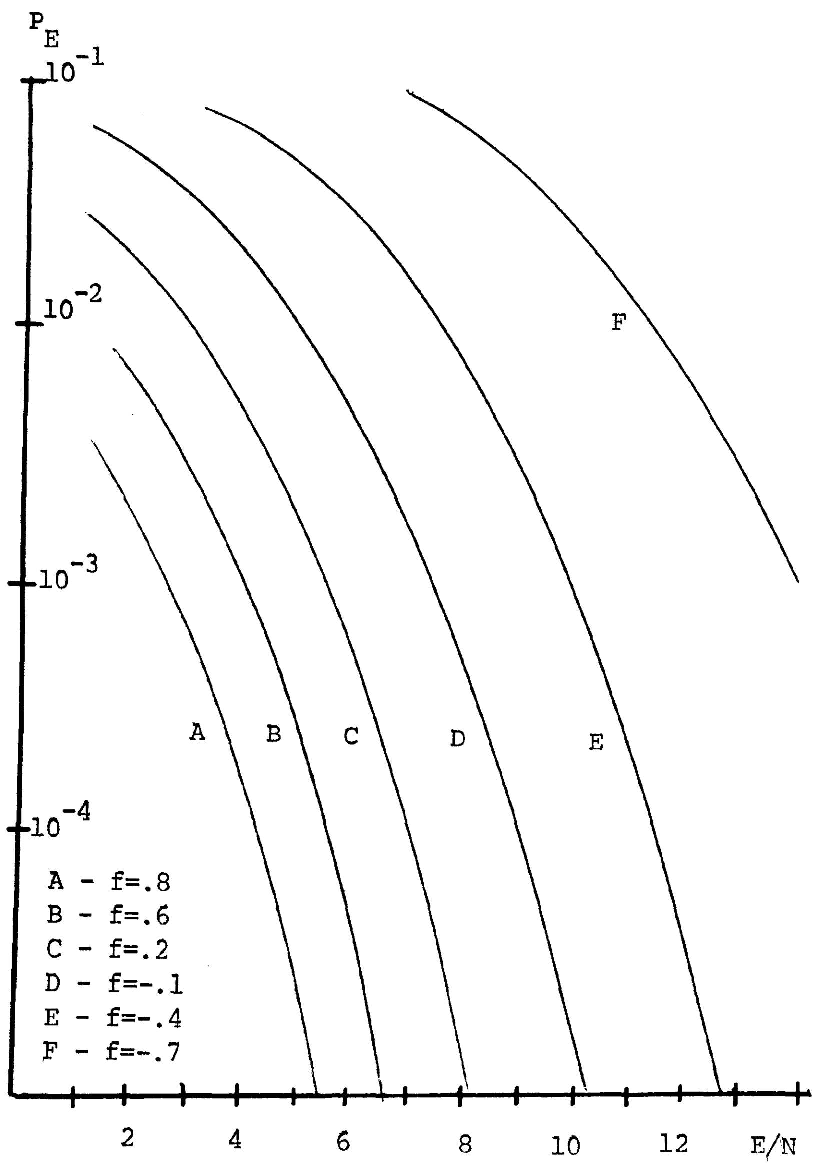

Figure 8a. Bit Error Probability versus E/N For Switched Threshold Receiver: τ = .2T

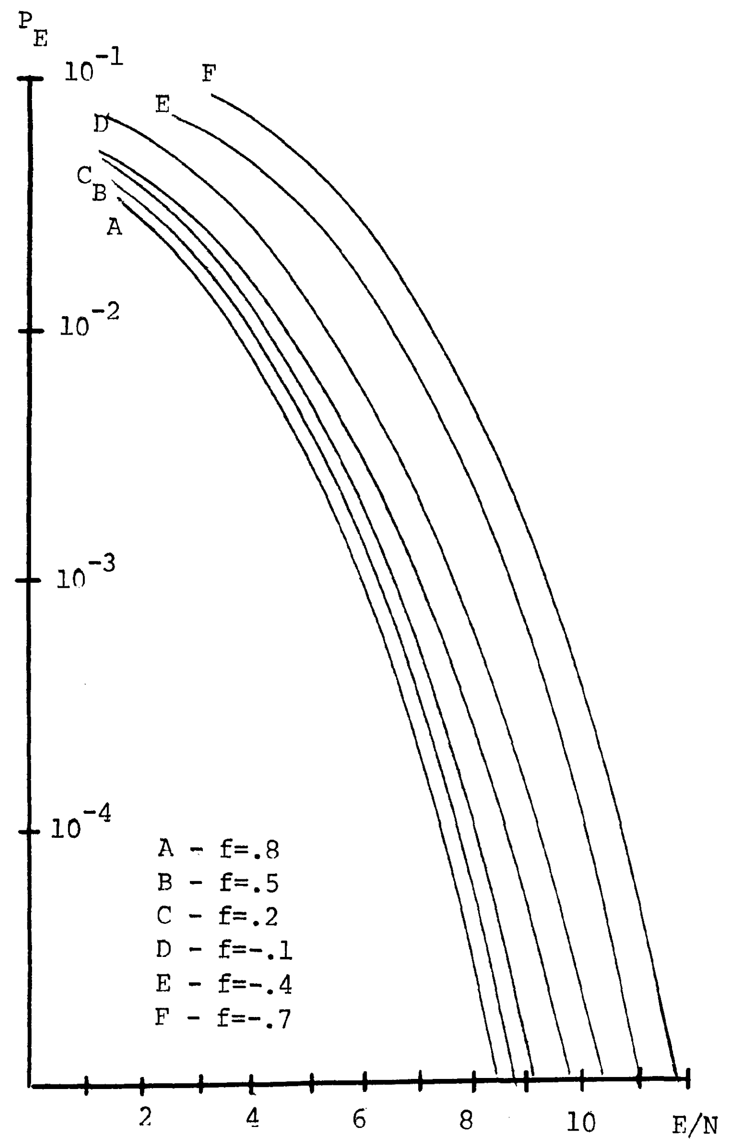

Figure 8b. Bit Error Probability versus E/N For Switched Threshold Receiver: τ = .8T

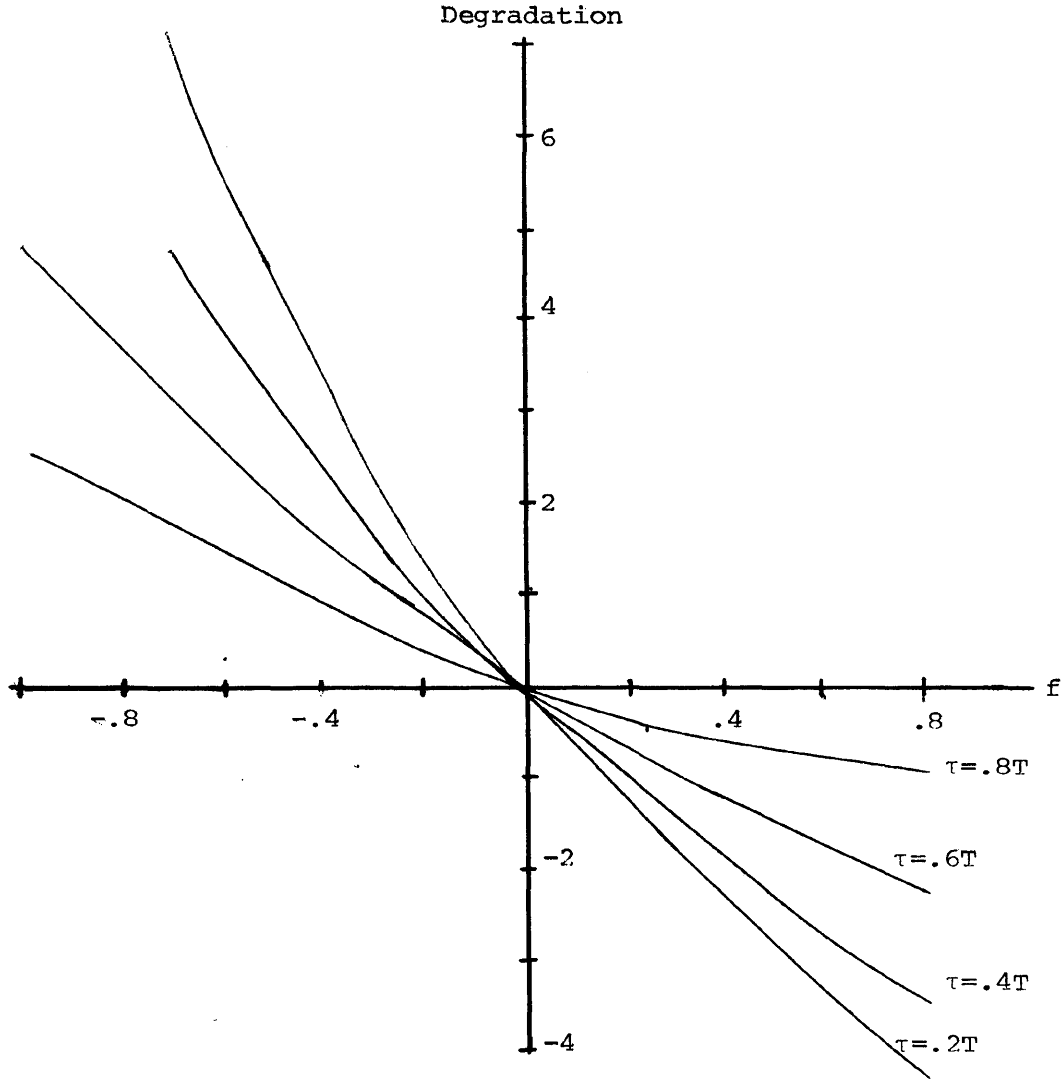

Figure 9. Degradation versus f For Switched Threshold Receiver

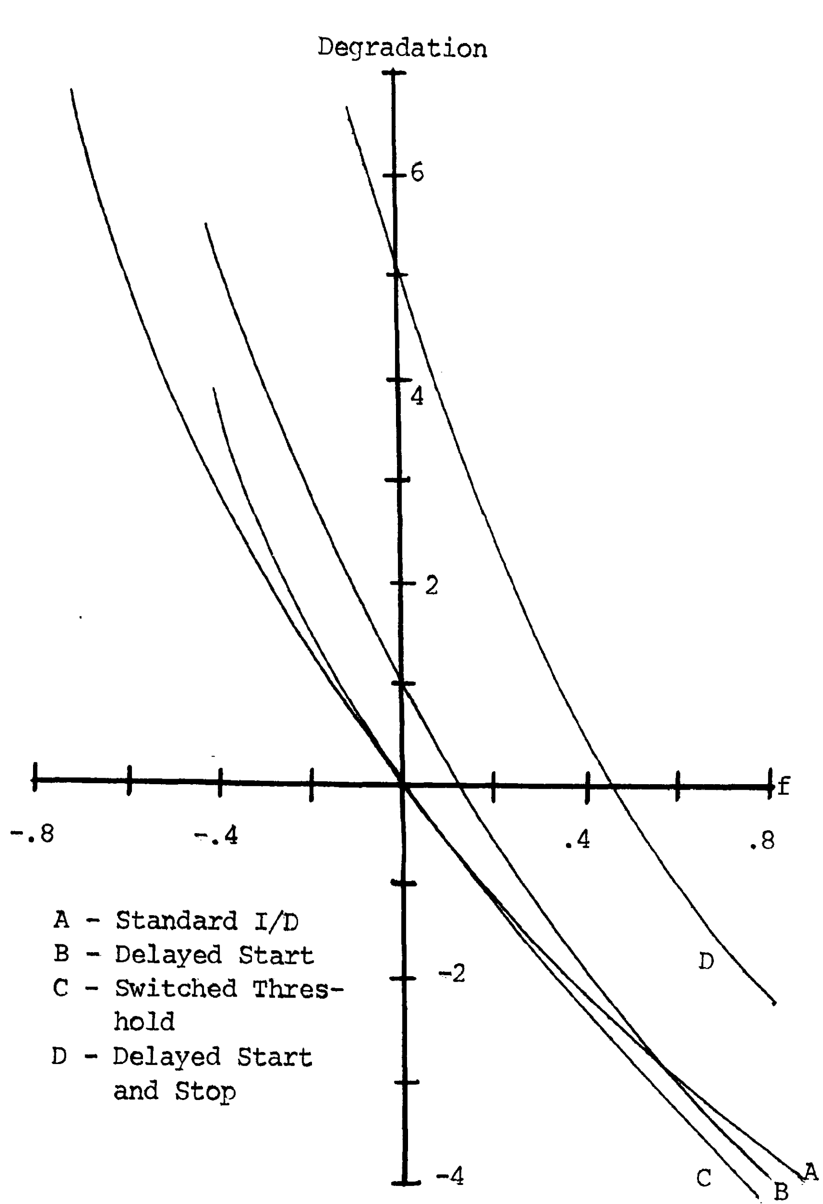

Figure 10a. Receiver Comparison: $\tau = .2T$

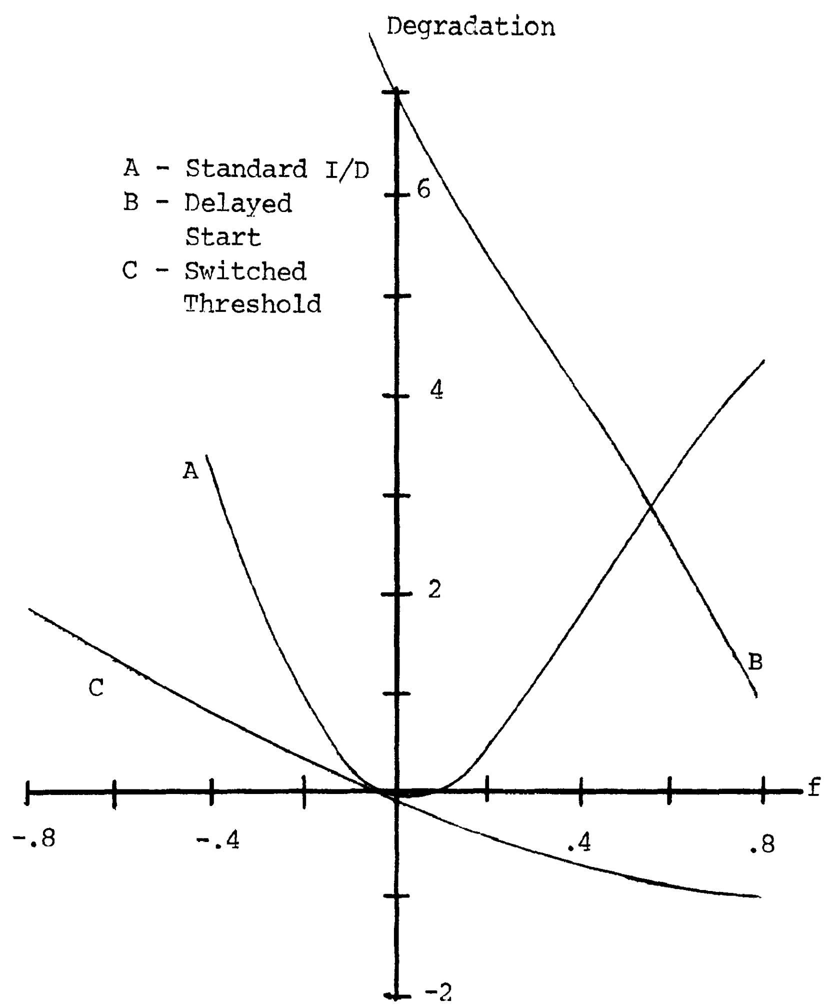

Figure 10b. Receiver Comparison: $\tau = .8T$

be subject to error, and the purpose of this section is to examine the performance degradation due to the error.

Appendix D is devoted to evaluating bit error probability in the receivers of interest, assuming that the receiver makes some specified error in its estimate of the required parameter(s). These error probabilities have been evaluated numerically and some typical results are presented in Figures 11 and 12.

Before interpreting these results it is necessary to give some consideration to the devices which will estimate the parameters. In particular, with respect to the switched threshold receiver, it should be noted that an estimator of f will require knowledge of τ. For example, the estimator might use its decisions from the previous two bits to estimate f by noting that, in the absence of noise,

$$f = \frac{k_{i-1}\tau I_2 - k_{i-1}(T - \tau)I_1}{k_{i-1}(T - \tau)I_1 - k_{i-2}I_2} , \qquad (6)$$

where

$$I_1 = \int_{(i-1)T}^{(i-1)T+\tau} S_T(t) \cos(\omega_0 t)\,dt , \qquad (7)$$

$$I_2 = \int_{(i-1)T+\tau}^{iT} S_T(t) \cos(\omega_0 t)\,dt . \qquad (8)$$

This result is easily verified by substituting Equations 1-3 directly into the above expressions and setting the noise variance to zero.

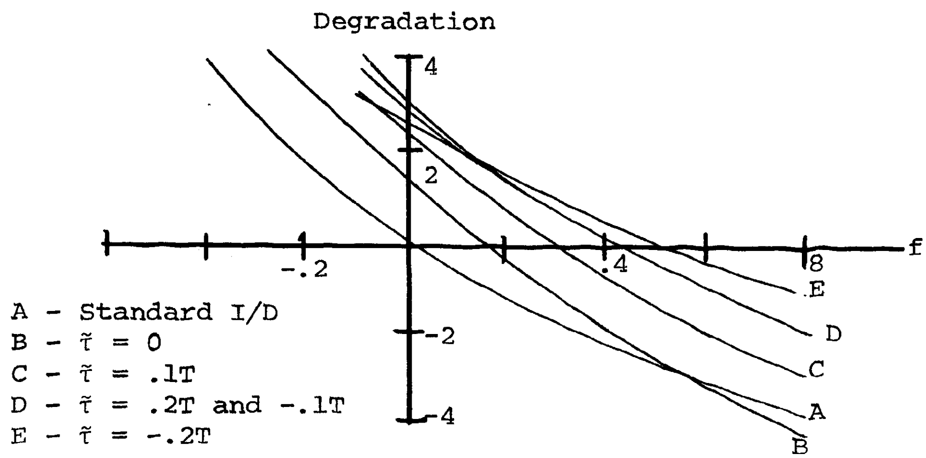

Figure 11a. Degradation versus f For Delayed Start Receiver With Error In Channel Estimate: τ = .2T

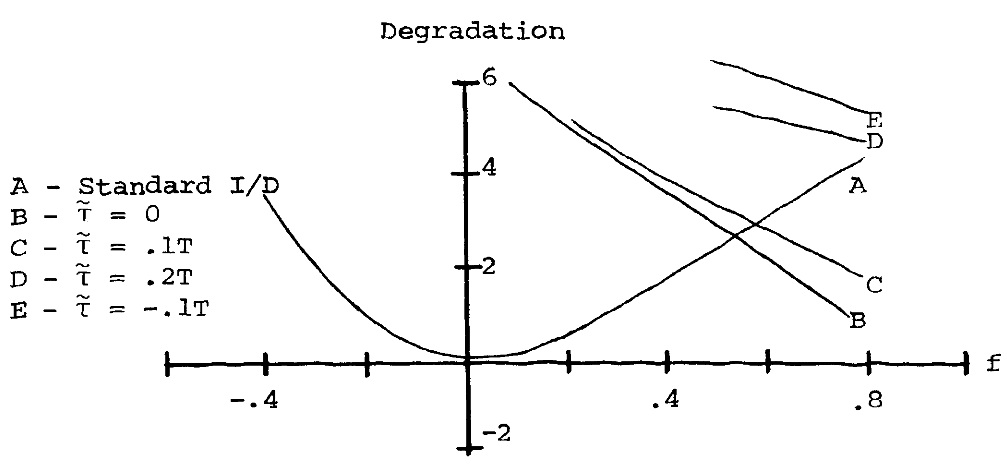

Figure 11b. Degradation versus f For Delayed Start Receiver With Error In Channel Estimate: τ = .8T

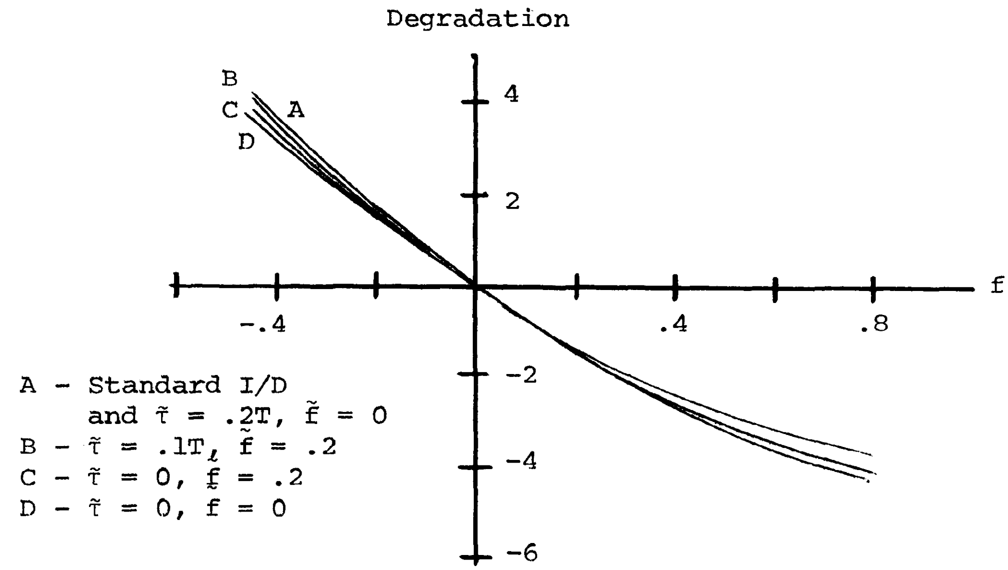

Figure 12a. Degradation versus f For Switched Threshold Receiver With Error In Channel Estimate: τ = .2T

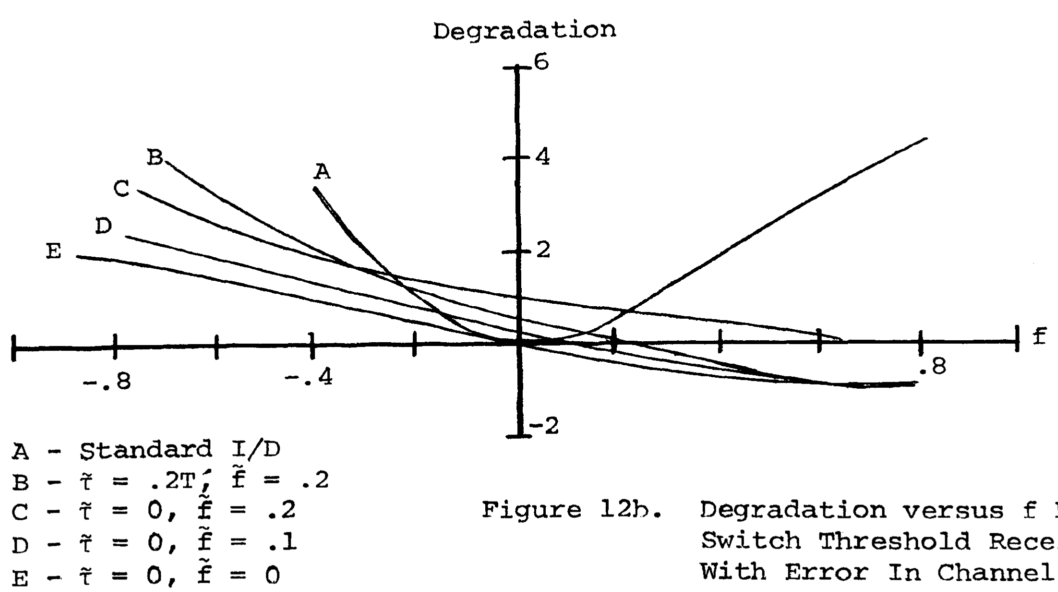

Figure 12b. Degradation versus f For Switch Threshold Receiver With Error In Channel Estimate: τ = .8T

Clearly, this would not be the optimum estimate of f, since the estimation is coupled to the Markov detection problem; however, this example illustrates that an estimate of τ will be a prerequisite to the estimate of f.

At this point it is also noted that f will be extremely sensitive to small changes in τ since $f = \alpha \cos(\omega_0\tau + \theta)$, so we can expect that f will fluctuate much more rapidly than τ.

Thus, since f is the most rapidly changing parameter, and since its estimate depends upon an estimate of τ, we expect that the estimate of f will be considerably less reliable than estimates of τ, so the performance sensitivity to estimates of f are of primary significance.

The following points relating to Figures 11 and 12 should be noted:

1) The delayed start receiver is considerably more sensitive to errors in the estimate of τ than the switched threshold receiver. In addition, the delayed start receiver is about twice as sensitive to negative errors in $\hat{\tau}$ (i.e., $\hat{\tau}-\tau<0$) as it is to positive errors.
2) The delayed start receiver results in improved performance only if f and τ are near 1, and if the standard deviation of $\frac{\hat{\tau}}{T}$ is less than about .05.
3) For small delays, the small performance advantages afforded by the switched threshold receiver are virtually lost if the error in

$\hat{f}$ is of the order of .2. This fact is of signifacance when it is considered that even small absolute errors in $\hat{\tau}$ could represent significant percentage errors for small τ. This high percentage error might well be reflected into the estimate of f.

4) For large delays, the I/D receiver performs better than the switched threshold for small $|f|$ if errors in the estimate of f are present. For large $|f|$ and large delay, the switched threshold receiver performs considerably better.

III. CONCLUSIONS

It was found that intersymbol interference due to specular multipath can be a significant source of performance degradation in digital communication links if the multipath has a delay comparable to the bit period, has an amplitude comparable to the direct signal, and is in phase with the direct component.

The delayed start integrate-and-dump receiver, which begins integrating only after the delayed component of the previous bit has terminated, affords improved performance only in the very limited case where the delay is fairly large and the in-phase component of the delayed signal is nearly the same amplitude as the direct component. If the delayed component amplitude is less, performance degrades quite rapidly. Furthermore, the receiver requires an estimate of the delay, and if this estimate is in error by as much as .15T, where T is the bit period, the receiver performs worse than the standard integrate-and-dump receiver for all f. Thus, the disadvantages of this receiver apparently make it impractical for any general purpose application, and not even particularly attractive as a solution for the special case for which the amplitude of the interfering component is large ($f \cong 1$).

The switched threshold receiver, which subtracts an estimate of the previous bit's tail before making a decision, performs better than the integrate-and-dump receiver in all cases of two component specular multipath if the channel is known. Not only does it eliminate the intersymbol interference, but it also improves performance under conditions of fading. However, significant performance improvement occurs only for large delay and/or amplitude of the delayed component. In addition, the receiver must estimate two parameters before making a decision: τ, the delay, and f, which includes the amplitude and phase of the delayed component. Errors of about 10% in these estimates cause a loss of much of the improvement in performance for small delays. For large delays, performance is actually degraded if $|f|$ is small and poorly known. When $|f|$ is large, performance gains can be considerable even if the parameter estimates are in error by 20%; however, loss of bit synchronization and phase lock will probably pose formidable problems when both f and τ are large. Furthermore, f can be expected to fluctuate rapidly at high carrier frequencies, making its estimation difficult and costly. Since these comments can reasonably be expected to apply to the optimum receiver of Gonsalves, we must conclude that, except in very special cases, alleviation of degradation due to two component specular multipath by modifying the receiver structure should be approached only after cost-effectiveness has been evaluated for alternative solutions; for example, space diversity and efficient coding. In those cases where the delay is large, the delayed component is of amplitude comparable to the direct component, and the channel is fairly stable, a modified receiver structure might be useful. However, more work would need to be done on the phase lock and bit synchronization problems in this channel before definite conclusions could be drawn. Other possibilities for future work include the performance analysis on the receivers considered here in the presence of more general types of interference, such as Rayleigh or Rician. In addition, experimental surveys to determine the prevalence of the two-component specular multipath channel would be of interest.

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VITA

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APPENDIX A

Bit Error Probability For The Integrate And Dump Receiver In Two Component Specular Multipath

Bit error probability is calculated in this appendix for an ideal correlation receiver for the PSK signals defined by Equation (2) and shown in Figure 1. After the result is obtained for arbitrary start and stop times on the integrator, simplifications are made for the special cases of interest.

We first consider the expected value of the output of the integrator due to the signal component. Note that this is the expected value averaged over the noise and conditioned upon the bit sign, k_i.

$$\overline{S}_{s0} = \int_{T_0}^{T_f} S_s(t) \cos(\omega_0 t)\,dt \tag{A.1}$$

$$= \sqrt{\frac{2E}{T}} \int_{T_0}^{T} \sin\left(\omega_0 t + k_i \frac{\pi}{2}\right) \cos(\omega_0 t)\,dt$$

$$+ U(T_f - T)\sqrt{\frac{2E}{T}} \int_{T}^{T_f} \sin\left(\omega_0 t + k_{i+1} \frac{\pi}{2}\right) \cos(\omega_0 t)\,dt \tag{A.2}$$

$$= \sqrt{\frac{2E}{T}} \left[k_i (T - T_0) + U(T_f - T)\, k_{i+1}\, (T_f - T)\right], \tag{A.3}$$

where $(i-1)T \leq T_0 \leq iT$

$$iT \leq T_f \leq (i+1)T$$

$$U(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \tag{A.4}$$

The expected value of the integrator output due to the multipath component is now considered.

$$\overline{S}_{r0} \triangleq \int_{T_0}^{T_f} S_r(t) \cos(\omega_0 t)dt \tag{A.5}$$

$$= U(\tau - T_0)\alpha\sqrt{\frac{2E}{T}} \int_{T_0}^{T} \sin\left[\omega_0(t-\tau) - \theta + k_i \frac{\pi}{2}\right] \cos\omega_0 t \, dt \tag{A.6}$$

$$+ U(T_0 - \tau)\alpha\sqrt{\frac{2E}{T}} \int_{T_0}^{T} \sin\left[\omega_0(t-\tau) - \theta + k_i \frac{\pi}{2}\right] \cos\omega_0 t \, dt$$

$$+ U(T + \tau - T_f)U(T_f - T)\alpha\sqrt{\frac{2E}{T}} \int_{T}^{T_f} \sin\left[\omega_0(t-\tau) - \theta + k_i \frac{\pi}{2}\right]$$

$$\cos\omega_0 t \, dt$$

$$+ \quad U(\tau - T_0)\alpha\sqrt{\frac{2E}{T}} \int_{\tau}^{T} \sin\left[\omega_0(t-\tau) - \theta + k_i \frac{\pi}{2}\right] \cos \omega_0 t \, dt$$

$$+ \quad U(T_f - T - \tau)\alpha\sqrt{\frac{2E}{T}} \left[\int_{T+\tau}^{T_f} \sin\left[\omega_0(t-\tau) + k_{i+1} \frac{\pi}{2} - \theta\right] \right.$$

$$\left. \cos \omega_0 t \, dt + \int_{T}^{T+\tau} \sin\left[\omega_0(t-\tau) - \theta + k_i \frac{\pi}{2}\right] \cos (\omega_0 t) dt \right]$$

$$= \quad f\sqrt{\frac{E}{2T}} \left[U(\tau - T_0)k_{i-1}(\tau - T_0) + U(\tau - T_0)k_i(T - \tau)\right.$$

$$+ U(T_0 - \tau)k_i(T - T_0)$$

$$+ U(T + \tau - T_f)U(T_f - T)(T_f - T)k_i + U(T_f - T - \tau)$$

$$\left.(k_i\tau + (T_f - T - \tau)k_{i+1})\right] , \qquad \text{(A.7)}$$

where $f = \alpha \cos (\omega_0\tau + \theta)$. (A.8)

Consequently, the (conditional) expected value of the total output of the integrator is

$$\overline{S}_{T0} = \overline{S}_{s0} + \overline{S}_{r0} \tag{A.9}$$

$$= \sqrt{\frac{E}{2T}}\, [f\, k_{i-1}(\tau - T_0)U(\tau - T_0) + k_i\ (T - T_0 + f(T - \tau)U(\tau - T_0)$$

$$+ f(T - T_0)U(T_0 - \tau) + f(T_f - T)U(T + \tau - T_f)U(T_f - T)$$

$$+ f\tau\ U(T_f - T - \tau))$$

$$+ k_{i+1}(f(T_f - T - \tau)U(T_f - T - \tau) + (T_f - T)U(T_f - T))] \tag{A.10}$$

The bit error probability consists of 8 terms corresponding to the 8 possibilities of (k_{i-1}, k_i, k_{i+1}); however, from the symmetry of the problem, it is clear that

$$P_E(k_{i-1}, k_i, k_{i+1}) = P_E(-k_{i-1}, -k_i, -k_{i+1}) \ , \tag{A.11}$$

so the total P_E consists of 4 equally likely terms (since $P_r\{k_i\} = P_r\{-k_i\} = \frac{1}{2}$):

$$P_E = \frac{1}{4}[P_E(1,1,1) + P_E(1,1,-1) + P_E(-1,1,1) + P_E(-1,1,-1)] \ . \tag{A.12}$$

Since the noise is gaussian, the variance of S_{T0} is easily found:

$$\sigma^2 = E\{ \int_{T_0}^{T_f} n(t) \cos(\omega_0 t)dt \int_{T_0}^{T_f} n(t')\cos(\omega_0 t')dt' \} \tag{A.13}$$

$$= \int_{T_0}^{T_f} \int_{T_0}^{T_f} E\{n(t)n(t')\} \cos(\omega_0 t) \cos(\omega_0 t')dt\, dt'$$

$$= \int_{T_0}^{T_f} \int_{T_0}^{T_f} \frac{N}{2} \delta(t - t') \cos(\omega_0 t) \cos(\omega_0 t')dt\, dt'$$

$$= \frac{N}{2} \int_{T_0}^{T_f} \cos^2(\omega_0 t)dt$$

$$= \frac{N}{4}(T_f - T_0) \ . \tag{A.14}$$

This allows the straightforward calculation

$$P_E(k_{i-1}, 1, k_{i+1}) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[- \frac{(R - S_{T0})^2}{2\sigma^2} \right] dR \tag{A.15}$$

$$= \int_{-\infty}^{-\frac{S_{T0}}{\sqrt{2\sigma}}} \frac{1}{\sqrt{\pi}} e^{-x^2} \, dx$$

$$= \frac{1}{2} \operatorname{erfc} \left[\frac{S_{T0}}{\sqrt{2\sigma}} \right] . \qquad (A.16)$$

Thus,

$$P_E(k_{i-1}, 1, k_{i+1}) = \frac{1}{2} \operatorname{erfc} \{ \sqrt{\frac{E}{N}} [k_{i-1} f \, U(\tau - T_0) + (T - T_0)$$

$$+ f \, U(\tau - T_0)(T - \tau) \qquad (A.17)$$

$$+ fU(T_0 - \tau)(T - T_0) + fU(T + \tau - T_f)U(T_f - T)(T_f - T)$$

$$+ fU(T_f - T - \tau)\tau$$

$$+ k_{i+1} [fU(T_f - T - \tau)(T_f - T - \tau) + (T_f - T)U(T_f - T)]]\} .$$

The total probability of error may now be found easily for each case of interest by substituting appropriate values for t_0 and T_f.

1) Standard I/D Receiver: $\left\{ \begin{array}{l} T_0 = 0 \\ T_f = T \end{array} \right.$

$$P_E = \frac{1}{4} [\mathrm{erfc} \{\sqrt{\frac{E}{N}} [1 + f - 2f \frac{\tau}{T}]\} + \mathrm{erfc} \{\sqrt{\frac{E}{N}} (1 + f)\}]$$

(A.18)

2) Delayed Start I/D Receiver: $\left\{ \begin{array}{l} T_0 = \tau \\ T_f = T \end{array} \right.$

$$P_E = \frac{1}{2} \mathrm{erfc} \{\sqrt{\frac{E(T-\tau)}{NT}} (1 + f)\} \qquad \text{(A.19)}$$

3) Delayed Start and Stop Receiver: $\left\{ \begin{array}{l} T_0 = \tau \\ T_f = T + \tau \end{array} \right.$

$$P_E = \frac{1}{4} [\mathrm{erfc} \{\sqrt{\frac{E}{N}} (1 + f)\} + \mathrm{erfc} \{\sqrt{\frac{E}{N}} (1 + f - 2 \frac{\tau}{T})\}]$$

(A.20)

APPENDIX B

Condition For Improved Performance With The Delayed Start Receiver

In this appendix the conditions on f and τ are established for which P_E is decreased by shortening the integration period. Referring to Appendix A and substituting $T_f = T$, we find that

$$P_E = \frac{1}{4} [\text{erfc} \{[(f + 1)T + (f - 1)T_0 - 2f\tau]\sqrt{\frac{E}{NT(T - T_0)}} \}$$

$$+ \text{erfc} \{(1 + f)\sqrt{\frac{E(T - T_0)}{NT}} \}] \qquad \text{(B.1)}$$

We wish to minimize P_E with respect to T_0, or, more specifically, when will P_E be minimized by making $T_0 > 0$? To answer this question, consider the sign of the slope of P_E:

$$\frac{\partial P_E}{\partial T_0} = \frac{1}{4} \frac{\partial}{\partial T_0} \{ \int_{\frac{S_{TO_1}}{\sqrt{2\sigma}}}^{\infty} \frac{2}{\sqrt{\pi}} e^{-x^2} dx + \int_{\frac{S_{TO_2}}{\sqrt{2\sigma}}}^{\infty} \frac{2}{\sqrt{\pi}} e^{-x^2} dx\} \qquad \text{(B.2)}$$

where $$\frac{\overline{S}_{TO_1}}{\sqrt{2\sigma}} = \sqrt{\frac{E}{NT(T-T_0)}} [(1 + f)T + (-1 + f)T_0 - 2f\tau] \qquad \text{(B.3)}$$

$$\frac{\overline{S}_{TO_2}}{\sqrt{2}\sigma} = \sqrt{\frac{E(T-T_0)}{NT}}\ (1 + f) \tag{B.4}$$

$$= -\frac{e^{-\left[\frac{\overline{S}_{TO_1}}{\sqrt{2}\sigma}\right]^2}}{2\sqrt{2\pi}}\left[\frac{\partial}{\partial T_0}\left(\frac{\overline{S}_{TO_1}}{\sigma}\right) + e^{\frac{-(\overline{S}^2_{TO_2} - \overline{S}^2_{TO_2})}{2\sigma^2}}\ \frac{\partial}{\partial T_0}\left(\frac{\overline{S}_{TO_2}}{\sigma}\right)\right] \tag{B.5}$$

$$= -\frac{e^{-\left[\frac{\overline{S}_{TO_1}}{\sqrt{2}\sigma}\right]^2}}{4}\sqrt{\frac{E}{N\pi T(T-T_0)}}\left\{(f + 1)\frac{T}{T - T_0} + (f - 1)\frac{T_0}{T - T_0} - 2f\frac{\tau}{T - T_0} - 2 + 2f + e^{\frac{-(S^2_{TO_2} - S^2_{TO_1})}{2\sigma^2}}[f + 1]\right\} \tag{B.6}$$

This will be non-positive if

$$(f + 1)\frac{T}{T - T_0} + (f - 1)\frac{T_0}{T - T_0} - 2f\frac{\tau}{T - T_0} + 2(f - 1) + e^{\frac{-(S^2_{TO_1} - S^2_{TO_2})}{2\sigma^2}} \geq 0 \tag{B.7}$$

Letting $f \cong 1$ and $T_0 = \tau$, we find that this reduces to $f > \frac{1}{3}$, which is trivially satisfied if $f \cong 1$. Thus, reducing the integration period is beneficial in the case $f \cong 1$.

APPENDIX C

Bit Error Probability For The Switched Threshold Receiver

This appendix develops the optimum receiver which is independent of adjacent bit period information. This receiver is then extended to the switched threshold receiver.

If we assume that $k_i = \pm 1$ with equal probability, that k_i is independent of $k_{j \neq i}$, that either type of error in decision is equally "costly", and that the cost of a correct decision is zero, then the Bayes risk is minimized [7] by making the decision such that

$$\Lambda(R) = \frac{p_{r|H_1}(R|H_1)}{p_{r|H_0}(R|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1, \tag{C.1}$$

where the hypotheses H_0 and H_1 are defined by:

$$\sqrt{\frac{2E}{T}} \sin\left(\omega_0 t + \frac{\pi}{2}\right) + \alpha\sqrt{\frac{2E}{T}} \sin\left[\omega_0(t-\tau) - \theta + k_{i-1}\frac{\pi}{2}\right] + n(t) \tag{C.2}$$

when $(i-1)T < t \leq (i-1)T + \tau$

H_1: $s(t) =$

$$\sqrt{\frac{2E}{T}} \sin\left(\omega_0 t + \frac{\pi}{2}\right) + \alpha\sqrt{\frac{2E}{T}} \sin\left[\omega_0(t-\tau) - \theta + \frac{\pi}{2}\right] + n(t), \tag{C.3}$$

when $(i-1)T + \tau < t \leq iT$

$$\sqrt{\frac{2E}{T}} \sin \left(\omega_0 t - \frac{\pi}{2}\right) + \alpha\sqrt{\frac{2E}{T}} \sin \left[\omega_0 (t-\tau) - \theta + k_{i-1} \frac{\pi}{2}\right] + n(t), \tag{C.4}$$

when $(i-1)T < t \leq (i-1)T + \tau$

H_0: $S(t) =$

$$\sqrt{\frac{2E}{T}} \sin \left(\omega_0 t - \frac{\pi}{2}\right) + \alpha\sqrt{\frac{2E}{T}} \sin \left[\omega_0 (t-\tau) - \theta - \frac{\pi}{2}\right] + n(t), \tag{C.5}$$

when $(i-1)T + \tau < t \leq iT$.

Notice that k_i is an unwanted parameter with probability density function

$$P(k_i) = \frac{1}{2} [\delta(k_i + 1) + \delta(k_i - 1)] . \tag{C.6}$$

Consequently, the appropriate likelihood ratio is [7]

$$\Lambda(R) = \frac{P(R|H_1,k_{i-1} = -1) + P(R|H_1,k_{i-1} = +1)}{P(R|H_0,k_{i-1} = -1) + P(R|H_0,k_{i-1} = +1)} \tag{C.7}$$

Now note that correlation of the signal with the difference of the two possible transmitted signals will yield a sufficient statistic, as can easily be verified through Karhunen-Loeve expansion [7]. Thus, let

$$R = \int_{(i-1)T}^{T} \cos(\omega_0 t) S(t)\, dt . \tag{C.8}$$

We now proceed to calculate the required conditional a priori densities. Since all densities required are Gaussian, we require only the mean and variance of each:

$$E\{R|H_1,k_{i-1}=-1\} = \sqrt{\frac{2E}{T}}\,[(1+\alpha\cos(-\omega_0\tau-\theta-\pi))\int_0^T \sin^2(\omega_0 t+\tfrac{\pi}{2})\,dt$$

$$+\alpha\sin(-\omega_0\tau-\theta-\pi)\int_0^T \sin(\omega_0 t+\tfrac{\pi}{2})\cos(\omega_0 t+\tfrac{\pi}{2})\,dt]$$

$$+\sqrt{\frac{2E}{T}}\,[(1+\alpha\cos(-\omega_0\tau-\theta))\int_\tau^T \sin^2(\omega_0 t+\tfrac{\pi}{2})\,dt +$$

$$+\alpha\sin(-\omega_0\tau-\theta)\int_\tau^T \sin(\omega_0 t+\tfrac{\pi}{2})\cos(\omega_0 t+\tfrac{\pi}{2})\,dt] \qquad (C.9)$$

$$= [(1+f)T-2f\tau]\sqrt{\frac{E}{2T}}\,, \qquad (C.10)$$

where $\quad f = \alpha\cos(\omega_0\tau+\theta)$ (C.11)

Similarly,

$$E\{R|H_1,k_{i-1}=+1\} = (1+f)\sqrt{\frac{ET}{2}} \tag{C.12}$$

$$E\{R|H_0,k_{i-1}=-1\} = -(1+f)\sqrt{\frac{ET}{2}} \tag{C.13}$$

$$E\{R|H_0,k_{i-1}=+1\} = -[(1+f)T - 2f\tau]\sqrt{\frac{E}{2T}} \tag{C.14}$$

The noise variances are, of course, identical:

$$\sigma^2 = \int_0^T \int_0^T \cos(\omega_0 t)\cos(\omega_0 t')\, E\{n(t)n(t')\}dt\, dt' \tag{C.15}$$

$$= \frac{NT}{4} \tag{C.16}$$

The likelihood ratio becomes:

$$\Lambda(R) = \frac{\exp\{-\dfrac{(R-[(1+f)T - 2f\tau]\sqrt{\frac{E}{T2}})^2}{NT/2}\} + \exp\{-\dfrac{[R - (1+f)\sqrt{\frac{ET}{2}}]^2}{NT/2}\}}{\exp\{-\dfrac{(R+[(1+f)T - 2f\tau]\sqrt{\frac{E}{2T}})^2}{NT/2}\} + \exp\{-\dfrac{[R - (1+f)\sqrt{\frac{ET}{2}}]^2}{NT/2}\}} \tag{C.17}$$

$$= e^{\frac{8}{N}\sqrt{\frac{E}{2T}}(1-f\,\frac{\tau}{T})R}\;\frac{\cosh\left\{\frac{2f\tau E}{NT}\left[\sqrt{\frac{2}{ET}}\,R + \left(2f\,\frac{\tau}{T} - [1+f)\right)\right]\right\}}{\cosh\left\{\frac{2f\tau E}{NT}\left[\sqrt{\frac{2}{ET}}\,R - \left(2f\,\frac{\tau}{T} - [1+f]\right)\right]\right\}} \tag{C.18}$$

$$\overset{H_1}{\underset{H_1}{\gtrless}}\; 1 \tag{C.19}$$

The expected value of the argument of the upper cosh term ranges from about -2 to +1 under the conditions $f \cong 1$ and $\frac{1}{2} < \frac{\tau}{T} < 1$, so a Taylor expansion is of no value, nor can one of the exponentials of cosh be eliminated in the general case. Consequently, the decision criterion for this receiver requires a highly non-linear operation on the data, making the physical implementation of the receiver totally impractical.

Notice, however, that the source of the nonlinearity was the fact that it was necessary to average the a priori densities over k_{i-1}, since no knowledge of this parameter was assumed. In fact, this assumption is unrealistic, since most systems require bit error probabilities less than 10^{-4}, indicating that k_{i-1} is known with near certainty. This fact leads us to construct the optimum receiver assuming that the previous bit is known (with certainty). The result will be seen to be the switched threshold receiver.

Performing an analysis similar to the previous case, but leaving $k_{i-1} = k$ as a parameter yields

$$E\{R|H_1,k\} = \sqrt{\frac{E}{2T}}\,[(1+f)T - f(1-k)\tau] \tag{C.20}$$

$$E\{R|H_0,k\} = -\sqrt{\frac{E}{2T}}\,[(1+f)T - f(1+k)\tau] \tag{C.21}$$

$$\sigma^2 = \frac{NT}{4} \tag{C.22}$$

$$\Lambda = \frac{P_{r|H_1}(R|H_1,k)}{P_{r|H_0}(R|H_0,k)} \tag{C.23}$$

$$= \frac{\exp\left\{ \dfrac{(R - \frac{E}{2T}[(1+f)T - f(1-k)\tau])^2}{NT/2} \right\}}{\exp\left\{ \dfrac{(R + \frac{E}{2T}[(1+f)T - f(1+k)\tau])^2}{NT/2} \right\}} \tag{C.24}$$

$$= \exp\left\{ \frac{8}{NT}\sqrt{\frac{E}{2T}}\,[(1+f)T - f\tau]R \right\} \exp\left\{ -\frac{4Efk}{NT^2}\,\tau[(1+f)T - f\tau] \right\} \tag{C.25}$$

$$= \exp\left\{ \frac{4}{NT}\sqrt{\frac{E}{2T}}\,[(1+f)T - f\tau]\left[2R - \sqrt{\frac{2E}{T}}\,k\tau f\right] \right\} \tag{C.26}$$

$$\underset{H_0}{\overset{H_1}{\gtrless}} \; 1 \tag{C.27}$$

Taking the natural log of the above, the decision rule becomes

$$R \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{E}{2T}}\; k_{i-1}\tau f = \Lambda_0 k_{i-1}\;, \quad \Lambda_0 = \sqrt{\frac{E}{2T}}\, f\tau \tag{C.28}$$

Thus, the receiver compares the correlator output to a threshold whose sign depends upon the previous bit, or, in effect, subtracts off the tail of the previous bit before making a decision.

In order to evaluate the receiver performance, it is necessary to consider the case where the previous decision was incorrect as well as the case where it is correct:

$$P_E = P_E\Big|_{\substack{\text{last bit}\\ \text{correct}}} \cdot P_r\,\{\text{last bit correct}\} + P_E\Big|_{\substack{\text{last bit}\\ \text{incorrect}}} \cdot P_r\,\{\text{last bit incorrect}\} \tag{C.29}$$

$$= P_E\Big|_{\substack{\text{last bit}\\ \text{correct}}} (1 - P_E) + P_E\Big|_{\substack{\text{last bit}\\ \text{incorrect}}} \cdot P_E \tag{C.30}$$

$$= \frac{P_E\big|_{\text{last bit correct}}}{1 + P_E\Big|_{\substack{\text{last bit}\\ \text{correct}}} - P_E\Big|_{\substack{\text{last bit}\\ \text{incorrect}}}} \tag{C.31}$$

It remains to evaluate the conditional error probabilities above. To simplify the notation, let

$$\overline{R}_{jk} = E\{R|H_j,\ k_{i-1}\}\ , \quad j = 0,\ 1 \qquad k = \pm 1 \tag{C.32}$$

$$= k_i \sqrt{\frac{E}{2T}}\ [(1+f)T - (1 - k_i k_{i-1}) f\tau] \tag{C.33}$$

$$P_E\Big|_{\substack{\text{last bit}\\ \text{correct}}} = \frac{1}{4}\ [P_E\Big|_{\substack{k_i=1\\ k_{i-1}=1}} + P_E\Big|_{\substack{k_i=1\\ k_{i-1}=-1}} + P_E\Big|_{\substack{k_i=-1\\ k_{i-1}=1}} + P_E\Big|_{\substack{k_i=-1\\ k_{i-1}=-1}}\ ] \tag{C.34}$$

$$P_E\Big|_{\substack{k_i=1\\ k_{i-1}=1}} = \int_{-\infty}^{\Lambda_0} \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{(R-\overline{R}_{1,1})^2}{2\sigma^2}}\ dR \tag{C.35}$$

$$= \frac{1}{2}\ \text{erfc}\ [\frac{1}{\sqrt{2}\ \sigma}\ (\overline{R}_{11} - \Lambda_0)] \tag{C.36}$$

$$= \frac{1}{2}\ \text{erfc}\ [\sqrt{\frac{E}{N}}\ (1 + f - f\ \frac{\tau}{T})] \tag{C.37}$$

$$P_E\Big|_{\substack{k_i=1 \\ k_{i-1}=-1}} = \int_{-\infty}^{-\Lambda_0} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(R-\overline{R}_{1,-1})^2}{2\sigma^2}}\, dR \tag{C.38}$$

$$= \frac{1}{2}\,\text{erfc}\,[\sqrt{\frac{E}{N}}\,(1 + f - f\,\frac{\tau}{T})] \tag{C.39}$$

$$= P_E\Big|_{\substack{k_i=1 \\ k_{i=1}=1}} \tag{C.40}$$

Similarly, it is easily shown that

$$P_E\Big|_{\substack{k_i=-1 \\ k_{i-1}=1}} = P_E\Big|_{\substack{k_i=-1 \\ k_{i-1}=-1}} = P_E\Big|_{\substack{k_{i=1} \\ k_{i-1}=1}}\ , \tag{C.41}$$

So

$$P_E\Big|_{\substack{\text{last bit} \\ \text{correct}}} = \frac{1}{2}\,\text{erfc}\,\{\sqrt{\frac{E}{N}}\,(1 + f - f\,\frac{\tau}{T})\}\ . \tag{C.42}$$

We now need to consider the probability of error assuming the previous bit was incorrect. The only difference which this introduces is that the k_{i-1} appearing in the threshold, $\Lambda_0 k_{i-1}$, is the negative of the k_{i-1} appearing in $\overline{R}$. In the following, the k_{i-1} subscript on P_E will refer to the k_{i-1} appearing in the threshold, so

$$P_E\Big|_{\substack{k_i=1 \\ k_{i-1}=1}} = \int_{-\infty}^{\Lambda_0} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(R-\overline{R}_{1,-1})^2}{2\sigma^2}}\, dR \tag{C.43}$$

$$= \frac{1}{2}\,\mathrm{erfc}\,\{\frac{\overline{R}_{1,-1} - \Lambda_0}{\sqrt{2}\,\sigma}\} \tag{C.44}$$

$$= \frac{1}{2}\,\mathrm{erfc}\,\{[(1+f) - 3f\,\frac{\tau}{T}]\,\sqrt{\frac{E}{N}}\,\} \tag{C.45}$$

$$P_E\Big|_{\substack{k_i=1 \\ k_{i-1}=-1}} = \int_{-\infty}^{-\Lambda_0} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(R-\overline{R}_{1,1})^2}{2\sigma^2}}\, dR \tag{C.46}$$

$$= \frac{1}{2}\,\mathrm{erfc}\,\{\frac{(R_{1,1} + \Lambda_0)}{\sqrt{2}\,\sigma}\} \tag{C.47}$$

$$= \frac{1}{2}\,\mathrm{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f + f\,\frac{\tau}{T}]\} \tag{C.48}$$

Since the other two cases again reflect the symmetry of the problem, there are only 2 distinct terms in the expression:

$$P_E\Big|_{\substack{\text{last bit}\\ \text{incorrect}}} = \frac{1}{4}\,[\text{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f - 3f\,\frac{\tau}{T}]\}$$

$$+ \text{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f + f\,\frac{\tau}{T}]\}] \qquad \text{(C.49)}$$

The total error probability may now be calculated as in Equation C.31 of this appendix.

APPENDIX D

Error Probabilities Assuming Error In Channel Estimate

The signal and channel models are identical to the previous appendices, but we assume now that the receiver makes an error in its estimate(s) of the channel parameter(s). Throughout the following, a circumflex over a symbol will denote the estimate of that parameter (e.g., $\hat{f}$), and a tilde will denote the error in the estimate; for example, $\tilde{f} = \hat{f} - f$.

The delayed start receiver is considered first. Here we must consider two cases of improperly chosen integration time:

1) $\hat{\tau} > \tau$. In this case the error probability is easily seen to be (see Equation A.19)

$$P_E = \frac{1}{2} \operatorname{erfc} \{\sqrt{\frac{E(T - \hat{\tau})}{NT}}(1 + f)\} \tag{D.1}$$

$$= \frac{1}{2} \operatorname{erfc} \{\sqrt{\frac{E}{NT} (T - \tau - \tilde{\tau}} (1 + f)\} \tag{D.2}$$

2) $\hat{\tau} < \tau$. This case requires reworking the analysis for the receiver:

$$\overline{R}_{i,i-1} = \int_{\hat{\tau}}^{T} \sqrt{\frac{2E}{T}} \sin (\omega_0 t + k_i \frac{\pi}{2}) dt \tag{D.3}$$

$$+ \int_{\hat{\tau}}^{\tau} \sqrt{\frac{2E}{T}}\, \alpha \sin [\omega_0 (t - \tau) + k_{i-1} \frac{\pi}{2} - \theta] \cos (\omega_0 t) dt$$

$$+ \int_{\tau}^{T} \sqrt{\frac{2E}{T}}\, \alpha \sin [\omega_0 (t - \tau) + k_i \frac{\pi}{2} - \theta] \cos (\omega_0 t) dt$$

$$R_{i,i-1} = \sqrt{\frac{E}{2T}}\, [k_i (1 + f)(T - \tau) + k_{i-1} f(\tau - \hat{\tau})] \qquad (D.4)$$

Performing the integrations as in the previous appendices, we find that

$$P_E = \frac{1}{4} [\text{erfc} \{\sqrt{\frac{E}{TN(T-\tau-\tilde{\tau})}}\, [(1 + f)(T - \tau) - f\tilde{\tau}]\} \qquad (D.5)$$

$$+ \text{erfc} \{\sqrt{\frac{E}{N(T-\tau-\tilde{\tau})T}}\, [(1 + f)(T - \tau) + f\tilde{\tau}]\} ,$$

when $\hat{\tau} < \tau$.

We now turn to the switched threshold receiver. Performing an analysis parallel to the known channel case we see

$$P_E\Big|_{\substack{\text{last bit}\\ \text{correct}}} = \frac{1}{4}\,[P_E\Big|_{\substack{k_i=1\\ k_{i-1}=1}} + P_E\Big|_{\substack{k_i=1\\ k_{i-1}=-1}} + P_E\Big|_{\substack{k_i=-1\\ k_{i-1}=1}} + P_E\Big|_{\substack{k_i=-1\\ k_{i-1}=-1}}\,] \qquad \text{(D.6)}$$

$$P_E\Big|_{\substack{k_i=1\\ k_i=1}} = \int_{-\infty}^{\hat{\Lambda}_0} \frac{1}{\sqrt{2\pi}\,\sigma}\,\exp\,[-\,\frac{(R-\overline{R}_{1,1})^2}{2\sigma^2}]\;dR\,, \qquad \text{(D.7)}$$

$$= \frac{1}{2}\,\text{erfc}\,\{\frac{1}{\sqrt{2}\,\sigma}\,(\overline{R}_{1,1} - \hat{\Lambda}_0)\} \qquad \text{(D.8)}$$

$$= \frac{1}{2}\,\text{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f - (f + \tilde{f})(\tau - \tilde{\tau})/T\} \qquad \text{(D.9)}$$

Similarly

$$P_E\Big|_{\substack{k_i=1\\ k_{i-1}=-1}} = \frac{1}{2}\,\text{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f - 2f\,\frac{\tau}{T} + (f + \tilde{f})(\tau - \tilde{\tau})/T]\} \qquad \text{(D.10)}$$

Thus, it is clear that

$$P_E\Big|_{\substack{\text{last bit}\\ \text{correct}}} = \frac{1}{4}\,[\text{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f + (f + \tilde{f})(\tau + \tilde{\tau})/T]\}$$

$$+\,\text{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f - 2f\,\tau/T + (f + \tilde{f})(\tau - \tilde{\tau})/T]\}]$$

Similarly, it is readily shown that

$$P_E\Big|_{\substack{\text{last bit}\\ \text{incorrect}}} = \frac{1}{4}\,[\mathrm{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f + (f + \tilde{f})(\tau + \tilde{\tau})/T]\} \tag{D.11}$$

$$+ \mathrm{erfc}\,\{\sqrt{\frac{E}{N}}\,[1 + f - 2f\tau/T - (f + \tilde{f})(\tau - \tilde{\tau})/T]\}] \tag{D.12}$$

Finally, recall that the total bit error probability is given by Equation C.31 of Appendix C.