Stresses in a plate having a relatively small circular hole and loaded parallel to the axis of the hole under various conditions of constraint with an application to deep oil wells

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STRESSES IN A PLATE HAVING A RELATIVELY SMALL CIRCULAR HOLE AND LOADED PARALLEL TO THE AXIS OF THE HOLE UNDER VARIOUS CONDITIONS OF CONSTRAINT WITH AN APPLICATION TO DEEP OIL WELLS

BY

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THESIS

submitted to the faculty of the

SCHOOL OF MINES AND METALLURGY OF THE UNIVERSITY OF MISSOURI

in partial fulfillment of the work required for the

Degree of

MASTER OF SCIENCE IN CIVIL ENGINEERING

Rolla, Missouri

1947

Approved by E. W. Carlton

Professor of Structural Engineering

ACKNOWLEDGMENT

I wish to express my thanks to Dr. A. J. Miles, Head of the Department of Mechanical Engineering, for suggesting the problem and for advice and encouragement during the investigation.

Thanks are due also to R. Z. Williams, Assistant Dean, Professor E. W. Carlton of the Civil Engineering Department, and Mr. N. C. Costakos, of the Mechanics Department, for advice and encouragement.

TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

LIST OF TABLES

INTRODUCTION

The object of this investigation is to determine the stresses about a hole in a thick plate of large dimensions compared with the radius of the hole when the plate is loaded parallel to the axis of the hole under various conditions of restraint, and to apply the results to the case of a deep well. The case of a plate with an initial stress in its plane is also considered.

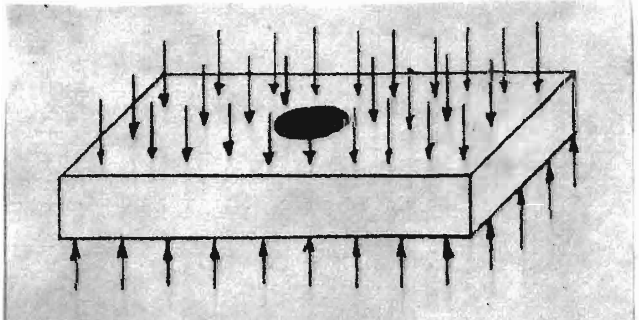

FIG. 1

It will be convenient to call the axis of the hole the Z-axis, and to refer to that axis as being vertical, although the results of the investigation would not be in the least affected if it had any other direction.

The conditions of lateral constraint considered are: (1) no lateral constraint; (2) constraint in one lateral direction; (3) constraint in both lateral directions. Each of these is combined with each of the following conditions of vertical constraint: a) no vertical constraint (case of plane stress) ; b) full vertical constraint (case of plane strain); c) uniform vertical expansion. Part I consists of an analysis of stress in a plane perpendicular to the axis of the hole. In Part II, stresses resulting from vertical loads are superposed on them under the conditions listed above.

The problem arose from difficulties encountered in the drilling of deep oil wells. After drilling has proceeded to a certain depth, the sides of the bore hole apparently collapse and fill up the hole, resulting not only in wasted time and effort but in the loss of the drill bits. By considering the region around the well as a thick plate of dimensions very large compared to the radius of the bore hole, the conditions will be seen to correspond to one of the cases investigated. Part III of this thesis is devoted to application of the results of the general investigation carried out in parts I and II to the special problem of stresses about a deep well and the causes of the failures described above.

All the cases investigated do not apply to deep wells, but it was felt that a complete investigation of the general problem might find application in other fields and would enhance the value of this thesis.

PART I

The problem of the distribution of coplanar stresses in a plate with a hole normal to the plane of the stresses has already been solved(1), though the detailed solution is not generally available. Since the assumptions made in the solution of such coplanar stress systems have an important bearing on the three-dimensional stress system the investigation of which is the object of this thesis, the analyses of

A) a symmetrical stress distribution around the hole, and

B) the effect of a circular hole on the stress distribution

in a semi-infinite plate loaded in the plane of the plate,

are given here. The determination of the maximum shearing stresses for these cases, which is important in the oil well application of Part III, is, however, original.

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(1) Lamé, "Lecons sur la théorie ....de L'élasticité," Paris, 1852;

G. Kirsch, V.D.I., vol. 42, 1898

SYMMETRICAL STRESS DISTRIBUTION

The general equations of equilibrium for an element are, in polar co-ordinates (with no body forces).[2]

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \tag{2}$$

For a symmetrical stress distribution, the stresses do not depend on θ and are functions of r only. Also, because of symmetry $\tau_{r\theta}$ must vanish. Hence the second of the above equations disappears, leaving the first as

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{3}$$

The general compatibility equation is[3]

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \tag{4}$$

For a symmetrical stress distribution, this becomes, by eliminating the functions of θ,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) = 0 \tag{5}$$

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(2) Timoshenko, S. Theory of Elasticity. N. Y., McGraw-Hill, 1934. p. 53.

(3) Ibid., p.55

Expanding:

$$\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right)=$$

$$\frac{\partial}{\partial r}\left(\frac{\partial^3\phi}{\partial r^3}+\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r^2}\frac{\partial\phi}{\partial r}\right)=$$

$$\frac{\partial^4\phi}{\partial r^4}+\frac{1}{r}\frac{\partial^3\phi}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r^3}\frac{\partial\phi}{\partial r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right)=$$

$$\frac{1}{r}\frac{\partial^3\phi}{\partial r^3}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r}\left(-\frac{1}{r^2}\right)\frac{\partial\phi}{\partial r}$$

So, equation (5) may be written

$$\frac{\partial^4\phi}{\partial r^4}+\frac{2}{r}\frac{\partial^3\phi}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r^3}\frac{\partial\phi}{\partial r}=0 \qquad (6)$$

This is an ordinary differential equation with variable coefficients. To change the variable coefficients to constant coefficients, let $r = e^t$. Then $\frac{dr}{dt} = e^t$ and $\frac{dt}{dr} = e^{-t}$, and:

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\times\frac{dt}{dr} = e^{-t}\frac{d\phi}{dt} \qquad (7)$$

$$\frac{d^2\phi}{dr^2}=\frac{d}{dr}\left(\frac{d\phi}{dr}\right)=\left[\frac{d}{dt}\left(\frac{d\phi}{dr}\right)\right]\left(\frac{dt}{dr}\right)=\left[\frac{d}{dt}\left(e^{-t}\frac{d\phi}{dt}\right)\right]\left(e^{-t}\right)$$

$$=e^{-t}\frac{d}{dt}\left(e^{-t}\frac{d\phi}{dt}\right)=e^{-t}\left(e^{-t}\frac{d^2\phi}{dt^2}-e^{-t}\frac{d\phi}{dt}\right)$$

$$=e^{-2t}\left(\frac{d^2\phi}{dt^2}-\frac{d\phi}{dt}\right) \qquad (8)$$

$$\frac{d^3\phi}{dr}=\frac{d}{dr}\left(\frac{d^2\phi}{dr^2}\right)=\left[\frac{d}{dt}\left(\frac{d^2\phi}{dr^2}\right)\right]\left(\frac{dt}{dr}\right)=e^{-t}\frac{d}{dt}\left[e^{-2t}\left(\frac{d^2\phi}{dt^2}-\frac{d\phi}{dt}\right)\right]$$

$$=e^{-t}\left[e^{-2t}\left(\frac{d^3\phi}{dt^3}-\frac{d^2\phi}{dt^2}\right)-2e^{-2t}\left(\frac{d^2\phi}{dt^2}-\frac{d\phi}{dt}\right)\right.$$

$$=e^{-3t}\left(\frac{d^3\phi}{dt^3}-3\frac{d^2\phi}{dt^2}+2\frac{d\phi}{dt}\right) \qquad (9)$$

$$\frac{d^4\phi}{dr^4} = \frac{d}{dr}\left(\frac{d^3\phi}{dr}\right) = \left[\frac{d}{dt}\left(\frac{d^3\phi}{dr^3}\right)\right]\frac{dt}{dr}$$

$$= e^{-t}\frac{d}{dt}\left[e^{-3t}\left(\frac{d^3\phi}{dt^3} - 3\frac{d^2\phi}{dt^2} + 2\frac{d\phi}{dt}\right)\right]$$

$$= e^{-t}\left[e^{-3t}\left(\frac{d^4\phi}{dt^4} - 3\frac{d^3\phi}{dt^3} + 2\frac{d^2\phi}{dt^2}\right) - 3e^{-3t}\left(\frac{d^3\phi}{dt^3} - 3\frac{d^2\phi}{dt^2} + 2\frac{d\phi}{dt}\right)\right]$$

$$= e^{-4t}\left(\frac{d^4\phi}{dt^4} - 6\frac{d^3\phi}{dt^3} + 11\frac{d^2\phi}{dt^2} - 6\frac{d\phi}{dt}\right) \qquad (10)$$

Writing equation (6) as an ordinary differential equation;

$$\frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0$$

Substituting values from (7), (8), (9), and (10), and also e^{-t} for $\frac{1}{r}$,

$$e^{-4t}\left(\frac{d^4\phi}{dt^4} - 6\frac{d^3\phi}{dt^3} + 11\frac{d^2\phi}{dt^2} - 6\frac{d\phi}{dt}\right)$$

$$+ e^{-4t}\left(2\frac{d^3\phi}{dt^3} - 6\frac{d^2\phi}{dt^2} + 4\frac{d\phi}{dt}\right)$$

$$+ e^{-4t}\left(-\frac{d^2\phi}{dt^2} + \frac{d\phi}{dt}\right)$$

$$+ e^{-4t}\left(\frac{d\phi}{dt}\right) = 0$$

$$e^{-4t}\left(\frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2}\right) = 0$$

Since e^{-4t} cannot equal zero for all values of t,

$$\frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0$$

Now let $\phi = e^{mt}$

Then $\frac{d\phi}{dt} = me^{mt}$, $\frac{d^2\phi}{dt^2} = m^2e^{mt}$, $\frac{d^3\phi}{dt^3} = m^3e^{mt}$,

$$\frac{d^4\phi}{dt^4} = m^4e^{mt}$$

Substituting,

$$e^{mt}(\ m^4 - 4m^3 + 4m^2) = 0$$

Factoring,

$$e^{mt}(\ m^2)\ (m-2)\ (m-2) = 0$$

which gives the solutions, $m_1 = 0$, $m_2 = 0$, $m_3 = 2$, $m_4 = 2$

Since $\phi = Ae^{m_1 t} + Bte^{m_2 t} + Ce^{m_3 t} + Dte^{m_4 t}$

$$\phi = A + Bt + Ce^{2t} + Dte^{2t}$$

Now, $e^t = r$; therefore $t = \log_e r$. So a general solution is

$$\phi = A + B\log_e r + Cr^2 + Dr^2\log_e r \tag{11}$$

The stresses in terms of this stress function, ϕ, are[4];

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} \\ \tau_{r\theta} &= \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} \end{aligned}\right\} \tag{12}$$

These are general equations; in the case of a symmetrical stress distribution, functions of θ vanish (ref. p. 4), and the equations become

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r} \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \tag{13}$$

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(4) Ibid., p.53.

Then, from (11)

$$\sigma_r = \frac{B}{r^2} + 2C + D(2 \log_e r + 1) \tag{14}$$

$$\sigma_\theta = -\frac{B}{r^2} + 2C + D(3 + 2 \log_e r) \tag{15}$$

$\tau_{r\theta} = 0$ (from (13))

For the case when no hole exists at the origin (i.e., r = 0), σ_r and σ_θ are infinite unless B and D are taken as zero. Since infinite stresses are not possible, this must be the only correct solution, and σ_r and σ_θ are seen to be constant.

When there is a hole of radius a at the origin and the outer radius is b , the solution is found by taking D = 0. This case can be shown to correspond to a hollow cylinder subjected to an internal pressure, p_i, and an external pressure, p_o .

If D = 0,

$$\sigma_r = \frac{B}{r^2} + 2C$$
$$\sigma_\theta = -\frac{B}{r^2} + 2C \tag{16}$$

and the boundary conditions are ;

$\sigma_r = -p_i$ when $r = a$

$\sigma_r = -p_o$ when $r = b$

Then

$$\frac{B}{a^2} + 2C = -p_i$$

$$\frac{B}{b^2} + 2C = -p_o$$

$$\frac{B}{a^2} - \frac{B}{b^2} = -p_i + p_o$$

$$(b^2 - a^2)B = a^2b^2(p_o - p_i)$$

$$B = \frac{(p_o - p_i)a^2b^2}{b^2 - a^2}$$

$$2C = -p_i - \frac{(p_o - p_i)a^2b^2}{a^2(b^2 - a^2)}$$

$$2C = \frac{-p_i b^2 + p_i a^2}{b^2 - a^2} - \frac{p_o b^2 - p_i b^2}{b^2 - a^2}$$

$$2C = \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

$$\sigma_r = \frac{(p_o - p_i)a^2b^2}{r^2(b^2 - a^2)} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \tag{17}$$

$$\sigma_\theta = \frac{-(p_o - p_i)a^2b^2}{r^2(b^2 - a^2)} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \tag{18}$$

For the case of $p_o = 0$.

$$\sigma_r = \frac{-a^2b^2p_i}{r^2(b^2 - a^2)} + \frac{a^2p_i}{b^2 - a^2}$$

$$\sigma_r = \frac{p_i a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \tag{19}$$

$$\sigma_\theta = \frac{p_i a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \tag{20}$$

The maximum shearing stress is also of interest. Since $\tau_{r\theta} = 0$, σ_r and σ_θ are principal stresses and since the maximum shearing stress is half the difference between the principal stresses[5],

$$\tau_{max.} = \frac{\sigma_r - \sigma_\theta}{2}$$

$$\tau_{max} = \frac{p_i a^2}{2(b^2 - a^2)}\left(1 - \frac{b^2}{r^2} - 1 - \frac{b^2}{r^2}\right)$$

$$\tau_{max.} = \pm \frac{b^2 a^2 p_i}{r^2(b^2 - a^2)} \tag{21}$$

If b is made very large in comparison with a , the stresses approach the values

$$\sigma_r = -\frac{p_i a^2}{r^2}$$

$$\sigma_\theta = +\frac{p_i a^2}{r^2} \tag{22}$$

$$\tau_{max.} = \pm \frac{p_i a^2}{r^2}$$

These stresses are plotted against r/a in Fig. 3 which shows that each stress —σ_r, σ_θ, and τ — has a maximum value equal to the internal pressure.

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(5) Ibid., p. 20.

For the case of $p_i = 0$,

$$\sigma_r = \frac{a^2 b^2 p_o}{r^2(b^2-a^2)} - \frac{p_o b^2}{b^2-a^2}$$

$$\sigma_r = \frac{p_o b^2}{b^2-a^2}\left(\frac{a^2}{r^2}-1\right) \quad (23)$$

$$\sigma_\theta = \frac{-p_o b^2}{b^2-a^2}\left(\frac{a^2}{r^2}+1\right) \quad (24)$$

$$\tau_{max.} = \frac{p_o b^2}{2(b^2-a^2)}\left(\frac{a^2}{r^2}-1+\frac{a^2}{r^2}+1\right)$$

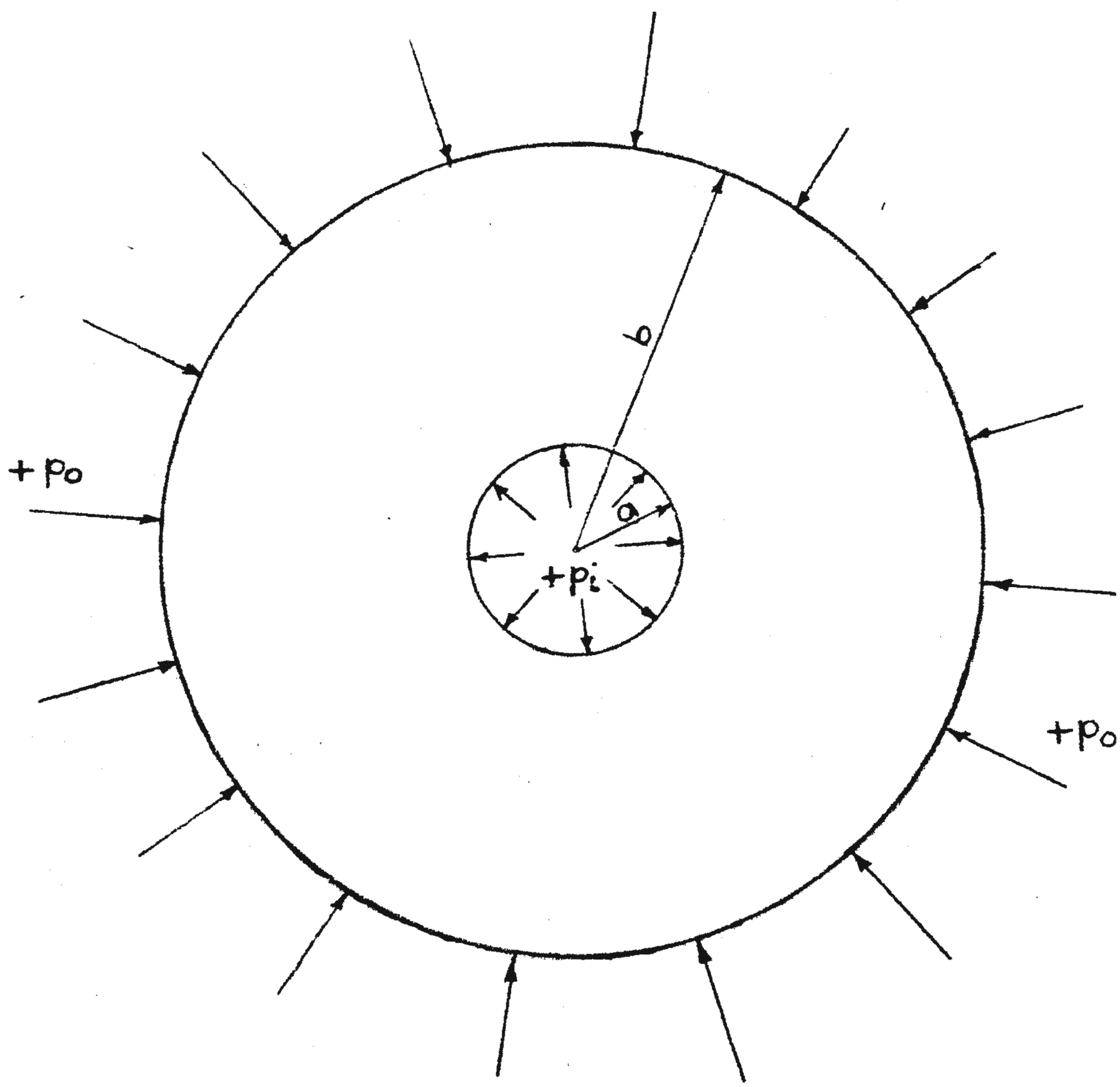
$$\tau_{max.} = \pm\frac{a^2 b^2 p_o}{r^2(b^2-a^2)} \quad (25)$$

If b is made very large in comparison with a, the stresses approach the values

$$\sigma_r = p_o\left(\frac{a^2}{r^2}-1\right)$$
$$\sigma_\theta = -p_o\left(\frac{a^2}{r^2}+1\right) \quad (26)$$
$$\tau_{max.} = \pm\frac{p_o a^2}{r^2}$$

These stresses are plotted against r/a in Fig. 4.

Note that the sum, $\sigma_r + \sigma_\theta$, (equations 17 and 18 and following) is constant through the wall of the cylinder. Therefore there is a uniform extension or contraction along the axis of the cylinder, cross-sections perpendicular to this axis remain plane, and the deformation of a thin disc cut out by planes perpendicular to the axis of the cylinder is not affected by the deformation of adjacent elements. Hence it is justifiable to consider the element in a state of plane stress, as was done in the above discussion. This fact will be seen later to be important in the analysis of stresses due to vertical loads.



Stress sign convention: $+\sigma$ = tension

$-\sigma$ = compression

FIG. 2

STRESS VARIATION IN A HOLLOW CYLINDER AS A FUNCTION OF THE DISTANCE FROM THE CENTER WHEN THE EXTERNAL PRESSURE IS ZERO, AND THE EXTERNAL RADIUS IS LARGE IN COMPARISON WITH THE INNER RADIUS

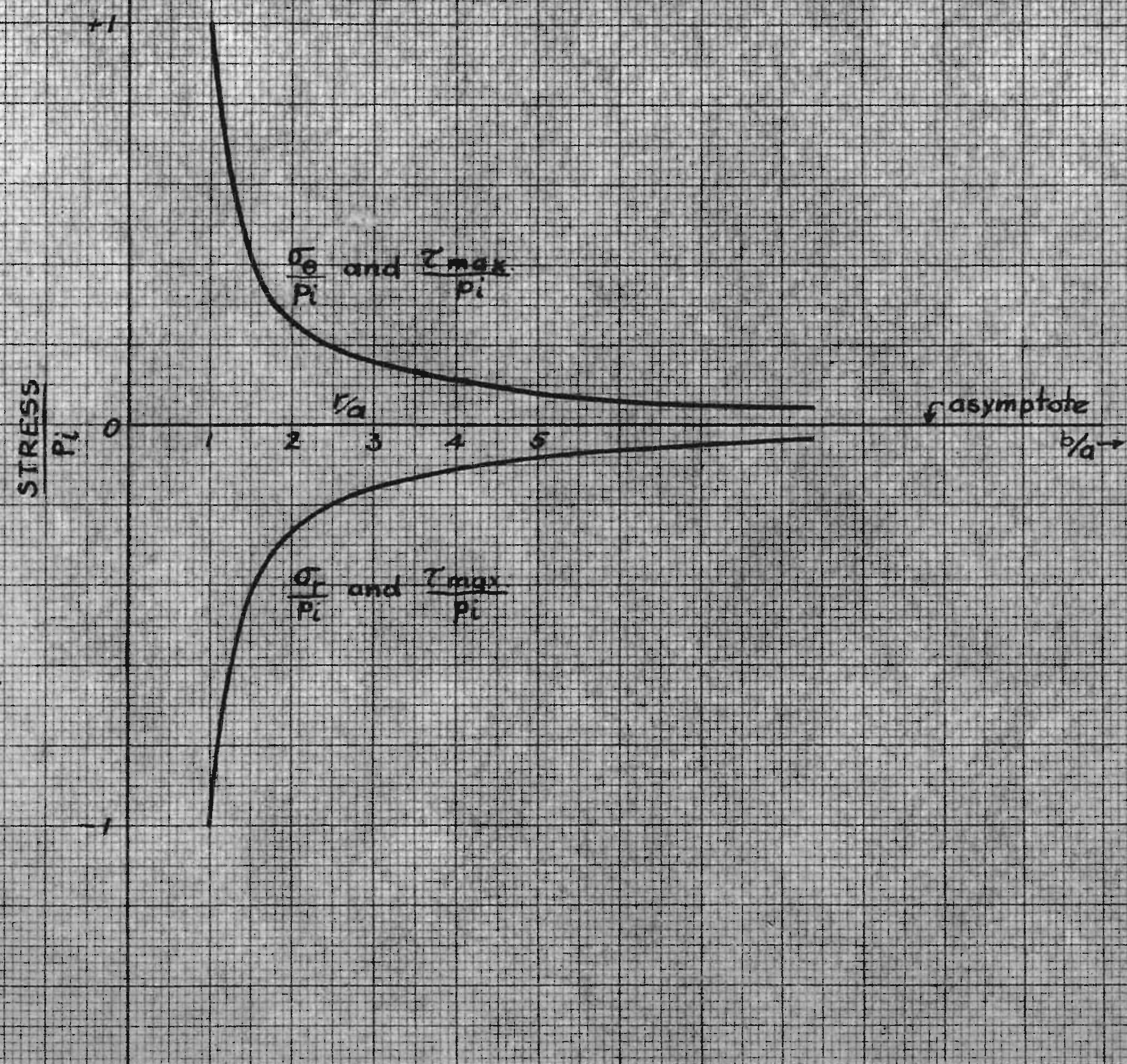


FIG. #3

STRESS VARIATION IN A HOLLOW CYLINDER AS A FUNCTION OF THE DISTANCE FROM THE CENTER WHEN THE INTERNAL PRESSURE IS ZERO, AND THE EXTERNAL RADIUS IS LARGE IN COMPARISON WITH THE INNER RADIUS

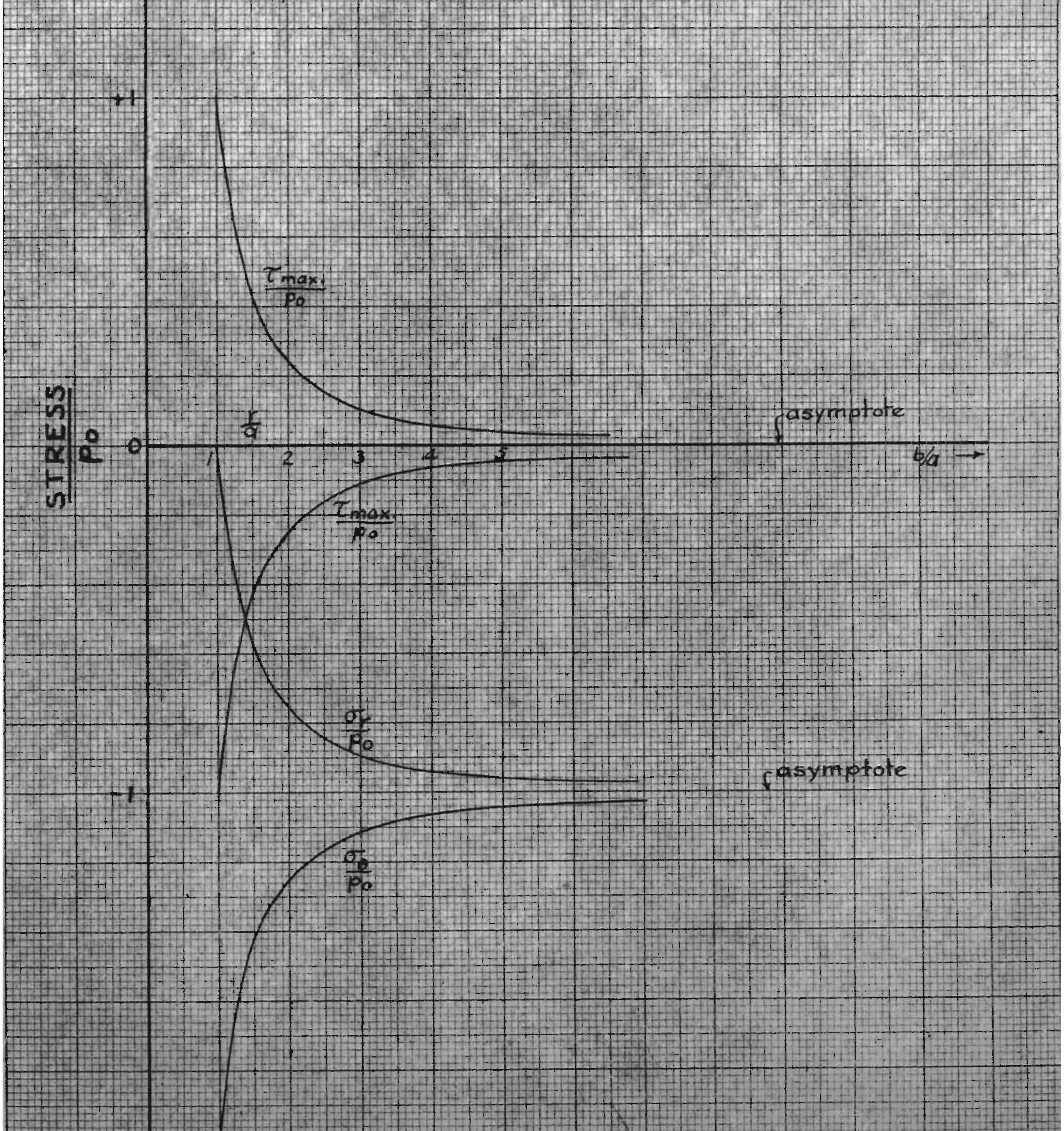


FIG. 4

UNIFORM STRESS IN ONE DIRECTION

Consider a plate subjected in one direction to a uniform tensile stress, S , and having a small hole of radius a . Stresses at radius b (b being large in comparison with a) are unaffected by the hole,

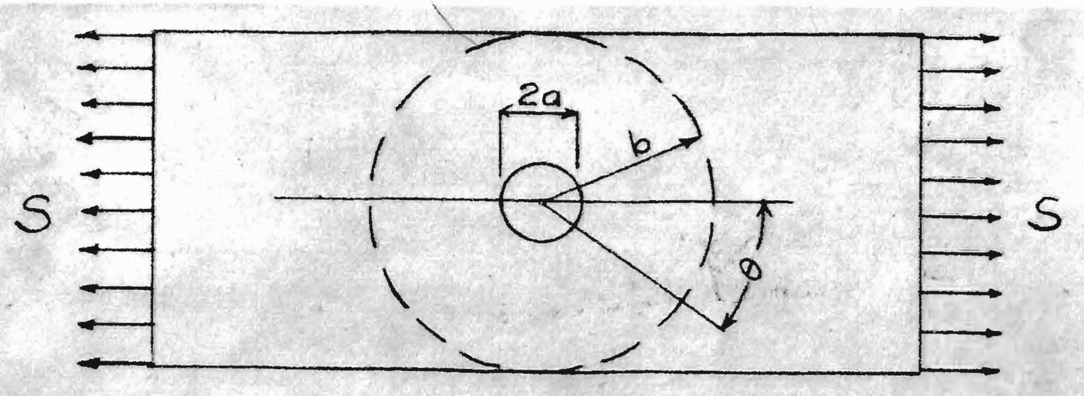


FIG. 5

from the principle of St. Venant[6]. Then

$$\left.\begin{aligned} \sigma_{r(r=b)} &= S\cos^2\theta = \tfrac{1}{2}S(1+\cos 2\theta) \\ \tau_{r\theta(r=b)} &= -S\sin\theta\cos\theta = -\tfrac{1}{2}S\sin 2\theta \end{aligned}\right\} \qquad (27)$$

Equation (4) is the general compatibility equation.

Expanding,

$$\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)$$

$$= \frac{\partial^3\phi}{\partial r^3} + \frac{1}{r}\frac{\partial^2\phi}{\partial r^2} + \left(-\frac{1}{r^2}\right)\frac{\partial\phi}{\partial r} + \left(\frac{-2}{r^3}\right)\left(\frac{\partial^2\phi}{\partial\theta^2}\right) + \frac{1}{r^2}\frac{\partial^3\phi}{\partial\theta^2\partial r}$$

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(6) Ibid., p. 31

$$\frac{\partial^2}{\partial r^2}\left(\quad\right) = \frac{\partial^4\phi}{\partial r^4} - r^{2}\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial^3\phi}{\partial r^3} + \frac{2}{r^3}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}$$

$$- \frac{2}{r^3}\frac{\partial^3\phi}{\partial\theta^2\partial r} + \frac{6}{r^4}\frac{\partial^2\phi}{\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3\phi}{\partial\theta^2\partial r} + \frac{1}{r^2}\frac{\partial^4\phi}{\partial\theta^2\partial r^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\quad\right) = \frac{1}{r}\frac{\partial^3\phi}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r^3}\frac{\partial\phi}{\partial r} - \frac{2}{r^4}\frac{\partial^2\phi}{\partial\theta^2}$$

$$+ \frac{1}{r^3}\frac{\partial^3\phi}{\partial\theta^2\partial r}$$

$$\frac{\partial}{\partial\theta}\left(\quad\right) = \frac{\partial^3\phi}{\partial r^2\partial\theta} + \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial^3\phi}{\partial\theta^3}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\quad\right) = \frac{1}{r^2}\frac{\partial^4\phi}{\partial r^2\partial\theta^2} + \frac{1}{r}\frac{\partial^3\phi}{\partial r\partial\theta^2} + \frac{1}{r^2}\frac{\partial^4\phi}{\partial\theta^4}$$

Collecting terms.

$$\frac{\partial^2}{\partial r^2}\left(\quad\right) = \frac{\partial^4\phi}{\partial r^4} + r^{-1}\frac{\partial^3\phi}{\partial r^3} - 2r^{-2}\frac{\partial^2\phi}{\partial r^2} + 2r^{-3}\frac{\partial\phi}{\partial r} - 4r^{3}\frac{\partial^3\phi}{\partial r\partial\theta^2}$$

$$+ r^{-2}\frac{\partial^4\phi}{\partial\theta^2\partial r^2} + 6r^{-4}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\quad\right) = r^{-1}\frac{\partial^3\phi}{\partial r^3} + r^{-2}\frac{\partial^2\phi}{\partial r^2} - r^{-3}\frac{\partial\phi}{\partial r} + r^{-3}\frac{\partial^3\phi}{\partial r\partial\theta^2}$$

$$- 2r^{-4}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\quad\right) = r^{-3}\frac{\partial^3\phi}{\partial r\partial\theta^2} + r^{-2}\frac{\partial^4\phi}{\partial\theta^2\partial r^2} + r^{-4}\frac{\partial^4\phi}{\partial\theta^4}$$

Adding.

$$\frac{\partial^4\phi}{\partial r^4} + 2r^{-1}\frac{\partial^3\phi}{\partial r^3} - r^{-2}\frac{\partial^2\phi}{\partial r^2} + r^{-3}\frac{\partial\phi}{\partial r} - 2r^{-3}\frac{\partial^3\phi}{\partial r\partial\theta^2} + 2r^{-2}\frac{\partial^4\phi}{\partial r^2\partial\theta^2}$$

$$+ 4r^{-4}\frac{\partial^2\phi}{\partial\theta^2} + r^{-4}\frac{\partial^4\phi}{\partial\theta^4} = 0$$

which is the compatibility equation expanded.

Now, the stresses at radius b given in equation (27) may be considered as forces acting externally on a ring of external radius b and internal radius a. They produce stresses within the ring which may be regarded as consisting of two parts. The first is due to the constant component, $\frac{1}{2}S$, of the normal forces, and can be obtained from equations (17) and (18). The other part is due to the remaining component of the normal forces, $\frac{1}{2}S\cos 2\theta$, and to the shearing force, $-\frac{1}{2}S\sin 2\theta$. The stresses due to these latter components may be found by using a stress function of the form

$$\phi = f(r)\cos 2\theta \tag{29}$$

Substituting this in the compatibility equation (28) (and writing f for f(r)):

$$\cos 2\theta\Big(\frac{\partial^4 f}{\partial r^4} + 2r^{-1}\frac{\partial^3 f}{\partial r^3} - r^{-2}\frac{\partial^2 f}{\partial r^2} + r^{-3}\frac{\partial f}{\partial r} + 8r^{-3}\frac{\partial f}{\partial r}$$
$$- 8r^{-2}\frac{\partial^4 f}{\partial r^4} - 16r^{-4} + 16r^{-4}\Big) = 0$$

$$\frac{\partial^4 f}{\partial r^4} + \frac{2}{r}\frac{\partial^3 f}{\partial r^3} - \frac{9}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{9}{r^3}\frac{\partial f}{\partial r} = 0 \tag{30}$$

This is an ordinary differential equation with variable coefficients which can be converted to constants by substituting e^t for r as was done for equation (6). The various derivatives correspond to equations

(7), (8), (9), and (10). Substituting in (30) and collecting terms,

$$e^{-4t}\left(\frac{d^4f}{dt^4} - 4\frac{d^3f}{dt^3} - 4\frac{d^2f}{dt^2} + 16\frac{df}{dt}\right) = 0$$

Since e^{-4t} cannot equal zero for all values of t,

$$\frac{d^4f}{dt^4} - 4\frac{d^3f}{dt^3} - 4\frac{d^2f}{dt^2} + 16\frac{df}{dt} = 0$$

Let $f = e^{mt}$. Then

$$\frac{df}{dt} = me^{mt}, \qquad \frac{d^2f}{dt^2} = m^2e^{mt},$$

$$\frac{d^3f}{dt^3} = m^3e^{mt}, \qquad \frac{d^4f}{dt^4} = m^4e^{mt}$$

$$e^{mt}(m^4 - 4m^3 - 4m^2 + 16m) = 0$$

$$me^{mt}(m-4)(m+2)(m-2) = 0$$

$$m_1 = 0, \quad m_2 = +4, \quad m_3 = -2, \quad m_4 = +2$$

Since $f = Ae^{mt} + Be^{m_2t} + Ce^{m_3t} + De^{m_4t}$

$$f = A + Be^{4t} + Ce^{-2t} + De^{2t}$$

and since $e^t = r$,

$$f = A + Br^4 + Cr^{-2} + Dr^2$$

$$\phi = (A + Br^4 + Cr^{-2} + Dr^2)\cos 2\theta \qquad (31)$$

Substituting this expression for ϕ in equations(12), the stresses become

$$\sigma_r = \frac{1}{r}(4Br^3 - 2Cr^{-3} + 2Dr)\cos 2\theta + \frac{1}{r^2}(-4\cos 2\theta)(A + Br^4 + Cr^{-2} + Dr^2)$$

$$\sigma_r = -\cos 2\theta(4Ar^{-2} + 6Cr^{-4} + 2D) \qquad (32)$$

$$\sigma_\theta = \left(12Br^2 + 6Cr^{-4} + 2D\right)(\cos 2\theta) \tag{33}$$

$$\tau_{r\theta} = \frac{1}{r^2}(-2\sin 2\theta)\left(A + Br^4 + Cr^{-2} + Dr^2\right) - \frac{1}{r}(-2\sin 2\theta)\left(4Br^3 - 2Cr^{-3} + 2Dr\right)$$

$$\tau_{r\theta} = -\sin 2\theta\left(2Ar^{-2} - 6Br^2 + 6Cr^{-4} - 2D\right) \tag{34}$$

The boundary conditions are;

$\sigma_r =$ 0 when $r = a$

$\tau_{r\theta} =$ 0 when $r = a$

$\sigma_r =$ $\frac{1}{2}S\cos 2\theta$ when $r = b$

$\tau_{r\theta} =$ $\frac{1}{2}S\sin 2\theta$ when $r = b$

$$\sigma_{r(r=a)} = 0 = -\cos 2\theta\left(4Aa^{-2} + 6Ca^{-4} + 2D\right)$$

$$4Aa^{-2} + 6Ca^{-4} + 2D = 0$$

$$\tau_{r\theta(r=a)} = 0 = 2Aa^{-2} - 6Ba^2 + 6Ca^{-4} - 2D = 0$$

$$\sigma_{r(r=b)} = \frac{S}{2}\cos 2\theta = -\cos 2\theta\left(4Ab^{-2} + 6Cb^{-4} + 2D\right)$$

$$-\frac{S}{2} = 4Ab^{-2} + 6Cb^{-4} + 2D$$

$$\tau_{r\theta(r=b)} = \frac{S}{2}\sin 2\theta = -\sin 2\theta\left(2Ab^{-2} - 6Bb^2 + 6Cb^{-4} - 2D\right)$$

$$-\frac{S}{2} = 2Ab^{-2} - 6Bb^2 + 6Cb^{-4} - 2D$$

Clearing of fractions. the equations to be solved simultaneously are

$$2a^2A \qquad + 3C + a^4D = 0$$
$$a^2A - 3a^6B + 3C - a^4D = 0$$
$$2b^2A \qquad + 3C + b^4D = \frac{-Sb^4}{4}$$
$$b^2A - 3b^6B + 3C - b^4D = \frac{-Sb^4}{4}$$

The solution of these equations gives

$$A = \frac{a^2S}{2}$$
$$B = 0$$
$$C = \frac{-a^4S}{4}$$
$$D = \frac{-S}{4}$$

Substituting these values in equations (32), (33), and (34).

$$\left.\begin{aligned} \sigma_r &= S\cos 2\theta\left(\frac{1}{2} - 2\frac{a^2}{r^2} + \frac{3a^4}{r^4}\right) \\ \sigma_\theta &= S\cos 2\theta\left(-\frac{1}{2} - \frac{3a^4}{2r^4}\right) \\ \tau_{r\theta} &= -S\sin 2\theta\left(\frac{a^2}{r^2} - \frac{3a^4}{r^4} + \frac{1}{2}\right) \end{aligned}\right\} \qquad (35)$$

The stresses due to the constant component of the normal forces when p_i is zero and b is large compared to a can be found from equations (26).

$$\left.\begin{aligned} \sigma_r &= \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) \\ \sigma_\theta &= \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \qquad (36)$$

Total stresses equal (35) plus (36).

$$\left.\begin{aligned}
\sigma_r &= \frac{S}{2}\left[\left(1-\frac{a^2}{r^2}\right)+\cos 2\theta\left(1-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\right]\\
\sigma_\theta &= \frac{S}{2}\left[\left(1+\frac{a^2}{r^2}\right)-\cos 2\theta\left(1+3\frac{a^4}{r^4}\right)\right]\\
\tau_{r\theta} &= -\frac{S}{2}\left(1+2\frac{a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta
\end{aligned}\right\} \quad (37)$$

At the edge of the hole, r a, which gives

$$\left.\begin{aligned}
\sigma_r &= 0\\
\sigma_\theta &= S-2S\cos 2\theta\\
\tau_{r\theta} &= 0
\end{aligned}\right\} \quad (38)$$

The tangential stress, σ_θ, may be seen to be a maximum when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, which gives $\sigma_\theta = 3S$. This stress occurs on a line through the center of the hole perpendicular to the direction of the stress S. This line is an axis of symmetry, which makes $\tau_{r\theta} = 0$ and σ_r and σ_θ principal stresses. The general equations for stress along this line are

$$\left.\begin{aligned}
\sigma_r &= \frac{3S}{2}\left(\frac{a^2}{r^2}-\frac{a^4}{r^4}\right)\\
\sigma_\theta &= \frac{S}{2}\left(2+\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\\
\tau_{r\theta} &= 0
\end{aligned}\right\} \quad (39)$$

It should also be noted that when $\theta = 0$ or π, $\sigma_\theta = -S$, which indicates a compressive stress at right angles to the direction of the original stress and equal to it in magnitude. If the initial stress were compressive, this stress would be tensile. The general equations for the stresses on the line $\theta = 0$ or π are:

$$\left.\begin{aligned} \sigma_r &= \frac{S}{2}\left(2 + 3\frac{a^4}{r^4} - 5\frac{a^2}{r^2}\right) \\ \sigma_\theta &= \frac{S}{2}\left(\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right) \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \quad (40)$$

The maximum shearing stress is also of interest. Fig. 6 shows that the difference between σ_r and σ_θ is greatest when $\theta = \frac{\pi}{2}$. Since σ_r and σ_θ are the principal stresses,

$$\tau_{max.} = \frac{\sigma_r - \sigma_\theta}{2} = -\frac{S}{2}\left(1 - \frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right) \quad (41)$$

from which $\tau_{max.} = -1.5S$ at the edge of the hole.

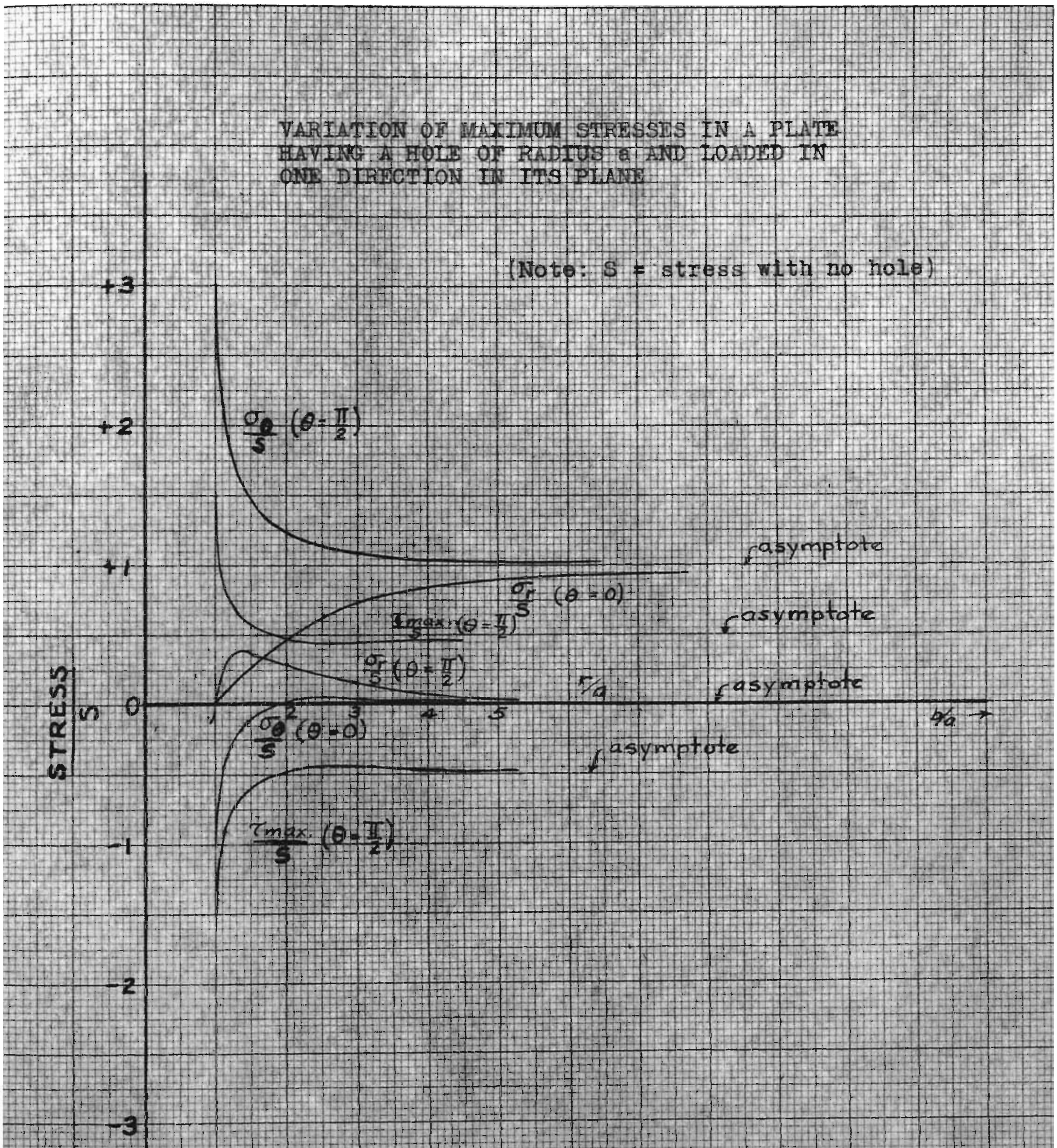
VARIATION OF MAXIMUM STRESSES IN A PLATE HAVING A HOLE OF RADIUS a AND LOADED IN ONE DIRECTION IN ITS PLANE
(Note: S = stress with no hole)
+3
+2
+1
0
-1
-2
-3
STRESS / S
$\frac{\sigma_\theta}{S}\ (\theta = \frac{\pi}{2})$
asymptote
$\frac{\sigma_r}{S}\ (\theta = 0)$
asymptote
$\frac{\tau_{max.}}{S}\ (\theta = \frac{\pi}{2})$
$\frac{\sigma_r}{S}\ (\theta = \frac{\pi}{2})$
r/a
asymptote
1
2
3
4
5
b/a
$\frac{\sigma_\theta}{S}\ (\theta = 0)$
asymptote
$\frac{\tau_{max.}}{S}\ (\theta = \frac{\pi}{2})$

FIG. 6

PART II

The effect of uniform loading perpendicular to the plane of the plate will now be investigated.

SYMMETRICAL STRESS DISTRIBUTION

In Part I, it was shown from equations 17 and 18 that the stresses σ_r and σ_θ produce uniform expansion parallel to the Z axis through the wall of the cylinder. Therefore a stress σ_z may be simply superposed without affecting the stresses σ_r and σ_θ, or being affected by them, where there is no constraint.

Consider now an isotropic plate very large in comparison with the size of the hole, and constrained at its boundaries from expansion or contraction in the plane perpendicular to the axis of the hole. According to the principle of St. Venant, the hole will affect the stresses in the plate only in the immediate vicinity of the hole. Then the stresses in the plate (except near the hole) due to a constant stress, σ_z, will be

$$\left.\begin{aligned} \sigma_r &= \nu\sigma_z + \nu\sigma_\theta \\ \sigma_\theta &= \nu\sigma_z + \nu\sigma_r \end{aligned}\right\} \qquad (42)$$

Getting σ_r and σ_θ in terms of σ_z :

$$\frac{\sigma_r}{\nu} = \sigma_z + \nu(\sigma_z + \sigma_r)$$

$$\sigma_r\left(\frac{1}{\nu} - \nu\right) = \sigma_z\,(1+\nu)$$

$$\sigma_r = \left(\frac{\nu^2+\nu}{1-\nu^2}\right)\sigma_z$$

$$\sigma_r = \frac{\nu}{1-\nu}\,\sigma_z = \sigma_\theta \tag{43}$$

These will be the stresses when $r = b$. For the cylindrical portion of the plate around the hole, this value of σ_r is equivalent to p_o in equations 23, 24, 25, and 26, except for sign.

$$p_o = -\frac{\nu}{1-\nu}\,\sigma_z \tag{44}$$

Substituting this value of p_o in equations 26, the stresses at the edge of the hole become

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= +2\sigma_z\left(\frac{\nu}{1-\nu}\right) \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \tag{45}$$

These are, of course, principal stresses. There are three maximum shearing stresses, occurring on planes bisecting the angle between each pair of principal stresses(7). The magnitude of each is equal to half

(7) Ibid. p. 187-188

the difference of the corresponding principal stresses.

Hence

$$\left.\begin{aligned} \tau &= \frac{\sigma_r - \sigma_\theta}{2} = \sigma_z\left(\frac{\nu}{1-\nu}\right) \\ \tau &= \frac{\sigma_z - \sigma_r}{2} = \frac{\sigma_z}{2} \\ \tau &= \frac{\sigma_z - \sigma_\theta}{2} = \frac{1}{2}\left(\sigma_z - \frac{2\nu}{1-\nu}\sigma_z\right) = \frac{1-3\nu}{2(1-\nu)}\sigma_z \end{aligned}\right\} \quad (46)$$

The largest of these is the maximum shearing stress at the hole. It may be seen that this will always be the difference between the largest and smallest principal stresses.

INITIAL UNIFORM STRESS IN ONE DIRECTION

PLANE STRESS

Where there is no constraint in any direction, a stress in the vertical direction may be simply superposed without affecting the stresses which may be calculated from equations (35).

If the plate is considered as constrained in the direction of the uniform stress S, S may be considered to be equal to $S_1 + \nu\sigma_z$ (S_1 being the uniform stress in one direction at r = b when σ_z is zero), and the case equivalent that above.

The stresses when r = a and $\theta = \frac{\pi}{2}$ are then (from equations 38):

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= 3S = 3(S_1 + \nu\sigma_{z_1}) \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \qquad (47)$$

When r = a and $\theta = 0$ or π,

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= -S = -(S_1 + \nu\sigma_{z_1}) \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \qquad (48)$$

The maximum shearing stress can always be calculated by taking half the difference of the largest and smallest principal stresses, which at the edge of the hole, are always (in this and all subsequent cases) σ_r , σ_θ , and σ_z .

If there is constraint in the horizontal plane perpendicular to the direction of S, the lateral stress at $r = b$ cannot be considered unidirectional — at least, not in the direction of the original stress, S_1 . The stresses when $r = b$ can be said to be

$$\left.\begin{aligned} \sigma_x &= S_1 \\ \sigma_y &= \nu(S_1 + \sigma_{z_1}) \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \tag{49}$$

where S_1 is considered to be acting along an X axis. The effects of σ_x , σ_y , and σ_z may now be simply superposed to get the stresses at the edge of the hole.

For $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$,

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= 3S_1 - \nu(S_1 + \sigma_{z_1}) \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \tag{50}$$

For $\theta = 0$ or π,

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= -S_1 + 3\sigma_y = (3\nu - 1)S_1 + 3\nu\sigma_{z_1} \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \tag{51}$$

The case of constraint in both horizontal directions may be solved by superposing the effects of the stress S_1 and the vertical stress, σ_z.

Considering the effect of S_1 along at $r = b$.

$$\left.\begin{aligned} \sigma_y &= \nu\sigma_x = \frac{\nu}{1-\nu^2} S_1 \\ \sigma_x &= S_1 + \nu\sigma_y = S_1 + \nu^2\sigma_x = \frac{S_1}{1-\nu^2} \end{aligned}\right\} \tag{52}$$

Then the stresses at the edge of the hole resulting from S_1 alone are.

for $\theta = \frac{\pi}{2}$.

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{3-\nu}{1-\nu^2} S_1 \end{aligned}\right\} \tag{53}$$

and for $\theta = 0$.

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{3\nu-1}{1-\nu^2} S_1 \end{aligned}\right\} \tag{54}$$

Superposing upon this the effects of the vertical stress, which for symmetrical lateral constraint are given by equations (45).

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2\nu}{1-\nu}\sigma_{z_1} + \frac{3-\nu}{1-\nu^2} S_1 \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \tag{55}$$

for $\theta = \frac{\pi}{2}$. For $\theta = 0$.

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2\nu}{1-\nu}\sigma_{z_1} + \frac{3\nu-1}{1-\nu^2} S_1 \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \tag{56}$$

PLANE STRAIN

Plane strain, of course, presumes full constraint in the vertical direction. In the case of symmetrical stress distribution, the results are no different from those obtained for plane stress since, as previously shown, any extension or contraction along the Z axis is uniform. This is not true in the case of a uniform stress in one direction. It can be seen, however, that the vertical stress is not affected by the lateral stresses except near the hole. That is, if σ_{z_1} is the vertical stress in the case of plane stress, it will also be σ_z when $r = b$, regardless of the conditions of lateral and vertical constraint; but at the edge of the hole, where $r = a$

$$\sigma_z = \sigma_{z_1} + \nu(\sigma_r + \sigma_\theta)$$

or, since $\sigma_r = 0$,

$$\sigma_z = \sigma_{z_1} + \nu\sigma_\theta \qquad (57)$$

The case of plane stress placed no restriction on the vertical expansion and contraction of the plate. If σ_{θ_1} is the tangential stress at the hole for the case of plane stress, then, for the case of plane strain

$$\sigma_\theta = \sigma_{\theta_1} + f(\sigma_z) \qquad (58)$$

In order to determine $f(\sigma_z)$, consider first the problem of stress in a body elastically constrained laterally. If there is <u>no</u>

lateral constraint, there is no lateral stress due to a vertical load, and the unit lateral deformation is

$$\epsilon_x = \frac{\nu \sigma_x}{E}$$

If there is full constraint, the stress is $\nu\sigma_x$ and the deformation is zero. The case of elastic constraint lies between these two cases.

The deformation of the constraining medium must equal that of the body. Let E_1 = the modulus of elasticity of the body material and E_2 = the modulus of elasticity of the constraining medium. Let the body be some unit volume surrounded on all sides by the constraining medium and loaded vertically. Equating deformations in the X direction,

$$\frac{\nu\sigma_z + \nu\sigma_y - \sigma_x}{E_1} = \frac{\sigma_x}{E_2} \qquad (59)$$

Equating them in the Y direction,

$$\frac{\nu\sigma_z + \nu\sigma_x - \sigma_y}{E_1} = \frac{\sigma_y}{E_2} \qquad (60)$$

Now consider as the body an element at the edge of a hole in a large plate. For the case of a uniform stress in one direction and plane strain, if the ratio a/b is very small, it may be considered elastically supported tangent to the hole and free to expand or contract in the radial direction, regardless of the constraint conditions at the outer boundary of the plate.

To apply equations (59) and (60) to this case, let

$$\sigma_x = \sigma_\theta$$

$$\sigma_y = \sigma_r = 0$$

$$E_1 = E_2$$

Then

$$\frac{\nu \sigma_z - \sigma_\theta}{E} = \frac{\sigma_\theta}{E}$$

$$\sigma_\theta = \frac{\nu}{2} \sigma_z$$

Thus

$$f(\sigma_z) = \frac{\nu}{2} \sigma_z$$

Substituting this in equation (58),

$$\sigma_\theta = \sigma_{\theta_1} + \frac{\nu}{2} \sigma_z \qquad (61)$$

Getting σ_z and σ_θ in terms of σ_{z_1} and σ_{θ_1} .

$$\sigma_\theta = \sigma_{\theta_1} + \frac{\nu}{2}\left(\sigma_{z_1} + \nu\sigma_\theta\right)$$

$$\sigma_\theta = \frac{2\sigma_{\theta_1} + \nu\sigma_{z_1}}{2-\nu^2} \tag{62}$$

$$\sigma_z = \sigma_{z_1} + \frac{\nu\left(2\sigma_{\theta_1} + \nu\sigma_{z_1}\right)}{2-\nu^2}$$

$$\sigma_z = \frac{2\left(\sigma_{z_1} + \nu\sigma_{\theta_1}\right)}{2-\nu^2} \tag{63}$$

By substituting the appropriate value for σ_{θ_1} from the results previously derived for the case of plane stress, the stresses at the hole may be obtained.

When there is no lateral constraint, the stresses at the edge of the hole for $\theta = \frac{\pi}{2}$ are

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{6S_1 + \nu\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2\left(\sigma_{z_1} + 3\nu S_1\right)}{2-\nu^2} \end{aligned}\right\} \tag{64}$$

and for $\theta = 0$ or π .

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{-2S_1 + \nu\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2\left(\sigma_{z_1} - \nu S_1\right)}{2-\nu^2} \end{aligned}\right\} \tag{65}$$

When there is constraint in the direction of S, the stresses at the edge of the hole are, for $\theta = \frac{\pi}{2}$ (refer equations 47),

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{6S_1 + 7\nu\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{6\nu S_1 + 2(1+3\nu^2)\sigma_{z_1}}{2-\nu^2} \end{aligned}\right\} \qquad (66)$$

and for $\theta = 0$ (ref. equations 48),

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{-2S_1 - \nu\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2(1-\nu^2)\sigma_{z_1} - 2\nu S_1}{2-\nu^2} \end{aligned}\right\} \qquad (67)$$

When there is constraint perpendicular to the direction of S in the horizontal plane, the stresses at the edge of the hole are, for $\theta = \frac{\pi}{2}$, (ref. equations 50)

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2(3-\nu)S_1 + \nu\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2(1-\nu)\sigma_{z_1} + 2\nu(3-\nu)S_1}{2-\nu^2} \end{aligned}\right\} \qquad (68)$$

and for $\theta = 0$, (ref. equations 51)

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2(3\nu-1)S_1 + 7\nu\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2(1+3\nu^2)\sigma_{z_1} + 2\nu(3\nu-1)S_1}{2-\nu^2} \end{aligned}\right\} \qquad (69)$$

When there is lateral constraint in all directions, the stresses at the edge of the hole become, for $\theta = \frac{\pi}{2}$, (ref. equations 55)

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2\left(\dfrac{2\nu\sigma_{z_1}}{1-\nu} + \dfrac{3-\nu}{1-\nu^2} S_1\right) + \nu\sigma_{z_1}}{2-\nu^2} \\ &= \frac{(5\nu + 4\nu^2 - \nu^3)\sigma_{z_1} + 2(3-\nu)S_1}{(1-\nu^2)(2-\nu^2)} \\ \sigma_z &= \frac{2(1+\nu^2+2\nu^3)\sigma_{z_1} + 2\nu(3-\nu)S_1}{(1-\nu^2)(2-\nu^2)} \end{aligned}\right\} \quad (70)$$

and for $\theta = 0$, (ref. equations 56)

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{(5\nu + 4\nu^2 - \nu^3)\sigma_{z_1} + 2\nu(3\nu-1)S_1}{(1-\nu^2)(2-\nu^2)} \\ \sigma_z &= \frac{2(1+\nu^2+2\nu^3)\sigma_{z_1} + 2\nu(3\nu-1)S_1}{(1-\nu^2)(2-\nu^2)} \end{aligned}\right\} \quad (71)$$

UNIFORM EXPANSION

In this case there is neither plane strain nor plane stress; the extension or contraction in the Z direction is uniform, and there is no other restriction. The stresses at $r = b$ will be the same as those for the corresponding condition of constraint in the case of plane stress. The lateral stresses at the hole may be expressed in the same form as in the case of plane strain; so the stresses for $r = a$ are

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \sigma_{\theta_1} + \frac{\nu}{2}\sigma_z \\ \sigma_z &= \sigma_{z_1} + \nu(\sigma_\theta + \sigma_r - \sigma_x - \sigma_y) \end{aligned}\right\} \quad (72)$$

(where σ_x and σ_y are the lateral stresses at $r = b$).

When there is no lateral constraint, the stresses at $r = b$ are

$$\left.\begin{aligned} \sigma_x &= S_1 \\ \sigma_y &= 0 \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \quad (73)$$

So, when $r = a$ and $\theta = \frac{\pi}{2}$,

$$\sigma_r = 0$$

$$\sigma_\theta = 3S_1 + \frac{\nu}{2}\sigma_z$$

$$\sigma_z = \sigma_{z_1} + \nu(\sigma_\theta - S_1)$$

which reduces to

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{(6-\nu^2)S_1 + \nu\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2(\sigma_{z_1} + 2\nu S_1)}{2-\nu^2} \end{aligned}\right\} \qquad (74)$$

For no constraint, $r = a$, and $\theta = 0$.

$$\sigma_r = 0$$
$$\sigma_\theta = -S_1 + \frac{\nu}{2}\sigma_z$$
$$\sigma_z = \sigma_{z_1} + \nu(\sigma_\theta - S_1)$$

which reduce to

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{\nu\sigma_{z_1} - (2+\nu^2)S_1}{2-\nu^2} \\ \sigma_z &= \frac{2(\sigma_{z_1} - 2\nu S_1)}{2-\nu^2} \end{aligned}\right\} \qquad (75)$$

When there is lateral constraint in the direction of S, the stresses at $r = b$ are

$$\left.\begin{aligned} \sigma_x &= S_1 + \nu\sigma_{z_1} \\ \sigma_y &= 0 \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \qquad (76)$$

and when $r = a$ and $\theta = \frac{\pi}{2}$.

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{(6-\nu^2)S_1 + (6+\nu-\nu^2)\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2(1+2\nu^2)\sigma_{z_1} + 4\nu S_1}{2-\nu^2} \end{aligned}\right\} \qquad (77)$$

For $\theta = 0$,

$$\left.\begin{aligned}
\sigma_r &= 0 \\
\sigma_\theta &= \frac{(\nu - 2 - \nu^2)\sigma_{z_1} - (2+\nu^2)S_1}{2-\nu^2} \\
\sigma_z &= \frac{2(1-2\nu^2)\sigma_{z_1} - 4\nu S_1}{2-\nu^2}
\end{aligned}\right\} \qquad (78)$$

When there is lateral constraint perpendicular to the direction of S, the stresses at $r = b$ are given by equations (49). The stresses when $r = a$ for $\theta = \frac{\pi}{2}$ are then (ref. equations 50).

$$\sigma_r = 0$$

$$\sigma_\theta = 3S_1 - \nu(S_1 + \sigma_{z_1}) + \frac{\nu}{2}\sigma_z$$

$$\sigma_z = \sigma_{z_1} + \nu(\sigma_\theta - S_1 - \nu S_1 - \nu\sigma_{z_1})$$

which reduce to

$$\left.\begin{aligned}
\sigma_r &= 0 \\
\sigma_\theta &= \frac{(6 - 2\nu - \nu^2 - \nu^3)S_1 - \nu(1+\nu^2)\sigma_{z_1}}{2-\nu^2} \\
\sigma_z &= \frac{2(1-2\nu^2)\sigma_{z_1} + 4\nu(1-\nu)S_1}{2-\nu^2}
\end{aligned}\right\} \qquad (79)$$

and for $r = a$ and for $\theta = 0$ (ref. equations 51).

$$\sigma_r = 0$$

$$\sigma_\theta = (3\nu - 1)S_1 + 3\nu\sigma_{z_1} + \frac{\nu}{2}\sigma_z$$

$$\sigma_z = \sigma_{z_1} + \nu(\sigma_\theta - \nu S_1 - \nu\sigma_{z_1} - S_1)$$

which reduce to

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{(6\nu - 2 - \nu^2 - \nu^3)S_1 + \nu(7-\nu)\sigma_{z_1}}{2-\nu^2} \\ \sigma_z &= \frac{2(1+2\nu^2)\sigma_{z_1} + 4\nu(\nu-1)S_1}{2-\nu^2} \end{aligned}\right\} \qquad (80)$$

When there is lateral constraint in all directions, the stresses at $r = b$ are

$$\sigma_x = S_1 + \nu(\sigma_{z_1} + \sigma_y)$$
$$\sigma_y = \nu(\sigma_{z_1} + \nu\sigma_x)$$
$$\sigma_z = \sigma_{z_1}$$

which reduce to

$$\left.\begin{aligned} \sigma_x &= \frac{S_1 + \nu(1+\nu)\sigma_{z_1}}{1-\nu^2} \\ \sigma_y &= \frac{\nu S_1 + \nu(1+\nu)\sigma_{z_1}}{1-\nu^2} \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \qquad (81)$$

The stresses at $r = a$ for $\theta = \frac{\pi}{2}$ are then (ref. equations 55).

$$\sigma_r = 0$$
$$\sigma_\theta = \frac{2\nu}{1-\nu}\sigma_{z_1} + \frac{3-\nu}{1-\nu^2}S_1 + \frac{\nu}{2}\sigma_z$$
$$\sigma_z = \sigma_{z_1} + \nu\left(\sigma_\theta - \frac{S_1 + 2\nu\sigma_{z_1}}{1-\nu}\right)$$

which reduce to

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{\nu(5+4\nu-3\nu^2-2\nu^3)\sigma_{z_1} + (6-2\nu-\nu^2-\nu^3)S_1}{(1-\nu^2)(2-\nu^2)} \\ \sigma_z &= \frac{2\sigma_{z_1} + 4\nu(1+\nu)S_1}{2-\nu^2} \end{aligned}\right\} \qquad (82)$$

and for $\theta = 0$,

$$\sigma_r = 0$$

$$\sigma_\theta = \frac{2\nu}{1-\nu^2}\sigma_{z_1} + \frac{3\nu-1}{1-\nu^2}S_1 + \frac{\nu}{2}\sigma_z$$

$$\sigma_z = \sigma_{z_1} + \nu\left(\sigma_\theta - \frac{S_1 + 2\nu\sigma_{z_1}}{1-\nu}\right)$$

which reduce to

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{\nu(5+4\nu-3\nu^2-2\nu^3)\sigma_{z_1} + (6\nu-2-\nu^2-\nu^3)S_1}{(1-\nu^2)(2-\nu^2)} \\ \sigma_z &= \frac{2\sigma_{z_1} - 4\nu(1+\nu)S_1}{2-\nu^2} \end{aligned}\right\} \quad (83)$$

PART III

APPLICATION TO A DEEP OIL WELL

CONDITIONS ENCOUNTERED

The results of Part II will now be applied to the case of a deep oil well. Any stratum or layer of rock through which the well is drilled may be considered as a plate of large lateral dimensions with a relatively small hole. The stresses due to the overburden may be considered symmetrical. However, large lateral stresses may exist in the earth's crust in some areas, and therefore the case of a uniform lateral stress should be of interest also.

The lateral expansion of the "plate" caused by the vertical compression due to the weight of the overburden is resisted by a constraint which must be considered neither as rigid nor as negligible but as elastic, for the "plate" is bounded only by more of the material of which the plate consists. This constraint presumably exists in all directions. As regards vertical constraint, it does not affect a symmetrical stress distribution. In the case of lateral stress, plane stress evidently does not exist since the ground around the well would in this case rise an amount which, for materials of a thickness of several thousand feet, would be quite perceptible. Such a rise does not occur. On the other hand, the earth above a given stratum is not perfectly rigid. Since an assumption of elastic vertical constraint can be shown to be open to an objection similar to that made to the assumption of a condition of plane stress, the case seems to correspond most closely to what has been called in Part II uniform expansion.

The two cases of interest appear to be symmetrical stress distribution with elastic constraint on all sides, and uniform stress in one direction with elastic constraint on all sides and uniform vertical expansion.

DERIVATION OF FORMULAE

The case of a body elastically constrained has already been discussed on page 31. Where the constraining medium is of the same material as the body, $E_1 = E_2$ in equation (59) and it becomes

$$\nu\sigma_z + \nu\sigma_y - \sigma_x = \sigma_x$$

$$\sigma_x = \frac{\nu}{2}(\sigma_z + \sigma_y) \tag{84}$$

If $\sigma_x = \sigma_y = \sigma_r = \sigma_\theta$ at $r = b$,

$$\sigma_r = \sigma_\theta = \frac{\nu\sigma_z}{2-\nu} \tag{85}$$

This compares with equation (43) for rigid constraint, so by the same reasoning used there, the stresses when $r = a$ are

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2\nu\sigma_{z_1}}{2-\nu} \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \tag{86}$$

These are the stresses for the case of symmetrical stress distribution with elastic constraint laterally.

The other case to be investigated is that of a uniform horizontal stress in one direction with elastic constraint in all lateral directions and uniform expansion vertically. The stresses at $r = b$ are

$$\sigma_x = S_1 + \frac{\nu}{2}(\sigma_{z_1} + \sigma_y)$$

$$\sigma_y = \frac{\nu}{2}(\sigma_x + \sigma_{z_1})$$

$$\sigma_z = \sigma_{z_1}$$

which reduce to

$$\left.\begin{aligned} \sigma_x &= \frac{4S_1 + \nu(2+\nu)\sigma_{z_1}}{4-\nu^2} \\ \sigma_y &= \frac{2\nu S_1 + \nu(2+\nu)\sigma_{z_1}}{4-\nu^2} \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \qquad (87)$$

The stresses at the edge of the hole could now be found from equations (72) if the stresses for the corresponding case of plane stress were known, so that case will be investigated here. In the case of plane stress, the stresses at $r = b$ due to S_1 alone are

$$\left.\begin{aligned} \sigma_x &= S_1 + \frac{\nu}{2}\sigma_y = \frac{4S_1}{4-\nu^2} \\ \sigma_y &= \frac{\nu}{2}\sigma_x = \frac{2\nu S_1}{4-\nu^2} \end{aligned}\right\} \qquad (88)$$

Then the stresses at the edge of the hole due to S_1 alone are, when $\theta = \frac{\pi}{2}$,

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2(6-\nu)S_1}{4-\nu^2} \end{aligned}\right\} \qquad (89($$

and when $\theta = 0$,

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2(3\nu-2)S_1}{4-\nu^2} \end{aligned}\right\} \qquad (90)$$

Superposing stresses due to σ_{z_1}, as given by equations (86), for $\theta = \frac{\pi}{2}$,

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{2\nu\sigma_{z_1}}{2-\nu} + \frac{2(3\nu-2)S_1}{4-\nu^2} \\ \sigma_z &= \sigma_{z_1} \end{aligned}\right\} \qquad (91)$$

and for $\theta = 0$,

$$\begin{aligned}
\sigma_r &= 0 \\
\sigma_\theta &= \frac{2\nu\sigma_{z_1}}{2-\nu} + \frac{2(3\nu-2)S_1}{4-\nu^2} \\
\sigma_z &= \sigma_{z_1}
\end{aligned} \tag{92}$$

These are stresses for the case of plane strain with elastic constraint in all lateral directions.

The stresses at the edge of the hole for the case of uniform expansion and elastic lateral constraint may now be found. Substituting equations (91) and (87) into equations for $\theta = \frac{\pi}{2}$.

$$\sigma_r = 0$$

$$\sigma_\theta = \frac{2\nu(\nu+2)\sigma_{z_1} + 2(6-\nu)S_1}{4-\nu^2} + \frac{\nu}{2}\sigma_z$$

$$\sigma_z = \sigma_{z_1} + \nu\left(\sigma_\theta - \frac{2(2+\nu)S_1 + 2\nu(2+\nu)\sigma_{z_1}}{4-\nu^2}\right)$$

which reduce to

$$\left.\begin{aligned}
\sigma_r &= 0 \\
\sigma_\theta &= \frac{\nu(12+4\nu-5\nu^2-2\nu^3)\sigma_{z_1} + 2(12-2\nu-2\nu^2-\nu^3)S_1}{(4-\nu^2)(2-\nu^2)} \\
\sigma_z &= \frac{2(2+\nu)\sigma_{z_1} + 4\nu S_1}{(2+\nu)(2-\nu^2)}
\end{aligned}\right\} \tag{93}$$

For $r = a$ and $\theta = 0$, substitute equations (93) and (88) in equations (93) and (88) in equations (72). Then

$$\sigma_r = 0$$

$$\sigma_\theta = \frac{2\nu\sigma_{z_1}}{2-\nu} + \frac{2(3\nu-2)S_1}{4-\nu^2} + \frac{\nu}{2}\sigma_z$$

$$\sigma_z = \sigma_{z_1} + \nu\left(\sigma_\theta - \frac{2(2+\nu)S_1 + 2\nu(2+\nu)\sigma_{z_1}}{4-\nu^2}\right)$$

which reduce to

$$\left.\begin{aligned}
\sigma_r &= 0 \\
\sigma_\theta &= \frac{\nu(12+4\nu-5\nu^2-2\nu^3)\sigma_{z_1}+2(6\nu-4-2\nu^2-\nu^3)S_1}{(4-\nu^2)(2-\nu^2)} \\
\sigma_z &= \frac{2(2+\nu)\sigma_{z_1}-8\nu S_1}{(2+\nu)(2-\nu^2)}
\end{aligned}\right\} \quad (94)$$

The effect of an internal pressure, p_i , within the bore hole , may be simply superposed on any case since stresses due to such a pressure are symmetrical. The stresses to be added would be, for all values of θ and when $r = a$,

$$\left.\begin{aligned}
\sigma_r &= -p_i \\
\sigma_\theta &= p_i \\
\sigma_z &= 0
\end{aligned}\right\} \quad (95)$$

The particular phenomena which prompted this investigation of the stresses around an oil well are; 1) Sudden pressure drops in the fluid pressure in the well during drilling, and 2) spalling of material from the sides of the bore hole. If the expressions for the stress around a well which have been derived are considered together with the mechanical properties of the rock, an explanation may appear. A brief summary of pertinent mechanical properties as given in available references is here presented.

MECHANICAL PROPERTIES OF ROCKS[8]

Table I

Rock Type	Average Ultimate Compressive Strength psi	Average Ultimate Tensile Strength psi	Average Ultimate Shearing Strength psi	Poisson's Ratio (ν)
Marble	13000	450	1300	.27
Limestone	9000	300	1350	.26
Sandstone	11000	200	1200	.23
Granite	20000	650	2000	.21

On p. 258 of Johnson's Materials of Construction is another table, taken from Bauschinger, showing that the shearing strength parallel to the bedding plane of the rock runs about 60 to 70% of the value perpendicular to the bedding plane.

Stress-strain diagrams on pp. 256-257, Johnson, show that stone does not strictly follow Hooke's law; however, the tangent modulus actually increases with load, rather than decreases, the curves being quite straight near the ultimate. Lewis' results[9] show fairly

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(8) Johnson, J. B., Withey, M. O., and Aston, James. Johnson's materials of construction. 8th ed. N. Y., Wiley, c. 1939, pp. 254-258 Values given in the table are approximate averages of values from Table 4, p. 255.

(9) Lewis, Walter E. The mechanical properties of mine rocks and a standardized test procedure for their determination. Thesis, Mo. School of Mines and Metallurgy, Rolla, Mo.

straight stress-strain curves with a proportional elastic limit very close to the ultimate. In either case, the assumption of an elastic material seems reasonably close.

Values for all properties of stone vary widely with different specimens; the compressive strength of limestone, for instance, may vary from 4000 psi to 20000 psi. Consequently, for accurate results, sample tests must be made of the rock at the place in question. With regard to the compressive strength of stone, it is undoubtedly considerably greater when confined on all sides than when unconfined. The values in Table I were from tests on unconfined stone.

SYMMETRICAL STRESS DISTRIBUTION

An average value of ν from Table I is about .25. Substituting this in equations (86),

$$\sigma_\theta = \frac{.5\,\sigma_{z_1}}{1.75} = .286\,\sigma_{z_1}$$

Using a weight of overburden, w, of one lb. per sq. in. per foot of depth and letting σ_{z_1} = wz, the stresses around a well at a depth of 10,000 feet (for example) are

$$\sigma_r = 0$$
$$\sigma_\theta = -2860 \text{ psi}$$
$$\sigma_z = -10000 \text{ psi}$$

if it is assumed that there is no internal pressure inside the bore hole. Oil wells are customarily drilled, however, with a liquid mud, having a specific gravity of about 2.0, being pumped into the hole to keep up a pressure. At a depth of 10000 feet, the static head would be about 8700 psi. Neglecting any pumping pressure, the stresses at 10000 feet would be (using equations (95) superposed on equations (86);

$$\sigma_r = -8700 \text{ psi}$$
$$\sigma_\theta = 5840 \text{ psi}$$
$$\sigma_z = -10000 \text{ psi}$$

The maximum shearing stress is evidently

$$\frac{\sigma_\theta - \sigma_z}{2} = 7920 \text{ psi}$$

A comparison of these results with Table I shows that the rock around the well should be expected to fail in tension and in shear long before the compressive strength of the material has been exceeded.

While tensile stresses exist only very close to the hole, they might cause vertical cracks which could be opened wider by the pressure of the fluid seeping into the cracks. The lateral pressure in the rock some distance from the hole, from equation (85) would be about 1430 psi, which is much less than the corresponding fluid pressure, so there is nothing in theory to prevent the crack from extending indefinitely except a drop in fluid pressure. Such a drop, of course, would be the natural result of the escape of the fluid into cracks.

In the case of a low fluid pressure, as in the case of water as the fluid or hypothesizing a pressure drop as suggested above, the shearing stresses in the rock become critical, since the tensile stresses are affected more than the shearing stresses by an internal pressure.

Consider a small pyramidal element on the boundary of the hole, the base of the pyramid being on the surface of the hole. It is seen that, with the tensile stress σ_θ small and the compressive stress σ_z large, the resultant of the normal forces on the faces of the pyramid tend to expel the element into the well, this tendency being resisted by the shearing forces on these faces. When the sides of the elemental pyramid make 45° with the base, the shearing stresses on these faces are maximum shearing stresses. When the shearing strength of the rock is exceeded on these planes, bits of rock may be expected to pop off the sides of the bore hole and the process of spalling to continue until the broken material can give the walls some support.

UNIFORM STRESS IN ONE DIRECTION

The probable magnitude of a lateral stress in a stratum 10000 ft. below the surface of the earth is difficult to estimate, but for purposes of example suppose S_1 to be equal to the vertical stress, σ_{z_1}, or 10000 psi compression, to use the same value as in the previous example. Then the stresses at the edge of the hole (with no internal pressure) are, from equations (93),

$$\sigma_r = 0$$
$$\sigma_\theta = -34000 \text{ psi}$$
$$\sigma_z = -12650 \text{ psi}$$

at $\theta = \frac{\pi}{2}$, and at $\theta = 0$,

$$\sigma_r = 0$$
$$\sigma_\theta = -1050 \text{ psi}$$
$$\sigma_z = -8050 \text{ psi}$$

Superposing stresses from an internal pressure of 8700 psi, for $\theta = \frac{\pi}{2}$,

$$\sigma_r = -8700 \text{ psi}$$
$$\sigma_\theta = -25300 \text{ psi}$$
$$\sigma_z = -12650 \text{ psi}$$

and for $\theta = 0$,

$$\sigma_r = -8700 \text{ psi}$$
$$\sigma_\theta = +7650 \text{ psi}$$
$$\sigma_z = -8050 \text{ psi}$$

The maximum shearing stress for $\theta = \frac{\pi}{2}$ is 8300 psi and 8130 psi at $\theta = 0$.

These calculations show that the existence of a lateral compressive stress accentuates the tendencies discussed for the case of symmetrical stress distribution. The tensile stress is higher for at least part of the surface of the hole, and the shearing stresses are also higher. The shearing stresses are a great deal higher for the case of a low fluid pressure, increasing to a value of 17000 psi at $\theta = \frac{\pi}{2}$ for a zero pressure as compared to a decrease to 5000 psi for the same limiting pressure in the case of symmetry. The tendency toward spalling would be concentrated on the sides of the hole at the ends of a diameter transverse to the direction of S_1, which might further contribute to the spalling tendency since the stress concentration around an elliptical or slotted hole with the major axis perpendicular to the direction of the applied stress (which shape the hole would tend to assume) is greater than for a circular hole(10). In fact, the tangential stress at $\theta = \frac{\pi}{2}$ approaches infinity as the length of the major axis is increased.

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(10) Timoshenko, S. Theory of Elasticity. N. Y., McGraw-Hill, 1934 p.81

SUMMARY AND CONCLUSIONS

The stresses in a plate with a relatively small hole have been discussed for various conditions of loading and constraint, and the results applied in the investigation of the phenomena of pressure drops and spalling during the drilling of deep oil wells. Sample calculations of the stresses around a well were made.

The general conclusions are stated in the various formulas which have been derived for stresses in a plate with a small hole in it. As regards the case of an oil well, it is concluded that the phenomena of pressure drops and spalling may be plausibly explained by consideration of the stress conditions in the rock surrounding the well.

APPENDIX A

Definitions of Terms

Compatibility equation — an equation correlating the strain or stress components at a point.

Constraint — confinement; resistance to expansion or contraction.

Plane strain — Deformation restricted to directions parallel to a single plane.

Plane stress — Complete freedom of expansion and contraction perpendicular to a plane, so that stresses parallel to the plane do not affect stresses perpendicular to the plane.

Principal stresses — Normal stresses corresponding to directions of zero shear.

Spalling — splitting off of pieces from a rock face.

Stress function — A mathematical function relating the stresses in a body, used in solving the equilibrium and compatibility equations.

Uniform expansion — Deformation perpendicular to a plane restricted only to the extent of being held to the same magnitude over the given area.

Vertical — In this paper, parallel to the axis of the hole.

APPENDIX B

Symbols

x, y, z	Rectangular coordinates.
r, θ, z	Cylindrical coordinates.
σ_x, σ_y, σ_z	Normal componets of stress parallel to x-, y-, and z-axes.
σ_r, σ_θ, σ_z	Radial, tangential, and vertical normal stresses in cylindrical coordinates.
τ	Shearing stress.
$\tau_{r\theta}$, τ_{rz}, $\tau_{\theta z}$	Shearing stress componets in cylindrical coordinates.
ϕ, f	Stress functions.
A, B, C, D, m	Constants.
a	Inside radius of a hollow cylinder.
b	Outside radius of a hollow cylinder.
P_o	External pressure on a hollow cylinder.
P_i	Internal pressure in a hollow cylinder.
S	Uniform stress perpendicular to the axis of the hole.
S_1	Initial uniform stress perpendicular to the axis of the hole.
σ_{θ_1}, σ_{z_1}	Stresses for the case of plane stress.
ϵ_x, ϵ_y, ϵ_z	Unit elongations in rectangular components.
u, v, w	Components of displacements.
w	Weight of overburden per foot of depth.
psi	Pounds per square inch.
ν	Poisson's ratio.

BIBLIOGRAPHY

1. Books;

a. Timoshenko, S. Theory of elasticity, N. Y., McGraw-Hill, 1934, 416 pp.

b. Johnson, J. B., Withey, M. O., and Aston, James. Johnson's materials of construction. 8th ed. N. Y. 1939, pp. 254-258

2. Unpublished Material

a. Lewis, Walter E. The mechanical properties of mine rocks and a standardized test procedure for their determination. Thesis, Missouri School of Mines and Metallurgy, Rolla, Mo.

INDEX

INDEX (cont.)

INDEX (cont.)

INDEX (cont.)