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CONVERTING SOME GLOBAL OPTIMIZATION PROBLEMS TO MIXED INTEGER LINEAR PROBLEMS USING PIECEWISE LINEAR APPROXIMATIONS

by

MANISH KUMAR

A THESIS

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI–ROLLA

In Partial Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE IN ENGINEERING MANAGEMENT

2007

Approved by

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ABSTRACT

Some global optimization problems are converted to mixed-integer linear problems (MILP) using piecewise-linear approximations in this thesis so that they can be solved using commerical MILP solvers, such as CPLEX. Special attention is given to approximating two-term log-sum functions, which appears frequently in generalized geometric programming problems. Numerical results indicate the proposed approach is sound and efficient.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. Chung-Li Tseng, for his continuous guidance during my graduate research and studies. It was his suggestions and timely help that kept my research progressing in the right direction. I would also like to thank my other committee members, Dr. David Spurlock and Dr. Vincent Yu, for their input and advice.

Finally, I would like to thank my parents, brother, grandparents and close friends for their immense encouragement and continued guidance to me. Without them, I would not have been able to achieve so much.

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1. INTRODUCTION

A global optimization problem may consist of a nonlinear objective function subject to linear and/or nonlinear constraints. Methods have been proposed to tackle this difficult class of optimization problems. In this thesis, piecewise linearization is considered. This approach is intuitive and many researches have been done in early literature, including approximation formulations and approximation errors. Generally, the more linear segments used to approximate a non-linear function, the more accurate results can be expected. However, an approximation with more linear segments will also require more variables and constraints, which increases the computational complexity as well. For example, approximating a small segment of a smooth nonlinear function to an acceptable error may easily incur hundreds of linear segments of approximation. This was forbidden before recent progress in computational power, which was surmised as the key reason why this approach only received limited applications. In recent years, the progress in computational power and storage capability of personal computers has finally enabled us to revisit this traditional approach. Commercial mixed integer linear program (MILP) solvers, such as CPLEX, are powerful. They are now employed to solve large-scale MILPs in various real applications worldwide on daily basis. Such real applications include, to name a few, airline crew scheduling, rail/fleet scheduling, revenue managment, and financial investments. MILP, in general, is NP-hard. Converting an optimization problem, which may not be NP-hard, to an NP-hard formulation may seem a reverse process. In this thesis, the computational complexity issue is set aside. The emphasis here is merely to make use available tools to obtain optimal solutions effectively and efficiently.

Section 2 provides a summary of literature on piecewise linear approximations and the approximation error. Section 3 covers general formulations and approximation error. Special attention will be given to approximating two-term log-sum functions, which appears frequently in generalized geometric progamming problems. Section 4 presents numerical results, including some test functions of global optimization problems and a floor planning problem. Section 5 concludes this thesis.

2. LITERATURE REVIEW

This section lists important findings in literature on the conversion of nonlinear problems to mixed integer linear problems using piecewise linear approximation. Geoffrion (1977) has emphasized the importance of approximating the objective functions in mathematical programming. The author successfully showed that the maximum overshoot (undershoot) of equal values gives the approximation that is best in the sense of the natural criterion. The paper shows that the natural criterion is equivalent to the Chebyshev approximation criterion. Magnitude of the errors imposed by piecewise linear approximations on nonlinear functions has been established in Thakur (1978). The error, i.e., the difference between the approximated function and its piecewise linear approximating function was analyzed. The author further explained how to obtain the lower and upper bounds on the optimal objective value. Thakur (1980) discussed about the lower and upper bounds on the optimal and dual optimal solutions and how they could be used to solve the nonlinear, convex separable programs.

Güder (1994) presented how to determine the minimum number of linear segments required to approximate the nonlinear problems. Thakur (1984) provided analysis of the bounds introduced by the given approximation, which was then used to solve highly nonlinear convex separable programs. Thakur (1986) determines the objective function error bound by solving a series of piecewise-linear problems.

3. PIECEWISE LINEAR APPROXIMATION

3.1. GENERAL FORMULATIONS

Some global optimization problems involve nonlinear objective functions and constraints which can be solved by breaking the function into predetermined number of segments. This is the piecewise linear approximation of the objective function or constraints. A function $f(\cdot)$ is divided using k linear segments as shown in Figure 3.1. These linear segments have slope m_i and value x_i, where $i = 1, \ldots, k$. Each segment is of length $\bar{x}_i (= x_i - x_{i-1})$. Now this function will be approximated using these segments and the function will be expressed as

$$f(x) \approx f(0) + \sum_{i=1}^{k} m_i x_i \tag{1}$$

with

$$x = \sum_{i=1}^{k} x_i \tag{2}$$

However, the above formulation is still not sufficient. For example if one would like to approximate $f(x)$ at $x = 12$, and $\bar{x}_i = 5,\ i = 1, \cdots, 7$. The correct answer would be $\{x_i\} = \{5, 5, 2, 0, 0, 0, 0\}$. However, $\{x_i\} = \{0, 0, 3, 5, 0, 4, 0\}$ could also be a solution to the above formulation. Therefore, additional constraints are needed to resolve this issue. The purpose is to make sure that the values of x_i are assigned sequentially and a subsequent segement is called upon only when its precedent segments have all been completely filled. There are, at least, two formulations that can serve this purpose.

3.1.1. Method 1.

Let binary variable u_i $(\in \{0, 1\})$ indicate whether the i-th segment will be chosen. That is, $u_i = 1$ iff $x_i > 0$. This method imposes the following constraints:

$$\bar{x}_i u_{i+1} \leq x_i \leq \bar{x}_i u_i, i = 1, \ldots, k-1 \tag{3}$$

$$x_k \leq \bar{x}_k u_k \tag{4}$$

In this method, the number of variables used is $2k$ and number of constraints used is $2k-1$. Equations (3) and (4) imply that if $u_{i+1} = 1$, $x_i = \bar{x}_i$. That is, if the $(i+1)$-th segment is used, all its precedent segment, the i-th segment, has to be completely filled.

3.1.2. Method 2. Let binary variable v_i ($\in \{0, 1\}$) decide whether the $(i+1)$-th segment will be chosen. That is, $v_i = 1$ iff $x_{i+1} > 0$. Method 2 imposes the following constraints:

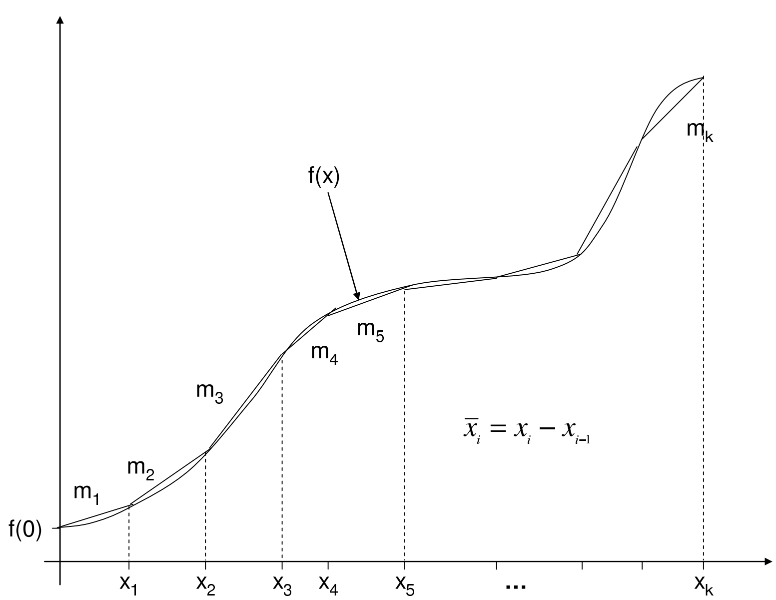

Figure 3.1. Piecewise-linear formulation

$$\bar{x}_i v_i \leq x_i \leq \bar{x}_i v_{i-1}, i = 1, \ldots, k-1 \tag{5}$$

$$0 \leq x_1 \leq \bar{x}_1 \tag{6}$$

Number of variables used is $2k - 1$ and number of constraints used $2k - 1$. Equations (5) and (6) imply that if $v_i = 1$, $x_i = \bar{x}_i$. The interpretation is same as in Method 1.

Method 1 and Method 2 have same number of constraints but Method 1 has one variable more than that of Method 2. Although Method 2 uses less variables, Method 1 will be used in the remainder of this thesis because it is more intuitive and easier to understand (by associating (u_i, x_i) with the i-th segment, instead of (u_{i-1}, x_i)).

3.2. APPROXIMATION ERROR

Approximating nonlinear functions with piecewise linear functions results in approximation errors. If one requires to reduce the approximation error, the size of the approximating interval must be reduced, which increases the total number of linear segments required for approximating the function. Consider a function to be approximated as a convex function and represent it by $f(x)$. As shown in Figure 3.2, the tangent to the curve at $x = x^*$ (parallel to the segment that is approximating this function) has slope $m = \frac{f(b)-f(a)}{b-a}$. If $[a, b]$ is the specified interval on x-axis, the error ϵ at $x = x^*$ is then given by

$$\epsilon = m(x^* - a) + f(a) - f(x^*) \tag{7}$$

If $(b-a)$ approaches to zero, ϵ will also approach to zero. Small approximation errors will have smaller length of the segment joining $(a, f(a))$ and $(b, f(b))$. Thus, decreasing the error value while approximating the functions will result in increasing the number of segments required to approximate the functions.

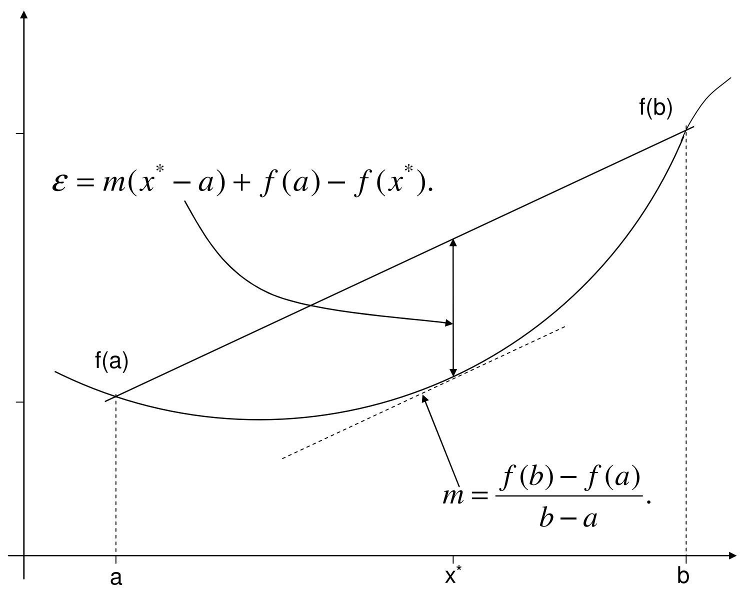

Figure 3.2. Approximation of a convex function

If the function is not convex, the segment may intersect the function at more than one point and hence will have more than one "subintervals", on which the function is either convex or concave (as shown in Figure 3.3).

In Figure 3.3, the segment intersects $f(x)$ at $(a, f(a))$, $(c, f(c))$, $(d, f(d))$ and $(b, f(b))$, then the tangent lines that are parallel to this approximating line segment (connecting $(a, f(a))$ and $(b, f(b))$) will meet the function at three points x_1^*, x_2^* and x_3^*. The slope of these tangent lines will be definedby the slope of the line segments joining $(a, f(a))$ and $(c, f(c))$, $(c, f(c))$ and $(d, f(d))$, and $(d, f(d))$ and $(b, f(b))$, respectively. Correspondingly, there will be three errors ϵ_1, ϵ_2 and ϵ_3 over three subintervals, $[a, c]$, $[c, d]$ and $[d, b]$, respectively. The approximation error will be the minimum of ϵ_1, ϵ_2

and ϵ_3. Therefore, if the function to be approximated is not convex, the approximation error will be more difficult to estimate. However, making $\bar{x}_i$ sufficiently small can solve the above problem but how small it should be is not trivial.

3.3. APPROXIMATING 2-TERM LOG-SUM FUNCTIONS

In generalized geometric programming, nonlinear objective functions and constraints may be "linearized" by taking logarithm of the objective function and each of the constraints. This idea can be illustrated with one simple example:

$$\begin{aligned} (P)\ \min \quad & xy \\ \text{s.t.} \quad & x \le y \le w \\ & x + y \ge z \end{aligned}$$

(P) has a nonlinear objective function that involves a product. A common approach in geometric programming is to take logarithm of the objective and constraints. Let $X = \log x$, $Y = \log y$, $W = \log w$ and $Z = \log z$. Problem (P) becomes

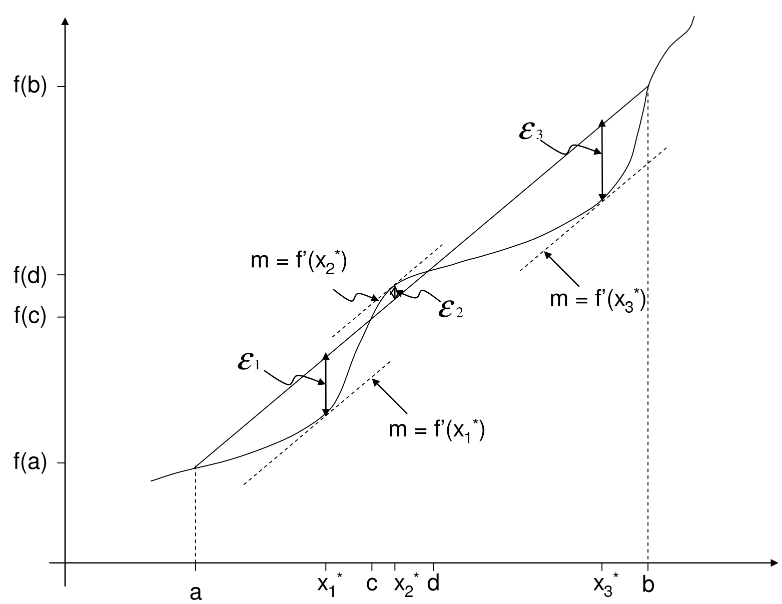

Figure 3.3. Approximation of non-convex function

(P') as shown below:

$$\begin{aligned} (P')\ \min \quad & X+Y \\ \text{s.t.} \quad & X \le Y \le W \\ & \log(x+y) \ge Z \qquad (8) \end{aligned}$$

It can be seen that (P') has one constraint (8) that cannot be linearlized. The left-handside of (8) is called a (two-term) log-sum function. With the formulation in (P'), no advantage has been gained by taking the logarithm unless (8) can also be "linearlized". The following material is partially taken from Tseng et al. (2007), a working paper that I co-author. I appreciate the other coauthors, Dr. Tseng and Dr. Tsu-Shuan Chang, who agree me to adopt it in my thesis.

Consider 2-term log-sum functions $\log(A+B)$, where A and B are the two positive variables inside the log function. If A is taken out of the brackets, then

$$\log(A+B) = \log(A(1+B/A)) = \log A + \log(1+B/A) \qquad (9)$$

It is of the form $\log(1+x)$, where $x = B/A$ and $x > 0$. $\log(1+x)$ is plotted against $\log x$ and the curve obtained is $f(\log x)$ and then $\log(1+x) = f(\log x)$. The approach is to approximate $f(\log x)$ as a piecewise linear formulation of $\log x$.

The Approach. The approach is using the whole $f(\cdot)$ curve. Starting from origin, the piecewise linear approximation can be done either in the positive or negative direction. $f(\log x)$ can be approximated using piecewise linear formulation using either of the segments, $[\log b, \log a]$ or $[\log d, \log c]$ (as shown in Figure 3.4). This approximation will have different approximating errors in each direction and will be given by

$$\epsilon = m(\log x^* - \log b) + f(\log b) - f(\log x^*) \tag{10}$$

and

$$\epsilon' = m'(\log x_1^* - \log c) + f(\log c) - f(\log x_1^*) \tag{11}$$

Since starting point is $[0, \log 2]$ and $f(\log x)$ is approximated using n segments in either positive or negative direction, these segments have value x_j (≥ 0) and y_j (≥ 0) in the negative and positive direction. The binary decision variables for deciding on which segment to select are u_j ($\in \{0,1\}$) and v_j ($\in \{0,1\}$), respectively. The length of each segment in either direction is given by $\bar{x}_j$ and $\bar{y}_j$, respectively. Then the log-sum function can be written as

$$\log(A+B) \approx \log A + \log 2 + \sum_{j=1}^{n} [m'_j y_j - m_j x_j] \tag{12}$$

Constraints required to support the log-sum functions are

$$\log(B/A) = -\textstyle\sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_j \tag{13}$$

$$\bar{x}_j u_{j+1} \leq x_j \leq \bar{x}_j u_j, j = 1, \ldots, n \tag{14}$$

$$\bar{y}_j v_{j+1} \leq y_j \leq \bar{y}_j v_j, j = 1, \ldots, n \tag{15}$$

$$u_1 + v_1 = 1 \tag{16}$$

At $j = n$, $u_{n+1} = 0$ and $v_{n+1} = 0$.

Consider the function $\log(1+x) = f(\log x)$ to be approximated using n segments in either positive $\log x$-direction or negative $\log x$-direction.

Proposition. Refer to Figure 3.4, if the segments are chosen in such a way that $\log a = -\log d$, $\log b = -\log c$, $\bar{x}_j = \log a - \log b$ and $\bar{y}_j = \log c - \log d$, then $\bar{x}_j = \bar{y}_j$. The slopes of the segment are m and m' and errors as ϵ and ϵ'. For the function $\log(1+x) = f(\log x)$ as described above,

1. $m' = (1-m)$
2. $\epsilon' = \epsilon$

Proof.

1. Slope of the segment chosen in negative direction,

$$m = \frac{\log(a+1) - \log(b+1)}{\log a - \log b}$$

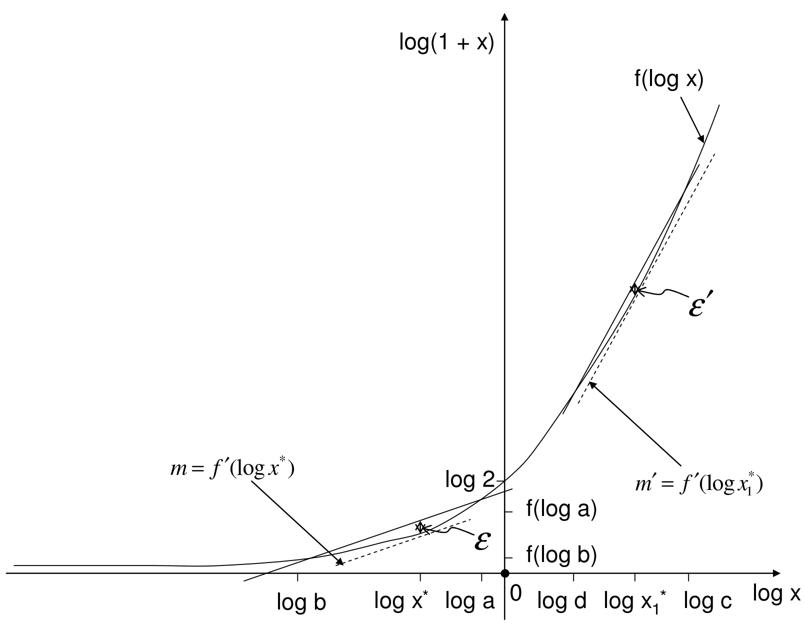

Figure 3.4. Piecewise-linear formulation of $f(\log x)$ in $(-\infty, \infty)$

and, Slope of the segment chosen in positive direction,

$$m' = \frac{\log(c+1) - \log(d+1)}{\log c - \log d}$$

Substituting $c = (1/b)$, $d = (1/a)$ and $(\log c - \log d) = (\log a - \log b)$ in the equation involving m',

$$m' = \frac{\log((1/b)+1) - \log((1/a)+1)}{\log a - \log b}$$

$$\Rightarrow m' = \frac{\log(b+1) - \log(a+1)}{\log a - \log b} + 1$$

$$\Rightarrow m' = (1-m)$$

2. Substituting $m' = (1-m)$, $x_1^* = (1/x^*)$, $c = (1/b)$ and $d = (1/a)$ in (11),

$$\epsilon' = (1-m)(-\log x^* + \log b) + \log(1+b) - \log b - \log(1+x^*) + \log x^*$$

$$\Rightarrow \epsilon' = (m-1)(\log x^* - \log b) + f(\log b) - f(\log x^*) + (\log x^* - \log b)$$

$$\Rightarrow \epsilon' = \epsilon$$

4. NUMERICAL RESULTS

Case Studies presented in this section utilize OPL 4.0 / CPLEX 9.0 for global search and then Matlab 7.0 for local search.

4.1. GLOBAL OPTIMIZATION

Four functions will be tested here. All of them are taken from Törn et al. (1989). The first three test functions are one-dimensional; the fourth one is two-dimensional.

4.1.1. Function 1. The first function to be minimized is

$$f_1(x) = \sin x + \sin(\frac{10x}{3}) + \ln x - 0.84x,\ 2.7 \leq x \leq 7.5$$

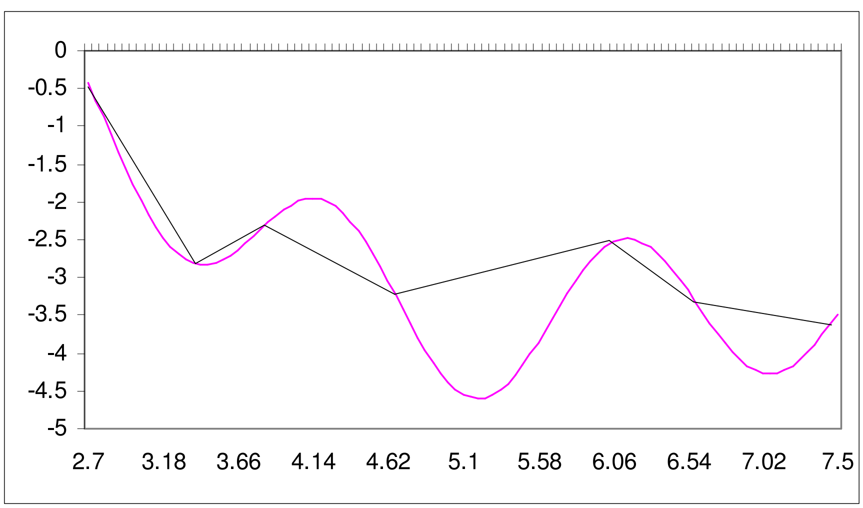

Figure 4.1. Illustration of the test function $f_1(x)$ and its approximation function with 6 segments

This function is formulated as piecewise linear and the number of segments (n) are increased by decreasing the size of interval. The plot of this function is shown in Figure 4.1. Constraints from Section 3.1.1 are used. Starting value is taken at $x = 2.7$ and the slopes of each segment is evaluated. The piecewise linear formulation is

$$\begin{aligned} \min \quad & f_1(2.7) + \textstyle\sum_{j=1}^{n} m_j x_j \\ \text{s.t.} \quad & \bar{x}_j u_{j+1} \leq x_j \leq \bar{x}_j u_j, j = 1, \ldots, n \\ & x_j \geq 0 \\ & u_j = \{0, 1\}. \end{aligned}$$

At $j = n$, $u_{n+1} = 0$. The above problem was solved for interval size of 0.4, 0.2, 0.1 and 0.048 resulting in the number of segments, $n = 12, 24, 48$ and 100. The results are shown in Table 4.1.

Table 4.1. Piecewise-linear formulation results for the test function $f_1(x)$

Number of Segments	Solution Time	Minimum Point	Minimum Value
12	1.74 sec	5.10	-4.5419
24	1.96 sec	5.30	-4.5422
48	2.07 sec	5.20	-4.6013
100	2.73 sec	5.196	-4.6012

4.1.2. Function 2. Second function to be minimized is

$$f_2(x) = \sin x + \sin(\frac{2x}{3}),\ 3.1 \leq x \leq 20.4$$

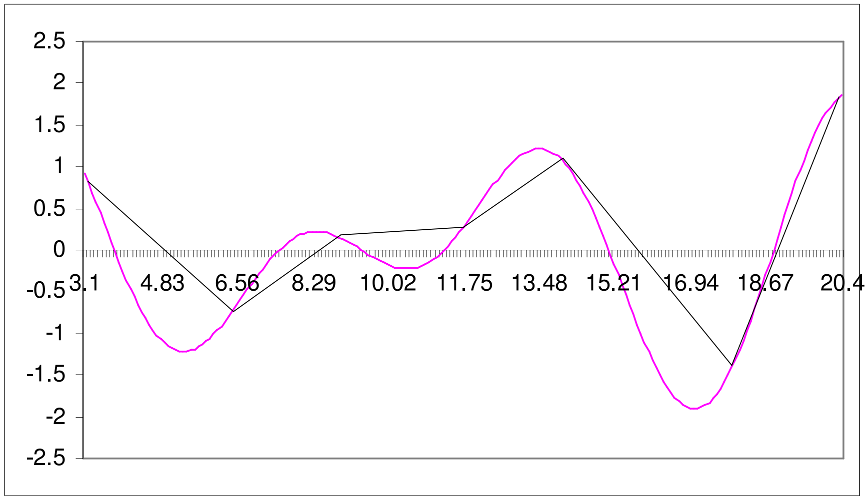

Figure 4.2. Illustration of the test function $f_2(x)$ and its approximation function with 6 segments

This function is formulated as piecewise linear and the number of segments (n) are increased by decreasing the size of interval. The plot of this function is shown in Figure 4.2. Constraints from Section 3.1.1 are used. Starting value is taken at $x = 3.1$ and the slopes of each segment is evaluated. The piecewise linear formulation is

$$
\begin{aligned}
\min \quad & f_2(3.1) + \textstyle\sum_{j=1}^{n} m_j x_j \\
\text{s.t.} \quad & \bar{x}_j u_{j+1} \le x_j \le \bar{x}_j u_j, j = 1, \dots, n \\
& x_j \ge 0 \\
& u_j = \{0, 1\}.
\end{aligned}
$$

At $j = n$, $u_{n+1} = 0$. The above problem was solved for interval size of 0.692, 0.346, 0.173 and 0.0865 resulting in the number of segments, $n = 25, 50, 100$ and 200. The results are shown in Table 4.2.

Table 4.2. Piecewise-linear formulation results for the test function $f_2(x)$

Number of Segments	Solution Time	Minimum Point	Minimum Value
25	2.51 sec	16.94	-1.8992
50	2.73 sec	16.94	-1.8992
100	2.89 sec	17.113	-1.9022
200	3.60 sec	17.0265	-1.9058

4.1.3. Function 3. Third function to be minimized is

$$f_3(x) = -\sum_{i=1}^{5} \sin((i+1)x + i),\ -10 \le x \le 10$$

This function is formulated as piecewise linear and the number of segments (n) are increased by decreasing the size of interval. The plot of this function is shown in Figure 4.3. This figure also shows the effect of less number of linear segments on missing the various global minimums. It can be seen that this function plot has three global minimum points. Constraints from Section 3.1.1 are used. Starting value is taken at $x = -10$. The piecewise linear formulation is

$$\begin{aligned} \min \quad & f_3(-10) + \textstyle\sum_{j=1}^{n} m_j x_j \\ \text{s.t.} \quad & \bar{x}_j u_{j+1} \le x_j \le \bar{x}_j u_j, j = 1, \ldots, n \\ & x_j \ge 0 \\ & u_j = \{0, 1\}. \end{aligned}$$

At $j = n$, $u_{n+1} = 0$. The above problem was solved for interval size of 0.8, 0.4, 0.2, 0.05 and 0.02 resulting in the number of segments, $n = 25, 50, 100, 400$ and 1000. The results shown in Table 4.3 indicate that increasing the number of segments changes the global minimum point. The results show the three global minimum point as the number of segments are changed from 100 to 400 and finally to 1000.

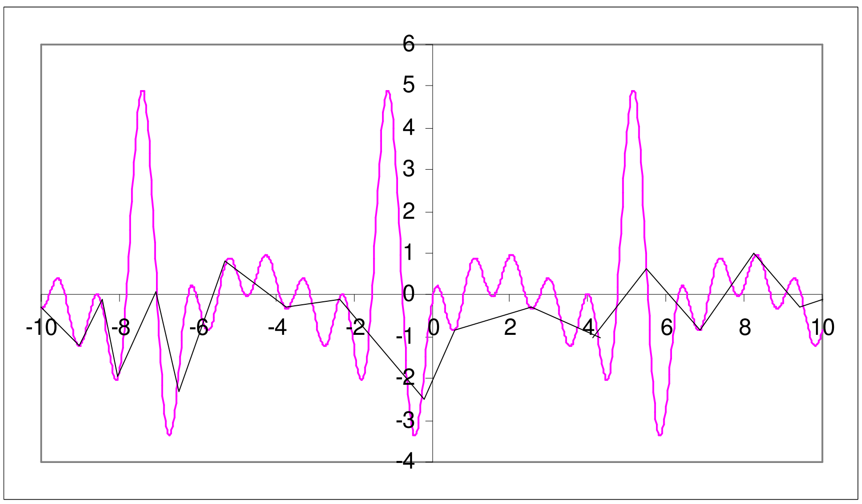

Figure 4.3. Illustration of the test function $f_3(x)$ and its approximation function with 17 segments

Table 4.3. Piecewise-linear formulation results for the test function $f_3(x)$

Number of Segments	Solution Time	Minimum Point	Minimum Value
25	2.18 sec	-0.4	-3.3263
50	2.40 sec	-0.4	-3.3263
100	3.06 sec	-0.4	-3.3263
400	6.49 sec	5.85	-3.3724
1000	11.70 sec	-6.72	-3.3729

4.1.4. Function 4. Fourth function to be minimized is

$$f_4(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4,\ -5 \le x_i \le 5,\ i = 1, 2$$

This function is called Six-hump camel-back function. Its detail can be found in Branin (1972). It is a two-dimensional problem. The nonlinear form is

$$\begin{array}{ll} \min & t \\ \text{s.t.} & 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4 \le t,\ -5 \le x_i \le 5,\ i = 1, 2 \end{array}$$

The above problem can be formulated in two ways: one for first and third quadrant ($0 \le x_1, x_2 \le 5$ and $-5 \le x_1, x_2 \le 0$) and another for second and fourth quadrant ($0 \le x_1 \le 5, -5 \le x_2 \le 0$ and $0 \le x_2 \le 5, -5 \le x_1 \le 0$). For each formulation logarithm is taken so that the problem becomes "linearized". The problem is then solved using OPL 4.0 / CPLEX 9.0 and then the local search using Matlab 7.0. It was found that the global minimum point lies in the second and fourth quadrant even if the problem is taken in first or third quadrant. So, the MILP form of this problem is written for second and fourth quadrant (where $Y_2 = \ln(y_2)$ and $y_2 = -x_2$) as shown below:

$$\begin{array}{ll} \min & P \\ \text{s.t.} & P_4 - P_5 \le 0 \\ & -P_4 + P_1 + \sum_{j=1}^{n}((1 - m_j)y_{4j} - m_jx_{4j}) = -\ln 2 \\ & -P_5 + P_2 + \sum_{j=1}^{n}((1 - m_j)y_{5j} - m_jx_{5j}) = -\ln 2 \end{array}$$

$$-P_1 + 2 * X_1 + \sum_{j=1}^{n}((1-m_j)y_{1j} - m_j x_{1j}) = -\ln 8$$

$$-P_2 + P + \sum_{j=1}^{n}((1-m_j)y_{2j} - m_j x_{2j}) = -\ln 2$$

$$-P_3 + 2 * Y_2 + \sum_{j=1}^{n}((1-m_j)y_{3j} - m_j x_{3j}) = -\ln 8$$

$$4 * Y_2 - P_1 + \sum_{j=1}^{n}(x_{4j} - y_{4j}) = -\ln 4$$

$$P_3 - P_2 + \sum_{j=1}^{n}(x_{5j} - y_{5j}) = 0$$

$$6 * X_1 - 4 * X_1 + \sum_{j=1}^{n}(x_{1j} - y_{1j}) = \ln 12$$

$$X_1 + Y_2 - P + \sum_{j=1}^{n}(x_{2j} - y_{2j}) = 0$$

$$4 * X_1 - 2 * Y2 + \sum_{j=1}^{n}(x_{3j} - y_{3j}) = -\ln(2.1/4)$$

$$\bar{x}_j u_{i(j+1)} \leq x_{ij} \leq \bar{x}_j u_{ij},\ i = 1, \ldots, 5,\ j = 1, \ldots, n$$

$$\bar{x}_j v_{i(j+1)} \leq y_{ij} \leq \bar{x}_j v_{ij},\ i = 1, \ldots, 5,\ j = 1, \ldots, n$$

$$u_{i1} + v_{i1} = 1,\ i = 1, \ldots, 5$$

Global optimum solution for the above formulation is shown in Table 4.4. It can be seen that two global minimum points are obtained with one global minimum solution. So, the log-sum formulation also works for more than one dimension global optimization problems.

4.2. FLOOR PLANNING PROBLEMS

The two problems discussed in this section belong to a class of Floor Planning Problems defined in Moh et al. (1996). Two cases will be tested, one contains four cells on the (circuit) floor, the other contains nine cells.

4.2.1. Four-Cell Floor Planning. Four cells on the circuit floor are to be designed (or layouted). Each cell has a box with width w and height z. The length of the rectangular cells is x_2 and y_3. The layout is depicted in Figure 4.4. So, objective is to minimize the rectangular area of the entire layout. The nonlinear problem formulation is

Table 4.4. Piecewise-linear formulation results for the test function $f_4(x)$

Cases	Minimum Point (OPL)	Minimum Point (Matlab)	OPL Solution	Matlab Solution	Solution Time (OPL)
10^{-3} error	$(0.3010, -0.2671)$	$(0.0899, -0.7127)$	$-1.1\,10^{-5}$	-1.0316	3.07 sec
10^{-4} error	$(0.1656, -0.1464)$	$(0.0899, -0.7127)$	$-1.1\,10^{-6}$	-1.0316	13.42 sec
10^{-5} error	$(-0.0100, 0.0882)$	$(-0.0899, 0.7127)$	$4.82\,10^{-7}$	-1.0316	219.46 sec

$$
\begin{array}{ll}
\min & x_2 y_3 \\
\text{s.t.} & x_1 \leq x_2 \\
& y_1 \leq y_2 \leq y_3 \\
& w_i z_i \geq a_i,\ i = 1,2,3,4 \\
& r_i \leq \frac{z_i}{w_i} \leq R_i,\ i = 1,2,3,4 \\
& x_1 \geq w_1,\ y_1 \geq z_1 \\
& x_1 \geq w_3,\ y_2 \geq z_2 \\
& x_2 - x_1 \geq w_2,\ y_3 - y_1 \geq z_3 \\
& x_2 - x_1 \geq w_4,\ y_3 - y_2 \geq z_4 \\
& x_1, x_2 \geq 0,\ y_1, y_2, y_3 \geq 0,\ w_i, z_i \geq 0,\ i = 1,2,3,4
\end{array}
$$

The above formulation has nonlinear objective fuction and few nonlinear constraints. It was converted into a linear problem by taking natural logarithm of the whole problem. After taking the logarithm, four log-sum functions ($P = \ln(x_1 + w_2)$, $Q = \ln(y_1 + z_3)$, $R = \ln(x_1 + w_4)$, and $S = \ln(y_2 + z_4)$) were created, which can be further converted to linear constraints using the procedure described in Section 3.3. The resultant MILP was solved using OPL 4.0 / CPLEX 9.0 for global search, followed by a final local search to arrive at the global optimum solution using Matlab 7.0. Using the approach described in Section 3.3 and Proposition 1 and 2, the MILP formulation is as follows:

$$\begin{aligned}
\text{min} \quad & X_2 + Y_3 \\
\text{s.t.} \quad & X_1 \leq X_2 \\
& Y_1 \leq Y_2 \leq Y_3 \\
& W_i + Z_i \geq A_i,\ i = 1,2,3,4 \\
& ar_i \leq Z_i - W_i \leq AR_i,\ i = 1,2,3,4 \\
& X_1 \geq W_1,\ Y_1 \geq Z_1 \\
& X_1 \geq W_3,\ Y_2 \geq Z_2
\end{aligned}$$

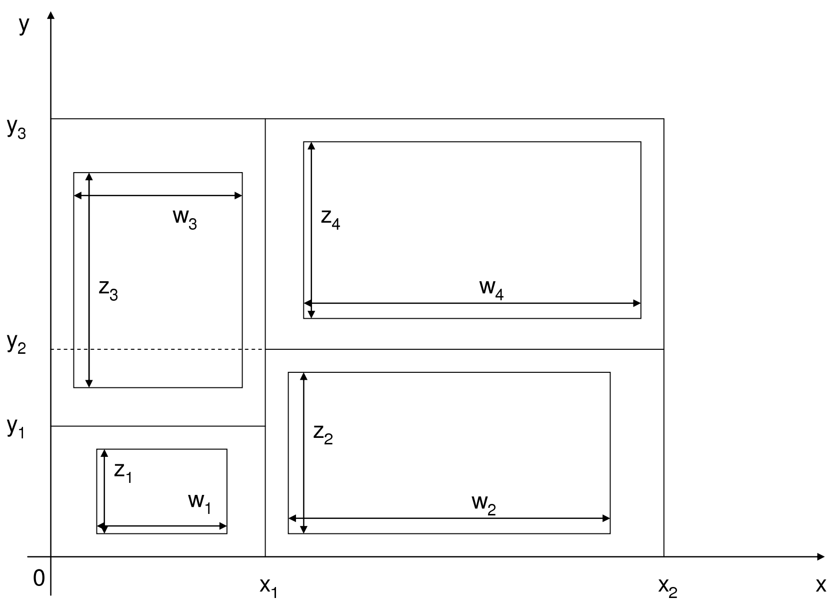

Figure 4.4. Illustration of the four-cell floor planning problem

$$X_2 - P \geq 0,\ Y_3 - Q \geq 0$$

$$X_2 - R \geq 0,\ Y_3 - S \geq 0$$

$$-P + X_1 + \textstyle\sum_{j=1}^{n}((1-m_j)y_{1j} - m_j x_{1j}) = -\ln 2$$

$$-Q + Y_1 + \textstyle\sum_{j=1}^{n}((1-m_j)y_{2j} - m_j x_{2j}) = -\ln 2$$

$$-R + X_1 + \textstyle\sum_{j=1}^{n}((1-m_j)y_{3j} - m_j x_{3j}) = -\ln 2$$

$$-S + Y_2 + \textstyle\sum_{j=1}^{n}((1-m_j)y_{4j} - m_j x_{4j}) = -\ln 2$$

$$W_2 - X_1 + \textstyle\sum_{j=1}^{n}(x_{1j} - y_{1j}) = 0$$

$$Z_3 - Y_1 + \textstyle\sum_{j=1}^{n}(x_{2j} - y_{2j}) = 0$$

$$W_4 - X_1 + \textstyle\sum_{j=1}^{n}(x_{3j} - y_{3j}) = 0$$

$$Z_4 - Y_2 + \textstyle\sum_{j=1}^{n}(x_{4j} - y_{4j}) = 0$$

$$\bar{x}_j u_{i(j+1)} \leq x_{ij} \leq \bar{x}_j u_{ij},\ i = 1,2,3,4,\ j = 1,\ldots,n$$

$$\bar{x}_j v_{i(j+1)} \leq y_{ij} \leq \bar{x}_j v_{ij},\ i = 1,2,3,4,\ j = 1,\ldots,n$$

$$u_{i1} + v_{i1} = 1,\ i = 1,2,3,4$$

All X's, Y's, W's, Z's, P, Q, R and S are real numbers as they are the natural logarithm of the positive values. Test results are summarized in the Table 4.5.

4.2.2. Nine-Cell Floor Planning. The problem is similar to that in the previous section, but it has more cells to layout. Each cell has a box with width w and height h. The length of the rectangular cells is x_6 and y_4. The layout is depicted in Figure 4.5. The objective is to minimize the rectangular area of the entire layout. The nonlinear problem formulation is

Table 4.5. Solving four-cell problem using Section 3.3

Cases	OPL Solution	Matlab Solution	Solution Time (OPL)
10^{-3} error	23.348	23.317	3.10 sec
10^{-4} error	23.322	23.317	3.68 sec
10^{-5} error	23.317	23.317	6.12 sec

$$
\begin{aligned}
\text{min} \quad & x_6 y_4 \\
\text{s.t.} \quad & x_1 \leq x_2 \leq x_6 \\
& x_1 \leq x_3 \leq x_2 \\
& x_2 \leq x_4 \leq x_6 \\
& x_3 \leq x_5 \leq x_4 \\
& y_1 \leq y_3 \leq y_2 \leq y_4 \\
& w_i h_i \geq a_i,\ i = 1, \ldots, 9 \\
& r_i \leq \tfrac{h_i}{w_i} \leq R_i,\ i = 1, \ldots, 9 \\
& x_1 \geq w_1,\ x_3 \geq w_4,\ x_3 \geq w_6 \\
& y_1 \geq h_1,\ y_1 \geq h_2,\ y_1 \geq h_3 \\
& x_2 - x_1 \geq w_2,\ x_6 - x_2 \geq w_3 \\
& x_4 - x_3 \geq w_5,\ x_5 - x_3 \geq w_7 \\
& x_4 - x_5 \geq w_8,\ x_6 - x_4 \geq w_9 \\
& y_2 - y_1 \geq h_4,\ y_3 - y_1 \geq h_5 \\
& y_4 - y_2 \geq h_6,\ y_4 - y_3 \geq h_7 \\
& y_4 - y_3 \geq h_8,\ y_4 - y_1 \geq h_9 \\
& x_j \geq 0,\ y_k \geq 0,\ w_i, z_i \geq 0,\ i = 1, \ldots, 9, j = 1, \ldots, 6, k = 1, 2, 3, 4
\end{aligned}
$$

The above formulation has nonlinear objective fuction and few nonlinear constraints. These can be converted into linear form by taking natural logarithm of the whole problem. There are 12 constraints that are linear in their original form and taking natural logarithm will make them nonlinear. These log-sum functions can be formulated using Section 3. Approach described in Section 3.3 is used on each log-sum function to generate 12 mixed integer linear formulations. The whole problem can be solved using OPL 4.0 / CPLEX 9.0 for global search and then Matlab 7.0 for final local search to arrive at the global optimum solution. Using the approach described in Section 3.3 and Proposition 1 and 2, the MILP of the nonlinear problem described above is as follows:

$$
\begin{aligned}
\text{min} \quad & X_6 + Y_4 \\
\text{s.t.} \quad & X_1 \le X_2 \le X_6 \\
& X_1 \le X_3 \le X_2 \\
& X_2 \le X_4 \le X_6 \\
& X_3 \le X_5 \le X_4 \\
& Y_1 \le Y_3 \le Y_2 \le Y_4 \\
& W_i + H_i \ge A_i,\ i = 1, \dots, 9 \\
& ar_i \le H_i - W_i \le AR_i,\ i = 1, \dots, 9 \\
& X_1 \ge W_1,\ X_3 \ge W_4,\ X_3 \ge W_6 \\
& Y_1 \ge H_1,\ Y_1 \ge H_2,\ Y_1 \ge H_3 \\
& X_2 - P_1 \ge 0,\ X_6 - P_2 \ge 0,\ X_4 - P_3 \ge 0 \\
& X_5 - P_4 \ge 0,\ X_4 - P_5 \ge 0,\ X_6 - P_6 \ge 0 \\
& Y_2 - P_7 \ge 0,\ Y_3 - P_8 \ge 0,\ Y_4 - P_9 \ge 0
\end{aligned}
$$

$$Y_4 - P_{10} \geq 0,\ Y_4 - P_{11} \geq 0,\ Y_4 - P_{12} \geq 0$$

$$-P_1 + X_1 + \sum_{j=1}^{n}((1-m_j)y_{1j} - m_j x_{1j}) = -\ln 2$$

$$-P_2 + X_2 + \sum_{j=1}^{n}((1-m_j)y_{2j} - m_j x_{2j}) = -\ln 2$$

$$-P_3 + X_3 + \sum_{j=1}^{n}((1-m_j)y_{3j} - m_j x_{3j}) = -\ln 2$$

$$-P_4 + X_3 + \sum_{j=1}^{n}((1-m_j)y_{4j} - m_j x_{4j}) = -\ln 2$$

$$-P_5 + X_5 + \sum_{j=1}^{n}((1-m_j)y_{5j} - m_j x_{5j}) = -\ln 2$$

$$-P_6 + X_4 + \sum_{j=1}^{n}((1-m_j)y_{6j} - m_j x_{6j}) = -\ln 2$$

$$-P_7 + Y_1 + \sum_{j=1}^{n}((1-m_j)y_{7j} - m_j x_{7j}) = -\ln 2$$

$$-P_8 + Y_1 + \sum_{j=1}^{n}((1-m_j)y_{8j} - m_j x_{8j}) = -\ln 2$$

$$-P_9 + Y_2 + \sum_{j=1}^{n}((1-m_j)y_{9j} - m_j x_{9j}) = -\ln 2$$

$$-P_{10} + Y_3 + \sum_{j=1}^{n}((1-m_j)y_{10j} - m_j x_{10j}) = -\ln 2$$

$$-P_{11} + Y_3 + \sum_{j=1}^{n}((1-m_j)y_{11j} - m_j x_{11j}) = -\ln 2$$

$$-P_{12} + Y_1 + \sum_{j=1}^{n}((1-m_j)y_{12j} - m_j x_{12j}) = -\ln 2$$

$$W_2 - X_1 + \sum_{j=1}^{n}(x_{1j} - y_{1j}) = 0$$

$$W_3 - X_2 + \sum_{j=1}^{n}(x_{2j} - y_{2j}) = 0$$

$$W_5 - X_3 + \sum_{j=1}^{n}(x_{3j} - y_{3j}) = 0$$

$$W_7 - X_3 + \sum_{j=1}^{n}(x_{4j} - y_{4j}) = 0$$

$$W_8 - X_5 + \sum_{j=1}^{n}(x_{5j} - y_{5j}) = 0$$

$$W_9 - X_4 + \sum_{j=1}^{n}(x_{6j} - y_{6j}) = 0$$

$$H_4 - Y_1 + \sum_{j=1}^{n}(x_{7j} - y_{7j}) = 0$$

$$H_5 - Y_1 + \sum_{j=1}^{n}(x_{8j} - y_{8j}) = 0$$

$$H_6 - Y_2 + \sum_{j=1}^{n}(x_{9j} - y_{9j}) = 0$$

$$H_7 - Y_3 + \sum_{j=1}^{n}(x_{10j} - y_{10j}) = 0$$

$$H_8 - Y_3 + \sum_{j=1}^{n}(x_{11j} - y_{11j}) = 0$$

$$H_9 - Y_1 + \sum_{j=1}^{n}(x_{12j} - y_{12j}) = 0$$

$$\bar{x}_j u_{i(j+1)} \leq x_{ij} \leq \bar{x}_j u_{ij},\ i = 1, \ldots, 9,\ j = 1, \ldots, n$$

$$\bar{x}_j v_{i(j+1)} \leq y_{ij} \leq \bar{x}_j v_{ij},\ i = 1, \ldots, 9,\ j = 1, \ldots, n$$

$$u_{i1} + v_{i1} = 1,\ i = 1, \ldots, 9$$

All X's, Y's, W's, Z's, P's are real numbers as they are the natural logarithm of the positive values. The test results are summarized in the Table 4.6.

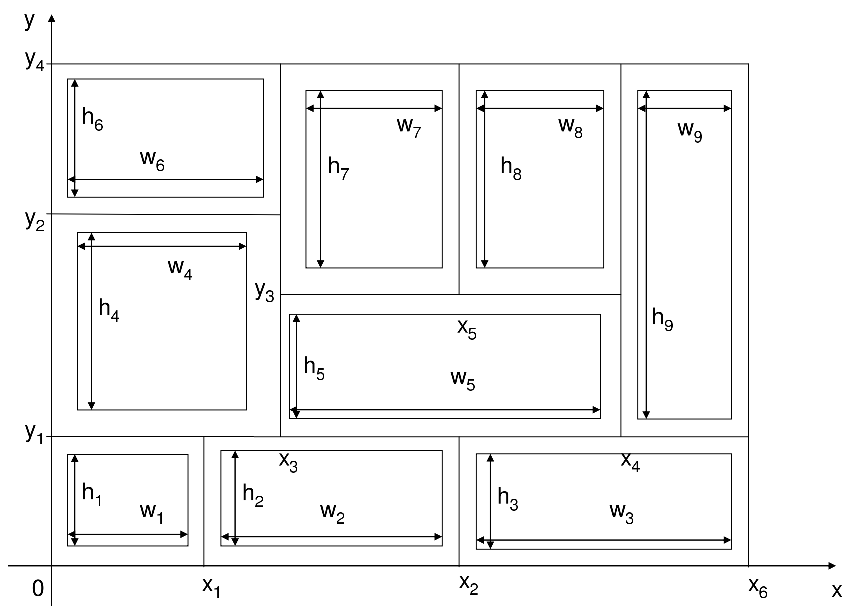

Figure 4.5. Illustration of the nine-cell floor planning problem

Table 4.6. Solving nine-cell problem using Section 3.3

Cases	OPL Solution	Matlab Solution	Solution Time (OPL)
10^{-3} error	55.645	55.543	2.56 sec
10^{-4} error	55.551	55.543	4.82 sec
10^{-5} error	55.543	55.543	53.15 sec

5. CONCLUSION

The methodology described in Section 3 has been used to test problems described in Section 4 successfully. The floor planning problem was formulated using piecewise linear formulation described in Section 3.3. The test results suggest that although the number of segments increases by increasing the size of the error and the number of variables and constraints increase as a result, the computational time to solve the mixed integer linear problem does not increase significantly. The results also suggest that the optimal objective value does not change much if predetermined error for linear approximation is decreased from 10^{-3} to 10^{-5}. Hence, approximation error of 10^{-3} is sufficient to decide that how many segments are to be used to linearize the two-term log-sum functions. Numerical tests suggest a strategy to use less linear segments to approximate (so it can be solved faster), followed by a local minimization search to locate the global optimum.

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While an undergraduate student at IIT Kanpur, he did his summer internship with Tata Steel Ltd., Jamshedpur, India (Integrated Steel Plant in India). After graduation, he decided to join the same company to gain practical experience in the field that he learnt during his undergraduate studies. After working in this company for four years and seeing the blend of engineering and management, he decided to pursue a Master of Science Degree in Engineering Management.

He worked along with Dr. Chung-Li Tseng on "Piecewise Linearization" concept and solved some Global Optimization Problems. On this topic, he co-authored a paper which was written by Dr. Chung-Li Tseng and Dr. Chang Tsu-Shuan entitled "A New Approach for Generalized Geometric Programming using Piecewise-Linear Approximations". He completed his research under the advising of Dr. Tseng and received his Master of Science Degree in December of 2007.