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Comparative analysis of numerical integration techniques

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COMPARATIVE ANALYSIS OF

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ABSTRACT

When integrating numerically, if the integrand can be expressed exactly as a polynomial of degree n, over a finite interval; then either Simpson's rule, Romberg integration, Legendre-Gauss or Jacobi-Gauss quadrature formulas provide good results. However, if the integrand can not be expressed exactly as an n^{th} degree polynomial, then perhaps it can be expressed as a function f(x) divided by $\sqrt{1-x^2}$, or as a function g(x) times $(1-x)^{\alpha}$ $(1+x)^{\beta}$, where α and β are some real numbers >-1, or as a function h(x) times one. If this is the case then the Chebyshev-Gauss, Jacobi-Gauss, and Legendre-Gauss quadrature are respectively quite useful. If the integrand can not be expressed as $f(x)/\sqrt{1-x^2}$ or as $g(x)\cdot(1-x)^{\alpha}\cdot(1+x)^{\beta}$ or as $h(x)\cdot(1)$ then the Romberg method should be used.

If the interval of integration is $[0,\infty]$ or $[-\infty,\infty]$ then the Laguerre-Gauss and the Hermite-Gauss methods respectively are generally quite useful.

The results of this study indicate that the quadrature formula to use in a given situation is dependent upon the interval of integration and the integrand. However, the results also indicate certain guide lines for choosing the type of quadrature formula to use in a given situation.

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INTRODUCTION

There are several problems encountered when integrating an expression over a definite interval. The problems encountered divide themselves into two basic classes. The first of these may be observed when an expression may not be analytically integrable and consequently the solution must be found numerically. This problem is evident when considering the following integral:

$$\int_{-1}^{1} \frac{x^7 \sqrt{1-x^2}}{(2-x)^{13/2}} \, dx.$$

Even though the analytical form of an integral may be available, the second type of problem presents itself in that it is sometimes preferable to compute the integral by numerical methods because the algebraic calculations involved are laborious. The crux of this problem is evident when considering the following integral which is to be evaluated many times with varying limits of integration.

$$\int_{A}^{B} \sqrt{ax^2+c} \, dx =$$

$$\left. \frac{x}{2} \sqrt{ax^2+c} + \frac{c}{2\sqrt{a}} \log \left(x\sqrt{a} + \sqrt{ax^2+c}\right) \right]_A^B$$

when $a > 0$

Or

$$= \frac{x}{2} \sqrt{ax^2+c} + \frac{c}{2\sqrt{-a}} \sin^{-1} \left(x\sqrt{\frac{-a}{c}}\right) \Bigg]_A^B$$

when $a < 0$.

Considering these problems, the purpose of this paper is to study and compare the use of various numerical integration techniques. The techniques considered in the course of this paper will be: Simpson's rule, Romberg integration, and Gaussian-type formulas. The necessity of comparing these methods is comprehended when considering the various methods of approach that may be taken when attempting to solve the following type of integral.

$$\int_0^{\infty} e^{-x}\, dx = 1.$$

In using the three methods, one might consider the problems involved when the interval of integration is $-\infty$ to ∞.

It seems evident therefore, that there is a need for a comparison of these methods, a comparison not only of speed and accuracy, but also of computer storage for the different techniques. However, at the outset one must bear in mind that there are no absolute rules to use in choosing which quadrature techniques are "best". But, in this study some factors in selecting the quadrature technique in various situations will be discussed in an effort to determine which method should be used for speed, which should be considered as the most accurate, which is most efficient in computer storage, or, in a broader prospective, which is most beneficial concerning any combination of these elements.

It should be noted that tabulated functions were not considered in this study. Only analytical functions, that were continuous over the interval of integration, were considered.

REVIEW OF LITERATURE

Before we can talk about Gaussian quadrature formulas; we must have an understanding of orthogonal polynomials. Hildebrand (1)*, Krylov (2), Ralston (3), Kopal (4) and Szego (5) define and discuss properties of orthogonal polynomials. Szego (5) presents a rigorous discussion of orthogonal polynomials and their properties.

The type of integrals used in this study is of the form

$$\int_a^b w(x)f(x)\,dx \qquad (2.01)$$

where w(x) is a given fixed function and f(x) is an arbitrary function. The function w(x) is called a weight function.

We will restrict ourselves to nonnegative weight functions, and let the closed interval [a,b] be any finite or infinite segment. According to Krylov (2), the weight function w(x) must satisfy the two conditions

1. w(x) is nonnegative, measurable, and not identically zero on the segment [a,b].
2. The product $w(x)x^m$, for any nonnegative integer m, are summable on [a,b].

The functions f(x) and g(x) are said to be orthogonal on the

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* All numbers (x) refer to the bibliography while the numbers (x.y) refer to equations.

segment [a,b] with respect to the weight function $w(x)$ if the product $w(x)f(x) \cdot g(x)$ is summable and

$$\int_a^b w(x)f\ (x)g(x)\ dx = 0 \qquad (2.02)$$

Using the notation of Hildebrand (1) equation (2.02) becomes

$$\int_a^b w(x)\phi_r(x)\ q_{r-1}(x)\ dx = 0, \qquad (2.03)$$

Where $\phi_r(x)$ is a polynomial of degree r and $q_{r-1}(x)$ is an arbitrary polynomial of degree r-1 or less. Hildebrand introduces the notation

$$w(x)\phi_r(x) = \frac{d^rU_r(x)}{dx^r} \equiv U_r^r(x)\ . \qquad (2.04)$$

Equation (2.03) now becomes

$$\int_a^b U_r^r\ (x)q_{r-1}(x)\ dx = 0. \qquad (2.05)$$

Now let: $u = q_{r-1}(x)$ $\qquad dv = U_r^r\ (x)$

$du = q'_{r-1}(x)$ $\qquad v = U_r^{r-1}(x)$

and using integration by parts equation (2.05) becomes

$$U_r^{r-1}(x)q_{r-1}(x)\Big]_a^b - \int_a^b U_r^{r-1}(x)q'_{r-1}(x)\ dx = 0\ . \qquad (2.06)$$

After repeated use of integration by parts equation (2.05) has the form

$$[U_r^{r-1}(x)q_{r-1}(x) - U_r^{r-2}(x)q'_{r-1}(x) + U_r^{r-3}(x)q''_{r-1}(x) \ldots$$

$$+(-1)^{r-1}U_r(x)q_{r-1}^{r-1}(x)]_a^b = 0 . \qquad (2.07)$$

From equation (2.04) we have that

$$\phi_r(x) = \frac{1}{w(x)} \frac{d^r U_r(x)}{dx^r} \qquad (2.08)$$

must be a polynomial of degree r which implies that $U_r(x)$ satisfy the differential equation

$$\frac{d^{r+1}\phi_r(x)}{dx^{r+1}} = \frac{d^{r+1}}{dx^{r+1}}\left[\frac{1}{w(x)} \frac{d^r U_r(x)}{dx^r}\right] = 0 \qquad (2.09)$$

in the interval [a,b]. The requirement of (2.07) must be satisfied for any value of $q_{r-1}(a)$, $q_{r-1}(b)$, $q'_{r-1}(a)$, $q'_{r-1}(b)$ and so forth which leads to the 2r boundary conditions

$$U_r(a) = U'_r(a) = U''_r(a) = \ldots = U_r^{r-1}(a) = 0 \qquad (2.10)$$

$$U_r(b) = U'_r(b) = U''_r(b) = \ldots = U_r^{r-1}(b) = 0 . \qquad (2.11)$$

Thus, if for each integer r, a solution of (2.09) which satisfies (2.10) and (2.11) can be obtained, the r th member of the required set of functionsis given by (2.08). From the homogeneous property of the above condition it can be seen that each solution will contain an arbitrary multiplicative constant. The linear transformation $y = \frac{1}{(b-a)}(2x - a - b)$ transforms the interval of interest [a,b] to the interval [-1,1].

Consider the case when the interval is [-1,1] and the weight function is $w(x) = 1$. The differential equation (2.09) becomes

$$\frac{d^{2r+1}U_r}{dx^{2r+1}} = 0 \tag{2.12}$$

and the boundary conditions (2.10) and (2.11) becomes

$$U_r(x) = U_r(1) = U_r''(1) = \cdots = U_r^{r-1}(1) = 0 \tag{2.13}$$

$$U_r(-1) = U_r'(-1) = U_r''(-1) = \cdots = U_r^{r-1}(-1) = 0 \tag{2.14}.$$

The solution of (2.12) with the boundary conditions (2.13) and (2.14) is

$$U_r = C_r(x^2 - 1)^r \tag{2.15}$$

where C_r is an arbitrary constant. From (2.08) the orthogonal polynomials $\phi_r(x)$ have the form

$$\phi_r(x) = C_r \frac{d^r (x^2 - 1)^r}{dx^r}. \tag{2.16}$$

If we let C_r be any arbitrary constant we will have a set of orthogonal polynomials. However the recurrence relationship may not be so readily found. For example if we let $C_r = 1$, the orthogonal polynomials are

$$\phi_r(x) = \frac{d^r(x^2 - 1)^r}{dx^r}.$$

By letting r take on integer values (r = 0, 1, 2, 3, 4, 5) the following orthogonal polynomials are obtained.

$$\phi_0(x) = 1$$

$$\phi_1(x) = 2x$$

$$\phi_2(x) = 12x^2 - 4$$

$$\phi_3(x) = 120x^3 - 72x$$

$$\phi_4(x) = 1680x^4 - 144x^2 + 144$$

$$\phi_5(x) = 30240x^5 - 33600x^3 + 7200x$$

We have, not to easily, the following recurrence relationship

$$\phi_n(x) = (4n - 2)x\, \phi_{n-1} - 4(n-1)^2\, \phi_{n-2}$$

for n = 2, 3, 4 ... .

With the proper choice of $C_r = \frac{1}{2^r r!}$ we have the well known Legendre polynomials usually noted by $P_r(x)$. The Legendre polynomials can be expressed as

$$P_r(x) = \frac{1}{2^r r!} \frac{d^r (x^2 - 1)^r}{dx^r} . \qquad (2.17)$$

Therefore by letting r take on integer values (r = 0, 1, 2, 3, 4, 5) the following Legendre polynomials are obtained.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

Szego (5) derives the following recurrence formula for the Legendre polynomials

$$P_{r+1}(x) = \frac{2r+1}{r+1}\, x\, P_r(x) - \frac{r}{r+1} P_{r-1}(x) \qquad (2.18)$$

for r = 1,2,3, ... Where $P_0(x) = 1$ and $P_1(x) = x$.

It can be seen that both sets of orthogonal polynomials have the same roots. Therefore, it is more convenient to use $C_r = \frac{1}{2^r r!}$ because of the fact that the recurrence relationship for the known Legendre orthogonal polynomials is of a simpler form.

Stroud and Secreset (6) and the Handbook of Mathematical Functions (7) have the Legendre polynomials expressed up to and including the 16^{th} degree polynomial. A table of coefficients for the Legendre polynomials are listed in the appendix.

It can be readily seen that for the Legendre polynomials the following is true.

$$\int_{-1}^{1} w(x)\ P_m(x) P_n(x)\ dx = \begin{cases} 0 & \text{when } m \neq n \\ \frac{2}{2n+1} & \text{when } m = n \end{cases} \qquad (2.19)$$

Some properties of the Legendre polynomials are as follows:

Property 1. $P_m(x)$ is a polynomial of degree m in x. This can be readily seen from the definition of the Legendre polynomials.

Property 2. The zeros of $P_m(x)$ are all real and distinct and lie on the interval -1,1. This property will be proven in this study.

Property 3. The zeros of $P_m(x)$ are symmetrically placed with respect to the origin. If m is odd, one zero of $P_m(x)$ is always x = 0.

Ralston (3) and Krylov (2) prove Property 2. Ralston's proof of Property 2 is as follows:

Proof of Property 2.

Let $a_1, a_2, \ldots a_m$ be the points of a,b where $p_n(x)$ changes sign. Then

$$(x-a_1)(x-a_2) \ \ldots \ (x-a_m)p_n(x)$$

does not change sign in [a,b.] Since $p_n(x)$ is orthogonal to all polynomials of degree less than n with respect to w(x) over [a,b,] we have

$$\int_{-1}^{1} w(x)(x-a_1)(x-a_2)\ldots(x-a_m)p_n(x)\ dx = 0$$

unless m = n. But the integral does not change sign. Therefore, the integral cannot be zero and so m = n, which proves the zeros are real, distinct, and lie within [a,b.]

Krylov (2), Stroud and Secrest (6), and the Handbook of Mathematical Functions (7) all give the zeros for the Legendre polynomials. The Handbook of Mathematical Functions (7) have tables for the zeros of the Legendre polynomials for degrees 2 through 10, 12, 16, 20, 24, 32, 40, 48, 64, 80, and 96th degree accurate to 21 decimal place accuracy. A table of the roots of the first twenty Legendre polynomials are listed in the appendix.

Now let $w(x) = e^{-\alpha x}$ where α is a positive constant on the interval of $[0,\infty]$. The relevant orthogonal polynomials from equation (2.04) now becomes

$$\phi_r = e^{\alpha x} \frac{d^r U_r}{dx^r} \tag{2.20}$$

The differential equation (2.09) becomes

$$\frac{d^{r+1}}{dx^{r+1}}\left[e^{\alpha x} \frac{d^r U_r}{dx^r} \right] = 0 \qquad (2.21)$$

with boundary conditions

$$U_r(0) = U_r'(0) = \ldots = U_r^{r-1}(0) = 0 \qquad (2.22)$$

$$U_r(\infty) = U_r'(\infty) = \ldots = U_r^{r-1}(\infty) = 0. \qquad (2.23)$$

The general solution of equation (2.21) has the following form.

$$U_r = e^{-\alpha x}(c_0 + c_1 x + \ldots + c_r x^r) +$$

$$d_0 + d_1 x + \ldots + d_{r-1} x^{r-1} \qquad (2.24)$$

where the c's and the d's are arbitrary constants. The boundary condition (2.22) gives that $c_0 = c_1 = \ldots = c_{r-1} = 0$ and the boundary condition (2.23) requires that all the d's must become zero. Therefore, equation (2.24) reduces to

$$U_r(x) = c_r x^r e^{-\alpha x} \qquad (2.25)$$

and equation (2.20) becomes

$$\phi_r = c_r e^{\alpha x} \frac{d^r}{dx^r} (x^r e^{-\alpha x}). \qquad (2.26)$$

By letting $C_r = 1$ and $\alpha = 1$ we get the well known Laguerre polynomials usually denoted by $L_r(x)$

$$L_r(x) = \frac{e^x d^r (x^r e^{-x})}{dx^r}. \qquad (2.27)$$

By letting r take on positive integer values (r = 0,1,2,3, 4, 5) the following Laguerre polynomials can be obtained.

$$L_0(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = 2 - 4x + x^2$$
$$L_3(x) = 6 - 18x + 9x^2 - x^3$$
$$L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4$$
$$L_5(x) = 120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5$$

Szego (5) establishes the recurrence relationship

$$L_{r+1}(x) = (1 + 2r - x)\; L_r(x) - r^2\; L_{r-1}(x) \qquad (2.28)$$

for r = 1, 2, 3, ... , with $L_0(x) = 1$ and $L_1(x) = 1 - x$.

It is seen that the following relationship holds true for the Laguerre polynomials.

$$\int_0^\infty e^{-x}\, L_m(x)\, L_n(x)\, dx = \begin{cases} 0 & \text{when } m \neq n \\ (m!)^2 & \text{when } m = n \end{cases}$$

Krylov (2), Stroud and Secrest (6), and the Handbook of Mathematical Function (7) list the coefficients for the Laguerre polynomials and also the zeros for the Laguerre polynomials. A table of coefficients and zeros of the Laguerre are listed in the appendix. It should be noted that the Laguerre coefficients in the appendix are listed with the coefficient of the highest power of x first.

With the interval $[-\infty, \infty]$ and the weight function $w(x) = e^{-\alpha^2 x^2}$, ϕ_r is defined as

$$\phi_r = e^{\alpha^2 x^2} \frac{d^r U_r}{dx^r} \qquad (2.29)$$

and U_r satisfies the differential equation (2.09) which reduces to

$$\frac{d^{r+1}}{dx^{r+1}}\left[e^{\alpha^2 x^2} \quad \frac{d^r U_r}{dx^r}\right] = 0 \tag{2.30}$$

with the following boundary conditions

$$U_r(\infty) = U_r'(\infty) = \ldots = U_r^{r-1}(\infty) = 0 \tag{2.31}$$

$$U_r(-\infty) = U_r'(-\infty) = \ldots = U_r^{r-1}(-\infty) = 0. \tag{2.32}$$

The solution of the differential equation (2.30) with boundary conditions (2.31) and (2.32) is

$$U_r(x) = C_r e^{-\alpha^2 x^2} \tag{2.33}$$

where C_r is an arbitrary constant. The orthogonal polynomial relationship of (2.29) is now

$$\emptyset_r = C_r e^{\alpha^2 x^2} \quad \frac{d^r}{dx^r}\,(e^{-\alpha^2 x^2}). \tag{2.34}$$

If we let $C_r = (-1)^r$ and let $\alpha = 1$, we will get the Hermite polynomials which is usually denoted by $H_r(x)$. The Hermite polynomials can be expressed as

$$H_r(x) = (-1)^r\, e^{x^2} \frac{d^r}{dx^r}\,(e^{-x^2}) \tag{2.35}$$

By letting r take on positive integer values (r=0,1,2,3,4,5) and using equation (2.35) we can obtain the following Hermite polynomials.

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 16x^3 + 120$$

Szego (5) establishes the recurrence relationship

$$H_{r+1}(x) = 2xH_r(x) - 2rH_{r-1}(x) \qquad (2.36)$$

with $H_0 = 1$ and $H_1(x) = 2x$.

It is seen that the following relationship holds true for the Hermite polynomial.

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = \begin{cases} 0 \text{ when } m \neq n \\ 2^n \sqrt{\pi}\, n\,! \text{ when } m = n \end{cases}$$

Kopal (4), Salzer, Zucker, and Capuano (8), Krylov (2), Stroud and Secrest (6) and The Handbook of Mathematical Functions (7) all give the coefficients and the zeros of the Hermite polynomials. A list of the coefficient and the zeros are given in the appendix.

Consider next the interval of [-1,1] and the weight function $w(x) = 1/\sqrt{1-x^2}$.

By the definition of (2.03) we want a polynomial $\emptyset_r(x)$ of degree r and any arbitrary polynomial $q_{r-1}(x)$ of degree r-1 or less such that

$$\int_{-1}^{1} \frac{\emptyset_r(x)\ q_{r-1}(x)dx}{\sqrt{1-x^2}} = 0. \qquad (2.37)$$

If we let $\cos\theta = x$, then equation (2.37) becomes

$$\int_0^{\pi} \emptyset_r(\cos\theta)\ q_{r-1}(\cos\theta)\ d\theta = 0. \qquad (2.38)$$

We have from trigonometry identities that

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1. \qquad (2.39)$$

Cos $k\theta$ can be expressed as a polynomial of degree k in terms of $\cos\theta$; also any k^{th} degree polynomial of $\cos\theta$ can be expressed as a linear combination of 1, $\cos\theta$, $\cos 2\theta, \ldots \cos k\theta$. Now for equation (2.38) to be satisfied the

$$\phi_r(\cos\theta)\ \cos k\theta\ d\theta = 0, \text{ for } (k=0,1..r-1)$$

Therefore $\phi_r(\cos\theta)$ must equal $C_r \cos r\theta$

$$\phi_r(\cos\theta) = C_r(\cos r\theta). \qquad (2.40)$$

Now changing back to our original variable

$$\phi_r(x) = C_r \cos(r \cos^{-1} x). \qquad (2.41)$$

With $C_r = 1$ we have the well known Chebyshev polynomials of the first kind, denoted by $T_r(x)$. Equation (2.41) can be written as

$$\phi_r(x) = T_r(x) = \cos(r \cos^{-1} x). \qquad (2.42)$$

By letting r take on positive integer values ($r = 0,1,2,3,4,5$) the following Chebyshev polynomial can be obtained.

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

From trigonometry identities for the sum of angles we have

$$\cos(k+1)\ \theta = \cos k\theta\ \cos\theta - \sin k\theta\ \sin\theta \tag{2.43}$$

$$\cos(k-1)\ \theta = \cos k\theta\ \cos\theta + \sin k\theta\ \sin\theta. \tag{2.44}$$

Adding equations (2.43) and (2.44) and rearrange we get

$$\cos(k+1)\theta = 2\cos k\theta\ \cos\theta - \cos(k-1)\theta. \tag{2.45}$$

We now let $\cos\theta = x$ and $\cos n\theta = T_n(x)$. Applying our substitution to equation (2.45) we get

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \tag{2.46}$$

which is the recurrence relationship for the Chebyshev polynomials.

It is seen that the following relationship is true

$$\int_{-1}^{1} \frac{T_m(x)\ T_n(x)\ dx}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{when } m \neq n \\ \pi & \text{when } m = n \text{ and } m = 0 \\ \pi/2 & \text{when } m = n \text{ and } m \neq 0 \end{cases}$$

Krylov (2), Stroud and Secrest (6) and the Handbook of Mathematical Functions (7), all give the coefficients and zeros of the Chebyshev polynomial of first kind. A list of the coefficients and zeros are given in the appendix.

The last orthogonal polynomials to be discussed are the Jacobi polynomials denoted by $P_r^{(\alpha,\beta)}(x)$. The weight function is $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ where α and β are > -1. The interval is from $[-1,1]$. Krylov (2) states that the $\phi_r(x)$ is

$$\phi_r(x) = \frac{(-1)^r}{2^r\ r!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^r}{dx^r}\left[(1-x)^{\alpha+r}(1+x)^{\beta+r}\right] \tag{2.48}$$

or

$$P_r^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^r}{dx^r} [(1-x)^{\alpha+r}(1+x)^{\beta+r}] \tag{2.49}$$

Szego (5) gives the following general recurrence relationship for the Jacobi polynomials.

$$2n(n+\alpha+\beta)(2n+\alpha+\beta-2)\, P^{(\alpha,\beta)}(x)$$

$$= (2n+\alpha+\beta-1)\{(2n+\alpha+\beta)(2n+\alpha+\beta-2)x + \alpha^2 - \beta^2\}P_{n-1}^{(\alpha,\beta)}(x)$$

$$-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)\, P_{n-2}^{(\alpha,\beta)}(x) \tag{2.50}$$

for $n = 2, 3, 4, \ldots$

where $P_0^{(\alpha,\beta)}(x) = 1$

$$P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta)$$

Davis (9) states that the following relationship holds true:

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} [P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)]\, dx$$

$$= \begin{cases} 0 \text{ when } m \neq n \\ \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)} \\ \qquad \text{when } m = n \end{cases} \tag{2.51}$$

When $\alpha=\beta=0$, we have the Legendre polynomials. When

$\alpha=\beta=-1/2$ we get the Chebyshev of the first kind. A list of the coefficients and zeros are listed in the appendix for various α and β.

In addition to the reference cited above pertaining to orthogonal polynomials, Shohat (10) gives a detailed bibliography on orthogonal polynomials.

Now we can consider the different numerical integration techniques. The well known Simpson's rule is based upon equally spaced intervals and, according to Hildebrand (1), defined over a finite interval. Also Simpson's rule requires an even number of subintervals.

Romberg integration is a recent technique in the field of numerical integration. Bauer (11) and Welsh (12) and others have developed algorithms for Romberg integration. Ralston (3) explains how Romberg integration works. Ralston (3) shows that the trapezoidal rule may be written as follows:

$$\int_a^b f(x)\ dx = h(1/2\ f_0 + f_1 + f_2 + .. + f_{n-1} + 1/2\ f_n)$$

$$+ \sum_{j=1}^{\infty} a_j h^{2j} \qquad (2.52)$$

where $h = \frac{b-a}{m}$ and the a_j's depend upon a, b, and f(x).

It should be noted that f_m is defined to be equal to the following

$$f_m = f(x_0+mh) \text{ where m is some integer}$$

and x_0 is the lower limit. In equation (2.52) for example

$f_0 = f(a + 0 \cdot h)$.

Now let

$$J = \int_a^b f(x)\ dx \tag{2.53}$$

and also let

$$T_{0,k} = \frac{b-a}{2} (1/2\ f_0 + f_1 + f_2 + \ldots + f_{2^k-1} + 1/2 f_{2^k}) \tag{2.54}$$

be the trapezoidal rule approximation for 2^k subintervals.

Then $$T_{0,k} = J - \sum_{j=1}^{\infty} a_j \left(\frac{b-a}{2^k}\right)^{2j} \tag{2.55}$$

Now define

$$T_{1,k} = \frac{1}{3} (4\ T_{0,k+1} - T_{0,k}) \tag{2.56}$$

for $k = 0, 1, \ldots$.

Using equation (2.55) we have that

$$T_{1,k} = \frac{1}{3} \left[4J - 4 \sum_{j=1}^{\infty} a_j \left(\frac{b-a}{2^{k+1}}\right)^{2j}\right] - \frac{1}{3} \left[J - \sum_{j=1}^{\infty} a_j \left(\frac{b-a}{2^k}\right)^{2j}\right]$$

or

$$T_{1,k} = J - \sum_{j=1}^{\infty} \frac{1}{3} \left(\frac{4}{2^{2j}} - 1\right) a_j \left(\frac{b-a}{2^k}\right)^{2j} \tag{2.57}$$

Equation (2.57) states that the leading term in the error is of order of $\left(\frac{b-a}{2^k}\right)^4$. $T_{1,k}$ is just Simpson's rule for 2^k subintervals. We now define in general

$$T_{m,k} = \frac{1}{4^m - 1} (4^m T_{m-1,k+1} - T_{m-1,k}) \tag{2.58}$$

for $k = 0, 1, \ldots$

and $m = 1, 2, \ldots$

$T_{m,k}$ is a linear combination of trapezoidal rules using 2^k, 2^{k+1}, ... 2^{k+m} subintervals. We usually arrange the results as indicated in the following table:

$$
\begin{array}{llll}
T_{0,0} & & & \\
T_{0,1} & T_{1,0} & & \\
T_{0,2} & T_{1,1} & T_{2,0} & \\
T_{0,3} & T_{1,2} & T_{2,1} & T_{3,0} \\
\vdots & \vdots & & \ddots \\
T_{0,m} & T_{1,m-1} & \cdots & T_{m,0}
\end{array}
$$

The first column is just the trapezoidal rule using 2^0, 2^1, 2^2, 2^3, ··· 2^m subintervals respectively. Ralston (3) proves that if the second derivative of f(x) is bounded in [a,b], then as $m\to\infty$, $T_{0,m}$ converges to J. Ralston (3) also proves that $T_{m,0}$ converges to J much more rapidly than does $T_{0,m}$. The fact that the technique converges as $m\to\infty$ is one reason why Romberg is considered so useful. However Thacker (13) states that there are some functions in which Romberg is not very useful.

A Gaussian quadrature formula is defined by Ralston (3) as any quadrature formula whose abscissas and weights are subject to no constraints and which are determined so as to achieve a maximum order of accuracy. Ralston (3) and Hildebrand (1) gives an analytical approach to the problem of finding abscissas and weights that will achieve maximum order of accuracy. First we start with the Hermite interpolation formula

$$f(x) = \sum_{j=1}^{n} h_j f(a_j) + \sum_{j=1}^{n} \bar{h}_j f'(a_j) + \frac{p_n^2(x)}{(2n)!} f^{(2n)}(\xi) \quad (2.59)$$

with $h_j(x) = [1-2(x-a_j)\ell_j'(a_j)]\, l_j^2(x)$

$\bar{h}_j(x) = (x-a_j)\ell_j^2\,(x)$ for $j = 1,\cdots n$. $\ell_j(x)$ is defined to be

$$\ell_j(x) = \frac{(x-a_1)\cdots(x-a_{j-1})(x-a_{j+1})\cdots(x-a_m)}{(a_j-a_1)\cdots(a_j-a_{j-1})(a_j-a_{j+1})\cdots(a_j-a_m)}$$

$\ell_j'(x)$ is just the first derivative of $\ell_j(x)$.

a_j $j = 1,2,\cdots m$ is defined to be tabular points of the m^{th} degree polynomial.

If $f(x)$ is a polynomial of degree $2n-1$ or less the error term in equation (2.59) is zero. That is,

$$\frac{p_n^2(x)}{(2n)!} f^{(2n)}(\xi) = 0.$$

We can see that

$$\int_a^b w(x) f(x) \, dx = \sum_{j=1}^{n} H_j f(x_j) + \sum_{j=1}^{n} \bar{H}_j f'(x_j) + E \qquad (2.60)$$

with $H_j = \int_a^b w(x) \, h_j(x) \, dx$

$$\bar{H}_j = \int_a^b w(x) \, \bar{h}_j(x) \, dx$$

and

$$E = \int_a^b \frac{w(x) \, p_n^2(x)}{(2n)!} \, f^{(2n)} (\xi) \, dx.$$

Since E is zero if f(x) is a polynomial of degree 2n-1 or less, we want to choose abscissas so that $\bar{H}_j = 0$ for $j = 1, \cdots n$ then equation (2.60) will have desired accuracy of order 2n-1. We have that

$$\bar{H}_j = \int_a^b w(x) \, \bar{h}_j(x) \, dx$$

$$= \int_a^b w(x)\ (x - a_j)\ \ell_j^2(x)\, dx$$

$$= \int_a^b w(x)\ p_n(x)\ \frac{\ell_j(x)}{p_n'(a_j)}\ dx \quad .$$

Since $p_n(x)$ is a polynomial of degree n and $l_j(x)$ is a polynomial of degree n-1, a sufficient condition for $\overline{H}_j = 0$, for j = 1, ... n is for $p_n(x)$ to be orthogonal to all polynomials of degree n-1 or less over [a,b]. With $\overline{H}_j = 0$ equation (2.60) has the form

$$\int_a^b w(x)\ f(x)dx = \sum_{j=1}^{n} H_j f(x_j)\ + E \qquad (2.61)$$

It was seen by Gauss that formulas of greater accuracy with a fixed number of points n could be obtained if both the abscissas and weights in the quadrature formula were unrestricted. An integration formula of the form

$$\int_a^b w(x)\ f(x)\ dx \approx \sum_{j=1}^{n} A_i\ f(x_i) \qquad (2.62)$$

was sought. Where A_i are called the weights and the x_i's are the abscissas.

We want to integrate the definite integral

$$\int_a^b y\ dx$$

where y = f(x). Using a change of variables the limits of integration become -1 and 1. The new value of y is

$$y = f(x) = f\left(\frac{(b-a)\ u + a + b}{2}\right) = \phi\ (u).$$

Now $dx = \frac{b-a}{2}$ du and the integral becomes

$$\int_a^b y\ dx = \frac{b-a}{2} \int_{-1}^{1} \phi(u)\ du \quad .$$

Gauss's formula (Scarborough (14)) is

$$I = \int_{-1}^{1} \phi(u)\,du = R_1\phi(u_1) + R_2\phi(u_2) + \ldots R_n\phi(u_n) \quad (2.63)$$

where $u_1, u_2, \ldots u_n$ are the points of the subdivision of the interval $u = -1$ to $u = 1$. We now assume that $\phi(u)$ can be expanded in a convergent power series in the interval $u = -1$ to $u = 1$. Therefore we have

$$\phi(u) = a_0 + a_1u + a_2u^2 + a_3u^3 + \ldots a_mu^m + \ldots \quad . \quad (2.64)$$

We also assume that the integral can be expressed as a linear function of the ordinates of the form (2.63). Integrating equation (2.64) between the limits -1 and 1 we get

$$I = \int_{-1}^{1} \phi(u)\ du = \int_{-1}^{1} (a_0 + a_1u + \ldots + a_mu^m + \ldots)\ du$$
$$= 2a_0 + 2/3\ a_2 + 2/5\ a_4 + 2/7\ a_6 + \ldots \quad (2.65)$$

From (2.64) we also have

$$\phi(u_1) = a_0 + a_1u_1 + a_2u_1^2 + \ldots + a_mu_1^m + \ldots$$

$$\phi(u_2) = a_0 + a_1u_2 + a_2u_2^2 + \ldots + a_mu_2 + \ldots$$

....

$$\phi(u_n) = a_0 + a_1 u_n + a_2 u_n^2 + \dots + a_m u_n^m + \dots$$

Substituting into equation (2.63) these values of

$\phi(u_1)$, $\phi(u_2)$, $\phi(u_3)$ ... $\phi(u_n)$ we get

$$\begin{aligned} I = {} & R_1(a_0 + a_1 u_1 + a_2 u_1^2 + \dots + a_m u_1^m + \dots) \\ & + R_2(a_0 + a_1 u_1 + a_2 u_2^2 + \dots + a_m u_2^m + \dots) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + R_n(a_0 + a_1 u_n + a_2 u_n^2 + \dots + a_m u_n^m + \dots) \end{aligned}$$

or rearranging

$$\begin{aligned} I = {} & a_0(R_1 + R_2 + R_3 + \dots R_n) \\ & + a_1(R_1 u_1 + R_2 u_2 + R_3 u_3 + \dots R_n u_n) \\ & + a_2(R_1 u_1^2 + R_2 u_2^2 + R_3 u_3^2 + \dots + R_n u_n^2) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + a_n(R_1 u_1^m + R_2 u_2^m + \dots + R_n u_n^m) \end{aligned} \tag{2.66}$$

Now if the integral I of equation (2.66) is to be identically the same as the integral I of equation (2.65) for all values of a_0, a_1, ... a_n then corresponding coefficients of a_0, a_1, ... a_n must be equal. That is to say that the following must be true.

$$R_1 + R_2 + R_3 + \dots + R_n = 2$$

$$R_1u_1 + R_2u_2 + R_3u_3 + \cdots R_nu_n = 0$$

$$R_1u_1^2 + R_2u_2^2 + R_3u_3^2 + \cdots + R_nu_n^2 = 2/3$$

$$R_1u_1^3 + R_2u_2^3 + R_3u_2^3 + \cdots + R_nu_n^3 = 0 \qquad (2.67)$$

....

....

By taking 2n of these equations and solving them simultaneously, it would be possible to find the 2n quantities $u_1, u_2, \ldots u_n$ and $R_1, R_2 \ldots R_n$. However, the task of solving these equations is quite tedious and difficult. Scarborough (14) states that it can be shown that if $\phi(u)$ is a polynomial of degree not higher than 2n-1, then $u_1, u_2, \ldots u_n$ are the zeros of the Legendre polynomials $P_n(u) = 0$. Once we have the u_i's we can solve for the R_i's.

If we let n = 3 for example, we find by solving $P_3(u) = 0$ that we will get

$$u_1 = -\sqrt{3/5}$$

$$u_2 = 0$$

$$u_3 = \sqrt{3/5}.$$

Now once these values are substituted into equation (2.67) we obtain that $R_1 = R_3 = 5/9$ and $R_2 = 8/9$.

Hildebrand (1) and Ralston (3) derives a formula for obtaining the weights H_i and error term

$$H_i = \frac{2(1-x_i^2)}{(n+1)^2[P_{n+1}(x_i)]^2} \qquad (2.68)$$

$$E = \frac{2^{2n+1}(n!)^4}{(2n+1)\ [(2n)!]^3}\ f^{(2n)}(\xi) \qquad (2.69)$$

where $-1<\xi<1$.

Therefore equation (2.62) can now be written (with proper change in variables) with $w(x) = 1$ as

$$\int_{-1}^{1} f(x)\ dx = \sum_{i=1}^{n} H_i f(x_i) + E \qquad (2.70)$$

where the H_i's are defined by (2.68) and the x_i's are the zeros of the Legendre polynomials. Equation (2.70) is usually called the Legendre-Gauss quadrature formula.

Equation (2.62) with $w(x) = e^{-x}$ and the interval of $[0,\infty]$ can be written as

$$\int_{0}^{\infty} e^{-x}\ f(x)\ dx = \sum_{i=1}^{n} H_i f(x_i) + E \qquad (2.71)$$

where according to Ralston (3) and Hildebrand (1)

$$H_i = \frac{(n!)^2}{L_n'(x_i)\ L_{n+1}(x_i)} \qquad (2.72)$$

or (2.72) can be expressed in a simpler form

$$H_i = \frac{(n!)^2\ x_i}{[L_{n+1}(x_i)]^2} \qquad (2.73)$$

where x_i are the zeros of the Laguerre polynomials. The error term is expressed as follows

$$E = \frac{(n!)^2}{(2n)!}\ f^{(2n)}(\xi) \qquad (2.74)$$

where $0<\xi<\infty$. Equation (2.71) is usually called the Laguerre-Gauss quadrature formula.

Equation (2.62) with $w(x) = e^{-x^2}$ and the interval of $[-\infty, \infty]$ can be written as

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = \sum_{i=1}^{n} H_i f(x_i) + E \qquad (2.75)$$

where according to Ralston (3) and Hildebrand (1)

$$H_i = \frac{-2^{n+1}\, n!\sqrt{\pi}}{H_n'(x_i) H_{n+1}(x_i)} \qquad (2.76)$$

or in simpler form

$$H_i = \frac{2^{n+1}\, n!\sqrt{\pi}}{[H_{n+1}(x_i)]^2} \qquad (2.77)$$

Where the x_i's are the zeros of the Hermite polynomials. The error term for equation (2.75) is

$$E = \frac{n!\,\sqrt{\pi}}{2^n (2n)!}\, f^{(2n)}(\xi) \qquad (2.78)$$

where $-\infty<\xi<\infty$.

Equation (2.75) is usually called the Hermite-Gauss quadrature formula.

Equation (2.62) with $w(x) = \frac{1}{\sqrt{1-x^2}}$ and the interval of $[-1,1]$ can be written as

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\, dx = \sum_{i=1}^{n} H_i f(x_i) + E \qquad (2.79)$$

where according to Ralston (3) and Hildebrand (1)

$$H_i = \pi/n \tag{2.80}$$

and

$$E = \frac{2\pi}{2^{2n}(2n)!} f^{(2n)}(\xi) \tag{2.81}$$

where $-1<\xi<1$.

Equation (2.70) is usually called the Chebyshev-Gauss quadrature formula.

Equation (2.62) with $w(x) = (1-x)^{\alpha} (1+x)^{\beta}$ with $\alpha>-1$ and $\beta>-1$ over the interval of -1 to 1, can be written as

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x)\, dx = \sum_{i=1}^{n} H_i f(x_i) + E \tag{2.82}$$

where according to Ralston (3) and Hildebrand (1)

$$H_i = \frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)\ 2^{2n+\alpha+\beta+1} n!}{\Gamma(n+\alpha+\beta+1)\ (1-x_i^2)\ P_n'^{(\alpha,\beta)}(x_i) P_{n+1}^{(\alpha,\beta)}(x_i)} \tag{2.83}$$

where the x_i's are the zeros of the Jacobi polynomials for the particular α and β.

The error term is as follows

$$E = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)\ 2^{2n+\alpha+\beta+1} m!}{(2n+\alpha+\beta+1)\ [\Gamma(2n+\alpha+\beta+1]^2\ (2n)!} f^{(2n)}(\xi) \tag{2.84}$$

where $-1<\xi<1$.

Equation (2.82) is usually called the Jacobi-Gauss quadrature formula or sometimes is called Mehler quadrature.

In some numerical integration problems it is sometimes desirable to prescribe one or more of the n abscissas to be involved in a quadrature formula. In the Gaussian-type quadrature already mentioned, the formulas do not include the end points. It sometimes is important to use one or both of the end points.

In the case of a finite interval with weighting function $w(x) = 1$, when one end point of the interval is to be assigned, we transform the interval of interest to the interval $[-1,1]$. We now assign $x = -1$ as the fixed abscissa. We now get, according to Ralston (3) and Hildebrand (1) a set of orthogonal polynomials as follows:

$$\phi_r(x) = \frac{C_r}{x+1} \frac{d^r}{dx^r} [(x+1) (x^2-1)^r]. \tag{2.85}$$

With the proper choice of $C_r = \frac{1}{2^r r!}$ and noting that $r = n-1$ since only one abscissa is preassigned. We note that the n-1 free abscissas are the zeros of equation (2.85) with r being equal to n-1. Equation (2.85) can be written as

$$\phi_{n-1}(x) = \frac{P_{n-1}(x) + P_n(x)}{1+x} \tag{2.86}$$

where $P_{n-1}(x)$ and $P_n(x)$ are the n-1 and nth degree Legendre polynomials respectively. Ralston (3) and Hildebrand (1) derive the weights formula and error term. The weights formula is

$$H_i = \frac{1}{1-x_i} \frac{1}{[P'_{n-1}(x_i)]^2} \qquad \begin{matrix}(x_i \neq -1) \\ i = 1, ..n-1\end{matrix}$$

and

$$H = \frac{2}{n^2} \text{ for } (x = -1) \tag{2.87}$$

where x_i's are the zeros of the orthogonal polynomials defined by equation (2.86). The error term is given as

$$E = \frac{2^{2n-1}\, n\, [(n-1)!]^4}{[(2n-1)!]^3} f^{(2n-1)}(\xi)$$

where $-1<\xi<1$.

The recurrence relationship for equation (2.86) given by Hildebrand (1) is as follows:

$$\phi_{r+1}(x) = \frac{[(2r+1)(2r+3)x - 1]\, \phi_r(x) - r(2r+3)\phi_{r-1}(x)}{(r+2)\quad(2r+1)}$$

The integral $\int_a^b f(x)\,dx$ can be written, with proper transformation as

$$\int_{-1}^{1} f(x)\,dx = \frac{2\,f(-1)}{n^2} + \sum_{i=1}^{n-1} H_i f(x_i) + E \tag{2.88}$$

where the x_i's are the zeros of equation (2.86) and H_i is defined by equation (2.87). Equation (2.88) is known as Radau quadrature formula.

Now consider the case when both end points are to be assigned over the interval of [-1, 1] and weighting function $w(x) = 1$. We assign $x = -1$ and $x = 1$ as the fixed abscissas. We now get, according to Ralston (3) and Hildebrand (1), a set of orthogonal polynomials as follows:

$$\phi_r(x) = \frac{C_r}{x^2-1} \frac{d^r(x^2-1)^{r+1}}{dx^r} \tag{2.89}$$

With the proper choice of $C_r = \frac{r+2}{2^{r+1} r!}$ and noting that r=n-2 since two abscissas are preassigned, equation (2.89) becomes

$$\phi_r(x) = P'_{r+1}(x) \tag{2.90}$$

where $P'_{r+1}(x)$ if the first derivative of the $(r+1)^{th}$ degree Legendre polynomial. Hildebrand (1) and Ralston (3) shows that the weights corresponding to the free abscissas are as follows:

$$H_i = \frac{2}{n(n-1)[P_{n-1}(x_i)]^2} \qquad (x_i \neq \pm 1) \tag{2.91}$$

and the weights corresponding to the fixed abscissas $x = -1$ and $x = 1$ are as follows:

$$H = \frac{2}{n(n-1)} \tag{2.92}$$

The error term is given by

$$E = \frac{-n(n-1)^3 \, 2^{2n-1} \, [(n-2)!]^4}{(2n-1) \, [(2n-2)!]^3} \, f^{(2n-2)}(\xi) \tag{2.93}$$

where $-1<\xi<1$.

Therefore the integral $\int_a^b f(x)\, dx$ can be written with proper transformation as

$$\int_{-1}^{1} f(x)\, dx = \frac{2}{n(n-1)} [f(-1)+f(1)] + \sum_{i=1}^{n-2} H_i f(x_i) + E \tag{2.94}$$

where the x_i's are the zeros of equation (2.90) and H_i's defined by equation (2.91). Equation (2.94) is known as the Lobatto's quadrature formula.

DISCUSSION

There are several numerical techniques available to use in solving integration problems numerically, where the integrand is some analytical function. The purpose of this study was to see which method or methods were applicable to certain types of integrals. The author also wanted to find out if one method would be "better" than another in integrating a particular type of integral.

In discussing numerical integration, three intervals are usually considered. They are the finite interval; the interval of 0 to ∞; and the interval of $-\infty$ to ∞. Several examples have been selected to show the various method in integrating over a finite interval.

Before discussing integration problems over a finite interval; an explanation of how to evaluate a Gaussian-type quadrature should be discussed.

Suppose we wanted to evaluate the integral $\int_{-1}^{1} x^2 dx = 2/3$ using the Legendre-Gauss formula with n = 4. Using equation (2.70) we have that

$$\int_{-1}^{1} x^2\, dx = \sum_{i=1}^{4} H_i f(a_i) + E,$$

where $f(x) = x^2$, or

$$\int_{-1}^{1} x^2 dx = \sum_{i=1}^{4} H_i f(a_i) + \frac{2^5 (4!)^4}{(9)[(8)!]^3} f^{(viii)}(\xi).$$

But the 8^{th} derivative of x^2 evaluated at $\xi = 0$, therefore

$$\int_{-1}^{1} x^2\, dx = \sum_{i=1}^{4} H_i f(a_i).$$

Using Appendix II to obtain the abscissas and weights for the Legendre-Gauss formula we have that

$$\int_{-1}^{1} x^2 dx = (.2378548451) \times f(.8611363115)$$
$$+ (.3478548451) \times f(-.8611363115)$$
$$+ (.6521451548) \times f(.3399810435)$$
$$+ (.6521451548) \times f(-.3399810435) = .6666666666.$$

Let us now consider some integrals over a finite interval using various techniques. Consider the integral $\int_1^3 dx/x$ which analytically is equal to ln 3 ≃ 1.098612. Simpson's rule using 32 subintervals gives a result of 1.098615, Romberg integration with m = k = 6, (m and k are some positive integers used in the Romberg formula) yields 1.098612, and Legendre-Gauss formula using n = 6 yields 1.098613.

Listed in Table 3.1 are results in integrating $\int_1^3 dx/x$. The heading "method" means what method was used. The heading "error" means the error from the true answer, if known, and the answer obtained by the various methods. When the true answer is an irrational number, such as π and ln 3, they were approximated by using the Handbook of Mathematical Functions (7). The heading "core storage" is the total amount of computer storage needed to solve the problem in question. The heading "time" is the time from execution of the computer program to the first printed result. All times are just approximate times; they could be in error as much as five seconds. The heading "order of error term" is the numerical value of the error term excluding the derivative. For Simpson's rule the error term is

$$\frac{n\ h^5}{180}\ f^{(iv)}\ (\xi),$$

where n is an even integer of subintervals and h is the width of the subinterval. The "order of error term" would be the numerical value of

$$\frac{n\ h^5}{180}$$

for some particular value of n and some particular value of h. The "order of error term" for Romberg integration is

$$\left(\frac{b\ -\ a}{2^k}\right)^{2(m+1)}$$

where a is the lower limit, b the upper limit of integration and m and k are the particular values used in Romberg integration. The error term for the Legendre-Gauss quadrature is given by equation (2.69). The "order of error term" is the numerical value of

$$\frac{2^{2n+1}\ (n!)^4}{(2n+1)\ [(2n)!]^3}$$

for a particular value of n. For the other Gaussian-type formulas the "order of error term" can be obtained from the error term of the particular type of method.

Table 3.1

Method	Error	Core Storage	Time	Order of Error Term
Simpson	-3×10^{-6}	17170	12 sec.	1.6×10^{-7}
Romberg	0×10^{-6}	19000	1 min.	1.5×10^{-5}
Legendre-Guass	-1×10^{-6}	55678	2 min. 20 sec.	1.5×10^{-12}

Consider next the integral $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$ which analytically is equal to $\pi \simeq 3.14159265$. Simpson's rule can not be used because of the singularity at both end points. However, if Simpson's rule was desired there are some formulas called open-type formulas that could be used. Romberg integration with m=k=9 yields 3.10081357, Jacobi-Gauss with $\alpha = \beta = -.5$ and using n = 6 yields 3.14159315. Chebyshev-Gauss using n = 6 yields 3.141592653.

Listed in table 3.2 are results in integrating

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx.$$

Table 3.2

Method	Error	Core Storage	Time	Order of Error Term
Romberg	4077908×10^{-8}	23518	3 min.	2.5×10^{-10}
Jacobi-Gauss	-50×10^{-8}	57086	2 min. 40 sec.	1.1×10^{-9}
Chebyshev-Gauss	0×10^{-8}	53658	2 min. 20 sec.	3.2×10^{-12}

The reason for error in the Jacobi-Gauss method is believed due to the error in approximating the gamma function.

Consider the integral $\int_0^{4.3} e^{-x^2}\,dx$, which according to Scarborough (14) is approximately .88622692. This example shows the power of Romberg integration. Simpson's rule using 64 subintervals yields .88622692. Romberg with m=k=2 and with m=k=6 yields a result of .88622692. Legendre-Gauss with n = 10 yields .88622691.

Listed in table 3.3 are results in integrating $\int_0^{4.3} e^{-x^2}\,dx$.

Table 3.3

Method	Error	Core Storage	Time	Order of Error Term
Simpson	0×10^{-8}	15666	26 sec.	4.2×10^{-6}
Romberg	0×10^{-8}	23164	53 sec.	2.0×10^{-25}
Legendre-Gauss	1×10^{-8}	57658	2 min. 30 sec.	1.2×10^{-25}

Let us now consider the integral

$$\int \frac{x^7\sqrt{1-x^2}}{(2-x)^{6.5}}\,dx$$

where according to Scarborough (14) is approximately equal to .0235. Simpson's rule using 512 subintervals yields .0238. Romberg integration with m=k=7 yields .0236. The Jacobi-Gauss method with $\alpha = \beta = .5$ with $n = 10$ yields .0238.

Listed in table 3.4 are results in integrating

$$\int_{-1}^{1} \frac{x^7\sqrt{1-x^2}}{(2-x)^{6.5}}\,dx$$

Table 3.4

Method	Error	Core Storage	Time	Order of Error Term
Simpson	-3×10^{-4}	17170	3 min. 15 sec.	5.3×10^{-8}
Romberg	-1×10^{-4}	24842	2 min.	5.9×10^{-8}
Jacobi-Gauss	-3×10^{-4}	57418	3 min. 40 sec.	2.5×10^{-18}

The results of other integrals over a finite interval are listed in Appendix III.

With the interval of 0 to ∞, we see some problems in numerical integration. Simpson's rule and Romberg integration are finite methods; therefore we must somehow transform the interval 0 to ∞ to some finite interval. By letting $y=x/x+1$ we see that the interval $[0,\infty]$ in the limit becomes the interval $[0,1]$. We now can transform the integral

$$\int_0^\infty f(x)\,dx,$$

by taking a limit, to the following integral

$$\lim_{A\to 1^-} \int_0^A f\left(\frac{-y}{y-1}\right) \frac{dy}{(y-1)^2}$$

$\lim\limits_{A\to 1^-}$ means to take a limit as A approaches +1 from the left.

We now can integrate the above integral by letting A take on different values until there is no significant change in the result from two different values of A. Consider the following integral

$$\int_0^\infty e^{-x}\, dx = 1.$$

By using the previous described transformation the above integral becomes:

$$\lim_{A\to 1^-} \int_0^A \frac{e^{\left(\frac{y}{y-1}\right)}}{(y-1)^2}\, dy.$$

Because of the time involved and the fact that Romberg method converges when the second derivative is bounded; Simpson's rule was not used to integrate this type of problem. Romberg integration with m=k=5 was used. The initial value of A was .9 and the final value of A found with no significant change in the results from previous value of A was .995. With this value of A the numerical result obtained was .99999778. For each value of A it took approximately one minute and thirty seconds to evaluate. It took five values of A to get the final result; therefore, the total time was approximately seven minutes and thirty seconds.

Using the Laguerre-Gauss method, using only two points yields a result of 1.0000000.

Listed in Table 3.5 are the results in integrating the integral

$$\int_0^\infty e^{-x}\,dx$$

Table 3.5

Method	Error	Core Storage	Time	Order of Error Term
Romberg	222×10^{-8}	24198	7 min 30 sec	9.5×10^{-5}
Laguerre-Gauss	0×10^{-8}	53334	2 min 15 sec	1.6×10^{-1}

Finally, consider the interval of $-\infty$ to ∞. Again we see the problems using Romberg integration. From calculus, the integral $\int_{-\infty}^{\infty} f(x)\,dx$ can be expressed as $\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$.

We already have a transformation for the interval of 0 to ∞, so all we need is one from $-\infty$ to 0. By letting $y = x/(1-x)$ we see that the interval $[-\infty,0]$ in the limit becomes the interval $[-1,0]$. Now transform the integral $\int_{-\infty}^{0} f(x)\,dx$, by taking a limit, to the integral

$$\lim_{B\to -1^+} \int_B^0 f\left(\frac{y}{y+1}\right)\frac{dy}{(y+1)^2}$$

$\lim_{B \to -1^+}$ means the limit as B approaches -1 from the right.

Let us consider the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \simeq 1.77245385.$

By using the two transformations, the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

becomes

$$\lim_{B \to -1^+} \int_B^0 \frac{e^{-\left(\frac{y}{y+1}\right)^2}}{(y+1)^2}\,dy \quad + \quad \lim_{A \to +1^-} \int_0^A \frac{e^{\left(\frac{y}{y-1}\right)^2}}{(y-1)^2}\,dy.$$

We now have two separate integrals to integrate . The method of using different values of A can also be applied for the integral involving B. Romberg integration with m=k=5 was used. The initial value for B was -.75 and the final value was -.90 with a result of .88622659. The initial value of A was .75 and final value was .90 with a result of .88622665. Adding the two results we get a final result of 1.77245324.

For each value of A and B it took approximately one minute and fifty seconds to evaluate. It took four values of A and B to get the final result. Therefore, the total time for the Romberg method to integrate the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ was fourteen minutes and forty seconds.

Using the Hermite-Gauss method, using only two points yields a result of 1.77245385.

Listed in Table 3.6 are the results in integrating the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx.$

Table 3.6

Method	Error	Core Storage	Time	Order of Error Term
Romberg	61×10^{-8}	24098	14 min 40 sec	1.1×10^{-5}
Hermite-Gauss	0×10^{-8}	56734	2 min 10 sec	3.6×10^{-2}

All computation was done on an IBM 1620 Model II with 60,000 core storage using Fortran II computer language. The gamma function used in the Jacobi-Gauss method was numerically approximated to seven decimal place accuracy by using Hasting's (15) approximation.

It can be seen that the Gaussian-type formulas takes a considerable amount of computer storage, and time. The reason for this is that each different method had to first generate the coefficients of the orthogonal polynomials; solve for the zeros of the different polynomials; store these results; then compute the weights for each particular zeros stored. Once the weights were stored, it was a simple matter to calculate

$$\sum_{j=1}^{m} H_j \cdot f(a_j).$$

If the zeros and associated weights could be permanently stored on disk and retrieved when needed, the computer storage and time would probably be reduced by as much as ten times.

CONCLUSION

How do we choose a quadrature formula to use in numerical integration? This question is hard to answer because one must take into consideration many factors. First of all what kind of interval do we have? How accurate do we want the results to be? How fast do we need the results? That is can we compute for one minute, one hour, one day, or what. How large of computer storage do we have available?

If we have a finite interval, is there any singularity at the end points? If there is singularity at the end points, then Simpson's rule would usually be omitted, and possible Romberg integration. Gaussian-type would then probably be used.

However, if there are no singular points what then? We then could use Simpson's rule, Romberg integration, Legendre-Gauss, Chebyshev-Gauss, or Jacobi-Gauss. If speed is desired and computer storage is limited then the Gaussian-type methods would be eliminated. If we wanted accuracy and disregarded all other factors, then Romberg integration would be used.

It is believed that if the expression to be integrated can be expressed exactly as a polynomial of degree n, then Romberg integration should be used. If the expression can not be expressed exactly as a n^{th} degree polynomial, then perhaps it can be expressed as a function

$$\frac{f(x)}{\sqrt{1-x^2}}$$

if so, use the Chebyshev-Gauss. If the expression can be expressed as a function $g(x)\cdot(1-x)^{\alpha}\cdot(1+x)^{\beta}$ where α and $\beta > -1$, then the Jacobi-Gauss should be used. If the expression can not be expressed exactly as a n^{th} degree polynomial, or as a function $\frac{f(x)}{\sqrt{1-x^2}}$, or as a function $g(x)\cdot(1-x)^{\alpha}\cdot(1+x)^{\beta}$, then the Romberg method should be used.

It is believed that the "best" overall method for a finite interval is Romberg integration; because Romberg integration does converge for any continuous functions.

What about the interval of $[0,\infty]$ and $[-\infty,\infty]$? It was seen that the transformations used were not too difficult but could very easily become cumbersome. The most important factor in these type of intervals is speed and computer storage. As was shown in the discussion, using Romberg integration it took approximately three to six times longer than the Laguerre-Gauss and Hermite-Gauss.

If computer storage is not a factor, then Laguerre-Gauss should be used when using an interval of $[0,\infty]$, and Hermite-Gauss should be used when using an interval of $[-\infty,\infty]$.

It is seen from the results of this study that the quadrature formula to use in a given situation is a function of the interval of integration and the integrand.

Further work in the area of numerical integration could be pursued by using a composite rule. The interval of integration could be divided into desired subintervals and each subinterval approximated by a n point Gaussian quadrature formula.

APPENDIX I

A polynomial of degree n is usually expressed in the form

$$S_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

where a_i, $i = 0,1,2,\cdots n$ are complex numbers (real or imaginary) and n is a positive integer. The numbers a_i, $i=0,1,\cdots n$ are called the coefficients of the polynomial. The coefficients of the following orthogonal polynomials are all real. An explanation of how to obtain a given n^{th} degree polynomial might be necessary. Take the following example of the Legendre coefficients:

P(2)

.150000000000E+01 .000000000000E-99 -.500000000000E-00

P(2) means that the coefficients are for a 2nd degree Legendre polynomial. (Note, the Jacobi and Legendre coefficients have the same notation of P(n).)

The coefficients are expressed in what is called E format. The expression .xxxxxxxxxxxxE+yy means to take the decimal number .xxxxxxxxxxxx times 10 to the positive yy power. If the expression has and E-yy, it means to take the decimal number times 10 to the negative yy power. A decimal number with an E-99 is equivalent to zero. Therefore the 2nd degree Legendre polynomial would be:

$$1.50000000000\ x^2 - .500000000000$$

LEGENDRE COEFFICIENTS

P(0)

.100000000000E+01

P(1)

.100000000000E+01 .000000000000E-99

P(2)

.150000000000E+01 .000000000000E-99 -.500000000000E-00

P(3)

.249999999999E+01 .000000000000E-99 -.149999999999E+01 .000000000000E-99

P(4)

.437499999999E+01 .000000000000E-99 -.374999999999E+01 .000000000000E-99

.375000000000E-00

P(5)

.787499999999E+01 .000000000000E-99 -.874999999999E+01 .000000000000E-99

.187499999999E+)1 .000000000000E-99

P(6)

.144374999999E+02 .000000000000E-99 -.196874999999E+02 .000000000000E-99

.656249999999E+01 .000000000000E-99 -.312499999999E-00

P(7)

.268124999999E+02	.000000000000E-99	-.433124999999E+02	.000000000000E-99
.196874999999E+02	.000000000000E-99	-.218749999999E+01	.000000000000E-99

P(8)

.502734374999E+02	.000000000000E-99	-.938437499999E+02	.000000000000E-99
.541406249999E+02	.000000000000E-99	-.984374999999E+01	.000000000000E-99
.273437499999E-00			

P(9)

.949609374999E+02	.000000000000E-99	-.201093749999E+03	.000000000000E-99
.140765624999E+03	.000000000000E-99	-.360937499999E+02	.000000000000E-99
.246093749999E+01	.000000000000E-99		

P(10)

.180425781249E+03	.000000000000E-99	-.427324218749E+03	.000000000000E-99
.351914062499E+03	.000000000000E-99	-.117304687499E+03	.000000000000E-99
.135351562499E+02	.000000000000E-99	.246093749999E-00	

P(11)

.344449218749E+03	.000000000000E-99	-.902128906249E+03	.000000000000E-99

LEGENDRE (con't)

.854648437399E+03	.000000000000E-99	-.351914062499E+03	.000000000000E-99
.586523437499E+02	.000000000000E-99	.270703124999E+01	.000000000000E-99

P(12)

.660194335937E+03	.000000000000E-99	-.189447070312E+04	.000000000000E-99
.202979003906E+04	.000000000000E-99	-.997089843749E+03	.000000000000E-99
.219946289062E+03	.000000000000E-99	.175957031249E+02	.000000000000E-99
.225585937499E-00			

P(13)

.126960449218E+04	.000000000000E-99	-.396116601562E+04	.000000000000E-99
.473617675781E+04	.000000000000E-99	-.270638671874E+04	.000000000000E-99
.747817382812E+03	.000000000000E-99	.879785156249E+02	.000000000000E-99
.293261718749E+01	.000000000000E-99		

P(14)

.244852294921E+04	.000000000000E-99	-.825242919921E+04	.000000000000E-99
.108932065429E+05	.000000000000E-99	-.710426513671E+04	.000000000000E-99
.236808837890E+04	.000000000000E-99	.373908691406E+03	.000000000000E-99

.219946289062E+02	.000000000000E+02	-.209472656249E-00	

P(15)

.472281103515E+04	.000000000000E-99	-.171396606445E+05	.000000000000E-99
.247572875976E+05	.000000000000E-99	-.181553442382E+05	.000000000000E-99
.710426513671E+04	.000000000000E-99	.142085302734E+04	.000000000000E-99
.124636230468E+03	.000000000000E-99	-.314208984374E+01	.000000000000E-99

P(16)

.917175888061E+04	.000000000000E-99	-.355035827636E+05	.000000000000E-99
.557038970947E+05	.000000000000E-99	-.453883605957E+05	.000000000000E-99
.204247622680E+05	.000000000000E-99	.497298559570E+04	.000000000000E-99
.592022094726E+03	.000000000000E-99	-.267077636718E+02	.000000000000E-99
.196380615234E-00			

P(17)

.178040025329E+05	.000000000000E-99	-.733740710449E+05	.000000000000E-99
.124262539672E+06	.000000000000E-99	-.111407794189E+06	.000000000000E-99
.567354507446E+05	.000000000000E-99	.163398098144E+05	.000000000000E-99

LEGENDRE (con't)

.248649279785E+04	.000000000000E-99	-.169149169921E+03	.000000000000E-99
.333847045898E+01	.000000000000E-99		
P(18)			
.346188938140E+05	.000000000000E-99	-.151334021530E+06	.000000000000E-99
.275152766418E+06	.000000000000E-99	-.269235502624E+06	.000000000000E-99
.153185717010E+06	.000000000000E-99	.510619056701E+05	.000000000000E-99
.953155572509E+04	.000000000000E-99	-.888033142089E+03	.000000000000E-99
.317154693603E+02	.000000000000E-99	-.185470581054E-00	
P(19)			
.674157405853E+05	.000000000000E-99	-.311570044326E+06	.000000000000E-99
.605336086120E+06	.000000000000E-99	-.642023121643E+06	.000000000000E-99
.403853253936E+06	.000000000000E-99	.153185717101E+06	.000000000000E-99
.340412704467E+05	.000000000000E-99	-.408495245361E+04	.000000000000E-99
.222008285522E+03	.000000000000E-99	-.352394104003E+01	.000000000000E-99
P(20)			
.131460694141E+06	.000000000000E-99	-.640449535560E+06	.000000000000E-99

LEGENDRE (con't)

.132417268838E+07	.000000000000E-99	-.151334021530E+07	.000000000000E-99
.104328757266E+07	.000000000000E-99	.444238579330E+06	.000000000000E-99
.114889287757E+06	.000000000000E-99	-.170206352233E+05	.000000000000E-99
.127654764175E+04	.000000000000E-99	-.370013809204E+02	.000000000000E-99
.176197052001E-00			

LAGUERRE COEFFICIENTS

L(0)

10.000000000000E-01

L(1)

-10.000000000000E-01 10.000000000000E-01

L(2)

10.000000000000E-01 -40.000000000000E-01 20.000000000000E-01

L(3)

-10.000000000000E-01 90.000000000000E-01 -18.000000000000E-00 60.000000000000E-01

L(4)

10.000000000000E-01 -16.000000000000E-00 72.000000000000E-00 -96.000000000000E-00

24.000000000000E-00

L(5)

-10.000000000000E-01 25.000000000000E-00 -20.000000000000E+01 60.000000000000E+01

-60.000000000000E+01 12.000000000000E+01

L(6)

10.000000000000E-01 -36.000000000000E-00 45.000000000000E+01 -24.000000000000E+02

54.000000000000E+02 -43.200000000000E+02 72.000000000000E+01

L(7)

-10.000000000000E-01 49.000000000000E-00 -88.200000000000E+01 73.500000000000E+02

-29.400000000000E+03 52.920000000000E+03 -35.280000000000E+03 50.400000000000E+02

L(8)

10.000000000000E-01 -64.000000000000E-00 15.680000000000E+02 -18.816000000000E+03

11.760000000000E+04 -37.632000000000E+04 56.448000000000E+04 -32.256000000000E+04

40.320000000000E+03

L(9)

-10.000000000000E-01 81.000000000000E-00 -25.920000000000E+02 42.336000000000E+03

-38.102400000000E+04 19.051200000000E+05 -50.803200000000E+05 65.318400000000E+05

-32.659200000000E+05 36.288000000000E+04

L(10)

10.000000000000E-01 -10.000000000000E+01 40.500000000000E+02 -86.400000000000E+03

10.584000000000E+05 -76.204800000000E+05 31.752000000000E+06 -72.576000000000E+06

81.648000000000E+06 -36.288000000000E+06 36.288000000000E+05

L(11)

-10.000000000000E-01	12.100000000000E+01	-60.500000000000E+02	16.335000000000E+04
-26.136000000000E+05	25.613280000000E+06	-15.267968000000E+07	54.885600000000E+07
-10.977120000000E+08	10.977120000000E+08	-43.908480000000E+07	39.916800000000E+06

L(12)

10.000000000000E-01	-14.400000000000E+01	87.120000000000E+02	-29.040000000000E+04
58.806000000000E+05	-75.271680000000E+06	61.471872000000E+07	-31.614105600000E+08
98.794080000000E+08	-17.563392000000E+09	15.807052800000E+09	-57.480192000000E+08
47.900160000000E+07			

L(13)

-10.000000000000E-01	16.900000000000E+01	-12.168000000000E+03	49.077600000000E+04
-12.269400000000E+06	19.876428000000E+07	-21.201523200000E+08	14.841066240000E+09
-66.784798080000E+09	18.551332800000E+10	-29.682132480000E+10	24.285381120000E+10
-80.951270400000E+09	62.270208000000E+08		

L(14)

10.000000000000E-01	-19.600000000000E-01	16.562000000000E+03	-79.497600000000E+04

24.048024000000E+06	-48.096048000000E+07	64.929664800000E+08	-59.364264960000E+09
36.360612288000E+10	-14.544244915200E+11	36.360612288000E+11	-52.888163328000E+11
39.666122496000E+11	-12.204960768000E+11	87.178291200000E+09	

L(15)

-10.000000000000E-01	22.500000000000E+01	-22.050000000000E+03	12.421500000000E+05
-44.717400000000E+06	10.821610800000E+08	-18.036018000000E+09	20.870249400000E+10
-16.696199520000E+11	90.901530720000E+11	-32.724551059200E+12	74.373979680000E+12
-99.165306240000E+12	68.652904320000E+12	-19.615115520000E+12	13.076743680000E+11

L(16)

10.000000000000E-01	-25.600000000000E+01	28.800000000000E+03	-18.816000000000E+05
79.497600000000E+06	-22.895308800000E+08	46.172206080000E+09	-65.960294400000E+10
66.784798080000E+11	-47.491411968000E+12	23.270791864320E+13	-76.158955192320E+13
15.866448998400E+14	-19.527937228800E+14	12.553673932800E+14	-33.476463820800E+13
20.922789888000E+12			

L(17)

-10.000000000000E-01	28.900000000000E+01	-36.992000000000E+03	27.744000000000E+05

-13.594560000000E+07	45.949612800000E+08	-11.027907072000E+10	19.062525081600E+11
-23.828156352000E+12	21.445340716800E+13	-13.725018058752E+14	61.138716807168E+14
-18.341615042150E+15	35.272336619520E+15	-40.311241850880E+15	24.186745110528E+15
-60.466862776320E+14	35.568742809600E+13		

L(18)

10.000000000000E-01	-32.400000000000E+01	46.818000000000E+03	-39.951360000000E+05
22.472640000000E+07	-88.092748800000E+08	24.812790912000E+10	-51.043455590400E+11
77.203226580480E+12	-85.781362867200E+13	69.482903922432E+14	-40.426416827596E+15
16.507453537935E+16	-45.712948258898E+16	81.630264748032E+16	-87.072282397900E+16
48.978158848819E+16	-11.524272670310E+16	64.023737057280E+14	

L(19)

-10.000000000000E-01	36.100000000000E+01	-58.482000000000E+03	56.337660000000E+05
-36.056102400000E+07	16.225246080000E+09	-53.002470528000E+10	12.796310741760E+12
-23.033359335168E+13	30.967071995059E+14	-30.967071995059E+15	22.803025741816E+16
-12.161613728968E+17	45.839928670728E+17	-11.787410229615E+18	19.645683716026E+18
-19.645683716026E+18	10.400656084955E+18	-23.112569077678E+17	12.164510040883E+16

LAGUERRE (con't)

L(20)

10.000000000000E-01	-40.000000000000E+01	72.200000000000E+03	-77.976000000000E+05
56.337660000000E+07	-28.844881920000E+09	10.816830720000E+11	-30.287126016000E+12
63.981553708800E+13	-10.237048593408E+15	12.386828798023E+16	-11.260753452748E+17
76.010085806053E+17	-37.420349935288E+18	13.097122477350E+19	-31.433093945642E+19
49.114209290066E+19	-46.225138155355E+19	23.112569077678E+19	-48.658040163532E+18
24.329020081766E+17			

HERMITE COEFFICIENTS

H(0)

10.000000000000E-01

H(1)

20.000000000000E-01 00.000000000000E-99

H(2)

40.000000000000E-01 00.000000000000E-99 -20.000000000000E-01

H(3)

80.000000000000E-01 00.000000000000E-99 -12.000000000000E-00 00.000000000000E-99

H(4)

16.000000000000E-00 00.000000000000E-99 -48.000000000000E-00 00.000000000000E-99

12.000000000000E-00

H(5)

32.000000000000E-00 00.000000000000E-99 -16.000000000000E+01 00.000000000000E-99

12.000000000000E+01 00.000000000000E-99

64.000000000000E-00 00.000000000000E-99 -48.000000000000E+01 00.000000000000E-99

72.000000000000E+01 00.000000000000E-99 -12.000000000000E+01

HERMITE (con't)

H(7)

12.800000000000E+01 00.000000000000E-99 -13.440000000000E+02 00.000000000000E-99

33.600000000000E+02 00.000000000000E-99 -16.800000000000E+02 00.000000000000E-99

H(8)

25.600000000000E+01 00.000000000000E-99 -35.840000000000E+02 00.000000000000E-99

13.440000000000E+03 00.000000000000E-99 -13.440000000000E+03 00.000000000000E-99

16.800000000000E+02

H(9)

51.200000000000E+01 00.000000000000E-99 -92.160000000000E+02 00.000000000000E-99

48.384000000000E+03 00.000000000000E-99 -80.640000000000E+03 00.000000000000E-99

30.240000000000E+03 00.000000000000E-99

H(10)

10.240000000000E+02 00.000000000000E-99 -23.040000000000E+03 00.000000000000E-99

16.128000000000E+04 00.000000000000E-99 -40.320000000000E+04 00.000000000000E-99

30.240000000000E+04 00.000000000000E-99 -30.240000000000E+03

H(11)

20.480000000000E+02 00.000000000000E-99 -56.320000000000E+03 00.000000000000E-99

50.688000000000E+04 00.000000000000E-99 -17.740800000000E+05 00.000000000000E-99

22.176000000000E+05 00.000000000000E-99 -66.528000000000E+04 00.000000000000E-99

H(12)

40.960000000000E+02 00.000000000000E-99 -13.516800000000E+04 00.000000000000E-99

15.206400000000E+05 00.000000000000E-99 -70.963200000000E+05 00.000000000000E-99

13.305600000000E+06 00.000000000000E-99 -79.833600000000E+05 00.000000000000E-99

66.528000000000E+04

H(13)

81.920000000000E+02 00.000000000000E-99 -31.948800000000E+04 00.000000000000E-99

43.929600000000E+05 00.000000000000E-99 -26.357760000000E+06 00.000000000000E-99

69.189120000000E+06 00.000000000000E-99 -69.189120000000E+06 00.000000000000E-99

17.297280000000E+06 00.000000000000E-99

H(14)

16.384000000000E+03 00.000000000000E-99 -74.547200000000E+04 00.000000000000E-99

12.300288000000E+06 00.000000000000E-99 -92.252160000000E+06 00.000000000000E-99

32.288256000000E+07 00.000000000000E-99 -48.432384000000E+07 00.000000000000E-99

24.216192000000E+07 00.000000000000E-99 -17.297280000000E+06

H(15)

32.768000000000E+03	00.000000000000E-99	-17.203200000000E+05	00.000000000000E-99
33.546240000000E+06	00.000000000000E-99	-30.750720000000E+07	00.000000000000E-99
13.837824000000E+08	00.000000000000E-99	-29.059430400000E+08	00.000000000000E-99
24.216192000000E+08	00.000000000000E-99	-51.891840000000E+07	00.000000000000E-99

H(16)

65.536000000000E+03	00.000000000000E-99	-39.321600000000E+05	00.000000000000E-99
89.456640000000E+06	00.000000000000E-99	-98.402304000000E+07	00.000000000000E-99
55.351296000000E+08	00.000000000000E-99	-15.498362880000E+09	00.000000000000E-99
19.372953600000E+09	00.000000000000E-99	-83.026944000000E+08	00.000000000000E-99
51.891840000000E+07			

H(17)

13.107200000000E+04	00.000000000000E-99	-89.128960000000E+05	00.000000000000E-99
23.396352000000E+07	00.000000000000E-99	-30.415257600000E+08	00.000000000000E-99
20.910489600000E+09	00.000000000000E-99	-75.277762560000E+09	00.000000000000E-99

13.173608448000E+10	00.000000000000E-99	-94.097203200000E+09	00.000000000000E-99
17.643225600000E+09	00.000000000000E-99		

H(18)

26.214400000000E+04	00.000000000000E-99	-20.054016000000E+06	00.000000000000E-99
60.162048000000E+07	00.000000000000E-99	-91.245772800000E+08	00.000000000000E-99
75.277762560000E+09	00.000000000000E-99	-33.874993152000E+10	00.000000000000E-99
79.041650688000E+10	00.000000000000E-99	-84.687482880000E+10	00.000000000000E-99
31.757806080000E+10	00.000000000000E-99	-17.643225600000E+09	

H(19)

52.428800000000E+04	00.000000000000E-99	-44.826624000000E+06	00.000000000000E-99
15.241052160000E+08	00.000000000000E-99	-26.671841280000E+09	00.000000000000E-99
26.005045248000E+10	00.000000000000E-99	-14.302774886400E+11	00.000000000000E-99
42.908324659200E+11	00.000000000000E-99	-64.362486988800E+11	00.000000000000E-99
40.226554368000E+11	00.000000000000E-99	-67.044257280000E+10	00.000000000000E-99

H(20)

10.485760000000E+05	00.000000000000E-99	-99.614720000000E+06	00.000000000000E-99

HERMITE (con't)

38.102630400000E+08	00.000000000000E-99	-76.205260800000E+09	00.000000000000E-99
86.683484160000E+10	00.000000000000E-99	-57.211099545600E+11	00.000000000000E-99
21.454o62329600E+12	00.000000000000E-99	-42.908324659200E+12	00.000000000000E-99
40.226554368000E+12	00.000000000000E-99	-13.408851456000E+12	00.000000000000E-99
67.044257280000E+10			

CHEBYSHEV COEFFICIENTS

T(0)

10.000000000000E-01

T(1)

10.000000000000E-01 00.000000000000E-99

T(2)

20.000000000000E-01 00.000000000000E-99 -10.000000000000E-01

T(3)

40.000000000000E-01 00.000000000000E-99 -30.000000000000E-01 00.000000000000E-99

T(4)

80.000000000000E-01 00.000000000000E-99 -80.000000000000E-01 00.000000000000E-99

10.000000000000E-01

T(5)

16.000000000000E-00 00.000000000000E-99 -20.000000000000E-00 00.000000000000E-99

50.000000000000E-01 00.000000000000E-99

T(6)

32.000000000000E-00 00.000000000000E-99 -48.000000000000E-00 00.000000000000E-99

18.000000000000E-00 00.000000000000E-99 -10.000000000000E-01

CHEBYSHEV (con't)

T(7)			
64.000000000000E-00	00.000000000000E-99	-11.200000000000E+01	00.000000000000E-99
56.000000000000E-00	00.000000000000E-99	-70.000000000000E-01	00.000000000000E-99
T(8)			
12.800000000000E+01	00.000000000000E-99	-25.600000000000E+01	00.000000000000E-99
16.000000000000E+01	00.000000000000E-99	-32.000000000000E-00	00.000000000000E-99
10.000000000000E-01			
T(9)			
25.600000000000E+01	00.000000000000E-99	-57.600000000000E+01	00.000000000000E-99
43.200000000000E+01	00.000000000000E-99	-12.000000000000E+01	00.000000000000E-99
90.000000000000E-01	00.000000000000E-99		
T(10)			
51.200000000000E+01	00.000000000000E-99	-12.800000000000E+02	00.000000000000E-99
11.200000000000E+02	00.000000000000E-99	-40.000000000000E+01	00.000000000000E-99
50.000000000000E-00	00.000000000000E-99	-10.000000000000E-01	

CHEBYSHEV (con't)

T(11)

10.240000000000E+02 00.000000000000E-99 -28.160000000000E+02 00.000000000000E-99

28.160000000000E+02 00.000000000000E-99 -12.320000000000E+02 00.000000000000E-99

22.000000000000E+01 00.000000000000E-99 -11.000000000000E-00 00.000000000000E-99

T(12)

20.480000000000E+02 00.000000000000E-99 -61.440000000000E+02 00.000000000000E-99

69.120000000000E+02 00.000000000000E-99 -35.840000000000E-02 00.000000000000E-99

84.000000000000E+01 00.000000000000E-99 -72.000000000000E-00 00.000000000000E-99

10.000000000000E-01

T(13)

40.960000000000E+02 00.000000000000E-99 -13.312000000000E+03 00.000000000000E-99

16.640000000000E+03 00.000000000000E-99 -99.840000000000E+02 00.000000000000E-99

29.120000000000E+02 00.000000000000E-99 -36.400000000000E+01 00.000000000000E-99

13.000000000000E-00 00.000000000000E-99

T(14)

81.920000000000E+02 00.000000000000E-99 -28.672000000000E+03 00.000000000000E-99

CHEBYSHEV (con't)

39.424000000000E+03 00.000000000000E-99 -26.880000000000E+03 00.000000000000E-99

94.080000000000E+02 00.000000000000E-99 -15.680000000000E+02 00.000000000000E-99

98.000000000000E-00 00.000000000000E-99 -10.000000000000E-01

T(15)

16.384000000000E+03 00.000000000000E-99 -61.440000000000E+03 00.000000000000E-99

92.160000000000E+03 00.000000000000E-99 -70.400000000000E+03 00.000000000000E-99

28.800000000000E+03 00.000000000000E-99 -60.480000000000E+02 00.000000000000E-99

56.000000000000E+01 00.000000000000E-99 -15.000000000000E-00 00.000000000000E-99

T(16)

32.768000000000E+03 00.000000000000E-99 -13.107200000000E+04 00.000000000000E-99

21.299200000000E+04 00.000000000000E-99 -18.022400000000E+04 00.000000000000E-99

84.480000000000E+03 00.000000000000E-99 -21.504000000000E+03 00.000000000000E-99

26.880000000000E+02 00.000000000000E-99 -12.800000000000E+01 00.000000000000E-99

10.000000000000E-01

T(17)

65.536000000000E+03 00.000000000000E-99 -27.852800000000E+04 00.000000000000E-99

CHEBYSHEV (con't)

48.742400000000E+04	00.000000000000E-99	-45.260800000000E+04	00.000000000000E-99
23.936000000000E+04	00.000000000000E-99	-71.808000000000E+03	00.000000000000E-99
11.424000000000E+03	00.000000000000E-99	-81.600000000000E+01	00.000000000000E-99
17.000000000000E-00	00.000000000000E-99		

T(18)

13.107200000000E+04	00.000000000000E-99	-58.982400000000E+04	00.000000000000E-99
11.059200000000E+05	00.000000000000E-99	-11.182080000000E+05	00.000000000000E-99
65.894400000000E+04	00.000000000000E-99	-22.809600000000E+04	00.000000000000E-99
44.352000000000E+03	00.000000000000E-99	-43.200000000000E+02	00.000000000000E-99
16.200000000000E+01	00.000000000000E-99	-10.000000000000E-01	

T(19)

26.214400000000E+04	00.000000000000E-99	-12.451840000000E+05	00.000000000000E-99
24.903680000000E+05	00.000000000000E-99	-27.238400000000E+05	00.000000000000E-99
17.704960000000E+05	00.000000000000E-99	-69.555200000000E+04	00.000000000000E-99
16.051200000000E+04	00.000000000000E-99	-20.064000000000E+03	00.000000000000E-99
11.400000000000E+02	00.000000000000E-99	-19.000000000000E-00	00.000000000000E-99

CHEBYSHEV (con't)

T(20)

52.428800000000E+04	00.000000000000E-99	-26.214400000000E+05	00.000000000000E-99
55.705600000000E+05	00.000000000000E-99	-65.536000000000E+05	00.000000000000E-99
46.592000000000E+05	00.000000000000E-99	-20.500480000000E+05	00.00000000000E-99
54.912000000000E+04	00.000000000000E-99	-84.480000000000E+03	00.00000000000E-99
66.000000000000E+02	00.000000000000E-99	-20.000000000000E+01	00.00000000000E-99
10.000000000000E-01			

JACOBI COEFFICIENTS WITH

Alpha = 1 and Beta = 0

P(0)

10.000000000000E-01

P(1)

15.000000000000E-01 50.000000000000E-02

P(2)

25.000000000000E-01 10.000000000000E-01 -50.000000000000E-02

P(3)

43.750000000000E-01 18.750000000000E-01 -18.750000000000E-01 -37.500000000000E-02

P(4)

78.750000000000E-01 35.000000000000E-01 -52.500000000000E-01 -15.000000000000E-01

37.500000000000E-02

P(5)

14.437500000000E-00 65.625000000000E-01 -13.125000000000E-00 -43.750000000000-01

21.875000000000E-01 31.250000000000E-02

JACOBI (con't)

P(6)

26.812500000000E-00 12.375000000000E-00 -30.937500000000E-00 -11.250000000000E-00

84.375000000000E-01 18.750000000000E-01 -31.250000000000E-02

P(7)

50.273437500000E-00 23.460937500000E-00 -70.382812500000E-00 -27.070312500000E-00

27.070312500000E-00 73.828125000000E-01 -24.609375000000E-01 -27.343750000000E-02

P(8)

94.960937500000E-00 44.687500000000E-00 -15.640625000000E+01 -62.562500000000E-00

78.203125000000E-00 24.062500000000E-00 -12.031250000000E-00 -21.875000000000E-01

27.343750000000E-02

P(9)

18.042578125000E+01 85.464843750000E-00 -34.185937500000E+01 -14.076562500000E+01

21.114843750000E+01 70.382812500000E-00 -46.921875000000E-00 -10.828125000000E-00

27.070312500000E-01 24.609375000000E-02

P(10)

34.444921875000E+01 16.402343750000E+01 -73.810546875000E+01 -31.078125000000E+01

54.386718750000E+01 19.195312500000E+01 -15.996093750000E+01 -43.656250000000E-00

15.996093750000E-00 24.609375000000E-01 -24.609375000000E-02

P(11)

66.019433593750E+01 31.574511718750E+01 -15.787255859375E+02 -67.659667968750E+01

13.531933593750E+02 49.854492187500E+01 -49.854492187500E+01 -14.663085937500E+01

73.315429687500E-00 14.663085937500E-00 -29.326171875000E-01 -22.558593750000E-02

P(12)

12.696044921875E+02 60.941015625000E+01 -33.517558593750E+02 -14.572851562500E+02

32.788916015625E+02 12.491015625000E+02 -14.572851562500E+02 -46.019531250000E+01

28.762207031250E+01 67.675781250000E-00 -20.302734375000E-00 -27.070312500000E-01

22.558593750000E-02

P(13)

24.485229492187E+02 11.789184570312E+02 -70.735107421874E+02 -31.123447267624E+02

77.808618164062E+02 30,446850585937E+02 -40.595800781250E+02 -13.531933593750E+02

10.148950195312E+02 26.707763671874E+01 -10.683105468750E+01 -18.852539062500E-00

31.420898437500E-01 20.947265625000E-02

P(14)

47.338110351562E+02	22.852880859374E+02	-14.854372558593E+03	-66.019433593749E+02
18.155344238281E+03	72.621376953124E+02	-10.893206542968E+03	-37.889414062500E+02
33.153237304687E+02	94.723535156249E+01	-47.361767578125E+01	-99.708984375000E-00
24.927246093750E-00	29.326171875000E-01	-20.947265625000E-02	

P(15)

91.717588806151E+02	44.379478454589E+02	-31.065634918212E+03	-13.925974273681E+03
41.777922821044E+03	17.020635223388E+03	-28.367725372314E+03	-10.212381134033E+03
10.212381134033E+03	31.081159973144E+02	-18.648695983886E+02	-44.401657104492E+01
14.800552368164E+01	23.369293212890E-00	-33.384704589843E-01	-19.638061523437E-02

P(16)

17.804002532958E+03	86.322436523435E+02	-64.741827392577E+03	-29.238244628905E+03
95.024295043944E+03	39.320397949218E+03	-72.087396240233E+03	-26.699035644530E+03
30.036415100097E+03	96.116528320311E+02	-67.281569824217E+02	-17.551713867187E+02
73.132141113280E+01	13.929931640624E+01	-29.849853515625E-00	-31.420898437499E-01
19.638061523437E-02			

P(17)

34.618893814086E+03	16.814891281127E+03	-13.451913024902E+04	-61.145059204098E+03
21.400770721435E+04	89.745167541501E+03	-17.949033508300E+04	-68.082540893552E+03
85.103176116941E+03	28.367725372314E+03	-22.694180297851E+03	-63.543704833983E+02
31.771852416991E+02	69.069244384764E+01	-19.734069824218E+01	-28.191528320312E-00
35.239410400390E-01	18.547058105468E-02		

P(18)

67.415740585325E+03	32.796846771239E+03	-27.877319755553E+04	-12.743917602538E+04
47.789691009520E+04	20.274414367675E+04	-43.927897796630E+04	-17.--4347534179E+04
23.380977859496E+04	80.624061584470E+03	-72.561655426024E+03	-21.499749755859E+03
12.541520690917E+03	30.099649658202E+02	-10.749874877929E+02	-18.695434570312E+01
35.053939819335E-00	33.384704589843E-01	-18.547058105468E-02	

P(19)

13.146069414138E+04	64.044953556058E+03	-57.640458200453E+04	-26.483453767775E+04
10.593381507110E+05	45.400206459043E+04	-10.593381507110E+05	-41.731502906797E+04
62.597254360198E+04	22.211928966521E+04	-22.211928966521E+04	-68.933572654722E+03

JACOBI (con't)

45.955715103148E+03 11.914444656371E+03 -51.061905670165E+02 -10.212381134033E+02

25.530952835082E+01 33.301242828368E-00 -37.001380920409E-01 -17.619705200195E-02

P(20)

25.666135522841E+04 12.520066108703E+04 -11.894062803268E+05 -54.895674476620E+04

23.330661652564E+05 10.088934768676E+05 -25.222336921691E+05 -10.088934768676E+05

16.394518999099E+05 59.616432723996E+04 -65.578075996397E+04 -21.154218063353E+04

15.865663547515E+04 43.767347717284E+03 -21.883673858642E+03 -48.630396352538E+02

15.196995735168E+02 24.315193176269E+01 -40.525321960448E-00 -35.239410400390E-01

17.619705200195E-02

JACOBI COEFFICIENTS WITH

Alpha = 0 and Beta = 1

P(0)

10.000000000000E-01

P(1)

15.000000000000E-01 -50.000000000000E-02

P(2)

25.000000000000E-01 -10.000000000000E-01 -50.000000000000E-02

P(3)

43.750000000000E-01 -18.750000000000E-01 -18.750000000000E-01 37.500000000000E-02

P(4)

78.750000000000E-01 -35.000000000000E-01 -52.500000000000E-01 15.000000000000E-01

37.500000000000E-02

P(5)

14.437500000000E-00 -65.625000000000E-01 -13.125000000000E-00 43.750000000000E-01

21.875000000000E-01 -31.250000000000E-02

JACOBI (con't)

P(6)

26.812500000000E-00 -12.375000000000E-00 -30.937500000000E-00 11.250000000000E-00

84.375000000000E-01 -18.750000000000E-01 -31.250000000000E-02

P(7)

50.273437500000E-00 -23.460937500000E-00 -70.382812500000E-00 27.070312500000E-00

27.070312500000E-00 -73.828125000000E-01 -24.609375000000E-01 27.343750000000E-02

P(8)

94.960937500000E-00 -44.687500000000E-00 -15.640625000000E+01 62.562500000000E-00

78.203125000000E-00 -24.062500000000E-00 -12.031250000000E-00 21.875000000000E-01

27.343750000000E-02

P(9)

18.042578125000E+01 -85.464843750000E-00 -34.185937500000E+01 14.076562500000E+01

21.114843750000E+01 -70.382812500000E-00 -46.921875000000E-00 10.828125000000E-00

27.070312500000E-01 -24.609375000000E-02

P(10)

34.444921875000E+01 -16.402343750000E+01 -73.810546875000E+01 31.078125000000E+01

JACOBI (con't)

54.386718750000E+01 -19.195312500000E+01 -15.996093750000E+01 42.656250000000E-00

15.996093750000E-00 -24.609375000000E-01 -24.609375000000E-02

P(11)

66.019433593750E+01 -31.574511718750E+01 -15.787255859375E+02 67.659667968750E+01

13.531933593750E+02 -49.854492187500E+01 -49.854492187500E+01 14.663085937500E+01

73.315429687500E-00 -14.663085937500E-00 -29.326171875000E-01 22.558593750000E-02

P(12)

12.696044921875E+02 -60.941015625000E+01 -33.517558593750E+02 14.572851562500E+02

32.788916015625E+02 -12.491015625000E+02 -14.572851562500E+02 46.019531250000E+01

28.762207031250E+01 -67.675781250000E-00 -20.302734375000E-00 27.070312500000E-01

22.558593750000E-02

P(13)

24.485229492187E+02 -11.789184570312E+02 -70.735107421874E+02 31.123447265624E+02

77.808618164062E+02 -30.446850585937E+02 -40.595800781250E+02 13.531933593750E+02

10.148950195312E+02 -26.707763671874E+01 -10.683105468750E+01 18.852539062500E-00

31.420898437500E-01 -20.947265625000E-02

P(14)

47.228110351562E+02	-22.852880859374E+02	-14.854372558593E+03	66.019433593749E+02
18.155344238281E+03	-72.621376953124E+02	-10.893206542968E+03	37.889414062500E+02
33.153237304687E+02	-94.723535156249E+01	-47.361767578125E+01	99.708984375000E-00
24.927246093750E-00	-29.326171875000E-01	-20.947265625000E-02	

P(15)

91.717588806151E+02	-44.379478454589E+02	-31.065634918212E+03	13.925974273681E+03
41.777922821044E+03	-17.020635223388E+03	-28.367725372314E+03	10.212381134033E+03
10.212381134033E+03	-31.081159973144E+02	-18.648695983886E+02	44.401657104492E+01
14.800552368164E+01	-23.369293212890E-00	-33.384704589843E-01	19.638061523437E-02

P(16)

17.804002532958E+03	-86.322436523435E+02	-64.741827392577E+03	29.238244628905E+03
95.024295043944E+03	-39.320397949218E+03	-72.087396240233E+03	26.699035644530E+03
30.036415100097E+03	-96.116528320311E+02	-67.281569824217E+02	17.551713867187E+02
73.132141113280E+01	-13.929931640624E+01	-29.849853515625E-00	31.420898437499E-01
19.630861523437E-02			

JACOBI (con't)

P(17)

34.618893814086E+03 -16.814891281127E+03 -13.451913024902E+04 61.145059204098E+03

21.400770721435E+04 -89.745167541501E+03 -17.949033508300E+04 68.082540893552E+03

85.103176116941E+03 -28.367725372314E+03 -22.694180297851E+03 63.543704833983E+02

31.771852416991E+02 -69.069244384764E+01 -19.734069824218E+01 28.191528320312E-00

35.239410400390E-01 -18.547058105468E-02

P(18)

67.415740585325E+03 -32.796846771239E+03 -27.877319755553E+04 12.743917602538E+04

47.789691009520E+04 -20.274414367675E+04 -43.927897796630E+04 17.004347534179E+04

23.380977859496E+04 -80.624061584470E+03 -72.561655426024E+03 21.499749755859E+03

12.541520690917E+03 -30.099649658202E+02 -10.749874877929E+02 18.695434570312E+01

35.053939819335E-00 -33.384704589843E-01 -18.547058105468E-02

P(19)

13.146069414138E+04 -64.044953556058E+03 -57.640458200453E+04 26.483453767775E+04

10.593381507110E+05 -45.400206459043E+04 -10.593381507110E+05 41.731502906797E+04

62.597254360198E+04 -22.211928966521E+04 -22.211928966521E+04 68.933572654722E+03

45.955715103148E+03	-11.914444656371E+03	-51.061905670165E+02	10.212381134033E+02
25.530952835082E+01	-33.301242828368E-00	-37.001380920409E-01	17.619705200195E-02

P(20)

25.666135522841E+04	-12.520066108703E+04	-11.894062803268E+05	54.895674476620E+04
23.330661652564E+05	-10.088934768676E+05	-25.222336921691E+05	10.088934768676E+05
16.394518999099E+05	-59.616432723996E+04	-65.578075996397E+04	21.154218063353E+04
15.865663547515E+04	-43.767347717284E+03	-21.883673858642E+03	48.630386352538E+02
15.196995735168E+02	-24.315193176269E+01	-40.525321960448E-00	35.239410400390E-01
17.619705200195E-02			

APPENDIX II

The zeros of a polynomial S(x) are the values of x for which the curve y = S(x) touches the x-axis. That is, we want all the values of x that satisfy the equation S(x) = 0. For example, take the polynomial $S(x) = 1.5\,x^2 - .5$, which is the second degree Legendre polynomial. If we solve the following equation

$$1.5\,x^2 - .5 = 0$$

we get that

$$x = \pm\,\frac{1}{\sqrt{3}} \simeq \pm\,.57735026918962 \quad .$$

The weights for the Legendre-Gauss, Laguerre-Gauss, Hermite-Gauss, Chebyshev-Gauss, and Jacobi-Gauss, are defined by equations (2.68), (2.73), (2.77), (2.80), and (2.83) respectively. All the weights are positive real numbers.

The following pages give the zeros and associated weights for the different types and degree, of the previously described orthogonal polynomials. The zeros were obtained by using a subroutine called ROTPOL.

ZEROS OF LEGENDRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

n	Abscissas X_i	Weights H_i
2	±57.735026918962E-02	10.000000000000E-01
3	±77.459666924148E-02	55.555555555554E-02
	±00.000000000000E-99	88.888888888888E-02
4	±86.113631159409E-02	34.785484513781E-02
	±33.998104358474E-02	65.214515486232E-02
5	±90.617984593866E-02	23.692688505611E-02
	±53.846931010568E-02	47.862867049937E-02
	±00.000000000000E-99	56.888888888889E-02
6	±93.246951420314E-02	17.132449237901E-02
	±66.120938646626E-02	36.076157304816E-02
	±23.861918608319E-02	46.791393457270E-02
7	±94.910701234277E-02	12.948496616898E-02
	±74.153118559937E-02	27.970539148916E-02
	±40.584515137739E-02	38.183005050512E-02
	±00.000000000000E-99	41.795918367347E-02
8	±96.028985650372E-02	10.122853641391E-02
	±79.666647740617E-02	22.238103439554E-02
	±52.553240991632E-02	31.370664587787E-02
	±18.343464249565E-02	36.268378337837E-02
9	±96.816023950760E-02	81.274388360880E-03
	±83.603110732666E-02	18.064816069510E-02
	±61.337143270058E-02	26.061069640289E-02

ZEROS OF LEGENDRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
9	±32.425342340380E-02	31.234707704000E-02
	±00.000000000000E-99	33.023935500126E-02
10	±97.390652851715E-02	66.671344308086E-03
	±86.506336668894E-02	14.945134915012E-02
	±67.940956829913E-02	21.908636251659E-02
	±43.339539412918E-02	26.926671930981E-02
	±14.887433898163E-02	29.552422471476E-02
11	±97.822865814728E-02	55.668567150347E-03
	±88.706259976674E-02	12.558036944929E-02
	±73.015200557406E-02	18.629021092781E-02
	±51.909612920680E-02	23.319376459198E-02
	±26.954315595234E-02	26.280454451025E-02
	±00.000000000000E-99	27.292508677790E-02
12	±98.156063426123E-02	47.175336827924E-03
	±90.411725633678E-02	10.693932556703E-02
	±76.990267422349E-02	16.007832875533E-02
	±58.731795427617E-02	20.316742667739E-02
	±36.783149899785E-02	23.349253653759E-02
	±12.523340851147E-02	24.914704581341E-02
13	±98.418305471372E-02	40.484004604642E-03
	±91.759839923163E-02	92.121499958120E-03
	±80.157809072943E-02	13.887351018842E-02

ZEROS OF LEGENDRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
13	±64.234933944027E-02	17.814598076158E-02
	±44.849275103644E-02	20.781604753689E-02
	±23.045831595513E-02	22.628318026290E-02
	±00.000000000000E-00	23.255155323087E-02
14	±98.628380874806E-02	35.119462156149E-03
	±92.843488355031E-02	80.158085449565E-03
	±82.720131516011E-02	12.151857149330E-02
	±68.729290478126E-02	15.720316698375E-02
	±51.524863636018E-02	18.553839748535E-02
	±31.911236892708E-02	20.519846371966E-02
	±10.805494870734E-02	21.526385346316E-02
15	±98.799251802032E-02	30.753241989599E-03
	±93.727339240159E-02	70.366047502452E-03
	±84.820658340891E-02	10.715922045247E-02
	±72.441773136125E-02	16.626920581580E-02
	±57.097217260825E-02	16.626920581580E-02
	±39.415134707754E-02	18.616100001551E-02
	±20.119409399743E-02	19.843148532712E-02
	±00.000000000000E-99	20.257824192556E-02
16	±98.940093500506E-02	27.152459958347E-03
	±94.457502304491E-02	62.253523444163E-03
	±86.561320240834E-02	95.158511898202E-03

ZEROS OF LEGENDRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
16	±75.540440835168E-02	12.462897123220E-02
	±61.787624439784E-02	14.959598879363E-02
	±41.801677766067E-02	16.915651940582E-02
	±28.160355077788E-02	18.260341504248E-02
	±95.012509837638E-03	18.945061045507E-02
17	±99.057547537642E-02	24.148305556368E-03
	±95.067552165945E-02	55.459527337226E-03
	±88.023915380736E-02	85.036149225578E-03
	±78.151400384486E-02	11.188384679670E-02
	±65.767115923988E-02	13.513636859201E-02
	±51.269053708083E-02	15.404576105626E-02
	±35.123176345385E-02	16.800410215641E-02
	±17.848418149584E-02	17.656270536700E-02
	±00.000000000000E-99	17.944647035621E-02
18	±99.156516840450E-02	21.616012771722E-03
	±95.582394960436E-02	49.714549547706E-03
	±89.260246647605E-02	76.425729995109E-03
	±80.370495897483E-02	10.094204412533E-02
	±69.168704306714E-02	12.255510675119E-02
	±55.977083106657E-02	14.064291464021E-02
	±41.175116146749E-02	15.468467513967E-02
	±25.188622568960E-02	16.427648374282E-02

ZEROS OF LEGENDRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
18	±84.775013041736E-03	16.914238296315E-02
19	±99.240684434915E-02	19.461812756076E-03
	±96.020815095253E-02	44.814201980325E-03
	±90.315590470852E-02	69.044556798974E-03
	±82.271465600663E-02	91.490016926636E-03
	±72.096617742904E-02	11.156664614446E-02
	±60.054530470736E-02	12.875396274928E-02
	±46.457074134828E-02	14.260670208475E-02
	±31.656409996356E-02	15.276604206573E-02
	±16.035864564022E-02	15.896884339396E-02
	±00.000000000000E-99	16.105444984879E-02
20	±99.312859933787E-02	17.614014944386E-03
	±96.397192693096E-02	40.601422123940E-03
	±91.223442855514E-02	62.672052473948E-03
	±83.911697168489E-02	83.276740279412E-03
	±74.633190648620E-02	10.193011999617E-02
	±63.605368073385E-02	11.819453199855E-02
	±51.086700194084E-02	13.168863841282E-02
	±37.370608872164E-02	14.209610933376E-02
	±22.778585113930E-02	14.917298646926E-02
	±76.526521133498E-03	15.275338713073E-02

ZEROS OF LAGUERRE POLYNOMIALS AND CORRESPONDING WEIGHTS

n	Abscissas X_i	Weights H_i
2	58.578643762690E-02	85.355339059328E-02
	34.142135623730E-01	14.644660940672E-02
3	62.899450829386E-01	10.389256501572E-03
	41.577455678349E-02	71.109300992898E-02
	22.942803602778E-01	27.851773357028E-02
4	45.366202969211E-01	38.887908515001E-03
	93.950709123010E-01	53.929470556140E-05
	32.254768961939E-02	60.315410434161E-02
	17.457611011583E-01	35.741869243780E-02
5	12.640800844357E-00	23.369972384124E-06
	35.964257710429E-01	75.942449681199E-03
	70.858100057754E-01	36.117586803894E-04
	26.356031971814E-02	52.175561058270E-02
	14.134030591064E-01	39.866681108341E-02
6	98.374674183855E-01	26.101720281390E-05
	15.982873980601E-00	89.854790643015E-08
	29.927363260659E-01	11.337338207079E-02
	57.751435690984E-01	10.399197453290E-03
	22.284660417926E-02	45.896467394992E-02
	11.889321016698E-01	41.700083078467E-02
7	19.395727862260E-00	31.703154789996E-09
	81.821534445591E-01	10.740101432881E-04

ZEROS OF LAGUERRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
7	12.734180291802E-00	15.865464348471E-06
	25.678767449506E-01	14.712634865759E-02
	49.003530845272E-01	20.633514468672E-03
	19.304367656036E-02	40.931895170126E-02
	10.266648953391E-01	42.183127786186E-02
8	15.740678641283E-00	84.857467162189E-08
	22.863131736888E-00	10.480011748718E-10
	70.459054024013E-01	27.945362351725E-04
	10.758516010170E-00	90.765087735130E-06
	22.510866298665E-01	17.579498663664E-02
	42.667001702850E-01	33.343492261566E-03
	17.027963230510E-02	36.918858934159E-02
	90.370177679934E-02	41.878678081469E-02
9	26.374071891148E-00	32.908740298269E-12
	13.466236912440E-00	65.921230135376E-07
	18.833597788010E-00	41.107693344174E-09
	62.049567778771E-01	55.996266107860E-04
	93.729852510983E-01	30.524976745781E-05
	20.051351556193E-01	19.928752537095E-02
	37.834739733322E-01	47.460562765419E-03
	15.232222773180E-02	33.612642179810E-02
	80.722002274224E-02	41.121398042419E-02

ZEROS OF LAGUERRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
10	21.996585812112E-00	18.395648237491E-10
	29.920697012232E-00	99.118272198942E-14
	11.843785838044E-00	28.259233488748E-06
	16.279257831207E-00	42.493139858974E-08
	55.524961401132E-01	95.015169734043E-04
	83.301527466631E-01	75.300838877999E-05
	18.083429017405E-01	21.806828761126E-02
	34.014336978422E-01	62.087456103520E-03
	13.779347054049E-02	30.844111576543E-02
	72.945454950315E-02	40.111992915573E-02
12	28.487967251580E-00	30.616016334322E-13
	37.099121044354E-00	81.480774680447E-17
	17.116855188368E-00	16.684938743329E-08
	22.151090378298E-00	13.423910321790E-10
	96.213168425451E-01	20.323159261638E-05
	13.006054992964E-00	83.650558623196E-07
	45.992276394346E-01	20.102381152856E-03
	68.445254530656E-01	26.639735423472E-04
	15.126102697767E-01	24.408201131868E-02
	28.337513377403E-01	90.449222214235E-03
	11.572211735802E-02	26.473137105583E-02
	61.175748451512E-02	37.775927587335E-02

ZEROS OF LAGUERRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
14	35.149443674073E-00	42.213523935651E-16
	44.366081710019E-00	60.523750266347E-20
	22.723381759926E-00	68.193135466882E-11
	28.272981663362E-00	32.313079497588E-13
	14.210805106464E-00	24.095852954578E-07
	18.104892064761E-00	58.015454262979E-09
	81.402491422919E-01	73.989037595162E-05
	10.916499482798E-00	54.907198251840E-06
	39.321028223144E-01	33.192092154140E-03
	58.255362181238E-01	61.928694424899E-04
	13.006291212515E-01	25.873461024514E-02
	24.308010787298E-01	11.548289355836E-02
	99.747507032597E-03	23.181557714662E-02
	52.685764885191E-02	35.378469159737E-02
16	41.940452658793E-00	50.504736559059E-19
	51.701160338417E-00	41.614623733626E-23
	28.578729875424E-00	21.270787076804E-13
	34.583398652877E-00	62.979672994472E-16
	19.180157127314E-00	18.810239654468E-09
	23.515905461474E-00	28.623511769384E-11
	12.214223461357E-00	14.844583164461E-06
	15.441527165032E-00	68.283223041374E-08

ZEROS OF LAGUERRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
16	70.703385372324E-01	18.490709246760E-04
	94.383143146490E-01	20.427193083737E-05
	34.370866339134E-01	47.328928684945E-03
	50.780186141975E-01	11.299900106206E-03
	11.410577748314E-01	26.579577764281E-02
	21.292836450925E-01	13.629693430877E-02
	87.649410478928E-03	20.615171495841E-02
	46.269632891508E-02	33.105785495183E-02
18	48.833923125314E-00	54.053967756352E-22
	59.090546394244E-00	26.916533394092E-26
	34.627930061116E-00	53.866475075936E-16
	41.041815226405E-00	10.498674416688E-18
	24.440683093281E-00	10.717084536700E-11
	29.168205487708E-00	10.891031000705E-13
	16.689306256520E-00	19.146699645336E-08
	20.310767273873E-00	58.360990151374E-10
	10.737990051240E-00	56.169649837968E-06
	13.513656207772E-00	40.153078813166E-07
	62.567250760845E-01	36.601797215389E-04
	83.278251536825E-01	54.062279402894E-05
	30.543531136577E-01	61.434917522276E-03
	45.042055376575E-01	17.687213259825E-03

ZEROS OF LAGUERRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
18	10.165201796237E-01	26.786656714679E-02
	18.948885098907E-01	15.297974770418E-02
	78.169166669705E-03	18.558860314736E-02
	41.249008525912E-02	31.018176637061E-02
20	55.810795975734E-00	52.864418491645E-25
	66.524416506743E-00	16.564566317662E-29
	40.833060583976E-00	11.550102488338E-18
	47.619992923798E-00	15.395236298495E-21
	29.932573221301E-00	47.674078582187E-14
	35.013425463769E-00	33.728789075160E-16
	21.478827931655E-00	17.578479929495E-10
	25.451671242490E-00	37.256917580063E-12
	14.814313789314E-00	10.864251875855E-07
	17.948860531780E-00	53.305469377700E-09
	95.943938991341E-01	15.574108782568E-05
	12.038795571998E-00	15.401806722734E-06
	56.151749745837E-01	62.025506760030E-04
	74.590174775048E-01	11.449622364787E-04
	27.491992602391E-01	74.826059167702E-03
	40.489253063720E-01	24.964419199887E-03
	91.658210286914E-02	26.668609826403E-02
	17.073065290478E-01	16.600246116467E-02

ZEROS OF LAGUERRE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
20	70.539889691989E-03	16.874680185473E-02
	37.212681800154E-02	29.125436200913E-02

ZEROS OF HERMITE POLYNOMIALS

AND CORRESPONDING WEIGHTS

n	Abscissas X_i	Weights H_i
2	± .707106581186	88.622692545275E-02
3	±1.224748871391	29.540897515091E-02
	±0.000000000000	11.816359006036E-01
4	±1.650680123885	81.312835447244E-03
	± .524647623275	80.491409000551E-02
5	±2.020182870456	19.953242059046E-03
	± .958572464613	39.361932315224E-02
	±0.000000000000	94.530872048293E-02
6	±2.350604973674	45.300099055088E-04
	±1.335849074013	15.706732032285E-02
	± .436077411927	72.462959522439E-02
7	±2.651961356835	97.178124509952E-05
	±1.673551628767	54.515582819127E-03
	± .816287882858	42.560725261013E-02
	±0.000000000000	81.026461755680E-02
8	±2.930637420257	19.960407221136E-05
	±1.981656756695	17.077983007413E-03
	±1.157193712446	20.780232581489E-02
	± .381186990207	66.114701255823E-02
9	±3.190993201781	39.606977263263E-06
	±2.266580584531	49.436242755369E-04
	±1.468553289216	88.474527394375E-03

ZEROS OF HERMITE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
	± .723551018752	43.265155900255E-02
	±0.000000000000	72.023521560604E-02
10	±3.436159118837	76.404328552323E-07
	±2.532731674232	13.436457467812E-04
	±1.756683649299	33.874394455481E-03
	±1.036610829789	24.013861108231E-02
	± .342901327223	61.086263373532E-02
11	±3.668470846557	14.395603937141E-07
	±2.783290099787	34.681946632334E-05
	±2.025948015821	11.911395444911E-03
	±1.326557084494	11.722787516771E-02
	± .656809566882	42.935975235613E-02
	±0.000000000000	65.475928691458E-02
12	±3.889724897872	26.585516843563E-08
	±3.020637025102	85.736870435879E-06
	±2.279507080537	39.053905846290E-04
	±1.597682635128	51.607985615885E-03
	± .947788391240	26.049231026416E-02
	± .314240376254	57.013523626248E-02
13	±4.101337596178	48.257318500732E-09
	±3.246608978372	20.430360402707E-06
	±2.519735685677	12.074599927193E-04

ZEROS OF HERMITE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
	±1.853107651602	20.862775296170E-03
	±1.220055036590	14.032332068702E-02
	± .605763879170	42.161629689855E-02
	±0.000000000000	60.439318792115E-02
14	±4.304448570474	86.285911681241E-10
	±3.462656933600	47.164843550185E-07
	±2.748470724986	35.509261355190E-05
	±2.095183258507	78.500547264576E-04
	±1.476682731141	68.505534223460E-03
	± .878713787329	27.310560906423E-02
	± .291745510672	53.640590971205E-02
15	±4.499990707314	15.224758042534E-10
	±3.669950373392	10.591155477110E-07
	±2.967166927923	10.000444123249E-05
	±2.325732486152	27.780688429126E-04
	±1.719991575206	30.780033872546E-03
	±1.136115585200	15.848891579593E-02
	± .565069583255	41.202868749889E-02
	±0.000000000000	56.410030872640E-02
16	±4.688738939319	26.548074740111E-11
	±3.869447904857	23.209808448649E-08
	±3.176999161938	27.118600925377E-06

ZEROS OF HERMITE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
	±2.546202157898	93.228400862409E-05
	±1.951787990891	12.880311535509E-03
	±1.380258539198	83.810041398979E-03
	± .822951449144	28.064745852852E-02
	± .273481046138	50.792947901658E-02
17	±4.871345193675	45.805789307986E-12
	±4.061946675873	49.770789816307E-09
	±3.378932091145	71.122891400212E-07
	±2.757762915697	29.864328669775E-05
	±2.173502826673	50.673499576275E-04
	±1.612924314218	40.920034149757E-03
	±1.067648725743	17.264829767009E-02
	± .531633001342	40.182646947041E-02
	±0.000000000000	53.091793762485E-02
18	±5.048364008875	78.281997721155E-13
	±4.248117873564	10.467205795791E-09
	±3.573769068491	18.106544810933E-07
	±2.961377505527	91.811268679289E-06
	±2.386299089167	18.885226302682E-04
	±1.835531604261	18.640042387543E-03
	±1.300920858389	97.301747641312E-03
	± .776682919267	28.480728566996E-02

ZEROS OF HERMITE POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
	± .258267750519	48.349569472545E-02
19	±1.000000000000	18.441912560820E-02
	±5.000000000000	44.374960535029E-13
	±2.347363542497	68.810669316280E-04
	±0.000000000000	50.297488827618E-02
	± .637324199346	94.078786072845E-02
	± .496581175197	39.236244369029E-02
	±0.000000000000	50.297488827618E-02
20	±5.387480890001	22.293936455340E-14
	±4.603682449649	43.993409922730E-11
	±3.944764039827	10.860693707692E-08
	±3.347854567718	78.025564785316E-07
	±2.788806058298	22.833863601634E-05
	±2.254974002077	32.437733422377E-04
	±1.738537712117	24.810520887462E-03
	±1.234076215395	10.901720602001E-02
	± .737473728545	28.667550536282E-02
	± .245340708300	46.224366960061E-02

ZEROS OF CHEBYSHEV POLYNOMIALS AND CORRESPONDING WEIGHTS

n	Abscissas X_i	Weights H_i
2	±70.710678118654E-02	15.707963267948E-01
3	±86.602540378443E-02	10.471975511965E-01
	±00.000000000000E-99	
4	±92.387953251128E-02	78.539816339744E-02
	±38.268343236508E-02	
5	±95.105651629516E-02	62.831853071795E-02
	±58.778525229245E-02	
	±00.000000000000E-99	
6	±96.592582628908E-02	52.359877559829E-02
	±70.710678118653E-02	
	±25.889104510252E-02	
7	±97.492791218194E-02	44.879895051282E-02
	±78.183148246789E-02	
	±43.388373911755E-02	
	±00.000000000000E-99	
8	±98.078528040323E-02	39.269908169872E-02
	±83.146961230254E-02	
	±55.557023301959E-02	
	±19.509032201612E-02	
9	±98.480775301224E-02	34.906585039886E-02
	±86.602540378440E-02	
	±64.278760968654E-02	

ZEROS OF CHEBYSHEV POLYNOMIALS
AND CORRESPONDING WEIGHTS
(con't)

n	Abscissas X_i	Weights H_i
	±34.202014332566E-02	34.906585039886E-02
	±00.000000000000E-99	
10	±98.768834059546E-02	31.415926535897E-02
	±89.100652418801E-02	
	±70.710678118668E-02	
	±45.399049973934E-02	
	±15.643446504923E-02	
11	±98.982144188120E-02	28.559933214452E-02
	±90.982144188120E-02	
	±75.574957435430E-02	
	±54.064081745560E-02	
	±28.173255684142E-02	
	±00.000000000000E-99	
12	±99.144486137268E-02	26.179938779914E-02
	±92.387953251392E-02	
	±79.335334028852E-02	
	±60.876142901057E-02	
	±38.268343236431E-02	
	±13.052619222005E-02	
13	±99.270887409800E-02	24.166097335306E-02
	±93.501624268557E-02	
	±82.298386589352E-02	

ZEROS OF CHEBYSHEV POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
	±66.312265824082E-02	24.166097335306E-02
	±46.472317204376E-02	
	±23.931566428755E-02	
	±00.000000000000E-99	
14	±99.371220989519E-02	22.439947525641E-02
	±94.388333030359E-02	
	±84.672419923421E-02	
	±70.710678118092E-02	
	±53.203207651916E-02	
	±33.027906195363E-02	
	±11.196447610330E-02	
15	±99.452189536196E-02	20.943951023931E-02
	±95.105651631323E-02	
	±86.602540376619E-02	
	±74.314482548544E-02	
	±58.778525229064E-02	
	±40.673664307577E-02	
	±20.791169081775E-02	
	±00.000000000000E-99	
16	±99.518472670836E-02	19.634954084936E-02
	±95.694033561652E-02	
	±88.192126447851E-02	

ZEROS OF CHEBYSHEV POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
	±63.439328415432E-02	19.634954084936E-02
	±47.139673683189E-02	
	±29.028467725220E-02	
	±98.017140329560E-03	
17	±99.573417631193E-02	18.479956785822E-02
	±96.182564311617E-02	
	±89.516329144824E-02	
	±79.801722718550E-02	
	±67.369564370363E-02	
	±52.643216286108E-02	
	±36.124166618709E-02	
	±18.374951781656E-02	
	±00.000000000000E-99	
18	±99.619469808657E-02	17.453292519943E-02
	±96.592582629376E-02	
	±90.630778704469E-02	
	±81.915204427245E-02	
	±70.710678120323E-02	
	±57.357643633810E-02	
	±42.261826174838E-02	
	±25.881904509966E-02	
	±87.155742747658E-03	

ZEROS OF CHEBYSHEV POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
19	±91.577332678483E-02	16.534698176788E-02
	±96.940026573367E-02	
	±99.658449311510E-02	
	±83.716647822630E-02	
	±83.716647822641E-02	
	±99.658449311499E-02	
	±73.572391067886E-02	
	±61.421271268711E-02	
	±47.594739303753E-02	
	±32.469946920457E-02	
	±16.459459028073E-02	
	±00.000000000000E-99	
20	±85.264016432921E-02	15.707963267948E-02
	±92.387953265769E-02	
	±97.236992017907E-02	
	±99.691733384264E-02	
	±76.040596556825E-02	
	±99.691733384367E-02	
	±64.944804836039E-02	
	±52.249856469893E-02	
	±52.249856469894E-02	
	±38.268343237421E-02	

ZEROS OF CHEBYSHEV POLYNOMIALS

AND CORRESPONDING WEIGHTS

(con't)

n	Abscissas X_i	Weights H_i
	±23.344536385262E-02	15.707963267948E-02
	±78.459095727844E-03	

ZEROS OF JACOBI POLYNOMIALS

AND CORRESPONDING WEIGHTS

With Alpha = 0 and Beta = 1

n	Abscissas X_i	Weights H_i
2	-28.989794855663E-02	72.783447302409E-02
	68.989794855663E-02	12.721655269758E-01
3	18.106627111853E-02	91.696442543837E-02
	-57.531892352169E-02	27.930791960581E-02
	82.282408097459E-02	80.372765495578E-02
4	-16.718086473783E-02	51.939019043293E-02
	44.631397272375E-02	81.385827204109E-02
	-72.048027131243E-02	12.472388380003E-02
	88.579160777096E-02	54.202765372592E-02
5	12.405037950522E-02	58.554794833869E-02
	-39.092854670727E-02	29.563548029047E-02
	60.397316425278E-02	66.869855237747E-02
	-80.292982840234E-02	62.991658086768E-03
	92.038028589706E-02	38.712636090660E-02
6	-53.846772406010E-02	17.582066220203E-02
	70.384280066302E-02	54.216998892603E-02
	-85.389134263948E-02	34.953207254438E-03
	94.136714568043E-02	28.924132290206E-02
	-11.734303754310E-02	39.464460356262E-02
	32.603061943769E-02	56.317021515280E-02
7	94.307252661110E-03	42.850026278351E-02

ZEROS OF JACOBI POLYNOMIALS

AND CORRESPONDING WEIGHTS

With Alpha = 0 and Beta = 1

(con't)

n	Abscissas X_i	Weights H_i
7	-29.475056577365E-02	26.553878586197E-02
	-63.951861652623E-02	10.963342688749E-02
	77.064189367828E-02	44.203703276400E-02
	-88.747487892612E-02	20.857448811218E-03
	95.504122712244E-02	22.386945369183E-02
	46.842035443081E-02	50.956358919834E-02
8	-42.635048571113E-02	18.175727801880E-02
	57.138304120874E-02	45.002319788356E-02
	-71.126748591571E-02	71.371610623947E-03
	81.735278420041E-02	36.447609454549E-02
	-91.073208942005E-02	13.180765768991E-03
	96.444016970525E-02	17.820321744575E-02
	-90.373369606853E-03	31.679839796928E-02
	25.613567083345E-02	42.418943774373E-02
9	64.776668767404E-02	39.413496868954E-02
	-76.384204242004E-02	48.240017139158E-03
	85.122522058134E-02	30.429702043506E-02
	38.066484014471E-02	40.123523677345E-02
	-52.564603037004E-02	12.721928596420E-02
	-92.748437423356E-02	87.233883430862E-04
	97.117518070250E-02	14.511201409853E-02

ZEROS OF JACOBI POLYNOMIALS

AND CORRESPONDING WEIGHTS

With Alpha = 0 and Beta = 1

(con't)

n	Abscissas X_i	Weights H_i
9	76.059197837977E-03	33.743328737970E-02
	-23.623446939058E-02	23.360478118066E-02
10	-35.188892335331E-02	17.360762562760E-02
	-60.195784207389E-02	91.098365813098E-03
	70.577710071442E-02	34.484520115944E-02
	-80.342197557966E-02	33.677279131635E-03
	87.653585624244E-02	25.714861800556E-02
	-73.477531431321E-03	26.421230225341E-02
	21.072030622842E-02	33.822843876332E-02
	-93.994193567779E-02	59.965624098627E-04
	97.616477313830E-02	12.039803217064E-02
	47.768064798306E-02	37.078757471085E-02
12	78.629101823835E-02	26.642700254128E-02
	-85.788420252840E-02	17.779231182474E-03
	91.110707368117E-02	18.971140782669E-02
	-95.687587366860E-02	31.007288838417E-04
	98.292189003869E-02	86.659443997472E-03
	-51.919777905039E-02	99.507121637032E-03
	61.569789093368E-02	30.758641069516E-02
	-29.920130055453E-02	16.180661482766E-02
	40.923823147486E-02	31.078526726217E-02

ZEROS OF JACOBI POLYNOMIALS

AND CORRESPONDING WEIGHTS

With Alpha = 0 and Beta = 1

(con't)

n	Abscissas X_i	Weights H_i
12	-70.910508752995E-02	49.744036665738E-03
	-61.901698625635E-03	22.645537485468E-02
	17.890983759708E-02	28.043735999053E-02
14	93.299719095175E-02	14.518431244598E-02
	-77.868561763837E-02	29.185936532465E-03
	83.802900061306E-02	20.978381382036E-02
	-63.077947888697E-02	60.129214975730E-03
	70.639026465616E-02	25.285210239078E-02
	-45.535290577847E-02	10.169386162856E-02
	54.755046819688E-02	27.132663839715E-02
	-96.755046819688E-02	17.588548348989E-04
	98.716647840924E-02	65.307017115623E-03
	-89.260540012142E-02	10.216405271154E-03
	-26.007337674080E-02	14.977238583594E-02
	35.745651202211E-02	26.556940250433E-02
	-53.475722679746E-03	19.807781809282E-02
	15.541068538485E-02	23.914223609013E-02
16	99.000540065739E-02	50.957242036870E-03
	-91.604883538251E-02	62.687538950728E-04
	94.771297265607E-02	11.444981067322E-02
	-82.621675307370E-02	18.156929544222E-03

ZEROS OF JACOBI POLYNOMIALS

AND CORRESPONDING WEIGHTS

With Alpha = 0 and Beta = 1

(con't)

n	Abscissas X_i	Weights H_i
16	87.313810883370E-02	16.851772464309E-02
	-70.827629555040E-02	38.136671538652E-03
	76.880312483366E-02	20.891437757390E-02
	-56.624510792974E-02	66.158337422551E-03
	63.825888068592E-02	23.301696084637E-02
	-40.496027585008E-02	10.063693242588E-02
	48.595038430943E-02	24.000341955186E-02
	-97.470694125104E-02	10.699746599501E-04
	-22.991430375297E-02	13.859126631714E-02
	31.706414490435E-02	23.087551843708E-02
	-47.068232050336E-03	17.598952875698E-02
	13.735131280874E-02	20.825655137523E-02
18	-36.432607822722E-02	97.950360005131E-03
	43.865641987984E-02	21.378115121168E-02
	58.058592081786E-02	21.276933851195E-02
	-64.721485955880E-02	44.527480395796E-03
	70.665979273637E-02	19.960890813811E-02
	-76.403965059839E-02	25.224155447685E-03
	-86.004177997930E-02	11.849772581716E-03
	89.801192008724E-02	13.788451253995E-02
	-20.597872111969E-02	12.853922451052E-02

ZEROS OF JACOBI POLYNOMIALS

AND CORRESPONDING WEIGHTS

With Alpha = 0 and Beta = 1

(con't)

n	Abscissas X_i	Weights H_i
18	28.474270004672E-02	20.366771542550E-02
	-51.275438681233E-02	69.233201923620E-03
	-93.260150286121E-02	40.521002867093E-04
	95.807269765265E-02	92.425295395141E-03
	-97.973369697615E-02	68.747647520239E-05
	99.199704427768E-02	40.857493229978E-03
	81.343920731285E-02	17.429562209691E-02
	-42.031641741552E-03	15.831664915444E-02
	12.304310155172E-02	18.432960396097E-02
20	91.625715424925E-02	11.467316133188E-02
	-70.787350619867E-02	30.910671808677E-03
	75.762026595482E-02	17.156118701990E-02
	84.639832111355E-02	14.700091854374E-02
	-88.493108889446E-02	80.458326885731E-04
	-94.471388687566E-02	27.322465731634E-04
	96.563753769451E-02	76.139324488454E-03
	-59.455501667009E-02	48.816184504568E-03
	65.190577469338E-02	18.736676729715E-02
	-46.796771647080E-02	70.388267516507E-03
	53.161622737637E-02	19.406442025006E-02
	-98.339991486852E-02	46.152837474211E-05

ZEROS OF JACOBI POLYNOMIALS

AND CORRESPONDING WEIGHTS

With Alpha = 0 and Beta = 1

(con't)

n	Abscissas X_i	Weights H_i
20	99.344780034164E-02	33.484056008733E-03
	-80.539174244142E-02	17.292875717331E-03
	-33.093939781108E-02	94.501184694229E-03
	39.943864632751E-02	19.194184643861E-02
	-18.653106696700E-02	11.960696737052E-02
	25.832563883118E-02	18.188235576540E-02
	-37.968577407147E-03	14.385967661546E-02
	11.142942607397E-02	16.527082726667E-02

APPENDIX III

The following tables are results obtained from integrals using the various numerical integration techniques. Only the "error" and "speed" were compared in these examples.

$$\int_{-1}^{1} e^{-x^2}\, dx \simeq 1.49364826$$

Method	Error	Time
Simpson	25×10^{-8}	27 sec.
Romberg	2×10^{-8}	1 min. 45 sec.
Legendre-Gauss	65×10^{-8}	2 min. 20 sec.

$$\int_{0}^{\pi} \sin x\, dx = 2$$

Method	Error	Time
Simpson	-6×10^{-8}	32 sec.
Romberg	1×10^{-8}	1 min. 15 sec.
Legendre-Gauss	-9×10^{-8}	2 min. 15 sec.

$$\int_{-4}^{4} \frac{1}{1+x^2} = 2 \tan^{-1} 4 \simeq 2.651635$$

Method	Error	Time
Simpson	8×10^{-6}	23 sec.
Romberg	4×10^{-6}	50 sec.
Legendre-Gauss	34×10^{-6}	2 min. 40 sec.

$$\int_{-1}^{1} \frac{dx}{(1-x^2)^{1/3}} = \frac{\sqrt{\pi}\ \Gamma(2/3)}{\Gamma(7/6)} \simeq 2.5551$$

Method	Error	Time
Simpson	45×10^{-4}	50 sec.
Romberg	10×10^{-4}	1 min.
Jacobi-Gauss	-22×10^{-4}	2 min. 20 sec.

$$\int_{-1}^{1} \sqrt{1+x}\ dx = \frac{2}{3}\sqrt{8} \simeq 1.8853$$

Method	Error	Time
Simpson	-2×10^{-4}	25 sec.
Romberg	-2×10^{-4}	45 sec.
Jacobi-Gauss	3×10^{-4}	2 min. 20 sec.

BIBLIOGRAPHY

1. Hildebrand, F. B., (1956) Introduction to Numerical Analysis, McGraw-Hill, New York. 511 pages.

2. Krylov, Vladimir Ivanovich, (1962) Approximate Calculation of Integrals, (translated by Arthur H. Stroud), MacMillian, New York. 357 pages.

3. Ralston, Anthony, (1965) A first course in Numerical Analysis, McGraw-Hill, New York. p. 40-158.

4. Kopal, Zdenek, (1955) Numerical Analysis, John Wiley and Sons, Inc., New York. p. 347-439.

5. Szegö, Gabor, (1959) Orthogonal Polynomials, American Mathematical Society, New York. 421 pages.

6. Stroud, Arthur H., and Secrest, Don, (1966) Gaussian Quadrature Formulas, Prentice-Hall, Englewood Cliffs, New Jersey. 374 pages.

7. Handbook of Mathematical Functions, (1964) edited by Milton Abramowitz and Irene A. Stegun, National Bureau of Standards Applied Mathematics Series, U.S. Government Printing Office, Washington, D. C. 1046 pages.

8. Salzer, Herbert E., Zucker, Ruth, and Capuano, Ruth, (1952) Table of the Zeros and Weight Factor of the First Twenty Hermite Polynomials, Journal of Research of the National Bureau of Standards 48, February, p. 111-116.

9. Davis, Philip J., (1963) Interpolation and Approximation Blaisdell Publishing Company, New York. p. 365-372.

10. Shohat, J. A., (1940) A Bibliography on Orthogonal Polynomials, Bulletin of the National Research Council No. 103, National Research Council of the National Academy of Sciences, Washington, D. C. 203 pages.

BIBLIOGRAPHY (con't)

11. Bauer, F. L., (1961) Algorithm 60 Romberg Integration Communication of the ACM, 4 June (1961). p. 255.

12. Welsh, John H., (1966) Algorithm 281 Abscissas and Weights for Romberg Quadrature, Communications of the ACM, 9 April (1966). p. 271-272.

13. Thacher, Henry C., (1964) Remark on Algorithm 60 Romberg Integration, Communication of the ACM, 7 July 1964. p. 420.

14. Scarborough, James B., (1962) Numerical Mathematical Analysis, The Johns Hopkins Press., Baltimore. p. 133-191.

15. Hastings, Cecil Jr., (1955) Approximations for Digital Computers, Princeton University Press, Princeton, New Jersey. p. 155-158.

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