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# Transfer function of the modified Kramer system

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TRANSFER FUNCTION OF THE MODIFIED

KRAMER SYSTEM

BY

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A

THESIS

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INTRODUCTION

The use of operational mathematics in the solution of engineering problems has grown tremendously in the last two decades, particularly during the last war. This has come about because of the facility with which problems which would be almost impossibly complicated or tedious to solve by classical methods can be solved using operational calculus.

The operational calculus, particularly the Laplace transform method, has become especially useful in the field of servomechanisms and controls. Here the constantly increasing complexity of systems used has resulted in very high order differential equations. The fact that the Laplace transform method makes it possible to inject the boundary conditions into the initial equations is often a great advantage in their solution. In addition, the fact that much can often be learned about the behavior of a system by a study of its operational equations, without having to resort to an actual solution, makes this approach even more valuable.

This paper proposes to develop the complete operational solution, or transfer function, of a specific control system, namely, the modified Kramer system for speed control of induction motors. This transfer function is derived as a basis for further study of the performance of the system, or of its components.

First, a brief qualitative discussion of the original Kramer system seems in order. This system has been in use for many years in steel mill power applications. It has been applied chiefly in situations where the amount of power wasted in the rotor circuit of an induction motor by use of resistances to control speed became large enough that it was economically feasible to recover this power and return it to the line.

Figure 1 shows a block diagram of the Kramer system. In operation the power developed in the rotor of the main machine, A, is converted to DC by the synchronous converter, B, and fed into the DC motor, C which in turn drives machine D as an induction generator; returning to the system the otherwise wasted rotor power. This is of course minus the losses of machines B, C, and D.

Speed variation is accomplished in the following manner. It is well known from induction motor theory that the speed is a function of the resistance of the rotor circuit. This rotor resistance of course represents a voltage drop. Whether this drop is caused by a passive impedance, or by an active element in opposition to the induced rotor voltage makes no difference as far as the induction machine is concerned. In the Kramer system a synchronous converter is inserted in the rotor circuit of the induction machine. The voltage developed by the converter affects the main machine's speed just as would a rotor resistance. The DC machine represents a load supplied by the DC side of the converter. By varying the excitation of the DC machine, therefore, the voltage of the DC side of the converter can be caused to vary. The voltage of the AC side of the converter is in fixed ratio to that of the DC side. Thus a variable voltage is inserted in the rotor circuit of the induction machine by varying the excitation of the DC machine, thereby causing variation in the induction motor's speed. Since the frequency of the converter must always equal the slip frequency of the induction motor, i.e., the frequency of the supply for the converter stator, it is apparent that the speed variation of B will be equal and opposite to that of machine A. Automatic speed control can be accomplished by providing the proper feedback link from the speed of machine A to the field of machine

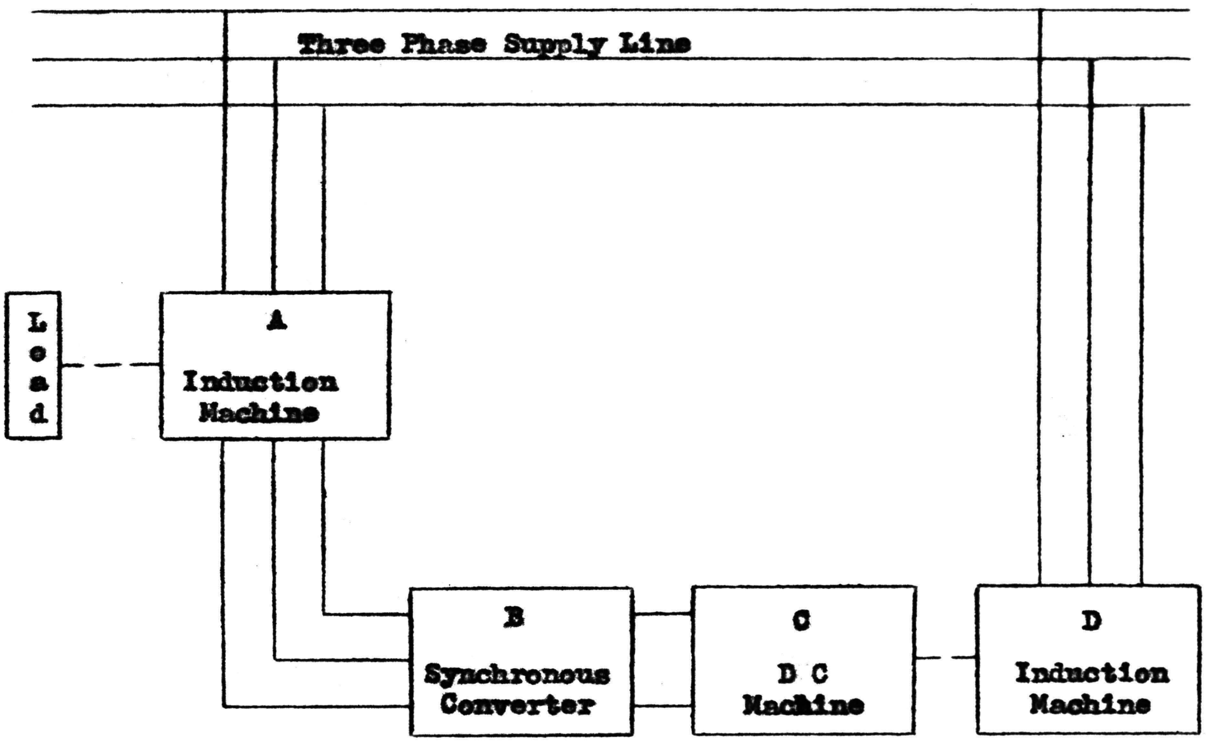


FIGURE 1 - Block Diagram of Kramer System

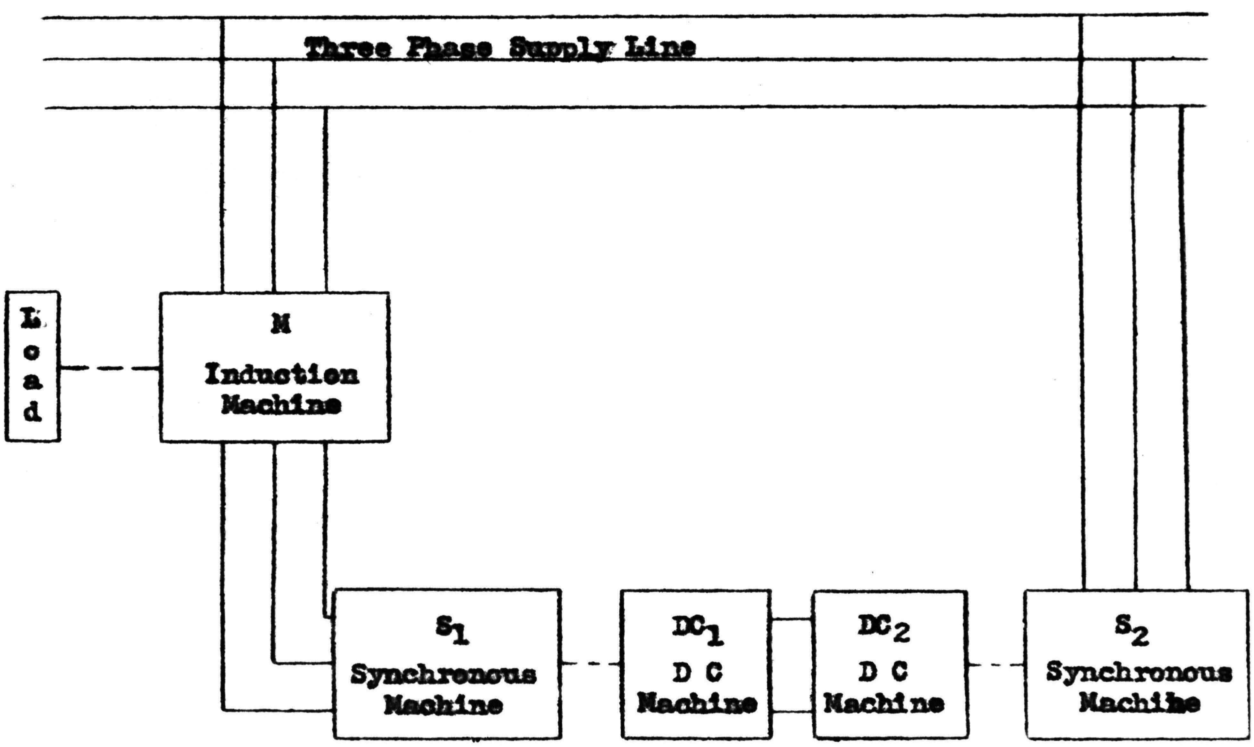


FIGURE 2 - Block Diagram of Modified Kramer System

c.

A block diagram of the modified Kramer system is shown in Figure 2. In operation it is very similar to the Kramer system. The modified system, however, utilizes a synchronous motor-generator set instead of the synchronous converter; and a second synchronous motor-generator set instead of the induction generator-DC motor set of the original Kramer system. The chief advantage of the modified system is that in general synchronous motor-generator sets are more readily available, cheaper, and require less auxiliary equipment than, for example, a converter. Also, the DC side of the MG set $S_1$-$DC_1$ is no longer rigidly tied, voltage-wise, to the AC side, as in the converter; and set $S_2$-$DC_2$ is of necessity a constant speed set.

Due to saturation, hysteresis, and non-uniform air gaps, electrical machinery is inherently non-linear in behavior. Therefore in order to obtain a mathematical analysis of its performance, it is conventional to make certain simplifying assumptions. These will be made in the development which follows, and will be tabulated at the very beginning of the discussion.

The development will be carried forward in three main sections, the last two of which will be further subdivided. In the first section the general assumptions will be discussed. In the second, relations for each of the individual machine groups will be developed. In the third, these relations will be matched at their boundaries to obtain the final solution.

# REVIEW OF LITERATURE

There is very little in the literature concerning the specific subject of this paper. Very brief qualitative mention of the Kramer system is made in most of the standard college texts on electrical machinery. The modified Kramer system made its first appearance about 1941, in a Wright Air Force Base wind tunnel application. [(1),(2)] The

(1) Dickey, A. D., Laffon, C. M., and Kilgore, L. A., Variable Speed Drive For USAAC Wind Tunnel, Wright Field, Dayton, Ohio, AIEE Transactions, Vol. 61, 1942, pp. 126-130.
(2) Clymer, C. C., Large Adjustable-Speed Wind Tunnel Drive, AIEE Transactions, Vol. 61, 1942, pp. 156-158.

only publication dealing directly with the problem in this paper is an article by M. M. Liwschitz and L. A. Kilgore. [(3)] That article deals

(3) Liwschitz, M. M., and Kilgore, L. A., A Study of the Modified Kramer or Asynchronous-Synchronous Cascade Variable-Speed Drive, AIEE Transactions, Vol. 61, 1942, pp. 256-260.

primarily with the steady-state stability of the system, and develops complete equations only for the double-fed machine. This paper approaches the problem from a somewhat different angle, and extends the analysis to obtain a transfer function for the complete system. The Liwschitz-Kilgore article works in the Heaviside calculus, whereas this paper uses the Laplace transforms.

Although as stated the literature dealing specifically with the subject of this paper is very limited in quantity, this review would be incomplete without mention of the classic article on the operational analysis of electrical machinery, namely, the "Park article". [(4)] Vir-

(4) Park, R. H., Two-Reaction Theory of Synchronous Machines-Generalized Method of Analysis-Part I, AIEE Transactions, Vol. 48, 1929, pp. 716-727.

tually all the works listed in the Bibliography of this paper in some way stem from or are based upon the above paper by Park. Noteworthy among these succeeding papers are: the Waring and Crary [(5)] article,

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(5) Waring, M. L., and Crary, S. B., The Operational Impedances of a Synchronous Machine, G. E. Review, Vol. 35, 1932, pp. 578-582.

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which presents Park's work on the basis of reciprocal mutual reactances; the Stanley article, [(6)] which solves the induction machine by a trans-

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(6) Stanley, H. C., An Analysis of the Induction Machine, AIEE Transactions, Vol. 57, 1938, pp. 751-755.

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formation to a set of stationary $\alpha-\beta$ axes; and the papers by Rankin [(7),(8)] which further correlate the work of Park with Waring and Crary

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(7) Rankin, A. W., The Equations of the Idealized Synchronous Machine, G. E. Review, Vol. 47, June, 1944, pp. 31-36.

(8) Rankin, A. W., Per-Unit Impedances of Synchronous Machines, AIEE Transactions, Vol. 64, 1945, part I, pp. 569-573.

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and others. This paper draws on all the above listed publications, particularly the Stanley article.

# DISCUSSION

## 1. Discussion of Complete Schematic Diagram and Tabulation of General Assumptions

Refer to Fig. 3. The symbols used in the figure are defined thereon. It should be noted that the masses and frictional effects of each complete motor-generator set have been lumped together as $J_2$, $B_2$, $J_3$, $B_3$, respectively. All shafts have been assumed to be rigid. This is generally valid because of their relatively large diameters and short lengths in most electrical machines.

For simplicity in writing the equations, and because it is very probably so in an actual application, synchronous machine $S_1$ and $S_2$ as well as machines $DC_1$ and $DC_2$, will be assumed to be identical .

Since it is desired to control the system by varying the excitation of one of the DC machines, three of the four excitation voltages shown on Fig. 3 can be taken as constants. $E_2$ has arbitrarily been chosen as the variable excitation. Therefore, $E_{f1}$, $E_{f2}$, and $E_1$ are constants. Furthermore, since the system is tied to a very large source of supply, the line voltage and frequency applied to motor M and synchronous machine $S_2$ are constants.

Finally, the presence of the two extra short circuited windings in the rotors of $S_1$ and $S_2$ is an approximate attempt to account for the effect of amortisseur windings. More will be said of this last assumption later on.

The general assumptions discussed above are listed below:

1. Masses and frictions of mechanically coupled elements are lumped
2. All shafts are rigid.
3. Machines $S_1$ and $S_2$ are identical, as are $DC_1$ and $DC_2$.

4. $E_{f1}$, $E_{f2}$, and $E_1$ are constant.

5. Line voltage and frequency to M and $S_2$ are constant.

6. Amortisseur windings can be represented by two short - circuited rotor windings.

## 2. Transfer Functions of Individual Machine Groups

### Induction Machine, M

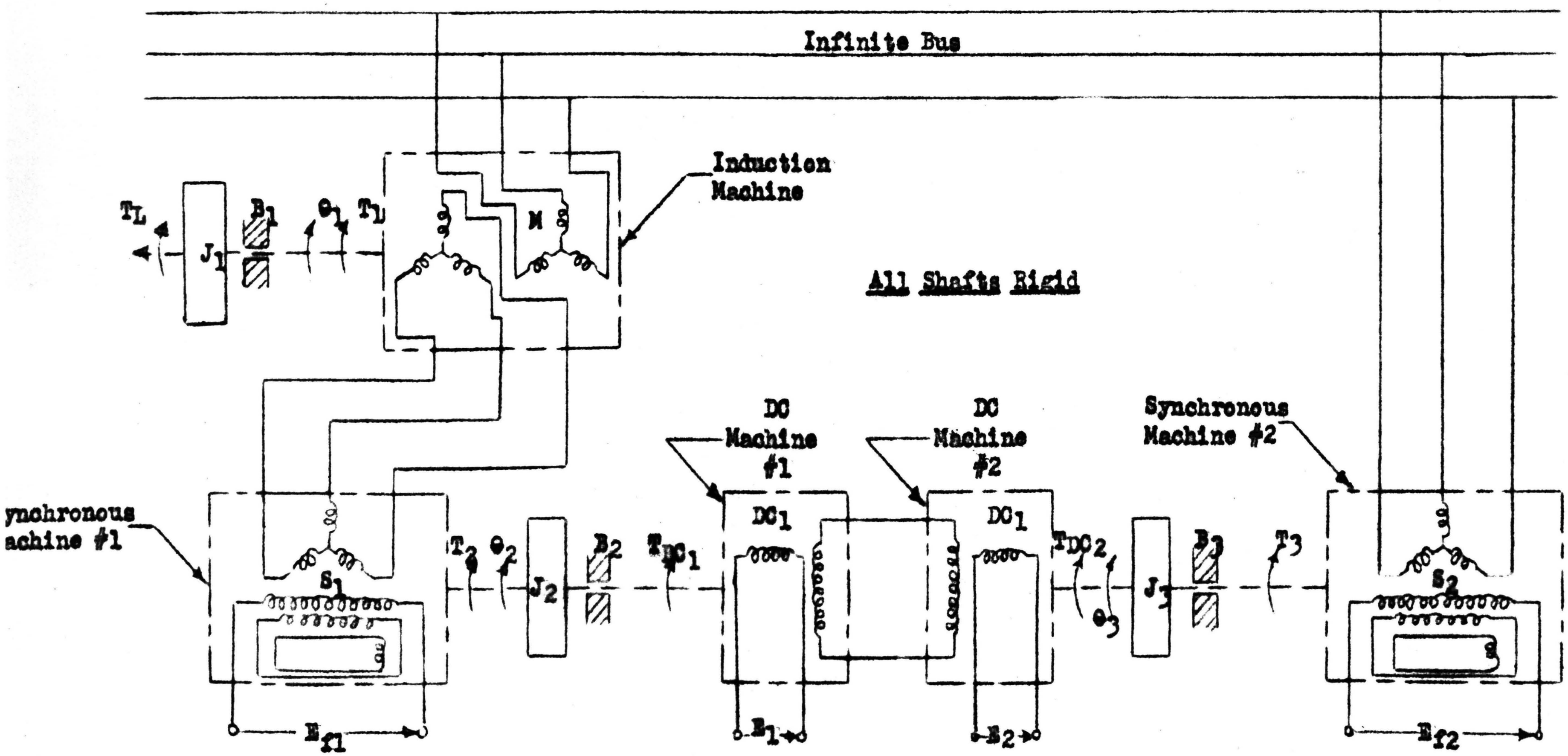
The development in this section follows that of Stanley [9] very closely in form and notation, and draws on an earlier paper by Levine [10] for the initial equations. However, there is an important difference from the Stanley solution. This is that a reference frame rotating at synchronous speed has been used rather than Stanley's fixed frame. As will be brought out later in the devlopment, this change of reference is the key to the Laplace transformable solution obtained.

(9) Stanley, H. C., op. cit., pp. 751-755.

(10) Levine, S. J., An Analysis of the Induction Motor, AIEE Transactions, Vol. 54, 1935, pp. 526-529.

In addition to those already mentioned, the following assumptions will be made:

1. Hysteresis, saturation, and eddy currents will be neglected.




$J_1$ = Moment of Inertia of M
$J_2$ = " " " " $S_1$ + $DC_1$
$J_3$ = " " " " $S_2$ + $DC_2$
$B_1$ = Friction Coefficient of M
$B_2$ = " " " $S_1$ + $DC_1$
$B_3$ = " " " $S_2$ + $DC_2$

$T_1$ = Electrical Torque of M
$T_2$ = " " " $S_1$
$T_{DC_2}$ = " " " $DC_2$
$T_L$ = Mechanical Torque to Load
$T_{DC_1}$ = " " " $DC_1$
$T_3$ = " " " $S_2$

$\theta_1$ = Angle of Rotor Axis of M
$\theta_2$ = " " " " " $S_1$
$\theta_3$ = " " " " " $DC_1$
$E_{f1}$ = Field Voltage of $S_1$
$E_{f1}$ = " " " $S_2$
$E_1$ = " " " $DC_1$
$E_2$ = " " " $DC_2$

FIGURE 3 - Schematic Diagram of Modified Kramer System

2. Rotor and stator windings are balanced.

3. Rotor is smooth, i.e., self inductances of all windings are independent of rotor position.

4. Coefficient of mutual inductance between rotor and stator windings is a cosine function of the angle between the axes of the two windings.

It can be stated as a general relation for any electric circuit that:

$$e = ir + \frac{d\psi}{dt}$$

where

$e$ = instantaneous applied voltage

$i$ = instantaneous positive current

$r$ = resistance

$\psi$ = total instantaneous flux linkages with the circuit, including those with coupled circuits, such linkages being positive when they produce positive drop in the circuit.

By the fundamental definition for induced voltages

$$\frac{d\psi}{dt} = L\frac{di}{dt} + \sum M_n \frac{di_n}{dt} = \frac{d\psi_{self}}{dt} + \frac{d\psi_{mut}}{dt}$$

where

$L$ = circuit self-inductance

$M_n$ = mutual inductance with the nth circuit

$i_n$ instantaneous current in the nth circuit

Then

(1) $$e = ir + L\frac{di}{dt} + \sum_{0}^{n} M_n \frac{di_n}{dt}$$

and

$$\Psi_n = M_n i_n$$

$$\Psi_{self} = Li$$

Under the assumptions made, the three-phase induction motor can be represented by six circuits, each having self and mutual inductances as shown in Fig. 4, but with the rotor circuit free to rotate with reference to the stator. On the basis of assumptions (2) and (3) the self inductances of each phase and the mutual inductances between phases on the stator and rotor are equal, i.e.,

$$L_{sa} = L_{sb} = L_{sc}$$

$$M_{sab} = M_{sbc} = M_{sac}$$

$$L_{ra} = L_{rb} = L_{rc}$$

$$M_{rab} = M_{rbc} = M_{rac}$$

The following is actually an explanation of assumption (4). The $M_{3ph}$ represents the maximum value of mutual inductance between one rotor phase and one stator phase, i.e., that inductance which occurs when the magnetic axes of the given phases are opposite each other. Referring to phases $a$ and 1, Fig. 5, it can be seen that the mutual coupling between any stator phase and a given rotor phase is a function of the cosine of the angle between them. With this in mind, and referring to Fig. 4 again, it may be written that

$$\Psi_a = L_1 i_a + M_1(i_b + i_c) + M_{3ph}\left[i_1 \cos\theta + i_2 \cos(\theta+120) + i_3 \cos(\theta-120)\right]$$

$$\Psi_b = L_1 i_b + M_1(i_a + i_c) + M_{3ph}\left[i_1 \cos(\theta-120) + i_2 \cos\theta + i_3 \cos(\theta+120)\right]$$

(2) $$\Psi_c = L_1 i_c + M_1(i_a + i_b) + M_{3ph}\left[i_1 \cos(\theta+120) + i_2 \cos(\theta-120) + i_3 \cos\theta\right]$$

$$\Psi_1 = L_2 i_1 + M_2(i_2 + i_3) + M_{3ph}\left[i_a \cos\theta + i_b \cos(\theta-120) + i_c \cos(\theta+120)\right]$$

$$\Psi_2 = L_2 i_2 + M_2(i_1 + i_3) + M_{3ph}\left[i_a \cos(\theta+120) + i_b \cos\theta + i_c \cos(\theta-120)\right]$$

$$\Psi_3 = L_2 i_3 + M_2(i_1 + i_2) + M_{3ph}\left[i_a \cos(\theta-120) + i_b \cos(\theta+120) + i_c \cos\theta\right]$$

But it is also true that

$$i_a + i_b + i_c = 0$$

(3)

$$i_1 + i_2 + i_3 = 0$$

from which, referring to equations (2) above

$$\Psi_a + \Psi_b + \Psi_c = 0$$

(4)

$$\Psi_1 + \Psi_2 + \Psi_3 = 0$$

The above equations quickly become unwieldy because of the involved trigonometric relations they contain. Therefore two successive simplifying transformations will be made. First, replace the three phase machine of Fig. 4 with its equivalent two phase machine. To make this equivalence it is necessary only to adjust the inductance values so that the same flux linkage relations are maintained. In the equivalent machine of Fig. 6, let the following current relations be assumed: (11)

(11) Kimbark, E. W., Two Phase Coordinates of a Three-Phase Circuit, AIEE Transactions, Vol. 58, 1939, p. 894.

(5) $i_x = i_a = \dfrac{2i_a - i_b - i_c}{3}$

(6) $i_y = j i_a = \dfrac{i_b - i_c}{\sqrt{3}}$

(7) $i_d = i_1 = \dfrac{2i_1 - i_2 - i_3}{3}$

(8) $i_q = j i_1 = \dfrac{i_2 - i_3}{\sqrt{3}}$

Then

(9)
$$\Psi_x = L_s i_x + M(i_d \cos\theta - i_q \sin\theta)$$
$$\Psi_y = L_s i_y + M(i_d \sin\theta + i_q \cos\theta)$$

(10)
$$\Psi_d = L_R i_d + M(i_x \cos\theta + i_y \sin\theta)$$
$$\Psi_q = L_R i_q + M(i_x \sin\theta + i_y \cos\theta)$$

From the symmetry of equations (3) and (4) it may also be stated, analagous to equations (5) through (8) that

(11) $\Psi_x = \Psi_a$

(12) $\Psi_y = \dfrac{\Psi_b - \Psi_c}{\sqrt{3}}$

(13) $\Psi_d = \Psi_1$

$$(14) \quad \Psi_q = \frac{\Psi_2 - \Psi_3}{\sqrt{3}}$$

Now, the values of $L_S$, $L_R$, and M can be determined by substituting in (11) and (13) from (2), (9), and (10:

$$L_s i_x + M[i_d \cos\theta - i_q \sin\theta] = L_1 i_a + M_1 (i_b + i_c)$$

(15)

$$+ M_{3ph} [i_1 \cos\theta + i_2 \cos(\theta + 120) + i_3 \cos(\theta - 120)]$$

$$L_R i_d + M[i_x \cos\theta + i_y \sin\theta] = L_2 i_1 + M_2 (i_2 + i_3)$$

(16)

$$+ M_{3ph} [i_a \cos\theta + i_b \cos(\theta - 120) + i_c \cos(\theta + 120)]$$

Using equations (3) and certain trigonometric identities, the right hand sides of (15) and (16) simplify, so that

$$L_s i_x + M[i_d \cos\theta - i_q \sin\theta] = (L_1 - M_1) i_a$$

$$+ \frac{3}{2} M_{3ph} \left[i_1 \cos\theta - \frac{1}{\sqrt{3}} (i_2 - i_3) \sin\theta\right]$$

$$L_R i_d + M[i_x \cos\theta + i_y \sin\theta] = (L_2 - M_2) i_1$$

$$+ \frac{3}{2} M_{3ph} \left[i_a \cos\theta + \frac{1}{\sqrt{3}} (i_b - i_c) \sin\theta\right]$$

which by the definition of the two phase quantities becomes

$$L_s i_x + M[i_d \cos\theta - i_q \sin\theta] = (L_1 - M_1) i_x$$

$$+ \frac{3}{2} M_{3ph} [i_d \cos\theta - i_q \sin\theta]$$

$$L_R i_d + M[i_x \cos\theta + i_y \sin\theta] = (L_2 - M_2) i_d$$

$$+ \frac{3}{2} M_{3ph} [i_x \cos\theta + i_y \sin\theta]$$

Inspection of these last two equations shows that the equivalent two phase inductances must be

$$L_S = L_1 - M_1$$

$$(17) \quad L_R = L_2 - M_2$$

$$M = \frac{3}{2} M_{3ph}$$

A complete family of equations can now be written for the two phase machine. These will include equations (9) and (10) and the following:

$$e_x = i_x r_s + \frac{d\psi_x}{dt}$$

$$e_y = i_y r_s + \frac{d\psi_y}{dt}$$

$$(18)$$

$$e_d = i_d R_R + \frac{d\psi_d}{dt}$$

$$e_q = i_q R_R + \frac{d\psi_q}{dt}$$

A further simplification, consisting of a transformation to a set of right angle axes, $\alpha$ and $\beta$ , rotating at synchronous speed, will next be made. Fig. 7-a shows the actual two phase machine at a given instant. In the steady state the flux, current, and voltage in each phase of the stator pulsate at synchronous frequency, and are in space and time quadrature with the corresponding quantities in the other phase. The quantities in phase x lead those in phase y. It can readily be shown that the resultant of each pair of pulsating quantities, e.g., $i_x$ and $i_y$, referred to a fixed reference is a vector of constant magnitude rotating clockwise at synchronous speed, $\omega_f$ . The rotor turns clockwise at a speed $\omega_1$ , so that the stator vectors sweep past it at

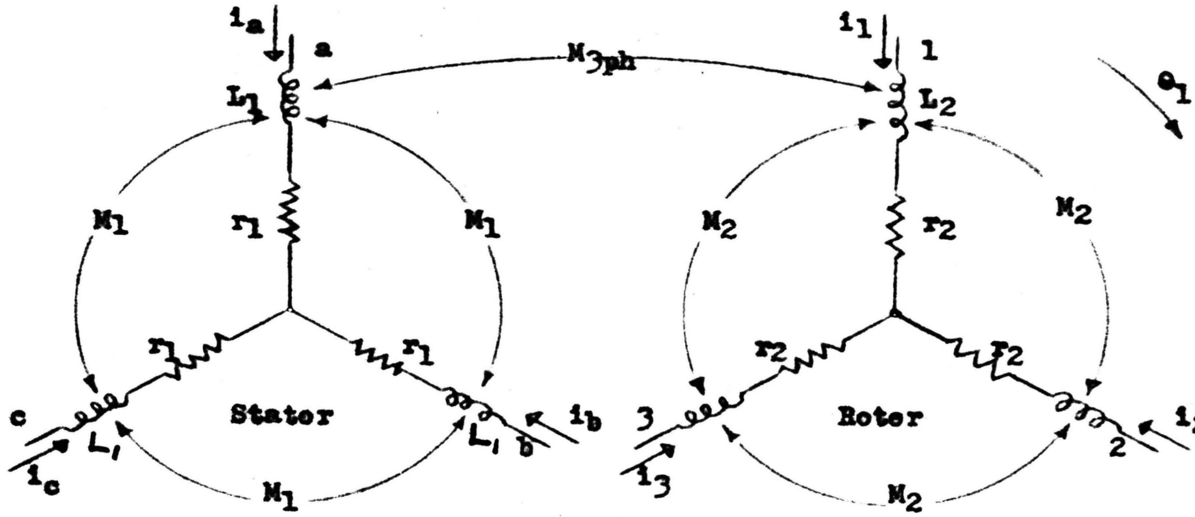



FIGURE 4 - Induction Motor Circuits.

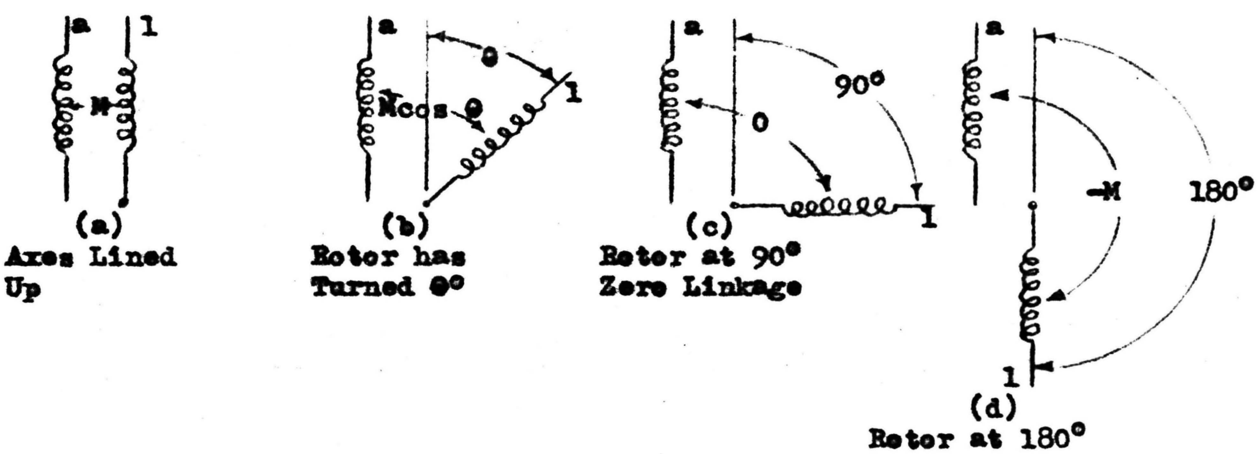



FIGURE 5 - Variation of M with Rotor Angle.

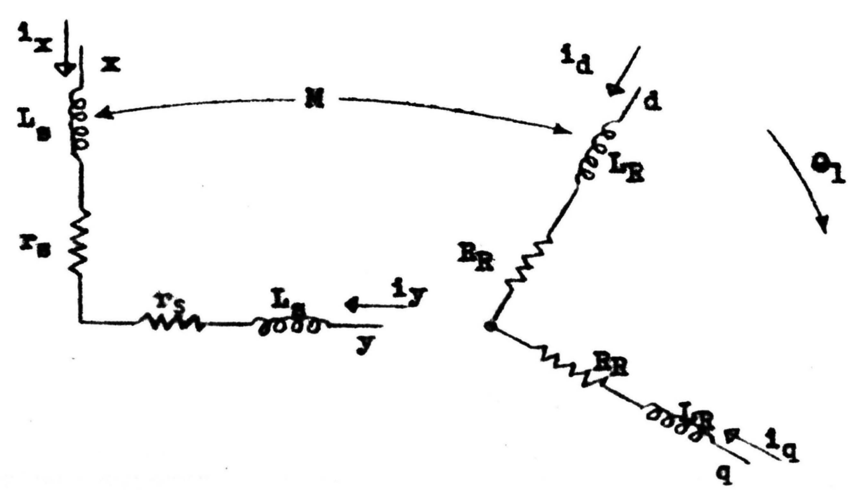
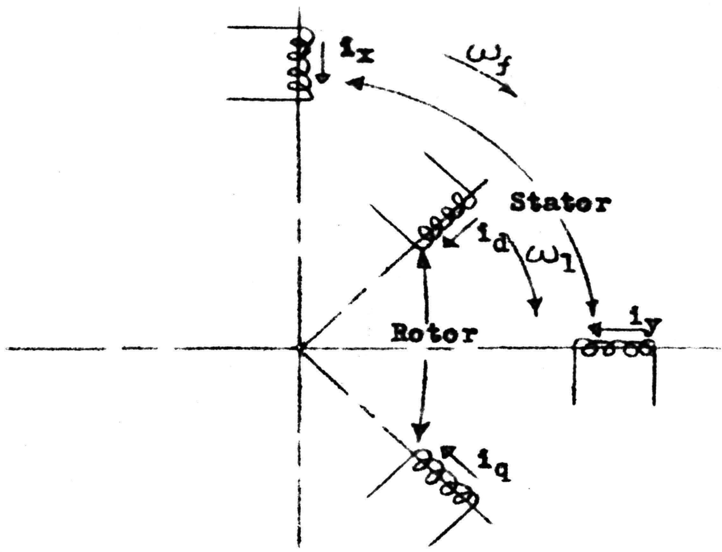
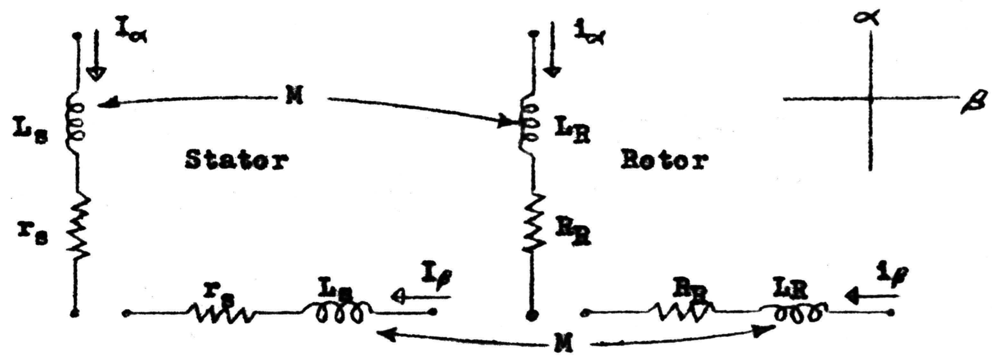



FIGURE 6 - Equivalent Two Phase Machine.

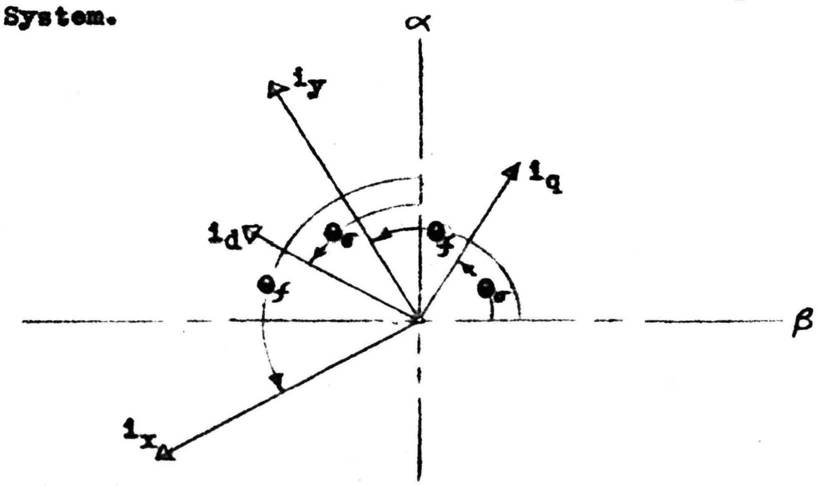
$\omega_f - \omega_1 = \omega_s$ or slip speed. By the converse of the relation which produces the rotating stator vectors, the rotating vectors set up slip frequency pulsating fluxes, currents and voltages in the rotor phases. These quantities in turn may be resolved into constant amplitude vectors, rotating clockwise at slip speed with reference to the rotor, or at synchronous speed relative to a fixed reference frame. Thus it is seen that both the stator and rotor quantities resolve into constant magnitude synchronously rotating vectors. Then if the system is viewed from a reference frame which is also rotating clockwise at synchronous speed, it reduces to a set of stationary constant magnitude vectors. If a set of right angle coordinate axes, $\alpha$ and $\beta$ is set up on this synchronously rotating reference frame, each vector can be broken up into components along these axes. There will then be in the $\alpha$ axis, for instance, constant components of the stator vectors, $I_\alpha$, $\Psi_\alpha$, and $E_\alpha$, respectively. A little consideration will show that three identical quantities would be present in a hypothetical coil oriented in the $\alpha$ axis with voltage $E_\alpha$ applied, current $I_\alpha$ flowing, and flux linkages $\Psi_\alpha$ present due to $I_\alpha L_s$ and to currents in any mutually coupled coil. Identical reasoning can be used for stator components in the $\beta$ axis, $I_\beta$, $\Psi_\beta$, and $E_\beta$; and for corresponding $\alpha$ and $\beta$ axis rotor components, $i_\alpha$, $\psi_\alpha$, $e_\alpha$, and $i_\beta$, $\psi_\beta$, $e_\beta$. Thus the actual two phase machine can be replaced, using the synchronously rotating $\alpha-\beta$ axes, by four stationary coils as shown in Fig. 7-b. In the steady state these coils will carry direct current and be linked by steady fluxes. The linkage relations are immediately very much simplified, in each case reducing to a self and mutual direct current component. (See equations 21). As will be seen when the specific transformation relations are introduced and the solution carried out, all unwieldy trigonometric relations are ultimately




(a)
Actual Two Phase Machine, Showing Rotor and Stator Coils, and Rotation of Rotor and of Stator Flux.




(b)
Equivalent Circuit of Induction Machine in the Rotating Coordinate System.




(c)
Instantaneous Diagram, Showing Rotor and Stator Vectors and their Rotational Angles Relative to the $\alpha$-$\beta$ Axes

eliminated.

The preceding has been a more or less qualitative discussion of the $\alpha - \beta$ axis transformation. It will now be necessary to develop specific relations between the two phase quantities and the $\alpha - \beta$ quantities. These can be brought out by referring to Fig. 7-c, which is an instantaneous picture of the stator and rotor current vectors in reference to the synchronously rotating $\alpha - \beta$ axes. Each current vector is actually aligned with the physical axis of the coil indicated by its subscript. Thus vectors $i_x$ and $i_y$ rotate counterclockwise at synchronous speed $\omega_f$ ; vectors $i_d$ and $i_q$ rotate counterclockwise at slip speed $\omega_\sigma$ . $\theta_f$ and $\theta_\sigma$ are defined respectively as the instantaneous angles between $i_x$ and the $\alpha$ axis and $i_d$ and the $\alpha$ axis. The stator current $I_\alpha$ is then the projection of $i_x$ and $i_y$ on the $\alpha$ axis; $I_\beta$ is the projection of $i_x$ and $i_y$ on the $\beta$ axis, so that

$$I_\alpha = i_x \cos\theta_f + i_y \sin\theta_f$$

(19-a)

$$I_\beta = i_x \sin\theta_f + i_y \cos\theta_f$$

Also, by exact analogy

$$E_\alpha = e_x \cos\theta_f + e_y \sin\theta_f$$

$$E_\beta = -e_x \sin\theta_f + e_y \cos\theta_f$$

(19-b)

$$\Psi_\alpha = \Psi_x \cos\theta_f + \Psi_y \sin\theta_f$$

$$\Psi_\beta = -\Psi_x \sin\theta_f + \Psi_y \cos\theta_f$$

Similarly, the rotor current $i_\alpha$ is the projection of $i_d$ and $i_q$ on the $\alpha$ axis and $i_\beta$ the projection of $i_d$ and $i_q$ on the $\beta$ axis. Then

$$i_\alpha = i_d \cos\theta_\sigma + i_q \sin\theta_\sigma$$

(20-a)

$$i_\beta = -i_d \sin\theta_\sigma + i_q \cos\theta_\sigma$$

and, as for the stator,

$$e_\alpha = e_d \cos\theta_\sigma + e_q \sin\theta_\sigma$$

$$e_\beta = -e_d \sin\theta_\sigma + e_q \cos\theta_\sigma$$

(20-b)

$$\psi_\alpha = \psi_d \cos\theta_\sigma + \psi_q \sin\theta_\sigma$$

$$\psi_\beta = -\psi_d \sin\theta_\sigma + \psi_q \cos\theta_\sigma$$

As was stated earlier, very simple flux linkage relations exist in this reference system. Referring to Fig. 7-b, and remembering that there can be no linkage between coils oriented 90° apart it can be written that

$$\Psi_\alpha = I_\alpha L_S + M i_\alpha$$

$$\Psi_\beta = I_\beta L_S + M i_\beta$$

(21)

$$\psi_\alpha = i_\alpha L_R + M I_\alpha$$

$$\psi_\beta = i_\beta L_R + M I_\beta$$

As the first step in solving relations (19), (20) and (21), take the time derivative of $\psi_\alpha$ :

$$\frac{d\psi_\alpha}{dt} = -\psi_d \sin\theta_\sigma \frac{d\theta_\sigma}{dt} + \frac{d\psi_d}{dt}\cos\theta_\sigma + \psi_q \cos\theta_\sigma \frac{d\theta_\sigma}{dt} + \frac{d\psi_q}{dt}\sin\theta_\sigma$$

$$= \frac{d\theta_\sigma}{dt}\left(-\psi_d \sin\theta_\sigma + \psi_q \cos\theta_\sigma\right) + \frac{d\psi_d}{dt}\cos\theta_\sigma + \frac{d\psi_q}{dt}\sin\theta_\sigma$$

But from equation (18)

$$(18)\quad e_d = \frac{d\psi_d}{dt} + R_R i_d$$
$$e_q = \frac{d\psi_q}{dt} + R_R i_q$$

so that, since $\psi_\beta = -\psi_d \sin\theta_\sigma + \psi_q \cos\theta_\sigma$

$$\frac{d\psi_\alpha}{dt} = \psi_\beta \frac{d\theta_\sigma}{dt} + (e_d - R_R i_d)\cos\theta_\sigma + (e_q - R_R i_q)\sin\theta_\sigma$$

$$= \psi_\beta \frac{d\theta_\sigma}{dt} + (e_d \cos\theta_\sigma + e_q \sin\theta_\sigma) - R_R(i_d \cos\theta_\sigma + i_q \sin\theta_\sigma)$$

$$= \psi_\beta \frac{d\theta_\sigma}{dt} + e_\alpha - R_R i_\alpha$$

or, solving for

$$(22\text{-a})\quad e_\alpha = \frac{d\psi_\alpha}{dt} - \psi_\beta \frac{d\theta_\sigma}{dt} + R_R i_\alpha$$

By an identical process, it can be shown that

$$e_\beta = \frac{d\psi_\beta}{dt} + \psi_\alpha \frac{d\theta_\sigma}{dt} + R_R i_\beta$$

$$(22\text{-b})\quad E_\alpha = \frac{d\Psi_\alpha}{dt} - \Psi_\beta \frac{d\theta_f}{dt} + r_s I_\alpha$$

$$E_\beta = \frac{d\Psi_\beta}{dt} + \Psi_\alpha \frac{d\theta_f}{dt} + r_s I_\beta$$

To get equations (22) into terms of currents and voltages only, substitute equations (21) into them:

$$e_\alpha = \frac{d}{dt}(L_R i_\alpha + M I_\alpha) - (L_R i_\beta + M I_\beta)\frac{d\theta_\sigma}{dt} + R_R i_\alpha$$

$$e_\beta = \frac{d}{dt}(L_R i_\beta + M I_\beta) + (L_R i_\alpha + M I_\alpha)\frac{d\theta_\sigma}{dt} + R_R i_\beta$$

(23)

$$E_\alpha = \frac{d}{dt}(L_s I_\alpha + M i_\alpha) - (L_s I_\beta + M i_\beta)\frac{d\theta_s}{dt} + r_s I_\alpha$$

$$E_\beta = \frac{d}{dt}(L_s I_\beta + M i_\beta) + (L_s I_\alpha + M i_\alpha)\frac{d\theta_s}{dt} + r_s I_\beta$$

One more equation can be written relating to the performance of the machine, namely, the electrical torque equation. Maxwell's equation for mechanical force between electric circuits can be stated for two circuits as

(24) $$T = \frac{1}{2} I_1^2 \frac{dL_1}{d\theta} + \frac{1}{2} I_2^2 \frac{dL_2}{d\theta} + I_1 I_2 \frac{dM_{12}}{d\theta}$$

Since the circuits are rigid, and since the rotor and stator have been assumed to be smooth, the self inductances, $L_1$ and $L_2$, are not functions of the angular position. Therefore $\frac{dL_1}{d\theta} = \frac{dL_2}{d\theta} = 0$ and

(25) $$T = I_1 I_2 \frac{dM_{12}}{d\theta}$$

or, expressed in terms of the actual machine quantities

(26) $$T_1 = \sum I_R I_S \frac{dM}{d\theta}$$

It has been shown (see Fig. 5) that M is a cosine function of $\theta$.

Therefore $\frac{dM}{d\theta}$ is a sine function of $\theta$, or

(27) $$T_1 = \sum I_R I_S M(-\sin\theta) \angle_{I_S}^{I_R}$$

Then from Fig. 6, and equation (27)

$$T_1 = M \left\{ \begin{array}{l} -i_x i_d \sin\theta_1 - i_x i_q \sin(\theta_1 + 90^\circ) \\ \qquad - i_y i_d \sin(\theta_1 - 90^\circ) - i_y i_q \sin\theta_1 \end{array} \right\}$$

$$= M\left\{-i_x i_d \sin\theta_1 - i_x i_q \cos\theta_1 + i_y i_d \cos\theta_1 - i_y i_q \sin\theta_1\right\}$$

$$= M \left\{ \begin{array}{l} -i_x i_d \sin(\theta_f - \theta_\sigma) - i_x i_q \cos(\theta_f - \theta_\sigma) \\ \qquad + i_y i_d \cos(\theta_f - \theta_\sigma) - i_y i_q \sin(\theta_f - \theta_\sigma) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} M \cos(\theta_f - \theta_\sigma)\left[-i_x i_q + i_y i_d\right] \\ \qquad + M \sin(\theta_f - \theta_\sigma)\left[-i_x i_d - i_y i_q\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{l} M(\cos\theta_f - \cos\theta_\sigma + \sin\theta_f \sin\theta_\sigma)\left[-i_x i_q + i_y i_d\right] \\ \qquad + M(\sin\theta_f \cos\theta_\sigma - \cos\theta_f \sin\theta_\sigma)\left[-i_x i_d - i_y i_q\right] \end{array} \right\}$$

$$= M \left\{ \begin{array}{l} -i_x \sin\theta_f (i_d \cos\theta_\sigma + i_q \sin\theta_\sigma) \\ \qquad - i_x \cos\theta_f (-i_d \sin\theta_\sigma + i_q \cos\theta_\sigma) \end{array} \right\}$$

$$+ M \left\{ \begin{array}{l} -i_y \sin\theta_f (i_d \sin\theta_\sigma + i_q \cos\theta_\sigma) \\ \qquad + i_y \cos\theta_f (i_d \cos\theta_\sigma + i_q \sin\theta_\sigma) \end{array} \right\}$$

$$= M \left\{ \begin{array}{l} (i_d \cos\theta_\sigma + i_q \sin\theta_\sigma)(-i_x \sin\theta_f + i_y \cos\theta_f) \\ \qquad - (-i_d \sin\theta_\sigma + i_q \cos\theta_\sigma)(i_x \cos\theta_f + i_y \sin\theta_f) \end{array} \right\}$$

and

$$(28) \quad T_1 = M\,(I_\beta i_\alpha - I_\alpha i_\beta)$$

from equations (19-a) and (20-a). This relation could have been written directly, using equation (27) and referring to Fig. 7-(b).

Inspection of equations (23) and (28) shows that each equation contains terms involving the product of two time functions, i.e., $i_\beta(t)\,\theta_\sigma(t)$, $I_\beta(t)\,i_\alpha(t)$, etc. In general it is not practical to attempt to take the Laplace transform of such products. Unless both time functions are known and their product is directly transformable, the transform of a product is a complex convolution integral. This is an implicit transform, and the integration may be difficult or impossible to carry out. However, the input and output disturbances encountered in control systems, as opposed to those in servomechanisms, are usually small. Therefore it will be permissible to carry out the analysis on the basis of incremental variations of currents, voltages, and speeds. Then each of the variables in equations (23) and (28) can be written as a steady state plus an incremental term. Accordingly

$$e_{\alpha o}+\Delta e_{\alpha}=\frac{d}{dt}\left[L_R(i_{\alpha o}+\Delta i_{\alpha})+M(I_{\alpha o}+\Delta I_{\alpha})\right]$$

$$-\left[L_R(i_{\beta o}+\Delta i_{\beta})+M(I_{\beta o}+\Delta I_{\beta})\right]\frac{d}{dt}(\theta_{\sigma o}+\Delta\theta_{\sigma})$$

$$+R_R(i_{\alpha o}+\Delta i_{\alpha})$$

$$=\left[\frac{d}{dt}(L_R i_{\alpha o}+MI_{\alpha o})-(L_R i_{\beta o}+MI_{\beta o})\frac{d\theta_{\sigma o}}{dt}+R_R i_{\alpha o}\right]$$

$$+\left[\frac{d}{dt}(L_R\Delta i_{\alpha}+M\Delta I_{\alpha})-\frac{d\theta_{\sigma o}}{dt}(L_R\Delta i_{\beta}+M\Delta I_{\beta})-\frac{d\Delta\theta_{\sigma}}{dt}(L_R i_{\beta o}+MI_{\beta o})+R_R\Delta i_{\alpha}\right]$$

$$+\left[\frac{d\Delta\theta_{\sigma}}{dt}(L_R\Delta i_{\beta}+M\Delta I_{\beta})\right]$$

The steady state terms can be cancelled directly. Then, since increments are very small changes, the products of increments become negligibly small, and may be dropped. The first term of the above equation is the steady state portion, and the third is the product of two incremental quantities. Therefore

$$\Delta e_{\alpha}=\frac{d}{dt}(L_R\Delta i_{\alpha}+M\Delta I_{\alpha})-\frac{d\theta_{\sigma o}}{dt}(L_R\Delta i_{\beta}+M\Delta I_{\beta})-\frac{d\Delta\theta_{\sigma}}{dt}(L_R i_{\beta o}+MI_{\beta o})+R_R\Delta i_{\alpha} \quad \text{(29-a)}$$

By the same process

$$\Delta e_\beta = \frac{d}{dt}(L_R\Delta + M\Delta I_\beta) + \frac{d\theta_{\sigma o}}{dt}(L_R\Delta i_\alpha + M\Delta I_\alpha)$$

$$+ \frac{d\Delta\theta_\sigma}{dt}(L_R i_{\alpha o} + MI_{\alpha o}) + R_R\Delta i_\beta$$

(29-b)

$$\Delta E_\alpha = \frac{d}{dt}(L_s\Delta I_\alpha + M\Delta i_\alpha) - \frac{d\theta_{fo}}{dt}(L_s\Delta I_\beta + M\Delta i_\beta)$$

$$- \frac{d\Delta\theta_f}{dt}(L_s I_{\beta o} + M i_{\beta o}) + r_s\Delta I_\alpha$$

$$\Delta E_\beta = \frac{d}{dt}(L_s\Delta I_\beta + M\Delta i_\beta) + \frac{d\theta_{fo}}{dt}(L_s\Delta I_\alpha + M\Delta i_\alpha)$$

$$- \frac{d\Delta\theta_f}{dt}(L_s I_{\alpha o} + M i_{\alpha o}) + r_s\Delta I_\beta$$

But $\frac{d\theta_{\sigma o}}{dt} = \omega_{\sigma o}$ , and $\frac{d\Delta\theta_\sigma}{dt} = \Delta\omega_\sigma$ . Furthermore, since the line voltage and frequency are constant, $\frac{d\theta_{fo}}{dt} = \omega_f$ , and $\Delta E_\alpha = \Delta E_\beta = \frac{d\Delta\theta_f}{dt} = 0$ . Also

(30)

$$\omega_\sigma = \omega_f - \omega_1$$

$$\omega_{\sigma o} + \Delta\omega_\sigma = \omega_f - \omega_{1o} - \Delta\omega_1$$

and

$$\Delta\omega_\sigma = -\Delta\omega_1$$

which is to say, the variation of slip speed is equal in magnitude and opposite in sign to the variation of rotational speed. Also, from equation (21)

$$L_R i_{\alpha o} + M I_{\alpha o} = \Psi_{\alpha o}$$

(31)

$$L_R i_{\beta o} + M I_{\beta o} = \Psi_{\beta o}$$

Then equations (29) reduce to

$$\Delta e_x = \frac{d}{dt}(L_R \Delta i_\alpha + M \Delta I_\alpha) - \omega_{\sigma o}(L_R \Delta i_\beta + M \Delta I_\beta)$$

$$+ \Psi_{\beta o} \Delta \omega_1 + R_R \Delta i_\alpha$$

$$\Delta e_\beta = \frac{d}{dt}(L_R \Delta i_\beta + M \Delta I_\beta) + \omega_{\sigma o}(L_R \Delta i_\alpha + M \Delta I_\beta)$$

(32)

$$- \Psi_{\alpha o} \Delta \omega_1 + R_R \Delta i_\beta$$

$$0 = \frac{d}{dt}(L_S \Delta I_\alpha + M \Delta i_\alpha) - \omega_f (L_S \Delta I_\beta + M \Delta i_\beta) + r_s \Delta I_\alpha$$

$$0 = \frac{d}{dt}(L_S \Delta I_\beta + M \Delta i_\beta) + \omega_f (L_S \Delta I_\alpha + M \Delta i_\beta) + r_s \Delta I_\beta$$

Inspection of equations (29) and (32) will now show that time function products are no longer present, since as was shown in the discussion of transformation to the $\alpha - \beta$ axis system, steady state currents, voltages, and fluxes are constants in this system. These equations may now be Laplace transformed very simply. The notation (S) to indicate that the variables are now functions of S will be omitted for convenience in writing. Then

(33) $$\Delta e_\alpha = (L_R S + R_R)\Delta i_\alpha + M S \Delta I_\alpha - \omega_{\sigma o}(L_R \Delta i_\beta + M \Delta I_\beta) + \Psi_{\beta o} \Delta \omega_1$$

(34) $$\Delta e_\beta = (L_R S + R_R)\Delta i_\beta + M_S \Delta I_\beta + \omega_{\sigma o}(L_R \Delta i_\alpha + M \Delta I_\alpha) + \Psi_{\alpha o} \Delta \omega_1$$

$$(35) \quad 0 = (L_s s + r_s)\Delta I_\alpha + Ms\Delta i_\alpha - \omega_f(L_s \Delta I_\beta + M\Delta i_\beta)$$

$$(36) \quad 0 = (L_s s + r_s)\Delta I_\beta + Ms\Delta i_\beta + \omega_f(L_s \Delta I_\alpha + M\Delta i_\alpha)$$

The torque equation (28) can be handled in the same manner. Thus

$$T_{10} + \Delta T_1 = M[(I_{\beta o} + \Delta I_\beta)(i_{\alpha o} + \Delta i_\alpha) - (I_{\alpha o} + \Delta I_\alpha)(i_{\beta o} + \Delta i_\beta)]$$

$$= M(I_{\beta o} i_{\alpha o} - I_{\alpha o} i_{\beta o}) + M(I_{\beta o}\Delta i_\alpha + i_{\alpha o}\Delta I_\beta)$$

$$+ M(I_{\alpha o}\Delta i_\beta - i_{\beta o}\Delta I_\alpha) + M(\Delta I_\beta \Delta i_\alpha - \Delta I_\alpha \Delta i_\beta)$$

from which

$$(37) \quad \Delta T_1 = M(I_{\beta o}\Delta i_\alpha + i_{\alpha o}\Delta i_\beta - I_{\alpha o}\Delta i_\beta - i_{\beta o}\Delta I_\alpha)$$

The transformed equation has the same form as (37), therefore it will not be necessary to rewrite it.

Since the rotor of the induction machine feeds the stator of the synchronous machine $S_1$, the equations of the two machines must be matched through the common parameters, $e_\alpha$, $e_\beta$, $i_\alpha$, $i_\beta$. Therefore $\Delta I_\alpha$ and $\Delta I_\beta$ are of no interest, and may be eliminated from equations (33) through (37). Accordingly, regroup equations (35) and (36)

$$(35) \quad (L_s s + r_s)\Delta I_\alpha - \omega_f L_s \Delta I_\beta = \omega_f M \Delta i_\beta - Ms\Delta i_\alpha$$

$$(36) \quad \omega_f L_s \Delta I_\alpha + (L_s s + r_s)\Delta I_\beta = \omega_f M\Delta i_\alpha - Ms\Delta i_\beta$$

Then, by determinants

$$(38)\quad \Delta I_\alpha = \frac{\begin{vmatrix} M(\omega_f \Delta i_\beta - s\Delta i_\alpha) & -\omega_f L_s \\ -M(\omega_f \Delta i_\alpha + s\Delta i_\beta) & (L_s s + r_s) \end{vmatrix}}{\begin{vmatrix} (L_s s + r_s) & -\omega_f L_s \\ \omega_f L_s & (L_s s + r_s) \end{vmatrix}}$$

$$= \frac{M(\omega_f \Delta i_\beta - s\Delta i_\alpha)(L_s s + r_s) - \omega_f L_s M(\omega_f \Delta i_\beta - s\Delta i_\alpha)}{(L_s s + r_s)^2 + \omega_f^2 L_s^2}$$

$$= \frac{(X_{Mf} Z_s - X_{sf} Ms)\Delta i_\beta - (X_{sf} X_{Mf} + M Z_s s)\Delta i_\alpha}{\delta_1}$$

where

$$Z_s \triangleq L_s s + r_s$$

$X_{sf} \triangleq \omega_f L_s =$ 60-cycle stator self reactance.

$X_{Mf} \triangleq \omega_f M =$ 60-cycle rotor-stator mutual reactance.

$$\delta_1 \triangleq Z_s^2 + X_{sf}^2$$

Similarly

$$(39)\quad \Delta I_\beta = \frac{-(X_{sf} X_{Mf} + M Z_s s)\Delta i_\beta - (X_{Mf} Z_s - X_{sf} Ms)\Delta i_\alpha}{\delta_1}$$

The above values for $\Delta I_\alpha$ and $\Delta I_\beta$ may be substituted in equations (33), (34) and (37), resulting in three equations involving rotor currents, voltages, speed, and electrical torque. After some regrouping, those equations are given below.

$$\Delta e_\alpha = \left\{ Z_R + \frac{X_{M60}(X_{Mf} Z_s - X_{sf} Ms) - Ms(X_{sf} X_{Mf} + M Z_s s)}{\delta_1} \right\} \Delta i_\alpha +$$

(40)
$$+\left\{\frac{Ms\left(X_{Mf}Z_s - X_{sf}Ms\right) + X_{M\sigma o}\left(X_{sf}X_{Mf} + MZ_s s\right)}{\delta_1} - X_{R\sigma o}\right\}\Delta i_\beta + \psi_{\beta o}$$

$$\Delta e_\beta = -\left\{\frac{Ms\left(X_{Mf}Z_s - X_{sf}Ms\right) + X_{M\sigma o}\left(X_{sf}X_{Mf} + MZ_s s\right)}{\delta_1} - X_{R\sigma o}\right\}\Delta i_\alpha$$

(41)
$$+\left\{Z_R + \frac{X_{M\sigma o}\left(X_{Mf}Z_s - X_{sf}Ms\right) - Ms\left(X_{sf}X_{Mf} + MZ_s s\right)}{\delta_1}\right\}\Delta i_\beta - \psi_{\alpha o}\Delta\omega_1$$

where

$X_{R\sigma o} \triangleq \omega_{\sigma o} L_R$ = steady-state slip frequency rotor self reactance.

$X_{M\sigma o} \triangleq \omega_{\sigma o} M$ = steady-state slip frequency rotor-stator mutual reactance.

(42)
$$\Delta T_1 = M\left\{I_{\beta o} + \frac{i_{\beta o}\left(X_{sf}X_{Mf} + MZ_s s\right) - i_{\alpha o}\left(X_{Mf}Z_s - X_{sf}Ms\right)}{\delta_1}\right\}\Delta i_\alpha -$$

$$-M\left\{I_{\alpha o}+\frac{i_{\alpha o}(X_{sf}X_{Mf}+MZ_s s)+i_{\beta o}(X_{Mf}Z_s-X_{sf}Ms)}{\delta_1}\right\}\Delta i_\beta$$

In order to simplify subsequent manipulations, equations (40), (41), and (42) are now rewritten as follows:

(43) $\Delta e_\alpha = g_1(s)\Delta i_\alpha + g_2(s)\Delta i_\beta + \Psi_{\beta o}\Delta\omega_1$

(44) $\Delta e_\beta = -g_2(s)\Delta i_\alpha + g_1(s)\Delta i_\beta - \Psi_{\alpha o}\Delta\omega_1$

(45) $\Delta T_1 = g_3(s)\Delta i_\alpha - g_4(s)\Delta i_\beta$

where $g_1$ (S), $g_2$ (S), $g_3$ (S), $g_4$ (S) are defined by the brackets in equations (40), (41), and (42).

By reference to Fig. 3, it is seen that $T_1$ is the electrical torque of machine M. However, since disturbances must originate at the load, $T_L$ is the variable of chief interest. It can then be written that

(46) $T_1 = B_1\frac{d\theta_1}{dt} + J_1\frac{d^2\theta_1}{dt^2} + T_L = B_1\omega_1 + J_1\frac{d\omega_1}{dt} + T_L$

Again taking increments

(47) $\Delta T_1 = B_1\Delta\omega_1 + J_1\frac{d\Delta\omega_1}{dt} + \Delta T_L$

Transformed, equation (47) becomes

$$\Delta T_1 = B_1\Delta\omega_1 + J_1 s\Delta\omega_1 + \Delta T_L = (J_1 s + B_1)\Delta\omega_1 + \Delta T_L$$

(48)

$$= Z_1\Delta\omega_1 + \Delta T_L$$

where

$$Z \triangleq J_1 s + B_1 = \text{mechanical impedance of machine M.}$$

Substituting equation (45) into (48) and rearranging, there results

(49) $\Delta T_L = g_3(s)\Delta i_\alpha - g_4(s)\Delta i_\beta - Z_1\Delta\omega_1$

Equations (43), (44), and (49) completely describe the performance of the main machine, M, in terms of load torque and rotor quantities. These equations are perfectly general for small variations of input and output, and can be applied to an ordinary induction motor tied to a large source simply by making $\Delta e_{\alpha}$ and $\Delta e_{\beta}$ equal to zero in equations (43) and (44). If the source cannot be considered infinite, $\Delta E_{\alpha}$ and $\Delta E_{\beta}$ cannot be considered equal to zero and slightly different equations will result.

Synchronous Machine, $S_1$

The assumptions made for the induction machine will again be made. In addition, it will be assumed that all rotor circuits in addition to the main field winding can be adequately represented by two short-circuited windings, one in the direct, or main-field axis, the other in the quadrature axis. The ordinary synchronous machine may have, in addition to the main field winding, an amortisseur winding, a field collar, and circuits through the spider iron. In general, the amortisseur winding is a continuous ladder-like circuit completely encircling the rotor. Each bar forms a loop with every other bar, and all loops are mutually linked with each other and with all other rotor and stator circuits. The result is an extremely complex group of short circuited windings, each of which under transient conditions contributes a decaying exponential term to the disturbance effect. Exact analysis of such a circuit would be almost impossibly complicated. A notion of the complexity of the circuits encountered can be gotten by reference to a paper by Linville [12], in

(12) Linville, T. M., Starting Performance of Salient-Pole Synchronous Motors, AIEE Transactions, Vol. 49, 1930, pp. 531-547.

which some relatively exact rotor equivalent circuits are developed. Fig. 8 shows the decrement curve of a simple RL circuit superimposed on a typical decrement curve of an actual machine. The two curves actually coincide at only three times, 0, T, and $\infty$. However, by proper selection of R and L the approximation can be made fair enough to make the mathematical simplification thus achieved more than worthwhile.

Making the assumption just discussed, the three-phase equivalent circuit of the synchronous machine can be represented as in Fig. 9. Impedances, currents, etc., used in writing the initial equations will be as indicated in the figure. $M_{sf}$, $M_{sd}$, and $M_{sq}$ are the maximum values of mutual inductance between the stator and the main field, the short-circuited direct axis circuit, and the short-circuited quadrature axis circuit, respectively. Rotation is clockwise and at a speed $\omega_2$ which in the steady state is equal to the $\omega_{\sigma o}$ of induction machine M since the synchronous machine is fed by the induction machine rotor.

Then, in a manner exactly similar to that for the induction machine, the following equations may be written:

$$\Psi_{s1} = L_{s3} i_1 + M_{s3}(i_2 + i_3) + I_f M_{sf3} \cos\theta_2 + I_d M_{sd3} \cos\theta_2 - I_q M_{sq3} \sin\theta_2$$

$$\Psi_{s2} = L_{s3} i_2 + M_{s3}(i_1 + i_3) + I_f M_{sf3} \cos(\theta_2 - 120) + I_d M_{sd3} \cos(\theta_2 - 120) - I_q M_{sq3} \sin(\theta_2 - 120)$$

(50) $$\Psi_{s3} = L_{s3} i_3 + M_{s3}(i_1 + i_2) + I_f M_{sf3} \cos(\theta_2 + 120) + I_d M_{sd3} \cos(\theta_2 + 120) - I_q M_{sq3} \sin(\theta_2 + 120)$$

$$\Psi_f = L_f I_f + M_{fd} I_d + i_1 M_{af3} \cos\theta_2$$
$$+ i_2 M_{af3} \cos(\theta_2 - 120) + i_3 M_{af3} \cos(\theta_2 + 120)$$

$$\Psi_d = L_d I_d + M_{fd} I_f + i_1 M_{ad3} \cos\theta_2$$
$$+ i_2 M_{ad3} \cos(\theta_2 - 120) + i_3 M_{ad3} \cos(\theta_2 + 120)$$

$$\Psi_q = L_q I_q - i_1 M_{aq3} \sin\theta_2 - i_2 M_{aq3} \sin(\theta_2 - 120)$$
$$- i_3 M_{aq3} \sin(\theta_2 + 120)$$

where $\theta_2$ is the angle between phase a and the rotor direct axis. The same two simplifying transformations will again be made. Accordingly the stator is first replaced by its equivalent two phase stator, resulting in the circuit of Fig. 10, in which

(51)
$$i_d = \frac{2i_1 - i_2 - i_3}{3} = i_1$$
$$i_q = \frac{i_2 - i_3}{\sqrt{3}} = -j i_1$$

(52)
$$\psi_{ad} = \frac{2\psi_{a1} - \psi_{a2} - \psi_{a3}}{3} = \psi_{a1}$$
$$\psi_{aq} = \frac{\psi_{a2} - \psi_{a3}}{\sqrt{3}} = -j\psi_{a1}$$

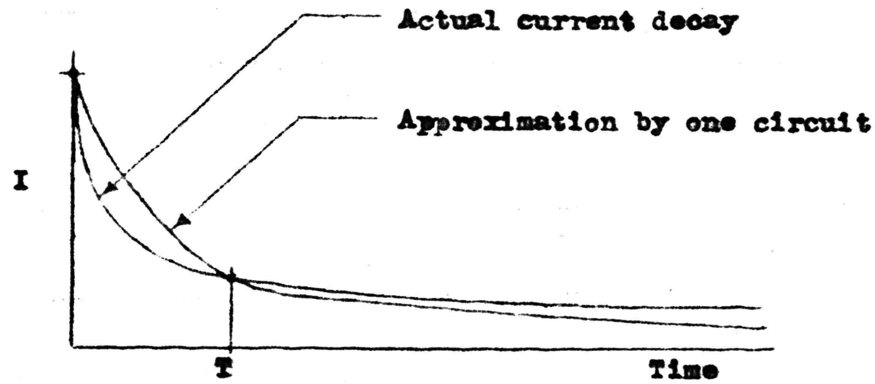



FIGURE 8 - Comparison of Transient Current Curve for One Rotor Circuit with the Curve of a Possible Actual Circuit.

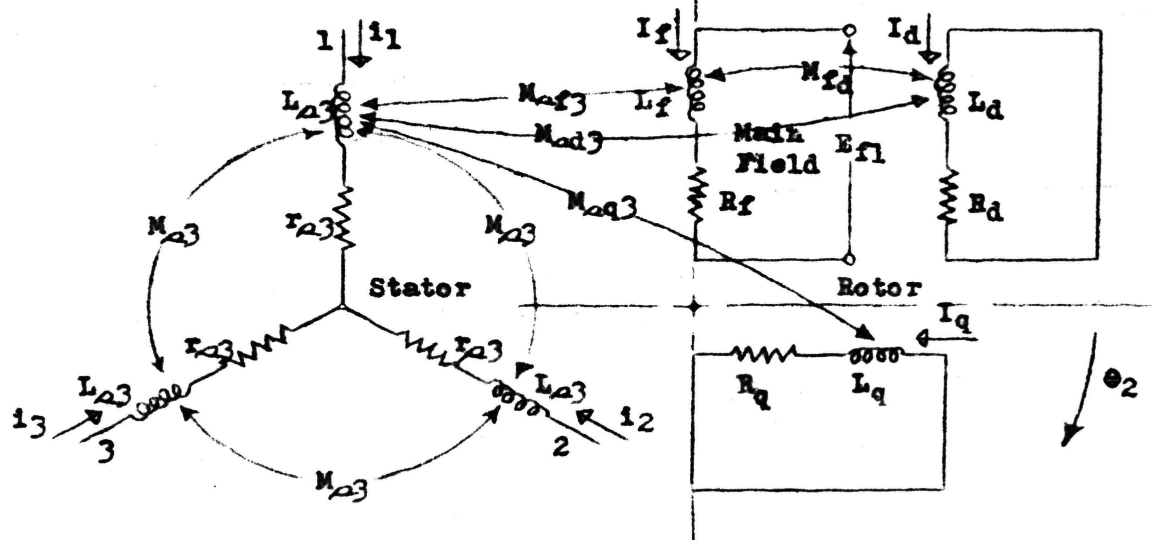



FIGURE 9 - Synchronous Machine Circuits.

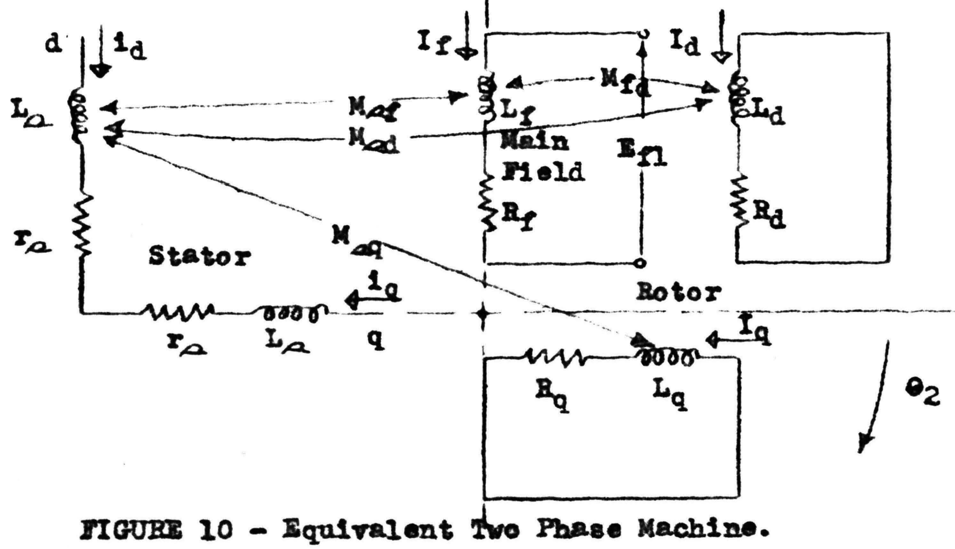



FIGURE 10 - Equivalent Two Phase Machine.

Then the linkage equations become

$$\Psi_{ad} = L_a i_d + I_f M_{af} \cos\theta_2 + I_d M_{ad} \cos\theta_2 - I_q M_{aq} \sin\theta_2$$

$$\Psi_{aq} = L_a i_q + I_f M_{af} \sin\theta_2 + I_d M_{ad} \sin\theta_2 + I_q M_{aq} \cos\theta_2$$

$$(53)\quad \Psi_f = L_f I_f + I_d M_{fd} + i_d M_{af} \cos\theta_2 + i_q M_{af} \sin\theta_2$$

$$\Psi_d = L_d I_d + I_f M_{fd} + i_d M_{ad} \cos\theta_2 + i_q M_{ad} \sin\theta_2$$

$$\Psi_q = L_q I_q - i_d M_{aq} \sin\theta_2 + i_q M_{aq} \cos\theta_2$$

As was the case for the induction machine, the values of the inductances must be altered in order to make the two phase machine exactly equivalent. From equations (52), $\Psi_{ad} = \Psi_{a1}$ . Therefore substituting the proper values from equations (50) and (53) for $\Psi_{ad}$ and $\Psi_{a1}$ , there is, after cancellations

$$L_a i_d = L_{a3} i_1 + M_{a3}(i_2 + i_3)$$

But from equations (51)

$$i_1 = i_d$$

$$i_2 + i_3 = -i_d$$

then

$$L_a i_d = L_{a3} i_d - M_{a3} i_d$$

and

$$(54)\quad L_a = L_{a3} - M_{a3}$$

Since the field circuits are unchanged, it follows that $\Psi_f$ , $\Psi_d$ , and $\Psi_q$ must remain unchanged. Therfore again equating the appropriate terms from equations (50) and (53), there results

$$M_{af}(i_d \cos\theta_2 + i_q \sin\theta_2)$$
$$= M_{af3}\left[i_1 \cos\theta_2 + i_2 \cos(\theta_2 - 120) + i_3 \cos(\theta_2 + 120)\right]$$
$$= \frac{3}{2} M_{af3}\left[i_1 \cos\theta_2 + \left(\frac{i_2 - i_3}{\sqrt{3}}\right) \sin\theta_2\right]$$
$$= \frac{3}{2} M_{af3}(i_d \cos\theta_2 + i_q \sin\theta_2)$$

from equations (51). Thus

(55) $M_{af} = \frac{3}{2} M_{af3}$

and since the $\psi_d$ equations are identical in form

(56) $M_{ad} = \frac{3}{2} M_{ad3}$

In a similar manner

$$-i_d M_{aq} \sin\theta_2 + i_q M_{aq} \cos\theta_2$$
$$= -M_{aq3}\left[i_1 \sin\theta_2 + i_2 \sin(\theta_2 - 120) + i_3 \sin(\theta_2 + 120)\right]$$

The right hand side of the above equation can be shown to be equal to

$$-M_{aq3}\left\{\left[i_1 - \frac{(i_2 + i_3)}{2}\right] \sin\theta_2 - \left[\frac{\sqrt{3}}{2}(i_2 - i_3)\right] \cos\theta_2\right\}$$

which, again from equations (51) reduces so that

$$-M_{aq}(i_d \sin\theta_2 - i_q \cos\theta_2) = -\frac{3}{2} M_{aq3}(i_d \sin\theta_2 - i_q \cos\theta_2)$$

and

(57) $M_{aq} = \frac{3}{2} M_{aq3}$

Equations (54) through (57) completely prescribe the relations which must hold between the three phase and the equivalent two phase machine.

Referring everything to a rotating reference frame, this time rotating clockwise at rotor speed $\omega_2$, the synchronous machine reduces to a static network in the same manner as did the induction machine. This network is shown in Fig. 11. Since in this case the rotor is turning at reference frame speed, the direct and quadrature axes of the rotor may be taken directly as references, and the rotor quantities remain unchanged.

The stator of the original machine is rotating counterclockwise at a speed $\omega_2$ in the new reference system. Then the new stator equations are

(58)

$$i_\alpha = i_d \cos\theta_2 + i_q \sin\theta_2$$

$$i_\beta = -i_d \sin\theta_2 + i_q \cos\theta_2$$

$$e_\alpha = e_d \cos\theta_2 + e_q \sin\theta_2$$

$$e_\beta = -e_d \sin\theta_2 + e_q \cos\theta_2$$

$$\Psi_{a\alpha} = \Psi_{ad} \cos\theta_2 + \Psi_{aq} \sin\theta_2$$

$$\Psi_{a\beta} = -\Psi_{ad} \sin\theta_2 + \Psi_{aq} \cos\theta_2$$

The rotor relations are unchanged, but will be repeated here for easy reference:

(59)

$$I_{f\alpha} = I_f$$

$$E_{f\alpha} = E_f$$

$$\Psi_{f\alpha} = \Psi_d$$

$$I_{d\alpha} = I_d$$

$$\Psi_{d\alpha} = \Psi_d$$

$$I_{q\beta} = I_q$$

$$\Psi_{q\beta} = \Psi_q$$

$$I_{f\beta} = E_{f\beta} = \Psi_{f\beta} = I_{d\beta} = E_{d\alpha} = E_{d\beta} = \Psi_{d\beta} = I_{q\alpha} = E_{q\alpha} = E_{q\beta} = \Psi_{q\alpha} = 0$$

By a procedure identical to that for the induction machine, it can be shown that

$$
\text{(60)} \qquad
\begin{aligned}
e_\alpha &= \frac{d\Psi_{s\alpha}}{dt} - \Psi_{s\beta}\frac{d\theta_2}{dt} + r_s i_\alpha \\
e_\beta &= \frac{d\Psi_{s\beta}}{dt} + \Psi_{s\alpha}\frac{d\theta_2}{dt} + r_s i_\beta
\end{aligned}
$$

Next the loop equation for each rotor circuit can be written,

$$
\text{(61)} \qquad
\begin{aligned}
E_f &= \frac{d\Psi_f}{dt} + R_f I_f \\
0 &= \frac{d\Psi_d}{dt} + R_d I_d \\
0 &= \frac{d\Psi_q}{dt} + R_q I_q
\end{aligned}
$$

and referring to Fig. 11, the following simple flux relations are seen to hold:

$$
\begin{aligned}
\Psi_{s\alpha} &= L_s i_\alpha + M_{sf} I_f + M_{sd} I_d \\
\Psi_{s\beta} &= L_s i_\beta + M_{sq} I_q
\end{aligned}
$$

$$
\text{(62)} \qquad
\begin{aligned}
\Psi_f &= L_f I_f + M_{fd} I_d + M_{sf} i_\alpha \\
\Psi_d &= L_d I_d + M_{fd} I_f + M_{sd} i_\alpha \\
\Psi_q &= L_q I_q + M_{sq} i_\beta
\end{aligned}
$$

The flux terms can now be eliminated from equations (60) and (61) by substitution of equations (62) into them:

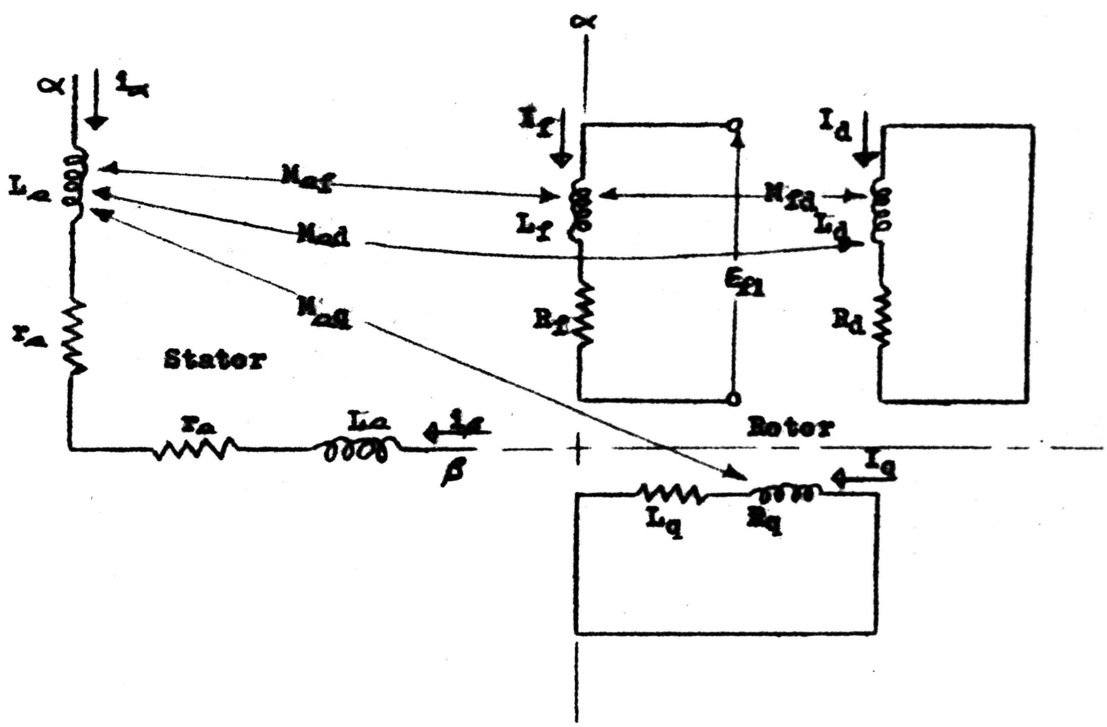



FIGURE 11 - Equivalent Machine Reduced to Static Network Fixed to Rotating $\alpha$-$\beta$ Reference Frame.

$$e_\alpha = \frac{d}{dt}(L_a i_\alpha + M_{af} I_f + M_{ad} I_d) - (L_a i_\beta + M_{aq} I_q)\frac{d\theta_2}{dt} + r_a i_\alpha$$

$$e_\beta = \frac{d}{dt}(L_a i_\beta + M_{aq} I_q) + (L_a i_\alpha + M_{af} I_f + M_{ad} I_d)\frac{d\theta_2}{dt} + r_a i_\beta$$

(63) $$E_f = \frac{d}{dt}(L_f I_f + M_{fd} I_d + M_{af} i_\alpha) + R_f I_f$$

$$0 = \frac{d}{dt}(L_d I_d + M_{ad} I_f + M_{ad} i_\alpha) + R_d I_d$$

$$0 = \frac{d}{dt}(L_q I_q + M_{aq} i_\beta) + R_q I_q$$

Using equation (27) and Fig. 11, the torque equation is next obtained:

(64) $$T_2 = i_\beta (M_{af} I_f + M_{ad} I_d) - M_{aq} i_\alpha I_q$$

Torques tending to turn the stator circuit in the direction of the actual stator's rotation with respect to the reference frame have been taken as positive, and vice versa.

The six equations (63) and (64) describe the performance of the synchronous machine. However it will again be necessary to take increments in order to make them transformable, i.e., to consider only small deviations in input and output. Thus

$$\Delta e_\alpha = \frac{d}{dt}(L_a \Delta i_\alpha + M_{af}\Delta I_f + M_{ad}\Delta I_d) - \frac{d\theta_{20}}{dt}(L_a \Delta i_\beta + M_{aq}\Delta I_q)$$
$$- \frac{d\Delta\theta_2}{dt}(L_a i_{\beta 0} + M_{aq} I_{q0}) + r_a \Delta i_\alpha$$

(65)

$$\Delta e_\beta = \frac{d}{dt}(L_a \Delta i_\beta + M_{aq}\Delta I_q) + \frac{d\theta_{20}}{dt}(L_a \Delta i_\alpha + M_{af}\Delta I_f + M_{ad}\Delta I_d)$$
$$+ \frac{d\Delta\theta_2}{dt}(L_a i_{\alpha 0} + M_{af} I_{f0} + M_{ad} I_{d0}) + r_a \Delta i_\beta$$

$$\Delta E_f = \frac{d}{dt}(L_f \Delta I_f + M_{fd}\Delta I_d + M_{af}\Delta i_\alpha) + R_f \Delta I_f$$

$$0 \quad = \frac{d}{dt}(L_d \Delta I_d + M_{fd}\Delta I_f + M_{ad}\Delta i_\alpha) + R_d \Delta I_d$$

$$0 \quad = \frac{d}{dt}(L_q \Delta I_q + M_{aq}\Delta i_\beta) + R_q \Delta I_q$$

But $\dfrac{d\theta_{2o}}{dt} = \omega_{2o} = \omega_{6o}$; $\dfrac{d\Delta\theta_2}{dt} = \Delta\omega_2$ and from equations (62)

$$L_a i_{\beta o} + M_{aq} I_{qo} = \Psi_{a\beta o}$$

(66)

$$L_a i_{\alpha o} + M_{af} I_{fo} + M_{ad} I_{do} = \Psi_{a\alpha o}$$

Also, it was originally assumed that $E_f$ was to be constant. Therefore, $\Delta E_f = 0$, so that equations (65) can be rewritten as

$$\Delta e_\alpha = \frac{d}{dt}(L_a \Delta i_\alpha + M_{af}\Delta I_f + M_{ad}\Delta I_d)$$

$$- \omega_{6o}(L_a \Delta i_\beta + M_{aq}\Delta I_q) - \Psi_{a\beta o}\Delta\omega_2 + r_a \Delta i_\alpha$$

$$\Delta e_\beta = \frac{d}{dt}(L_a \Delta i_\beta + M_{aq}\Delta I_q)$$

(67)

$$+ \omega_{6o}(L_a \Delta i_\alpha + M_{af}\Delta I_f + M_{ad}\Delta I_d) + \Psi_{a\alpha o}\Delta\omega_2 + r_a \Delta i_\beta$$

$$0 = \frac{d}{dt}(L_f \Delta I_f + M_{fd}\Delta I_d + M_{af}\Delta i_\alpha) + R_f \Delta I_f$$

$$0 = \frac{d}{dt}(L_d \Delta I_d + M_{fd}\Delta I_f + M_{ad}\Delta i_\alpha) + R_d \Delta I_d$$

$$0 = \frac{d}{dt}(L_q \Delta I_q + M_{aq}\Delta i_\beta) + R_q \Delta I_q$$

Taking increments in equation (64) there is obtained

$$\Delta T_2 = i_{\beta o}(M_{af}\Delta I_f + M_{ad}\Delta I_d) + (M_{af} I_{fo} + M_{ad} I_{do})\Delta i_\beta$$

(68)

$$- M_{aq}(i_{\alpha o}\Delta I_q + I_{qo}\Delta i_\alpha)$$

Equations (67) and (68) may now be transformed:

$$\Delta e_\alpha = Z_s \Delta i_\alpha + s(M_{sf}\Delta I_f + M_{sd}\Delta I_d)$$
$$- \omega_{60}(L_s \Delta I_\beta + M_{sq}\Delta I_q) - \Psi_{s\beta o}\Delta\omega_2$$

$$\Delta e_\beta = Z_s \Delta i_\beta + sM_{sq}\Delta I_q + \omega_{60}(L_s \Delta i_\alpha + M_{sf}\Delta I_f + M_{sd}\Delta I_d)$$
$$- \Psi_{s\alpha o}\Delta\omega_2$$

(69) $$0 = Z_f \Delta I_f + s(M_{fd}\Delta I_d + M_{sf}\Delta i_\alpha)$$

$$0 = Z_d \Delta I_d + s(M_{fd}\Delta I_f + M_{sd}\Delta i_\alpha)$$

$$0 = Z_q \Delta I_q + sM_{sq}\Delta i_\beta$$

$$\Delta T_2 = i_{\beta o}(M_{sf}\Delta I_f + M_{sd}\Delta I_d) + (M_{sf}I_{fo} + M_{sd}I_{do})\Delta i_\beta$$
$$- M_{sq}(i_{\alpha o}\Delta I_q + I_{qo}\Delta i_\alpha)$$

where

$$Z_s \triangleq L_s s + r_s$$

$$Z_f \triangleq L_f s + R_f$$

$$Z_d \triangleq L_d s + R_d$$

$$Z_q \triangleq L_q s + R_q$$

Again the rotor currents are of no interest. Accordingly they may be eliminated from equations (69). From the quadrature axis equation

(70) $$\Delta I_q = -\frac{sM_{sq}\Delta i_\beta}{Z_q}$$

and from the main field and direct axis damper equations solved simultaneously:

$$(71)\quad \Delta I_f = \frac{S(SM_{fd}M_{ad} - M_{af}Z_d)}{\delta_2}$$

$$(72)\quad \Delta I_d = \frac{S(SM_{fd}M_{af} - M_{ad}Z_f)}{\delta_2}$$

where

$$\delta_2 = Z_f Z_d - S^2 M_{fd}^2$$

Substituting equations (70) through (72) into equations (69) there is obtained

$$\Delta e_\alpha = \frac{S^2\left[M_{af}(SM_{fd}M_{ad} - M_{af}Z_d) + M_{ad}(SM_{fd}M_{af} - M_{ad}Z_f)\right]}{\delta_2}\Delta i_\alpha$$
$$+ Z_a \Delta i_\alpha - \omega_{50}\left(L_a - \frac{SM_{aq}^2}{Z_q}\right)\Delta i_\beta - \Psi_{a\beta o}\Delta\omega_2$$

(73)

$$\Delta e_\beta = \omega_{50}\left\{L_a + \frac{S\left[\begin{array}{l} M_{af}(SM_{fd}M_{ad} - M_{af}Z_d) \\ \quad + M_{ad}(SM_{fd}M_{af} - M_{ad}Z_f)\end{array}\right]}{\delta_2}\right\}\Delta i_\alpha$$
$$+ \left(Z_a - \frac{S^2M_{aq}^2}{Z_q}\right)\Delta i_\beta + \Psi_{a\alpha o}\Delta\omega_2$$

$$\Delta T_2 = i_{\beta o} S\left[\frac{M_{af}(SM_{fd}M_{ad} - M_{af}Z_d) + M_{ad}(SM_{fd}M_{af} - M_{ad}Z_f)}{\delta_2}\right]\Delta i_\alpha$$
$$- M_{aq}I_{qo}\Delta i_\alpha + \left(M_{af}I_{fo} + M_{ad}I_{do} + \frac{i_{\alpha o}M_{aq}^2}{Z_q}\right)\Delta i_\beta$$

After some regrouping, equations (73) may be written as

$$(74)\quad \Delta e_\alpha = \left\{ Z_a + \frac{s^2\left[M_{af}(sM_{fd}M_{ad} - M_{af}Z_d) + M_{ad}(sM_{fd}M_{af} - M_{ad}Z_f)\right]}{\delta_2} \right\} \Delta i_\alpha$$

$$- \left( X_{a60} - \frac{sX_{Maq60}M_{aq}}{Z_q} \right) \Delta i_\beta - \Psi_{a\beta o} \Delta \omega_2$$

$$(75)\quad \Delta e_\beta = \left\{ X_{a60} + \frac{s\left[X_{Maf60}(sM_{fd}M_{ad} - M_{af}Z_d) + X_{Mad60}(sM_{fd}M_{af} - M_{ad}Z_f)\right]}{\delta_2} \right\} \Delta i_\alpha$$

$$- \left( Z_a - \frac{s^2 M_{aq}}{Z_q} \right) \Delta i_\beta + \Psi_{a\alpha o} \Delta \omega_2$$

$$(76)\quad \Delta T_2 = \left\{ \frac{i_{\beta o}\, s\left[M_{af}(sM_{fd}M_{ad} - M_{af}Z_d) + M_{ad}(sM_{fd}M_{af} - M_{ad}Z_f)\right]}{\delta_2} - M_{aq}I_{qo} \right\} \Delta i_\alpha$$

$$+ \left( M_{af}I_{fo} + M_{ad}I_{do} + \frac{sM_{aq}^2\, i_{\alpha o}}{Z_q} \right) \Delta i_\beta$$

Where

$X_{a60} = \omega_{60} L_a =$ Slip frequency stator self reactance

$X_{Maq60} = \omega_{60} M_{aq} =$ Slip frequency stator-quadrature axis mutual reactance

$X_{Maf60} = \omega_{60} M_{af} =$ Slip frequency stator-main field mutual reactance

$X_{Mad60} = \omega_{60} M_{ad} =$ Slip frequency stator-direct axis mutual reactance

Then for simplification the above equations are rewritten as follows:

$$(77)\quad \Delta e_\alpha = h_1(s)\Delta i_\alpha - h_2(s)\Delta i_\beta - \Psi_{a\beta o}\Delta\omega_2$$

(78) $\Delta e_\beta = h_3(s)\Delta i_\alpha + h_4(s)\Delta i_\beta + \Psi_{a\alpha o}\,\Delta\omega_2$

(79) $\Delta T_2 = h_5(s)\Delta i_\alpha + h_6(s)\Delta i_\beta$

Where $h_1(S)$, etc., are defined by the appropriate brackets in equations (74) through (76). $\Delta T_2$ is the electrical torque of the synchronous machine. As a final step, the effects of inertia and friction of this machine plus machine DC, will be included. Thus

(80) $\Delta T_2 = B_2\Delta\omega_2 + J_2 s\Delta\omega_2 + \Delta T_{DC_1} = Z_2\,\Delta\omega_2 + T_{DC_1}$

where

$Z_2 = J_2 s + B_2$ = mechanical impedance of machines $S_1$ and $DC_1$.

Finally, from equations (79) and (80)

(81) $\Delta T_{DC_1} = h_5(s)\Delta i_\alpha + h_6(s)\Delta i_\beta - Z_2\Delta\omega_2$

Equations (77), (78), and (81) completely describe the performance of synchronous machine $S_1$ working into a mechanical load $DC_1$. These equations are general for small variations of input and output. They have been derived on the assumption that the machine is being supplied from a non-infinite source, i.e., variations in load may change the input voltage. To apply them to a machine supplied from a large source and having variable excitation, $\Delta e_\alpha$ and $\Delta e_\beta$ would be equated to zero, whereas $\Delta E_f$ would no longer be zero.

Solution for the Double-Fed Machine

Since the rotor of machine M and the stator of machine $S_1$ are tied together, obviously the quantities $\Delta e_\alpha$, $\Delta e_\beta$, $\Delta i_\alpha$, and $\Delta i_\beta$ are common in the two machines. It is now desired to obtain a relation bridging the two machines, i.e., between $\Delta T_1$ and $\Delta T_{DC}$. This can be done by equating $\Delta e_\alpha$ and $\Delta e_\beta$ from equations (43) and (44) to the corresponding quantities from equations (78) and (79). This eliminates $\Delta e_\alpha$ and $\Delta e_\beta$. Since the speed control variations are to come through $\Delta T_{D<_1}$, it is also apparent that $\Delta i_\alpha$ and $\Delta i_\beta$ are no longer of interest. Accordingly from equations (43) and (77)

$$g_1(s)\Delta i_\alpha + g_2(s)\Delta i_\beta + \Psi_{\beta o}\Delta\omega_1 = h_1(s)\Delta i_\alpha - h_2(s)\Delta i_\beta - \Psi_{a\beta o}\Delta\omega_2 \tag{82}$$

and from equations (44) and (78)

$$-g_2(s)\Delta i_\alpha + g_1(s)\Delta i_\beta - \Psi_{\alpha o}\Delta\omega_1 = h_3(s)\Delta i_\alpha + h_4(s)\Delta i_\beta + \Psi_{a\alpha o}\Delta\omega_2 \tag{83}$$

Equations (82) and (83) may be rearranged as follows:

$$\text{(84)}\quad [h_1(s) - g_1(s)]\Delta i_\alpha - [h_2(s) + g_2(s)]\Delta i_\beta = \Psi_{\beta o}\Delta\omega_1 + \Psi_{a\beta o}\Delta\omega_2$$

$$\text{(85)}\quad -[g_2(s) + h_3(s)]\Delta i_\alpha + [g_1(s) - h_4(s)]\Delta i_\beta = \Psi_{\alpha o}\Delta\omega_1 + \Psi_{a\alpha o}\Delta\omega_2$$

Solving (84) and (85) for $\Delta i_\alpha$ and $\Delta i_\beta$, there is obtained, after regrouping

$$\text{(86)}\quad \Delta i_\alpha = \left[\frac{\{\Psi_{\beta o}[g_1(s) - h_4(s)] + \Psi_{\alpha o}[g_2(s) + h_2(s)]\}\Delta\omega_1 + \{\Psi_{a\beta o}[g_1(s) - h_4(s)] + \Psi_{a\alpha o}[g_2(s) + h_2(s)]\}\Delta\omega_2}{\delta_3}\right]$$

$$\text{(87)}\quad \Delta i_\beta = \left[\frac{\{\Psi_{\beta o}[g_2(s) + h_3(s)] - \Psi_{\alpha o}[g_1(s) - h_1(s)]\}\Delta\omega_1 + \{\Psi_{a\beta o}[g_2(s) + h_3(s)] - \Psi_{a\alpha o}[g_1(s) - h_1(s)]\}\Delta\omega_2}{\delta_3}\right]$$

where

$$\delta_3 = [h_1(s) - g_1(s)][g_1(s) - h_4(s)] + [g_2(s) + h_3(s)][g_2(s) + h_2(s)]$$

For convenience in writing define

$$f_1(s) = \Psi_{\beta o}\left[g_1(s) - h_4(s)\right] + \Psi_{\alpha o}\left[g_2(s) + h_2(s)\right]$$

$$f_2(s) = \Psi_{\Delta\beta o}\left[g_1(s) - h_4(s)\right] + \Psi_{\Delta\alpha o}\left[g_2(s) + h_2(s)\right]$$

$$f_3(s) = \Psi_{\beta o}\left[g_2(s) + h_3(s)\right] - \Psi_{\alpha o}\left[g_1(s) - h_1(s)\right]$$

$$f_4(s) = \Psi_{\Delta\beta o}\left[g_2(s) + h_3(s)\right] - \Psi_{\Delta\alpha o}\left[g_1(s) - h_1(s)\right]$$

Then

(88) $$\Delta i_\alpha = \frac{f_1(s)\Delta\omega_1 + f_2(s)\Delta\omega_2}{\delta_2}$$

(89) $$\Delta i_\beta = \underline{f_3(s)\Delta\omega_1 + f_4(s)\Delta\omega_2}$$

Using equations (88) and (89), $\Delta i_\alpha$ and $\Delta i_\beta$ may now be eliminated from equations (49) and (81):

(90) $$\Delta T_L = g_3(s)\left[\frac{f_1(s)\Delta\omega_1 + f_2(s)\Delta\omega_2}{\delta_3}\right] - g_4(s)\left[\frac{f_3(s)\Delta\omega_1 + f_4(s)\Delta\omega_2}{\delta_3}\right] - Z_1\Delta\omega_1$$

(91) $$\Delta T_{DC_1} = h_5(s)\left[\frac{f_1(s)\Delta\omega_1 + f_2(s)\Delta\omega_2}{\delta_3}\right] + h_6(s)\left[\frac{f_3(s)\Delta\omega_1 + f_4(s)\Delta\omega_2}{\delta_3}\right] - Z_1\Delta\omega_2$$

or, rearranged

(90) $$\Delta T_L = \left[\frac{g_3(s)f_1(s) - g_4(s)f_3(s)}{\delta_3} - Z_1\right]\Delta\omega_1 + \left[\frac{g_3(s)f_2(s) - g_4(s)f_4(s)}{\delta_3}\right]\Delta\omega_2$$

$$\Delta T_{DC_1} = \left[ \frac{h_5(s) f_1(s) + h_6(s) f_3(s)}{\delta_3} \right] \Delta\omega_1 + \left[ \frac{h_5(s) f_2(s) + h_6(s) f_4(s)}{\delta_3} - Z_2 \right] \Delta\omega_2 \tag{91}$$

or, in more abbreviated form

$$\Delta T_L = G_1(s)\Delta\omega_1 + G_2(s)\Delta\omega_2 \tag{92}$$

$$\Delta T_{DC_1} = G_3(s)\Delta\omega_1 + G_4(s)\Delta\omega_2 \tag{93}$$

Eliminating $\Delta\omega_2$ from the above equations, the final equation is obtained.

$$\Delta T_{DC_1} = \frac{G_4(s)}{G_2(s)} \Delta T_L + \left[ G_3(s) - \frac{G_1(s) G_4(s)}{G_2(s)} \right] \Delta\omega_1 \tag{94}$$

Equation (94) is the complete solution of the double-fed machine in terms of load torques and induction motor speed. This solution can be tied through $\Delta T_{DC_1}$ to the solution for the direct current machine group.

## Direct Current Machines, DC and $DC_2$

The DC machine relations used in this development are well known, and may be found in any standard text on DC Machinery, e.g., Dawes.[13]

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(13) Dawes, C. L., Electrical Engineering, Vol. I., Direct Currents, McGraw-Hill Book Co., N. Y., 1936, pp. 397-495.

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The usual assumptions will be made neglecting hysteresis and eddy currents. The effect of armature slots will also be neglected.

Since the armature circuits of the two machines form one simple loop, they may conveniently be considered together. The combined electric circuit is as shown in Fig. 12. The parameter which ties the machines together is of course $I_A$.

In any DC machine the torque is proportional to flux and armature current. Therefore the following equations can be written:

(95) $T_{DC_1} = K_1 \Phi_1 I_A$

(96) $T_{DC_2} = K_1 \Phi_2 I_A$ (14)

where

$$K_1 = \frac{PN}{20\pi P'} \times \frac{1}{4.45 \times 10^5 \times 2.5 \times 12}$$

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(14) Dawes, C. L., op. cit., pages 401 and 484.

---

and

P = number of poles

N = number of conductors

P = number of parallel armature paths.

The field flux linking the armature at open circuit is

$$\Phi_1 = K_f I_{f1}$$

Where $K_f$ is a coefficient of such value that $(L_{f1} - K_f)\, I_{f1}$ represents the field leakage flux. When the armature is carrying current, however, the flux is decreased because of armature reaction, and as the current increases, because of saturation. When current flows in the armature, some of the flux it sets up acts in direct opposition to the field flux, i.e., is a demagnetizing flux. This demagnetizing flux is a function of current, brush position, and reluctance of the flux paths. In addition, the coefficient $K_f$ is a function of reluctance. The reluctance of any circuit containing iron is in turn a function of saturation, so that since $K_f$ has been defined at no load, it cannot be said to hold strictly when the armature carries current. No simple relation can be written to express the overall change in flux due to these factors. However, an ap-

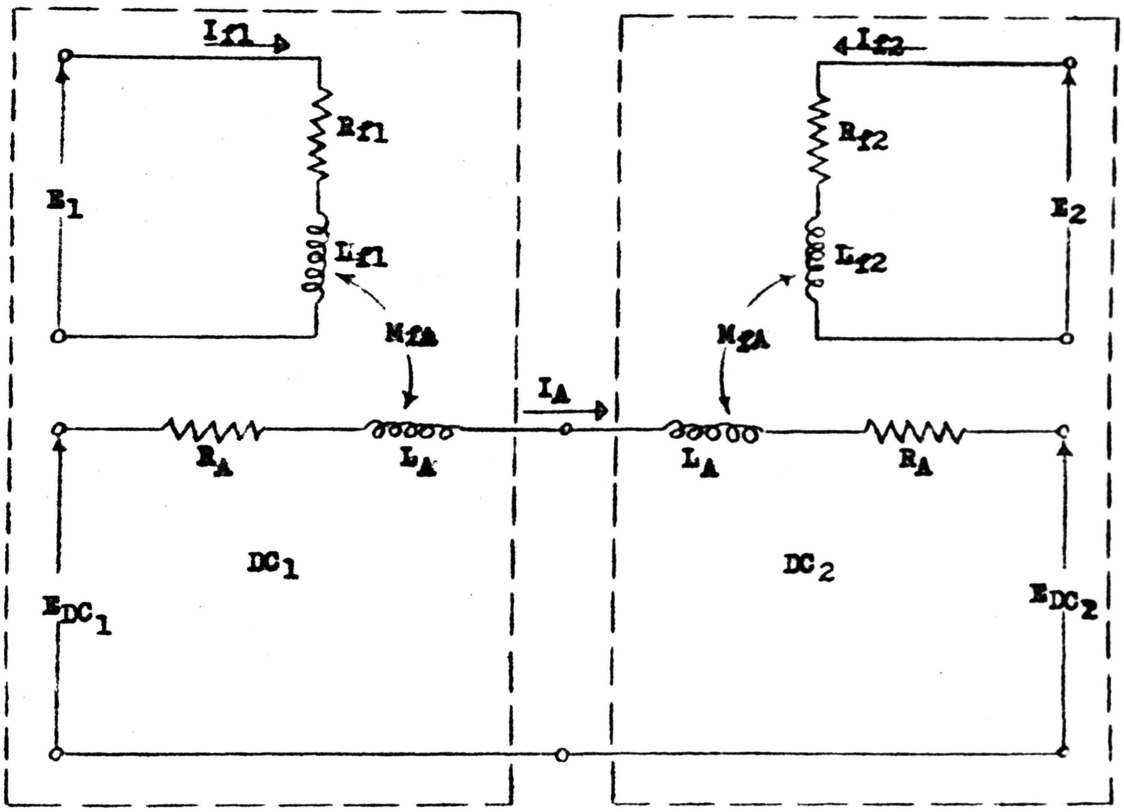


**FIGURE 12 – Electric Circuit of DC Machines**

proximation can be made by assuming the variation in flux is directly proportional to armature current. This is a good approximation over a considerable range because of the air gaps in the magnetic circuit. Then for machine $DC_1$ operating as a generator

(97) $\Phi_1 = K_f I_{f1} - A_1 I_A$

and for $DC_2$, operating as a motor

(98) $\Phi_2 = K_f I_{f2} - A_2 I_A$

Also, referring to the field circuits of $DC_1$ and $DC_2$, Fig. 12,

(99) $E_1 = L_{f2} \frac{dI_{f1}}{dt} + I_{f1} R_{f2} + M_{fA} \frac{dI_A}{dt}$

(100) $E_2 = L_{f2} \frac{dI_{f2}}{dt} + I_{f2} R_{f2} + M_{fA} \frac{dI_A}{dt}$

since $L_{f1} = L_{f2}$ and $R_{f1} = R_{f2}$. $M_{fA}$ is an ordinary mutual inductance coefficient, defining the voltage induced in the field by changes in armature current. Very strictly speaking, $M_{fA}$ is an approximation because it too is a function of saturation, hence of current. However, it will be considered constant for this solution.

From equations (95) through (100) a solution in terms of $T_{DC_1}$, $T_{DC_2}$, and $E_2$ can be obtained. Taking increments

(101) $\Delta T_{DC_1} = K_1 \left( \Phi_{10} \Delta I_A + I_{A0} \Delta \Phi_1 \right)$

and similarly

(102) $\Delta T_{DC_2} = K_1 \left( \Phi_{20} \Delta I_A + I_{A0} \Delta \Phi_2 \right)$

(103) $\Delta \Phi_1 = K_f \Delta I_{f1} - A_1 \Delta I_A$

Also

$$(104)\quad \Delta\Phi_2 = K_f \Delta I_{f2} - A_2 \Delta I_A$$

$$E_{2o} + \Delta E_2 = L_{f2}\frac{d}{dt}(I_{f2o} + \Delta I_{f2}) + R_{f2}(I_{f2o} + \Delta I_{f2}) + M_{fA}\frac{d}{dt}(I_{Ao} + \Delta I_A)$$

$$(105)\quad \Delta E_2 = L_{f2}\frac{d\Delta I_{f1}}{dt} + R_{f2}\Delta I_{f2} + M_{fA}\frac{d\Delta I_A}{dt}$$

and since $E_1$ is held constant

$$(106)\quad 0 = L_{f2}\frac{d\Delta I_{f1}}{dt} + R_{f2}\Delta I_{f1} + M_{fA}\frac{d\Delta I_A}{dt}$$

Taking the LaPlace transforms of equations (101) through (106) there is

$$(107)\quad \Delta T_{DC_1} = K_1(\Phi_{10}\Delta I_A + I_{Ao}\Delta\Phi_1)$$

$$(108)\quad \Delta T_{DC_2} = K_1(\Phi_{20}\Delta I_A + I_{Ao}\Delta\Phi_2)$$

$$(109)\quad \Delta\Phi_1 = K_f \Delta I_{f1} - A_1 \Delta I_A$$

$$(110)\quad \Delta\Phi_2 = K_f \Delta I_{f2} - A_2 \Delta I_A$$

$$(111)\quad \Delta E_2 = (L_{f2}S + R_{f2})\Delta I_{f2} + M_{fA}S\Delta I_A = Z_{f2}\Delta I_{f2} + M_{fA}S\Delta I_A$$

$$(112)\quad 0 = (L_{f2}S + R_{f2})\Delta I_{f1} + M_{fA}S\Delta I_A = Z_{f2}\Delta I_{f1} + M_{fA}S\Delta I_A$$

where

$$Z_{f2} = L_{f2}S + R_{f2}$$

Eliminating $\Delta I_{f1}$ from equation (109) with equation (112), then substituting $\Delta\Phi_1$ and $\Delta\Phi_2$ from equations (109) and (110) into equations (107) and (108) and regrouping:

$$(113)\quad \Delta T_{DC_1} = K_1\left[(\Phi_{10} - A_1 I_{Ao}) - \frac{K_f M_{fA} I_{Ao}}{Z_{f2}}\right]\Delta I_A$$

$$(114)\quad \Delta T_{DC_2} = K_1\left[(\Phi_{20} - A_2 I_{Ao})\Delta I_A + K_f I_{Ao}\Delta I_{f2}\right]$$

Solving equation (113) for $\Delta I_A$, and substituting into equations (111) and (114), there is obtained

$$(115)\quad \Delta E_2 = Z_{f2}\Delta I_{f2} + \frac{M_{fA}S}{K_1\left[(\Phi_{10} - A_1 I_{A0} - \frac{K_f M_{fA} I_{A0} S}{Z_{f2}}\right]} \Delta T_{DC_1}$$

$$(115\text{-a})\quad \Delta T_{DC_2} = \left\{\frac{(\Phi_{20} - A_2 I_{A0})}{(\Phi_{10} - A_1 I_{A0} - \frac{K_f M_{fA} I_{A0} S}{Z_{f2}}}\right\}\Delta T_{DC_1} + K_1 K_f I_{A0}\Delta I_{f2}$$

Then, eliminating $\Delta I_{f2}$ from equations (115) and (115-a) and rearranging

$$(115\text{-b})\quad \Delta T_{DC_1} = D_1(S)\Delta T_{DC_2} - D_2(S)\Delta E_2$$

where

$$D_1(S) = \left[\frac{Z_{f2}(\Phi_{10} - A_1 I_{A0}) - K_f M_{fA} I_{A0} S}{Z_{f2}(\Phi_{20} - A_2 I_{A0}) - K_f M_{fA} I_{A0} S}\right]$$

$$D_2(S) = \frac{K_1 K_f I_{A0}}{Z_{f2}} D_1(S)$$

Now, from Fig. 3

$$(115\text{-c})\quad \Delta T_{DC_2} = J_3 S\Delta\omega_3 + B_3\Delta\omega_3 + \Delta T_3 = Z_3\Delta\omega_3 + \Delta T_3$$

where

$Z_3 = J_3 S + B_3 =$ mechanical impedance of machines $DC_2$ and $S_2$.

Substituting equation (115-c) into (115-b) the final transfer function for the DC machines is obtained.

$$(116)\quad \Delta T_{DC_1} = D_1(S)\left[Z_3\Delta\omega_3 + \Delta T_3\right] - D_2(S)\Delta E_2$$

As expected, this solution states that the input torque of the generator is a function of the excitation, speed, and load torque of the motor. The load torque $T_3$ is of course the input to the synchronous machine $S_2$, feeding power back to the system. A relation for this torque will be obtained in the following section.

## Synchronous Machine $S_2$

The solution for synchronous machine $S_2$ will differ from that developed for $S_1$ only in that $S_2$ is tied to an infinite bus. Hence and $e_{\beta 3}$ will be constants, as will be the frequency, which now is synchronous frequency. Refer then to equations (69). Using the subscript (3) to distinguish terms in the $S_2$ equations which differ from corresponding terms in the $S_1$ equations, and remembering that $\Delta e_{\alpha 3}$ and $\Delta e_{\beta 3}$ equal zero and that $\omega_{60}$ is replaced by $\omega_f$, there can be written

$$0 = Z_a \Delta i_{\alpha 3} + s\left(M_{af} \Delta I_{f3} + M_{ad} \Delta I_{d3}\right) - \omega_f\left(L_a \Delta I_{\beta 3} + M_{aq} \Delta I_{q3}\right) - \Psi_{a\beta 30} \Delta \omega_3$$

$$0 = Z_a \Delta i_{\beta 3} + s M_{aq} \Delta I_{q3} + \omega_f\left(L_a \Delta i_{\alpha 3} + M_{af} \Delta I_{f3} + M_{ad} \Delta I_{d3}\right) + \Psi_{a\alpha 30} \Delta \omega_3$$

(117) $$0 = Z_f \Delta I_{f3} + s\left(M_{fd} \Delta I_{d3} + M_{af} \Delta i_{\alpha 3}\right)$$

$$0 = Z_d \Delta I_{d3} + s\left(M_{fd} \Delta I_{f3} + M_{ad} \Delta i_{\alpha 3}\right)$$

$$0 = Z_q \Delta I_{q3} + s M_{aq} \Delta i_{\beta 3}$$

$$\Delta T_3 = i_{\beta 30}\left(M_{af} \Delta I_{f3} + M_{ad} \Delta I_{d3}\right) + \left(M_{af} I_{f30} + M_{ad} I_{d30}\right) \Delta i_{\beta 3} - M_{aq}\left(i_{\alpha 30} \Delta I_{q3} + I_{q30} \Delta i_{\alpha 3}\right)$$

After eliminating $\Delta I_{f3}$ , $\Delta I_{d3}$ , and $\Delta I_{q3}$ and regrouping, there may be written, analagous to equations (74), (75) and (76)

$$0=\left\{Z_{a}+\frac{s^{2}\left[\begin{array}{l}M_{af}(sM_{fd}M_{ad}-M_{af}Z_{d})\\ \qquad +M_{ad}(sM_{fd}M_{af}-M_{ad}Z_{f})\end{array}\right]}{\delta_{2}}\right\}\Delta i_{\alpha 3}$$

$$-\left(X_{af}-\frac{sX_{M_{aqf}}M_{aq}}{Z_{q}}\right)\Delta i_{\beta 3}-\Psi_{a\beta 30}\,\Delta\omega_{3}$$

$$0=\left\{X_{af}+\frac{s\left[\begin{array}{l}X_{M_{aff}}(sM_{fd}M_{ad}-M_{af}Z_{d})\\ \qquad +X_{M_{fdf}}(sM_{fd}M_{af}-M_{ad}Z_{f})\end{array}\right]}{\delta_{2}}\right\}\Delta i_{\alpha 3}$$

$$+\left(Z_{a}-\frac{s^{2}M_{aq}^{2}}{Z_{q}}\right)\Delta i_{\beta 3}+\Psi_{a\alpha 30}\,\Delta\omega_{3}$$

$$\Delta T_{3}=\left\{\frac{i_{\beta 30}\,s\left[\begin{array}{l}M_{af}(sM_{fd}M_{ad}-M_{af}Z_{d})\\ \qquad +M_{ad}(sM_{fd}M_{af}-M_{ad}Z_{f})\end{array}\right]}{\delta_{2}}-M_{aq}I_{q30}\right\}\Delta i_{\alpha 3}$$

$$+\left(M_{af}I_{f30}+M_{ad}I_{d30}+\frac{sM_{aq}^{2}\,i_{\alpha 30}}{Z_{q}}\right)\Delta i_{\beta 3}$$

where

$X_{af}$ = 60-cycle stator self reactance $=\omega_{f}L_{a}$

$X_{M_{aff}}=\omega_{f}M_{af}$ = 60-cycle stator-main field mutual reactance

$X_{M_{adf}} = \omega_f M_{ad} =$ 60-cycle stator-direct axis mutual reactance

$X_{M_{aqf}} = \omega_f M_{aq} =$ 60-cycle stator-quadrature axis mutual reactance

or, in abbreviated form

(121) $0 = h_{13}(s)\Delta i_{\alpha 3} - h_{23}(s)\Delta i_{\beta 3} - \Psi_{a\beta 30}\Delta\omega_3$

(122) $0 = h_{33}(s)\Delta i_{\alpha 3} + h_{43}(s)\Delta i_{\beta 3} + \Psi_{a\alpha 30}\Delta\omega_3$

(123) $\Delta T_3 = h_{53}(s)\Delta i_{\alpha 3} + h_{63}(s)\Delta i_{\beta 3}$

where $h_{13}$ (S), etc., are defined by the appropriate brackets above. Equations (121) and (122) may now be solved simultaneously for $\Delta i_{\alpha 3}$ and $\Delta i_{\beta 3}$, and the results substituted in equation (123). By determinants

$$\Delta i_{\alpha 3} = \left[\frac{\Psi_{a\beta 30} h_{43}(s) - \Psi_{a\alpha 30} h_{23}(s)}{\delta_4}\right]\Delta\omega_3$$

$$\Delta i_{\beta 3} = -\left[\frac{\Psi_{a\beta 30} h_{33}(s) + \Psi_{a\alpha 30} h_{13}(s)}{\delta_4}\right]\Delta\omega_3$$

where

$$\delta_4 = h_{13}(s)h_{43}(s) + h_{33}(s)h_{23}(s)$$

from which

(124) $$\Delta T_3 = \left\{\frac{\Psi_{a\beta 30}\left[h_{53}(s)h_{43}(s) - h_{63}(s)h_{33}(s)\right] - \Psi_{a\alpha 30}\left[h_{53}(s)h_{23}(s) + h_{63}(s)h_{13}(s)\right]}{\delta_4}\right\}\Delta\omega_3$$

or, more simply

(125) $\Delta T_3 = H(s)\Delta\omega_3$

Equation (124) expresses the behavior of the synchronous machine tied to an infinite bus, with constant field excitation.

3. Overall Open-Loop Transfer Function of the Modified Kramer System

Having already derived the transfer functions for the three principle machine groups, i.e., the double-fed machine M; direct current machines $DC_1$ and $DC_2$; and synchronous machine S2; it remains necessary only to match them through their common terms. Those three solutions, equation (94), for the double-fed machine; equation (116), for the direct current machines; and equation (125), for synchronous machine $S_2$; are repeated below for convenient reference.

$$(94) \quad \Delta T_{DC_1} = \frac{G_4(s)}{G_2(s)} \Delta T_L + \left[ G_3(s) - \frac{G_1(s) G_4(s)}{G_2(s)} \Delta \omega_1 \right]$$

$$(116) \quad \Delta T_{DC_1} = D_1(s) \left[ Z_3 \Delta \omega_3 + \Delta T_3 \right] - D_2(s) \Delta E_2$$

$$(125) \quad \Delta T_3 = H(s) \Delta \omega_3$$

As would be expected, the common terms, $\Delta T_{DC_1}$ and $\Delta T_3$, arise at the actual physical boundaries of the group, the shafts which transmit the torque from one group to another. It is desired to obtain finally an expression for changes in induction machine speed. Accordingly, equating (94) and (116), eliminating $\Delta T_3$ with (125), and rearranging, the final overall solution is obtained:

$$(126) \quad \Delta \omega_1 = \left[ \frac{G_4(s)}{G_1(s) G_4(s) - G_2(s) G_3(s)} \right] \Delta T_L - \left\{ \frac{D_1(s) G_2(s) \left[ Z_3 + H(s) \right]}{G_1(s) G_2(s) - G_2(s) G_3(s)} \right\} \Delta \omega_3$$

$$+ \left[ \frac{D_2(s) G_2(s)}{G_1(s) G_4(s) - G_2(s) G_3(s)} \right]$$

or, in abbreviated form

$$(127) \quad \Delta \omega_1 = F_1(s) \Delta T_L - F_2(s) \Delta \omega_3 + F_3(s) \Delta E_2$$

where $F_1$ (S), etc., are defined by the proper brackets in equation (126). The speed of the induction motor is seen to be a function of its load, of the speed and excitation of the controlling DC machine, and the various machine parameters involved in the S functions. All the S functions defined are tabulated in the Appendix, in the order of their appearance in the paper.

## SUMMARY

The object of the study made in this thesis was to obtain a solution, accurate within the limits of the idealizing assumptions necessary, for the open-loop transfer function of the modified Kramer speed control system. The intention was to establish relations which would be usable as the basis for further work leading to synthesis of a speed controller to close the loop from the speed of the induction machine to the excitation of the controlling direct current machine. The steps taken to obtain such a solution, and some of the difficulties encountered, are summarized briefly in the following discussion.

The work proceeded literally from left to right in the block diagram of Fig. 2. A solution was first obtained for induction machine M, then for synchronous machine $S_1$. These were then combined to give the solution for the double-fed machine. To accomplish this, two major simplifications had to be made. First, the unwieldy trigonometric relations contained in the three phase equations, had to be eliminated, and, second, the resulting equations had to be Laplace transformable. Considerable trigonometric simplification was gained by the first step, replacing the three phase machines by equivalent two phase machines. The next, and most important, step was transformation to synchronously rotating axes. This entirely eliminated trigonometric quantities from the solution, and as was shown in the development, reduced the steady state currents, voltages, and flux linkages to steady direct current values. It should be mentioned in passing that the use of a synchronously rotating reference frame was not new in the solution for the synchronous machine, having been applied by Park (15) in the original article on the operation-

(15) Park, R. H., op. cit., pp. 716-730.

al solution of synchronous machinery. However, insofar as the author was able to ascertain in a survey of the literature, this reference frame has not been used before in the solution of the induction machine.

The third step was the treatment of the problem on an incremental basis. The incremental equations were found to be made up of products of steady state and incremental terms only. Because the steady state values in the rotating reference frame were constants, time function products were no longer present. Hence the equations were readily transformed, and solutions for the induction and the synchronous machines were obtained simply by algebraic manipulation. These are equations (43), (44), and (48) for the induction machine and equations (77), (78) and (81) for the synchronous machine. These six equations were combined through the voltages at the junction of the machines, and a solution for the double-fed machine obtained in terms of load torque, synchronous machine torque, and induction machine speed, as given in equation (94).

Solution of the direct current machines presented no particular difficulty. It was found convenient to lump the two together and solve them as one electric circuit, as in Fig. 12. The solution was obtained in terms of the torque input to $DC_1$ from synchronous machine $S_1$, the torque output of $DC_2$ to synchronous machine $S_2$, and the excitation and speed of $DC_2$, as given in equation (116).

The solution for the second synchronous machine was a repetition of that for the first except that $S_2$ operated in the steady state at synchronous speed $\omega_f$ and with constant stator voltage, whereas, $S_1$ operated in the steady state at induction machine slip speed $\omega_{\sigma o}$ and with variable stator voltage. Accordingly the solution, equation (125), was considerably simpler in form involving only the torque and speed of $S_2$.

Finally the group solutions were matched through their common torque terms to obtain the open-loop transfer function of the system. This is equation (127). This transfer function was not obtained as an explicit equation in descending powers of S because of the complexity of such a solution.

## CONCLUSIONS

As has been stated, the relations developed in this paper constitute an accurate solution for both the modified Kramer system and the individual machines contained in it. It would now theoretically be possible through the use of Nyquist diagrams and other implements of servomechanism theory to determine the system performance for variations in $E_2$ or $T_L$, and from this determination proceed toward synthesis of a controller which would maintain desirable performance or correct improper performance. However, this would involve plotting each of the S functions contained in the Appendix; then combining these functions, by properly adding, subtracting, multiplying and dividing graphically, to obtain the final plot for the system. This would seem to involve a prohibitive amount of time and tedious labor.

A more logical approach to extension of the study would be an attempt to simplify the relations and reduce the order of the equations by various approximations. This would necessitate a detailed study of all the parameters of the several machines. Some of these, for instance the reactances and resistances of the induction and synchronous machines, are pretty well limited in range in any realizable or economical design at a given rating. Such parameters could be calculated or estimated numerically and substituted into the S functions. Such substitutions might lead to the possibility of neglecting some terms, cancelling others, etc. Having reduced the complexity as much as possible by the above procedure, further simplification and a more approximate solution might be achieved by deliberately neglecting some small but not completely negligible quantities. Finally the simplified transfer function could be plotted, and steps taken to obtain desirable performance of the system through syn thesis of a controller and through adjustment of variable open-loop parameters such as the synchronous machine excitation voltage.

It should also be noted that the transfer functions of the components of the modified Kramer system could themselves be used in a completely independent study of the individual machines, or could serve as building blocks in setting up a solution for some altogether different combination of machines. The fact that synchronous and induction machines are by far the most common of all rotating electrical equipment renders their transfer functions doubly useful.

# APPENDIX I

## Tabulation of S Functions Used as Abbreviations in the Transfer Function Development

1. **Induction Machine:**

$$g_1(s) = Z_R + \frac{X_{M60}\left(X_{Mf}Z_s - X_{sf}MS\right) - MS\left(X_{sf}X_{Mf} + MZ_s S\right)}{Z_s^2 + X_{sf}^2}$$

$$g_2(s) = \frac{MS\left(X_{Mf}Z_s - X_{sf}MS\right) + X_{M60}\left(X_{sf}X_{Mf} + MZ_s s\right)}{Z_s^2 + X_{sf}^2} - X_{R60}$$

$$g_3(s) = M\left[I_{\beta 0} + \frac{i_{\beta 0}\left(X_{sf}X_{Mf} + MZ_s s\right) - i_{\alpha 0}\left(X_{Mf}Z_s - X_{sf}MS\right)}{Z_s^2 + X_{sf}^2}\right]$$

$$g_4(s) = M\left[I_{\alpha 0} + \frac{i_{\alpha 0}\left(X_{sf}X_{Mf} + MZ_s s\right) + i_{\beta 0}\left(X_{Mf}Z_s - X_{sf}MS\right)}{Z_s^2 + X_{sf}^2}\right]$$

2. **Synchronous Machine $S_1$:**

$$h_1(s) = Z_a + \frac{s^2\left[M_{af}\left(sM_{fd}M_{ad} - M_{af}Z_d\right) + M_{ad}\left(sM_{fd}M_{af} - M_{ad}Z_f\right)\right]}{Z_f Z_d - s^2 M_{fd}^2}$$

$$h_2(s) = X_{a60} - \frac{sX_{Maq60}M_{aq}}{Z_q}$$

$$h_3(s) = X_{a60} + \frac{s\left[X_{Maf60}\left(sM_{fd}M_{ad} - M_{af}Z_d\right) + X_{Mad60}\left(sM_{fd}M_{af} - M_{ad}Z_f\right)\right]}{Z_f Z_d - s^2 M_{fd}^2}$$

$$h_4(s) = Z_a - \frac{s^2 M^2_{aq}}{Z_q}$$

$$h_5(s) = \frac{i_{\beta o} s \begin{bmatrix} M_{af}(s M_{fd} M_{ad} - M_{af} Z_d) \\ + M_{ad}(s M_{fd} M_{af} - M_{ad} Z_f) \end{bmatrix}}{Z_f Z_d - s^2 M^2_{fd}} - M_{aq} I_{qo}$$

$$h_6(s) = M_{af} I_{fo} + M_{ad} I_{do} + \frac{s M^2_{aq} i_{\alpha o}}{Z_q}$$

3. **Double-Fed Machine:**

$$f_1(s) = \Psi_{\beta o}\left[g_1(s) - h_4(s)\right] + \Psi_{\alpha o}\left[g_2(s) + h_2(s)\right]$$

$$f_2(s) = \Psi_{a\beta o}\left[g_1(s) - h_4(s)\right] + \Psi_{a\alpha o}\left[g_2(s) + h_2(s)\right]$$

$$f_3(s) = \Psi_{\beta o}\left[g_2(s) + h_3(s)\right] - \Psi_{\alpha o}\left[g_1(s) - h_1(s)\right]$$

$$f_4(s) = \Psi_{a\beta o}\left[g_2(s) + h_3(s)\right] - \Psi_{a\alpha o}\left[g_1(s) - h_1(s)\right]$$

$$G_1(s) = \frac{f_1(s)\, g_3(s) - f_3(s)\, g_4(s)}{\delta_3} - Z_1$$

$$G_2(s) = \frac{f_2(s)\, g_3(s) - f_4(s)\, g_4(s)}{\delta_3}$$

$$G_3(s) = \frac{f_1(s)\, h_5(s) + f_3(s)\, h_6(s)}{\delta_3}$$

$$G_4(s) = \frac{f_2(s)\, h_5(s) + f_4(s)\, h_6(s)}{\delta_3}$$

Where

$$\delta_3 = [h_1(s) - g_1(s)][g_1(s) - h_4(s)] + [g_2(s) + h_3(s)][g_2(s) + h_2(s)]$$

4. **Direct Current Machines:**

$$D_1(s) = \frac{Z_{f2}(\Phi_{10} - A_1 I_{A0}) - K_f M_{fA} I_{A0} s}{Z_{f2}(\Phi_{20} - A_2 I_{A0}) - K_f M_{fA} I_{A0} s}$$

$$D_2(s) = \frac{K_1 K_f I_{A0}}{Z_{f2}} D_1(s)$$

5. **Synchronous Machine $S_2$:**

$$h_{13}(s) = Z_a + \frac{s_2 \left[ \begin{array}{l} M_{af}(s M_{fd} M_{ad} - M_{af} Z_d) \\ \qquad + M_{ad}(s M_{fd} M_{af} - M_{ad} Z_f) \end{array} \right]}{Z_f Z_d - s^2 M_{fd}^2}$$

$$h_{23}(s) = X_{af} - \frac{s M_{aqf} M_{aq}}{Z_q}$$

$$h_{33}(s) = X_{af} + \frac{s \left[ \begin{array}{l} X_{M_{Rf}f}(s M_{fd} M_{ad} - M_{af} Z_d) \\ \qquad + X_{M_{ad}f}(s M_{fd} M_{af} - M_{ad} Z_f) \end{array} \right]}{Z_f Z_d - s^2 M_{fd}^2}$$

$$h_{43}(s) = Z_a - \frac{s^2 M_{aq}^2}{Z_q}$$

$$h_{53}(s) = \frac{i_{\beta 30} s \begin{bmatrix} M_{\alpha f}(sM_{fd}M_{\alpha d} - M_{\alpha f}Z_d) \\ + M_{\alpha d}(sM_{fd}M_{\alpha f} - M_{\alpha d}Z_f) \end{bmatrix}}{Z_f Z_d - s^2 M^2_{fd}} - M_{\alpha q} I_{q30}$$

$$h_{63}(s) = M_{\alpha f} I_{f30} + M_{\alpha d} I_{d30} + \frac{sM^2_{\alpha q}\, i_{\alpha 30}}{Z_q}$$

$$H(s) = \left\{ \frac{\begin{matrix} \Psi_{\alpha\beta 30}[h_{53}(s) h_{43}(s) - h_{63}(s) h_{33}(s) \\ - \Psi_{\alpha\alpha 30}[h_{53}(s) h_{23}(s) + h_{63}(s) h_{13}(s)] \end{matrix}}{\delta_3} \right\}$$

6. **Final Solution:**

$$F_1(s) = \frac{G_4(s)}{G_1(s) G_4(s) - G_2(s) G_3(s)}$$

$$F_2(s) = \frac{D_1(s) G_2(s) [Z_3 + H(s)]}{G_1(s) G_4(s) - G_2(s) G_3(s)}$$

$$F_3(s) = \frac{D_2(s) G_2(s)}{G_1(s) G_4(s) - G_2(s) G_3(s)}$$

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# VITA

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