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EXPERIMENTAL EVALUATION OF THERMAL

STRAINS AND STRESSES

BY

TERRY FRANKLIN LEHNHOFF

A

THESIS

submitted to the faculty of the

SCHOOL OF MINES AND METALLURGY OF THE UNIVERSITY OF MISSOURI

in partial fulfillment of the work required for the

Degree of

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I. ABSTRACT

This investigation is the study of a method for the experimental measurement of thermal strains. Using four electric strain gages (two of one type and two of another) the method will yield the normal and shear strains as well as the temperature change in any biaxial stress field. The gage types must be chosen for the range of temperature in which it is desired to measure the thermal stresses.

The method described eliminates the need for extra resistors, thermocouples, unrestrained dummy gages at the same temperature as the active gages, special alloy gages for each type of metal and special dual element gages as are required by the various temperature compensating techniques.

An analytical solution to the test problem has been obtained for comparison with fair agreement considering the assumptions that are necessary for the solution.

ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. T. R. Faucett, Professor of Mechanical Engineering, for suggesting this problem and for his guidance and interest during the investigation.

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LIST OF SYMBOLS

- ϵ_x — strain in the x direction (in./in.)
 ϵ_y — strain in the y direction (in./in.)
 γ_{xy} — shear strain in the xy plane (in./in.)
 ϵ_{xL} — strain in the x direction due to load (in./in.)
 ϵ_{yL} — strain in the y direction due to load (in./in.)
 γ_{xyL} — shear strain in the xy plane due to load (in./in.)
 ϵ_{xTR} — strain in the x direction due to thermal restraint (in./in.)
 ϵ_{yTR} — strain in the y direction due to thermal restraint (in./in.)
 γ_{xyTR} — shear strain in the xy plane due to thermal restraint (in./in.)
 δ_x — deformation in the x direction (in./in.)
 δ_y — deformation in the y direction (in./in.)
 ϵ_{θ} — strain in the θ direction (in./in.)
 ϵ_{θ_1} — strain in the θ direction due to strain in the x direction
(in./in.)
 ϵ_{θ_2} — strain in the θ direction due to strain in the y direction
(in./in.)
 ϵ_{θ_3} — strain in the θ direction due to shear strain in the xy plane
(in./in.)
 α — coefficient of thermal expansion (in./in./°F)
 α_f — coefficient of thermal expansion of the gage filament (in./in./°F)
 α_A — coefficient of thermal expansion of the aluminum plate
(in./in./°F)
 α_S — coefficient of thermal expansion of the steel plate (in./in./°F)

- T --- temperature ($^{\circ}\text{F}$)
- ΔT --- temperature change or temperature above the condition of zero thermal stress ($^{\circ}\text{F}$)
- T_0 --- initial temperature at which zero thermal stress exists ($^{\circ}\text{F}$)
- T_1 --- final temperature at which thermal stress exists ($^{\circ}\text{F}$)
- G_n --- gage reading (microinches/inch) ($n = 1, 2, 3, 4$)
- K_T --- thermal coefficient of resistivity of the gage filament
(ohms/ohm/ $^{\circ}\text{F}$)
- GF_n --- gage factor or strain sensitivity of the gage filament
(ohms/ohm/in./in.) ($n = 1, 2, 3, 4$)
- R --- gage resistance (ohms)
- ΔR --- change in gage resistance (ohms)
- L --- gage length (in.)
- ΔL --- change in gage length (in.)
- D --- gage diameter (in.)
- ν --- Poisson's ratio (in./in./in./in.)
- ν_A --- Poisson's ratio for aluminum (in./in./in./in.)
- ν_S --- Poisson's ratio for steel (in./in./in./in.)
- F_A --- axial force in the aluminum plate (lb.)
- F_S --- axial force in the steel plate (lb.)
- F --- axial force (lb.)
- M --- bending moment (in.lb.)
- M_A --- bending moment in the aluminum plate (in.lb.)
- M_S --- bending moment in the steel plate (in.lb.)
- h --- plate thickness (in.)
- h_A --- thickness of the aluminum plate (in.)
- h_S --- thickness of the steel plate (in.)

- w --- separation between the plates (in.)
 r --- radius of curvature (in.)
 r_A --- radius of curvature of the aluminum plate (in.)
 r_S --- radius of curvature of the steel plate (in.)
 E --- modulus of elasticity (psi.)
 E_A --- modulus of elasticity of the aluminum plate (psi.)
 E_S --- modulus of elasticity of the steel plate (psi.)
 Δ --- change in fiber length due to bending (in.)
 $d\theta$ --- small angle formed by the intersection of lines extending from
the plate ends to the radius of curvature (radians)
 $2L_p$ --- plate length (in.)
 I --- moment of inertia about a base (in.⁴)
 I_A --- moment of inertia about a base of the aluminum plate (in.⁴)
 I_S --- moment of inertia about a base of the steel plate (in.⁴)
 B --- plate depth (in.)
 S_x --- stress in the x direction (psi.)
 S_A --- stress in the aluminum plate (psi.)
 S_S --- stress in the steel plate (psi.)
 A --- cross sectional area of the plates (in.²)
 A_A --- cross sectional area of the aluminum plate (in.²)
 A_S --- cross sectional area of the steel plate (in.²)

II. INTRODUCTION

Since their development in the late 1930's by E. E. Simmons at the California Institute of Technology and A. C. Ruge at the Massachusetts Institute of Technology electric strain gages have become a very important part of virtually every experimental stress analysis. Their effective use in thermal stress work, however, has developed in a slower manner. The problem of measuring thermal stresses is one that must be considered in any design study involving transfer of heat because stresses induced by temperature change are often the controlling factors. Analytical solutions to most thermal stress problems are very difficult, therefore an effective experimental approach is necessary to complete a thorough analysis of most problems. It is the purpose of this study to show how a rectangular array of four electric strain gages can be used to measure thermal stresses.

Arranging the gages in a rectangular rosette with gages of one type at 0° and 90° and gages of another type at 45° and 135° will give four readings that will permit a set of four independent equations to be written. these equations can be solved for the four unknown quantities

$$\epsilon_{x'}, \epsilon_{y'}, \gamma_{xy} \text{ and } \Delta T.$$

Using Mohr's circle for the strains will yield the principal strains in the most general case and from direct relationships the principal stresses may be determined.

This investigation is a result of the authors interest in analytical as well as experimental stress and strain analysis.

III. REVIEW OF LITERATURE

The analytical approach to the solution of thermal stress problems has been considered extensively by Boley and Weiner (1) and also by Timoshenko and Goodier (2) with the conclusion that problems with anything other than a simple configuration are too difficult to be solved analytically. This fact makes an effective experimental approach of great importance.

The measurement of thermal stresses has been attempted with varying amounts of success and almost exclusively the methods used involve some type of temperature compensating technique. As far as the author has been able to determine, the method of measuring thermal stresses which is described in this thesis has not been previously used.

Several temperature compensating techniques exist for electric strain gages by which it is possible to measure thermal stresses approximately. The use of some of these methods is described in papers by Brewer and Ingham (3), Meriam, Steidel, Brown and Lyman (4) and Thomson and Vergamini (5). The methods and related problems are described in books by Murray and Stein (6) and Perry and Lissner (7); also in papers by Hines and Weymouth (8), Gray, Crossman and Rubin (9) and DeMichele (10). The paper by Hines and Weymouth has an excellent description of five methods of temperature compensation.

Temperature compensation can be accomplished in several ways, one of these is by maintaining the dummy gage in an unrestrained condition and at the same temperature as the active gage. This probably is the most accurate of all types of compensation.

One type of temperature compensating strain gage is made up of two resistance elements connected in series. One of these elements is the main grid and usually has a negative temperature coefficient of resistance when mounted on a particular material. Temperature compensation is achieved by connecting a short length of an alloy having a high positive temperature coefficient of resistance in series with the strain sensitive main grid. This short strain insensitive length of alloy along with a balast resistor replaces the dummy gage.

Another type of compensation utilizes a gage having a high, but accurately known, temperature coefficient of resistance. Mounted physically close to the active gage and connected in series with the lead wires to the dummy arm of the strain gage circuit is a small variable resistor having a temperature coefficient of resistance much higher than the active gage. The compensating resistor is adjusted so as to balance out the change in resistance of the active gage due to temperature, by causing an equal and opposite change in the dummy arm.

A widely used temperature compensating strain gage is the single element type which is made of an alloy which does not produce a strain signal for unrestrained thermal expansion. The measurement of thermal stresses with this type of temperature compensated strain gage may best be visualized utilizing the principle of superposition. A compensated strain gage attached to an unrestrained bar (of the particular material for which the gage is compensating) heated over the temperature range of the gage will not produce a strain signal. Maintaining the temperature and compressing the bar back to its original size will produce a strain signal equal to the mechanical strain of compression. This strain may then be used to determine the thermal stresses.

A thermocouple can also be used for temperature compensation. The change in bridge unbalance caused by an uncompensated strain gage is a function of the gage temperature and for a fixed bridge excitation voltage, the output voltage of the bridge can be expressed in terms of millivolts per degree change in temperature. A change in the excitation voltage will produce a proportional change in the slope of the millivolt per degree output curve. With DC bridge excitation voltage the change in bridge output versus temperature has the same units and character as a thermocouple output. Thus, a thermocouple can be used to produce an equal and opposite voltage for cancellation purposes.

In 1935 Osgood (11) derived the relationships for determining principal stresses from strains on four intersecting gage lines 45° apart. This constitutes a rectangular rosette with four observations as is used in this paper to determine thermal stresses.

Murray and Stein (12), Perry and Lissner (13) and Hetenyi (14) also describe techniques for using four gage strain rosettes. The rectangular rosette with four observations and the tee delta rosette are described with derivations showing how the four observations in each case can be used to determine principal stresses even though one reading is redundant if conventional stress - strain relationships are used. The authors consider the fourth gage reading useful only as a check and they do not refer to the use of four gage rosettes for the determination of thermal stresses as described in this paper.

IV. DISCUSSION

The problem of measuring thermal strains with electric strain gages requires a knowledge of the basic fundamentals that make it possible to design and use the gages for measuring strains due to any type of applied force.

The principal of electric strain gages is that a small wire (usually .001 inches in diameter) fastened securely with some type of adhesive to almost any material will (when forces are applied) deform in exactly the same manner as the material with some change in the cross sectional area and length of the wire and therefore with a resultant change in its resistance to the flow of electricity. This change in resistance is proportional to the elongation of the wire which in turn is equal to the strain in the material to which the gage is attached. Any change in temperature will also affect the resistance of the gage wire and will appear as strain on a measuring instrument.

The gage wire resistance is a function of length, diameter and temperature.

$$R = f(L, D, T) \quad (1)$$

Using the chain rule of differentiation, this expression can be written as

$$dR = \frac{\partial R}{\partial L} dL + \frac{\partial R}{\partial D} dD + \frac{\partial R}{\partial T} dT. \quad (2)$$

The term $(\partial R/\partial D)dD$ can be shown to be negligible when compared to $(\partial R/\partial L)dL$. The definition of Poisson's ratio is

$$u = \frac{dD/D}{dL/L},$$

where

$$dD = \frac{D}{L} u dL$$

and since D is usually .001 in. the center term of equation (2) is of this order of magnitude smaller than the first term and can be neglected for all practical purposes. Considering the isothermal case $dT = 0$, expression (2) becomes

$$dR = \frac{\partial R}{\partial L} dL. \quad (3)$$

Dividing both sides by RL and rearranging terms gives

$$\frac{dR/R}{dL/L} = \frac{\partial R}{\partial L} \frac{L}{R}. \quad (4)$$

The right side of equation (4) has been called the gage factor, (GF).

Making this substitution the relation between resistance and strain becomes

$$\frac{dR}{R} = GF \frac{dL}{L}. \quad (5)$$

The gage factor is determined by previous calibration by the manufacturer. This calibration can be accomplished by measuring the strain optically and by measuring the resistance and change in resistance with a wheatstone bridge. Work done by Campbell (15) on several types of strain gages showed that the gage factor did not vary with temperature more than four percent over the range -73 °C to 93 °C.

DEVELOPMENT OF THE METHOD

Consider a rectangular coordinate system superimposed on the material on which it is desired to measure the thermal strains. The strain in the x direction, the strain in the y direction and the shear strain can have two components, one due to load, and the other due to restrained

thermal expansion.

$$\begin{aligned}\epsilon_x &= \epsilon_{xL} + \epsilon_{xTR} \\ \epsilon_y &= \epsilon_{yL} + \epsilon_{yTR} \\ \gamma_{xy} &= \gamma_{xyL} + \gamma_{xyTR}\end{aligned}\tag{6}$$

Total deformation in the x and y direction consists of the strains in the respective directions plus a component of free thermal expansion which is equal to the coefficient of thermal expansion multiplied by the temperature change.

$$\begin{aligned}\delta_x &= \epsilon_x + \alpha \Delta T \\ \delta_y &= \epsilon_y + \alpha \Delta T\end{aligned}\tag{7}$$

An expression for the strain in a general direction θ can be developed using the geometry of figure 1. Strain in a general direction can be considered as consisting of three parts, that due to strain in the x direction, that due to strain in the y direction and that part due to shear strain.

$$\epsilon_\theta = \epsilon_{\theta 1} + \epsilon_{\theta 2} + \epsilon_{\theta 3}\tag{8}$$

Considering figure 1-A the strain in the θ direction due to strain in the x direction can be determined as follows:

$$\delta_x = aa' = \epsilon_x + \alpha \Delta T$$

$$\epsilon_{\theta 1} = \frac{oc' - oc}{oc} \approx \frac{c'c''}{oc}$$

$$cc' = aa' \frac{bc}{bd} = (\epsilon_x + \alpha \Delta T) oc \cos \theta$$

$$cc' = \frac{c'c''}{\cos \theta}$$

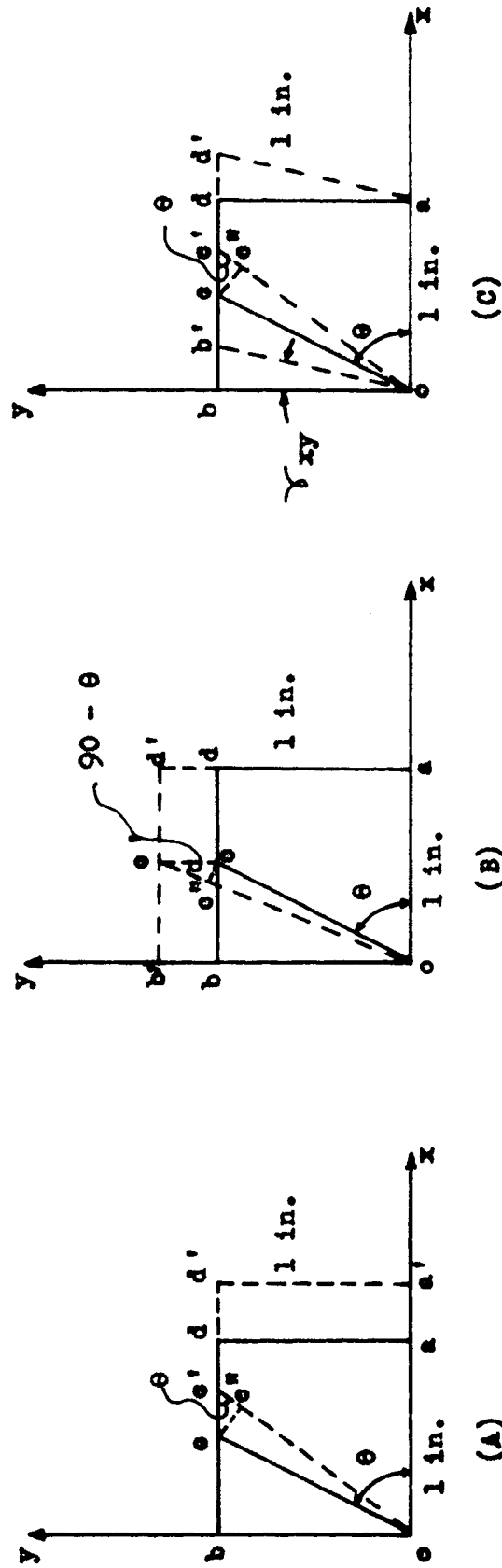


Figure 1
 STRAIN IN THE θ DIRECTION
 DUE TO
 NORMAL AND SHEAR STRAIN

$$\begin{aligned} c'c'' &= (\epsilon_x + \alpha \Delta T) oc \cos^2 \theta \\ \epsilon_{\theta 1} &= (\epsilon_x + \alpha \Delta T) \cos^2 \theta \end{aligned} \quad (9)$$

Similarly figure 1-B will give the strain in the θ direction due to strain in the y direction.

$$\epsilon_{\theta 2} = (\epsilon_y + \alpha \Delta T) \sin^2 \theta \quad (10)$$

The contribution by the shear strain to the strain in the θ direction can be determined from figure 1-C.

$$\begin{aligned} \epsilon_{\theta 3} &= \frac{oc' - oc}{oc} \approx \frac{c'c''}{oc} \\ c'c'' &= cc' \cos \theta = bb' \cos \theta \\ bb' &= \tan \gamma_{xy} \approx \gamma_{xy} \\ c'c'' &= \gamma_{xy} \cos \theta \\ \epsilon_{\theta 3} &= \frac{\gamma_{xy} \cos \theta}{oc} \end{aligned} \quad (11)$$

$$oc = \frac{1}{\sin \theta}$$

$$\epsilon_{\theta 3} = \gamma_{xy} \sin \theta \cos \theta$$

Replacing expressions (9), (10) and (11) in (8) and simplifying gives

$$\epsilon_{\theta} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta + \alpha \Delta T. \quad (12)$$

From equation (5), with the effect of temperature included, the strain gage reading is

$$G = \frac{1}{GF} \frac{dR}{R} = \epsilon_{\theta} + \frac{K_T \Delta T}{GF}, \quad (13)$$

where K_T is the thermal coefficient of resistivity of the gage filament.

Substituting expression (12) in (13) gives the simplified general expression

$$G_n = \left[\frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta_n + \frac{\gamma_{xy}}{2} \sin 2\theta_n \right] + \left[\alpha + \frac{K_T}{GF_n} \right] \Delta T. \quad (14)$$

It is noteworthy that the α of equations (13) and (14) is actually the difference between the α of the material to be tested and α_f of the gage filament, however, it is not necessary to separate the coefficients of thermal expansion as they can be included in the constant term

$$K_1 = \left[\alpha - \alpha_f + \frac{K_T}{GF_n} \right], \quad (15)$$

which is determined by calibration. Now considering the four equations that result from equation (14) when $n = 1, 2, 3, 4$ and using readings from gages arranged at $0^\circ, 45^\circ, 90^\circ$ and 135° let

$$P = \frac{\epsilon_x + \epsilon_y}{2},$$

$$Q = \frac{\epsilon_x - \epsilon_y}{2},$$

$$D = \left[\alpha + \frac{K_{T1}}{GF_1} \right]$$

and (16)

$$D' = \left[\alpha + \frac{K_{T2}}{GF_2} \right].$$

Two different gage factors are enough to make the equations independent.

Equation (14) then becomes

$$G_1 = P + Q + D \Delta T, \quad (17)$$

$$G_2 = P + \frac{\gamma_{xy}}{2} + D' \Delta T, \quad (18)$$

$$G_3 = P - Q + D \Delta T \quad (19)$$

and

$$G_4 = P - \frac{\gamma_{xy}}{2} + D' \Delta T. \quad (20)$$

These equations may now be solved for the four unknowns, ϵ_x , ϵ_y , γ_{xy} and ΔT . Adding and subtracting expressions (17) and (19) gives

$$G_1 + G_3 = 2P + 2 D \Delta T \quad (21)$$

$$G_1 - G_3 = 2Q. \quad (22)$$

Substituting for Q gives

$$\epsilon_x - \epsilon_y = G_1 - G_3. \quad (23)$$

Adding and subtracting expressions (18) and (20) gives

$$G_2 + G_4 = 2P + 2 D' \Delta T \quad (24)$$

and

$$G_2 - G_4 = \gamma_{xy} \quad (25)$$

Multiplying equation (24) by D/D' and subtracting equation (21) eliminates the temperature change.

$$(G_2 + G_4) \frac{D}{D'} - (G_1 + G_3) = 2P \left(\frac{D}{D'} - 1 \right)$$

Substituting for P yields

$$\epsilon_x + \epsilon_y = \frac{(G_2 + G_4) \frac{D}{D'} - (G_1 + G_3)}{\left(\frac{D}{D'} - 1 \right)}. \quad (26)$$

Adding and subtracting expressions (23) and (26) yields

$$\epsilon_x = \frac{G_1 - G_3}{2} + \frac{(G_2 + G_4) \frac{D}{D'} - (G_1 + G_3)}{2 \left(\frac{D}{D'} - 1 \right)} \quad (27)$$

and

$$\epsilon_y = \frac{G_3 - G_1}{2} + \frac{(G_2 + G_4) \frac{D}{D'} - (G_1 + G_3)}{2 \left(\frac{D}{D'} - 1 \right)}. \quad (28)$$

From equation (21),

$$\Delta T = \frac{G_1 + G_3 - 2P}{2D}. \quad (29)$$

Readings from the four gages may be used to solve for the unknowns

ϵ_x , ϵ_y , γ_{xy} and ΔT from equations (25), (27), (28) and (29).

EXPERIMENTAL VERIFICATION OF THE METHOD

Experimental verification of the method has been attempted by measuring the thermal strains in the bimetallic configuration of figures 3 and 4 for comparison with an analytical solution. The elastically restrained condition of the test problem subjected the method to conditions which it will encounter in actual practice, however, the strains produced by temperature change were lower than those which usually cause concern and more accurate results could be expected when measuring strains of greater magnitude.

The elastically restrained test actually began after a preliminary but important unrestrained thermal expansion calibration test to determine the constants of expressions (15) and (16). When these constants were determined the plates of figure 2 were pinned back to back as in figures 3 and 4 for the elastically restrained thermal expansion tests.

Heat for the tests was supplied by a Fisher Scientific Company oven with thermostatic control within $\pm .5$ °F. Temperature measurements, necessary only for calibration, were made with thermocouples monitored by a Leeds and Northrup semi-precision potentiometer. Gage readings were taken with a Baldwin-Lima-Hamilton type L strain indicator. The assembled apparatus is shown in figure 5.

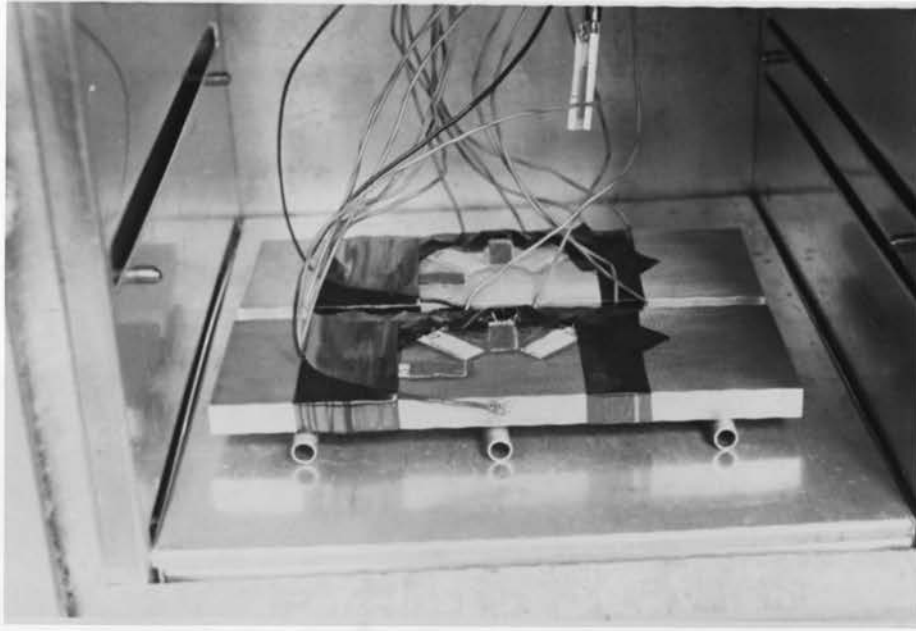


Figure 2

UNRESTRAINED THERMAL EXPANSION CALIBRATION TEST

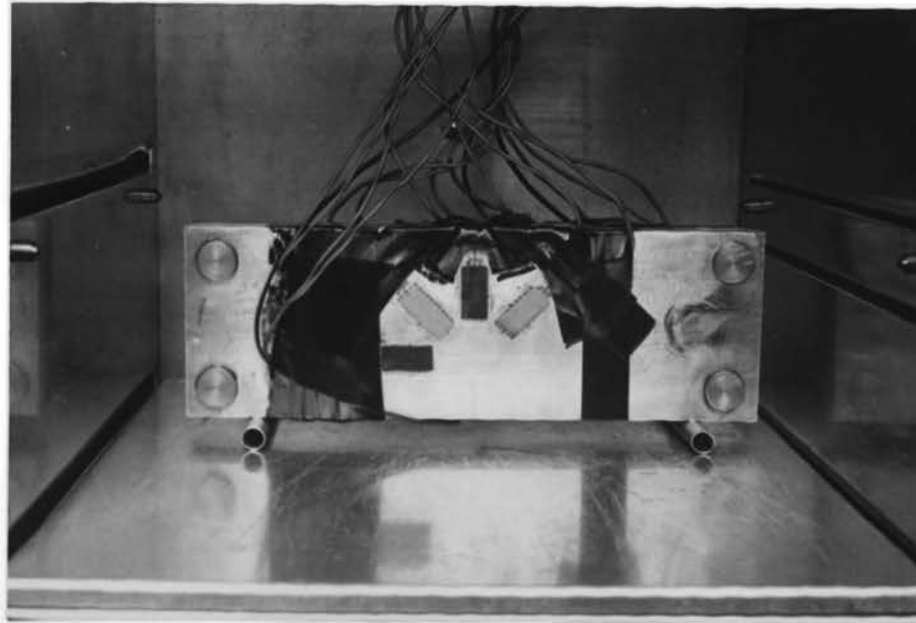


Figure 3

ELASTICALLY RESTRAINED THERMAL EXPANSION TEST

Test Specimen Data;

Effective Length - 8 in.

Thickness - 1/2 in.

Width - 3 in.

Separation - 1/16 in.

Gage Orientation;

Type A-5 at 0° and 90°

Type C-5 at 45° and 135°

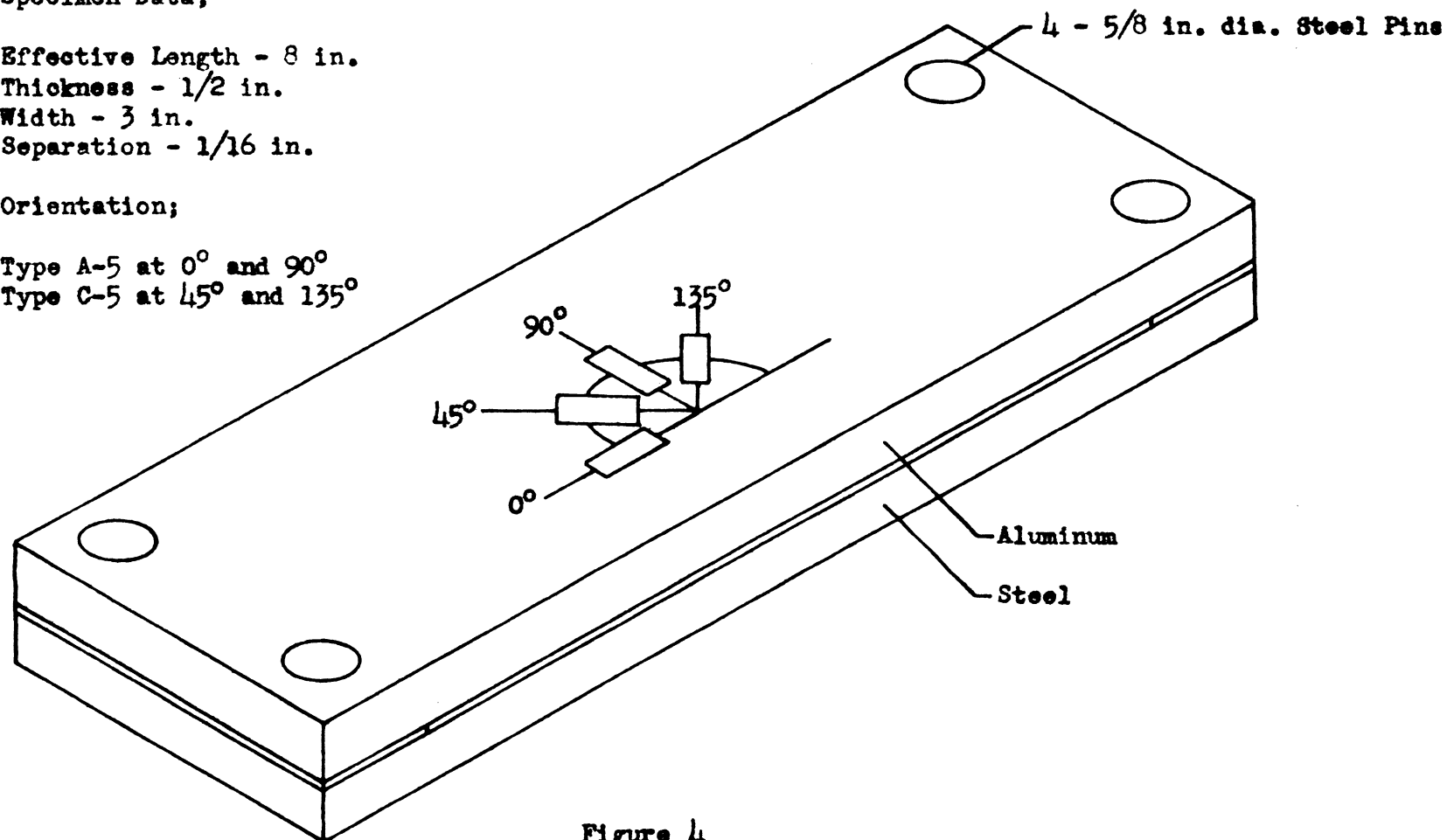


Figure 4

TEST SPECIMENS AND GAGE ORIENTATION

ELASTICALLY RESTRAINED THERMAL EXPANSION TEST

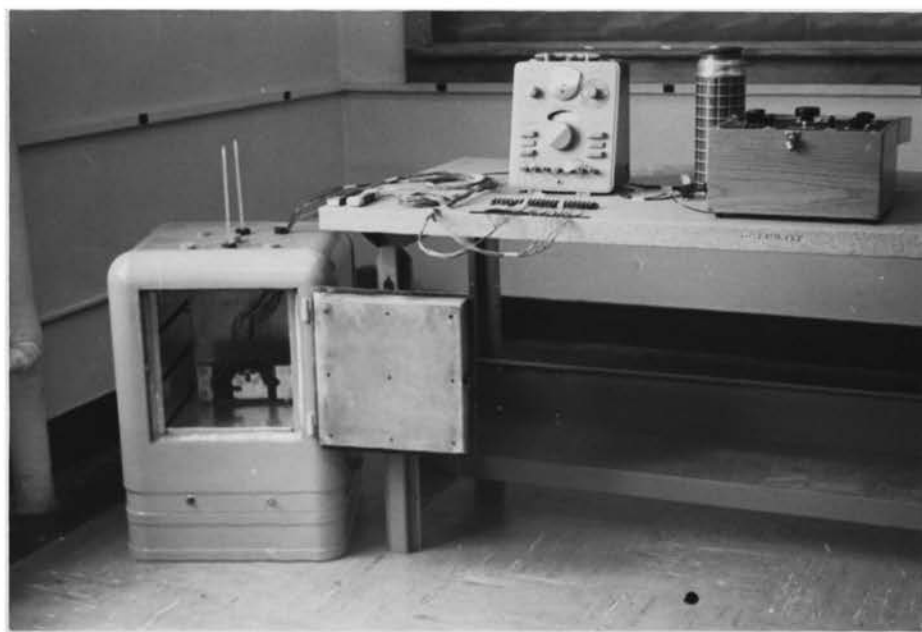


Figure 5

TEST APPARATUS ARRANGEMENT

In order to obtain strains from the four gage readings, using equations (17), (18), (19) and (20), it is necessary to know accurately the constant term of expressions (15) and (16). This term can be obtained from an unrestrained thermal expansion calibration test for the particular material being tested. This is possible because all load strains ϵ_x , ϵ_y , and γ_{xy} in expression (14) are zero during unrestrained thermal expansion. The general equation (14) is then reduced to

$$G_n = K_1 \Delta T \quad (30)$$

which can be solved for the constant K_1 . This constant is different for every combination of gage and test material, however, the term K_{Tn}/G_n is a property of the gage only and can be determined if the coefficient of thermal expansion is accurately known for the material on which the calibration test was made. Knowing the quantity K_{Tn}/G_n would make it possible to use the same type of gage (without calibration) on any material for which the coefficient of thermal expansion is known as the sum of these two quantities is the constant term K_1 .

It is desirable that extreme care be taken during the calibration test as the calculated strains are very sensitive to the magnitude of K_1 . For each temperature increment in this investigation 12 to 24 hours were allowed for all parts of the test specimens to come to an equilibrium condition.

For the purpose of this work, calibration was first accomplished using figure 6 and table I and then refined to greater accuracy by trial and error adjustment until the calculated temperatures from the first elastic restraint test compared most favorably with thermocouple temperatures which were originally taken for comparison only. The refined constants, when multiplied by a particular temperature change, according

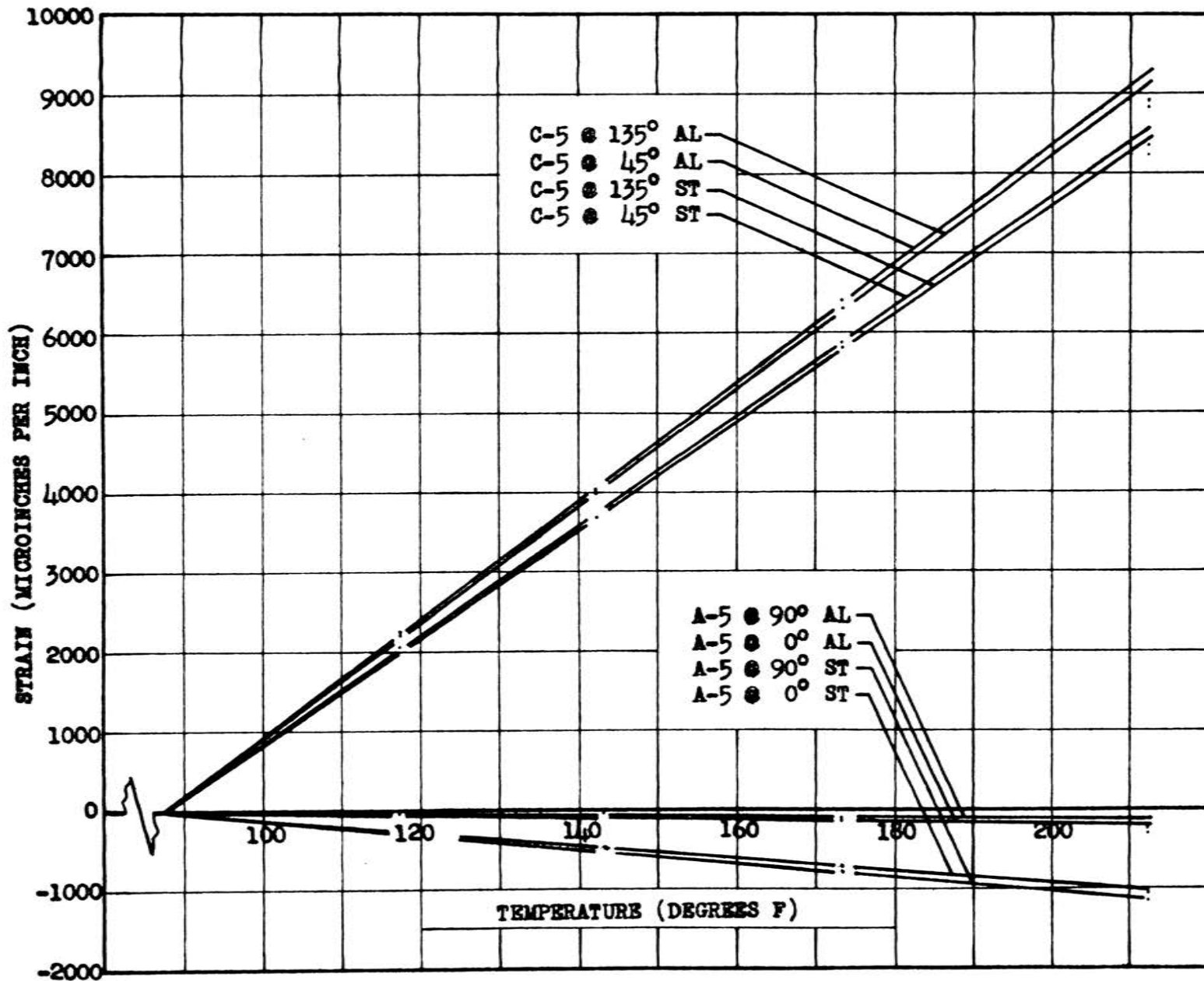


Figure 6

CALIBRATION CURVES FROM UNRESTRAINED EXPANSION TEST

Table I
 STRAIN VARIATION WITH TEMPERATURE
 FROM
 UNRESTRAINED EXPANSION CALIBRATION TEST

Aluminum				
Temp. °F	Strain 10 ⁻⁶ in./in. A-5 @ 0°	Strain 10 ⁻⁶ in./in. C-5 @ 45°	Strain 10 ⁻⁶ in./in. A-5 @ 90°	Strain 10 ⁻⁶ in./in. C-5 @ 135°
87	0	0	0	0
117	-2	2292	-1	2261
142	-29	4077	-12	4032
173	-102	6462	-87	6362
212	-295	9014	-266	8923
Steel				
Temp. °F	Strain 10 ⁻⁶ in./in. A-5 @ 0°	Strain 10 ⁻⁶ in./in. C-5 @ 45°	Strain 10 ⁻⁶ in./in. A-5 @ 90°	Strain 10 ⁻⁶ in./in. C-5 @ 135°
87	0	0	0	0
117	-200	2110	-205	2112
142	-398	3743	-417	3732
173	-688	5935	-742	5891
212	-1052	8423	-1112	8326

All strain values have been corrected by the ratio of an accurately known strain to the measured strain which was equal to 1.108 for the type A-5 gages and .583 for the type C-5 gages. Perry and Lissner (18) describe the procedure.

to expression (30), give gage readings that are very near the original readings which were taken from the calibration curves of figure 6. The difference is one that could result from reading the curves, therefore, a refining procedure such as is described above is advisable for greatest accuracy. The adjusted constants were verified by the second elastic restraint test. The values obtained for K_1 and K_{Tn}/G_n in this investigation are as follows:

Aluminum

$$\text{Type A-5, } K_1 = -.800 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

$$\text{Type C-5, } K_1 = 73.360 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

Steel

(31)

$$\text{Type A-5, } K_1 = -7.725 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

$$\text{Type C-5, } K_1 = 66.940 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

Using average values from Marks' Mechanical Engineers' Handbook (16) of $\alpha_A = 13 \times 10^{-6}$ in./in./ $^\circ\text{F}$ and $\alpha_S = 6.5 \times 10^{-6}$ in./in./ $^\circ\text{F}$, K_{Tn}/G_n may be determined.

Aluminum

$$\text{Type A-5, } K_{T1}/GF_1 = -13.800 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

$$\text{Type C-5, } K_{T2}/GF_2 = 60.360 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

Steel

(32)

$$\text{Type A-5, } K_{T1}/GF_1 = -14.225 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

$$\text{Type C-5, } K_{T2}/GF_2 = 60.440 \times 10^{-6} \text{ in./in./}^\circ\text{F}$$

The small differences in the two K_{T1}/GF_1 and K_{T2}/GF_2 factors are due to the fact that α_A and α_S were not accurately known. These factors however, compare favorably and emphasize the fact that they are independent of the material to which the gage is attached. Calibration on materials for which the exact coefficient of thermal expansion is known would

determine the K_{Tn}/GF_n factors more accurately.

Figures 7 and 8 as well as tables II and III show the strain variation with temperature for the elastically restrained thermal expansion tests. The stress in the x direction is calculated below at $T = 200$ °F for each plate, in both tests, for comparison with analytical values. Using equation (14), the strain curves of figures 7 and 8 and the calibration constants (31), a set of four simultaneous equations with four unknowns may be written for each plate.

The material constants used are average values which were taken from Marks' Mechanical Engineers' Handbook (17). They are as follows:

$$E_A = 10 \times 10^{-6} \text{ psi.}$$

$$E_S = 30 \times 10^{-6} \text{ psi.}$$

$$u_A = .332$$

$$u_S = .287$$

Aluminum plate - Test 1

$$0 = \epsilon_x - .800 \times 10^{-6} \Delta T$$

$$8680 \times 10^{-6} = \left[\frac{\epsilon_x + \epsilon_y}{2} + \frac{\gamma_{xy}}{2} \right] + 73.36 \times 10^{-6} \Delta T \quad (33)$$

$$-110 \times 10^{-6} = \epsilon_y - .800 \times 10^{-6} \Delta T$$

$$8580 \times 10^{-6} = \left[\frac{\epsilon_x + \epsilon_y}{2} - \frac{\gamma_{xy}}{2} \right] + 73.36 \times 10^{-6} \Delta T$$

Solving equations (33) simultaneously gives

$$\epsilon_x = 98.7 \times 10^{-6} \text{ in./in.,}$$

$$\epsilon_y = -16.4 \times 10^{-6} \text{ in./in.} \quad (34)$$

and

$$\Delta T = 117.1 \text{ °F.}$$

Stress in the x direction can be determined by

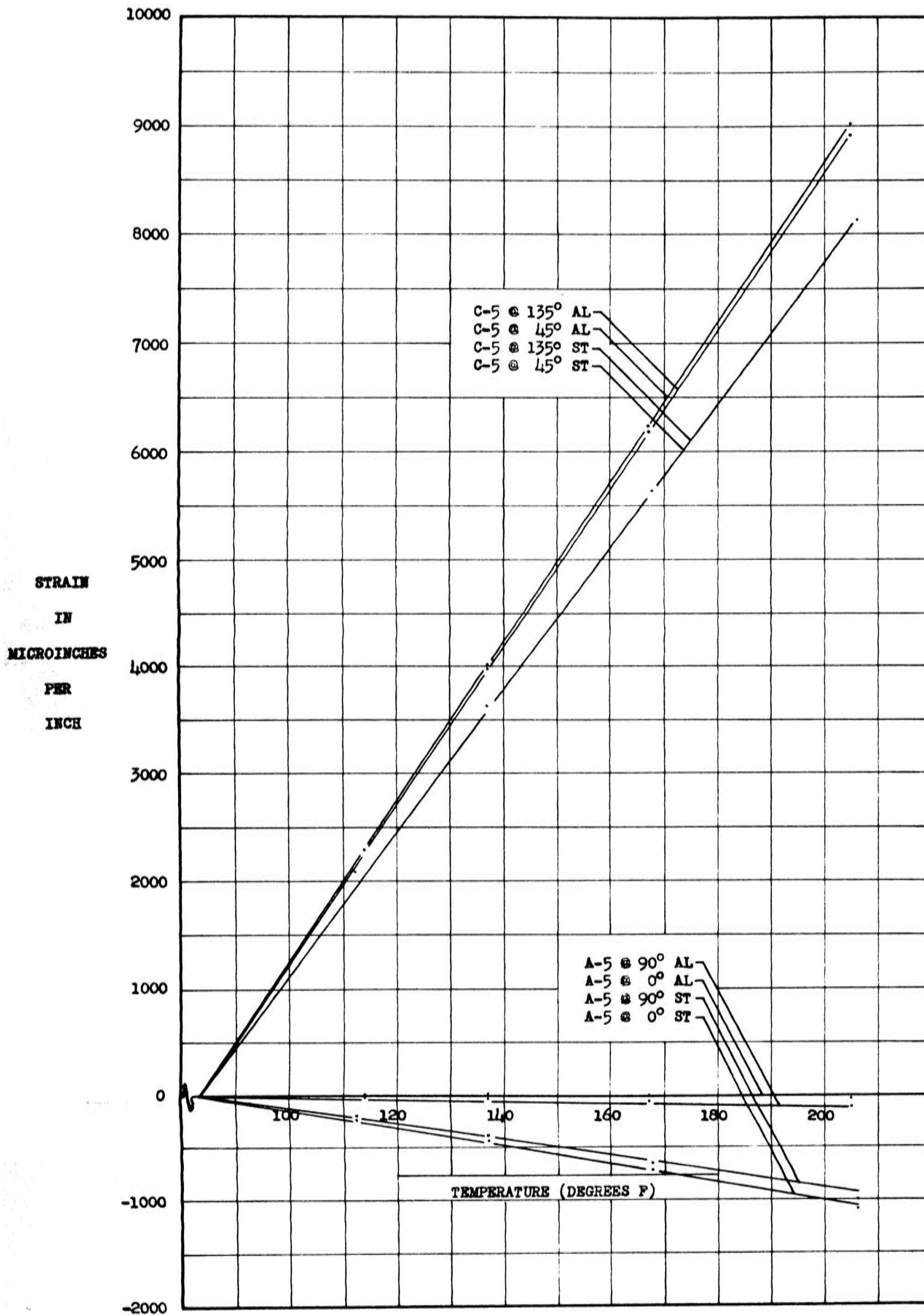


Figure 7
 STRAIN VARIATION WITH TEMPERATURE
 FROM
 ELASTICALLY RESTRAINED EXPANSION TEST 1

Table II
 STRAIN VARIATION WITH TEMPERATURE
 FROM
 ELASTICALLY RESTRAINED EXPANSION TEST 1

Aluminum				
Temp. °F	Strain 10 ⁻⁶ in./in. A-5 @ 0°	Strain 10 ⁻⁶ in./in. C-5 @ 45°	Strain 10 ⁻⁶ in./in. A-5 @ 90°	Strain 10 ⁻⁶ in./in. C-5 @ 135°
83	0	0	0	0
114	19	2290	-9	2297
139	26	4008	-21	3979
170	10	6242	-51	6181
209	-27	9021	-111	8918
Steel				
Temp. °F	Strain 10 ⁻⁶ in./in. A-5 @ 0°	Strain 10 ⁻⁶ in./in. C-5 @ 45°	Strain 10 ⁻⁶ in./in. A-5 @ 90°	Strain 10 ⁻⁶ in./in. C-5 @ 135°
83	0	0	0	0
114	-235	2096	-196	2103
139	-435	3621	-377	3621
170	-725	5633	-652	5627
209	-1096	8133	-1001	8120

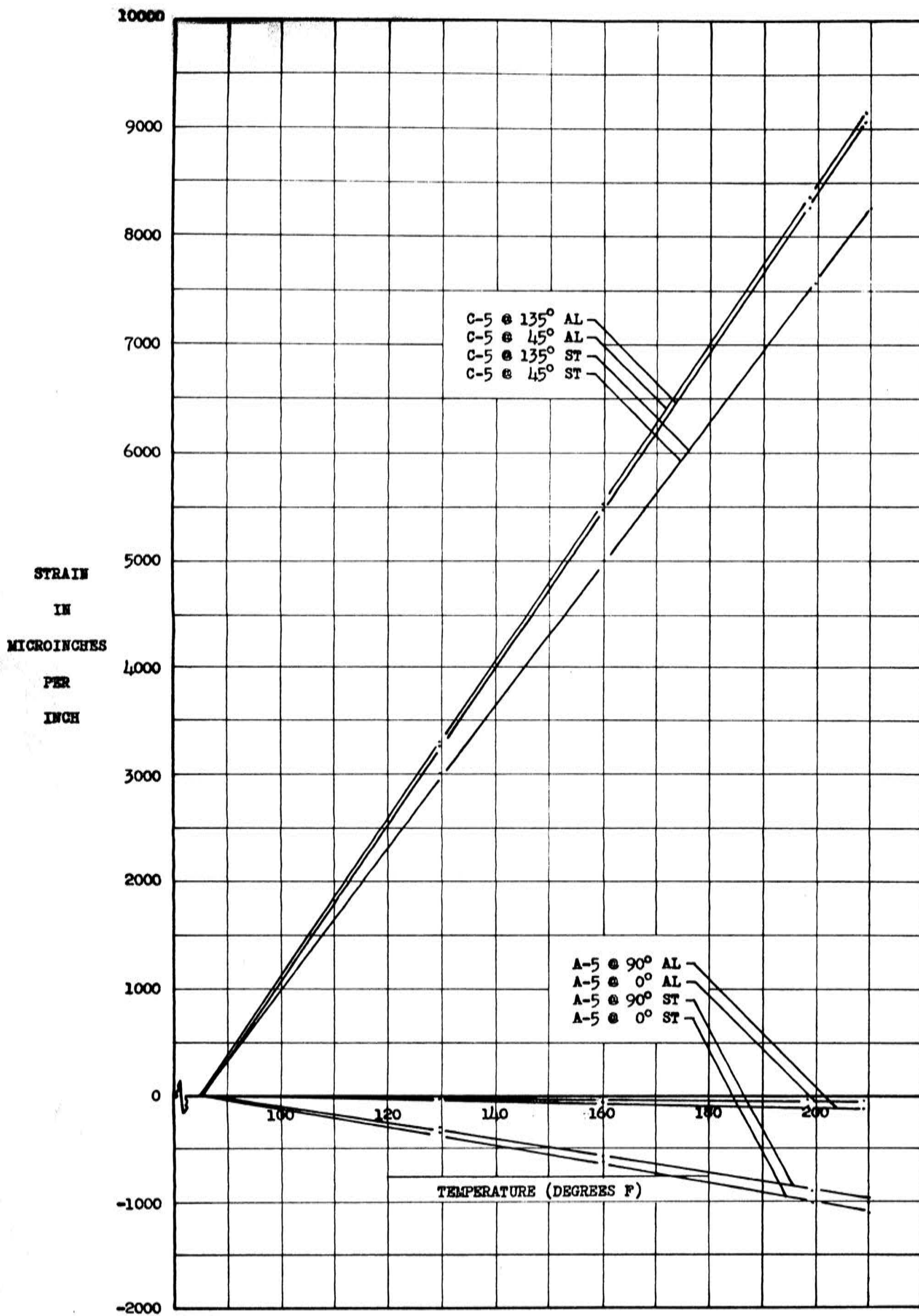


Figure 8
 STRAIN VARIATION WITH TEMPERATURE
 FROM
 ELASTICALLY RESTRAINED EXPANSION TEST 2

Table III
 STRAIN VARIATION WITH TEMPERATURE
 FROM
 ELASTICALLY RESTRAINED EXPANSION TEST 2

Aluminum				
Temp. °F	Strain 10 ⁻⁶ in./in. A-5 @ 0°	Strain 10 ⁻⁶ in./in. C-5 @ 45°	Strain 10 ⁻⁶ in./in. A-5 @ 90°	Strain 10 ⁻⁶ in./in. C-5 @ 135°
85	0	0	0	0
127	5	3307	-24	3283
160	4	5537	-46	5480
198	-35	8380	-100	8289
210	-48	9166	-124	9066
Steel				
Temp. °F	Strain 10 ⁻⁶ in./in. A-5 @ 0°	Strain 10 ⁻⁶ in./in. C-5 @ 45°	Strain 10 ⁻⁶ in./in. A-5 @ 90°	Strain 10 ⁻⁶ in./in. C-5 @ 135°
85	0	0	0	0
127	-343	3009	-298	3012
160	-632	5003	-573	5001
198	-1002	7569	-914	7561
210	-1103	8278	-1017	8270

$$S_x = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y). \quad (35)$$

Substituting the material constants and the values of equations (34) and solving gives

$$S_x = \frac{10 \times 10^6}{1-(.332)^2} [93.7 + .332(-16.4)] \times 10^{-6}$$

or

$$S_x = 992 \text{ psi.}$$

Experimental $S_x = 992 \text{ psi.}$

Analytical $S_x = 616 \text{ psi.}$

Experimental $\Delta T = 117.1 \text{ }^\circ\text{F}$

Measured (thermocouple) $\Delta T = 117 \text{ }^\circ\text{F}$

Steel plate - Test 1

$$-1010 \times 10^{-6} = \epsilon_x - 7.725 \times 10^{-6} \Delta T$$

$$7730 \times 10^{-6} = \left[\frac{\epsilon_x + \epsilon_y}{2} + \frac{\nu_{xy}}{2} \right] + 66.94 \times 10^{-6} \Delta T$$

$$-890 \times 10^{-6} = \epsilon_y - 7.725 \times 10^{-6} \Delta T$$

$$7730 \times 10^{-6} = \left[\frac{\epsilon_x + \epsilon_y}{2} - \frac{\nu_{xy}}{2} \right] + 66.94 \times 10^{-6} \Delta T$$

(36)

Solving equations (36) simultaneously gives

$$\epsilon_x = -111.6 \times 10^{-6} \text{ in./in.,}$$

$$\epsilon_y = 8.4 \times 10^{-6} \text{ in./in.}$$

(37)

and

$$\Delta T = 116.3 \text{ }^\circ\text{F.}$$

Substituting these values and the material constants into equation (35)

gives

$$S_x = \frac{30 \times 10^6}{1-(.287)^2} [-111.6 + .287(8.4)] \times 10^{-6}$$

or

$$S_x = -3571 \text{ psi.}$$

Experimental $S_x = -3571$ psi.

Analytical $S_x = -3479$ psi.

Experimental $\Delta T = 116.3$ °F

Measured (thermocouple) $\Delta T = 117$ °F

Aluminum plate - Test 2

$$\begin{aligned}
 -30 \times 10^{-6} &= \epsilon_x - .800 \times 10^{-6} \Delta T \\
 8500 \times 10^{-6} &= \left[\frac{\epsilon_x + \epsilon_y}{2} + \frac{\gamma_{xy}}{2} \right] + 77.36 \times 10^{-6} \Delta T \\
 -110 \times 10^{-6} &= \epsilon_y - .800 \times 10^{-6} \Delta T \\
 8410 \times 10^{-6} &= \left[\frac{\epsilon_x + \epsilon_y}{2} - \frac{\gamma_{xy}}{2} \right] + 77.36 \times 10^{-6} \Delta T
 \end{aligned} \tag{38}$$

Solving equations (38) simultaneously gives

$$\begin{aligned}
 \epsilon_x &= 62 \times 10^{-6} \text{ in./in.}, \\
 \epsilon_y &= -17.6 \times 10^{-6} \text{ in./in.}
 \end{aligned} \tag{39}$$

and

$$\Delta T = 115 \text{ °F}$$

Substituting these values and the material constants into equation (35)

gives

$$S_x = \frac{10 \times 10^6}{1 - (.332)^2} [62 + .332(-17.6)] \times 10^{-6}$$

or

$$S_x = 631 \text{ psi.}$$

Experimental $S_x = 631$ psi.

Analytical $S_x = 606$ psi.

Experimental $\Delta T = 115$ °F

Measured (thermocouple) $\Delta T = 115$ °F

Steel plate - Test 2

$$-1000 \times 10^{-6} = \epsilon_x - 7.725 \times 10^{-6} \Delta T$$

$$\begin{aligned}
 7640 \times 10^{-6} &= \left[\frac{\epsilon_x + \epsilon_y}{2} + \frac{\gamma_{xy}}{2} \right] + 66.94 \times 10^{-6} \Delta T \\
 -910 \times 10^{-6} &= \epsilon_y - 7.725 \times 10^{-6} \Delta T \\
 7640 \times 10^{-6} &= \left[\frac{\epsilon_x + \epsilon_y}{2} - \frac{\gamma_{xy}}{2} \right] + 66.94 \times 10^{-6} \Delta T
 \end{aligned} \tag{40}$$

Solving equations (40) simultaneously gives

$$\epsilon_x = -110.9 \times 10^{-6} \text{ in./in.,}$$

$$\epsilon_y = -20.9 \times 10^{-6} \text{ in./in.}$$

and

$$\Delta T = 115.1 \text{ }^\circ\text{F.}$$

Substituting these values and the material constants into equation (35) gives

$$S_x = \frac{30 \times 10^6}{1 - (.287)^2} [-110.9 + .287(-20.9)] \times 10^{-6}$$

or

$$S_x = -3822 \text{ psi.}$$

Experimental $S_x = -3822 \text{ psi.}$

Analytical $S_x = -3573 \text{ psi.}$

Experimental $\Delta T = 115.1 \text{ }^\circ\text{F}$

Measured (thermocouple) $\Delta T = 115 \text{ }^\circ\text{F}$

Stress in the x direction was also calculated for each of the experimental points of tables II and III. The results of the calculations did not always compare as favorably with the analytical results as those calculated from the curves of figures 7 and 8. This is reasonable as average curve values are almost always of greater value in experimental work than the individual measurements.

An analytical solution for obtaining the stress in the x direction due to elastically restrained thermal expansion was obtained in the

following manner. The symmetry of the plates permits a complete solution to be obtained by considering figure 9 below.

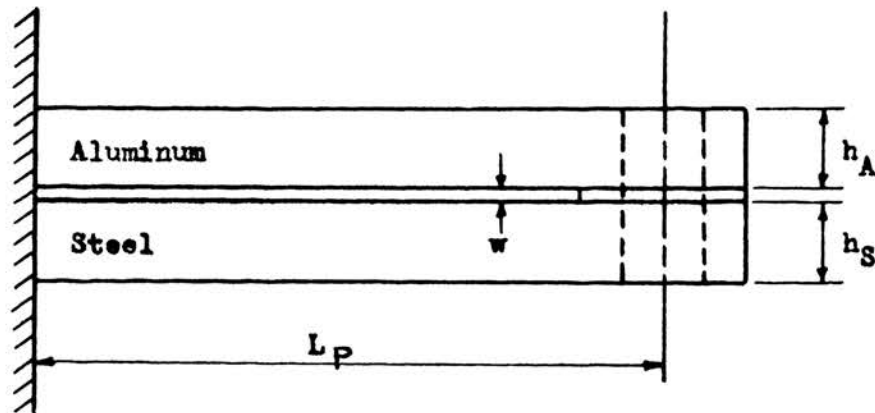
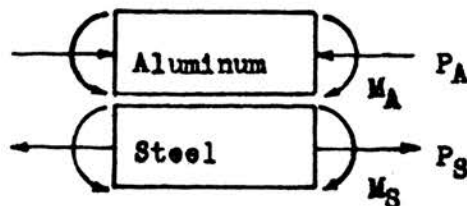


Figure 9

Initially the plates are straight and of the same length and after they have been subjected to an increase in temperature they will be curved downward because α_A is greater than α_S . Considering a section of figure 9,



the direct forces are equal.

$$F_A = F_S = F$$

and the direct stresses are

$$S_{DA} = \frac{-F}{A_A}$$

and (42)

$$S_{DS} = \frac{P}{A_S}$$

The bending moment equation may be written

$$M_A + M_S = P\left(\frac{h_A + h_S}{2} + w\right) \quad (43)$$

and since the reciprocal of the radius of curvature is approximately d^2y/dx^2 which in turn is equal to M/EI , the moments may be written

$$M_A = \frac{EAIA}{r_A}$$

and (44)

$$M_S = \frac{E_S I_S}{r_S}$$

where

$$r_A = r_S + (h_A + h_S)/2 + w. \quad (45)$$

The substitution for equations (44) may be verified as follows:

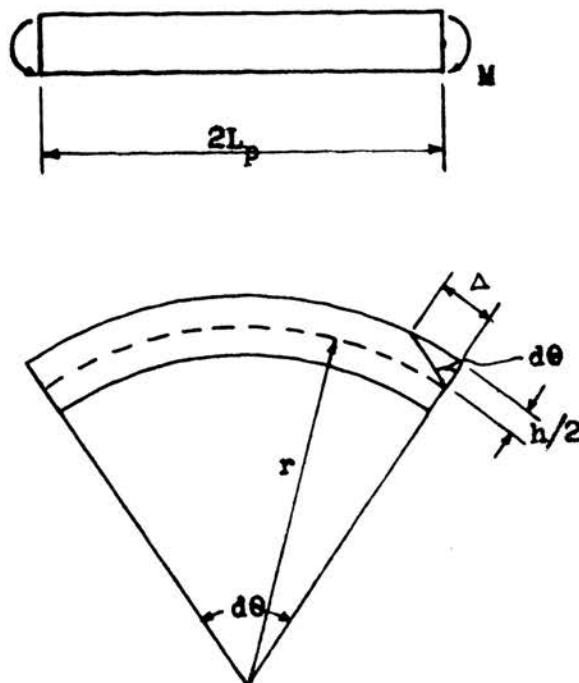


Figure 10

Strain is equal to the change in length divided by the original length in figure 10.

$$\epsilon = \frac{\Delta}{2L_p} \quad (46)$$

and

$$d\theta = \tan^{-1} \frac{\Delta}{h/2} \approx \frac{\Delta}{h/2}$$

Replacing Δ in equation (46) gives

$$\epsilon = \frac{hd\theta}{4L_p}$$

and since

$$rd\theta = 2L_p \quad (47)$$

$$\epsilon = \frac{h}{2r} \quad (48)$$

Also

$$\epsilon = \frac{S}{E} = \frac{Mh}{2EI} \quad (49)$$

From equations (48) and (49)

$$\frac{1}{r} = \frac{M}{EI}$$

and from calculus

$$\frac{1}{r} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \approx \frac{d^2y}{dx^2}$$

Now substituting equations (44), (45) and (47) in (43) and simplifying yields

$$\frac{E_A I_A}{2L_p/d\theta} + \frac{E_S I_S}{2L_p/d\theta - (h_A + h_S)/2 - w} = F[(h_A + h_S)/2 + w] \quad (50)$$

Also consider the unit strain in the lower fiber of the aluminum compared to that in the upper fiber of the steel. Three components are present in each case, the unit strain due to temperature, the unit strain

due to direct stress and the unit strain due to bending stress.

$$\epsilon_A(T_1 - T_0) - \frac{F}{AAE_A} - \frac{had\theta}{2L_p} = \epsilon_S(T_1 - T_0) + \frac{F}{ASE_S} + \frac{had\theta}{2L_p} + \frac{wd\theta}{2L_p} \quad (51)$$

The last term of equation (51) is the difference in the strains in the two fibers due to the separation w . That term was determined as follows:

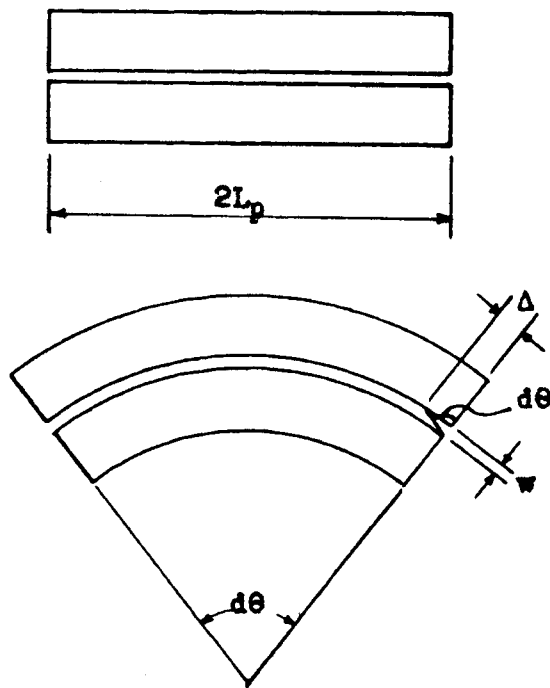


Figure 11

From figure 11

$$d\theta = \tan d\theta = \frac{\Delta}{w}$$

and Δ becomes $wd\theta/2L_p$ when expressed as unit strain.

Solving equation (51) for $d\theta$ and simplifying gives

$$d\theta = \frac{(\epsilon_A - \epsilon_S)(T_1 - T_0) - \frac{F}{B} \left[\frac{1}{h_A E_A} + \frac{1}{h_S E_S} \right]}{1/2L_p(h_A + h_S + w)} \quad (52)$$

where B is the width of the plates. Substituting $d\theta$ in equation (50) yields an expression with F as the only unknown. The equation cannot be solved algebraically and a trial and error solution is necessary. Equations (50) and (52) can be solved directly for F, if the approximation that $r_A \approx r_S$ is made. The approximation is good for large values of r_A and r_S . Solving for F gives

$$F = \frac{2(\alpha_A - \alpha_S)(T_1 - T_0)}{\frac{(h_A + h_S + 2w)(h_A + h_S + w)}{E_A I_A + E_S I_S} + \frac{2}{B} \left[\frac{1}{h_A E_A} + \frac{1}{h_S E_S} \right]} \quad (53)$$

and knowing the numerical value of F makes it possible to solve for $d\theta$ in expression (52) and also for the stresses in the following equations.

$$S_A = \frac{-F}{A_A} \pm \frac{M_A h_A}{2I_A}$$

and (54)

$$S_S = \frac{F}{A_S} \pm \frac{M_S h_S}{2I_S},$$

where

$$M_A = \frac{E_A I_A}{2L_p/d\theta}$$

and

$$M_S = \frac{E_S I_S}{2L_p/d\theta - (h_A + h_S)/2 - w} \quad (55)$$

Equations (52), (53) and (54) have been programmed in Act III language for the Royal McBee LGP-30 Digital Computer located in the Missouri School of Mines and Metallurgy Computer Center and the results are tabulated in table IV and represented graphically in figure 12.

Table IV
 THERMAL STRESS VARIATION WITH TEMPERATURE
 CALCULATED ANALYTICALLY

Temperature Change (°F)	Stress (psi.) Aluminum	Stress (psi.) Steel
0	0	0
10	52.7	-310.7
20	105.4	-621.3
30	158.0	-932.0
40	210.7	-1242.7
50	263.4	-1553.4
60	316.1	-1864.0
70	368.7	-2174.7
80	421.4	-2485.4
90	474.1	-2796.0
100	526.8	-3106.7
110	579.4	-3417.4
120	632.1	-3528.1
130	684.8	-4038.7
140	737.5	-4349.4
150	790.2	-4660.1
160	842.8	-4970.7

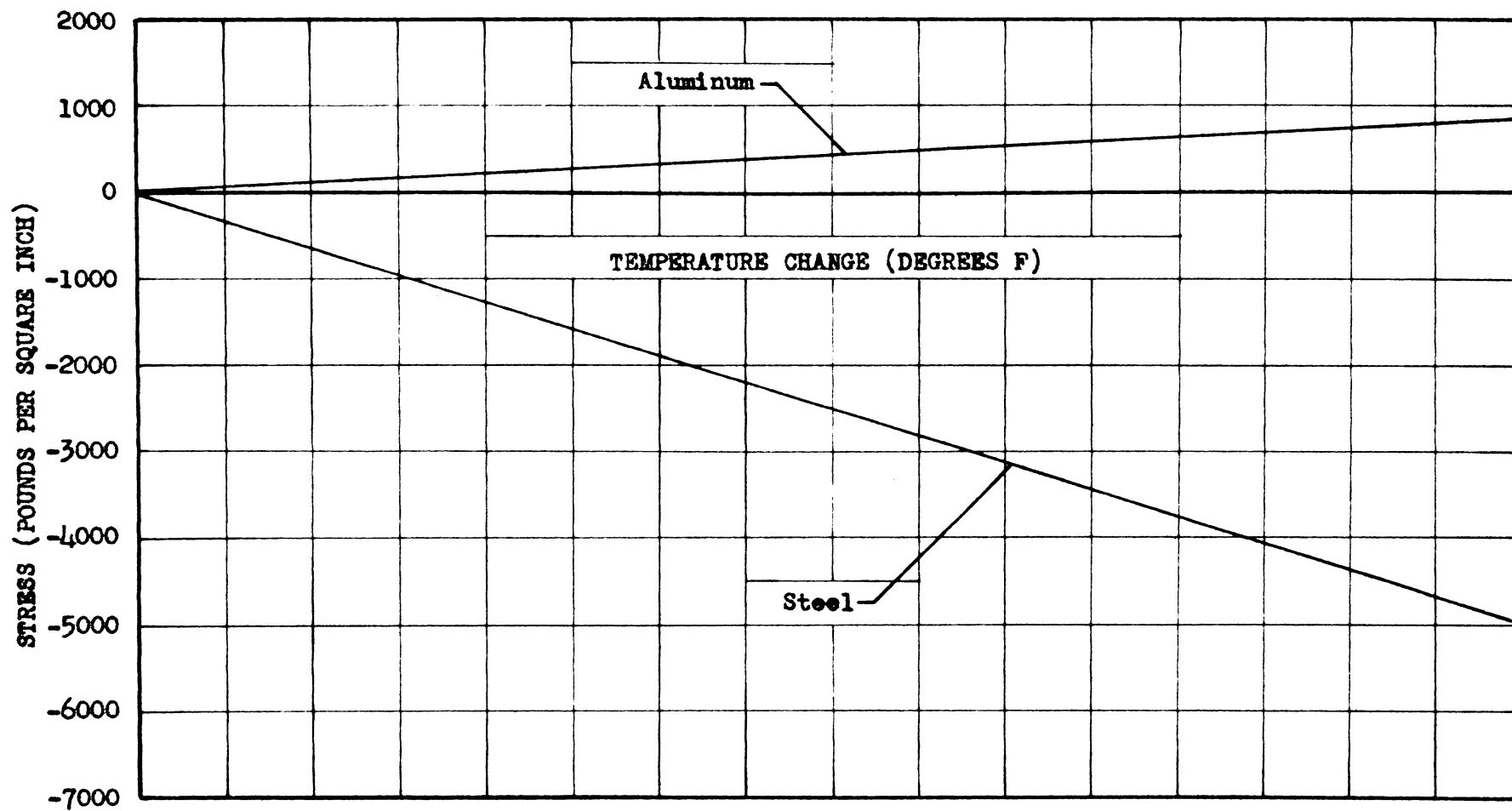


Figure 12

THERMAL STRESS VARIATION WITH TEMPERATURE

CALCULATED ANALYTICALLY

V. CONCLUSIONS

The experimental results of this investigation show that the method of measuring thermal strains which is described is a feasible method. The method offers advantages over every type of temperature compensating technique as described in the abstract. After more complete studies are made, a single type of four gage rosette may be used to measure thermal strains on any type of material for which the coefficient of thermal expansion is known.

Further investigation could yield accurate calibration gage constants K_{Tn}/GF_n for several types of gages which may be used in different temperature ranges. The use of different sets of gages as the temperature range increases or decreases is necessary in any thermal strain investigation. This is true because the gage factor remains essentially constant only in the range for which the gage is designed and also because some gage construction is unsatisfactory for elevated or cryogenic temperature measurements. In Campbell's study (19) the maximum deviation from a constant gage factor occurred at the ends of the temperature range for which the gages were designed.

Extended studies might also include a consideration of the effect of gage transverse sensitivity and also the effect of differences in Poisson's ratio. The errors which these characteristics contribute are usually small but they may account for part of the deviation between experimental and analytical solutions.

Strain analyzing equipment for which the zero drift, over a period of time in the instrument, may be corrected is desirable.

The analytical solution of the test problem for this investigation contains several assumptions that may also contribute small errors.

Most methods which are now being used to measure thermal strains contain even more possibilities for error. This fact and the fact that good results were obtained in this investigation make it reasonable to conclude that the method of measuring thermal strains which is described herein may prove very useful in the future.

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VITA

Terry Franklin Lehnhoff was born on July 7, 1939 in St. Louis, Missouri. His parents are Mr. and Mrs. Chester F. Lehnhoff of Bridgeton, Missouri.

The author received his elementary school education in Overland, Missouri and in 1957 he graduated from Pattonville High School in Bridgeton.

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The author is married to Donna Louise Hoecker Lehnhoff and has one son, Mark Franklin.

