Free vibrations of circular cylindrical shells

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FREE VIBRATIONS OF CIRCULAR CYLINDRICAL SHELLS

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THESIS

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ABSTRACT

In this report bending theory of shells is used to determine the natural frequencies and mode shapes of circular cylindrical shells. The governing eighth-order system of differential equations has been put in a form which is especially suitable for numerical integration and application of different sets of homogeneous boundary conditions. The Holzer method is used to solve the eigenvalue problem. During numerical integration of the differential equations, the exponentially growing solutions are suppressed whenever they become larger than previously selected values. Numerical results are obtained for various shell geometry parameters and for three different sets of homogeneous boundary conditions. These results are compared with the energy method solutions developed by Rayleigh and by Arnold and Warburton. The difference between the results obtained by the numerical integration method and the energy method has been found to be less than 10 percent for all the cases.

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TABLE OF CONTENTS

LIST OF SYMBOLS

a = radius of the cylinder

h = thickness of the cylinder wall

L = characteristic length of the shell

ε = axial coordinate

x = dimensionless axial coordinate ε/a

ϕ = circumferential coordinate

$\bar{u}, \bar{v}, \bar{w}$ = components of displacement at the middle surface in the axial, tangential and radial directions

u_n, v_n, w_n = middle surface displacement components for nth harmonic

$\bar{\theta}$ = $\partial\bar{w}/\partial x$

$\bar{M}_\phi, \bar{M}_x$ = bending moments

$\bar{M}_{\phi x}, \bar{M}_{x\phi}$ = twisting moments

$\bar{N}_\phi, \bar{N}_x$ = normal forces

$\bar{N}_{\phi x}, \bar{N}_{x\phi}$ = in-plane shear forces

$\bar{Q}_\phi, \bar{Q}_x$ = transverse shear forces

$\bar{T}_x$ = effective tangential shear stress resultant = $\bar{N}_{x\phi} - \bar{M}_{x\phi}/a$

$\bar{S}_x$ = effective radial shear stress resultant = $\bar{Q}_x + \frac{1}{a}\ \frac{\partial \bar{M}_{x\phi}}{\partial \phi}$

m = number of axial half-waves

n = number of circumferential waves

E = Young's modulus of elasticity

LIST OF SYMBOLS (continued)

ν = Poisson's ratio

ρ = mass density of shell material

ω = circular frequency

ω_n = natural circular frequency

ω_o = lowest extensional frequency of a ring in plane strain = $[E/\rho a^2(1-\nu^2)]^{1/2}$

ω_n/ω_o = frequency factor

U_i = fundamental shell variable

D = $Eh/(1 - \nu^2)$

K = $Eh^3/(1 - \nu^2)$

$(\ldots)'$ = $\partial(\ldots)/\partial x$

$(\dot{\ldots})$ = $\partial(\ldots)/\partial \phi$

$\overline{D}$ = matrix of partial solutions

M_i = suppression matrix at the ith suppression point

B_i = matrix of constants to be determined at the ith suppression point

Δ = characteristic determinant

LIST OF FIGURES

LIST OF TABLES

I. INTRODUCTION

Cylindrical shell structures are widely used for industrial applications such as for spacecraft , electric machinery, storage tanks, to cite but a few. Therefore, it is important to know the modal characteristics (natural frequencies and mode shapes) of such structural elements in order to predict their dynamic behavior. When natural modes are excited, excessive deformations and internal stresses may result. In the case of rotating machinery with structural shell frames, a knowledge of natural frequencies is desirable because of its direct relation to noise generation.

Since a general formulation of a free shell problem by bending theory leads to a set of eighth-order differential equations with four boundary conditions at each end, it is difficult to find an exact solution in closed form except in a few special cases. Therefore, to solve such problems most authors have used approximate techniques involving variational principles, finite element methods and simplified (Donnell) equations. The references for these methods can be found in Kraus (1)*.

An exact method, outlined by Flugge in 1934, has been used in a recent paper by Forsberg (2) in which he

*Numbers in parentheses refer to the list of references at the end of the thesis.

has described the influence of boundary conditions on the modal behavior of thin circular cylindrical shells. In another paper (3) he compares his exact solutions with solutions obtained by various approximate methods. Another method is the numerical integration of the differential equations with the aid of a suppression technique. This is also an exact method in the sense that no assumptions or simplifications are made except those already introduced in the theory of thin shells.

A. Objectives

While the above procedures for obtaining exact solutions are available in the literature, numerical results are not available in sufficient detail to allow critical comparison with results from the approximate methods commonly used in engineering practice. The objectives of this investigation of thin circular cylindrical shells are:

1. To determine exact natural frequencies by numerical integration aided by a suppression technique,

2. To compare the exact solutions with approximate solutions developed by Rayleigh (4) and Arnold and Warburton (5).

Ranges of shell length, radius and thickness are such that:

$$L/a = 1,2,3 \text{ and } h/a = 0.02, 0.03, 0.05.$$

Boundary conditions at the ends of these shells are:

(a) Free-Free

(b) Clamped-Clamped with axial constraint

(c) Simply supported without axial constraint

B. Review of Literature

An energy method has been used by Rayleigh (4) in the theory of inextensional vibration of circular cylindrical shells, and by Arnold and Warburton (5) for predicting approximate natural frequencies of freely supported and clamped-clamped cylinders. This method has the significant advantage of requiring very little computer time in determining frequency patterns for different cases. Frequencies obtained by this technique are commonly used as first approximations for exact but iterative types of solutions. One serious disadvantage of the above energy method is that all types of boundary conditions cannot be handled readily with the same displacement functions and, for each case, an entirely new solution has to be generated. Secondly, before using these solutions, one must determine their accuracy for the range of parameters of interest.

Accuracy may be determined by comparison with experimental results or results from some exact method such as the method of stepwise integration.

In the stepwise integration procedure, the governing equations of the problem are reduced to first-order differential equations of the type:

$$\frac{dU_i}{dx} = f_i(U_j, x), \qquad \begin{array}{l} i = 1, 2, \ldots, 8 \\ j = 1, 2, \ldots, 8. \end{array}$$

Zarghamee and Robinson (6) and Carter, Robinson and Schnobrich (7) use u, v, w, u', v', w', w" and w''' as the eight variables represented by U_i, i = 1, 2, ..., 8 respectively in the dynamic analysis of axisymmetric shells with one end fixed. Goldberg and Bogdanoff (8) and Kalnins (9) use a better choice of u, v, w, θ, N_x, M_x, T_x, and S_x, because these eight fundamental variables are directly involved in boundary conditions at the ends of the shells and these, upon integration, produce explicit point values of the quantities of immediate interest.

Many simple as well as sophisticated algorithms are available for the purpose of stepwise integration. Galletely (10) has used the fourth-order Runge-Kutta method while some other authors have used Adam's method. Adam's method gives more accurate results but it is not self starting as is the Runge-Kutta method.

One difficulty in the stepwise integration procedure encountered by Sepetoski, et al (11) and Galletely, et al (12) is that accuracy of the solutions is lost if the generator of the shell exceeds a critical length. The loss of accuracy is not caused by the cumulative errors in the integration process but is the result of subtraction of almost equal, very large numbers in the process of determining some of the unknown boundary values.

This problem was resolved by Kalnins (9) and Cohen (13) by dividing the shell into short segments along its generator. The differential equations were then integrated over each segment and the partial solutions were combined to satisfy the continuity at each segment junction.

Another technique, developed independently by Zarghamee and Robinson (6) and by Goldberg, Setlur and Alspaugh (14), is to suppress the rapidly growing solutions as many times as necessary while the integration proceeds along the generator. This method is preferred to the multisegment method because only half as many partial solutions of the differential equations are required. Results obtained by the stepwise integration method aided by the suppression technique compare well with the exact solutions of Forsberg (3). As this method can be easily applied to all shell configurations, it is considered one of the simplest and most accurate approaches to the numerical solutions of the equations describing the theory of thin elastic shells.

II. DYNAMIC CIRCULAR CYLINDRICAL SHELL EQUATIONS

A. Derivation

The equations of motion for free vibration of thin circular cylindrical shells and relations between stress resultants and displacements for this analysis have been developed by Flugge (15). Since the effects of shear distortion and rotatory inertia of the shell wall have been neglected, the results apply only for thin shells ($L/ma > 10h/a$, $\pi/n > 10h/a$).

Analysis of the forces and moments acting on the differential shell element in Figure 1 gives the equations of motion:

$$\bar{N}_x' + \dot{\bar{N}}_{\phi x} = \rho h a \frac{\partial^2 u}{\partial t^2} ,$$

$$\dot{\bar{N}}_\phi + \bar{N}_{x\phi}' - \bar{Q}_\phi = \rho h a \frac{\partial^2 v}{\partial t^2} ,$$

$$\dot{\bar{Q}}_\phi + \bar{Q}_x' + \bar{N}_\phi = - \rho h a \frac{\partial^2 w}{\partial t^2} , \qquad \text{(1a-f)}$$

$$\dot{\bar{M}}_\phi + \bar{M}_{x\phi}' - a\bar{Q}_\phi = 0 ,$$

$$\bar{M}_x' + \dot{\bar{M}}_{\phi x} - a\bar{Q}_x = 0 ,$$

$$a\bar{N}_{x\phi} - a\bar{N}_{\phi x} + \bar{M}_{\phi x} = 0 .$$

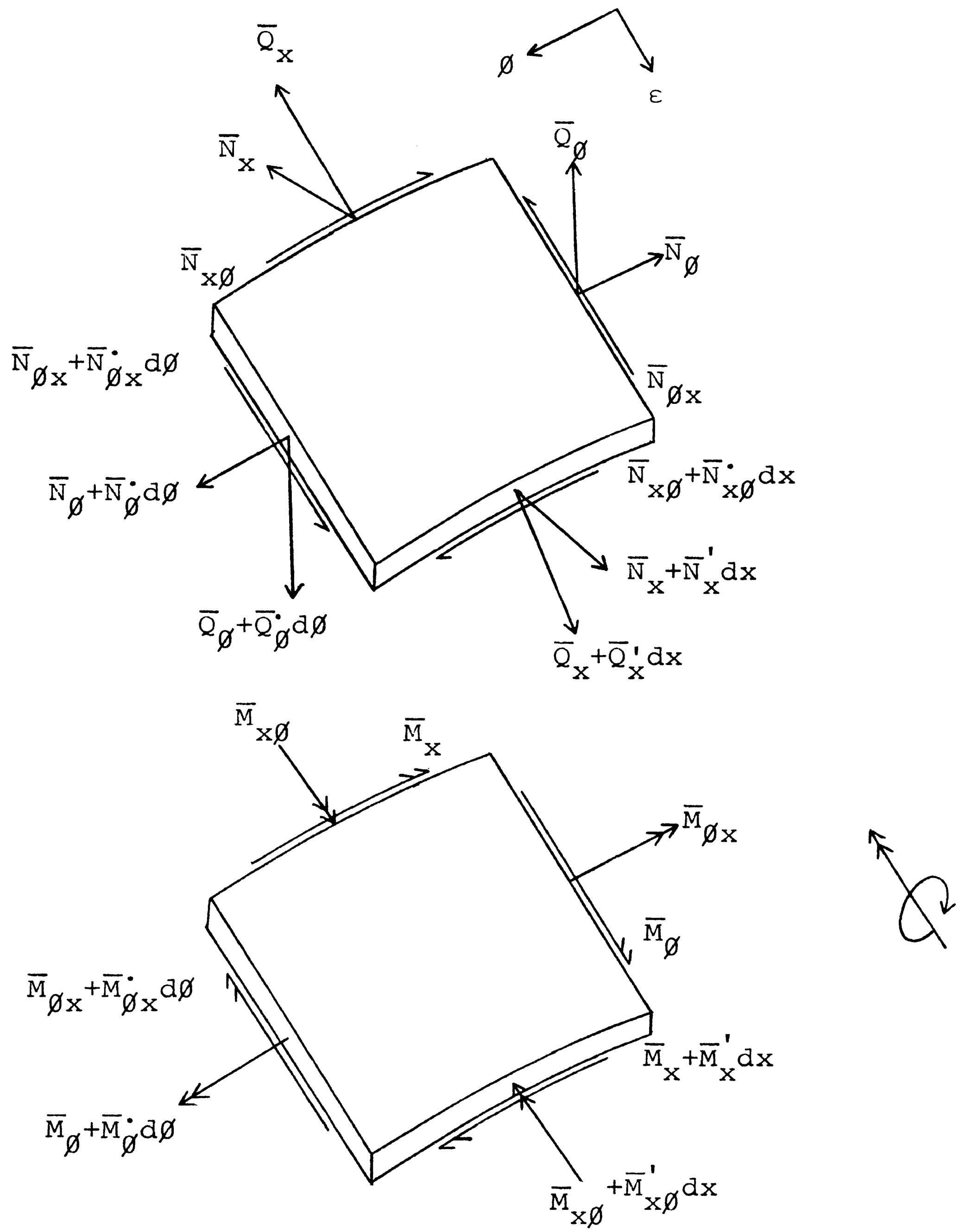


Figure 1: Stress resultants on differential shell element

From (1d) and (1e):

$$\bar{Q}_\phi = \frac{\bar{M}^{\cdot}_\phi + \bar{M}'_{x\phi}}{a} \;,$$
$$\bar{Q}_x = \frac{\bar{M}'_x + \bar{M}^{\cdot}_{\phi x}}{a} \;. \qquad \text{(2a-b)}$$

Substituting Equations (2) in Eqs. (1a), (1b) and (1c), the following system of equations is obtained:

$$\bar{N}'_x + \bar{N}^{\cdot}_{\phi x} = \rho h a \frac{\partial^2 u}{\partial t^2} \;,$$
$$a\bar{N}^{\cdot}_\phi + a\bar{N}'_{x\phi} - \bar{M}^{\cdot}_\phi - \bar{M}_{x\phi} = \rho h a^2 \frac{\partial^2 \nu}{\partial t^2} \;, \qquad \text{(3a-d)}$$
$$\bar{M}^{\cdot\cdot}_\phi + \bar{M}'^{\cdot}_{x\phi} + \bar{M}'^{\cdot}_{\phi x} + \bar{M}''_x + a\bar{N}_\phi = -\rho h a^2 \frac{\partial^2 w}{\partial t^2} \;,$$
$$a\bar{N}_{x\phi} - a\bar{N}_{\phi x} + \bar{M}_{\phi x} = 0 .$$

Assuming that the displacements are small and that all points lying on a normal to the middle surface before deformation remain on a normal after deformation, the relations between stress resultants and displacements for a linear, elastic shell are given by the following set of equations:

$$\bar{N}_\phi = \frac{D}{a}\,(\bar{v}^{\cdot} + \bar{w} + \nu\bar{u}') + \frac{K}{a^3}\,(\bar{W} + \bar{W}^{\cdot\cdot}),$$

$$\bar{N}_x = \frac{D}{a}\,(\bar{u}' + \nu\dot{\bar{v}} + \nu\bar{w}) - \frac{K}{a^3}\,\bar{w}'' ,$$

$$\bar{N}_{\phi x} = \frac{D}{a}\,(\frac{1-\nu}{2})\,(\dot{\bar{u}} + \bar{v}') + \frac{K}{a^3}\,(\frac{1-\nu}{2})\,(\dot{\bar{u}} + \dot{\bar{w}}') ,$$

$$\bar{N}_{x\phi} = \frac{D}{a}\,(\frac{1-\nu}{2})\,(\dot{\bar{u}} + \bar{v}') + \frac{K}{3}\,(\frac{1-\nu}{2})\,(\bar{v}' - \dot{\bar{w}}') ,$$

$$\bar{M}_{\phi} = \frac{K}{a^2}(\bar{w} + \ddot{\bar{w}} + \nu\bar{w}''), \qquad (4a\text{-}h)$$

$$\bar{M}_x = \frac{K}{a^2}\,(\bar{w}'' + \nu\ddot{\bar{w}} - \bar{u}' - \nu\dot{\bar{v}}),$$

$$\bar{M}_{\phi x} = \frac{K}{a^2}\,(1-\nu)\,(\dot{\bar{w}}' + \frac{1}{2}\,\dot{\bar{u}} - \frac{1}{2}\,\bar{v}'),$$

$$\bar{M}_{x\phi} = \frac{K}{a^2}\,(1-\nu)\,(\dot{\bar{w}}' - \bar{v}').$$

A general solution of eqs. (3) and (4) can be written as:

$$\bar{u} = \sum_{n=0}^{\infty} u_n \cos n\phi\; e^{i\omega t} ,$$

$$\bar{v} = \sum_{n=0}^{\infty} v_n \sin n\phi\; e^{i\omega t} ,$$

$$\bar{w} = \sum_{n=0}^{\infty} w_n \cos n\phi\; e^{i\omega t} ,$$

$$\bar{N}_{\phi} = \sum_{n=0}^{\infty} N_{\phi n} \cos n\phi\; e^{i\omega t},$$

$$\overline{N}_x = \sum_{n=0}^{\infty} N_{xn} \cos n\phi \, e^{i\omega t},$$

$$\overline{N}_{\phi x} = \sum_{n=0}^{\infty} N_{\phi xn} \sin n\phi \, e^{i\omega t}, \tag{5a-k}$$

$$\overline{N}_{x\phi} = \sum_{n=0}^{\infty} N_{x\phi n} \sin n\phi \, e^{i\omega t},$$

$$\overline{M}_{\phi} = \sum_{n=0}^{\infty} M_{\phi n} \cos n\phi \, e^{i\omega t},$$

$$\overline{M}_x = \sum_{n=0}^{\infty} M_{xn} \cos n\phi \, e^{i\omega t},$$

$$\overline{M}_{\phi x} = \sum_{n=0}^{\infty} M_{\phi xn} \sin n\phi \, e^{i\omega t},$$

$$\overline{M}_{x\phi} = \sum_{n=0}^{\infty} M_{x\phi n} \sin n\phi \, e^{i\omega t}.$$

The coefficients of these infinite series, u_n, v_n, w_n ... ..., $M_{x\phi n}$ are functions of x. Since these coefficients are independent of ϕ, their partial differentials with respect to x are also the total differentials.

For some special cases in which the boundary conditions depend on only one harmonic, or when the boundary conditions are homogeneous, the natural frequency will be a function of a single value of n. In this report only homogeneous boundary conditions are considered, therefore, summation over n is omitted and subscript n is dropped.

Substituting eqs. (5) into eqs. (3), the equations of motion become:

$$N_x' + nN_{\phi x} = -pu ,$$

$$-naN_\phi + aN_{x\phi}' + nM_\phi - M_{x\phi}' = -pav ,$$

$$-n^2M_\phi + nM_{x\phi}' + nM_{\phi x} + M_x'' + aN_\phi = paw ,$$

$$aN_{x\phi} - aN_{\phi x} + M_{\phi x} = 0 ,$$

where $p = \rho h a \omega^2$.

Similarly, substituting eqs. (5) into eqs. (4), the stress resultant coefficients are:

$$N_\phi = \frac{D}{a}(nv + w + \nu u') + \frac{K}{a^3}(w - n^2 w) ,$$

$$N_x = \frac{D}{a}(u' + n\nu v + \nu w) - \frac{K}{a^3} w'' ,$$

$$N_{\phi x} = \frac{D}{a}\left(\frac{1-\nu}{2}\right)(-nu + v') + \frac{K}{a^3}\left(\frac{1-\nu}{2}\right)(-nu - nw') ,$$

(7a-h)

$$N_{x\phi} = \frac{D}{a}\left(\frac{1-\nu}{2}\right)(-nu + v') + \frac{K}{a^3}\left(\frac{1-\nu}{2}\right)(v' + nw') ,$$

$$M_\phi = \frac{K}{a^2}(w - n^2 w + \nu w'') ,$$

$$M_x = \frac{K}{a^2}(w'' - n^2 \nu w - u' - \nu n v) ,$$

$$M_{\phi x} = \frac{K}{a^2} (1-\nu)(-nw' - \frac{n}{2} u - \frac{1}{2} v') ,$$

$$M_{x\phi} = \frac{K}{a^2} (1-\nu)(-nw'-v') .$$

B. Reduction to First-Order Ordinary Differential Equations

In this section the circular cylindrical shell equations are rearranged in the form:

$$\frac{dU_i}{dx} = \sum_{j=1}^{8} A_{ij} U_j , \qquad i = 1,2,\ldots,8, \tag{8}$$

where U_i, $i = 1,2,\ldots,8$ are u, v, w, $\theta = w'$, N_x, M_x, T_x and S_x respectively. A_{ij} are, in general, functions of shell dimensions, material properties and circular frequency, ω. The new quantities T_x and S_x are the effective tangential and radial shear coefficients contained in:

$$\bar{T}_x = \bar{N}_{x\phi} - \frac{\bar{M}_{x\phi}}{a} ,$$

$$\bar{S}_x = \bar{Q}_x + \frac{\dot{\bar{M}}_{x\phi}}{a} . \tag{9a-b}$$

$\bar{S}_x$ is analogous to Kirchoff's force in the theory of plates. Both $\bar{S}_x$ and $\bar{T}_x$ are used because of their significance in specifying the effective state of stresses at the ends (boundaries) for the cylindrical shells.

Differentiating eqs. (9a) and (9b) with respect to x and using eqs. (2) and (5), eqs. (9) become:

$$aT'_x = aN'_{x\phi} - M'_{x\phi} \quad ,$$
$$aS'_x = M''_x + n(M'_{x\phi} + M'_{\phi x}) \; . \qquad \text{(10a-b)}$$

For the sake of brevity in representing the equations, the following symbols are used to represent various combinations of geometric and elastic parameters:

$$D_1 = \frac{D}{a} \; ,$$
$$D_2 = \frac{D}{a}\left(\frac{1-\nu}{2}\right) \; ,$$
$$K_1 = \frac{K}{a^2} \; ,$$
$$K_2 = \frac{K}{a^3} \; , \qquad \text{(11a-f)}$$
$$K_3 = \frac{K}{a^2}(1-\nu) \; ,$$
$$K_4 = \frac{K}{a^3}\left(\frac{1-\nu}{2}\right) \; .$$

Substituting eqs. (11) in (7b) and (7f) and rearranging:

$$K_2\theta' - D_1u' = D_1n\nu v + D_1\nu w - N_x$$
$$\theta' - u' = n\nu v + n^2\nu w + M_x/K_1 \qquad \text{(12a-b)}$$

Solving eqs. (12):

$$u' = \frac{(k_2n^2\nu - D_1\nu)W + (k_2\nu n - D_1 n\nu)v + N_x + \frac{M_x}{a}}{D_1 - k_2} \tag{13}$$

and

$$\theta' = \frac{(D_1n^2\nu - D_1\nu)w + N_x + \frac{D_1}{k_1}M_x}{D_1 - k_2} \quad . \tag{14}$$

Therefore, from eq. (13):

$$A_{11} = A_{14} = A_{17} = A_{18} = 0,$$

and

$$A_{12} = -n\nu,$$

$$A_{13} = \frac{\nu(k_2n^2 - D_1)}{D_1 - k_2} \; ,$$

$$A_{15} = \frac{1}{D_1 - k_2} \; , \tag{15}$$

$$A_{16} = \frac{1}{a(D_1 - k_2)} \; .$$

Also, from eq. (14):

$$A_{41} = A_{42} = A_{44} = A_{47} = A_{48} = 0, \text{ and}$$

$$A_{43} = \frac{D_1\nu(n^2 - 1)}{D_1 - k_2} \; ,$$

$$A_{45} = \frac{1}{D_1 - k_2} ,$$

$$A_{46} = \frac{D_1}{k_1(D_1 - k_2)} . \tag{16}$$

To find the second equation represented by eqs. (8), the following equation is obtained by substituting eqs. (5) into eq. (9a):

$$T_x = N_{x\phi} - \frac{M_{x\phi}}{a}$$

Inserting the expressions for $N_{x\phi}$ and $M_{x\phi}$ given by eqs. (7) and using eqs. (11):

$$T_x = -D_2 nu + (D_2 + 3k_4)v' + 3k_4 n\theta .$$

Therefore,

$$v' = \frac{D_2 nu - 3k_4 n\theta + T_x}{D_2 + 3k_4} . \tag{17}$$

From eq. (17):

$$A_{22} = A_{23} = A_{25} = A_{26} = A_{28} = 0, \text{ and}$$

$$A_{21} = \frac{D_2 n}{D_2 + 3k_4} ,$$

$$A_{24} = - \frac{3k_4 n}{D_2 + 3k_4} ,$$

$$A_{27} = \frac{1}{D_2+3k_4} \quad . \tag{18}$$

The third equation represented by the eqs. (8) is:

$$w' = \theta \quad .$$

Therefore:

$$A_{31} = A_{32} = A_{33} = A_{35} = A_{36} = A_{37} = A_{38} = 0,$$

and $A_{34} = 1$. (19)

To obtain the fifth equation represented by eqs. (8), eqs. (7), (11) and (17) are substituted in the first equation of motion (6a):

$$N_x' = (-p + D_2n^2 + k_4n^2 - nD_2A_{21})u +$$
$$(k_4n^2 - D_2n\ A_{24})\theta - D_2n\ A_{27}\ T_x,$$

this gives:

$$A_{52} = A_{53} = A_{55} = A_{56} = A_{58} = 0, \text{ and}$$

$$A_{51} = -p + (D_2 + k_4)n^2 - D_2n\ A_{21},$$

$$A_{54} = k_4n^2 - D_2n\ A_{24} \ , \tag{20}$$

$$A_{57} = -D_2n\ A_{27} \ .$$

Similar algebraic manipulations with eqs. (10b), (6b) and (6c) yield:

$$A_{62} = A_{63} = A_{65} = A_{66} = 0,$$

$$A_{61} = \frac{k_3 n^2}{2} + \frac{3k_3 n \ A_{21}}{2} \ ,$$

$$A_{64} = 2k_3 n^2 + \frac{3k_3 n \ A_{24}}{2} \ , \qquad (21)$$

$$A_{67} = \frac{3k_3 n \ A_{27}}{2} \ ,$$

$$A_{68} = a, \quad \text{and}$$

$$A_{71} = A_{74} = A_{77} = A_{78} = 0,$$

$$A_{72} = -p + D_1 n^2 + D_1 n\nu \ A_{12},$$

$$A_{73} = D_1 n + n\nu(D_1 A_{13} - k_2 A_{43}), \qquad (22)$$

$$A_{75} = n\nu(D_1 A_{15} - k_2 \ A_{45}),$$

$$A_{76} = n\nu(D_1 \ A_{16} - k_2 \ A_{46}), \ \text{and}$$

$$A_{81} = A_{84} = A_{87} = A_{88} = 0,$$

$$A_{82} = -D_1 n - D_1 \nu \ A_{12},$$

$$A_{83} = p - D_1 - k_2(1-n^2) + k_2 n^2 \ (1-n^2) + \nu(-D_1 A_{13} + k_2 n^2 A_{43}), \qquad (23)$$

$$A_{85} = \nu(-D_1A_{15} + k_2n^2A_{45}),$$

$$A_{86} = \nu(-D_1A_{16} + k_2n^2A_{46}).$$

C. Boundary Conditions

The derivation of the homogeneous boundary conditions can be found in Kraus (1). Only the final results for the different cases are given here.

1. For free end:

$$N_x = 0, \quad M_x = 0, \quad T_x = 0, \quad S_x = 0. \tag{24}$$

2. For clamped end with axial constraint:

$$u = 0, \quad v = 0, \quad w = 0, \quad \theta = 0. \tag{25}$$

3. For simply supported end without axial constriant:

$$v = 0, \quad w = 0, \quad N_x = 0, \quad M_x = 0. \tag{26}$$

4. For clamped end without axial constriant:

$$v = 0,$$
$$w = 0,$$
$$\theta = 0, \tag{27}$$
$$N_x = 0.$$

5. For simply supported end with axial constaint:

$$u = 0,$$
$$v = 0,$$
$$w = 0, \tag{28}$$
$$M_x = 0.$$

III. METHOD OF SOLUTION

A generalization of the Holzer method is used to find the natural frequencies and mode shapes. In this method, the value of circular frequency, ω, is assumed and a solution is found which satisfies the equations of motion and the boundary conditions of the problem. A particular value of ω for which a non-trivial solution exists is an eigenvalue of the problem. The corresponding displacement functions are the mode shapes associated with that eigenvalue.

Any suitable method can be used to find the solution of the problem for the assumed value of ω. A stepwise integration method has been used in this report to determine the natural frequencies and mode shapes of the circular cylindrical shells because of the difficulty in finding an exact solution in closed form. The fourth-order Runge-Kutta method is used for integration as it is simple, self starting and gives results of good accuracy.

The problem of free vibration of shells is represented by a system of eighth-order differential equations. Therefore, eight boundary conditions are required to find all the constants of integration. A shell problem is a two point boundary value problem since the eight boundary conditions are distributed at the two ends.

In the previous section, the cylindrical shell equations have been put in the form:

$$\frac{dU_i}{dx} = \sum_{j=1}^{8} A_{ij}U_j, \quad i = 1,2,\ldots,8.$$

U_i, $i = 1,2,\ldots,8$, are the eight fundamental shell variables u, v, w, θ, N_x, M_x, T_x and S_x respectively. Any four of these eight variables are known at each end of the shell depending on the boundary conditions. But, to solve the above set of differential equations by stepwise integration, the initial values of all of the eight variables are required. Since only four boundary conditions are known at the near end ($x = 0$), four additional unknown boundary conditions must be assumed. These four unknown boundary conditions at the near end correspond to the four constants of integration that have to be determined from the boundary conditions at the far end ($x = L/a$). The procedure to obtain the natural frequencies and the unknown boundary conditions is explained below.

The four fundamental variables, which are known at the near end , are denoted by U_{n1}, U_{n2}, U_{n3}, and U_{n4}. The unknown variables are denoted by U_{n5}, U_{n6}, U_{n7} and U_{n8}. The subscripts n1, n2,...., n8 represent the numbers 1,2,..., 8, in an order depending on the boundary conditions at the near end. Similarly, U_{m1}, U_{m2}, U_{m3} and U_{m4} are used to denote the known variables at the far end. For example, for a cantilevered (fixed-free) shell, at the fixed near end n_1, n_2, n_3,..., n_7 and n_8 are 1,2,3,...7 and 8, respectively and at the free far end m_1 m_2, m_3 and m_4 are 5,6,7 and 8 respectively.

Four independent partial solutions of the problem are obtained by numerical integration. The initial conditions of the eight variables for different partial solutions are shown in Table I. Superscript 0 denotes the initial values of the variables.

Table I. Initial Conditions for Integration

Solution	U^o_{n1}	U^o_{n2}	U^o_{n3}	U^o_{n4}	U^o_{n5}	U^o_{n6}	U^o_{n7}	U^o_{n8}
1	0	0	0	0	1	0	0	0
2	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	1	0
4	0	0	0	0	0	0	0	1

The stepwise integration is carried out independently for the four solutions. After the integration is complete, four values of each of the eight variables are known at the selected points along the generator of the shell. The values of the variables that are obtained at the far end, are denoted by U_i^k, i = 1,2,...,8. Superscript k, k = 1, 2,3 and 4 represents the number assigned to the partial solutions.

Since the differential equations of the problem are linear, the total solution is obtained by linear combinations of the partial solutions. Therefore, the values of the fundamental variables at the far end can be given by matrix eq. (29).

In eq. (29), a_1, a_2, a_3 and a_4 are the unknown constants which are found from the boundary conditions at the far end of the shell.

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}_x = \begin{bmatrix} U_1^1 & U_1^2 & U_1^3 & U_1^4 \\ U_2^1 & U_2^2 & U_2^3 & U_2^4 \\ U_3^1 & U_3^2 & U_3^3 & U_3^4 \\ U_4^1 & U_4^2 & U_4^3 & U_4^4 \\ U_5^1 & U_5^2 & U_5^3 & U_5^4 \\ U_6^1 & U_6^2 & U_6^3 & U_6^4 \\ U_7^1 & U_7^2 & U_7^3 & U_7^4 \\ U_8^1 & U_8^2 & U_8^3 & U_8^4 \end{bmatrix}_x \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{29}$$

U_{m1}, U_{m2}, U_{m3} and U_{m4} represent those fundamental variables which are known at the far end, therefore, for homogeneous boundary conditions at that end:

$$\begin{bmatrix} U^1_{m1} & U^2_{m1} & U^3_{m1} & U^4_{m1} \\ U^1_{m2} & U^2_{m2} & U^3_{m2} & U^4_{m2} \\ U^1_{m3} & U^2_{m3} & U^3_{m3} & U^4_{m3} \\ U^1_{m4} & U^2_{m4} & U^3_{m4} & U^4_{m4} \end{bmatrix}_{x=L/a} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{30}$$

Equation (30) represents a set of four linear homogeneous algebraic equations for the unknowns a_1, a_2, a_3 and a_4. It has a non-trivial solution if the value of the determinant, Δ, is equal to zero, where Δ is given by eq. (31).

$$\Delta = \begin{vmatrix} U^1_{m1} & U^2_{m1} & U^3_{m1} & U^4_{m1} \\ U^1_{m2} & U^2_{m2} & U^3_{m2} & U^4_{m2} \\ U^1_{m3} & U^2_{m3} & U^3_{m3} & U^4_{m3} \\ U^1_{m4} & U^2_{m4} & U^3_{m4} & U^4_{m4} \end{vmatrix}_{x=L/a} \tag{31}$$

The value of the determinant, Δ, depends on the assumed value of circular frequency, ω. A value of ω, for which Δ is zero, is an eigenvalue of the problem since it gives a non-trivial solution.

A curve of frequency, ω, as abscissa versus the corresponding value of determinant, Δ, as ordinate is continuous. Theoretically it crosses the abscissa an infinite number of times representing the natural frequencies for various numbers of axial half-waves. A trial and error procedure, the false position method, is used to find a particular eigenvalue, ω_n. Giving successive increments to the value of ω, two values ω_o and ω_1 are found in the vicinity of ω_n such that the values of their corresponding determinants, Δ_o and Δ_1, are of opposite sign.

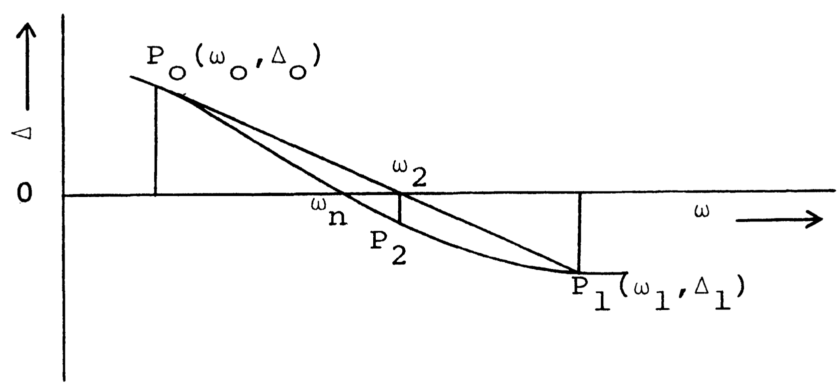


Figure 2: Graphical representation of linear interpolation

The next estimate of ω is given by eq. (32) which represents a point where the line joining $P_o(\omega_o,\Delta_o)$ and $P_1(\omega_1,\Delta_1)$ intersects the $\Delta = 0$ axis as shown in Figure 2.

$$\omega_2 = \frac{\Delta_1\omega_o-\Delta_o\omega_1}{\Delta_1-\Delta_o} . \tag{32}$$

The value of Δ_2 is obtained corresponding to ω_2. Additional estimates are obtained by the recursion relationship:

$$\omega_k = \frac{\Delta_{k-1}\omega_{k-2} - \Delta_{k-2}\omega_{k-1}}{\Delta_{k-1} - \Delta_{k-2}} , \tag{33}$$

until a root ω_k which has a desired accuracy is found.

Once a value of ω is found for which Δ is zero, the matrix eq. (30) can be solved for a_1, a_2, a_3 and a_4. One of these four unknowns can have an arbitrary value. The total solution at any point along the generator of the shell is given by the linear combinations of the four partial solutions as represented by the eq. (34):

$$\left\{ U_i \right\}_x = \begin{bmatrix} U_i^1 & U_i^2 & U_i^3 & U_i^4 \end{bmatrix}_x \left\{ A \right\} , \tag{34}$$

where U_i is a column matrix denoting eight fundamental variables and A is a column matrix of four elements a_1, a_2, a_3 and a_4.

Using eq. (34) at the near end ($x = 0$), it can be shown that the four unknown boundary conditions, U_{n5}^o, U_{n6}^o, U_{n7}^o and U_{n8}^o, are given by a_1, a_2, a_3 and a_4 respectively.

Some numerical difficulties are encountered in the integration process due to the rapid growth of exponential solutions. A suppression technique has been used to resolve this problem.

IV. SUPPRESSION TECHNIQUE

The general solution of a shell problem contains certain components that are rapidly growing functions of the dimensionless variable x, and others that are rapidly decaying functions. In the analysis of long thin cylindrical shells, the effect of any edge correction damps out rapidly and is negligible at the other end. In other words, the components of the solutions at the near end, which are associated with the end conditions at the far end, are very small. Therefore, the assumed initial values of these components that are used in the four independent partial solutions, will be many orders of magnitude too large. Furthermore, these components grow exponentially towards the far end. Therefore, after the integration has proceeded some distance along the generator of the shell, the growing parts of the solutions become extremely large compared to the decaying parts. This makes the problem numerically erroneous, because relatively small values must be obtained by the combinations of extremely large ones as required in the computation of Δ and in the solution of eq. (30).

This numerical problem is resolved by the suppression technique. This requires the recombining of the initial value problems several times as the integration proceeds along the axis of the shell. To this end, four linear combinations of the partial solutions are required to satisfy four sets of independent conditions with small magnitudes. The points at which these conditions are imposed are known as suppression points. At each suppression point the linear combinations of the partial solutions become the new set of partial solutions at that point and the solutions at all previous points are modified accordingly. The new partial solutions are propagated to the next suppression point where the above process is repeated.

The technique is explained in matrix notation in the following steps.

1. The partial solutions are represented in the matrix form as

$$\bar{D} = \begin{bmatrix} U_i^1 & U_i^2 & U_i^3 & U_i^4 \end{bmatrix} , \qquad (35)$$

where U_i is the column matrix of eight fundamental variables.

The partial solutions corresponding to those variables which are unknown at the near end are represented by another matrix M which is given by eq. (36).

2. The partial solutions are suppressed whenever they become large compared with their initial values. The four independent conditions can be

imposed arbitrarily on any four linear combinations of the partial solutions.

$$M = \begin{bmatrix} U^1_{n5} & U^2_{n5} & U^3_{n5} & U^4_{n5} \\ U^1_{n6} & U^2_{n6} & U^3_{n6} & U^4_{n6} \\ U^1_{n7} & U^2_{n7} & U^3_{n7} & U^4_{n7} \\ U^1_{n8} & U^2_{n8} & U^3_{n8} & U^4_{n8} \end{bmatrix} \qquad (36)$$

A scheme which has given good estimates of natural frequencies is as follows:

The solutions are suppressed whenever the value of any of the four variables represented by the diagonal elements of the matrix M exceeds ten times its value at the previous suppression point.

The four sets of independent conditions imposed on the four linear combinations of partial solutions at the suppression point i are represented by the eqs. (37).

$$\begin{bmatrix} M_i \end{bmatrix} \begin{bmatrix} b_1 & b_{21} & b_{31} & b_{41} \end{bmatrix}_i^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T \qquad (37a)$$

$$\begin{bmatrix} M_i \end{bmatrix} \begin{bmatrix} b_{12} & b_{22} & b_{32} & b_{42} \end{bmatrix}_i^T = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T \qquad (37b)$$

$$\begin{bmatrix} M_i \end{bmatrix} \begin{bmatrix} b_{13} & b_{23} & b_{33} & b_{43} \end{bmatrix}_i^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T \qquad (37c)$$

$$\begin{bmatrix} M_i \end{bmatrix} \begin{bmatrix} b_{14} & b_{24} & b_{34} & b_{44} \end{bmatrix}_i^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \qquad (37d)$$

The elements b_{11}, b_{21},...,b_{44} are to be determined from eqs. (37). The above equations can be written as

$$\begin{bmatrix} & & & \\ & M_i & & \\ & & & \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (38)$$

or simply as $M_i B_i = I$, where I denotes the identity matrix.

Therefore,

$$B_i = M_i^{-1} \; . \quad (39)$$

3. After the B matrix for a suppression point is determined, the new sets of partial solutions at the point of suppression and at all the previous points are given by the relationship

$$\bar{D}_{new} = \bar{D}_{previous} B \; . \quad (40)$$

The new partial solutions are propagated to the next suppression point where they are required to satisfy the four independent conditions as before. This process is repeated until the far end of the shell is reached and then Δ is calculated.

4. To find the mode shapes, the above procedure requires that the partial solutions be retained at all the

points. It is often desirable to conserve the computer storage at the cost of slight increase in the amount of computation. To accomplish this, the partial solutions are retained and modified only at the initial point and each suppression point. The series of modifications is represented as follows:

The matrix of partial solutions at the suppression point j, after a total of i suppressions have been made, is denoted by $\bar{D}_j^i$. The value j = 0 corresponds to the near end of the shell. Therefore, the assumed initial conditions for the partial solutions are represented by $\bar{D}_o^o$.

At the first suppression point, B_1 is found from equation $B_1 = M_1^{-1}$. The modified values of the partial solutions at the initial point and first suppresion point become

$$\bar{D}_o^1 = \bar{D}_o^o B_1 , \tag{41a-b}$$

$$\bar{D}_1^1 = \bar{D}_1^o B_1 .$$

At the second suppression point $B_2 = M_2^{-1}$ and

$$\bar{D}_o^2 = \bar{D}_o^1 B_2 ,$$

$$\bar{D}_1^2 = \bar{D}_1^1 B_2 , \tag{42a-c}$$

$$\bar{D}_2^2 = \bar{D}_2^1 B_2 .$$

The values of $\bar{D}_o^1$ and $\bar{D}_1^1$ used in eqs. (42) are already known from eqs. (41).

Similarly at the suppression point p, $B_p = M_p^{-1}$ and

$$
\begin{aligned}
\bar{D}_o^p &= \bar{D}_o^{p-1} \; B_p \;, \\
\bar{D}_1^p &= \bar{D}_1^{p-1} \; B_p \;, \\
&\vdots \\
\bar{D}_p^p &= \bar{D}_p^{p-1} \; B_p \;.
\end{aligned}
\tag{43}
$$

When the far end of the shell is reached , the value of the determinant, Δ, is computed. This value of Δ and the corresponding value of ω are used in the recursion relationship (33) to find the next estimate of ω.

Assuming an eigenvalue which has desired accuracy is obtained and eqs. (30) are solved, the correct initial point and suppression point conditions are given by

$$\{U_i\} \;=\; \bar{D}_j^k \; A, \qquad j = 0,1,2,\ldots,k \tag{44}$$

where A is a column matrix of the elements a_1 , a_2, a_3 and a_4. Superscript k represents the total number of suppressions required for the value of natural frequency, ω_n. The equations of motion are then re-integrated from suppression point to suppression point to find the mode shapes. For instance, to propagate the solutions from suppression point j to the next suppression point j + 1, the initial values of the fundamental variables for step-wise integration are given by the matrix $\bar{D}_j^k$. The difference between the propagated values of the variables at the suppression point j + 1 and the values given by the matrix $\bar{D}_{j+1}^k$ gives an estimate of the accuracy of the results.

V. NUMERICAL RESULTS AND DISCUSSION

The frequency factors, ω_n/ω_o, have been obtained for three different sets of homogeneous boundary conditions by the method of stepwise integration aided by the suppression technique as well as by approximate methods developed by Rayleigh (4) and Arnold and Warburton (5). The displacement functions and frequency equations of the Rayleigh and the Arnold and Warburton solutions are given in Appendix A. The boundary conditions for the three different cases are as follows:

Case 1. Free-Free ends.

$$N_x = M_x = T_x = S_x = 0 \text{ at } x = 0 \text{ and } x = L/a.$$

Case 2. Clamped-Clamped ends with axial constraint.

$$u = v = w = \theta = 0 \text{ at } x = 0 \text{ and } x = L/a.$$

Case 3. Both ends simply supported without axial constraint.

$$v = w = N_x = M_x = 0 \text{ at } x = 0 \text{ and } x = L/a.$$

For each set of boundary conditions, all combinations of the following values of shell geometry parameters, circumferential wave numbers and axial half-wave numbers have been considered.

L/a = 1.0, 2.0 and 3.0;
h/a = 0.02, 0.03 and 0.05;
n = 2,3 and 4;
m = 1 and 2.

Only one value of Poisson's Ratio, $\nu = 0.3$, is used to obtain numerical results. The non-dimensional frequency factor, ω_n/ω_o, does not depend on the values of E and ρ.

For any fixed value of n and m, there are three frequencies corresponding to three mode shapes. In general, two of these frequencies are several orders of magnitude higher than the minimum value. These two higher frequencies have not been considered and the frequency factors have been obtained only for the minimum frequencies.

The results for various cases are shown in Tables II, III, IV, V, VI and VII. Some representative results for one axial half-wave are plotted in Figures 3, 4 and 5.

Figure 3 shows that for free-free shells the natural frequencies for one axial half-wave remain approximately constant with increasing length. The effect of change in length for other sets of boundary conditions is quite different as shown in Figures 4 and 5, where natural frequencies decrease sharply as the shell length increases. This is also true for the case of two axial half-waves, even for a free-free shell, as shown in Tables III, V and VII.

It is also observed from Figures 3, 4 and 5 that the natural frequencies increase with increased shell thickness. For a free-free shell, the natural frequencies for $m = 1$ vary almost linearly with thickness. A comparison of frequency factor values given in Tables shows that the

Table II. Frequency factors, ω_n/ω_o, for free-free shells, m = 1

n	L/a	h/a = 0.02		h/a = 0.03		h/a = 0.05	
		SUPPRESS.	RAYLEIGH	SUPPRESS.	RAYLEIGH	SUPPRESS.	RAYLEIGH
2	1.0	0.01540	0.01549	0.02300	0.02324	0.03806	0.03873
3	1.0	0.04341	0.04382	0.06492	0.06572	0.1079	0.1095
4	1.0	0.08317	0.08402	0.1245	0.1260	0.2070	0.2100
2	2.0	0.01548	0.01549	0.02315	0.02324	0.03836	0.03873
3	2.0	0.04363	0.04382	0.06530	0.06572	0.1087	0.1095
4	2.0	0.08357	0.08402	0.1252	0.1260	0.2084	0.2100
2	3.0	0.01552	0.01549	0.02319	0.02324	0.03845	0.03873
3	3.0	0.04371	0.04382	0.06543	0.06572	0.1089	0.1095
4	3.0	0.08370	0.08402	0.1254	0.1260	0.2086	0.2100

Table III. Frequency factors, ω_m/ω_o, for free-free shells, m = 2

n	L/a	h/a = 0.02		h/a = 0.03		h/a = 0.05	
		SUPPRESS.	RAYLEIGH*	SUPPRESS.		SUPPRESS.	
2	1.0	0.02787		0.04175		0.06931	
3	1.0	0.06938		0.1039		0.1728	
4	1.0	0.1170		0.1754		0.2921	
2	2.0	0.02065		0.03091		0.05138	
3	2.0	0.05190		0.07780		0.1286	
4	2.0	0.09081		0.1398		0.2332	
2	3.0	0.01811		0.02712		0.04506	
3	3.0	0.04755		0.07116		0.1187	
4	3.0	0.08804		0.1321		0.2904	

*Rayleigh's solution gives only one frequency for fixed values of shell geometry and circumferential wave numbers. This frequency is a good approximation of the exact frequency corresponding to m = 1 and not for other axial wave forms.

Table IV. Frequency factors, ω_n/ω_o, for clamped-clamped shells, m = 1.

n	L/a	h/a = 0.02		h/a = 0.03		h/a = 0.05	
		SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD
2	1.0	0.6747	0.7118	0.6927	0.7277	0.7445	0.7763
3	1.0	0.5376	0.5702	0.5667	0.5958	0.6472	0.6708
4	1.0	0.4509	0.4749	0.4986	0.5184	0.6239	0.6371
2	2.0	0.3807	0.4134	0.3850	0.4158	0.3962	0.4233
3	2.0	0.2694	0.2887	0.2808	0.2978	0.3120	0.3249
4	2.0	0.2157	0.2263	0.2457	0.2542	0.3224	0.3275
2	3.0	0.2466	0.2675	0.2489	0.2688	0.2555	0.2732
3	3.0	0.1653	0.1754	0.1759	0.1847	0.2051	0.2117
4	3.0	0.1404	0.1449	0.1740	0.1775	0.2530	0.2546

Table V. Frequency factors, ω_n/ω_o, for clamped-clamped shells, m = 2

n	L/a	h/a = 0.02		h/a = 0.03		h/a = 0.05	
		SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD
2	1.0	0.9414	0.9562	1.0271	1.0419	1.2649	1.2774
3	1.0	0.8686	0.8848	0.9722	0.9865	1.2464	1.2569
4	1.0	0.8012	0.8194	0.9297	0.9449	1.2549	1.2637
2	2.0	0.6638	0.6926	0.6747	0.7014	0.7053	0.7289
3	2.0	0.5108	0.5429	0.5310	0.5598	0.5875	0.6107
4	2.0	0.4133	0.4401	0.4517	0.4744	0.5542	0.5698
2	3.0	0.4748	0.5122	0.4793	0.5151	0.4918	0.5243
3	3.0	0.3342	0.3624	0.3461	0.3720	0.3799	0.4010
4	3.0	0.2610	0.2784	0.2915	0.3062	0.3714	0.3812

Table VI. Frequency factors, ω_n/ω_o, for simple supported shells, m = 1

n	L/a	h/a = 0.02		h/a = 0.03		h/a = 0.05	
		SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD
1	1.0	0.6561	0.6561	0.6611	0.6610	0.6767	0.6764
3	1.0	0.4911	0.4910	0.5035	0.5032	0.5414	0.5404
4	1.0	0.3802	0.3800	0.4109	0.4103	0.4963	0.4949
2	2.0	0.3283	0.3283	0.3298	0.3297	0.3346	0.3342
3	2.0	0.2018	0.2017	0.2115	0.2112	0.2398	0.2392
4	2.0	0.1569	0.1567	0.1908	0.1905	0.2723	0.2716
2	3.0	0.1827	0.1827	0.1842	0.1841	0.1887	0.1884
3	3.0	0.1095	0.1094	0.1225	0.1223	0.1572	0.1568
4	3.0	0.1075	0.1074	0.1471	0.1469	0.2321	0.2316

Table VII. Frequency factors, ω_n/ω_o, for simply supported shells, m = 1

n	L/a	h/a = 0.02		h/a = 0.03		h/a = 0.05	
		SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD	SUPPRESS.	ARNOLD
2	1.0	0.8960	0.8963	0.9375	0.9380	1.0594	1.0605
3	1.0	0.8176	0.8177	0.8730	0.8731	1.0304	1.0304
4	1.0	0.7413	0.7412	0.8196	0.8192	1.0300	1.0297
2	2.0	0.6561	0.6561	0.6611	0.6610	0.6767	0.6764
3	2.0	0.4911	0.4910	0.5035	0.5032	0.5414	0.5404
4	2.0	0.3802	0.3800	0.4108	0.4103	0.4962	0.4949
2	3.0	0.4615	0.4614	0.4636	0.4634	0.4702	0.4698
3	3.0	0.3040	0.3039	0.3132	0.3129	0.3409	0.3401
4	3.0	0.2259	0.2257	0.2560	0.2556	0.3346	0.3337

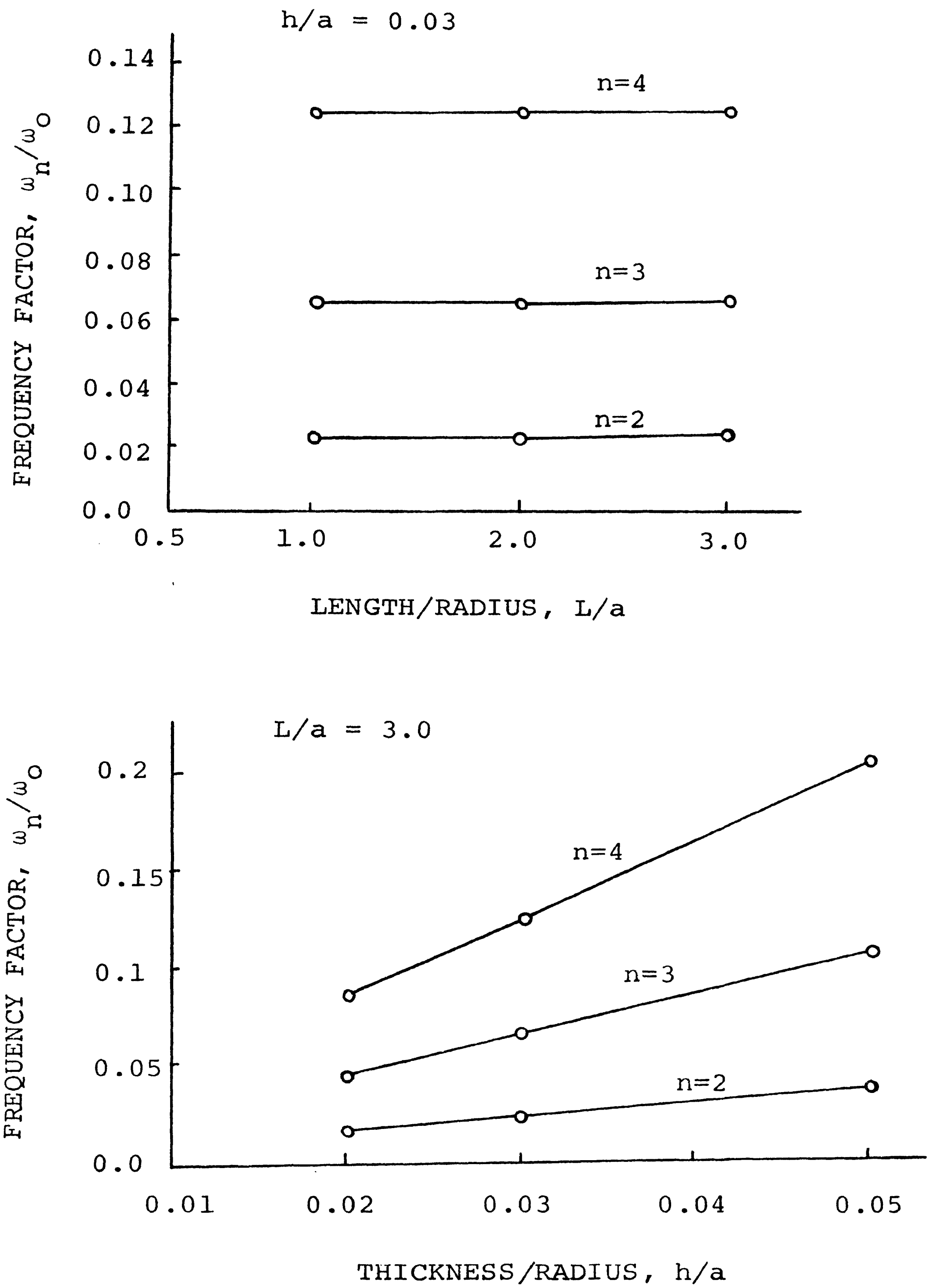


Figure 3: Influence of shell geometry on frequency factor, free-free ends, m = 1.

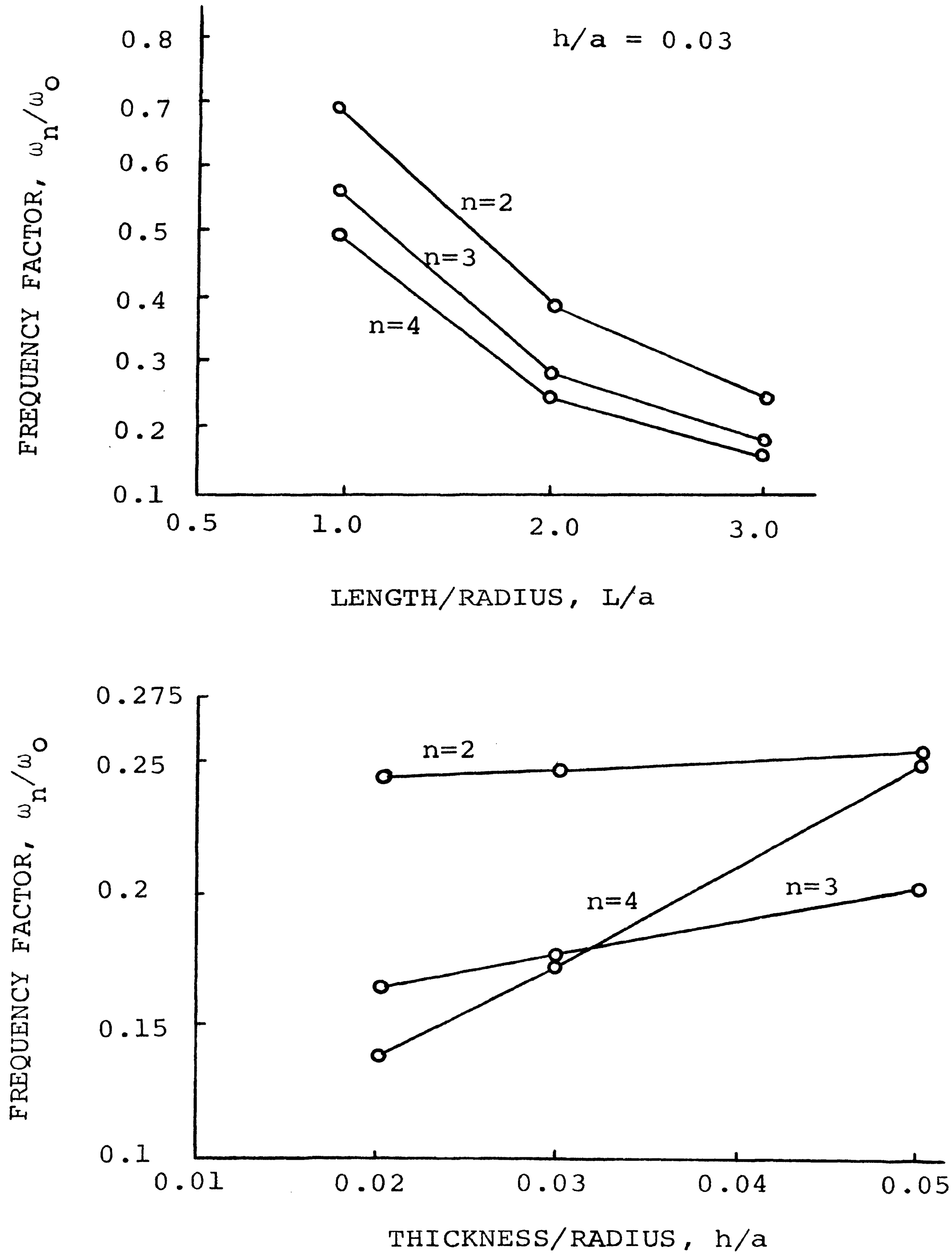
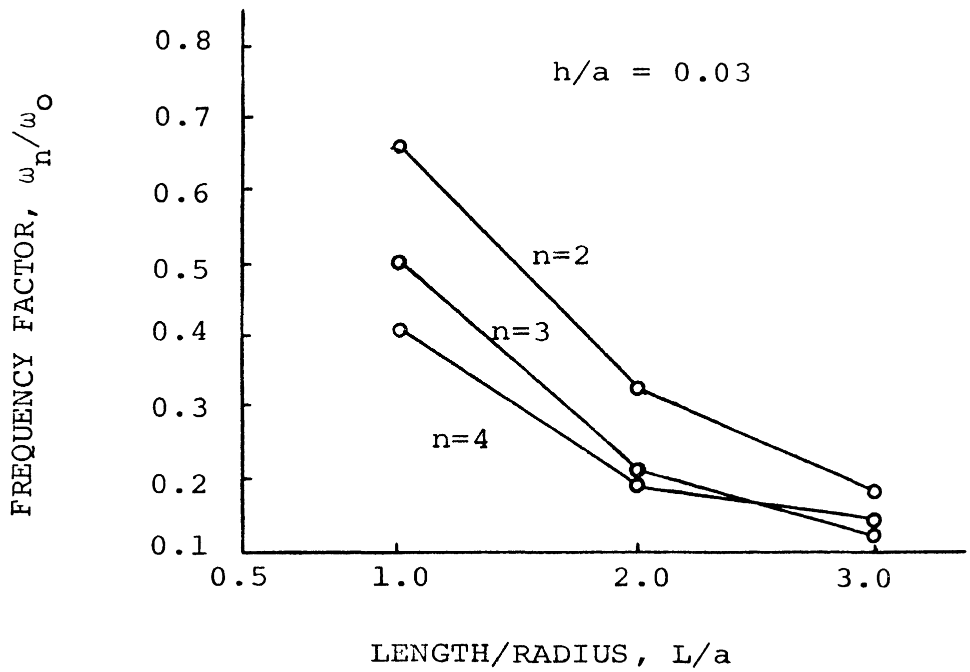


Figure 4: Influence of shell geometry on frequency factor, clamped-clamped ends, m = 1.



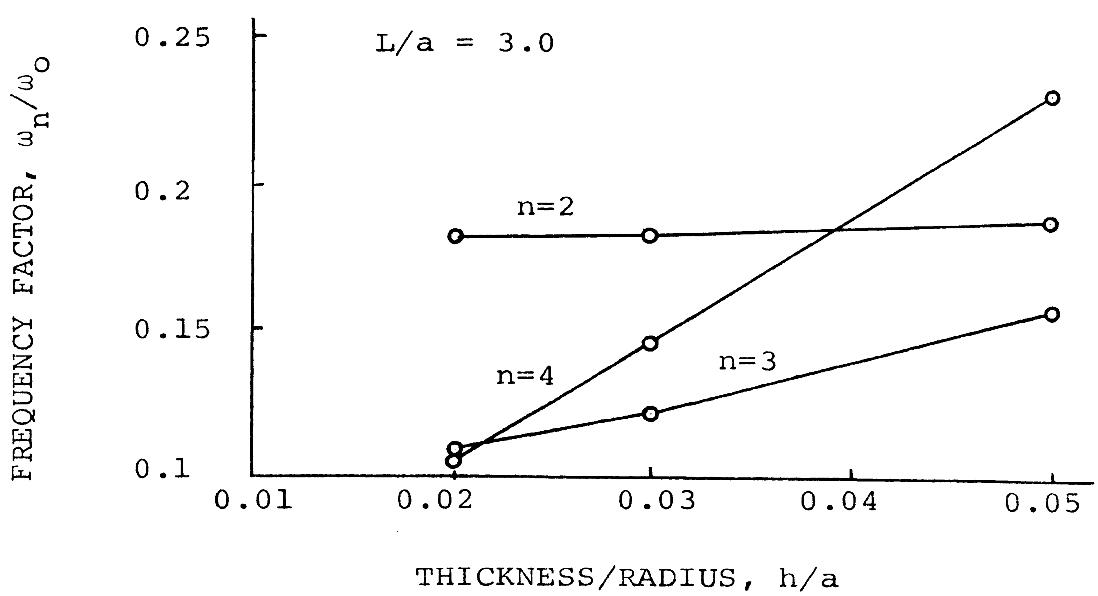


Figure 5: Influence of shell geometry on frequency factor, simply supported ends, m = 1.

natural frequency increases for higher axial half-wave numbers.

For fixed shell geometry and circumferential wave number, the minimum natural frequency always occurs for a mode having one axial half-wave (m = 1). Minimum frequency can occur for any circumferential wave number, n, depending on the values of shell geometry. Figures 4 and 5 show that there are ranges of shell geometry parameters in which the natural frequency associated with any one of the three circumferential wave numbers, n = 2, 3 and 4, is lower than those corresponding to the other two. Moreover, a comparison of Figures 4 and 5 shows that a change in the boundary conditions may also alter the value of n associated with minimum natural frequency.

The energy method solutions developed by Rayleigh (6) and Arnold and Warburton (7) give an excellent representation of frequency spectrum. In his original contribution to the free vibration analysis of cylinders, Rayleigh postulated that the cylinder undergoes bending but no stretching. He assumed that the length of any line on the middle surface of the shell remains unaltered during vibrations of small amplitude. Since the stretching energy is associated primarily with the end conditions and the number of axial waves, a theory which neglects the stretching energy will give results which are independent of end conditions and for which generators remain straight during vibration. For this reason, Rayleigh's solution gives good approximation to natural frequencies only for one axial half-wave and for very long cylinders or cylinders with free ends.

Arnold and Warburton (7) allow the stretching of the middle surface in the calculations of strain energy. The displacement functions used by them satisfy the boundary conditions of the problem. Therefore, their values of natural frequencies are expected to be higher than the exact values. This is the case for cylinders with clamped-clamped ends (with axial constraint). As shown in Tables IV and V, Arnold and Warburton's results are 1 to 10 percent higher than the results obtained by the stepwise integration method with suppression. But, for cylinders with simply supported ends (without axial constraint), the difference between the results obtained by these two methods is very small. Tables VI and VII show 0.3 percent maximum difference and, rather unexpectedly, Arnold and Warburton's results are lower in most cases. This is not consistant with vibration theory and may have resulted for the following two reasons. Firstly, for economical use of computer time, the accuracy of the final estimate of frequency by the Holzer method was held to the order of 0.2 percent. This accuracy was further diminished by the discretization and round-off errors in the integration process. Secondly, Arnold and Warburton use Timoshenko (16) strain-displacement relationships for the shell element in the calculation of strain energy. These relationships are somewhat different from those given by Flugge (15) and used in the stepwise integration method of this investigation.

VI. CONCLUSIONS

The energy method solutions developed by Rayleigh and Arnold and Warburton give excellent approximations to the natural frequencies. The maximum error for free-free, clamped-clamped and simply supported at both ends boundary conditions is always less than ten percent. The major advantage of these solutions is that they yield explicit algebraic expressions for natural frequencies. Therefore, frequency spectra for large numbers of cases can be computed with minimum computer time. The serious drawback of the energy approach is that entirely new solutions are needed for different boundary conditions, requiring an extensive amount of work for each case.

In the stepwise integration method with suppression as presented in this investigation, the differential equations are in terms of eight fundamental shell variables. Therefore, it can readily handle all types of homogeneous boundary conditions and can give explicit point values of displacements and stress resultants. But this method is much slower than the energy methods commonly used in engineering practice.

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VITA

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APPENDICES

APPENDIX A

RAYLEIGH SOLUTION AND ARNOLD AND WARBURTON SOLUTIONS

1. Rayleigh Solution

Rayleigh used the following displacement functions in developing an expression for the natural frequencies of inextensional vibrations of circular cylindrical shells.

$$\bar{u} = \left(-\frac{B_n a \sin n\phi}{n} - \frac{D_n a \cos n\phi}{n}\right) \cos \omega t,$$

$$\bar{v} = \{a(A_n+B_n x)\cos n\phi - a(C_n+D_n x)\sin n\phi\}\cos \omega t, \qquad \text{(1a-c)}$$

$$\bar{w} = \{na(A_n+B_n x)\sin n\phi + na(C_n+D_n x)\cos n\phi\}\cos \omega t.$$

A_n, B_n, C_n and D_n are constants. By equating kinetic energy at the mean position to strain energy at the maximum displacement, he obtained the following expression for the natural circular frequency, ω_n:

$$\omega_n = \sqrt{\frac{Eh^2}{12\rho a^4(1-\nu^2)}\left[\frac{n^2(n^2-1)^2}{n^2+1}\right]} \quad . \qquad (2)$$

2. Arnold and Warburton Solutions

(a) Clamped-Clamped Circular Cylindrical Shells.

The displacement functions used are given below. For this case the origin is chosen at mid-length.

For odd numbers of axial half-waves:

$$\bar{u} = A_n(-\sin\mu x + k \sinh \mu x)\cos n\phi \cos\omega t,$$

$$\bar{v} = B_n(\cos \mu x + k \cosh \mu x)\sin n\phi \cos\omega t, \qquad (3a\text{-}c)$$

$$\bar{w} = C_n(\cos \mu x + k \cosh \mu x)\cos n\phi \cos\omega t.$$

For even numbers of axial half-waves:

$$\bar{u} = A_n(\cos \mu x - k \cosh \mu x)\cos n\phi \cos\omega t,$$

$$\bar{v} = B_n(\sin \mu x - k \sinh \mu x)\sin n\phi \cos\omega t, \qquad (4a\text{-}c)$$

$$\bar{w} = C_n(\sin \mu x - k \sinh \mu x)\cos n\phi \cos\omega t.$$

For both odd and even numbers of axial half-waves, $k = \sin(\frac{\mu L}{2a})/\sinh(\frac{\mu L}{2a})$, and μ is given by $\tan(\mu L/2a) = (-1)^m \tanh(\mu L/2a)$; the roots of the equation for odd numbers of axial half-waves are $(\mu L/a) = 1.506\pi$, $7\pi/2$, $11\pi/2$, $15\pi/2$,...., corresponding to $m = 1, 3, 5, 7, \ldots$, respectively. For even numbers of axial half-waves, the roots are given by $(\mu L/a) = 5\pi/2$, $9\pi/2$, $13\pi/2$,..., for $m = 2, 4, 6, \ldots$, respectively.

Using these displacement functions Arnold and Warburton calculated the strain energy and kinetic energy at any instant. Then applying the Lagrange equation, they obtained the following expression for the frequency factor:

$$F^3 - R_2F^2 + R_1F - R_o = 0 \tag{5}$$

where F = frequency factor. R_o, R_1 and R_2 are given by the following equations:

$$\begin{aligned}
R_o = {} & 1/2(1-\nu)[1-\nu^2(\frac{\theta_2}{\theta_1})^2]\mu^4+\beta[\frac{1}{2}(1-\nu)(\mu^8+n^8) \\
& + \{(1-2\nu)\frac{\theta_2}{\theta_1} + \frac{\theta_1}{\theta_2}\}(\mu^6n^2+\mu^2n^6)+\{3-\nu-2\nu(\frac{\theta_2}{\theta_1})^2\}\mu^4n^4 \\
& - (2-\nu)\{2-(1+\nu)\nu(\frac{\theta_2}{\theta_1})^2\}\mu^4n^2 \\
& - \{2\frac{\theta_1}{\theta_2}+2(1-2\nu)\frac{\theta_2}{\theta_1}\}\mu^2n^4-(1-\nu)n^6+\{2(1-\nu) \\
& - 2\sigma^2(1-\nu)(\frac{\theta_2}{\theta_1})^2\}\mu^4+\{(1-2\nu)\frac{\theta_2}{\theta_1} + \frac{\theta_1}{\theta_2}\}\mu^2n^2 \\
& + \frac{1}{2}(1-\nu)n^4] ,
\end{aligned}$$

$$\begin{aligned}
R_1 = {} & \frac{1}{2}(1-\nu)(\mu^4+n^4)+(\frac{\theta_1}{\theta_2} -\nu \frac{\theta_2}{\theta_1})\mu^2n^2+\frac{1}{2}(1-\nu)n^2 \\
& + \{\frac{1}{2}(1-\nu-2\nu^2)\frac{\theta_2}{\theta_1} + \frac{\theta_1}{\theta_2}\}\mu^2+\beta[(\frac{1}{2}(1-\nu)\frac{\theta_2}{\theta_1} + \frac{\theta_1}{\theta_2}\}\mu^6 \\
& + \{\frac{1}{2}(7-\nu)+(1-\nu)(\frac{\theta_2}{\theta_1})^2\}\mu^4n^2+\{\frac{1}{2}(7-3\nu)\frac{\theta_2}{\theta_1} + \frac{\theta_1}{\theta_2}\}\mu^2n^4 \\
& + \frac{1}{2}(3-\nu)n^6+2(1-\nu)\mu^4-\{(3-\nu^2)\frac{\theta_2}{\theta_1} - \frac{\theta_1}{\theta_2}\}\mu^2n^2 \\
& - \frac{1}{2}(3+\nu)n^4+2(1-\nu)\frac{\theta_2}{\theta_1} \mu^2+n^2] ,
\end{aligned}$$

$$R_2 = [\frac{\theta_1}{\theta_2} + \frac{1}{2}(1-\nu)\frac{\theta_2}{\theta_1}]\mu^2+\frac{1}{2}(3-\nu)n^2+1+\beta[\mu^4+2\frac{\theta_2}{\theta_1}\mu^2 n^2$$

$$+n^4+2(1-\nu)\frac{\theta_2}{\theta_1}\mu^2+n^2] ,$$

where $\theta_1 = 1+(-1)^{m+1} k^2$, and

$$\theta_2 = 1+(-1)^{m+1}(\frac{2a}{\mu L} \sin \frac{\mu L}{a} -k^2) .$$

(b) Simply Supported Circular Cylindrical Shells.

Arnold and Warburton used the following displacement functions:

$$\bar{u} = A_n \cos \frac{m\pi ax}{L} \cos n\phi \cos\omega t,$$

$$\bar{v} = B_n \sin \frac{m\pi ax}{L} \sin n\phi \cos\omega t, \qquad (6a\text{-}c)$$

$$\bar{w} = C_n \sin \frac{m\pi ax}{L} \cos n\phi \cos\omega t.$$

The equation for the frequency factor, obtained by using these displacement functions, is given by:

$$F^3 - P_2F^2 + P_1F - P_o = 0 , \qquad (7)$$

where F is the frequency factor. P_o, P_1 and P_2 are given by the following expressions:

$$P_o = \frac{1}{2}(1-\nu)^2(1+\nu)\lambda^4+\frac{1}{2}(1-\nu)\beta[(\lambda^2+n^2)^4-2(4-\nu^2)\lambda^4n^2$$

$$-8\lambda^2n^4-2n^6+4(1-\nu^2)\lambda^4+4\lambda^2n^2+n^4] ,$$

$$P_1 = \frac{1}{2}(1-\nu)(\lambda^2+n^2)^2+\frac{1}{2}(3-\nu-2\nu^2)\lambda^2+\frac{1}{2}(1-\nu)n^2$$

$$+\beta[\frac{1}{2}(3-\nu)(\lambda^2+n^2)^3+2(1-\nu)\lambda^4-(2-\nu^2)\lambda^2n^2$$

$$-\frac{1}{2}(3+\nu)n^4+2(1-\nu)\lambda^2+n^2] ,$$

and

$$P_2 = 1+\frac{1}{2}(3-\nu)(\lambda^2+n^2)+\beta[(\lambda^2+n^2)^2+2(1-\nu)\lambda^2+n^2] ,$$

where $\lambda = \frac{m\pi a}{L}$ and $\beta = \frac{h^2}{12a^2}$.