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Photoelastic investigation of short beams

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PHOTOELASTIC INVESTIGATION OF SHORT BEAMS

BY

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A

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NOTATIONS

D	Depth of beam.
W	Width of beam.
P	Central load.
F.O.	Fringe order.
L-D	Length-depth ratio.
M	Bending moment.
I	Moment of inertia.
S	Unit stress.
c	Distance from neutral axis to outermost fiber.
Z	Section modulus.
x, y, z	Rectangular coordinates.
l, m	Direction cosines of the outer normal.
A	Cross-sectional area.
g	Gravitational acceleration.
ρ	Density.
X, Y	Components of a body force per unit volume.
$\bar{X}$, $\bar{Y}$	Components of a distributed surface force per unit area.
$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress parallel to x-, y-, and z- axes.
τ	Shearing stress.
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing stress components in rectangular coordinates.
u, v, w	Components of displacements.
ϵ	Unit elongation.
$\epsilon_x, \epsilon_y, \epsilon_z$	Unit elongations in x-, y-, and z- directions.
γ	Unit shear.

NOTATIONS

(Continued)

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ Shearing strain components in rectangular coordinates.

E Modulus of elasticity in tension and compression.

G Modulus of elasticity in shear.

ν Poisson's ratio.

ϕ Stress function.

PREFACE

In this investigation the author used beams made of Bakelite. They were of rectangular cross-section and centrally loaded. These beams were viewed in polarized light and the maximum fiber stresses observed in that manner were compared with those computed from the flexure formula. The purpose was to determine the minimum length-depth ratio for which the latter gave fairly accurate results.

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PART 1

A DISCUSSION OF THE OPTICS INVOLVED IN PHOTOELASTIC ANALYSIS

The strength and properties of materials under load play a very important part in engineering structures of every kind. Modern needs demand a more accurate solution to difficult problems in engineering and design. These solutions must be found by one means or another. Mathematicians are finding it impossible to keep pace with the ever increasing demand for solutions to these problems, so the engineer has resorted to laboratory methods. One of the most useful of these methods is the examination of the properties exhibited by a model of the proposed structure, in the same material or in different material, under loads bearing a proper scale relation to the loads to be carried by the full sized structure. Another method is that of viewing a model, made of a transparent material, under polarized light, and determining the stresses optically.

The starting point of all photo-elastic research was the discovery by Sir David Brewster, in 1816, that when a piece of glass is loaded and viewed under polarized light it shows brilliant color effects due to the internal stresses produced in the material.

Since his discovery, many materials possessing the same optical properties as glass have been discovered. Among these are Bakelite, Marbellette, and celluloid. These materials are isotropic and exhibit no sign of double refraction. When subjected to great strain, however, they become double refracting. If one of these materials is subjected to a strain and viewed between two crossed Polaroids, two beams of light are transmitted, and the strains are rendered evident by the interference of the light, resulting in color bands.

An ordinary beam of light may be considered as consisting of vibrations in the ether in all directions perpendicular to the direction of the ray. If a transparent material under load is viewed in such a light, there are no visible effects of the stress in the material. A more simple type of light vibration must be employed to reveal these stresses. If from the ordinary beam all vibrations are destroyed except those that vibrate in one plane, the resultant ray is uni-directional as regards its transverse vibrations, or, as it is commonly termed, is plane polarized.

There are many methods used to obtain polarized light. Two methods which are most generally used in the photo-elastic apparatus are (1) the use of the prism invented by Nicol and composed of two wedges of Iceland spar cemented together by Canada balsam, and (2) the use of Polaroid, a material which polarizes light by simple absorption of all vibrations except the ones parallel to its axis.

An ordinary beam of light, after passing through a Polaroid, emerges as a uni-directional ray. This ray then is passed through a transparent material under load. The latter causes the beam to break up into two systems of transverse waves, both of which have been retarded. These two waves execute their vibrations in planes at right angles to one another. As they leave the stressed specimen, they are out of phase an amount depending on the stress distribution within the specimen. Then they are passed through a second Polaroid whose axis is at right angles to the axis of the first Polaroid. This second Polaroid allows only those components to go through which are parallel to its optic plane. Two waves emerge, which are out of phase, and vibrate in planes parallel to one another. Because they are out of phase, they give interference effects which show brilliant color patterns when white light is used.

When a stressed specimen is placed between two Polaroids whose principal planes are at right angles to one another, there is, in addition to the color effects, a system of black bands known as iso-clinic bands. They only appear when the specimen is loaded, and change their shape when the type of loading is changed. These bands are useful in determining the directions of the principal stresses. They connect all the points at which one of the principal stress directions coincides with the axis of the second Polaroid.

After the directions of the principal stresses have been obtained, the iso-clinic bands can be removed by passing the polarized beam through a quarter wave plate. The latter, which is generally made of Mica, takes the plane polarized beam of light from the polarizer and breaks it up into two constituent rays. If the axis and the thickness of this plate are properly adjusted, two similar simple harmonic waves at right angles to one another and a quarter wave out of phase will emerge. The result will be a circularly polarized beam of light. This beam of light then is passed through the stressed specimen like a corkscrew and when it emerges, it is still circularly polarized but the two waves constituting this circular motion are retarded differently. However, the same amount of retardation is produced as for plane polarized light. The circularly polarized light is converted back to plane polarized light by passing it through a second quarter wave plate whose principal axis makes an angle of 90 degrees with the principal axis of the first plate. Then the analyzer picks out only those components which are parallel to its principal plane and again, two waves emerge out of phase but with their planes of vibration parallel to one another, thereby producing the interference patterns on the screen.

The interference patterns obtained are proportional to the difference in the principal stresses and therefore proportional to the shear.

It has been shown that stress can be made visible by the use of polarized light. Some of the facts pertaining to stress distribution in plates, for which the photo-elastic methods are suitable, will next be considered.

PART 2

THE DIFFERENTIAL EQUATIONS FOR TWO DIMENSIONAL STRESS PROBLEMS

Plane stresses are stresses that are parallel to one plane. This type of stress can always be obtained by subjecting a thin plate to the action of forces applied at the boundary, parallel to the plane of the plate and uniformly distributed over its thickness.

If a body is loaded with forces perpendicular to the longitudinal elements, plane deformation is obtained on parts at a considerable distance from the ends. The dimension perpendicular to the X-Y plane must be very large, and the forces applied must not vary along the length of the body.

In discussing the deformation of an elastic body, it will be assumed that the body does not move and that there are no displacements of particles of the body without a deformation of it. Only small deformations such as occur in engineering structures will be used. The small displacements of particles of a deformed body will be resolved into components, u, v and w parallel to the coordinate axes, X, Y and Z respectively. It will be assumed that these components are very small quantities varying continuously over the volume of the body.

Let's consider a small element dx dy dz of an elastic body as shown in Fig. 1. If the body undergoes a deformation and u, v, and w are the components of the displacement of point O, the displacement in the X- direction of an adjacent point A on the X- axis is $u + \frac{\partial u}{\partial x} dx$, due to the increase $\frac{\partial u}{\partial x} dx$ of the function u with increase of the coordinate X. The increase in length of the element OA due to deformation is therefore $\frac{\partial u}{\partial x}dx$. Hence the unit elongation at point 0 in the X- direction is $\frac{\partial u}{\partial x}$.

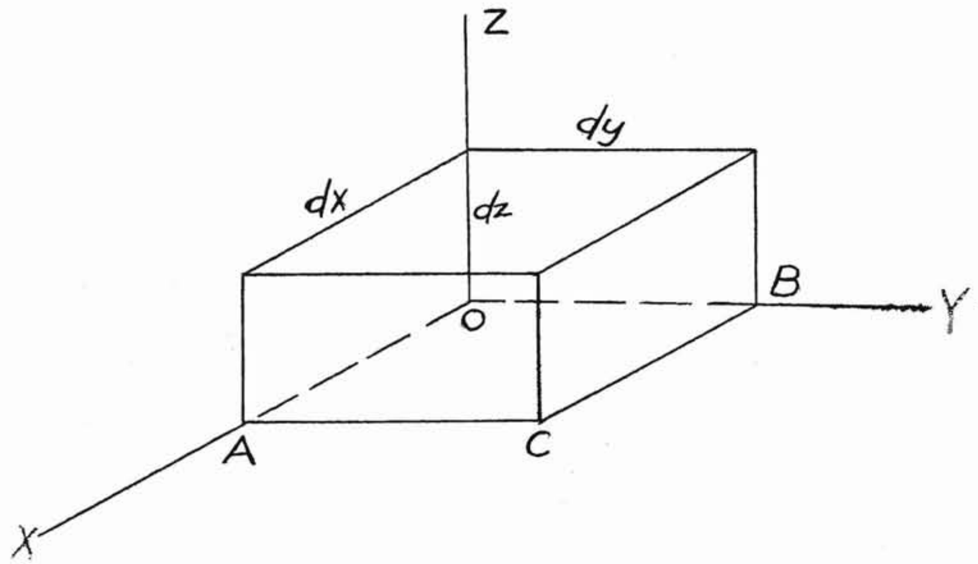


Fig. 1

In the same manner it can be shown that the unit elongations in the Y- and Z- directions are given by the derivatives $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$.

We will now consider the distortion of the angle between the elements OA and OB in Fig. 2.

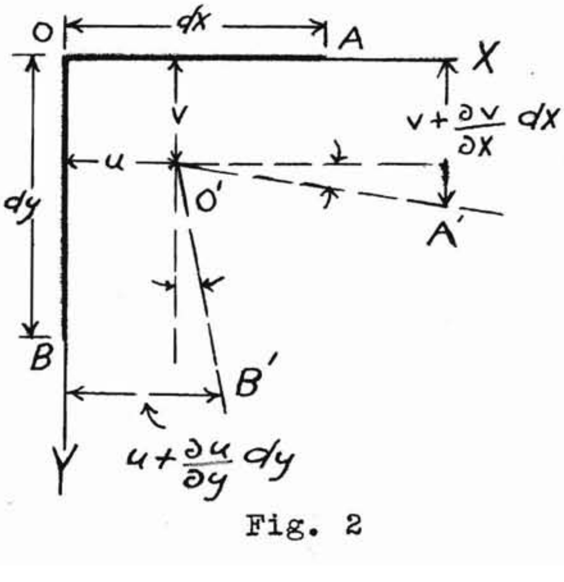


Fig. 2

If u and v are the components of the displacements of the point O in the X- and Y- directions, the displacements of the point A in the Y- direction and of the point B in the X- direction are $v + \frac{\partial v}{\partial x}dx$ and $u + \frac{\partial u}{\partial y}dy$, respectively. Due to these displacements, the new direction O'A' of the element OA is inclined to the initial direction by the small angle indicated in the figure equal to $\frac{\partial v}{\partial x}$. In the same manner the direction O'B' is inclined to OB by the small angle $\frac{\partial u}{\partial y}$. It can be seen from the figure that the right angle AOB between the two elements OA and OB is diminished by the angle $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. This is the shearing strain between the planes XZ and YZ. The shearing strains between the planes XY and XZ and the planes YX and YZ can be obtained in the same manner.

Using ϵ for unit elongation and γ for unit shearing strain, the following equations are obtained from the above discussion.

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = \frac{\partial w}{\partial z},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{1}$$

The relations between the components of stress and the components of strain have been established experimentally and are known as Hooke's Law. Imagine an elemental rectangular parallelopiped with the sides parallel to the coordinate axes and subjected to the action of normal stresses σ_x uniformly distributed over two opposite sides. The magnitude of the unit elongation of this element is given by the equation

$$\epsilon_x = \frac{\sigma_x}{E} \qquad (a)$$

where E is the modulus of elasticity in tension. The unit lateral contractions are given by the equations

$$\epsilon_y = \frac{\sigma_x}{E}\nu \qquad (b)$$

$$\epsilon_z = \frac{\sigma_x}{E}\nu \qquad (c)$$

in which ν is a constant called Poisson's ratio.

If the element is subjected to the action of normal stresses σ_x, σ_y, and σ_z, the resultant components of strain can be obtained from the equations (a), (b), and (c). In order to get these strain components we have to superpose the strain components produced by each of the three stresses.

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

These reduce to

$$\begin{aligned} \epsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\ \epsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \qquad (2) \\ \epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \end{aligned}$$

If the stress components σ_x, σ_y, and τ_{xy} are known for any point of a plate in a condition of plane stress, the stress acting on any plane through this point, perpendicular to the plate, and inclined to the X- and Y- axes can be calculated from the equations of statics.

Let O be the point (Fig. 3), and let the stress components σ_x, σ_y, and τ_{xy} be known. To find the stress for

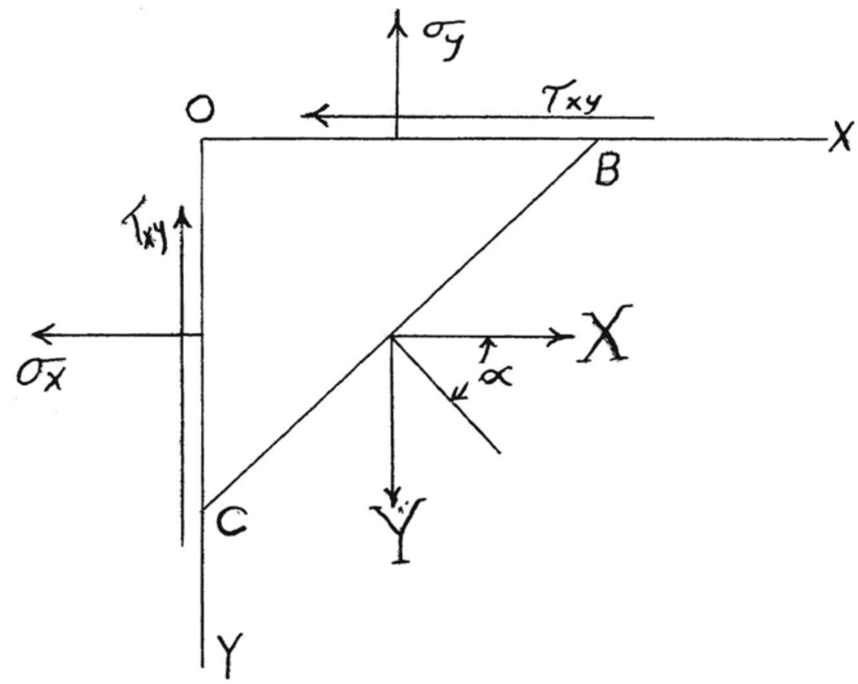


Fig. 3

any plane through the Z- axis and inclined to the X- and Y- axes, we take a plane BC parallel to it at a small distance from O, so that this plane BC, together with the coordinate planes, cuts out from the plate a very small triangular prism OBC. Since the stresses vary continuously over the volume of the body, the stress acting on the plane BC will approach the stress on the parallel plane through O as the element is made smaller.

Let A be the area of the side BC,

then Al is the area of the side OC, $(l = \cos\alpha)$

and Am is the area of the side OB, $[m = \cos(90 - \alpha)]$

X and Y are the components of stress on the side BC.

$\sum F_x = 0$

$AX = \tau xy\ (A \sin\alpha) + \sigma x\ (A \cos\alpha)$

$\Sigma Fy = 0$

$AY = \sigma y\ (A \sin\alpha) + \tau xy\ (A \cos\alpha)$

These reduce down to:

$$\begin{aligned} X &= m\,\tau xy + l\,\sigma x \\ Y &= m\,\sigma y + l\,\tau xy \end{aligned} \qquad (3)$$

Thus the components of stress on any plane defined by the direction cosines l and m can easily be calculated from equations (3), provided the three components of stress, σx, σy, and τxy, at the point 0 are known.

The shearing and normal components of stress on the plane BC are:

$$\sigma = X \cos\alpha + Y \sin\alpha$$

$$\tau = Y \cos\alpha + X \sin\alpha$$

but $X = l\,\sigma x + m\,\tau xy = \sigma x \cos\alpha + \tau xy \sin\alpha$

$Y = m\,\sigma y + l\,\tau xy = \sigma y \sin\alpha + \tau xy \cos\alpha$

$$\begin{aligned} \sigma &= [\sigma x \cos\alpha + \tau xy \sin\alpha] \cos\alpha + [\sigma y \sin\alpha + \tau xy \cos\alpha] \sin\alpha \\ &= \sigma x \cos^2\alpha + \tau xy \sin\alpha\cos\alpha + \sigma y \sin^2\alpha + \tau xy \sin\alpha\cos\alpha \\ &= \sigma x \cos^2\alpha + \sigma y \sin^2\alpha + 2\tau xy \sin\alpha\cos\alpha \qquad (4) \end{aligned}$$

$$\begin{aligned} \tau &= [\sigma y \sin\alpha + \tau xy \cos\alpha]\cos\alpha - [\sigma x \cos\alpha + \tau xy \sin\alpha] \sin\alpha \\ &= \sigma y \sin\alpha\cos\alpha + \tau xy \cos^2\alpha - \sigma x \sin\alpha\cos\alpha - \tau xy \sin^2\alpha \\ &= (\sigma y - \sigma x) \sin\alpha\cos\alpha + \tau xy (\cos^2\alpha - \sin^2\alpha) \qquad (4) \end{aligned}$$

Angle α can be chosen in such a manner that the shearing stress τ becomes 0.

For this case we have:

$$\tau xy (\cos^2\alpha - \sin^2\alpha) + (\sigma y - \sigma x) \sin\alpha\cos\alpha = 0$$

$$\frac{\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{\sin\alpha\cos\alpha}{\frac{1}{\sec^2\alpha} - \frac{1}{\csc^2\alpha}}$$

$$= \frac{(\sin\alpha\cos\alpha)(\sec^2\alpha\csc^2\alpha)}{\csc^2\alpha - \sec^2\alpha}$$

$$= \frac{\frac{\sin\alpha\cos\alpha}{\cos^2\alpha\sin^2\alpha}}{\csc^2\alpha - \sec^2\alpha} = \frac{1}{\frac{\cos\alpha\sin\alpha}{\sin^2\alpha} - \frac{\cos\alpha\sin\alpha}{\cos^2\alpha}}$$

$$= \frac{1}{\cot\alpha - \tan\alpha} = \frac{1}{\frac{1 - \tan^2\alpha}{\tan\alpha}}$$

$$= \frac{\tan\alpha}{1 - \tan^2\alpha} = \tfrac{1}{2}\tan 2\alpha \qquad (5)$$

From the preceding equations we see that two perpendicular directions can be found for which the shearing stress is zero. These directions are called principal directions, and the corresponding normal stresses are called principal stresses.

If the X- and Y- axes are taken as the principal directions, τ_{xy} is equal to zero and the equations used to obtain the normal and shearing components of stress along the plane BC become

$$\begin{aligned} \sigma &= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha \\ \tau &= \tfrac{1}{2}\sin 2\alpha(\sigma_y - \sigma_x) \end{aligned} \qquad (4')$$

Now we will consider the variation of the stress components σ_x, σ_y, and τ_{xy} as we change the position of the point. A small rectangular parallelopiped (Fig. 4) with sides dx and dy will be used for this discussion. Here we take into consideration the small changes of the components of stress due to the small increases dx and dy of the coordinates. The stresses acting at the centers of the sides of the small rectangular parallelopiped are shown along with their positive directions.

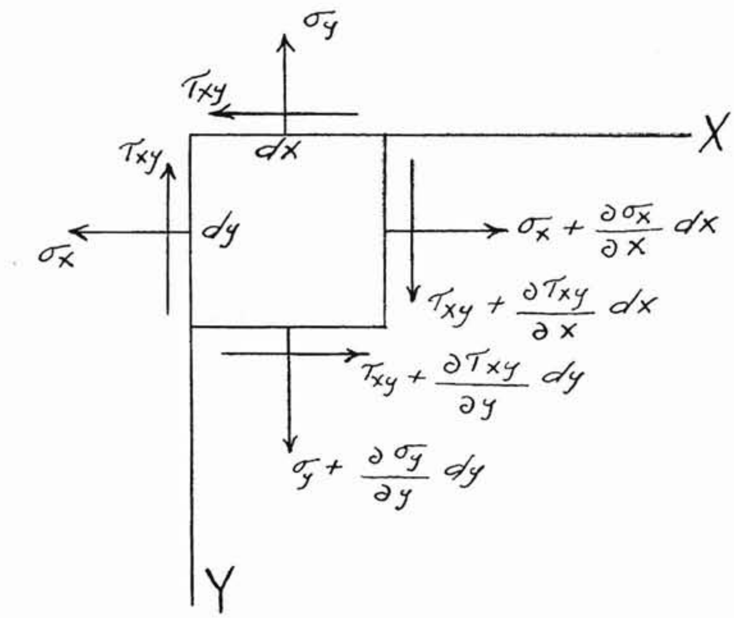


Fig. 4

In this discussion we must also consider the body force acting on the element because it is of the same order of magnitude as the terms due to the variation of the stress components. If X and Y denote the components of this force per unit volume of the element, then the equations of equilibrium obtained by summing forces in the X- and Y- directions become:

$\sum Fx = 0$

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy - \sigma_x \, dy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right) dx - \tau_{xy} \, dx + X \, dx \, dy = 0$$

$\sum Fy = 0$

$$\left(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy\right) dx - \sigma_y \, dx + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dy - \tau_{xy} \, dy + Y \, dx \, dy = 0$$

These equations reduce to:

$$\sigma x dy + \frac{\partial \sigma x}{\partial x} dx\ dy - \sigma x\ dy + \tau xy\ dx + \frac{\partial \tau xy}{\partial y} dy\ dx - \tau xy\ dx + X\ dx\ dy = 0$$

and dividing by dx dy

$$\frac{\partial \sigma x}{\partial x} + \frac{\partial \tau xy}{\partial y} + X = 0$$

$$\sigma y dx + \frac{\partial \sigma y}{\partial y} dy\ dx - \sigma y\ dx + \tau xy\ dy + \frac{\partial \tau xy}{\partial x} dx\ dy - \tau xy\ dy + Y\ dx\ dy = 0$$

and dividing by dx dy

$$\frac{\partial \sigma y}{\partial y} + \frac{\partial \tau xy}{\partial x} + Y = 0$$

In practical applications the weight of the body is usually the only body force. Let ρ = the mass per unit volume and taking the Y- axis downward as positive, the preceding equations become:

$$\begin{aligned} \frac{\partial \sigma x}{\partial x} + \frac{\partial \tau xy}{\partial y} &= 0 \\ \frac{\partial \sigma y}{\partial y} + \frac{\partial \tau xy}{\partial x} + \rho g &= 0 \end{aligned} \qquad (6)$$

These are the differential equations of equilibrium for two dimensional problems.

The equations just derived must be satisfied at all points throughout the volume of the body. The stress components vary over the volume of the plate, and when they arrive at the boundary they must be such as to be in equilibrium with the external forces on the boundary of the plate, so that external forces may be regarded as a continuation of the internal stress distribution. These conditions of equilibrium at the boundary can be obtained from equations (3). Taking the small triangular prism $\overline{OBC}$ (Fig. 3) so that the side BC coincides with the boundary of the plate as shown in Fig. 5, and letting $\overline{x}$ and $\overline{y}$ be the components of the surface forces per unit area at this point of the boundary, equations (3) become:

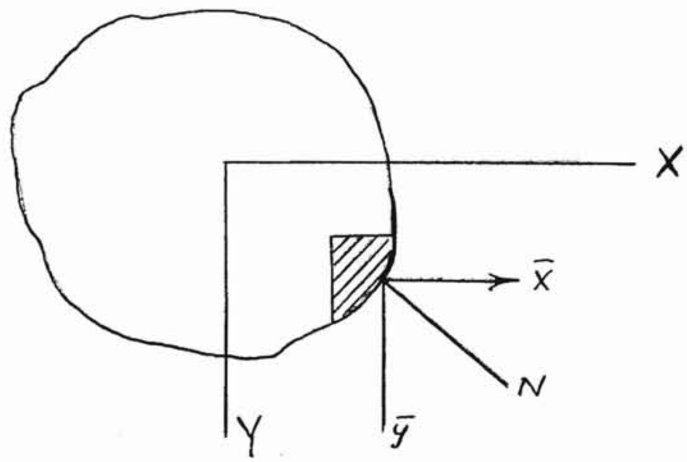


Fig. 5

$$\overline{x} = l\sigma_x + m\,\tau_{xy}$$
$$\overline{y} = m\sigma_y + l\,\tau_{xy} \qquad (7)$$

In which l and m are the direction cosines of the normal N to the boundary.

In the case of a rectangular plate the coordinate axes are usually taken parallel to the sides of the plate and the boundary conditions (7) can be simplified. Taking a side of the plate parallel to the X- axis, the normal N becomes parallel to the Y- axis. Hence $l = 0$, and $m = \pm 1$. Then equations (7) become:

$$\overline{x} = \pm\,\tau_{xy}\ , \text{ and } \overline{y} = \pm\,\sigma_y$$

The positive sign is taken if the normal N has the positive direction of the Y- axis and the negative sign should be taken if the normal N has the opposite direction.

In the case of a two dimensional problem it is necessary to solve the differential equations of equilibrium and the solution must be such as to satisfy the boundary conditions. These equations were all derived by application of the equations of statics for rigid bodies, and containing three stress components, σ_x, σ_y, and τ_{xy}. The problem is a statically indeterminate one and in order to obtain a solution, the elastic deformation of the body must be considered.

The mathematical formulation of the compatilibity of stress distribution with the existence of continuous functions u, v, and w defining the deformation will be obtained from equations (1). In the case of two dimensional problems only three strain components need be considered:

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (1')$$

Differentiating the first of the equations listed above twice with respect to y, the second twice with respect to x, and the third once with respect to x and once with respect to y:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^3 u}{\partial y^2 \partial x}$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^3 v}{\partial x^2 \partial y}$$

with respect to x

$$\frac{\partial \tau_{xy}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}$$

with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial \tau_{xy}}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}\right)$$

or

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

and from the preceding we obtain:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad (8)$$

This differential equation is called the condition of compatibility and must be satisfied by the strain components to secure the existence of functions u and v connected with the strain components by equations (1'). By using Hooke's law the condition (8) can be transformed into a relation between the components of stress.

In the case of plane stress distribution equations (2) can be reduced to:

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \quad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

substituting in equation (8), we find:

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \cdot 2(1+\nu) \qquad (9)$$

By using the equations of equilibrium this equation can be written in a different form. Differentiating the first of equations (6) with respect to x and the second with respect to y and adding them, we obtain:

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

and adding

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

substituting in equation (9), the compatability equation in terms of stress components become:

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = (1+\nu)\left(-\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}\right)$$

$$\frac{\partial^2 \sigma_x}{\partial y^2} - \frac{\partial^2 \nu\sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \frac{\partial^2 \nu\sigma_x}{\partial x^2} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \nu\sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \nu\sigma_y}{\partial y^2}$$

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma x + \sigma y) = 0 \qquad (10)$$

And in the same manner with the general equations of equilibrium:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma x + \sigma y) = -(1 + \nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \qquad (11)$$

It has been shown that a solution of two dimensional problems reduces to the intergration of the differential equations of equilibrium together with the compatability equation and the boundary conditions. In the case where the weight of the body is the only body force, the equations to be satisfied are:

$$\begin{aligned} \frac{\partial \sigma x}{\partial x} + \frac{\partial \tau xy}{\partial y} &= 0 \\ \frac{\partial \sigma y}{\partial y} + \frac{\partial \tau xy}{\partial x} + \rho g &= 0 \end{aligned} \qquad (6)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma x + \sigma y) = 0 \qquad (10)$$

and the boundary conditions (7). The usual method of solving these equations is by introducing a new function called the stress function. Equations (6) are satisfied by taking any function ϕ of x and y and putting the following expressions for the stress components:

$$\sigma x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau xy = -\frac{\partial^2 \phi}{\partial x \partial y} - \rho g x \qquad (12)$$

In this manner we can get a variety of solutions of the equations of equilibrium (6). The true solution is that which satisfies the compatability equation (11) also. Substituting expressions (12) for the stress components in equation (11) we find that the stress function ϕ must satisfy the equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma x + \sigma y) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0$$

$$\frac{\partial^4 \phi}{\partial y^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial x^4} = 0 \qquad (13)$$

Thus the solution of a two dimensional problem, when the weight of the body is the only body force, reduces to finding a solution of equation (13) which satisfies the boundary conditions (7) of the problem.

PART 3

PROCEDURE

Standard photoelastic apparatus was used in this investigation. It consisted of two Polaroids, four lenses, a source of monochromatic light, two quarter wave plates, a loading frame, and a camera arrangement. The monochromatic light was obtained by using a mercury vapor bulb and passing the light through a green filter.

The beams used were made of Bakelite. The optical constant, which is the value of unit stress represented by each fringe, was determined from a pure tension member and then checked with a long beam on end supports with a concentrated load at the center.

The photographs shown on plates 6-A, 7-A, ------15-A were made with safety process films requiring eight minute exposures. The reader will notice the center line and the short horizontal lines spaced equal distances apart on the photographs. The lines were put in 1/10 inch apart for the purpose of determining the magnification and to facilitate the plotting of the curves shown on plates 6 to 15 inclusive. These curves were used to determine the fringe order on the edge of the beams. The fringe order was plotted against the distance from the edge of the beam and the curve extended out to the edge, thereby obtaining a fairly accurate value of the fringe order at that point. Curve 1 on plate 6 was plotted from fig. 1 on plate 6-A, curve 2 from fig. 2, etc. The curves on plate 7 were plotted from the corresponding photographs on plate 7-A, 8 from 8-A, --------- 15 from 15-A.

Four tests were made in this investigation. For the first test the author used a beam 0.286 inches wide, 1.702 inches deep, and 11.15 inches long. The first observation on this first test was made with the supports spaced 11 inches apart. The supports were moved in one inch closer after each observation. The final observation was made with the supports one inch apart. In the second test the same beam was used with no overhanging ends. That is, the beam was cut down to the proper length after each observation. The third test was similar to the first except for the new dimensions of the beam. For this test the beam had the same width and length but a new depth of 1.141 inches. The same beam was used in the fourth test, and, like the second test, the beam was cut down to the proper length after each observation.

PART 4

RESULTS

DATA SHEET FOR FIRST TEST

W = 0.286 inches $\qquad\qquad$ $I = \frac{bh^3}{12} = 0.118$ inches^4

D = 1.702 " $\qquad\qquad$ $Z = I/c = 0.139$ inches^3

$$S = \frac{Mc}{I} = \frac{PLc}{4I} = \frac{PL}{.554}$$

O.C. = 315 psi.

TABLE I

L	P	S	L/D	F.O.	S′	S'/S
11	151.1	3005	6.45	9.58	3018	1.010
10	205.8	3718	5.87	11.70	3680	.990
9	212.2	3450	5.29	10.70	3440	.996
8	212.2	3065	4.70	9.60	3022	.986
7	212.2	2660	4.11	8.40	2650	.996
6	226.1	2450	3.52	7.65	2410	.984
5	226.1	2040	2.94	6.25	1970	.965
4	267.0	1930	2.35	5.84	1840	.954
3	335.2	1815	1.76	5.50	1730	.953
2	335.2	1210	1.18	3.90	1230	1.015
1	286.1	517	.59	2.12	668	1.290

L = Length between supports in inches

P = Load at center in pounds

S = Stress from flexure formula

L/D = Length-depth ratio

F.O. = Fringe order on edge of beam

S' = Stress from photoelastic analysis

S'/S = Stress concentration factor

O.C. = Optical constant in pounds per square inch

DATA SHEET FOR SECOND TEST

W = 0.286 inches

D = 1.702 "

$I = \frac{bh^3}{12} = 0.118$ inches^4

$Z = I/c = 0.139$ inches^3

$$S = \frac{Mc}{I} = \frac{PLc}{4I} = \frac{PL}{.554}$$

O.C. = 315 psi.

TABLE II

L	P	S	L/D	F.O.	S'	S'/S
10	202.2	3650	5.87	11.52	3630	.994
9	229.3	3725	5.29	11.77	3710	.995
8	256.8	3710	4.70	11.66	3680	.994
7	284.0	3590	4.11	11.60	3660	1.017
6	311.0	3379	3.52	10.90	3440	1.020
5	325.0	2930	2.94	8.85	2790	.952
4	325.0	2345	2.35	6.81	2145	.914
3	335.5	1820	1.76	5.58	1760	.968
2	335.5	1211	1.18	4.40	1384	1.140
1	335.5	606	.59	5.22	1645	2.710

L = Length of beam in inches

P = Load at center in pounds

S = Stress from flexure formula

L/D = Length-depth ratio

F.O. = Fringe order on edge of beam

S' = Stress from photoelastic analysis

S'/S = Stress concentration factor

O.C. = Optical constant in pounds per square inch

DATA SHEET FOR THIRD TEST

W = 0.286 inches

D = 1.141 "

$$I = \frac{bh^3}{12} = 0.0354 \text{ inches}^4$$

$$Z = I/c = 0.0625 \text{ inches}^3$$

$$S = \frac{Mc}{I} = \frac{PLc}{4I} = \frac{PL}{.25}$$

O.C. = 315 psi.

TABLE III

L	P	S	L/D	F.O.	S'	S'/S
11	87.86	3860	9.64	12.1	3810	.987
9	102.30	3690	7.89	11.95	3770	1.020
7	123.96	3465	6.14	10.70	3370	.974
5	160.06	3210	4.38	9.88	3110	.970
4	181.72	2910	3.51	8.80	2770	.953
3	210.62	2530	2.63	7.30	2300	.910
2	239.48	1915	1.75	5.50	1735	.905
1	282.80	1130	.88	3.55	1118	.990

L = Length between supports in inches

P = Load at center in pounds

S = Stress from flexure formula

L/D = Length-depth ratio

F.O. = Fringe order on edge of beam

S' = Stress from photoelastic analysis

S'/S = Stress concentration factor

O.C. = Optical constant in pounds per square inch

DATA SHEET FOR FOURTH TEST

W = 0.286 inches

D = 1.141 "

$I = \frac{bh^3}{12} = 0.0354$ inches^4

$Z = I/c = 0.0625$ inches^3

$$S = \frac{Mc}{I} = \frac{PLc}{4I} = \frac{PL}{.25}$$

O.C. = 315 psi.

TABLE IV

L	P	S	L/D	F.O.	S'	S'/S
11	87.86	3860	9.64	12.46	3920	1.015
9	102.30	3680	7.89	11.70	3680	1.000
7	131.18	3680	6.14	11.70	3680	1.000
5	174.50	3480	4.38	11.31	3560	1.021
4	217.82	3480	3.51	10.60	3340	.960
3	261.14	3140	2.63	9.30	2930	.932
2	304.46	2440	1.75	7.60	2390	.980
1	333.34	1330	.88	6.40	2015	1.515

L = Length of beam in inches

P = Load at center in pounds

S = Stress from flexure formula

L/D = Length-depth ratio

F.O. = Fringe order on edge of beam

S' = Stress from photoelastic analysis

S'/S = Stress concentration factor

O.C. = Optical constant in pounds per square inch

The curve on plate 1 was obtained by plotting the distance between supports against the stress concentration factor for the first test. Plates 2, 3, and 4 were obtained in the same manner for the second, third, and fourth tests respectively. Plate 5 shows the length-depth ratio plotted against the stress concentration factor for each of the four tests. The following is a description of plate 5:

o = First test

□ = Second test

Δ = Third test

x = Fourth test

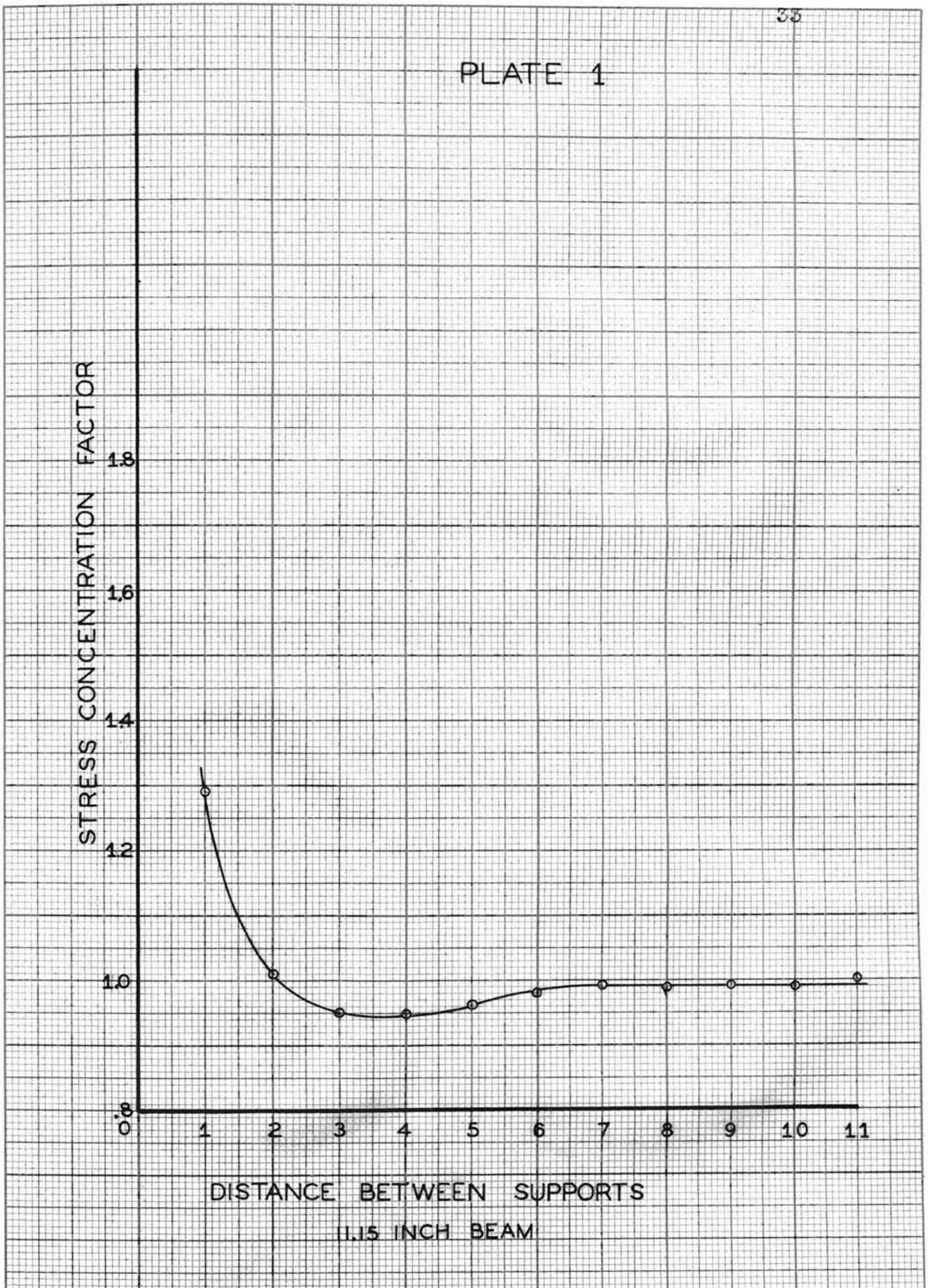


PLATE 1
STRESS CONCENTRATION FACTOR
1.8
1.6
1.4
1.2
1.0
.8
0
1
2
3
4
5
6
7
8
9
10
11
DISTANCE BETWEEN SUPPORTS
11.15 INCH BEAM

PLATE 2

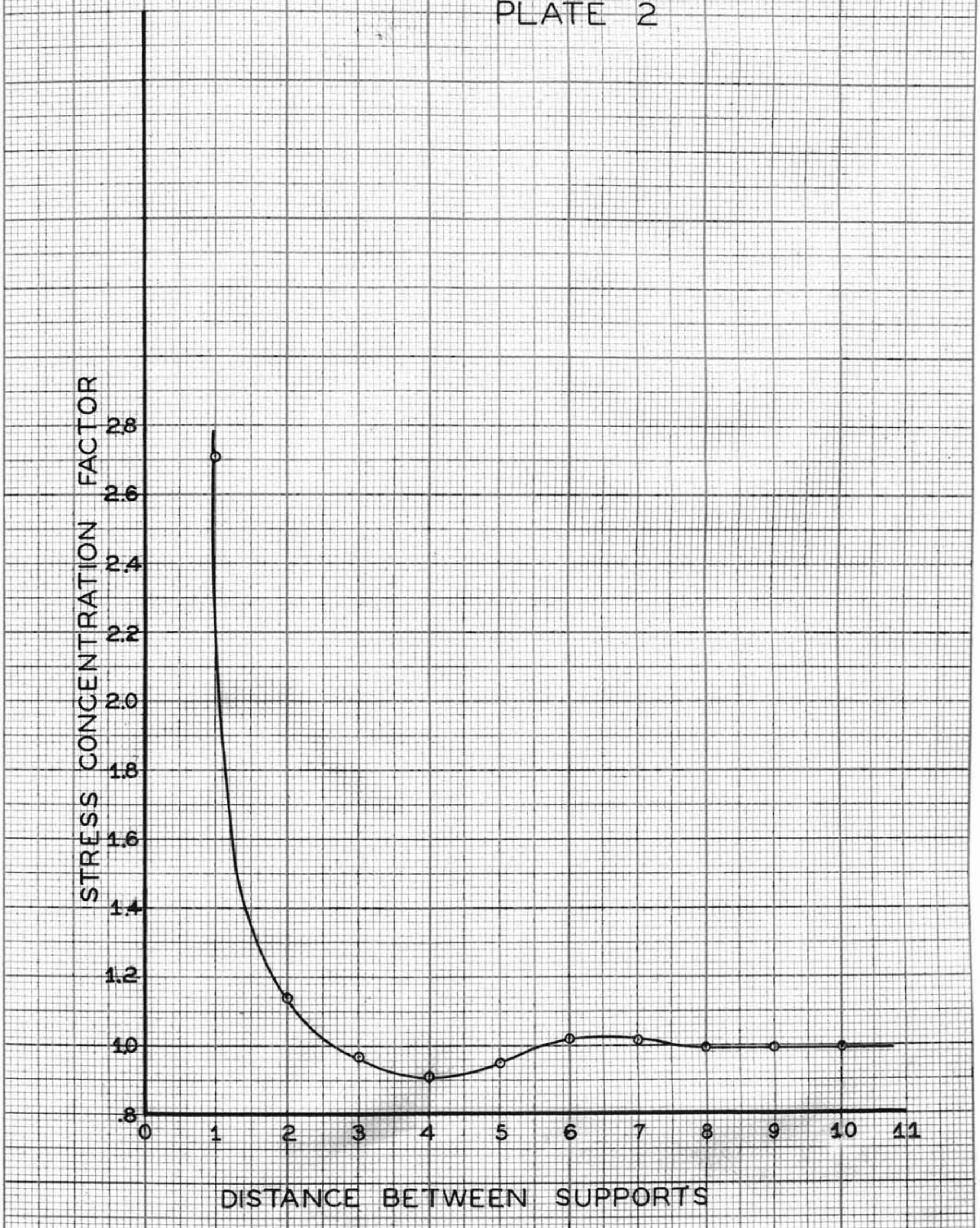

STRESS CONCENTRATION FACTOR
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
0
1
2
3
4
5
6
7
8
9
10
11
DISTANCE BETWEEN SUPPORTS

PLATE 3

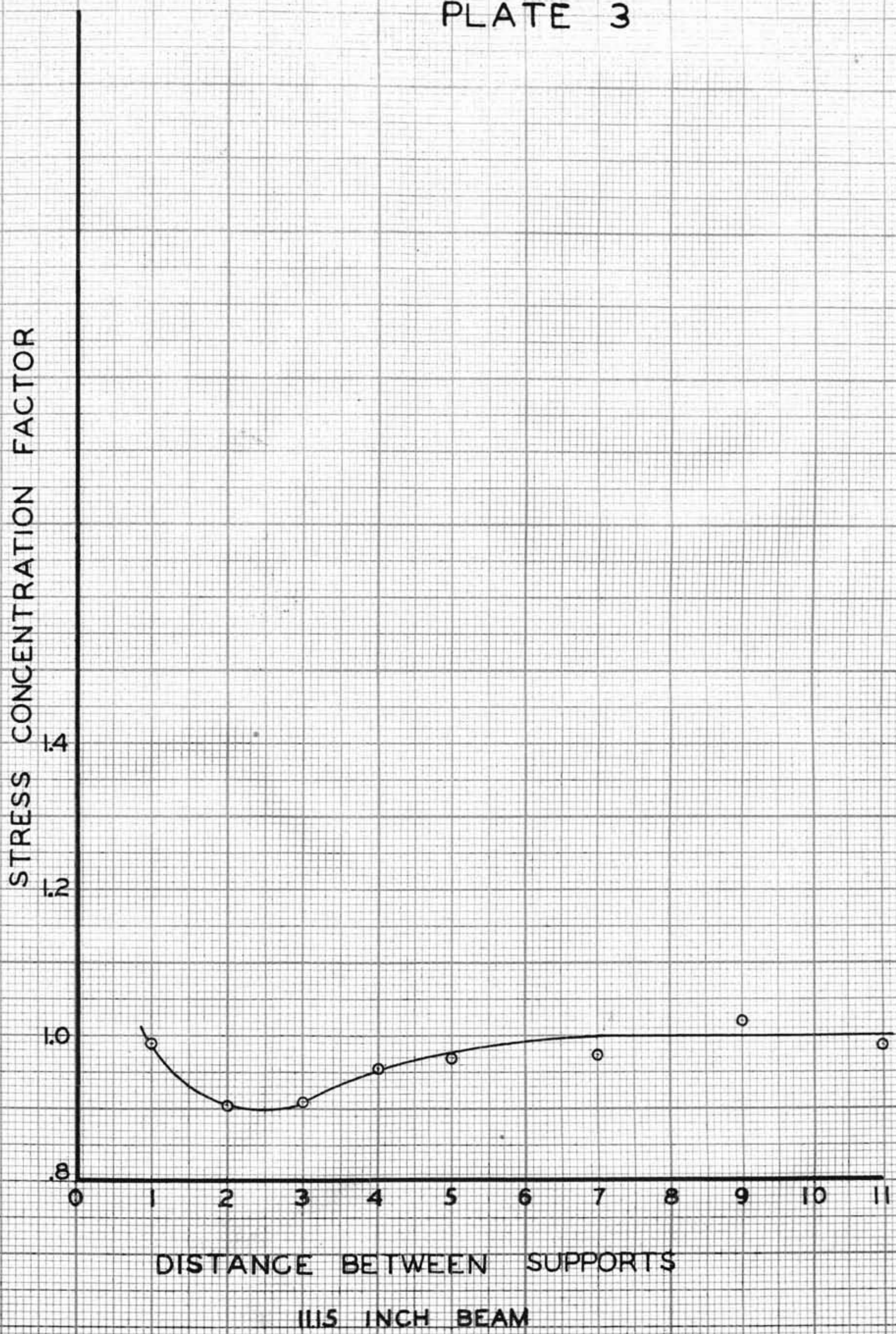

STRESS CONCENTRATION FACTOR
1.4
1.2
1.0
.8
0
1
2
3
4
5
6
7
8
9
10
11
DISTANCE BETWEEN SUPPORTS
11.15 INCH BEAM

PLATE 4

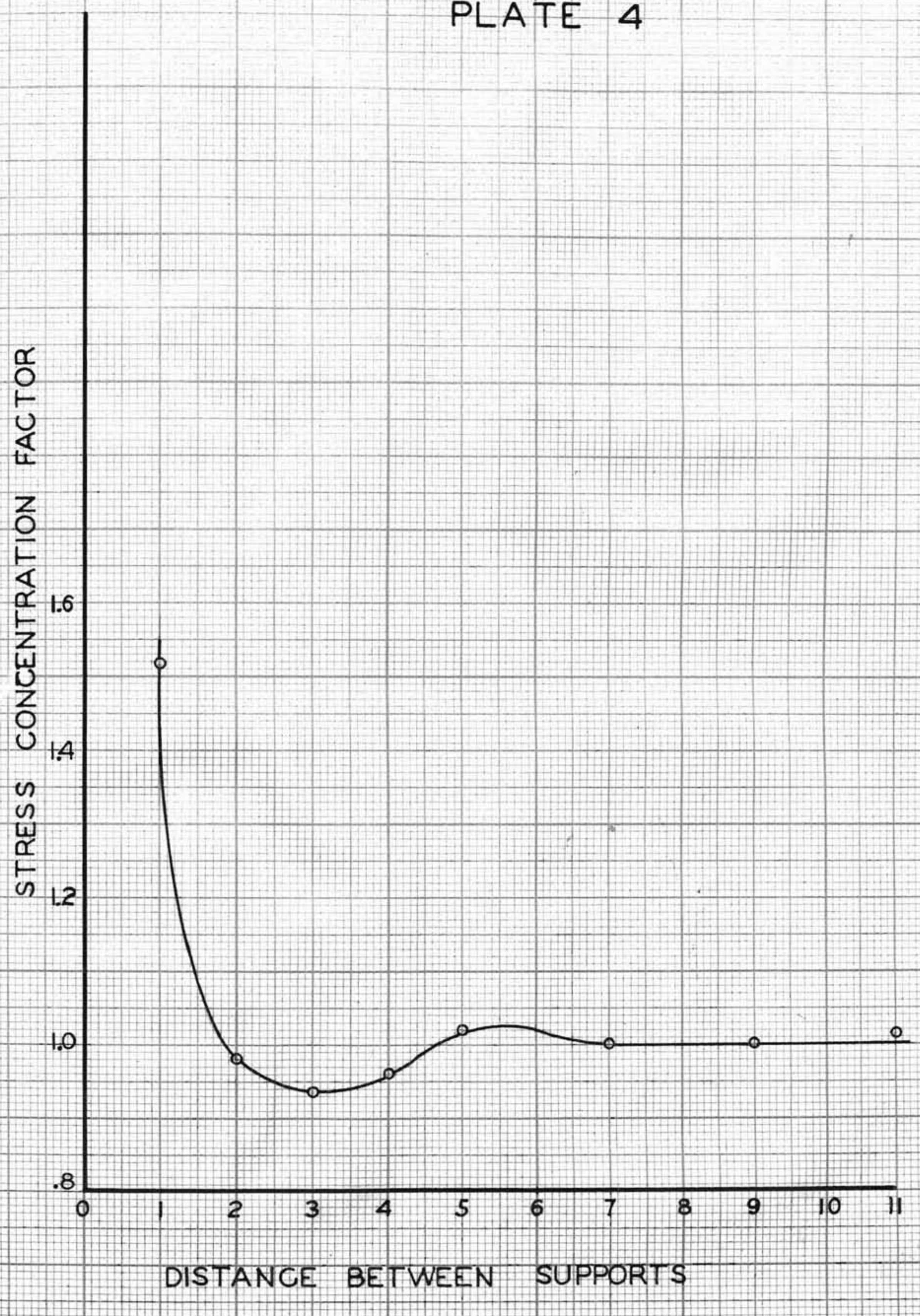
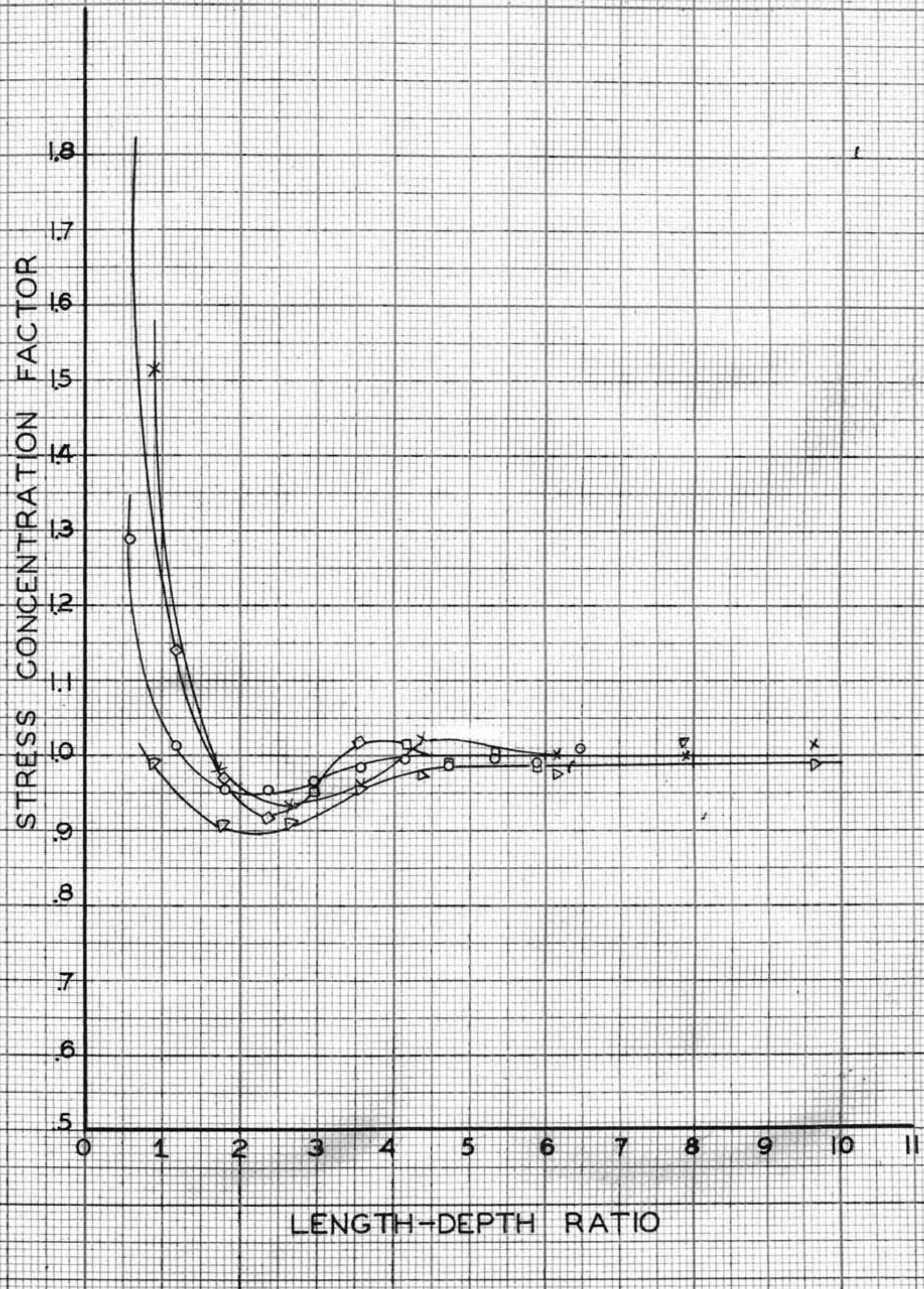

STRESS CONCENTRATION FACTOR
1.6
1.4
1.2
1.0
.8
0
1
2
3
4
5
6
7
8
9
10
11
DISTANCE BETWEEN SUPPORTS

PLATE 5



STRESS CONCENTRATION FACTOR
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
0
1
2
3
4
5
6
7
8
9
10
11
LENGTH-DEPTH RATIO

PART 5

CONCLUSIONS

Most engineering formulae are accurate to within two or three per cent. Bearing this in mind, the author arrived at the following conclusions: (1) The flexure formula may be used down to a length-depth ratio of 4, (2) for a length-depth ratio above 6 the formula gives results to within one per cent, (3) a higher stress concentration factor was obtained for short beams with no overhang than for the short beams with overhang, and (4) a stress concentration factor less than unity was obtained at a length-depth ratio of approximately $2\frac{1}{2}$.

PLATE 6

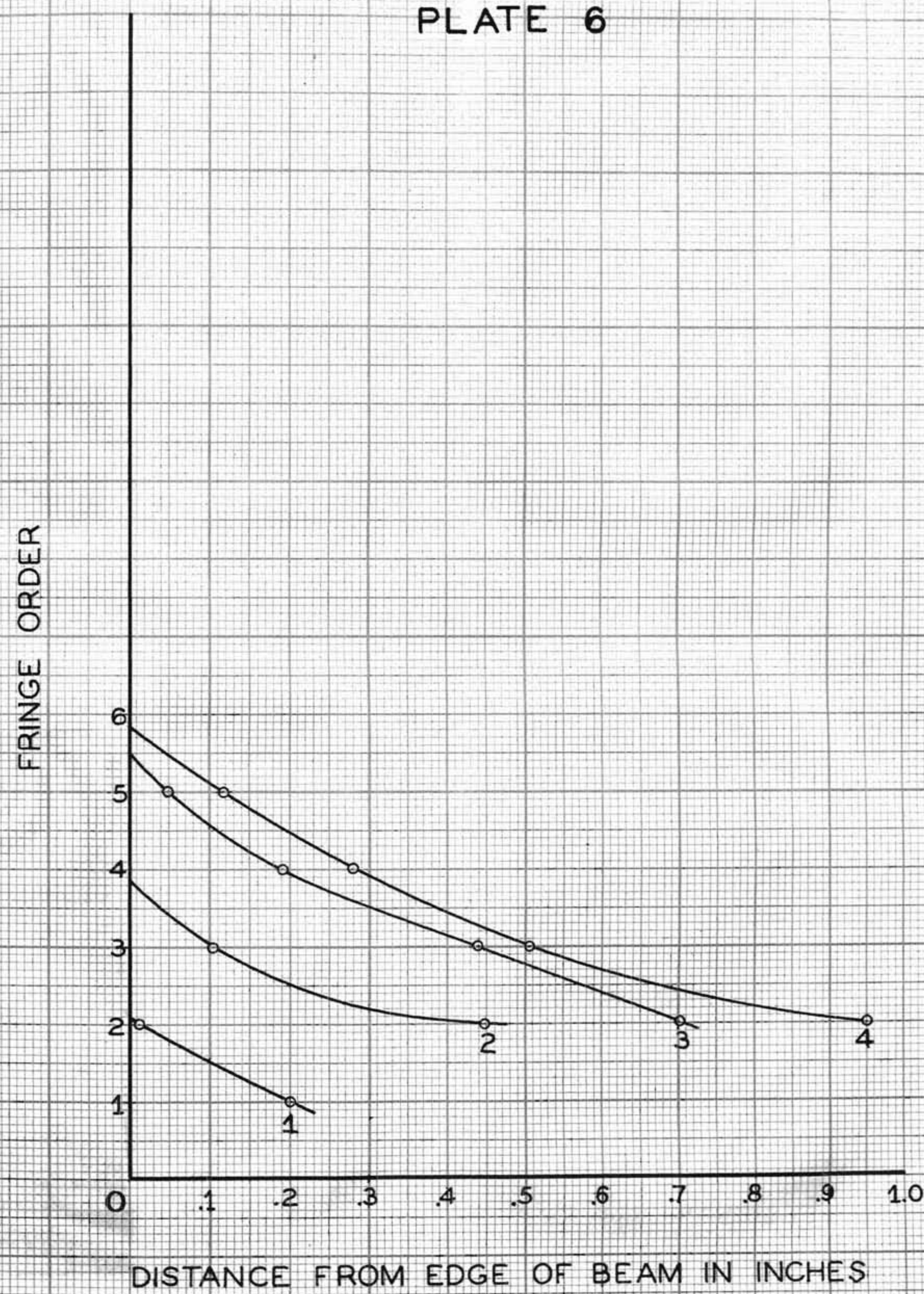

FRINGE ORDER
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
2
3
4
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 6-A

Fig. 1

L -- 1.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 286.100 lb.

Fig. 2

L -- 2.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 335.200 lb.

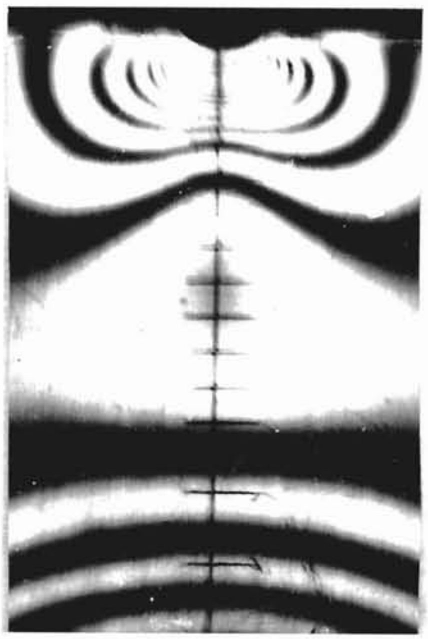


Fig. 3

L -- 3.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 335.200 lb.

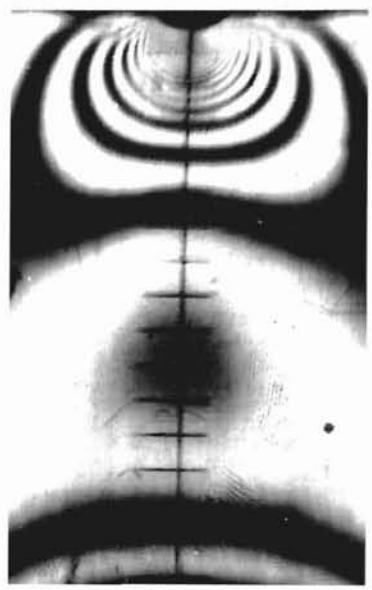
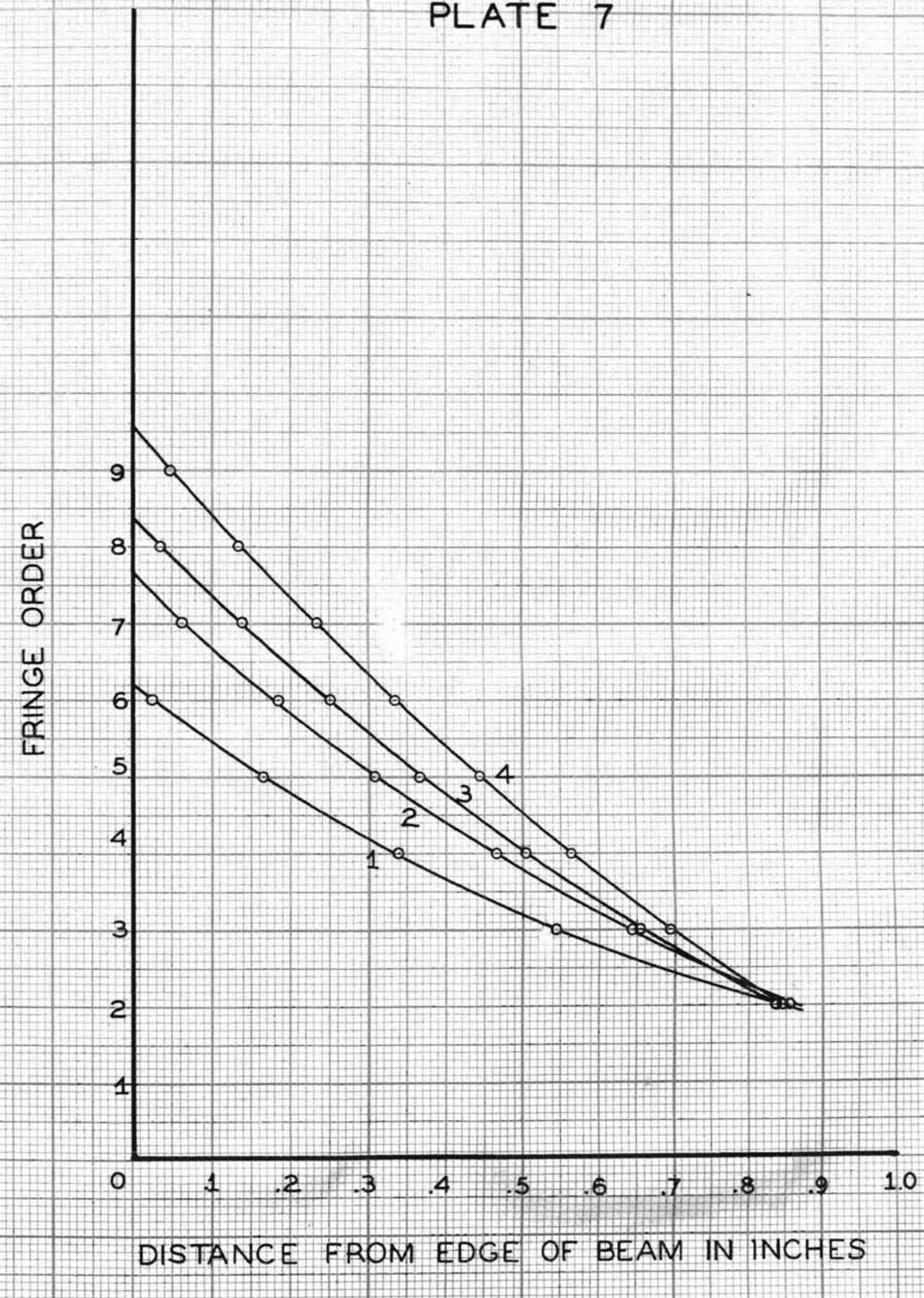


Fig. 4

L -- 4.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 267.000 lb.

PLATE 7



FRINGE ORDER
1
2
3
4
5
6
7
8
9
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 7-A

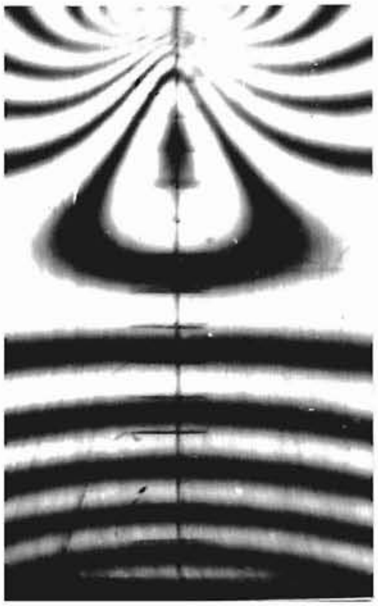


Fig. 1

L -- 5.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 226.100 lb.

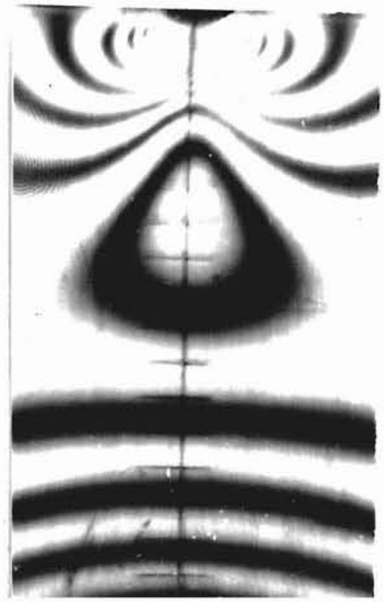


Fig. 2

L -- 6.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 226.100 lb.

Fig. 3

L -- 7.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 212.200 lb.

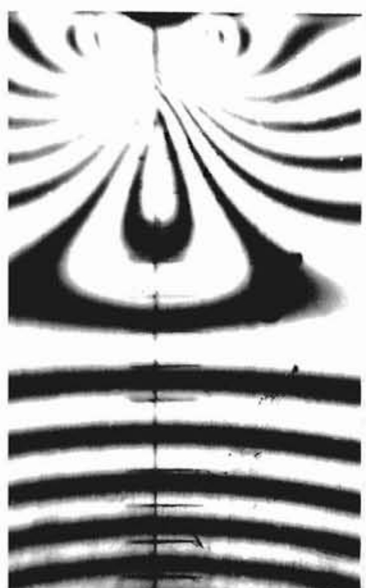


Fig. 4

L -- 8.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 212.200 lb.

PLATE 8

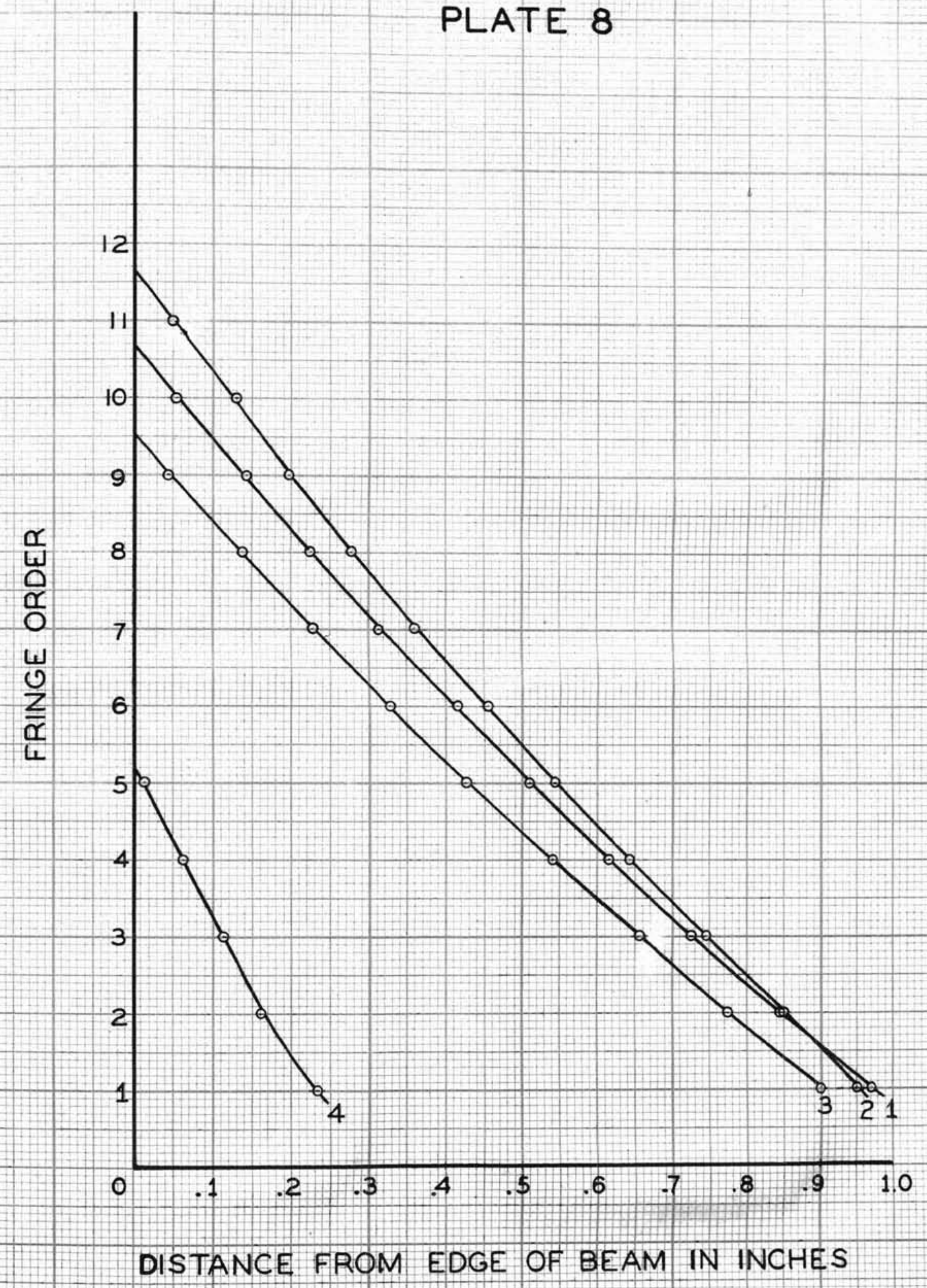

FRINGE ORDER
12
11
10
9
8
7
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
2
3
4
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 8-A

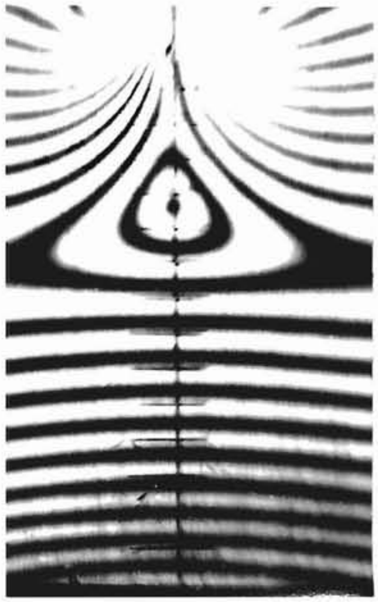


Fig. 1

L -- 9.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 212.200 lb.

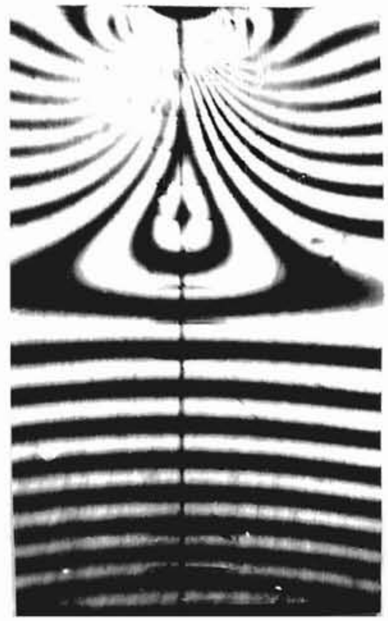


Fig. 2

L -- 10.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 205.800 lb.

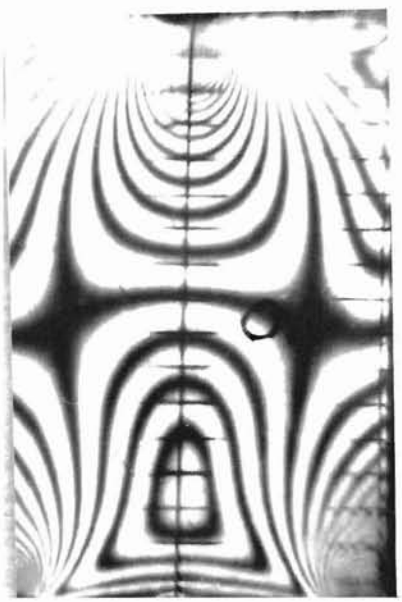


Fig. 3

L -- 11.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 151.100 lb.

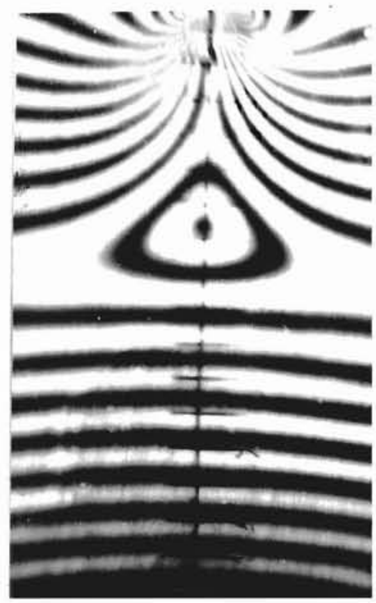


Fig. 4

L -- 1.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 335.500 lb.

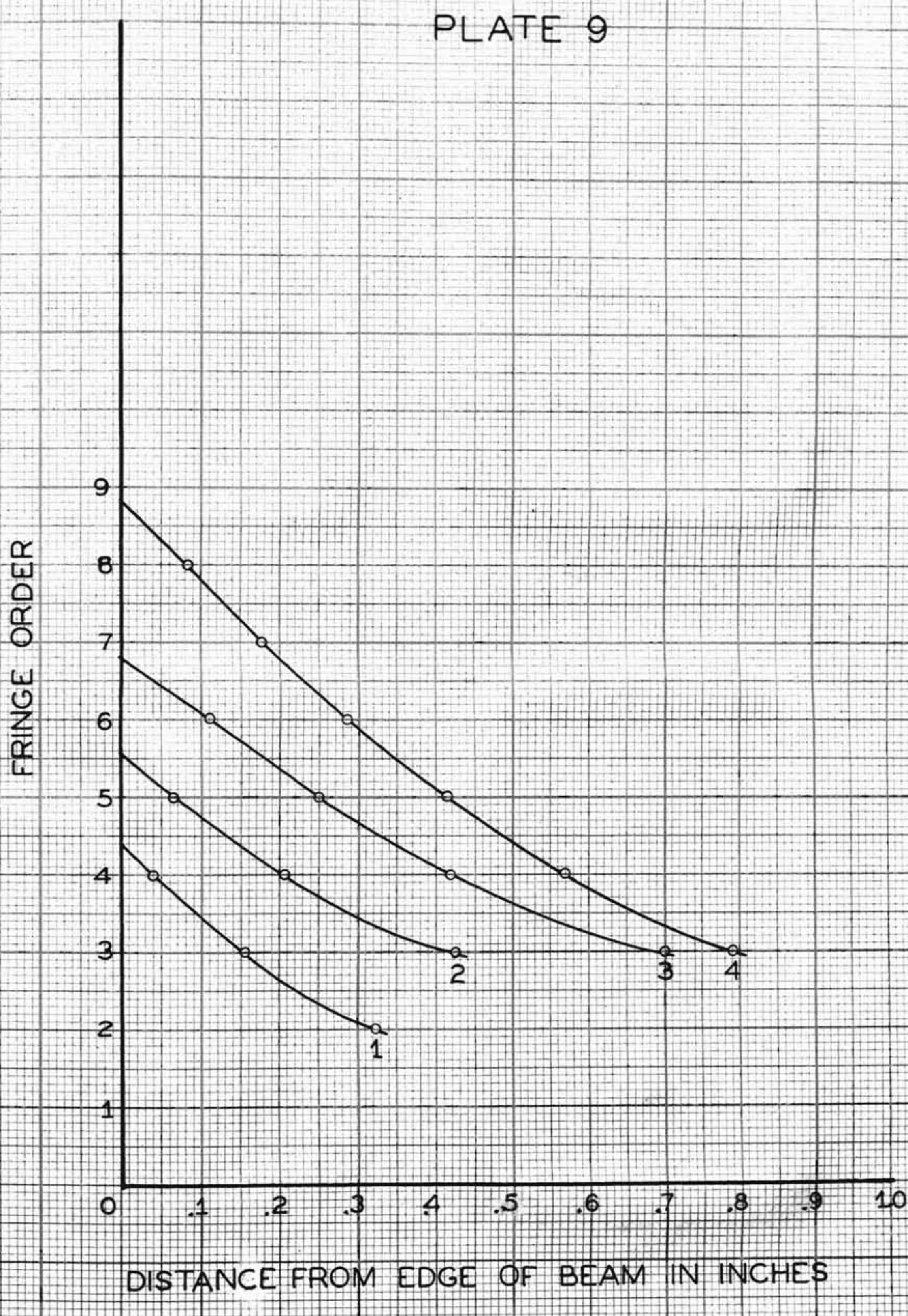


PLATE 9
FRINGE ORDER
9
8
7
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
2
3
4
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 9-A

Fig. 1

L -- 2.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 335.500 lb.

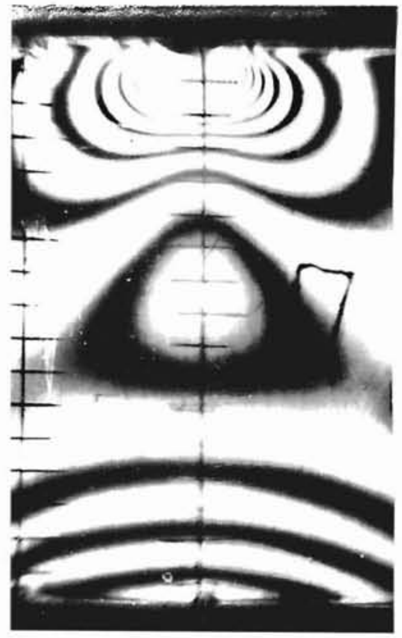


Fig. 2

L -- 3.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 335.500 lb.

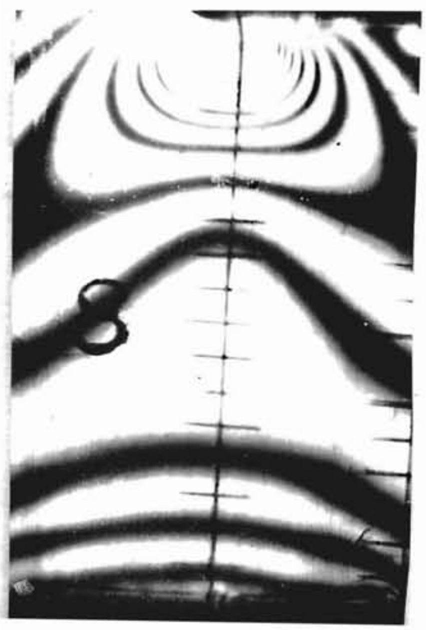


Fig. 3

L -- 4.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 325.000 lb.

Fig. 4

L -- 5.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 325.000 lb.

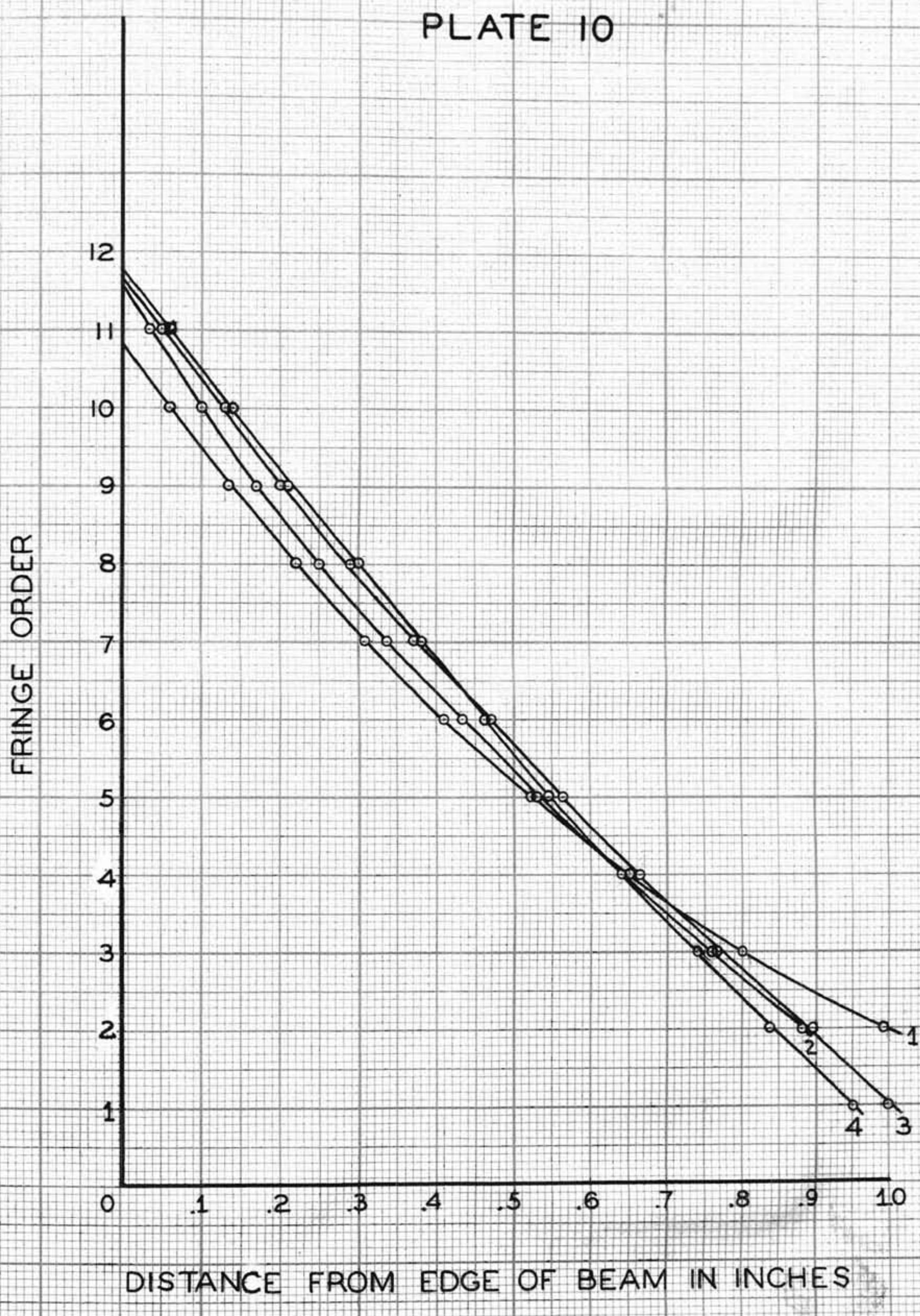


PLATE 10
FRINGE ORDER
12
11
10
9
8
7
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
2
3
4
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 10-A

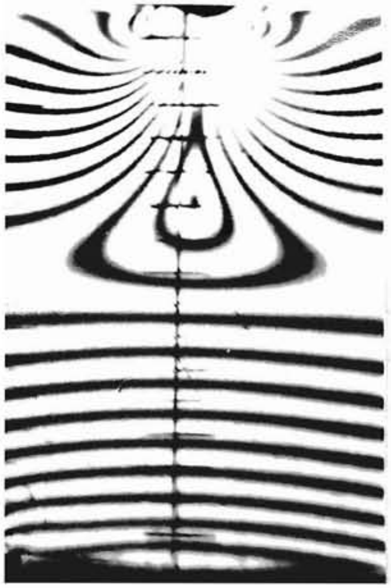


Fig. 1

L -- 6.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 311.000 lb.

Fig. 2

L -- 7.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 284.000 lb.

Fig. 3

L -- 8.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 256.800 lb.

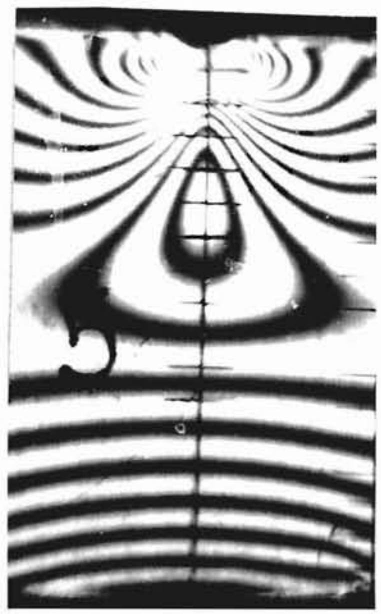


Fig. 4

L -- 9.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 229.300 lb.

PLATE 11

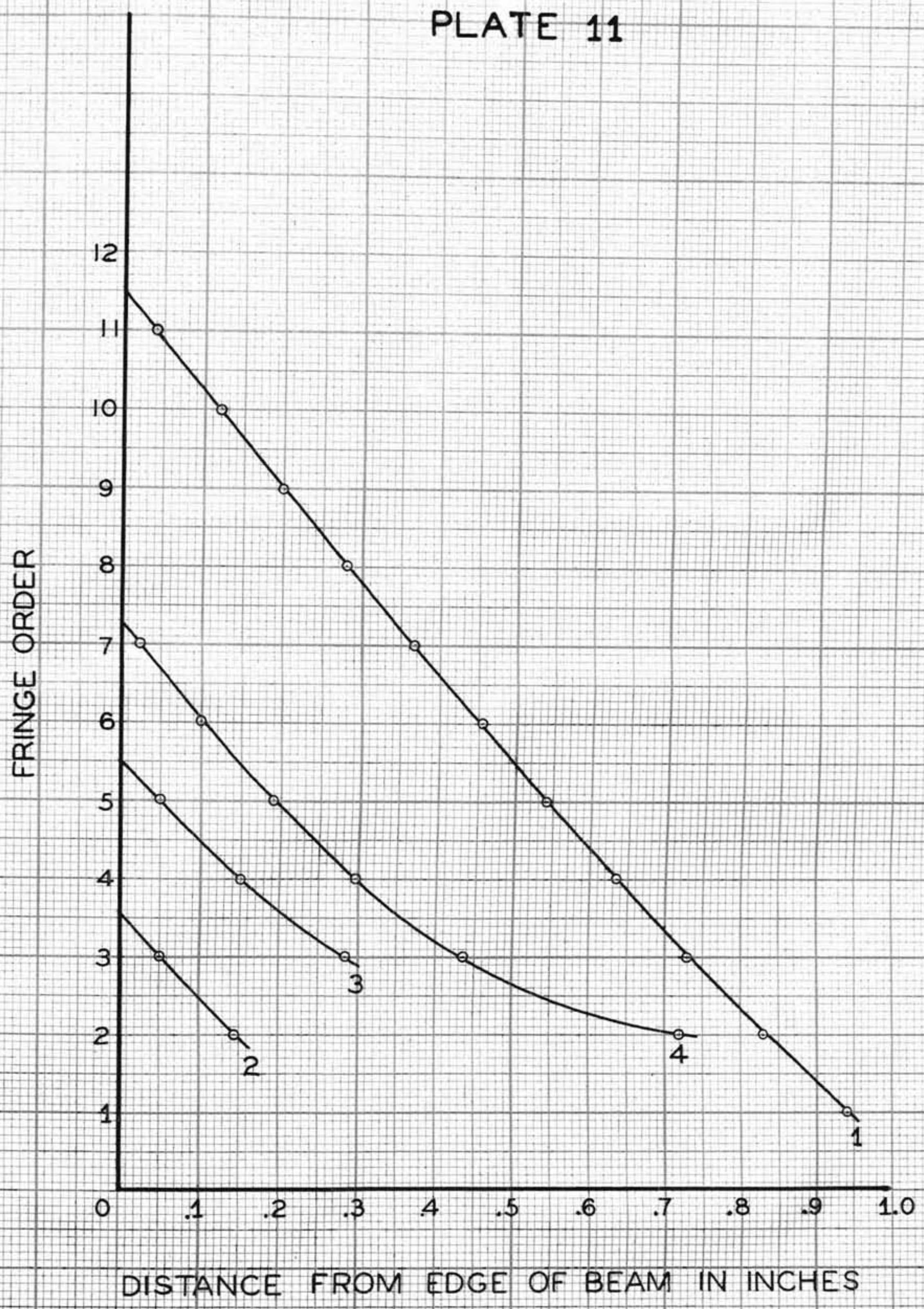

FRINGE ORDER
12
11
10
9
8
7
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
2
3
4
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 11-A

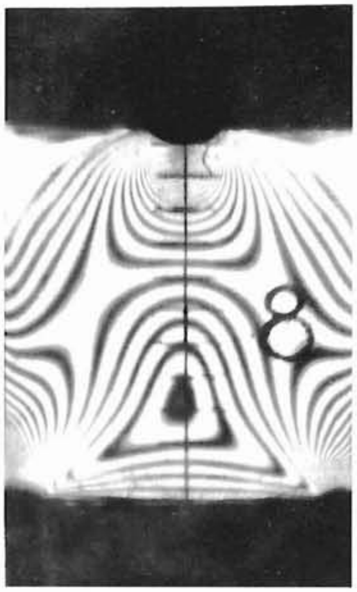


Fig. 1

L -- 10.000 in.
D -- 1.702 "
W -- 0.286 "
P -- 202.000 lb.

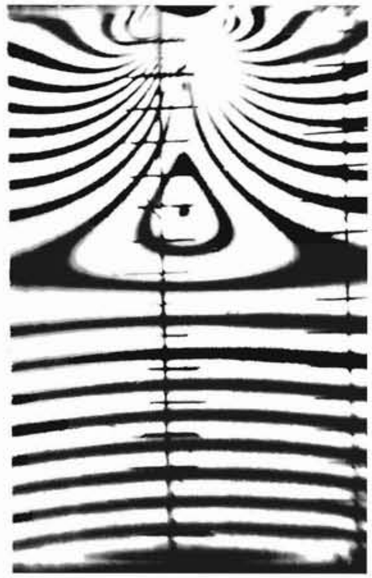


Fig. 2

L -- 1.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 282.800 lb.

Fig. 3

L -- 2.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 239.480 lb.

Fig. 4

L -- 3.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 210.620 lb.

PLATE 12

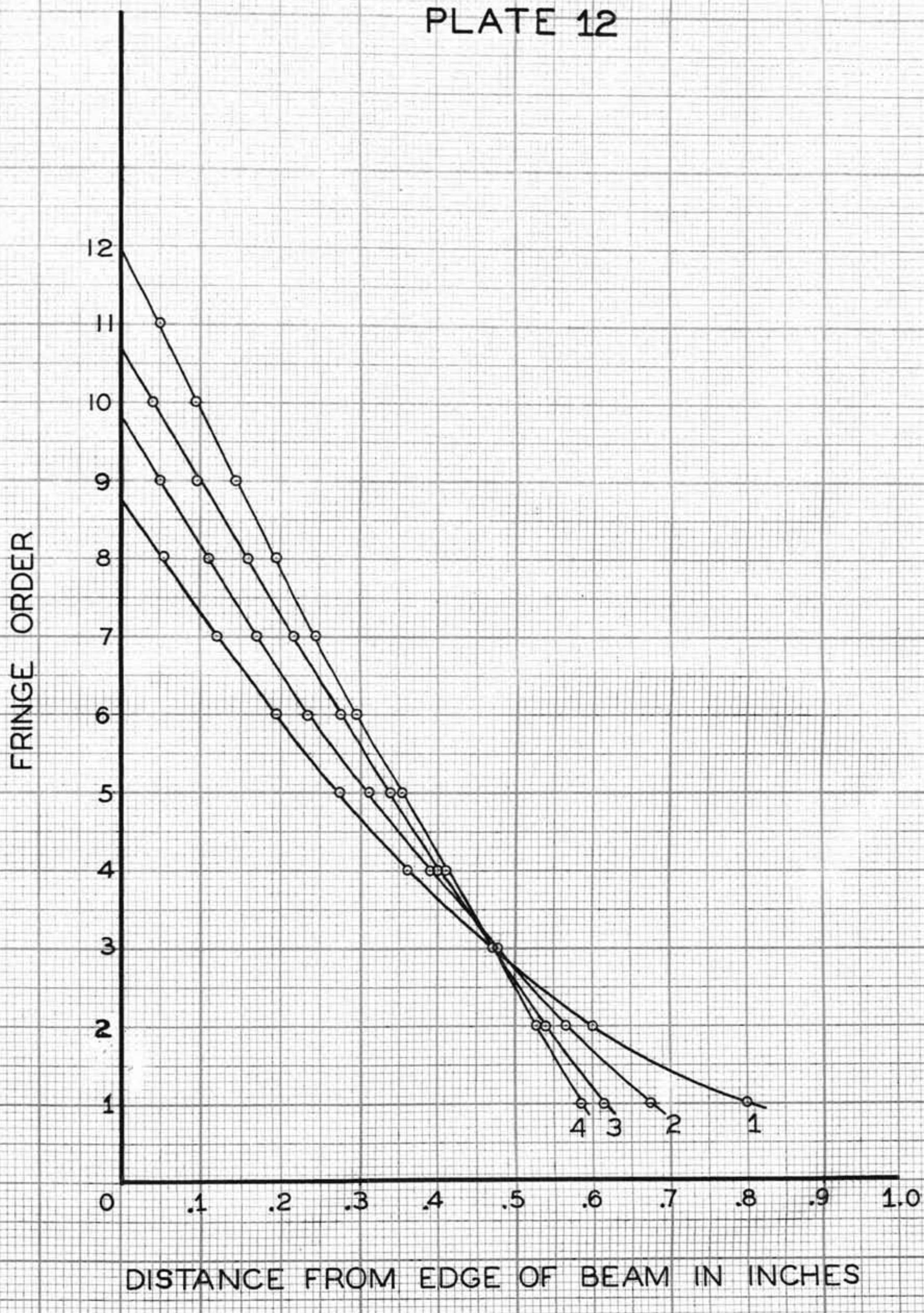

FRINGE ORDER
12
11
10
9
8
7
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
4
3
2
1
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 12-A

Fig. 1

L -- 4.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 181.720 lb.

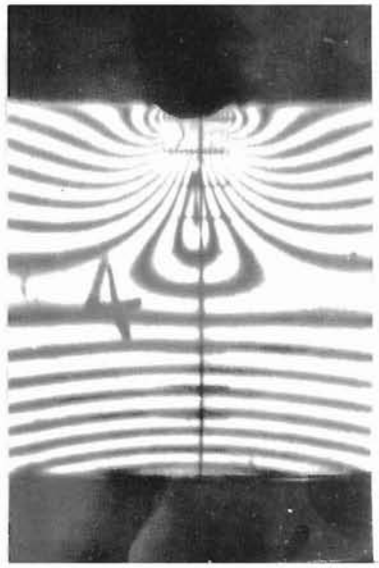


Fig. 2

L -- 5.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 160.060 lb.

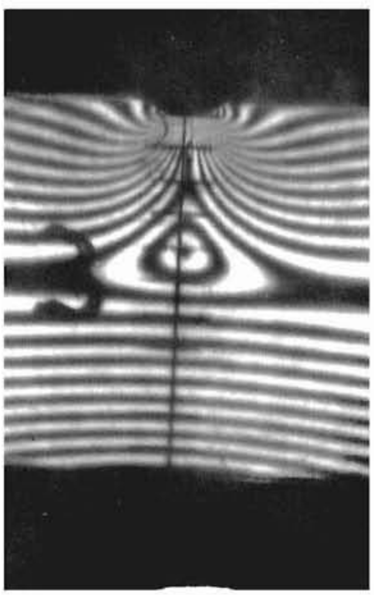


Fig. 3

L -- 7.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 123.960 lb.

Fig. 4

L -- 9.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 102.300 lb.

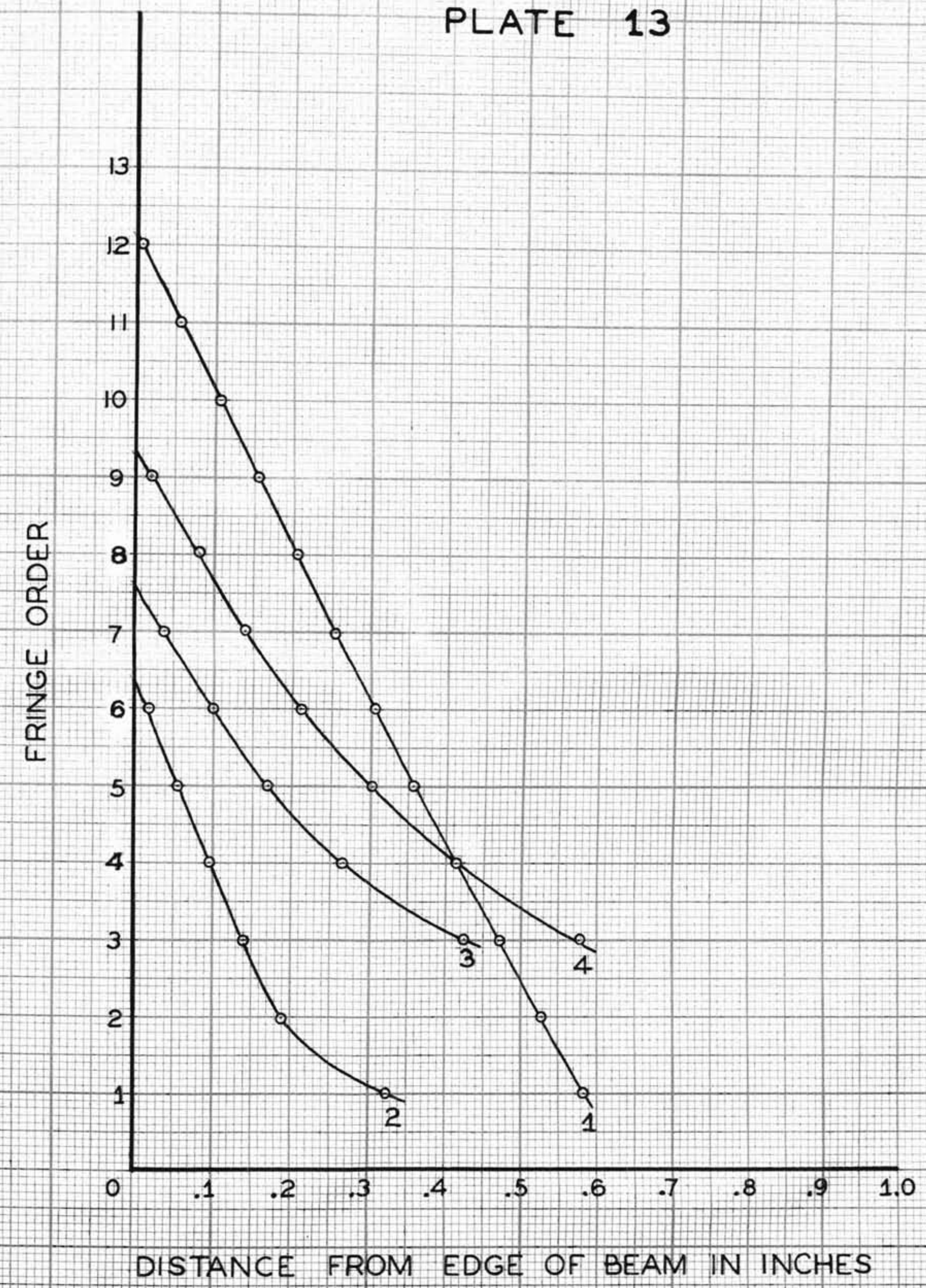


PLATE 13
FRINGE ORDER
DISTANCE FROM EDGE OF BEAM IN INCHES
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
2
3
4
5
6
7
8
9
10
11
12
13
1
2
3
4

PLATE 13-A

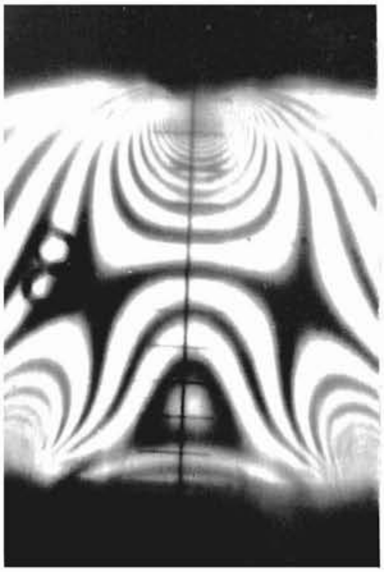


Fig. 1

L -- 11.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 87.860 lb.

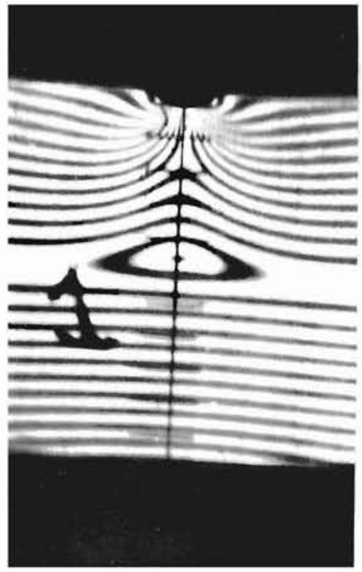


Fig. 2

L -- 1.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 333.340 lb.

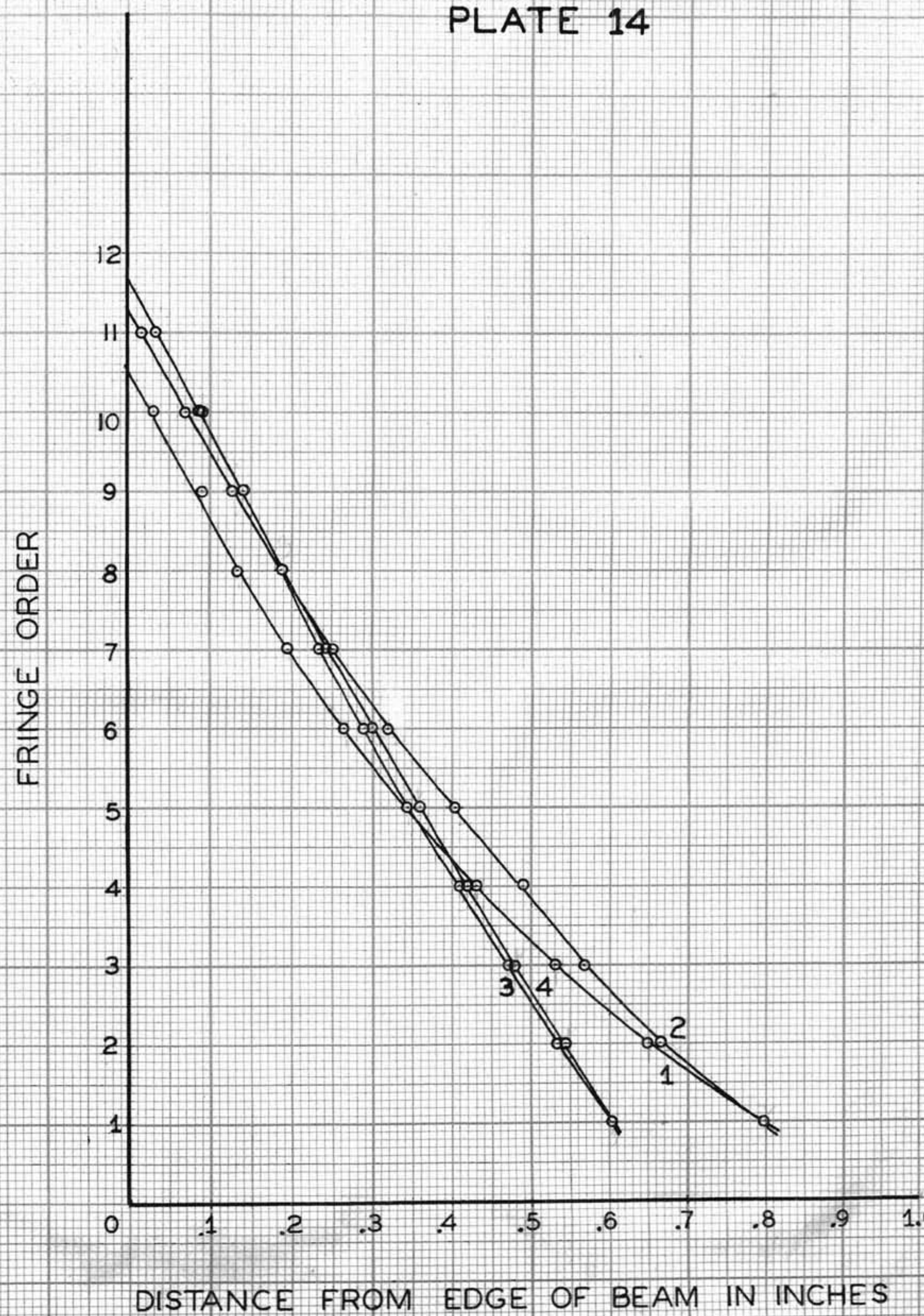
Fig. 3

L -- 2.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 304.460 lb.

Fig. 4

L -- 3.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 261.140 lb.

PLATE 14



12
11
10
9
8
7
6
5
4
3
2
1
0
FRINGE ORDER
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
2
3
4
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 14-A

Fig. 1

L -- 4.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 217.820 lb.

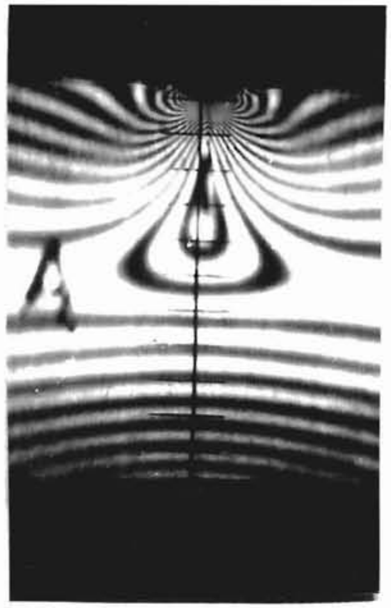


Fig. 2

L -- 5.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 174.500 lb.

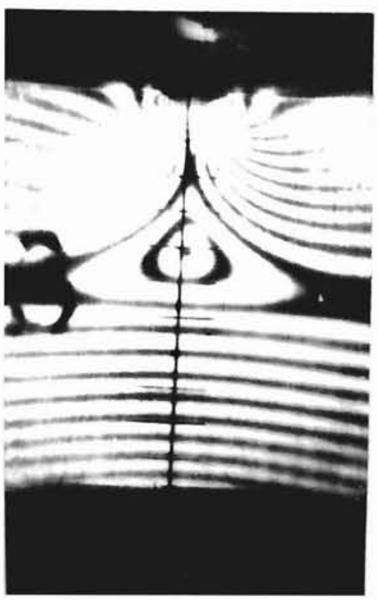


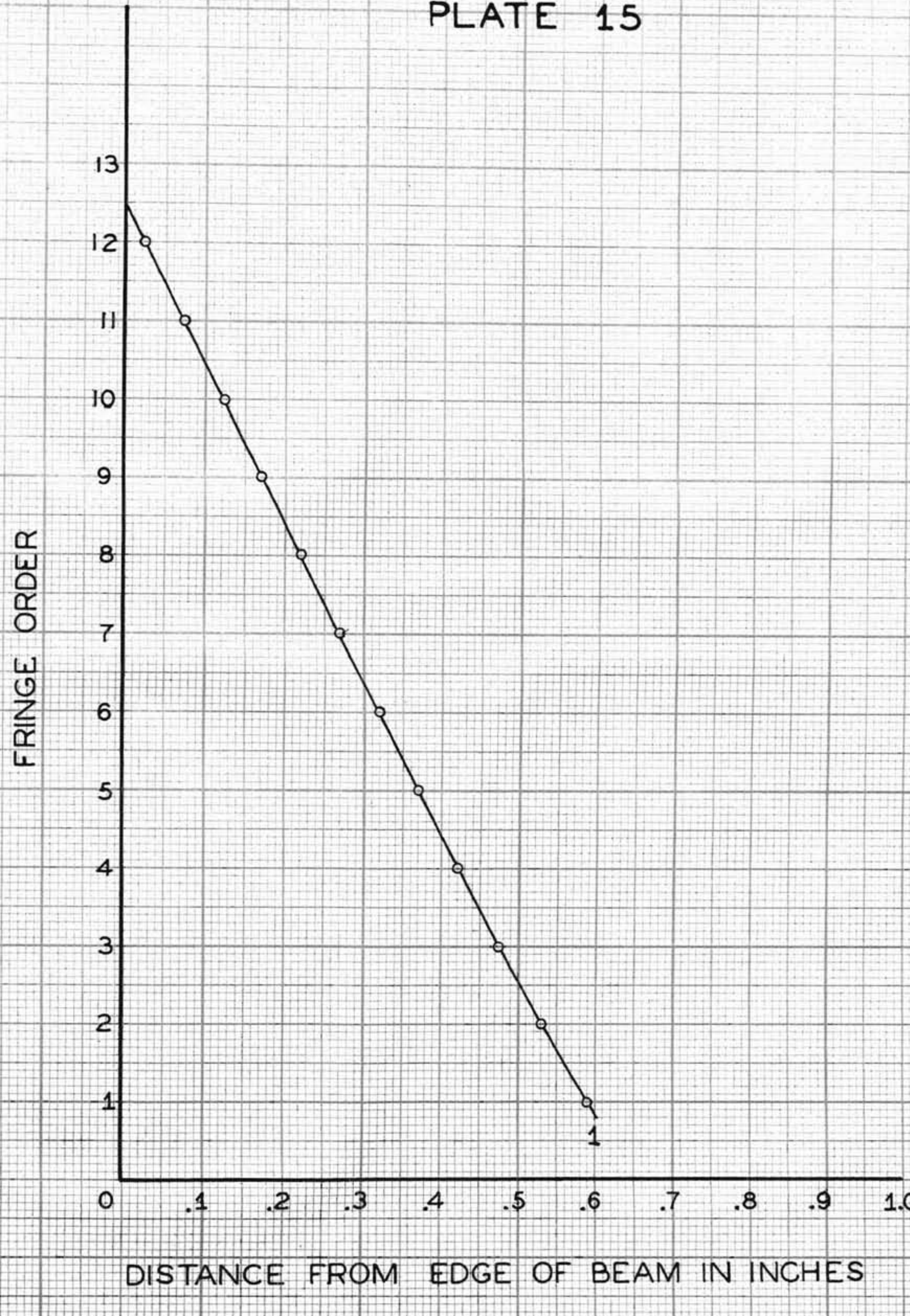
Fig. 3

L -- 7.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 131.180 lb.

Fig. 4

L -- 9.000 in.
D -- 1.141 "
W -- 0.286 "
P -- 102.300 lb.

PLATE 15



FRINGE ORDER
13
12
11
10
9
8
7
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1
DISTANCE FROM EDGE OF BEAM IN INCHES

PLATE 15-A

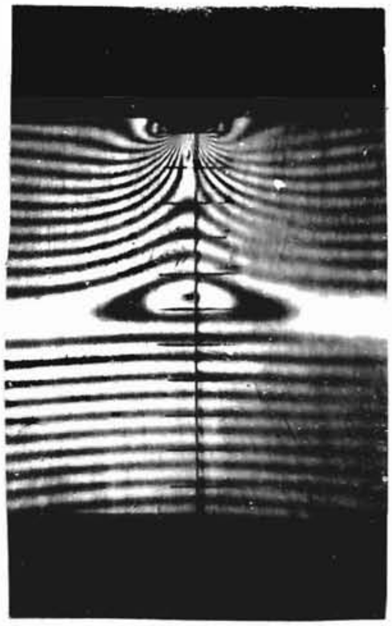


Fig. 1

L	--	11.000	in.
D	--	1.141	"
W	--	0.286	"
P	--	87.860	lb.

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