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Analysis of elastic thermal stresses by station-function collocation methods

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ANALYSIS OF ELASTIC THERMAL STRESSES BY
STATION-FUNCTION COLLOCATION METHODS

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THESIS

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ABSTRACT

An approximate method for the solution of thermal stress problems is presented. The method makes use of polynomial approximations to reduce the partial differential equation to a system of linear algebraic equations or a set of first-order ordinary differential equations. This results in satisfying the differential equation at a finite number of stations. The boundary conditions are satisfied identically.

Two examples of the method, presented in detail, indicate that the solutions of the biharmonic equation for the stress function and the Fourier equation for the temperature distribution have good accuracy with a minimum of labor.

A generalized method is derived for solving two-dimensional thermal-stress problems.

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The never ending encouragements of my parents is deeply appreciated.

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NOMENCLATURE

x, y	Rectangular coordinates
ξ, η	Rectangular coordinates
r, θ, z	Cylindrical coordinates
$A_{k\ell}^{ij}, B_{k\ell}^{ij}$	Dimensionless constants
E	Young's modulus of elasticity
k	Constant
m	Number of stations in x or z
n	Number of stations in y or r
P_i	Polynomial in x or z associated with i^{th} station
Q_j	Polynomial in y or r associated with j^{th} station
r_0	Outside radius of cylinder
T	Temperature
T_0	Reference temperature or initial temperature
T_{ij}	Temperature at ij^{th} station
∇^2	Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇^4	Biharmonic operator, $\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
α	Coefficient of linear thermal expansion or thermal diffusivity
ϕ	Stress function, function of x and y
ϕ_{ij}	Stress function associated with ij^{th} station
ψ	Approximate function of ϕ
σ	Normal stress
τ	Shear stress

ε Normal strain

γ Shear strain

ν Poisson's ratio

τ^* Fourier's modulus, $t\alpha/r_0^2$

$\underset{i \neq j}{\pi}$ Product of all values of i except i = j

Superscripts and Subscripts:

$*$ Dimensionless quantity

$'$ Derivative

i,j,k,ℓ Summation or multiplication dummy indices or refer to the i^{th}, j^{th}, k^{th}, or ℓ^{th} station

x,y Partial differentiation with respect to that subscript

I. INTRODUCTION

Current interest in thermal stress arises chiefly because of the many engineering components which fail because of it. Jet engines, high-speed airplanes and missiles, and nuclear power-plants are examples of modern devices in which large temperature gradients exist. Such temperature gradients can produce large thermal stresses which, by themselves or in conjunction with stresses produced by various external loads, can cause serious component failures. Thus serious thermal stresses must be of concern to the designer. A number of methods that can be effectively applied for determining thermal stresses will be helpful to the designer and practicing engineer in avoiding the deleterious effects of them.

In most cases, however, exact solution is not possible, and recourse to approximate methods must be taken. Among the methods for solving thermal-stress problems, the station-function collocation method is one of the more general methods. It requires a minimum of mathematical insight and ingenuity from the analyst, although it yields very accurate results. Its particular merit lies in the treatment of two-dimensional problems, since it permits the partial differential equations involved in such problems to be replaced by a series of readily solvable ordinary differential equations. In many cases no differential equations at all are involved.

Because of the generality of the method and because its use has thus far been limited, the purpose of this thesis is to develop a generalized station-function collocation method for solving two-dimensional thermal-stress problems.

II. REVIEW OF LITERATURE

The station-function collocation method is essentially an extension of the collocation procedure[1] applied to partial differential equations. This method was first developed in connection with thermal-shock problems, in which the principal detail was the determination of temperature distribution[2]. The method makes use of polynomial approximations to the temperature distribution by means of which the Fourier equation is reduced to a set of first-order ordinary differential equations.

It was later applied to the solution of the biharmonic equation[3,4,5]. Reference 3 makes use of polynomial approximations to a set of ordinary differential equations. This results in satisfying the differential equation everywhere in one direction and at a finite number of stations in the other direction. The boundary conditions are everywhere satisfied. Reference 5 provides intermediate information from a collocation procedure in tabular form that minimizes the effort required to determine the spanwise and chordwise stresses for a large variety of plate geometries and temperature distributions.

The solution of a set of ordinary differential equations by classical means, which results after collocating for one direction, can become somewhat cumbersome. Not only does the final solution contain many terms, but there are involved solutions for and manipulations of, complex numbers which do not degenerate into the real domain until the last stage of analysis. Furthermore, the determination of a particular integral may become difficult unless the temperature is specified in terms of a single function. Hence, Manson suggested a method of "double collocation"[4] which avoids all of these difficulties by applying a collocation procedure to the solution of ordinary differential equations.

Since Manson's work was done, the double collocation procedure has been applied to a finite cylinder[6] and a thin, flat plate[7] problem with some modifications. In those references, integrodifferential equations in the shear stresses are derived that lend themselves to solution by a two-dimensional collocation procedure requiring relatively little labor.

III. ANALYSIS

A. Solution by Double Collocations:

As discussed in references 4 and 9, the double-collocation method requires relatively little labor, and yields results nearly identical to that of the single-collocation method. The application of this method to the biharmonic equation and the Fourier equation will be described in some detail through the use of two simplified examples.

1. Biharmonic equation:

Consider a rectangular thin plate, 2 by 6 k, of constant thickness with temperature distribution $T = T(x,y)$, as shown in Fig. 1. The determination of thermal stresses requires the solution of the biharmonic equation[8]

$$\nabla^4\phi = -\nabla^2 E\alpha T \tag{1}$$

The stresses are given in terms of the stress function as follows

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}\ ,\ \sigma_y = \frac{\partial^2\phi}{\partial x^2}\ ,\ \tau_{xy} = \frac{-\partial^2\phi}{\partial x\partial y} \tag{2}$$

The boundary conditions at the edges of the plate are for $k = 1$:

$$\text{at}\quad x = \pm 3 \qquad \sigma_x = \frac{\partial^2\phi}{\partial y^2} = 0\ ,\quad \tau_{xy} = \frac{-\partial^2\phi}{\partial x\partial y} = 0 \tag{3}$$

$$\text{at} \quad y = \pm 1 \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0 \;, \quad \tau_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y} = 0 \tag{4}$$

Equation (1) will be solved approximately by a collocation procedure where by the differential equation is satisfied at m by n stations. m stations are taken along x, and n stations are taken along y. The stress function ϕ is then assumed to have the following form:

$$\phi = \sum_{i=1}^{m} \sum_{j=1}^{n} P_i(x) Q_j(y) \phi_{ij} \tag{5}$$

Where ϕ_{ij} is a numerical value of ϕ at particular stations as shown in Fig. 1. This constant is as yet unknown. P_i and Q_j are station functions and satisfy the following conditions in order to ensure that Eq. (5) holds for any ϕ_{ij}:

$$P_i(x_i) = 1 \;, \; Q_j(y_j) = 1 \tag{6}$$

$$\begin{aligned} P_i(x_k) &= 0 \;, \quad k \neq i \\ Q_i(x_k) &= 0 \;, \quad k \neq i \end{aligned} \tag{7}$$

In addition to satisfying the above conditions, the station functions are chosen to satisfy the boundary conditions, Eqs. (3) and (4):

$$P_i(\pm 3) = 0 \;, \quad Q_j(\pm 1) = 0 \tag{8}$$

$$P_i'(\pm 3) = 0 \ , \quad Q_j'(\pm 1) = 0 \tag{9}$$

Polynomials that have the desired properties can readily be obtained. For example:

$$P_i(x) = \frac{(x^2-9)^2}{(x_i^2-9)^2} \prod_{k\neq i} (x-x_k) / \prod_{k\neq i} (x_i-x_k) \ , \quad x_i \neq 1 \tag{10}$$

Where $\prod_{k\neq i}$ is the product for all values of k except k=i.

Equation (5) is now substituted into Eq. (1) and evaluated at each of the m stations in x and n stations in y to produce a system of m by n simultaneous linear algebraic equations in ϕ_{ij} of the following form:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} A_{k\ell}^{ij} \phi_{ij} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E\alpha T(x_k, y_\ell) \tag{12}$$

$$k=1,2,\ldots,m$$

$$\ell=1,2,\ldots,n$$

$$\text{where } A_{k\ell}^{ij}=P_i''''(x_k)Q_j(y_\ell)+2P_i''(x_k)Q_j''(y_\ell) + P_i(x_k)Q_j''''(y_\ell) \tag{13}$$

The stress function at any point can now be determined from Eq. (5), and the stresses can be determined from Eq. (2).

For a numerical example, the curves as shown in Figs. 2 and 3 are calculated for the following data:

$T=(y^2-\frac{1}{3})T_0$, $k=1$, $n=3$, $m=3$, $y_1=\frac{1}{6}$, $y_2=\frac{3}{6}$, $y_3=\frac{5}{6}$,

$x_1=0.5$, $x_2=1.5$, $x_3=2.5$.

Figures 2 and 3 show that the curves of the single-collocation method coincide with the curves of the double-collocation method.

2. Fourier equation:

The chief difficulty of many thermal shock problems is in evaluating the temperature distribution at a given time after conditions are suddenly changed. The solution of the Fourier equation in one dimension by the single-collocation method has been used in reference 2. For a two-dimensional problem, the double-collocation method can be applied.

For example, the Fourier equation for the transient temperature distribution in a finite solid cylinder, as shown in Fig. 4, with rotational symmetry is

$$\frac{\partial^2 T^*}{\partial z^2} + \frac{1}{r}\frac{\partial T^*}{\partial r} + \frac{\partial^2 T^*}{\partial r^2} = \frac{\partial T^*}{\partial \tau^*} \tag{14}$$

Boundary conditions are:

$$\left.\frac{\partial T^*}{\partial z}\right|_{z=0} = 0$$

$$\left.\frac{\partial T^*}{\partial r}\right|_{r=0} = 0 \qquad (15)$$

$$\left.\frac{\partial T^*}{\partial z}\right|_{z=1} = -\beta T^*_{z=1}$$

$$\left.\frac{\partial T^*}{\partial r}\right|_{r=1} = -\beta T^*_{r=1} \qquad (16)$$

$$T^*_0 = T^*_0(z,r,0) \text{ at } \tau^* = 0 \qquad (17)$$

$$\text{Assume} \quad T^* = \sum_{i=1}^{m} \sum_{j=1}^{n} P_i(z)Q_j(r)T^*_{ij} \qquad (18)$$

T^*_{ij} is a numerical value of T^* at station ij in the cylinder as shown in Fig. 4. P_i and Q_j are station functions and satisfy the following conditions:

$$P_i(z_i) = 1$$

$$Q_j(r_j) = 1 \qquad (19)$$

$$P_i(z_k) = 0 \quad k \neq i$$

$$Q_j(r_k) = 0 \quad k \neq j \qquad (20)$$

Furthermore, in order to satisfy the boundary conditions in the z direction, all the polynomials except P_m

have zero slope at $z=1$ whereas P_m has the proper slope at z equal to 1 to satisfy the boundary condition there, and P_1 has zero slope at $r=0$. Such polynomials can be written by inspection. For example,

$$P_1 = \frac{(z-1)^2}{(0-1)^2} \cdot \frac{(z-a_1)}{(0-a_1)} \cdot \prod_{i=2}^{m-1} \frac{(z-z_i)}{(0-z_i)}$$

$$\vdots$$

$$P_j = \frac{(z-1)^2}{(z_j-1)^2} \cdot \frac{z^2}{z_j^2} \cdot \prod_{\substack{i=2 \\ i\neq j}}^{m-1} \frac{(z-z_i)}{(z_j-z_i)}$$

$$\vdots$$

$$P_m = \frac{z^2}{1^2} \cdot \frac{(z-a_2)}{(1-a_2)} \prod_{i=2}^{m-1} \frac{(z-z_i)}{(1-z_i)}$$

If $m=4$, $z_1=0$, $z_2=\frac{1}{3}$, $z_3=\frac{2}{3}$, and $z_4=1$, then

$$P_1 = \frac{(z-\frac{1}{3})(z-\frac{2}{3})(z-1)^2(z-a_1)}{(-\frac{1}{3})(-\frac{2}{3})(-1)(-a_1)}$$

$$P_2 = \frac{(z)^2(z-\frac{2}{3})(z-1)^2}{(\frac{1}{3})^2(-\frac{1}{3})(\frac{2}{3})^2}$$

$$P_3 = \frac{(z)^2(z-\frac{1}{3})(z-1)^2}{(\frac{2}{3})^2(\frac{1}{3})(\frac{-1}{3})^2}$$

$$P_4 = \frac{(z)^2(z-\frac{1}{3})(z-\frac{2}{3})(z-a_2)}{(1)(\frac{2}{3})(\frac{1}{3})(1-a_2)}$$

Where a_1 and a_2 are used to satisfy Eqs. (15) and (16), respectively. Equation (18) is now substituted into Eq. (14) and evaluated at each of the m stations in z and n stations in r to produce a system of first order linear differential equations in T_{ij} of the following form:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} B_{k\ell}^{ij} \, \dot{T}^*_{ij} = \frac{\partial}{\partial \tau^*} T^*(z_k, r_\ell) \qquad \begin{array}{l} k=1,2,\ldots,m \\ \ell=1,2,\ldots,n \end{array} \tag{21}$$

where

$$B_{k\ell}^{ij} = P_i''(z_k)Q_j(r_\ell)+\frac{1}{r}P_i(z_k)Q_j'(r_\ell)+P_i(z_k)Q_j''(r_\ell) \tag{22}$$

These equations are solvable by many methods. In Fig. 5, the numerical data calculated by digital computer using the Runge-Kutta method is compared with the exact solution. It shows that the approximate method has good accuracy.

B. Application of the Station-Function Collocation Method to General Two-Dimensional Thermal-Stress Problems:

1. General equations:

With reference to the coordinate system shown in Fig. 6(a), the relations that define the elastic state of stress and strain for the case of plane stress for a solid section are given as follows[8]:

$$\varepsilon_\xi = \frac{1}{E}(\sigma_\xi - \nu\sigma_\eta) + \alpha T \tag{23a}$$

$$\varepsilon_\eta = \frac{1}{E}(\sigma_\eta - \nu\sigma_\xi) + \alpha T \tag{23b}$$

$$\gamma = \frac{2(1+\nu)}{E}\,\tau \tag{23c}$$

$$\frac{\partial}{\partial\xi}\,\sigma_\xi + \frac{\partial}{\partial\eta}\,\tau = 0 \tag{24a}$$

$$\frac{\partial}{\partial\eta}\,\sigma_\eta + \frac{\partial}{\partial\xi}\,\tau = 0 \tag{24b}$$

$$\frac{\partial^2}{\partial\eta^2}\,\varepsilon_\xi + \frac{\partial^2}{\partial\xi^2}\,\varepsilon_\eta - \frac{\partial^2}{\partial\xi\partial\eta}\,\gamma = 0 \tag{25}$$

Making all quantities dimensionless by dividing Eqs. (23) and (25) by αT_0 and Eq. (24) by $E\alpha T_0$ and transforming ξ and η to x and y (Fig. 6(b)) yields:

$$\varepsilon_x^* = \sigma_x^* - \nu\sigma_y^* + T^* \tag{26a}$$

$$\varepsilon_y^* = \sigma_y^* - \nu\sigma_x^* + T^* \tag{26b}$$

$$\gamma^* = 2(1+\nu)\tau^* \tag{26c}$$

$$\frac{1}{\beta}\,\frac{\partial}{\partial x}\,\sigma_x^* + \frac{\partial}{\partial y}\,\tau^* = 0 \tag{27a}$$

$$\frac{\partial}{\partial y}\,\sigma_y^* + \frac{1}{\beta}\,\frac{\partial}{\partial x}\,\tau^* = 0 \tag{27b}$$

$$\frac{\partial^2}{\partial y^2} \varepsilon_x^* + \frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} \varepsilon_y^* - \frac{1}{\beta} \frac{\partial^2}{\partial x \partial y} \gamma^* = 0 \tag{28}$$

For greatest generality, a transformation is made to another coordinate system having the following properties:

$$x = \frac{\xi}{b} ; \quad y = \frac{\eta}{a} ; \quad \beta = \frac{b}{a}$$

Expressing the compatibility Eq. (28) in terms of the stresses, first by substituting the stress-strain Eqs. (26) into Eq. (28) and then eliminating the shear terms by differentiating Eq. (27a) with respect to x and (27b) with respect to y and substituting yields:

$$\left(\frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) (\sigma_x^* + \sigma_y^*) = -\left(\frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T^* \tag{29}$$

If a stress function ϕ is defined as follows:

$$\sigma_x^* = \frac{\partial^2 \phi}{\partial y^2} \tag{30a}$$

$$\sigma_y^* = \frac{1}{\beta^2} \cdot \frac{\partial^2 \phi}{\partial x^2} \tag{30b}$$

$$\tau_{xy}^* = -\frac{1}{\beta} \cdot \frac{\partial^2 \phi}{\partial x \partial y} \tag{30c}$$

then Eqs. (27) are automatically satisfied. Substituting Eqs. (30) into Eq. (29) gives the equation to be solved for the stress function:

$$\left(\frac{1}{\beta^4}\frac{\partial^4}{\partial x^4} + \frac{2}{\beta^2}\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)(\phi) = -\left(\frac{1}{\beta^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T^* \tag{31}$$

The boundary conditions for the traction-free lateral surface become†

$$\begin{aligned} \frac{1}{\beta}\sigma_x^*\ell + \tau^* m &= 0 \\ \frac{1}{\beta}\tau^*\ell + \sigma_y^* m &= 0 \end{aligned} \tag{32}$$

Where $\ell = \cos\theta = -\frac{dy}{ds}$ and $m = \sin\theta = \frac{dx}{ds}$, as shown in Fig. 6(b). Upon substituting from Eqs. (30), Eq. (32) becomes

$$\frac{\partial^2\phi}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2\phi}{\partial x \partial y}\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial y}\right) = 0$$

$$\frac{\partial^2\phi}{\beta\partial x\partial y}\frac{\partial y}{\partial s} + \frac{\partial^2\phi}{\beta\partial x^2}\frac{\partial x}{\partial s} = \frac{1}{\beta}\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial x}\right) = 0$$

Thus

$$\frac{\partial\phi}{\partial x} = A \;, \quad \frac{\partial\phi}{\partial y} = B$$

and $\phi = \int_0^s \frac{\partial\phi}{\partial s}\, ds = \int_0^s \left(A\frac{\partial x}{\partial s} + B\frac{\partial y}{\partial s}\right) ds = Ax + By + C$

† If surface forces are present, an isothermal two-dimensional problem may be solved separately and the results superposed.

Where A, B, and C are constants. Since the stress components depend only on the second derivatives of ϕ, it is permissible to take these constants as zero. Then

$$\frac{\partial \phi}{\partial n} = A\frac{\partial x}{\partial n} + B\frac{\partial y}{\partial n}$$

also vanishes. With the boundary conditions for ϕ

$$\phi = \frac{\partial \phi}{\partial n} = 0 \tag{33}$$

2. Solution by collocation:

Equation (31) will be solved approximately by the method of collocation in two dimensions. It is assumed that there exists a function ψ that converges to the solution ϕ of Eq. (9) as n approaches infinity and that is given by

$$\psi = \sum_{i=1}^{n} P_i(x)Q_i(y)\phi_i \tag{34}$$

where $P_i(x)$ and $Q_i(y)$ are known polynomials selected in such a manner that the boundary conditions and certain other conditions to be defined are satisfied. The constant ϕ_i is a specific value of ϕ at station i, and n is the number of stations, as shown in Fig. 7, that are chosen for solving Eq. (31). The collocation method now requires that the error in replacing ϕ by ψ

in Eq. (31) vanish at n specified points. To do this, Eq. (34) is substituted into Eq. (31) and the ϕ_i determined so that Eq. (31) is satisfied at each of these n stations. This will result in n linear algebraic equations for the unknown ϕ_i.

In order to satisfy the conditions at the n stations the polynomials $P_i(x)$ and $Q_i(y)$ are chosen as follows:

$$P_i(x_i) = 1 \tag{35a}$$

$$P_i(x_j) = 0 \ , \ j \neq i \tag{35b}$$

The quantity $Q_i(y)$ is a polynomial in y associated with the i^{th} station and satisfies similar conditions

$$Q_i(y_i) = 1 \tag{35c}$$

$$Q_i(y_j) = 0 \ , \ j \neq i \tag{35d}$$

Hence, as seen from Eq. (34) and Fig. 7, if the value of y is fixed, the function ψ yields

$$\psi = P_{i-a}\phi_{i-a}+P_{i-a+1}\phi_{i-a+1}+\ldots+P_i\phi_i+\ldots+P_{i+c}\phi_{i+c} \tag{36}$$

when $x = x_i$, then $\psi=\phi_i$.

In addition to satisfying the station conditions, the polynomials P_i and Q_i are chosen to satisfy the boundary conditions. From Eq. (33), $\phi = \frac{\partial\phi}{\partial n} = 0$, we have

$$P_i(x_{Bi}) = 0, \; P_i(x_{bi}) = 0$$
$$Q_i(y_{Bi}) = 0, \; Q_i(y_{bi}) = 0 \tag{37}$$

and

$$\frac{\partial P_i(x_{Bi})}{\partial x} = 0, \; \frac{\partial P_i(x_{bi})}{\partial x} = 0$$
$$\frac{\partial Q_i(y_{Bi})}{\partial y} = 0, \; \frac{\partial Q_i(y_{bi})}{\partial y} = 0 \tag{38}$$

Where x_{Bi} and x_{bi} are the x coordinates of the boundary at $y = y_i$, and y_{Bi} and y_{bi} are the y coordinates of the boundary at $x = x_i$, as shown in Fig. 7.

Equations (35-38) are the desired properties of the polynomials $P_i(x)$ and $Q_i(y)$.

Equation (34) is now substituted into Eq. (31), and the equation is then evaluated at each of the n stations. This results in a set of algebraic equations of the following form:

$$\sum_{i=1}^{n} [\frac{1}{\beta^4} P_i^{''''}(x) Q_i(y) + \frac{2}{\beta^2} P_i^{''}(x) Q_i^{''}(y) + P_i(x) Q_i^{''''}(y)] \phi_i$$
$$= -\frac{1}{\beta^2} T_{xx}(x,y) - T_{yy}(x,y) \tag{39}$$

Substituting the values of coordinates x and y of each station into the above equation, we get n algebraic equations

$$\sum_{i=1}^{n}\left[\frac{1}{\beta^4}P_i''''(x_j)Q_i(y_j)+\frac{2}{\beta^2}P_i''(x_j)Q_i''(y_j)\right.$$

$$\left.+P_i(x_j)Q_i''''(y_j)\right]\phi_i=-\frac{1}{\beta^2}T_{xx}(x_j,y_j)-T_{yy}(x_j,y_j) \qquad (40)$$

where $j = 1,2,3,\ldots,n$.

3. The determination of station functions:

In the solution of a general two-dimensional thermal-stress problem, it is desirable to express the station functions satisfying Eqs. (35), (37), and (38). For any other particular problem, the station functions may be changed to satisfy the desired properties. In this section we will consider the general two-dimensional problem. Let us choose stations 1, 2, 3, and 4 as examples. For any other station we can follow the same steps as the examples to obtain the desired polynomial.

In order to satisfy equations (35), assume

$$P_1 = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$P_2 = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$P_3 = \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$P_4 = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

In order to satisfy the boundary conditions, Eqs. (37) and (38), at $x=x_{B1}$ and $x=x_{b1}$ (where x_{Bi} is defined as the right side boundary coordinate in the x direction of a line through station i, so $x_{B1}=x_{B2}=x_{B3}=x_{B4}$; similarly x_{bi} is the left side boundary coordinate in x direction), we have to add more factors into these station functions. Let us reassume the station functions:

$$P_1 = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} \cdot \frac{(x-x_{B1})^2}{(x_1-x_{B1})^2} \cdot \frac{(x-x_{b1})^2}{(x_1-x_{b1})^2}$$

$$P_2 = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \cdot \frac{(x-x_{B1})^2}{(x_2-x_{B1})^2} \cdot \frac{(x-x_{b1})^2}{(x_2-x_{b1})^2}$$

$$P_3 = \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} \cdot \frac{(x-x_{B1})^2}{(x_3-x_{B1})^2} \cdot \frac{(x-x_{b1})^2}{(x_3-x_{b1})^2} \qquad (41)$$

$$P_4 = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \cdot \frac{(x-x_{B1})^2}{(x_4-x_{B1})^2} \cdot \frac{(x-x_{b1})^2}{(x_4-x_{b1})^2}$$

Where the factors $\dfrac{(x-x_{B1})^2}{(x_i-x_{B1})^2}$, $i=1,2,3,4$, are used for satisfying the boundary conditions $\phi = \frac{\partial\phi}{\partial x} = 0$ at $x = x_{B1}$, and the factors $\frac{(x-x_{b1})^2}{(x_i-x_{b1})^2}$, $i = 1,2,3,4,$ for satisfying the boundary conditions $\phi = \frac{\partial\phi}{\partial x} = 0$ at $x = x_{b1}$. So the final station functions that satisfy all conditions are shown in Eqs. (41). For any other station we can write down its station function without much difficulty. For example, the station function at i in x direction is

$$P_i = \frac{(x-x_{i-a})\ldots(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-a})\ldots(x_i-x_{i-1})(x_i-x_{i+1})} \qquad (42)$$

$$\frac{\ldots(x-x_{i+c})(x-x_{Bi})^2(x-x_{bi})^2}{\ldots(x_i-x_{i+c})(x_i-x_{Bi})^2(x_i-x_{bi})^2}$$

Since the station functions have the same form, the calculation of each station can be simplified by using a high-speed digital computer. For each new station, it is simply necessary to change the coordinate constants. Furthermore, as shown in the previous work[1-6], it is unnecessary to use many stations to solve a simple problem.

4. Example:

Consider a circular disk, a = b = 1, subjected to a thermal gradient in the r-direction independent of

θ and z:

$$T = (r^2 - \tfrac{1}{2})T_0 \quad \text{or} \quad T = (x^2 + y^2 - \tfrac{1}{2})T_0$$

The determination of the thermal stresses requires the solution of Eq. (31). Since the temperature distribution is symmetric about x and y, only the first quadrant will be considered. Assume m = n = 6, as shown in Fig. 8, the polynomials that satisfy the Eqs. (35), (37), and (38) are chosen as, for example,

$$P_{15} = \frac{(x^2-.1^2)(x^2-.3^2)(x^2-.68^2)(x^2-.82^2)}{(.5^2-.1)(.5^2-.3^2)(.5^2-.68^2)(.5^2-.82^2)} \cdot \frac{[x^2-(1^2-.5^2)]}{[.5^2-(1^2-.5^2)]}$$

Similarly, the polynomials of each station can be obtained by the same method as described previously. Substituting these polynomials and their derivatives into Eqs. (40) produce a system of simultaneous algebraic equations, and the right sides of these equations are a constant, -4. The solution to the system is shown in Fig. 9 compared with the exact solution. Figures 10 and 11 are the stresses obtained from the stress function. These figures disclose that the approximate method has an acceptable degree of accuracy.

IV. DISCUSSION AND CONCLUSIONS

In order to determine the relative accuracy of the double and single collocation methods, comparisons are made in Figs. 2 and 3. It can be seen that for these examples the results of both methods are nearly identical.

The numerical example of the double-collocation procedure to the Fourier equation, as discussed previously, is shown in Fig. 5. Using a 4 by 4 station collocation, the temperature changes at stations T^*_{11}, T^*_{41}, and T^*_{44} are plotted with respect to time. The curves of the approximate solution and the exact solutions show good agreement.

As shown in Fig. 9, the stress function obtained by the approximate method compares well with the exact solution. The accuracy is generally improved by collocating at a larger number of stations. However, the increasing degree of the polynomials involved in the use of a large number of stations may result in poor approximations between the lines of the stations.

Using Eqs. (30) and (34) and the stress function values at all stations, we obtain the normal stresses. The results are shown in Fig. 10. The exact solution of a thin circular disk is given in Appendix A. A block diagram showing the procedure of computation of the approximate method is shown in Appendix B.

The shear stress calculated by the same method as normal stresses yields data which oscillates along the exact curves. The possible reasons are as follows: (1) The shear stress requires the differentiation of the stress function in both directions which results in the shear stress being more unstable; and (2) Near the boundary of the disk, the stress function ϕ_i at each station is small and the polynomials P_i' and Q_i' at each station are large. For a small difference of stress function, the error will exaggerate in the term containing their product $P_i'Q_i'\phi_i$. The dashed lines, shown in Fig. 11, have been calculated directly from the stress function values at the stations by the finite difference method.

There are some methods which can be applied for obtaining better results. For example, since the normal stresses have good accuracy, we may rotate the coordinate system through an angle, again determine the normal stresses and then calculate the principal stresses at each specific point with the two sets of normal stresses. The equation for finding the principal stresses is presented in Appendix C.

Since the collocation method is general in nature, the extension to other equations or three-dimensional problems is possible. For example, the station function at a specific point in a three-dimensional problem can be assumed as $P_i(x)Q_i(y)R_i(z)\phi_i$. The polynomial $R_i(z)$ could then be determined by the same method as P_i.

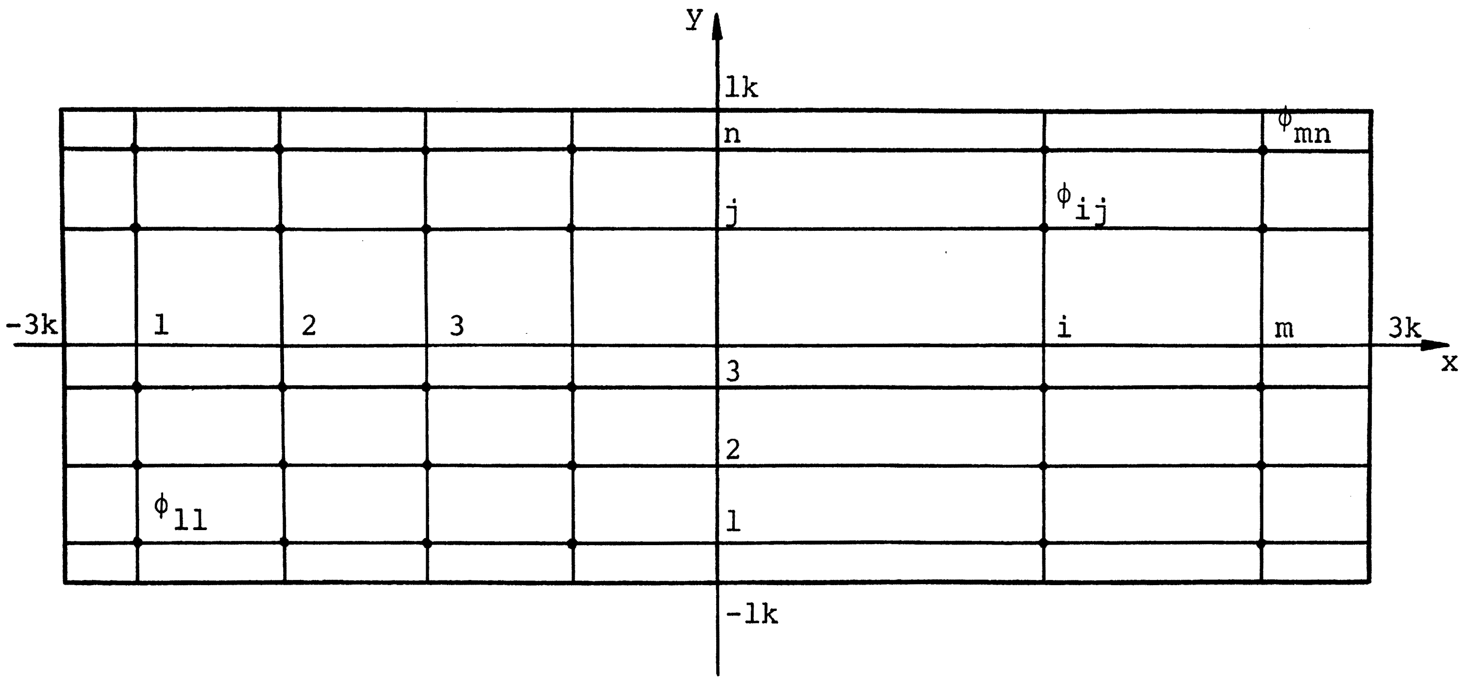


FIGURE 1. Thin Plate of Constant Thickness Showing the Stations at Which the Differential Equation is Satisfied

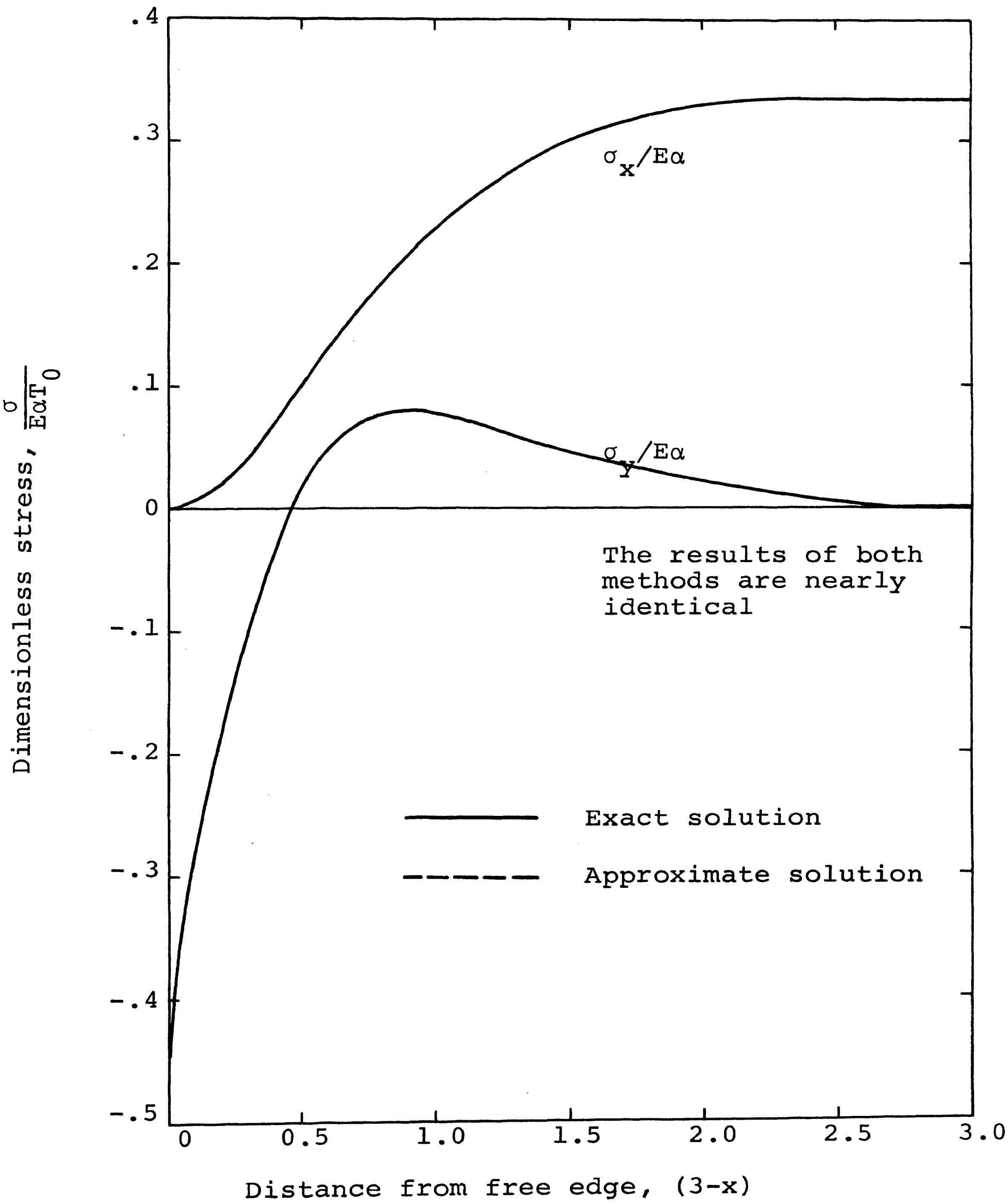


FIGURE 2. Comparison of Stresses in a Thin Rectangular Plate at y=0 for the Single-Collocation and Double-Collocation Methods

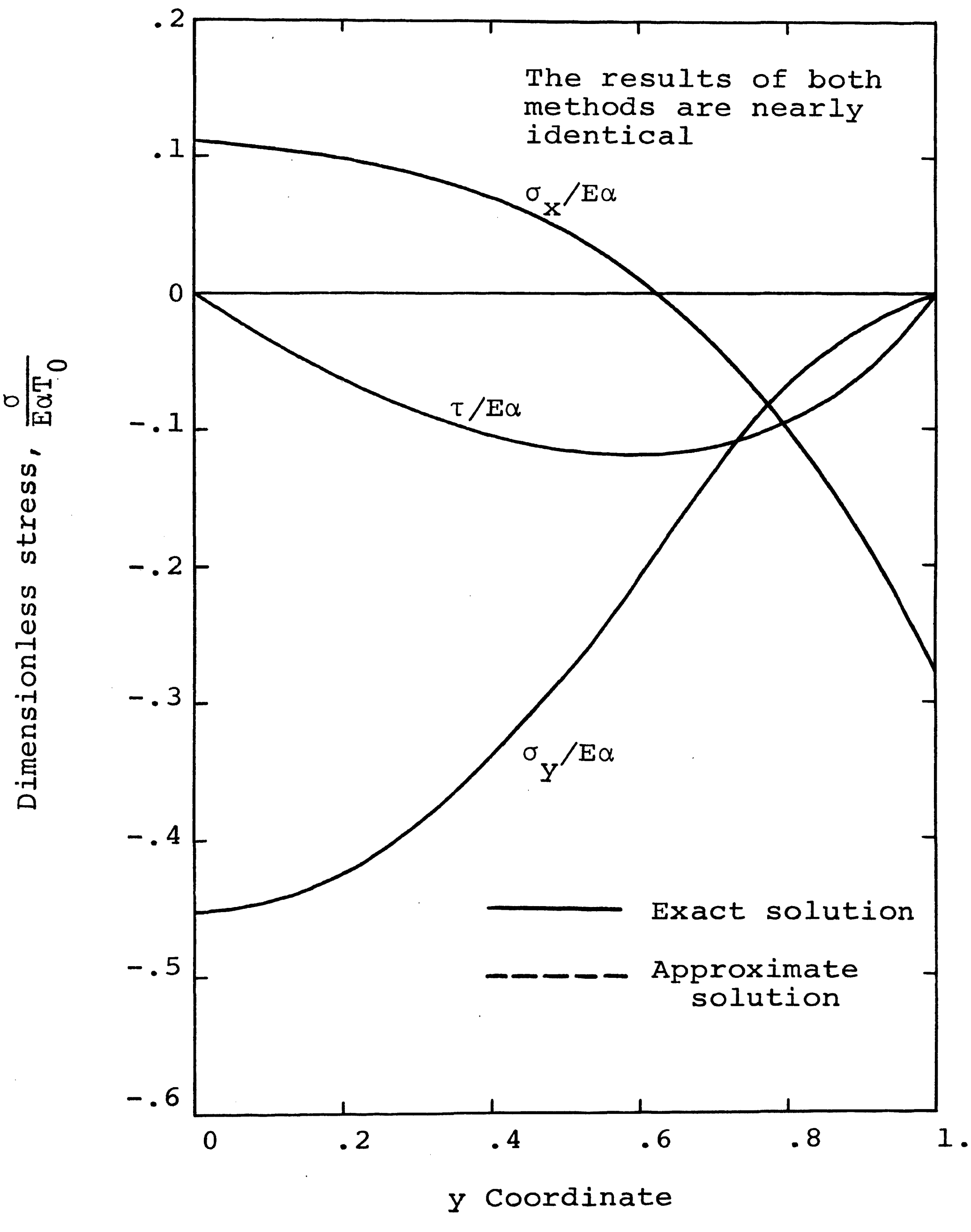


FIGURE 3. Comparison of the Stresses in a Thin Rectangular Plate for the Single-Collocation and Double-Collocation Methods. σ_y Plotted at Free End x=3, σ_x and τ Plotted at x=2½

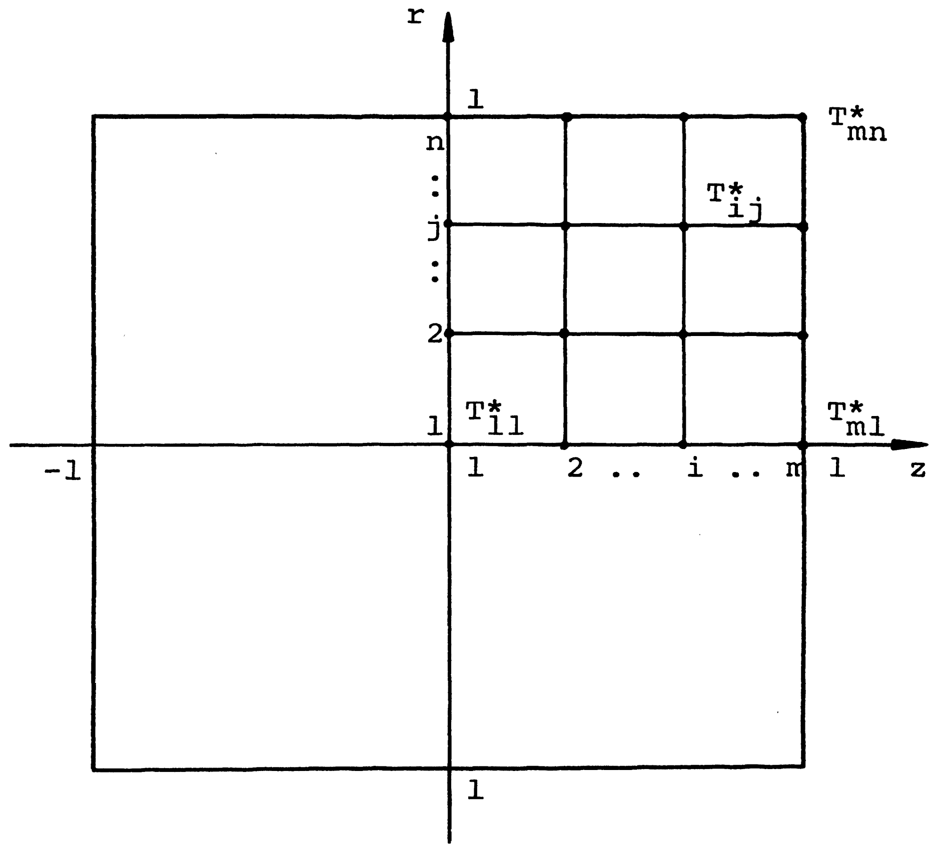


FIGURE 4. Finite Solid Cylinder Showing the Stations at Which the Differential Equation is Satisfied

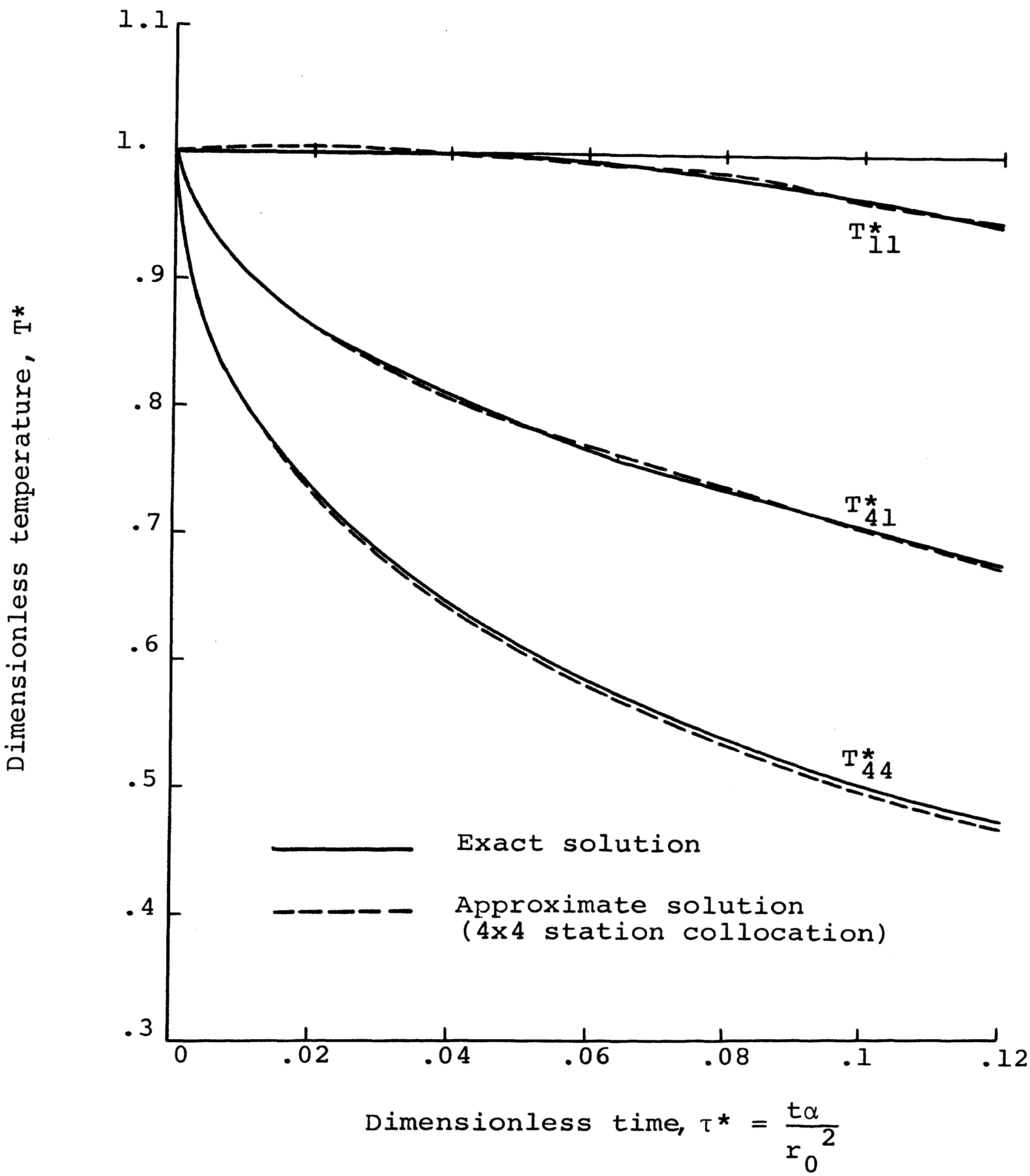
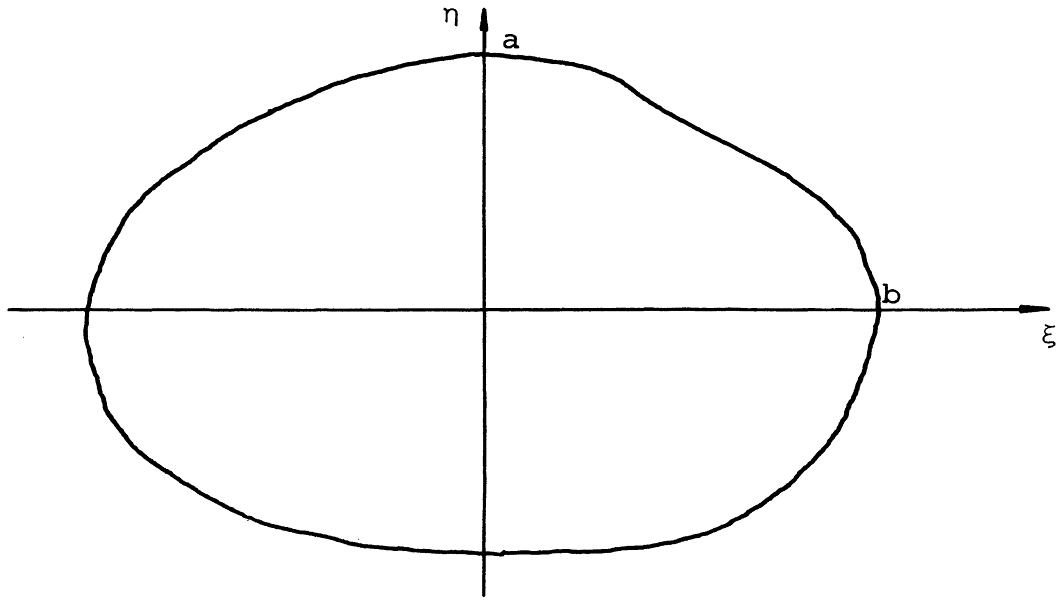
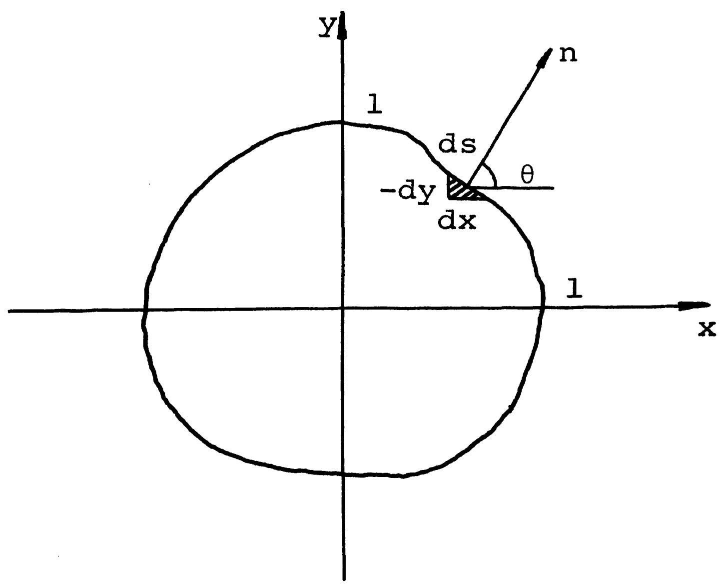


FIGURE 5. Comparison of Temperatures in a Finite Solid Cylinder for the Exact and Approximate Solutions

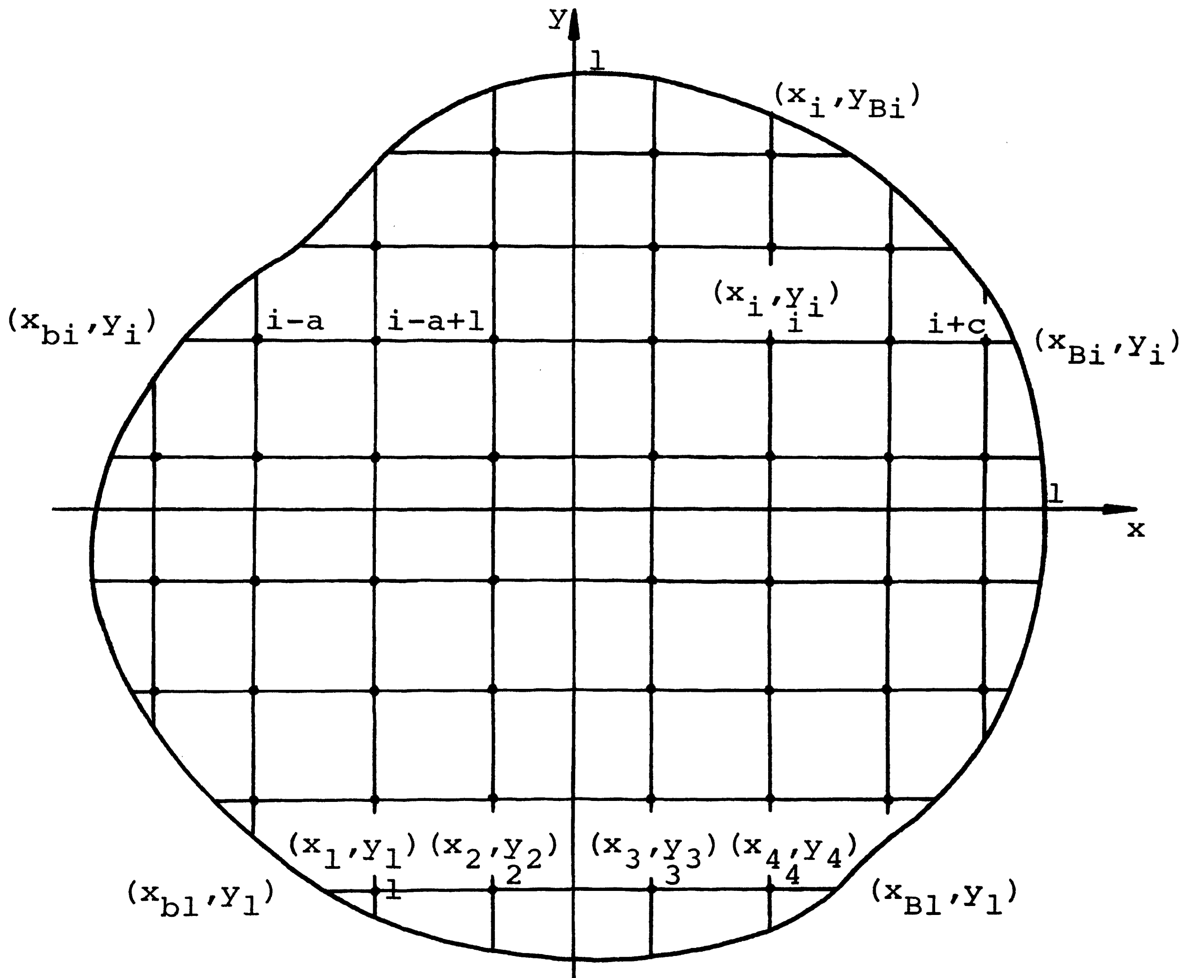


(a) ξ,η Coordinate system



(b) x,y Coordinate system

FIGURE 6. Coordinate System used to Calculate Two-Dimensional Stress Distribution Due to Thermal Loading



n=Number of stations where $y_1=y_2=y_3=y_4$

$x_{B1}=x_{B2}=x_{B3}=x_{B4}$

$x_{b1}=x_{b2}=x_{b3}=x_{b4}$

FIGURE 7. Station-Function Collocation Net for Two-Dimensional Thermal Stress Problems

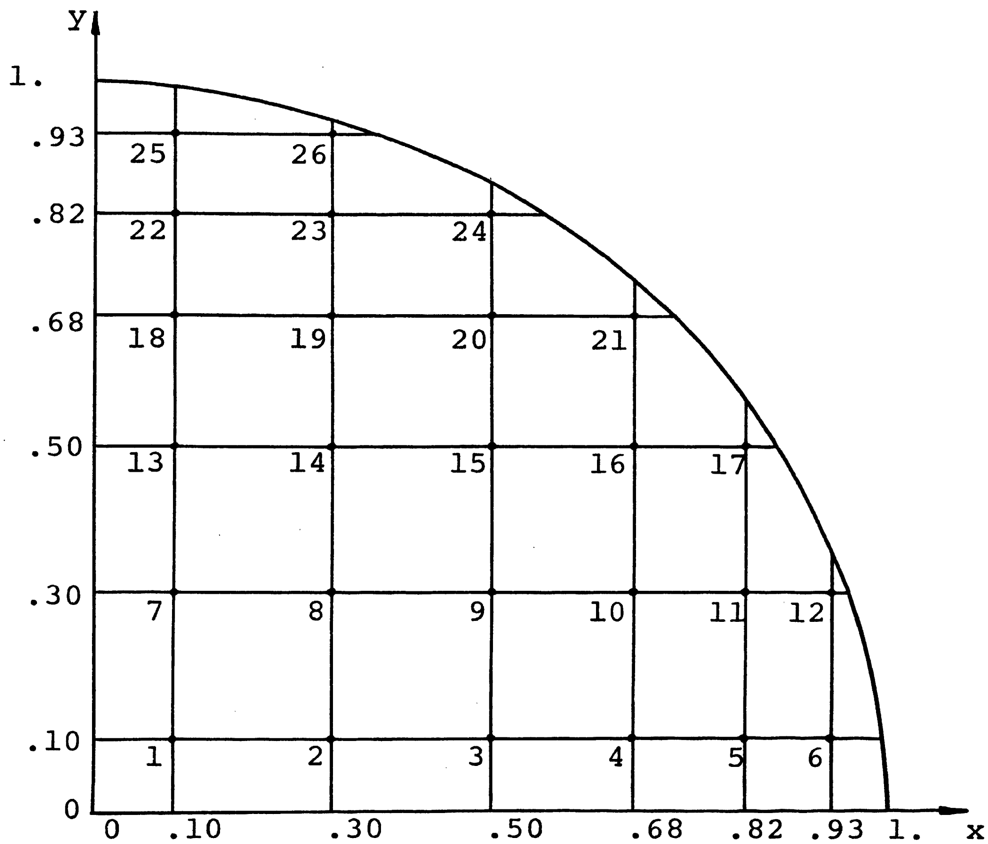


FIGURE 8. Thin Circular Plate Showing Stations

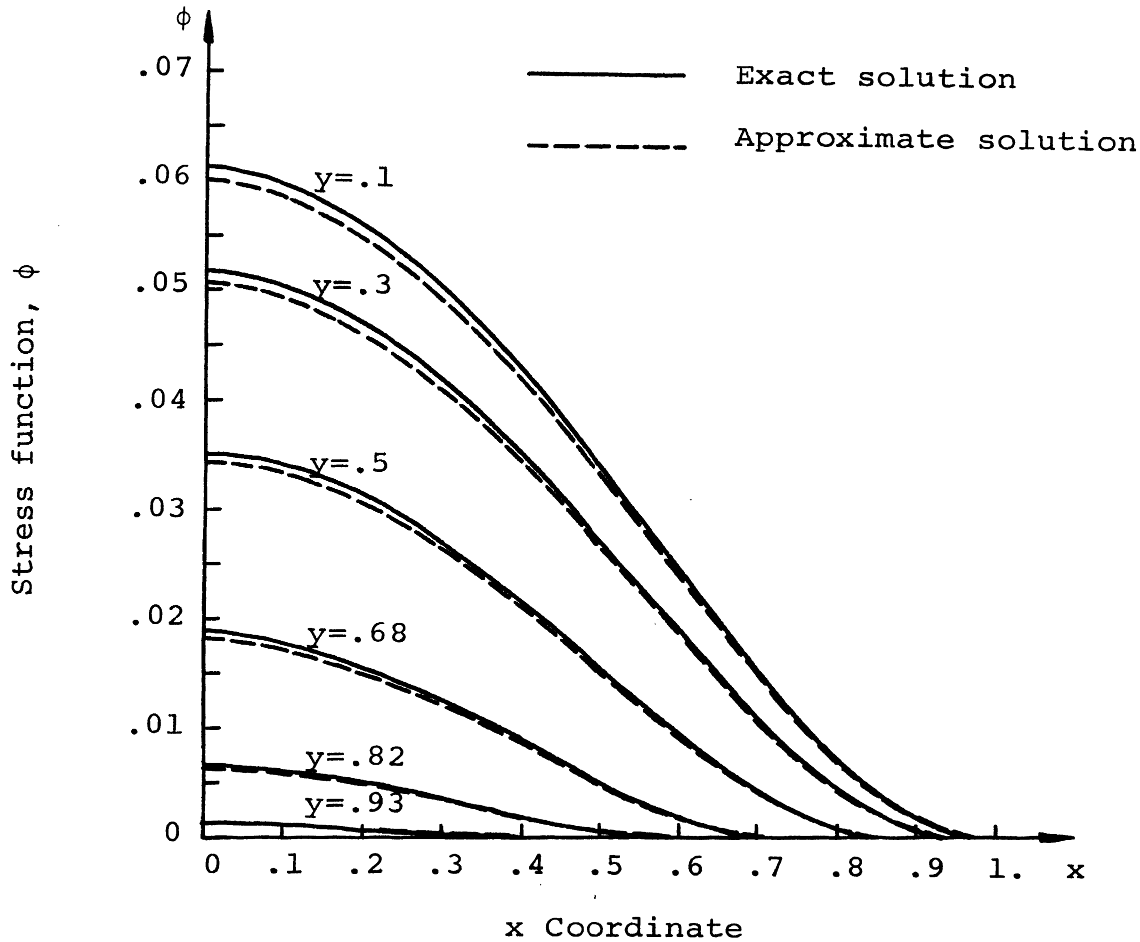


FIGURE 9. Comparison of Stress Functions of a Thin Circular Plate for the Exact and Approximate Solutions

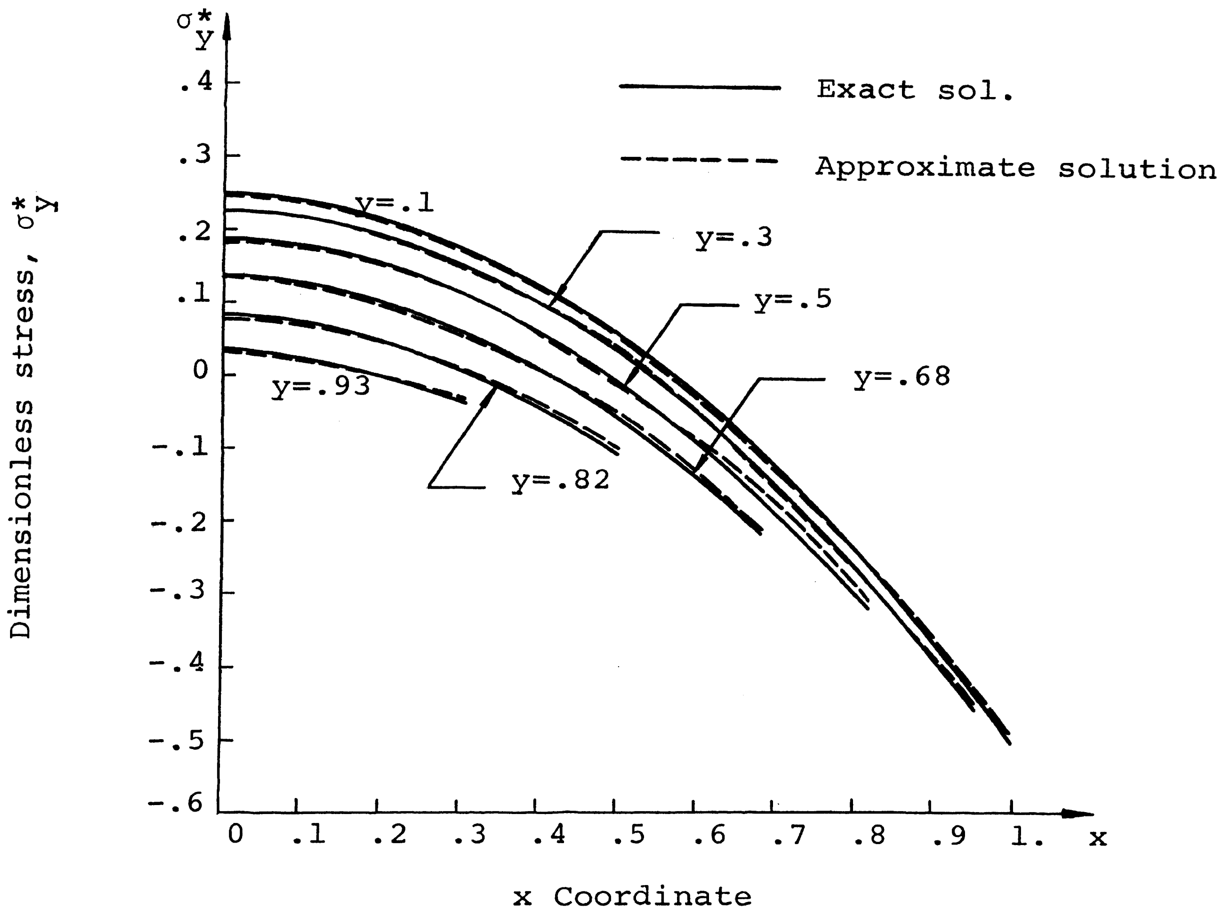


FIGURE 10. Comparison of Dimensionless Stress in a Thin Circular Plate for the Exact and Approximate Solutions

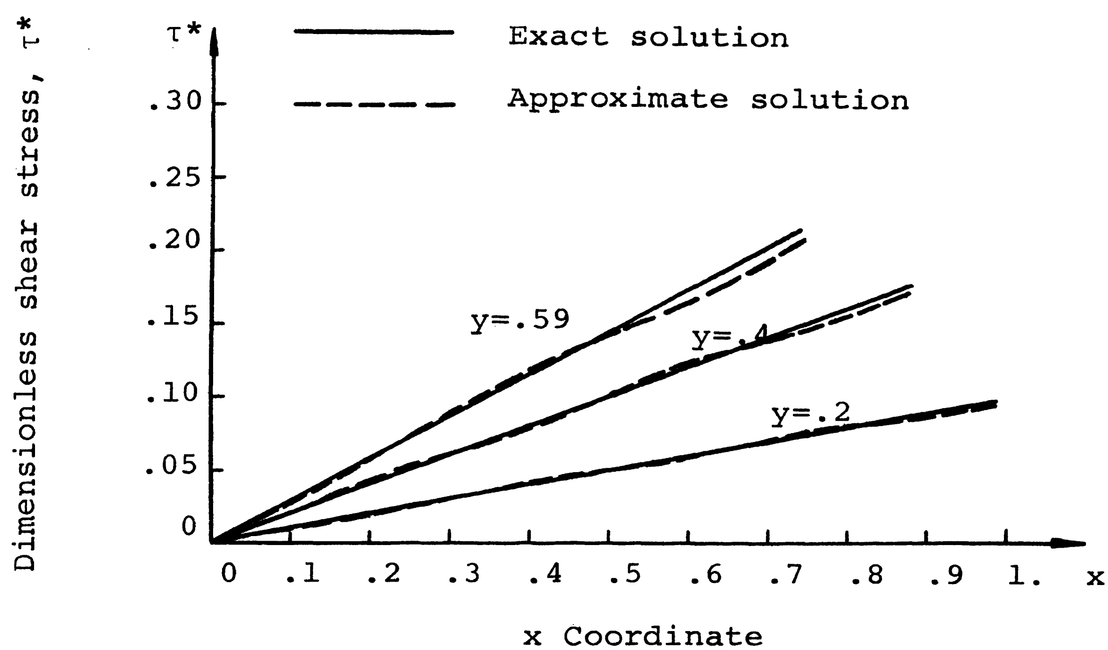


FIGURE 11. Comparison of Dimensionless Shear Stress in a Thin Circular Plate for the Exact and Approximate Solutions

APPENDIX A

THE EXACT SOLUTION FOR A THIN CIRCULAR DISK: TEMPERATURE SYMMETRICAL ABOUT CENTER

$$\sigma_r = \alpha E\left(\frac{1}{b^2}\int_0^b TRdR - \frac{1}{R^2}\int_0^R TRdR\right)$$

$$\sigma_\theta = \alpha E\left(-T + \frac{1}{b^2}\int_0^b TRdR + \frac{1}{R^2}\int_0^R TRdR\right)$$

Where b is the outside diameter, and R is the radius.

Assume

$$T = (R^2 - \tfrac{1}{2})T_0 \text{ and } b = 1, \text{ then}$$

$$\sigma_r = \alpha E T_0\left(\tfrac{1}{4} - \frac{R^2}{4}\right)$$

$$\sigma_\theta = \alpha E T_0\left(\tfrac{1}{4} - \tfrac{3}{4}R^2\right)$$

$$\text{or } \sigma_r^* = \tfrac{1}{4}(1 - r^2)$$

$$\sigma_\theta^* = \tfrac{1}{4}(1 - 3r^2)$$

For a one-dimensional problem,

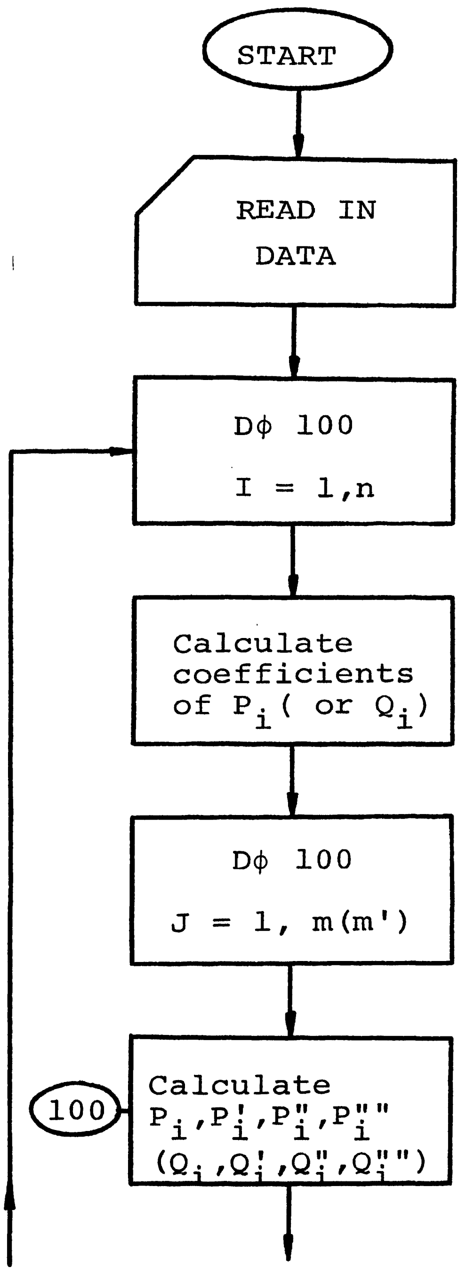
$$\sigma_r^* = \frac{1}{r}\frac{d\phi}{dr} \tag{A}$$

$$\text{and} \quad \phi = 0 \text{ at } r = 0 \tag{B}$$

Equations (A) and (B) yield: $\phi = \frac{-1}{16}(r^4 - 2r^2 + 1)$

APPENDIX B

BLOCK DIAGRAM FOR COMPUTATION OF TWO-DIMENSIONAL THERMAL-STRESS PROBLEMS



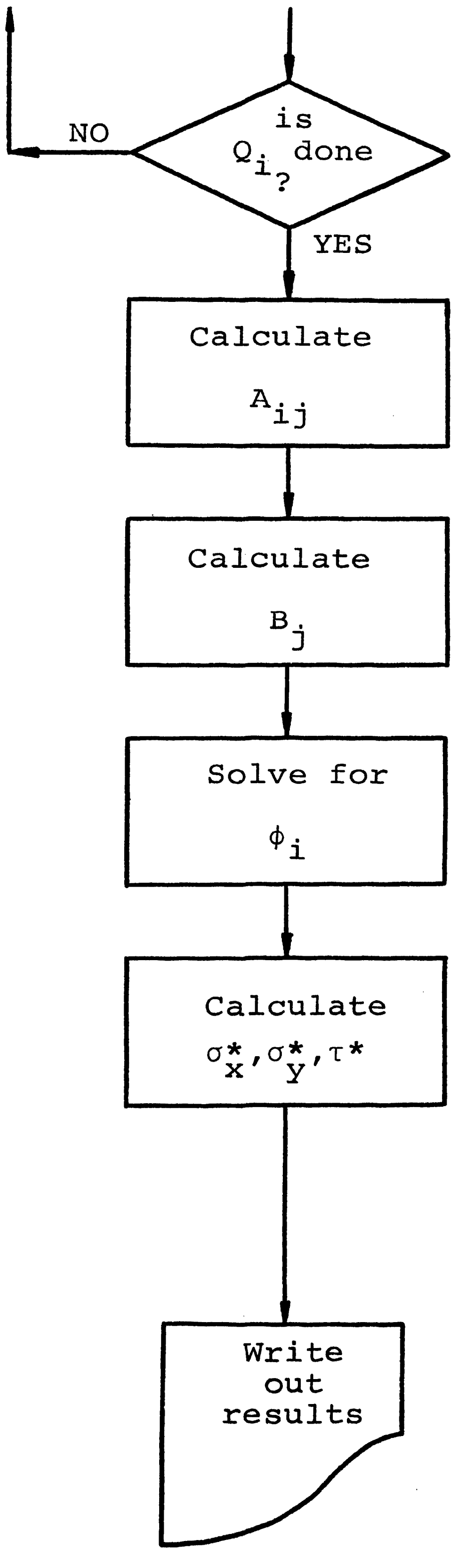
Read in data for each station.

DΦ loop for the n stations.

Calculate the coefficients of each station function.

DΦ loop for the m values of the abscissa (or m' values of the ordinate).

Calculate P_i, P_i', P_i'', and P_i''' for each value of the abscissa (or Q_i, Q_i', Q_i'', and Q_i''' for each value of the ordinate).



$$A_{ij}=\left[\frac{1}{\beta^4}P_j''''(x_j)Q_i(y_j)+\frac{2}{\beta^2}P_i''(x_j)Q_i''(y_j)+P_i(x_j)Q_i''''(y_j)\right] \quad \begin{matrix} i=1,2,\ldots n \\ j=1,2,\ldots n \end{matrix}$$

$$B_j=-\frac{1}{\beta^2}T_{xx}(x_j,y_j)-T_{yy}(x_j,y_j) \qquad j=1,2,\ldots n$$

From Eq. (40), we have

$$[A_{ij}][\phi_i] = [B_j]$$

Thus,

$$[\phi_i] = [A_{ij}]^{-1}[B_j]$$

Calculate stresses as follows:

$$\sigma_x^* = \frac{\partial^2\phi}{\partial y^2} = \sum_{i=1}^{n}[P_i''(x)Q_i(y)]\phi_i$$

$$\sigma_y^* = \frac{\partial^2\phi}{\partial x^2} = \sum_{i=1}^{n}[P_i(x)Q_i''(y)]\phi_i$$

$$\tau^* = \frac{-\partial^2\phi}{\partial x\partial y} = \sum_{i=1}^{n}[P_i'(x)Q_i'(y)]\phi_i$$

APPENDIX C

CALCULATION OF PRINCIPAL STRESSES WITH TWO SETS OF NORMAL STRESSES

Assume

$\sigma_x, \sigma_y, \sigma'_x, \sigma'_y$: two sets of known normal stresses

θ: angle of rotation

σ_1, σ_2: principle stresses, $\sigma_1 > \sigma_2$

θ_1: angle between σ_1 and σ_y

θ_2: angle between σ_1 and σ'_y

then $$\sigma_y = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2} \cos 2\theta_1 \quad \text{..........(1)}$$

$$\sigma_x = \frac{\sigma_1+\sigma_2}{2} - \frac{\sigma_1-\sigma_2}{2} \cos 2\theta_1 \quad \text{..........(2)}$$

$$\sigma'_y = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2} \cos 2\theta_2 \quad \text{..........(3)}$$

$$\sigma'_x = \frac{\sigma_1+\sigma_2}{2} - \frac{\sigma_1-\sigma_2}{2} \cos 2\theta_2 \quad \text{..........(4)}$$

and $$\theta_2 = \theta_1 + \theta \quad \text{.........................(5)}$$

From the above equations, we have

$$\sigma'_y = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2} [\cos 2\theta_1 \cos 2\theta - \sin 2\theta_1 \sin 2\theta] \quad \text{.....(6)}$$

$$\cos 2\theta_1 = \frac{2[\sigma_y - \frac{\sigma_1+\sigma_2}{2}]}{\sigma_1-\sigma_2}, \quad \sin 2\theta_1 = \pm \sqrt{1-\cos^2 2\theta_1} \quad \text{........(7)}$$

$$\sigma_1+\sigma_2 = \sigma_x+\sigma_y \text{ or } \sigma_1-\sigma_2 = -(\sigma_x+\sigma_y)+2\sigma_1 \quad \text{................(8)}$$

Substituting (7) and (8) into (6) yields

$$\sigma_1^2-\sigma_1(\sigma_x+\sigma_y)+\sigma_x\sigma_y+\left[\frac{-\sigma'_y+\frac{\sigma_x+\sigma_y}{2}+\cos 2\theta\left(\frac{\sigma_y-\sigma_x}{2}\right)}{\sin 2\theta}\right]^2=0 \quad \ldots\ldots\ldots(9)$$

Solving Eq. (9), we obtain σ_1.

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VII. VITA

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