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Steady state temperature distribution in a rotating disk

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STEADY STATE TEMPERATURE DISTRIBUTION IN A ROTATING DISK

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INTRODUCTION

The purpose of this investigation was to obtain an expression for the steady state temperature distribution of a disk of uniform thickness rotating at a constant speed in ambient air while absorbing heat through the periphery at a constant rate. Engineers encounter such problems in connection with brakes of several varieties. The general solution obtained here could be extended to similar physical problems such as grinding wheels, disk clutches, and so on.

A differential equation was obtained for the temperature in the disk by the application of Fourier's conduction law and Newton's convection law. A solution of this equation was obtained in the form of a Bessel function of the first kind and order zero and the constants of integration as determined by the boundary conditions.

ASSUMPTIONS

1. Steady state conditions prevail. (i. e. the temperature is constant with respect to time.
2. The disk material is homogenious and isotropic. (i. e. all intensive properties of the disk are symmetrical about the polar axis.
3. The disk is thin enough that the temperature variation other than radially can be neglected.
4. The thermal conductivity and the heat transfer film coefficient are constant over the region of the disk under consideration.
5. No heat is lost through the disk periphery.
6. The temperature of the surrounding air is constant.
7. Aerodynamic heating is neglected.
8. Heat is transferred through the edge at a uniform rate.
9. Angular speed is maintained constant.

NOTATIONS

Refer to Figure 1.

2Q = B.T.U./hr energy applied at the periphery of the disk by the brake.

h = Combined heat transfer film coefficient expressed in B.T.U./hr ft^2°F.

k = Thermal conductivity in B.T.U./hr ft^2°F/ft.

r = Radial distance in ft.

y = One-half of the disk thickness in ft.

w = The constant angular velocity of the disk in radians/sec.

θ = Temperature difference between the disk at a given point and the ambient air in °F.

i = $\sqrt{-1}$

m = $\sqrt{h/ky}$

$J_n(x)$ = Bessel function of the first kind and order "n".

$H_n(x)$ = Bessel function of the second kind and order "n", also known as the Hankel function. It is most commonly denoted as $Y_n(x)$ although it is denoted as $N_n(x)$ in the "Tables of Functions" of Jahnke-Emde. (1) The notation used here is that of Max Jakob in Volume I of "Heat Transfer". (2)

R = The Reynolds number, $w\ r^2/v$

v = Kinematic Viscosity

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(1) All references are in the bibliography.

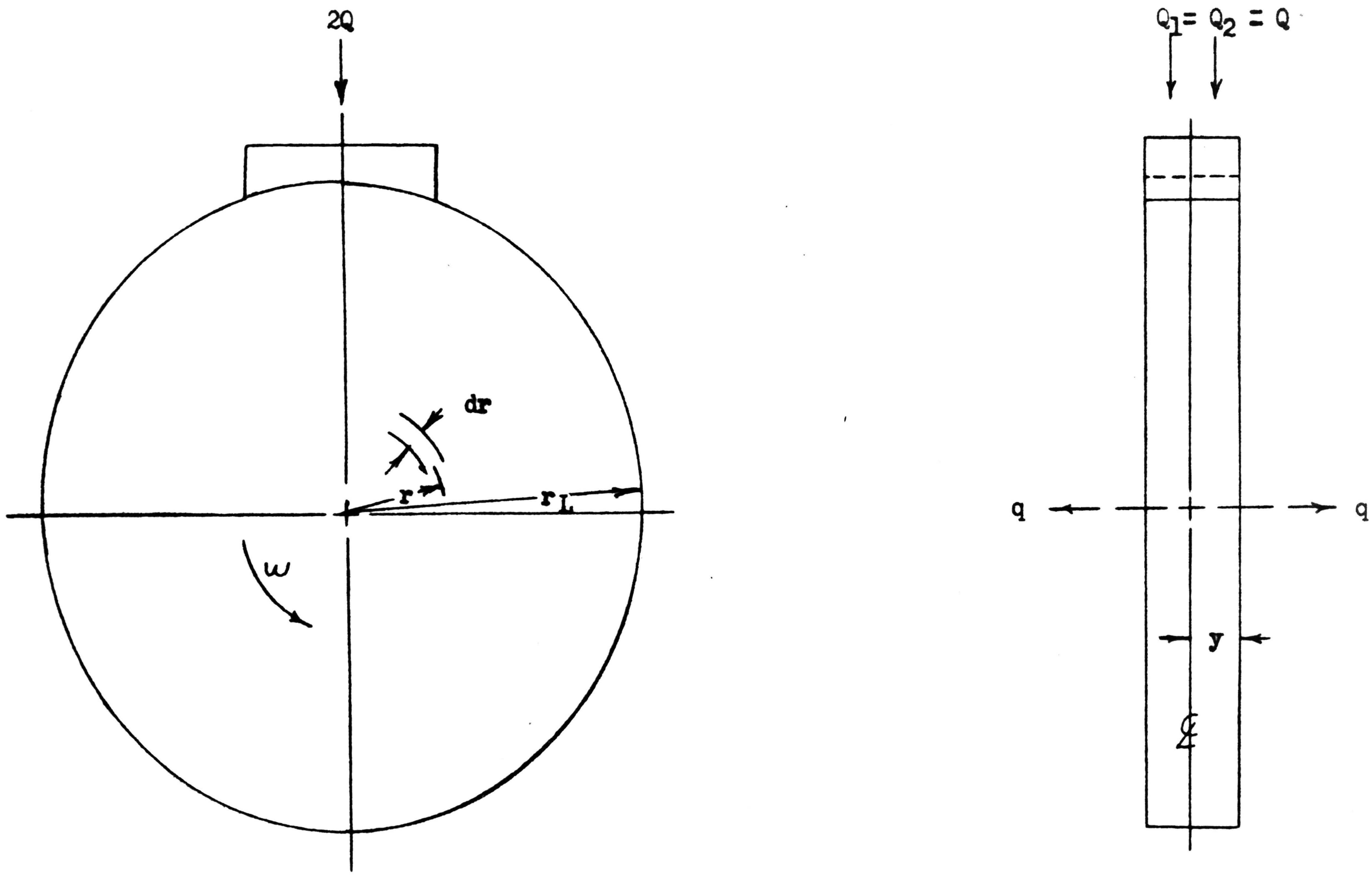


Figure 1. Rotating disk with heat added through the periphery

STEADY STATE HEAT CONDUCTION IN A ROTATING DISK WITH A CONSTANT FILM COEFFICIENT

Consider a disk of uniform thickness 2y rotating in ambient air at constant angular velocity, w, radians per second, while receiving heat uniformly at the rate of 2Q B.T.U./hr, through its periphery. The heat is dissipated by convection from the plane surfaces of the disk, according to Newton's law, at the rate given by $hA\theta$, where h is the film coefficient considered to be constant here, A is the area giving off the heat and θ is the temperature difference between the surface of the disk and the ambient air. The heat is conducted through the metal according to the Fourier equation at the rate given by $k A' \frac{d\theta}{dr}$ where k is the thermal conductivity of the metal considered to be a constant here, A' is the cross-sectional flow area, and $\frac{d\theta}{dr}$ is the temperature gradient.

A differential equation for the temperature difference in terms of the radius can be obtained as follows:

The heat flow through the cross-sectional area at any radial distance, r, is given according to the Fourier equation, by

$$q = -k\ (2\pi)\ ry\ \frac{d\theta}{dr}$$

The heat loss by convection at this cross-sectional area is, by Newton's law, $-dq = h\theta dA = h\theta\ (2\pi)\ r\ dr$

From $q = -k\ (2\pi)\ ry\ \frac{d\theta}{dr}$,

$$\frac{d}{dr}\ (q) = -k\ (2\pi)\ y\ \left(\frac{d\theta}{dr} + r\ \frac{d^2\theta}{dr^2}\right)$$

By equating this to $dq = -h\theta 2\ (\pi)\ r\ dr$,

$$\frac{dq}{dr} = -k\,(2\,\pi)\,y\,\left(\frac{d\theta}{dr} + \frac{rd^2\theta}{dr^2}\right) = -h\theta\,(2\,\pi)\,r\,dr$$

From this, $ky\left(\frac{d\theta}{dr} + \frac{rd^2\theta}{dr^2}\right) = h\theta\, r\, dr$

By transposing, this becomes the differential equation

$$r^2 \frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - \frac{hr^2}{ky}\,\theta = 0 \qquad (1)$$

An alternate form of this equation is:

$$r^2 \frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} + \left\{(im)^2\, r^2 - n^2\right\}\theta = 0 \qquad (2)$$

where $n = 0$ and $m^2 = \frac{h}{ky}$

This is the Bessel's equation of the first kind and order n with a parameter im. (2) In this particular case, n = 0.

The general solution for this equation (3) (4) is:

$$\theta = M\, J_o\,(imr) + Ni\, H_o\,(imr) \qquad (3)$$

where M and Ni are constants of integration to be determined by the boundary conditions.

One of the boundary conditions is obtained from the assumption that the temperature distribution is a continuous function over the entire disk with a minimum at r = 0. From this, at r = 0, $\frac{d\theta}{dr} = 0$.

To obtain $\frac{d}{dr}(\theta)$ from equation (3), two differential formulas are needed.

1. $\frac{d}{dr} J_o\,(imr) = -im\, J_1\,(imr)$ where m is constant (5)

2. $\frac{d}{dr} H_o\,(imr) = -im\, H_1\,(imr)$ where m is constant (6)

Using these, $\frac{d\theta}{dr} = -M\,(im)\,J_1\,(imr) - Ni\,(im)\,H_1\,(imr)$

Considering this equation as r approaches zero, J_1 (imr), as can be seen from a table of values, approaches zero, and $d\theta/dr$ becomes zero, but H_1 (imr) approaches minus infinity. From these conditions, N must be equal to zero for this particular problem.

Equation (3) then becomes: $\theta = M\, J_o\, (imr)$ (4)

Evaluating M by noting that at $r = 0$, when $\theta = \theta_o$, equation (4) becomes: $\theta_o = M\, J_o\, (0) = M$ since $J_o\, (0) = 1$

Equation (3) now becomes: $\theta = \theta_o\, J_o\, (imr)$ (5)

θ_o, the temperature at the center of the disk can be obtained in terms of the energy being added at the periphery of the wheel by both Fourier's law and Newton's law.

From the Fourier equation $q = -k\, (2\pi)\, ry\, d\theta/dr$, where q is the heat being conducted through the cross-sectional area $(2\pi)\, ry$ at the radial distance r. Since Q is the heat being conducted through one-half of the disk thickness, y, at the radial distance $r = r_L$ the Fourier equation gives the expression for Q as: $Q = (2\pi)\, k\, r_L\, y\, d\theta/dr\Big]_{r_L}$

where $d\theta/dr\Big]_{r_L}$ is the value of $d\theta/dr$ evaluated at r_L.

Since $d\theta/dr = -\theta_o\, (im)\, J_1\, (imr)$

then $d\theta/dr\Big]_{r_L} = -\theta_o\, (im)\, J_1\, (imr_L)$

and Q becomes $Q = -(2\pi)\, k\, r_L\, y\, \theta_o\, (im)\, J_1\, (imr_L)$.

From this, $\theta_o = \dfrac{Q}{(2\pi)\, r_L\, ky\, m\, (-i)\, J_1\, (imr_L)}$

Since $m^2 = h/ky$, $kym = \sqrt{hky}$ and O_o can be written as

$$O_o = \frac{Q}{(2\pi)\, r_L\, \sqrt{hky}\, (-i)\, J_1\, (imr_L)}\,. \qquad (6)$$

Also, under the assumption that one-half of the energy absorbed flows out of each face, Newton's equation can be integrated over the entire face of the disk to find Q.

$$Q = -\int_0^{r_L} dq = (2\pi)\, h \int_0^{r_L} \theta\, r\, dr$$

In evaluating this integral the following theorem will be utilized. (7)

"Let 'f' be a given function continuous on the closed interval (a,b). Suppose that 'F' is any differentiable function such that F' (x) = f (x) when $a \leqq x \leqq b$. Then $\int_a^b f(x)\,dx = f(b) - F(a)$.

Also, from the recurrence formula (8):

$$Z J_o (Z) = \frac{d\{Z J_1 (Z)\}}{dZ}$$

where $Z = x = (imr)$

Substituting,

$$Z J_o (Z) = (imr) J_o (imr) = \frac{d\{imr J_1 (imr)\}}{d(imr)}$$

or transposing,

$$(im)^2 r J_o (imr) dr = d\{(imr) J_1 (imr)\}$$

which leads to:

$$r J_o (imr) dr = \frac{1}{(imr)} d \{r J_1 (imr)\} \tag{7}$$

From Equation (5) $\theta = \theta_o J_o (imr)$

Therefore $Q = (2\pi) h \int_o^{rL} \theta r dr = (2\pi) h \theta_o \int_o^{r_L} J_o (imr) r dr$

From (7) $r J_o (imr) dr = \frac{1}{i m} d\{r J_1 (imr)\}$

So Q becomes $Q = (2\pi) h \theta_o \int_o^{r_L} \frac{1}{(im)} d\{r J_1 (imr)\}$

Integrating,

$$Q = \frac{(2\pi)}{i m} h \theta_o r_L J_1 (imr_L)$$

From this, $\theta_o = \frac{(im) Q}{(2\pi) h r_L J_1 (imr_L)} = \frac{\left\{\left(\frac{h}{ky}\right)\right\}^{1/2} Q}{(2\pi) h r_L (-i) J_1 (imr_L)} =$

$$\frac{Q}{(2\pi)(k h y)^{1/2} r_L (-i) J_1 (imr_L)}$$

This is the same expression as equation (6), so from equation (5):

$$\theta = \theta_o J_o (imr) = \frac{Q}{(2\pi)\sqrt{(hky)} r_L (-i) J_1 (imr_L)} J_o (imr) \tag{8}$$

Equation (8) is an exact solution only in cases where the heat transfer film coefficient and the thermal conductivity are constant over the entire disk. Theory predicts and experimental data shows that in the laminar flow region the heat transfer coefficient can be obtained for an isothermal disk. (9) (10)

Laminar flow exists at Reynolds numbers below about 2×10^5. For laminar flow across a rotating isothermal disk for the Reynolds number, R, between 100,000 to 200,000

$$N = .36\,(R)^{.5} \qquad (11)$$

where N is the local Nusselt number given by $\frac{h_c r}{k}$ where h_c is the local convection heat-transfer coefficient at r, the radial distance. This can be written as

$$N = \frac{h_c r}{k} = .36\,(w\, r^2/v)^{.5}$$

Upon simplifying, the following expression for the local heat transfer convection coefficient is obtained:

$$h_c = .36\,\frac{k}{r}\,(w\, r^2/v)^{.5} = .36\, k\,(w/v)^{.5}$$

An important point to note is that the local convection heat transfer coefficient for an isothermal rotating disk is not a function of the radius. (12) (13) For moderate temperature ranges, the effects of radiation can be neglected without appreciable error. The above expression for the convection heat transfer coefficient for an isothermal rotating disk can be used to approximate an average heat transfer coefficient for the disk, neglecting radiation. This approximate heat transfer coefficient can be used in equation (8) to approximate the temperature distribution across the disk for laminar flow conditions.

Obviously, the higher the temperatures involved, the greater the effects of radiation upon the actual disk. Where radiation does occur the results predicted by equation (8) are higher than the actual temperatures will be for the high temperature region of the disk, i.e., the disk periphery. The results obtained from equation (8) for various constant values of h, k, and y are shown in Fig. 2 and Fig. 3.

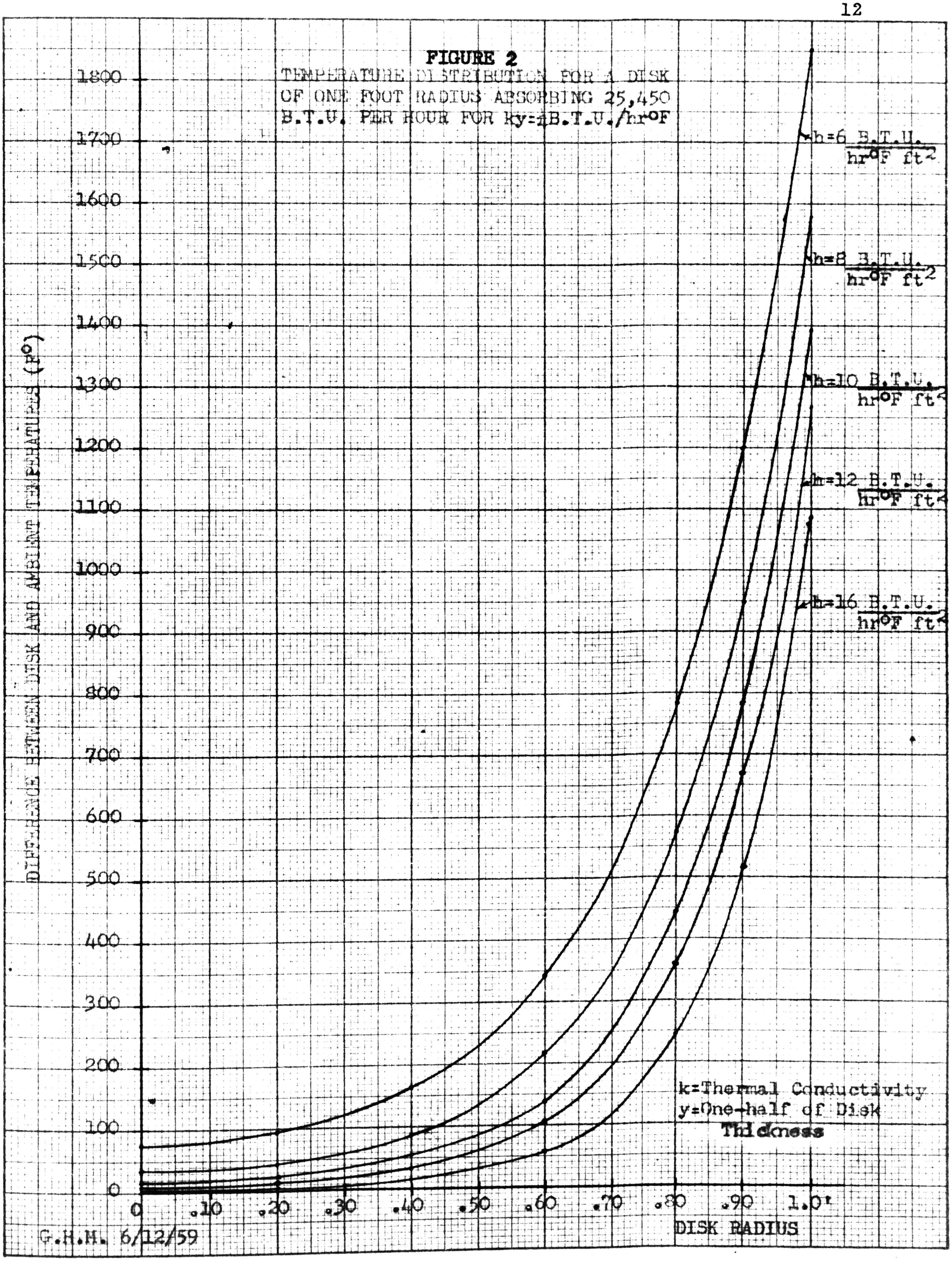


FIGURE 2
TEMPERATURE DISTRIBUTION FOR A DISK
OF ONE FOOT RADIUS ABSORBING 25,450
B.T.U. PER HOUR FOR ky=1B.T.U./hr°F
h=6 B.T.U./hr°F ft²
h=8 B.T.U./hr°F ft²
h=10 B.T.U./hr°F ft²
h=12 B.T.U./hr°F ft²
h=16 B.T.U./hr°F ft²
DIFFERENCE BETWEEN DISK AND AMBIENT TEMPERATURES (F°)
1800
1700
1600
1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
0
0
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.0'
DISK RADIUS
k=Thermal Conductivity
y=One-half of Disk
Thickness
G.H.M. 6/12/59

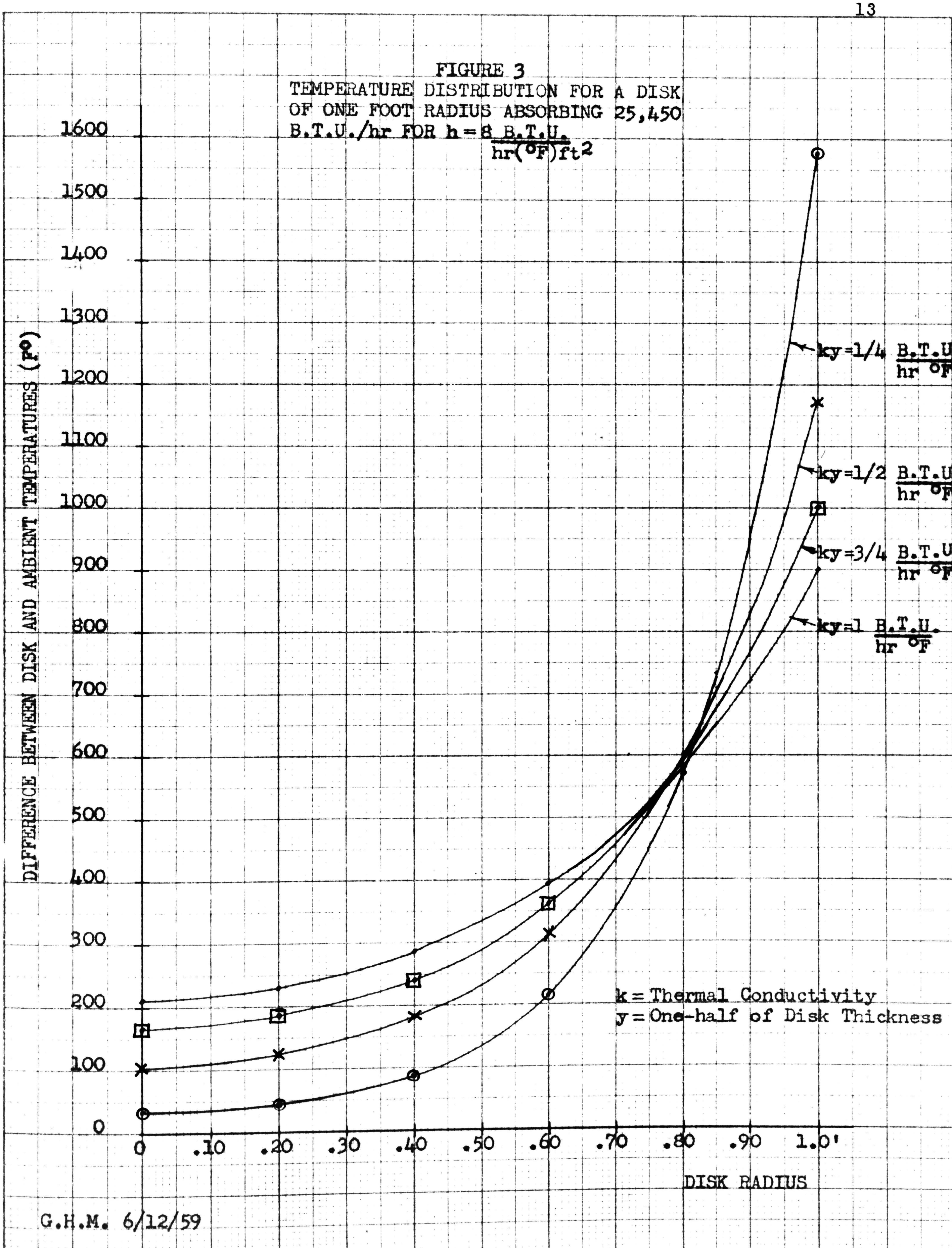


FIGURE 3
TEMPERATURE DISTRIBUTION FOR A DISK OF ONE FOOT RADIUS ABSORBING 25,450 B.T.U./hr FOR h = 8 B.T.U./hr(°F)ft²
DIFFERENCE BETWEEN DISK AND AMBIENT TEMPERATURES (F°)
1600
1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
0
0
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.0'
DISK RADIUS
ky = 1/4 B.T.U./hr °F
ky = 1/2 B.T.U./hr °F
ky = 3/4 B.T.U./hr °F
ky = 1 B.T.U./hr °F
k = Thermal Conductivity
y = One-half of Disk Thickness
G.H.M. 6/12/59

STEADY STATE HEAT CONDUCTION IN A ROTATING METAL DISK WITH A VARIABLE FILM COEFFICIENT AND THERMAL CONDUCTIVITY

Consider the case of heat conduction in a metal disk of uniform thickness rotating in ambient air at a constant angular velocity, w, radians per second, while receiving heat uniformly through its periphery where the heat transfer film coefficient and the thermal conductivity are not constant over the entire disk.

The disk can be split into n concentric rings of small enough area that the heat transfer film coefficient and the thermal conductivity can be considered to have a constant, but different value, for each of the given rings. As will be shown, the solution for the temperature distribution across each of the regions can be determined as a function of the given values of the heat transfer film coefficient and the thermal conductivity across each particular ring.

Within a given ring, the heat is dissipated from the surface in accordance with Newton's law at the rate given by

$$q = h \theta A$$

where h is the heat transfer film coefficient which is considered to be constant across the ring, but which may vary from ring to ring, θ is the temperature difference between the surface of the disk and the ambient air in Fahrenheit degrees, and A is the area dissipating the heat. The heat is conducted through the metal by the Fourier equation at the rate given by

$$q = k \quad A' \, d\theta/dr$$

where k is the thermal conductivity of the metal, A' is the cross-sectional flow area, and $d\theta/dr$ is the temperature gradient.

As in the previous part, a differential equation for the temperature difference between the disk and the ambient air as a function of the radius of the disk can be obtained. In this case the values of h, the film coefficient, and k, the thermal conductivity, may vary from ring to ring. The value of h is greatest near the periphery because of radiation effects and, in the case of turbulent flow, due to the increased air velocity at the periphery.

Within a given ring area the heat flow through the cross-sectional area at the radial distance r, is given by

$$q = -k\,(2\pi)\,r\,y\,d\theta/dr$$

The heat dissipated at this cross-sectional area is, by Newton's law,

$$-dq = h\,\theta\,dA = h\,\theta\,(2\pi)\,r\,dr$$

From $q = -k\,(2\pi)\,r\,y\,d\theta/dr$

$$\frac{d}{dr}\,(q) = -k\,(2\pi)\,y\,\left(d\theta/dr + r\,\frac{d^2\theta}{dr^2}\right)$$

By equating this expression for dq to the expression for dq obtained from Newton's law

$$dq/dr = -k\,(2\pi)\,y\,\left(\frac{d\theta}{dr} + r\,\frac{d^2\theta}{dr^2}\right) = -h\,\theta\,(2\pi)\,r\,dr$$

$$\text{or } ky\,\left(d\theta/dr + r\,\frac{d^2\theta}{dr^2}\right) = h\,\theta\,dr$$

By transposing, this becomes the differential equation obtained previously,

$$r^2\,\frac{d^2\theta}{dr^2} + r\,\frac{d\theta}{dr} - \left(\frac{hr^2}{ky}\right)\,\theta = 0$$

An alternate form of this equation is:

$$r^2\,\frac{d^2\theta}{dr^2} + r\,\frac{d\theta}{dr} + \left\{(im)^2\,r^2 - n^2\right\}\,\theta = 0$$

where $n = 0$ and $m^2 = h/ky$. Here m is a constant for a given ring. This is the Bessel's equation of the first kind and order n with a parameter <u>im</u>. Again, $n = 0$ for this particular equation. The important distinction between this and the previous case, where h and k were constant over the entire disk, is that in this case the value of the parameter m is a constant only for a particular ring and the value of the constant changes from ring to ring so now there are as many equations for the disk as there are rings being considered. The distinguishing characteristic of the equations representing the various rings lies entirely in the values of the parameter m.

The general solution for this equation is as before:

$$\theta_n = f\ (imr) = M_n\ J_o\ (im_n\ r) + N_n\ i\ H_o\ (im_n\ r) \qquad (9)$$

Where θ_n is the temperature at the radial distance r in ring <u>n</u>, M_n and N_n are constants of integration that have to be evaluated for each ring from the boundary conditions for the particular rings in question, and m_n is the particular parameter across ring <u>n</u>.

There must be no discontinuity in temperature or heat flow at a boundary common to adjacent rings. These two conditions will provide the necessary equations for the evaluation of the integration constants, M_n and N_n.

Q_n, the heat flow through the one-half thickness, y, into a ring <u>n</u> whose periphery is at r_n can be expressed in terms of the Fourier equation as

$$Q_n = (2\pi)\ r_n\ y\ k_n\ (d\theta/dr)_{r\ =\ r_n}$$

From the general solution for θ within a given ring,

$$d\theta/dr = \frac{d}{dr}\ f\ (imr) = -M_n\ (im_n)\ J_1\ (im_n\ r) - N_n\ i\ (im_n)\ H_1\ (im_n\ r)$$

(Refer page 6 of thesis)

Substituting $d\theta/dr$ at $r = r_n$, the expression for Q_n becomes

$$Q_n = (2\pi)\, r_n\, y\, k_n \left\{- M_n (im_n)\, J_1 (im_n r_n) - N_n\, i\, (im_n)\, H_1 (im_n r_n)\right\}$$

A more useful form of this equation, obtained by simplifying, is

$$Q_n = (2\pi)\, r_n \sqrt{h_n k_n y}\, \left\{M_n (-i)\, J_1 (im_n r_n) + N_n H_1 (im_n r_n)\right\}$$

ΔQ_n, the heat dissipated by one of the surfaces of a given ring, n, can be determined from Newton's law as

$$\Delta Q_n = (2\pi)\, h_n \int_{r_{(n-1)}}^{r_n} r\,\theta\, dr$$

where r_n and r_{n-1} are the radial distances to the boundaries of ring n.

Since within the ring n the temperature difference, θ, is given by

$$\theta = M_n J_o (im_n r) + N_n\, i\, H_o (im_n r)$$

the integral expression for ΔQ_n becomes

$$\Delta Q_n = (2\pi)\, h_n \int_{r_{(n-1)}}^{r_n} r \left\{M_n J_o (im_n r) + N_n\, i\, H_o (im_n r)\right\} dr$$

To evaluate this integral the following are utilized:

$$r J_o (im_n r)\, dr = \frac{i}{im_n} \left\{d \left[r J_1 (im_n r)\right]\right\} \quad \text{(Refer page 9 of thesis)}$$

$$r H_o (im_n r)\, dr = \frac{i}{im_n} \left\{d \left[r H_1 (im_n r)\right]\right\} \qquad (14)$$

Using these, ΔQ_n becomes

$$\Delta Q_n = \frac{(2\pi)\, h_n}{im_n} \int_{r_{n-1}}^{r_n} \left\{M_n\, d \left[r J_1 (im_n r)\right] + N_n\, i\, d \left[r H_1 (im_n r)\right]\right\}$$

Upon integration,

$$\triangle Q_n = \frac{(2\pi)\, h_n}{im_n} \Big\{ M_n \big[r_n J_1 (im_n r_n) - r_{n-1} J_1 (im_n r_{n-1}) \big] + N_n i \big[r_n H_1 (im_n r_n) - r_{n-1} H_1 (im_n r_{n-1}) \big] \Big\}$$

This equation can also be written as

$$\triangle Q_n = (2\pi) \sqrt{h_n k_n y} \Big\{ M_n \big[r_n (-i) J_1 (im_n r_n) - r_{n-1} (-i) J_1 (im_n r_n) \big] + N_n \big[r_n H_1 (im_n r_n) - r_{n-1} H_1 (im_n r_{n-1}) \big] \Big\}$$

There is one restriction to this integral. For the ring at the center of the disk, i.e., for n = 1, the value of N is zero because iH_1 (imr) approaches infinity as r goes to zero, and $\frac{d\theta}{dr}$ must be zero, so only the Bessel function has to be integrated between r = 0 and r = 1. (Refer page 6 of thesis)

An expression for the heat flow out of a ring n where the inner radius is r_{n-1} and the outer radius is r_n can be obtained by either of two different approaches. The first is to simply write the Fourier equation for the heat flow out of the ring. The constants will be those for ring n but the radial distance to the cross sectional area in question is r_{n-1} so in this case $d\theta/dr$ is evaluated at r_{n-1}.

$$Q_{n_{out}} = (2\pi)\, r_{n-1}\, y\, k_n\, (d\theta/dr)_{r = r_{n-1}}$$

where $Q_{n_{out}}$ is the heat flow leaving through the inner boundry of ring n where $r = r_{n-1}$

Upon substituting the value of $d\theta/dr$ at $r = r_{n-1}$, the expression for the heat flow out of the interior boundary of the ring in question becomes

$$Q_{n_{out}} = (2\pi)\, r_{n-1}\, y\, k_n \Big\{ M_n m_n (-i) J_1 (im_n r_{n-1}) + N_n m_n H_1 (im_n r_{n-1}) \Big\}$$

A more useful form of this expression is

$$Q_{n_{out}} = (2\pi)\, r_{n-1} \sqrt{h_n k_n y}\left\{M_n (-i)\, J_1 (im_n r_{n-1}) + N_n H_1 (im_n r_{n-1})\right\}$$

The second approach to obtaining the same expression is to write a heat balance for the ring in question. Since the heat conducted through the periphery on ring n at $r = r_n$ minus the heat dissipated to the air across the ring surface is equal to the heat being conducted through the inside boundary of the ring at $r = r_{n-1}$, a heat balance can be expressed as

$$Q_{n_{out}} = Q_n - \Delta Q_n$$

Where Q_n is the heat flow into the ring n at the periphery through y, the one-half thickness of the disk, and $-\Delta Q_n$ is the heat dissipated across one of the plane surfaces of ring n.

Using the two expressions

$$Q_n = (2\pi) r_n \sqrt{h_n k_n y}\left\{M_n (-i)\, J_1 (im_n r_n) + N_n H_1 (im_n r_n)\right\}$$

$$\Delta Q_n = (2\pi) \sqrt{h_n k_n y}\left\{M_n \left[r_n (-i) J_1 (im_n r_n) - r_{n-1} (-i) J_1 (im_n r_{n-1})\right] + N_n \left[r_n H_1 (im_n r_n) - r_{n-1} H_1 (im_n r_{n-1})\right]\right\}$$

The expression for $Q_{n_{out}}$ can now be obtained by subtracting ΔQ_n which gives the expression

$$Q_{n_{out}} = Q_n - \Delta Q_n = (2\pi)\, r_{n-1} \sqrt{h_n k_n y}\left\{M_n (-i)\, J_1 (im_n r_{n-1}) + N_n H_1 (im_n r_{n-1})\right\}$$

This is the same as the previously obtained expression for $Q_{n_{out}}$.

To obtain the temperature distribution across the disk, the disk is split into n concentric rings of small enough surface area that the heat transfer film coefficient and the thermal conductivity can be considered constant throughout the given ring. For the temperature difference anywhere in the disk, n equations will be required. This temperature difference for the disk can be expressed as

$$\theta_n = M_n J_o (im_n r) + N_n i H_o (im_n r) \qquad (9)$$

Where θ_n is the temperature difference at the radial difference, r, in the ring n where n varies from n = 0 to n = n. As before, m_n is the value of the parameter m for the given ring n under consideration.

With the n temperature difference equations, there are 2 n-1 integration constants to be evaluated. (i.e., M_1, M_2.....M_n and N_2, N_3,.....N_n) As explained previously N_1, the integration constant for the Hankel function in the region containing the center of the disk is zero. (Refer page of this thesis) These 2n-1 integration constants can be evaluated from the equations that can be written as a result of the temperature difference and the heat flow continuities at the ring boundaries and also from the equation that can be written from the known heat flow into the outer rim of the disk. The first of the 2n-1 equations can be obtained by equating the expression for the heat flow into the outer ring of the disk to the known value of the heat being absorbed through the periphery of the disk. In addition, n-1 equations can be written by virtue of the continuity of the temperature difference at the boundaries of the rings. The temperature at each of the n-1 common boundaries can be written in terms of either ring. The two expressions for each given common boundary temperature difference between the disk and the ambient air can then be equated, resulting in one equation for each of the n-1 common boundaries, or n-1 equations in all, each with only integration constants as unknowns. Since the heat flow across a given common ring boundary can also be written in terms of either of the adjoining rings, another equation in terms of the integration constants for the adjacent rings results for each of the common boundaries after the two expressions for the common heat flow through a given boundary are equated. Since there are n-1 common boundaries, another n-1 equations result. So far, 2n-2 equations can be obtained from the temperature and the heat flow continuities, and one equation can be obtained from the known heat energy absorbed by the disk; now there are 2n-1 possible equations whose only unknowns are the integration constants. This gives as many equations as there are unknowns, so the integration constants can be determined. After the integration constants are determined, the temperature at any point in the disk can be determined by using the general equation with the appropriate integration constants and the correct value of the parameter for each ring.

An example problem follows to illustrate the method.

Example Problem (Ref. Fig. 4)

Suppose a homogeneous disk of uniform thickness, 2y, with a one foot radius is absorbing heat energy through its periphery at a constant 25,450 B.T.U./hr while rotating with a constant angular velocity. Furthermore, suppose the disk was split into five concentric rings and the average combined heat transfer film coefficient across each ring was determined to be the value as listed in Table 1. In this case, the value for ky was assumed to be 1/4 B.T.U./hr °F for convenience.

Since there are n = 5 concentric rings, there will be five temperature equations required to express the temperature difference between the disk and the ambient air as a continuous function across the entire disk which has been split into five rings. These five temperature equations will have 2n-1 = 9 integration constants to be exaluated.

The first of the nine equations can be obtained by equating the expression for the heat flow into the outer ring to the known quantity of heat energy being absorbed by the disk. Therefore, for n = 5 for the outer ring,

$$Q_n = Q_5 = (2\pi) \sqrt{h_5\, ky} \left\{ M_5 (-i) J_1 (im_5 r_5) + N_5 H_1 (im_5 r_5) \right\}$$

Upon substituting the known values of Q_5 and the various parameters listed in Tables 1 and 2,

$$Q_5 = \frac{25,450}{2} = (2\pi) \sqrt{h5\ ky} \left\{ M_5 (532.8) - N_5 (.00005900) \right\}$$

Upon transposing, this becomes

(1.) $M_5 (532.8) - N_5 (.00005900) = 954.7081$

TABLE 1

PARAMETER VALUES

n	r_{n-1} (ft.)	r (ft.)	h BTU/hr °F	m_n (/ft.)	$m_n\ r_{n-1}$	$m_n\ r_n$
1	0	.2	8	5.656854	0	1.1313708
2	.2	.4	10	6.324555	1.2649110	2.5298220
3	.4	.6	12	6.928203	2.7712812	4.1569218
4	.6	.8	15	7.745967	4.6475802	6.1967736
5	.8	1.0	18	8.485281	6.7882248	8.4852810

TABLE 2

TABLE OF FUNCTIONS (15)

mr	J_o (imr)	iH_o (imr)	(-i) J_1 (imr)	$-H_1$ (imr)
1.1313708	1.3467	0.2227	0.6614	0.3084
1.2649110	1.4429	0.1882	0.7677	0.2502
2.5298220	3.3660	0.03831	2.586	0.04532
2.7712812	4.0640	0.02885	3.217	0.03370
4.1569218	12.955	0.005963	11.256	0.006621
4.6475802	19.805	0.003481	17.61	0.003817
6.1967736	80.48	0.0006400	73.66	0.0006903
6.7882248	138.62	0.0003392	127.96	0.0003634
8.4852810	673.80	0.00005578	532.80	0.00005900

Four more equations (n-1 = 4) can be written from the expressions for the temperature at each of the four common ring boundaries as follows: (All numerical values are taken from Tables 1 and 2)

At $r = r_1 = .2$

$$\theta = M_1 J_o (im_1 r_1) = M_2 J_o (im_2 r_1) + N_2 i H_O (im_2 r_1)$$

or,

(2) $M_1 (1.3467) = M_2 (1.4429) + N_2 (0.1882)$

At $r = r_2 = .4$

$$\theta = M_2 J_o (im_2 r_2) + N_2 iH_o (im_2 r_2) = M_3 J_o (im_3 r_2) + N_3 iH_o (im_3 r_2)$$

or,

(3) $M_2 (3.3660) + N_2 (0.03831) = M_3 (4.0640) + N_3 (0.02885)$

At $r = r_3 = .6$

$$\theta = M_3 J_o (im_3 r_3) + N_3 iH_o (im_3 r_3) = M_4 J_o (im_4 r_3) + N_4 iH_o (im_4 r_3)$$

or,

(4) $M_3 (12.955) + N_3 (0.005963) = M_4 (19.805) + N_4 (0.003481)$

At $r = r_4 = .8$

$$\theta = M_4 J_o (im_4 r_4) + N_4 iH_o (im_4 r_4) = M_5 J_o (im_5 r_4) + N_5 iH_o (im_5 r_4)$$

or,

(5) $M_4 (80.48) + N_4 (0.0006400) = M_5 (138.62) + N_5 (0.0003392)$

The remaining four required equations can be obtained from the expressions for the heat flow at each of the four common ring boundaries as follows: (Again numerical values are taken from Tables 1 and 2)

At $r = r_1 = .2$

$$Q_1 = (2\pi)\, r_1 \sqrt{h_1\ ky}\ \{M_1\ (-i)\ J_1\ (im_1\ r_1)\} = (2\pi)\, r_1 \sqrt{h_2\ \ ky}$$
$$\{M_2\ (-i)\ J_1\ (im_2\ r_2)\ - N_2\ (-)\ H_1\ (im_2\ r_2)\}$$

or, upon evaluating the constants and simplifying,

(6) $M_1\ (0.66.4) = .858315\ M_2 - .279732\ N_2$

At $r = r_2 = .4$

$$Q_2 = (2\pi)\, r_2 \sqrt{h_2\ ky}\ \{M_2\ (-i)\ J_1\ (im_2\ r_2) - N_2\ (-)\ H_1\ (im_2\ r_2)\}$$
$$= (2\pi)\, r_2 \sqrt{h_3\ ky}\ \{M_3\ (-i)\ J_1\ (im_3\ r_2) - N_3\ (-)\ H_1\ (im_3\ r_2)\}$$

or, upon evaluating the constants and simplifying,

(7) $M_2\ (2.586) - N_2\ (0.04532) = 3.52405\ M_3 - 0.0369165\ N_3$

At $r = r_3 = .6$

$$Q_3 = (2\pi)\, r_3 \sqrt{h_3\ ky}\ \{M_3\ (-i)\ J_1\ (im_3\ r_3) - N_3\ (-)\ H_1\ (im_3\ r_3)\}$$
$$= (2\pi)\, r_3 \sqrt{h_4\ ky}\ \{M_4\ (-i)\ J_1\ (im_4\ r_3) - N_4\ (-)\ H_1\ (im_4\ r_3)\}$$

or, upon evaluating the constants and simplifying,

(8) $M_3\ (11.256) - N_3\ (0.006621) = 19.6774\ M_4 - 0.0042675\ N_4$

At $r = r_4 = .8$

$$Q_4 = (2\pi)\, r_4 \sqrt{h_4\ ky}\ \{M_4\ (-i)\ J_1\ (im_4\ r_4)\ -N_4\ (-)\ H_1\ (im_4\ r_4)\}$$
$$= (2\pi)\, r_4 \sqrt{h_5\ ky}\ \{M_5\ (-i)\ J_1\ (im_5\ r_4)\ -N_5\ (-)\ H_1\ (im_5\ r_4)\}$$

or, upon evaluating the constants and simplifying,

(9) $M_4\ (73.66) - N_4\ (0.0006903) = 140.173\ M_5 - 0.000398085$

Now there are nine equations in terms of the nine unknown integration constants so the integration constants can be determined algebraically. Upon the determination of the integration constants, the five temperature equations become:

From $r = 0$ to $r = .2$

$$\theta = 7.4745\ J_o\ (im_1\ r)$$

From r = .2 to r = .4

$$\theta = 6.62890\ J_o\ (im_2\ r) + 2.66879\ iH_o\ (im_2\ r)$$

From r = .4 to r = .6

$$\theta = 5.23865\ J_o\ (im_3\ r) + 39.0033\ iH_o\ (im_3\ r)$$

From r = .6 to r = .8

$$\theta = 3.23483\ J_o\ (im_4\ r) + 1,158.74\ iH_o\ (im_4\ r)$$

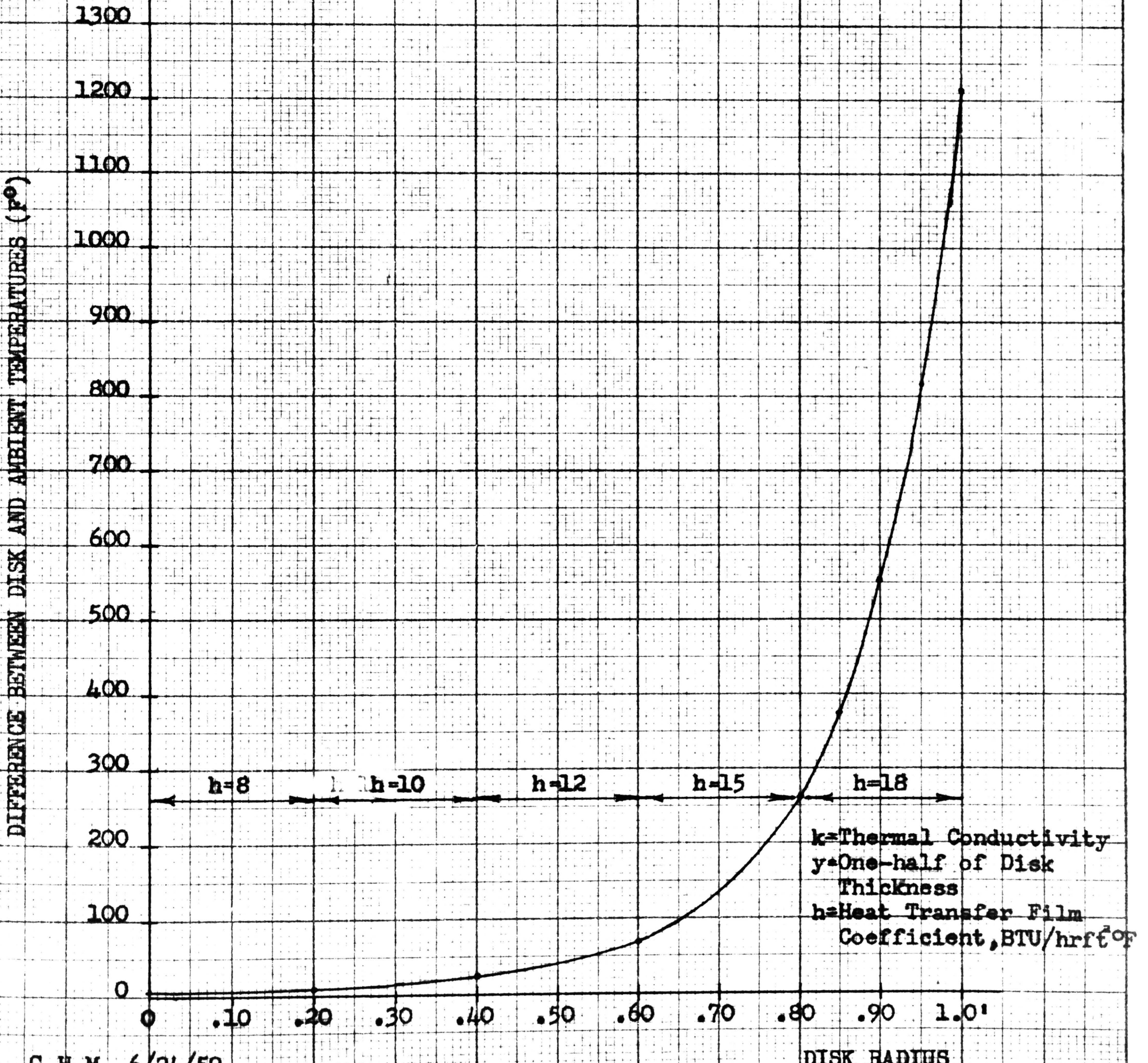
From r = .8 to r = 1.0

$$\theta = 1.79583\ J_o\ (im_5\ r) + 35,796.4\ iH_o\ (im_5\ r)$$

where the values of the parameter m_n are listed in Table 1.

The temperature difference distribution across the entire disk as shown by these equations is graphically illustrated in Fig. 4. It should be noted here that the last decimal place of the tabulated Bessel and Hankel functions is uncertain [16] so the end result is that the temperatures obtained from the temperature equations above are accurate only to three places. However, for computational consistancy in evaluating and cross-checking the various constants, several more places were used.

FIGURE 4
TEMPERATURE DISTRIBUTION FOR A DISK OF ONE FOOT RADIUS ABSORBING 25,450 B.T.U./hr FOR ky=¼B.T.U./hr°F AND h A VARIABLE



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CONCLUSIONS

In this investigation, the steady state temperature distribution for a disk of uniform thickness rotating at a constant speed in ambient air while absorbing heat at a constant rate through its periphery was obtained.

A simple Bessel function relationship, equation (8), was obtained for cases where h, the heat transfer film coefficient, and k, the thermal conductivity, can be considered a constant throughout the disk. Fig. 2 and 3 show the temperature distribution in a two-feet diameter disk for various constant values of the heat transfer film coefficient and the parameter ky where k is the thermal conductivity and y is one-half the disk thickness. These figures illustrate that most of the heat is dissipated within a few inches of the periphery. As a result of this heat dissipation, the temperature at the center of the disk for a given set of parameters is negligable compared to the temperature at the periphery. The majority of the engineering applications for a disk brake can be handled with this simple relationship. The author recommends that equation (8) be used to approximate the temperature distribution if the exact distribution is not required because of the simplicity of the computations required. By evaluating the variables h and k at the center and at the periphery of the disk, an upper and lower boundary of the possible temperature values across the disk can be quickly obtained with equation (8) for a given amount of heat added at the periphery of a given disk.

For the cases where an exact temperature distribution is required, the disk is split into a number of small concentric rings and equation

(9) is used to evaluate the temperature distribution across each ring. Equation (9) expresses the temperature distribution within each ring in terms of Bessel and Hankel functions of the parameter $m = \sqrt{h/ky}$ where m is evaluated specifically for each ring. For the disk of n concentric rings, n temperature equations are required which have constants of integration to be evaluated from the boundary conditions listed in the thesis. An example problem is included to illustrate the method. Fig. 4 illustrates the temperature distribution for the example problem featuring a disk two feet in diameter absorbing 25,450 B.T.U./hr. The disk was assumed to have a ky value 1/4 B.T.U./hr °F which corresponds to a one-fourth inch thick steel disk.

The above method is an exact solution, but the mathematic computations tend to encourage the use of equation (8) to approximate the resulting temperature distribution rather than the more exact solution given by equation (9).

APPENDIX

On the Evaluation of the Combined Heat Transfer Film Coefficient

The equations expressing the temperature difference distribution of the rotating disk are only as correct as the computed values of the required parameters. Tabulated values of the thermal conductivity are readily available and determining the thickness of the disk is no problem. However, determining the heat transfer film coefficient is much more involved.

At the present time, the best way to evaluate the heat transfer film coefficient seems to be to evaluate the convection film coefficient and then add to it a film coefficient to cover radiation effects.

Two empirical equations are available for evaluating the convection film coefficient. The one used for the laminar flow range for Reynolds number, R, from 100,000 to 240,000:

$$h_c = .36\ k\ (w/v)^{.5} \qquad (17)$$

where h_c is the local convection film coefficient, k is the thermal conductivity of the air, and v is the local kinematic viscosity of the air.

In the turbulent range for Reynolds number, R, greater than 240,000:

$$N_m = \frac{\bar{h}_c\ r}{k} = 0.015\ (R)^{.8} \qquad (18)$$

where N_m is an average Nusselt number based on $\bar{h}_c$, an average convection film coefficient for the ring in question, and average values of k and r.

This expression for the average Nusselt number can be rearranged to give:

$$\bar{h}_c = 0.015 \frac{k}{r} (R)^{.8}$$

These two equations were obtained from experimental data based on isothermal rotating disks. The expression for turbulent flow is probably only a rough approximation of the average convection film coefficient for a given ring, but it is all that is available at the present time. Note that the two equations were extended into the transition region where R = 200,000 to 280,000.

To account for the radiation effects, a fictitious radiation film coefficient can be computed from the following:

$$h_r = 0.173\ e \quad \frac{(T_d/100)^4 - (T_a/100)^4}{(T_d - T_a)} \qquad (19)$$

Where T_d = the average absolute temperature of the given ring of the disk, °Rankine

T_a = the absolute ambient temperature, °Rankine

e = emissivity constant

If the disk has been split into rings of small enough area, the combined heat transfer film coefficient for each ring can be obtained by combining the convection and the radiation film coefficients for the given ring.

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VITA

George H. Morgan was born July 23, 1935, in St. Louis, Missouri, the son of Mr. and Mrs. George N. Morgan. He received his elementary education at Jefferson City, Missouri, Plaquemine, Louisiana, Grove, Oklahoma, Rolla, Missouri and Lake Charles, Louisiana. He graduated May, 1953, as Valedictorian from the J. A. Landry Memorial High School conducted by the Christian Brothers at Lake Charles, Louisiana. Having volunteered for the draft, he entered the army two weeks later and was stationed at Fort Sill, Oklahoma, for the next two years. His duty station was with the Army Aviation School where he attended the Basic Cargo Helicopter Maintenance Course and later became a helicopter stage field chief for six months.

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