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The temperature within a homogeneous plate of finite thickness

Charles R. Remington

THE TEMPERATURE WITHIN A HOMOGENEOUS PLATE OF FINITE THICKNESS

BY

CHARLES ROY REMINGTON, JR.

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THESIS

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INTRODUCTION

The object of this thesis is to determine by an analytical method the temperature within a homogeneous plate of finite thickness subjected to a constant temperature on one side and a periodic fluctuating temperature, expressible in a Fourier series, on the opposite side. This problem in periodic temperature changes has not, to the author's knowledge, been solved previously. The flow of heat in the problem is considered in only one direction, the x-direction. Flow in the other two directions is considered negligible, or zero.

Periodic temperature changes in the earth are of importance. These temperature changes have been used by writers since Fourier in the determination of the thermal conductivity of the earth. The daily and annual variations of the earth's surface temperatures are noticeable only at points comparatively near the surface. Below a depth of 60 to 70 feet, they fade away and the temperature becomes constant.

In the field of engineering, periodic changes in temperature should be considered in construction features of heat engines and regenerators for blast furnaces, etc.

Another field of application of heat conduction is in the drying of pourous solids, such as wood.

The results of the study of heat conduction have been put to use in certain gravitational problems, elasticity, and in static and current electricity, and the methods developed are of very general application in mathematical physics. (1)

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(1) L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll, Heat Conduction with Engineering and Geological Applications, 1st Ed., N. Y., McGraw-Hill, 1948, p. 5.

The work of Biot on the settlement or consolidation of soils indicates that the heat conduction equation may play an important part in the theory of these phenomena. (2)

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(2) *Ibid.*

Also, the basic law of heat conduction is in complete analogy to the laws of electrical conduction. (3)

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(3) M. Jakob, Heat Transfer, Vol. 1, N. Y., John Wiley and Sons, 1949, p. 1.

REVIEW OF LITERATURE

From the literature available, it was seen that problems concerning periodic heat flow are generally solved by experimentation, or by mathematical or graphical methods which are at best approximate.

The mathematical theory of heat conduction is due principally to the work of Fourier. He was first to bring order out of confusion in which experimental physicists had left the subject. The most authoritative recent work on the subject is that of Carslaw and Jaeger. (4)

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(4) L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll, Heat Conduction with Engineering and Geological Application, 1st Ed., N. Y., McGraw-Hill, 1948, p. 3.

Ingersoll, Zobel, and Ingersoll, have put certain problems in heat conduction in a very simple and readable form.

Many papers have been published on heat conduction problems in the A.S.M.E. Transactions, but the majority of these are in the form of tables and curves which were compiled either by analytical or experimental methods.

Nowhere in the available literature was the problem studied in this thesis attempted by anyone.

LIST OF SYMBOLS

Symbol	Quantity	Units
t	Time	Hours
T, U, V, T_1	Temperature	Degrees (F)
k	Thermal Conductivity	BTU hr^{-1} ft^{-1} F^{-1}
c	Specific Heat	BTU lb^{-1} F^{-1}
L	Total thickness of Plate	ft
x	Direction normal to surface	ft
ρ	Density	lb ft^{-3}
α	Thermal Diffusivity	ft^2 hr^{-1}
i	Constant	$\sqrt{-1}$
A	Constant	
B	Constant	
γ	Constant	

DISCUSSION

In this thesis problem heat transfer by conduction was the only type considered. The temperature at the surface of the conductivity plate is a known function of time.

A periodic fluctuating temperature is one which has a certain value at a certain time and then repeats itself at equal time intervals. In other words, it travels through the plate in a wave motion. It is a form of an unsteady state of flow.

The general differential equation for unsteady heat conduction expressed in rectangular coordinates is as follows:

$$\frac{1}{\rho c}\left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right)\right] = \frac{dT}{dt} \quad (5)$$

where ρ = Density, c = Specific Heat, k = Thermal Conductivity, T = Temperature, t = time, and x, y, and z = directions of flow.

(5) W. H. McAdams, Heat Transmission, 2nd Ed., N. Y., McGraw-Hill, 1942, p. 29.

Neglecting the variations of k with the temperature and since the plate studied in this problem is homogeneous, the k is taken outside of the parenthesis and a new constant is formed which is called the thermal diffusivity, or, $\frac{k}{\rho c} = \alpha$ (6)

(6) Ibid.

Applying the conditions that the value of k is a constant and that heat conduction will be considered in the x-direction only, the general differential equation for heat conduction reduces to

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}. \qquad (1)$$

This is a modified form of Fourier's heat conduction equation, and the solution of any heat conduction problem, employing constant conductivity and heat flow in the x-direction only, must be a solution of this equation. The solution of such an equation must meet certain conditions which are characteristic to the particular problem. These conditions are known as boundary conditions. [7]

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(7) L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll, Heat Conduction with Engineering and Geological Applications, 1st Ed., N. Y., McGraw-Hill, 1948, pp. 14-15.

Carslaw shows in his book on heat conduction that any equation which satisfies the particular boundary conditions and the general differential equation for heat conduction, admits only one solution to the particular problem, or, in other words, it is unique. [8]

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(8) Carslaw, H. S., Introduction to the Mathematical Theory of the Conduction of Heat in Solids, N. Y., Dover Publications, 1945, pp. 13-16.

In this thesis the particular problem was to solve for the temperature, which is a function of time, at any given plane in a rectangular plate of thickness L, which is subjected to a periodically fluctuating temperature on one side, and a constant temperature on the opposite side. All other sides of the plate were considered to be perfectly insulated so that heat flow would take place in the x-direction only. The value of the thermal conductivity was considered constant.

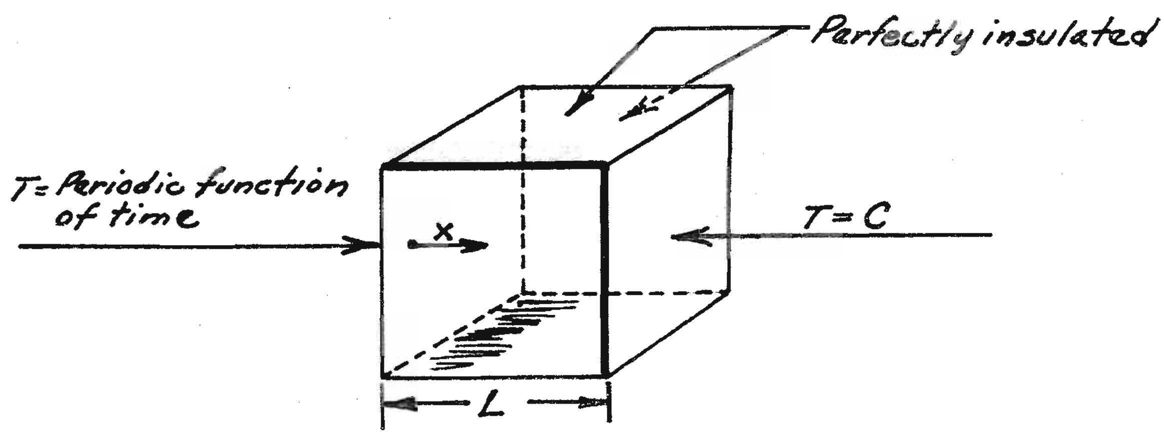

Fig. 1.
Sketch of rectangular plate showing applied temperatures.

On the side of the plate (See Fig. 1.) where x = 0, the temperature is a periodic function of time, and is uniform over the entire surface, and is also single valued. On the face the temperature may have a finite number of finite discontinuities and still be expressible either in a Fourier sine or cosine series. The limitation imposed above excludes no case of engineering interest.

Hence the boundary conditions are as follows:

At $x = 0$, $T =$ a function of time expressible in a Fourier series either of sines or cosines.

At $x = L$, $T = C$, a constant.

It is now necessary to find a solution of equation (1) which will meet these boundary conditions.

Equation (1) is linear and homogeneous with constant coefficients and has a solution

$$T = Ae^{bt+cx},\text{(9)} \qquad (2)$$

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(9) L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll, Heat Conduction with Engineering and Geological Application, 1st Ed., N. Y., McGraw-Hill, 1948, p. 46.

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where A, b, and c, are constants.

Differentiating equation (2) twice with respect to x and once with respect to t, and substituting in equation (1),

$$\alpha c^2 A e^{bt+cx} = Abe^{bt+cx}.$$

Cancelling like terms:

$$\alpha c^2 = b,$$

or

$$c = \pm\sqrt{\frac{b}{\alpha}}. \qquad (3)$$

Substituting equation (3) in equation (2),

$$T = Ae^{bt \pm \sqrt{\frac{b}{\alpha}}x}. \qquad (4)$$

If b is replaced by the constant $\pm ir$,

$$T = Ae^{\pm irt \pm \sqrt{\frac{\pm ir}{\alpha}}x}, \qquad (5)$$

where $i = \sqrt{-1}$.

But $\sqrt{i} = \pm \frac{(1+i)}{\sqrt{2}}$, and $\sqrt{-i} = \pm \frac{(1-i)}{\sqrt{2}}$.[10]

(10) Ibid, p. 46.

Substituting in equation (5),

$$T = Ae^{\pm irt \pm x\sqrt{\frac{r}{2\alpha}}(1 \pm i)}$$

$$= Ae^{\pm i\left[rt \pm x\sqrt{\frac{r}{2\alpha}}\right]} e^{\pm x\sqrt{\frac{r}{2\alpha}}}. \tag{6}$$

Several solutions are contained in equation (6), such as

$$\left.\begin{array}{l} T = Ae^{\pm x\sqrt{\frac{r}{2\alpha}}} e^{i(rt \pm x\sqrt{\frac{r}{2\alpha}})} \\ \text{and} \\ T = Ae^{\pm x\sqrt{\frac{r}{2\alpha}}} e^{-i(rt \pm x\sqrt{\frac{r}{2\alpha}})}. \end{array}\right\} \tag{7}$$

Since the sum of any number of solutions to a differential equation is also a solution, then

$$T = Ae^{\pm x\sqrt{\frac{r}{2\alpha}}} \left[e^{i(rt \pm x\sqrt{\frac{r}{2\alpha}})} \pm e^{-i(rt \pm x\sqrt{\frac{r}{2\alpha}})}\right].$$

Writing two solutions from this equation gives,

$$T = A_1 e^{\pm x\sqrt{\frac{r}{2\alpha}}} \left[e^{i(rt \pm x\sqrt{\frac{r}{2\alpha}})} + e^{-i(rt \pm x\sqrt{\frac{r}{2\alpha}})}\right] \tag{8}$$

and

$$T = A_2 e^{\pm x\sqrt{\frac{r}{2\alpha}}} \left[e^{i(rt \pm x\sqrt{\frac{r}{2\alpha}})} - e^{-i(rt \pm x\sqrt{\frac{r}{2\alpha}})}\right]. \tag{9}$$

Employing Euler's transformations,

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta;^{(11)}$$

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(11) I. S. Sokolnikoff, Advanced Calculus, 1st Ed., N. Y., McGraw-Hill, 1939, pp. 306-308.

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then

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \qquad (10)$$

and

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta.^{(12)} \qquad (11)$$

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(12) L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll, Heat Conduction with Engineering and Geological Application, 1st Ed., N. Y., McGraw-Hill, 1948, p. 31.

Substituting equation (10) in equation (8) and equation (11) in (9), we have

$$T = 2A_1 e^{\pm x\sqrt{\frac{\gamma}{2\alpha}}}\cos\left(\gamma t \pm x\sqrt{\frac{\gamma}{2\alpha}}\right) \qquad (12)$$

$$T = 2iA_2 e^{\pm x\sqrt{\frac{\gamma}{2\alpha}}}\sin\left(\gamma t \pm x\sqrt{\frac{\gamma}{2\alpha}}\right). \qquad (13)$$

By equating the constant $2iA_2 = B$, equation (13) becomes

$$T = Be^{\pm x\sqrt{\frac{\gamma}{2\alpha}}}\sin\left[\gamma t \pm x\sqrt{\frac{\gamma}{2\alpha}}\right]. \qquad (14)$$

Equation (13) was chosen instead of (12) because the author was solving for the temperature expressible in a sine series only of Fourier's series.

Since heat does flow through the plate, the temperature must decrease as the distance increases; therefore, the minus exponential value in equation (14) must be used and the same sign must preceed the term $x\sqrt{\frac{r}{2\alpha}}$.

Equation (14) now becomes

$$T = Be^{-x\sqrt{\frac{r}{2\alpha}}} \sin\left[rt - x\sqrt{\frac{r}{2\alpha}}\right]. \qquad (15)$$

Substituting the boundary conditions at x = 0,

$$T = B \sin rt,$$

where B and r are both undetermined constants of integration. Therefore, the solution may be written as a series of similar terms, or,

$$T = \sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-x\sqrt{\frac{n}{2\alpha}}} \sin\left[nt - x\sqrt{\frac{n}{2\alpha}}\right]. \qquad (16)$$

a_n is yet to be determined to meet the boundary conditions. This can be done at the plane x = 0, since this is a sine series of a form convenient for expressing any single valued function of time, with less than an infinite number of finite discontinuities, in a single Fourier sine series. Hence the a's are determined in the usual manner associated with the expansion of a function in a Fourier series.

For simplicity the temperature will be written as the sum of two temperatures: T = U + V, where:

For U, we may write:

$$U = a_1 \sin t + a_2 \sin 2t + a_3 \sin 3t + \ldots a_n \sin nt, \quad \text{where } x = 0$$

and $\alpha \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$:

and for V we may write:

$$V = 0, \text{ at } x = 0. \qquad \frac{\partial^2 V}{\partial x^2} = 0.$$
$$V = T_c, \text{ at } x = L.$$

The solution of the latter equation is:

$$V = A'x + B'$$

Substituting the boundary conditions,

$$T_c = A'L$$
$$A' = \frac{T_c}{L} \text{ and } B' = 0.$$
$$\therefore V = T_c\left(\frac{x}{L}\right) \tag{17}$$

It is now necessary to make the function U vanish at x = L. In order to do this a method of fictitious sources will be used. The medium is imagined to extend beyond the initial boundaries, x = 0 and x = L, and the temperature distribution is adjusted to the boundary conditions by fictitious sources and sinks symmetrically located in the extension of this space. (13) (14)

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(13) M. Jakob, Heat Transfer, Vol. 1, N. Y., John Wiley and Sons, 1949, p. 332.

(14) W. S. Hogan, Temperature Fluctuations within a Solid of Finite Thickness, Thesis, Missouri School of Mines, Rolla, Missouri, pp. 15-18.

Imagine two solids, one placed on each side of the original solid at the planes where x = 0 and x = L, which extend to infinity in both the positive and negative x-directions. As stated before, there will be no heat flow in the y and z directions.

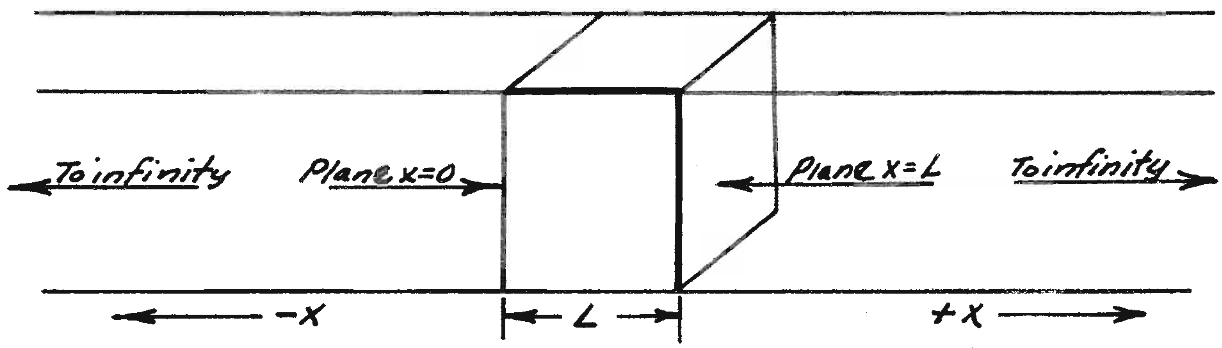

Fig. 2.
Sketch of original plate and two adjacent solids of infinite length.

Imagine applying a fictitious source at the plane $x = 0$ of such strength that the temperature at any point and at any time in the positive x-direction will be

$$T_1' = \sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-x\sqrt{\frac{n}{2\alpha}}} \sin\left[nt - x\sqrt{\frac{n}{2\alpha}}\right]. \qquad (16)$$

Equation (16) gives the solution for the temperature in the plate at the plane $x = 0$. By applying another fictitious source of equal strength at a distance of $+ 2\,L$ from the plane $x = 0$, but 180° out of phase with T_1', the temperature at the plane $x = L$ will be zero. The temperature function is

$$T_2' = -\sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-(2L-x)\sqrt{\frac{n}{2\alpha}}} \sin\left[nt - (2L-x)\sqrt{\frac{n}{2\alpha}}\right].$$

But, this source changes the temperature at the plane $x = 0$, therefore, another fictitious source of equal strength but 180° out of phase with T_2' is applied at a distance of $- 2\,L$ from the plane $x = 0$. This source

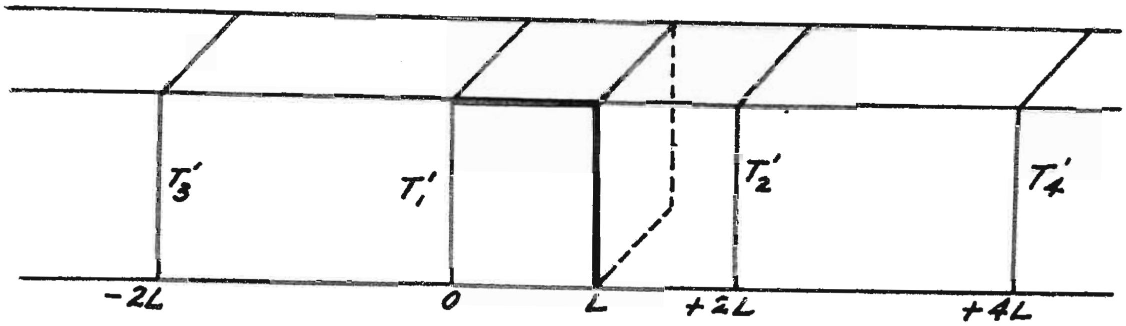

Fig. 3.
Sketch of original and infinite solids showing locations of fictitious heat sources.

will give the correct temperature at the plane x = 0, and the temperature at - 2 L may be shown by

$$T_3' = \sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-(2L+x)\sqrt{\frac{n}{2\alpha}}} \sin\left[nt-(2L+x)\sqrt{\frac{n}{2\alpha}}\right].$$

It is again necessary to apply another fictitious source at a distance + 4 L from the plane x = 0 in order to make the temperature equal to zero at the plane x = L.

$$T_4' = -\sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-(4L-x)\sqrt{\frac{n}{2\alpha}}} \sin\left[nt-(4L-x)\sqrt{\frac{n}{2\alpha}}\right].$$

The boundary conditions of the problem can be fulfilled if these opposing fictitious sources are placed at intervals of 2 L in both the positive and negative x-directions from the plane x = 0 to plus and minus infinity.

The sum of this infinite number of temperature expressions for the infinite number of sources will give an expression for the term U, which meets the boundary conditions for U.

$$U = \sum_{m=0}^{m=\infty} \sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-(2mL+x)\sqrt{\frac{n}{2\alpha}}} \sin\left[nt-(2mL+x)\sqrt{\frac{n}{2\alpha}}\right]$$

$$-\sum_{m=1}^{m=\infty} \sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-(2mL-x)\sqrt{\frac{n}{2\alpha}}} \sin\left[nt-(2mL-x)\sqrt{\frac{n}{2\alpha}}\right], \qquad (18)$$

where n and m have all integral values from 1 to ∞.

Combining equation (18) with the equation for V (equation (17)), we have

$$T = T_c\left(\frac{x}{L}\right) + \sum_{m=0}^{m=\infty} \sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-(2mL+x)\sqrt{\frac{n}{2\alpha}}} \sin\left[nt-(2mL+x)\sqrt{\frac{n}{2\alpha}}\right]$$

$$-\sum_{m=1}^{m=\infty} \sum_{n=1,2,3\ldots}^{n=\infty} a_n e^{-(2mL-x)\sqrt{\frac{n}{2\alpha}}} \sin\left[nt-(2mL-x)\sqrt{\frac{n}{2\alpha}}\right]. \qquad (19)$$

Equation (19) will give the temperature at any plane from x = 0 to x = L at any time in the plate, and since all the boundary conditions of the problem are met and the equation is a solution to the general heat conduction equation (1), it is a solution to this thesis problem.

It can be seen from the derived temperature equation (19) that calculation of the temperature is dependent upon the value of a_n, the constants of the material, and the thickness of the plate. With a small value of the thermal diffusivity, α, the negative exponent on the e-term becomes large, which reduces the number of terms to be evaluated.

Graphs were not deemed necessary, because of the relative simplicity of evaluation of the temperature in equation (19).

CONCLUSIONS

Equation (19) as derived is a means of determining the temperature at any point within a plate of finite thickness subjected to a periodic fluctuating temperature on one side, expressible in a Fourier sine series, and a constant temperature on the opposite side.

The equation also meets the conditions for a Fourier's series, which are:

1. The $T = f(t)$ is single valued, i.e., for every value of t there is one and only one value of T (save at discontinuities).
2. The $f(t)$ is finite. For example $T = \tan t$ cannot be expanded in Fourier series.
3. There are only a finite number of maxima and minima. For example $f(t) = 1/t$ cannot be so expanded.
4. The $f(t)$ is continuous, or at least has only a finite number of finite discontinuities.

The function that represents the initial state of temperature will satisfy these conditions, for there can be but a single value of the temperature at each point of a body, and this value must be finite. Furthermore, while there may exist initial discontinuities, as at a surface of separation between two bodies, such discontinuities will always be finite. This indicates the applicability as well as the importance of Fourier's series in the theory of heat conduction.[15]

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(15) L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll, Heat Conduction with Engineering and Geological Applications, 1st Ed., N. Y., McGraw-Hill, 1948, pp. 58-59.

The solution of the problem assumed heat flow by conduction only, but it could be extended to include the effect of convection by applying the method used by G. L. Scofield,[16] for convection.

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(16) G. L. Scofield, Periodic Heat Flow, Thesis, Missouri School of Mines, Rolla, Missouri.

Since the derived equation for the temperature contains two double infinite series, the degree of accuracy of the problem would be entirely dependent on the number of terms evaluated in both series. However for engineering usage, not too many terms would have to be evaluated to give results accurate enough.

SUMMARY

By using the solution for temperature of the plate at the plane $x = 0$ and applying an infinite number of fictitious sources, it is possible to solve for the temperature at any plane, and at any time, with a plate of finite thickness subjected to a periodic fluctuating temperature on one face and a constant temperature on the opposite face. The temperature can be determined when the time and the distance from the face of the plate are given.

Many problems are suggested from the solution to this thesis, such as the combined effect of conduction, convection, and radiation, or the combined effect in a two or three dimensional flow problem. However, such problems would probably be quite complicated and difficult.

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VITA

The author was born July 15, 1924, at Webster Groves, Missouri. He attended grade school in Webster Groves, Missouri, and graduated from high school in Springfield, Missouri. He attended the Missouri School of Mines for one semester before being inducted into the United States Army in February, 1943. He served three years in the Armed Forces and was honorably discharged in February, 1946. He then resumed his studies at the Missouri School of Mines and graduated with a B. S. in Mechanical Engineering in May, 1949. Since that time he has been doing graduate work toward his M. S. in Mechanical Engineering.