1969

Plastic analysis of beam - columns

Ning-Cheng Tsao

PLASTIC ANALYSIS OF BEAM - COLUMNS

by

NING - CHENG TSAO, 1933-

A

THESIS

submitted to the faculty of the

UNIVERSITY OF MISSOURI - ROLLA

in partial fulfillment of the requirements for the

Degree of

MASTER OF SCIENCE IN CIVIL ENGINEERING

Rolla, Missouri

1969

(advisor)

ABSTRACT

The results of an analytical study of the inelastic deformation of a wide-flange steel beam-column are presented. Emphasis has been placed on the influence of axial thrust on the pinned-end beam-column subjected to equal end moments with opposite direction. For this type of beam-column behavior, the relationship between the applied moment, axial load, and resultant curvature for the region in the plastic range has been studied in this investigation.

ACKNOWLEDGEMENT

The author is deeply indebted to Dr. Joseph H. Senne, Professor and Chairman of the Department of Civil Engineering, University of Missouri - Rolla, for his suggestions and guidance. He also wishes to thank Dr. Peter G. Hansen, Professor of the Department of Engineering Mechanics and Dr. Wei-Wen Yu, Associate Professor of Civil Engineering for their helpful suggestions and corrections.

The author would like to express his sincere appreciation to his wife, Julie Tsao, for her encouragement throughout this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

TABLE OF SYMBOLS

A	Area of cross-section
b	Flange width
c	Distance from neutral axis to the extreme fiber
d	Depth of cross-section
E	Young's modulus of elasticity
e	Eccentricity
f, f_1, f_2, f_a	Fiber stresses
f_y	Yield stress
f_{Rc}	Compression residual stress
f_{Rt}	Tension residual stress
H, J	Coefficients
I	Moment of inertia
L	Beam-column length
M	Bending moment
M_e	First yield bending moment
M_o	End bending moment
M_{oe}	First yield end bending moment
M_p	Plastic moment
M_{om}	Original moment
M_y	Initial yield moment under pure moment
P	Axial load
P_E	Euler buckling load
P_y	Compressive yield stress over entire section times the area of cross-section
r	Radius of gyration about the strong axis
S	Section modulus

t_w	Web thickness
t_f	Flange thickness
Y	Deflection
y	Distance from neutral axis to fiber stress
Y_c	Distance of compressive yield stress penetration
Y_t	Distance of tension yield stress penetration
$\varepsilon, \varepsilon_1, \varepsilon_2$	Unit strain
ε_y	Strain corresponding to initial yield point stress
ϕ	Angle change per unit length; curvature
ϕ_y	Curvature corresponding to initial yield under pure bending moment
Δ	Deflection at mid-height of a beam-column
α	Ratio of Y_c/d
β	Ratio of Y_t/d
λ	Length of segment of beam-column
L/r	Slenderness ratio
ψ	A factor by which end moments are modified to obtain the maximum moment

I. INTRODUCTION

(1) Introduction

Beam-columns are members in which both bending and thrust are present. Figure 1 shows the typical behavior of a beam-column. The analysis of this problem is more complicated than that of pure bending or single axial load, because the simultaneous action of bending and axial load produces an interaction effect on deflections, and the usual principle of superposition is not applicable; in other words, bending causes deflection, and the deflection multiplied by the axial load will represent an increase in bending moment. For a typical case shown in Figure 1 the bending moment at a distance x from the end support is represented as

$$M = M_{om} + P \cdot \delta$$

where, M_{om} = Original moment due to external load

and $P \cdot \delta$ = Moment due to axial load.

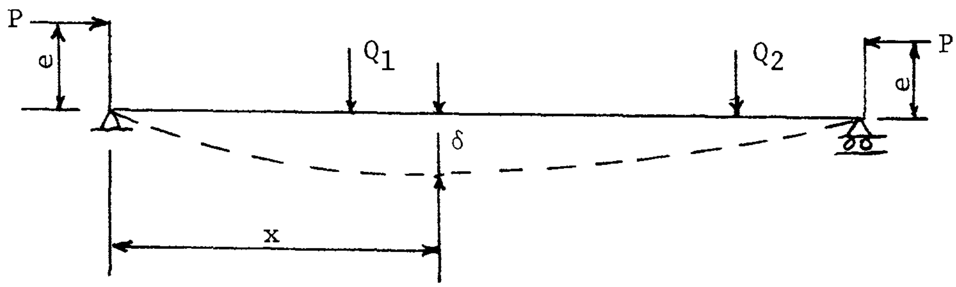
$P \cdot \delta$ cannot easily be determined because it depends on the unknown moment which is related to δ.

For analysis of the strength of beam-columns, equations in the elastic range can be found in any textbook on structure analysis. For a pinned-end column with an eccentricity e, the maximum compression stress is defined by the following equation (Secant formula)

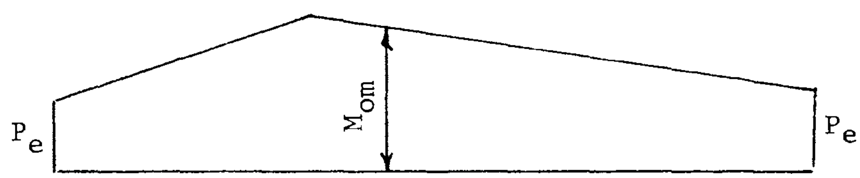
$$F_{max} = \frac{P}{A} [1 + \frac{ec}{r^2} \operatorname{Sec}(\frac{\pi}{2}\sqrt{\frac{P}{P_E}})]$$

and the maximum deflection is

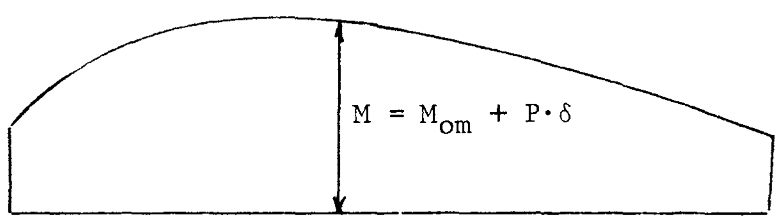
$$\delta_{max} = e[\operatorname{Sec} \frac{L}{2\sqrt{\frac{EI}{P}}} - 1]$$



Column under eccentric and lateral loads



Original moment due to external vertical load



Total moment combined V.L. and axial load

Figure 1. Typical moment in beam-columns

This equation is applicable only within the elastic limit. If the stress exceeds the elastic limit of the material, non-linear relationships will occur, and we have to use other assumptions to solve this problem.

(2) Purpose

The problem considered in this thesis is to determine the ultimate bending moment that a member can sustain for a given axial load. In the analysis, a pinned-end, wide-flange section is to be subjected to equal end moments about the strong axis while the weak axis is laterally braced. This way of treatment is shown in Figure 2.

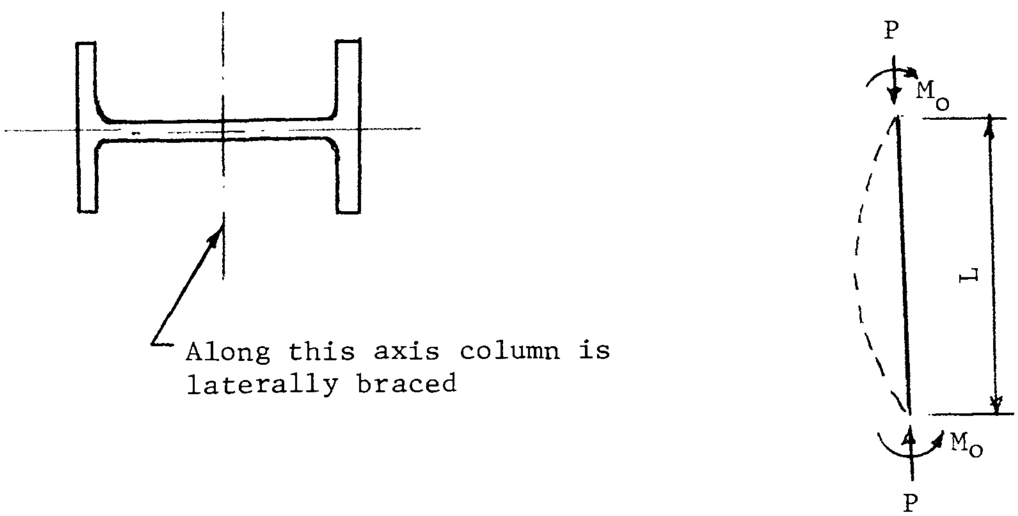


Figure 2. Column under combined bending and axial load

Two methods are presented in this paper, one is by numerical analysis, the other is done by moment-curvature curves. Both analyses are based on the assumption that the deflected shape of the beam-column after loading is a sine curve.

Residual stresses and bending along the unbraced weak axis will be discussed in Chapter III.

The basic assumptions in analyzing this problem are made as follows:

A. Bending is produced about the strong-axis of a wide-flange section; weak axis is completely braced.

B. The material is A-36 structural steel with an idealized strain-stress relationship; that is, a straight line follows the lower yield point (f_y = 36ksi) and the strain hardening is neglected, as shown in Figure 3.

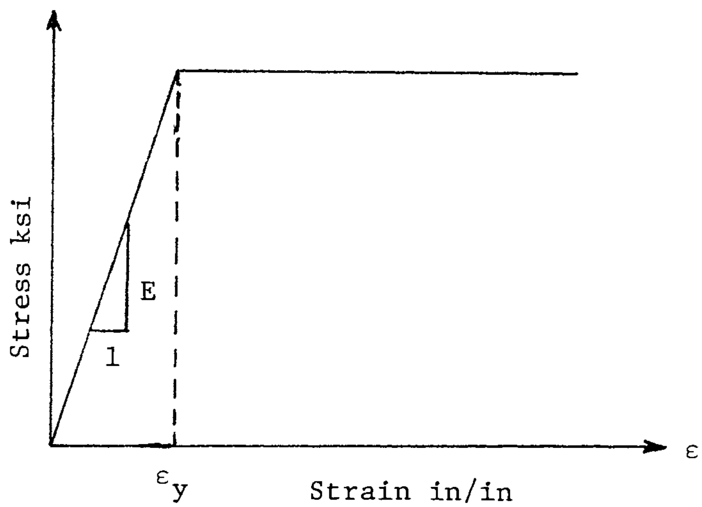


Figure 3. Idealized stress-strain diagram

C. Residual stress is not considered.

D. Lateral torsional buckling is prevented.

(3) General Review of Literature

A. Elastic analysis

The strength of a beam-column can be defined with reasonable accuracy for both axially and eccentrically loaded columns in the elastic range. A convenient and powerful method for determining the strength of such a column is that of interaction.

The general criterion for failure of a beam-column is expressed by a functional relationship in terms of the ratio of actual load to the strength of a member under pure axial or pure bending load as follows:

$$P/P_u = f_1(M/M_u) \text{ or } M/M_u = f_2(P/P_u)$$

where P = actual axial load

f_1 and f_2 = constants

M = maximum bending moment acting simultaneously with P

P_u = ultimate strength of the particular member when subjected to pure axial load. For long slender columns $P_u = P_{cr} = \pi^2 EI/L^2$; for intermediate columns $P_u = \pi^2 E_t I/L^2$* and for short columns $P_u = f_u A$ or $f_y \cdot A$

M_u = ultimate strength of the particular member when subjected to pure flexure.

The derivation of this type of failure criterion can be best illustrated by considering a short length of an ideally elastic prismatic bar which fails when the maximum fiber stress due to axial load, or bending moment, or a combination of the two reaches a value f_u, the maximum stress in such a bar. When the member is subjected to a combination of load P and M_{max} i.e.

$$f_{max} = P/A + \frac{M_{max}}{I/C}\text{**} \quad \text{and} \tag{1.1}$$

failure occurs when $f_{max} = f_u$. Dividing both sides by f_u, one can obtain the following equation:

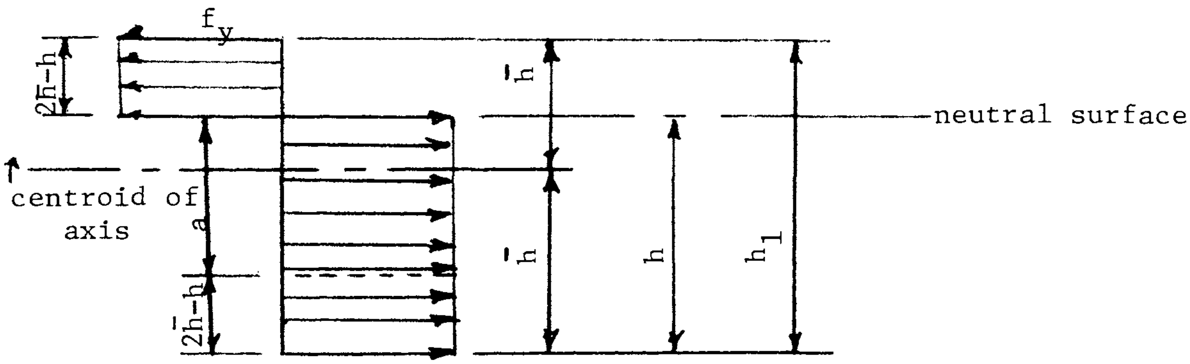
**The use of E_t for intermediate columns is based on the stress-strain curve affected by residual stress.*

***See Chapter III for the amplication factor due to secondary moment.*

$$1 = \frac{P}{f_u\ A} + \frac{M_{max}}{(I/C)/f_u} \quad \text{or}$$

$$\frac{P}{P_u} + \frac{M_{max}}{M_u} \leqq 1. \tag{1.2}$$

B. Plastic analysis



$$a = h - (h_1 - h) = h - (2\bar{h} - h) = 2(h - \bar{h})$$

Figure 4. Stress distribution for plastic analysis of a rectangular section

From Figure 4 it can be seen that the total tensile force,

$$(P) = 2f_yt(h-\bar{h}) \tag{1.3}$$

and the total moment,

$$(M) = tf_yh(2\bar{h}-h), \tag{1.4}$$

with respect to the neutral surface.

If $h = 2\bar{h}$, for pure tension

$$P_u = 2tf_y\bar{h}, \tag{1.5}$$

where, P_o = pure tension over the entire section.

If $h = \bar{h}$ then, we have pure bending or

$$M_u = tf_u\bar{h}^2 \tag{1.6}$$

Also from the above expressions,

$$(\frac{P}{P_u})^2 = \left[\frac{2fyt(h-\bar{h})}{2tfy\bar{h}}\right]^2 = (\frac{h-\bar{h}}{\bar{h}})^2 \qquad (1.7)$$

and

$$\frac{M}{M_u} = \frac{tfyh(2\bar{h}-h)}{tfy\bar{h}^2} = \frac{h(2\bar{h}-h)}{\bar{h}^2} \qquad (1.8)$$

If h is eliminated between these two equations by addition, then

$$(\frac{P}{P_u})^2 + (\frac{M}{M_u}) \leqq \frac{(h-\bar{h}^2)}{\bar{h}^2} + \frac{h(2\bar{h}-h)}{\bar{h}^2} = 1 \quad \text{or}$$

$$(\frac{P}{P_u})^2 + \frac{M}{M_u} \leqq 1. \qquad (1.9)$$

Equations (1.2) and (1.9) are shown graphically in Figure 5.

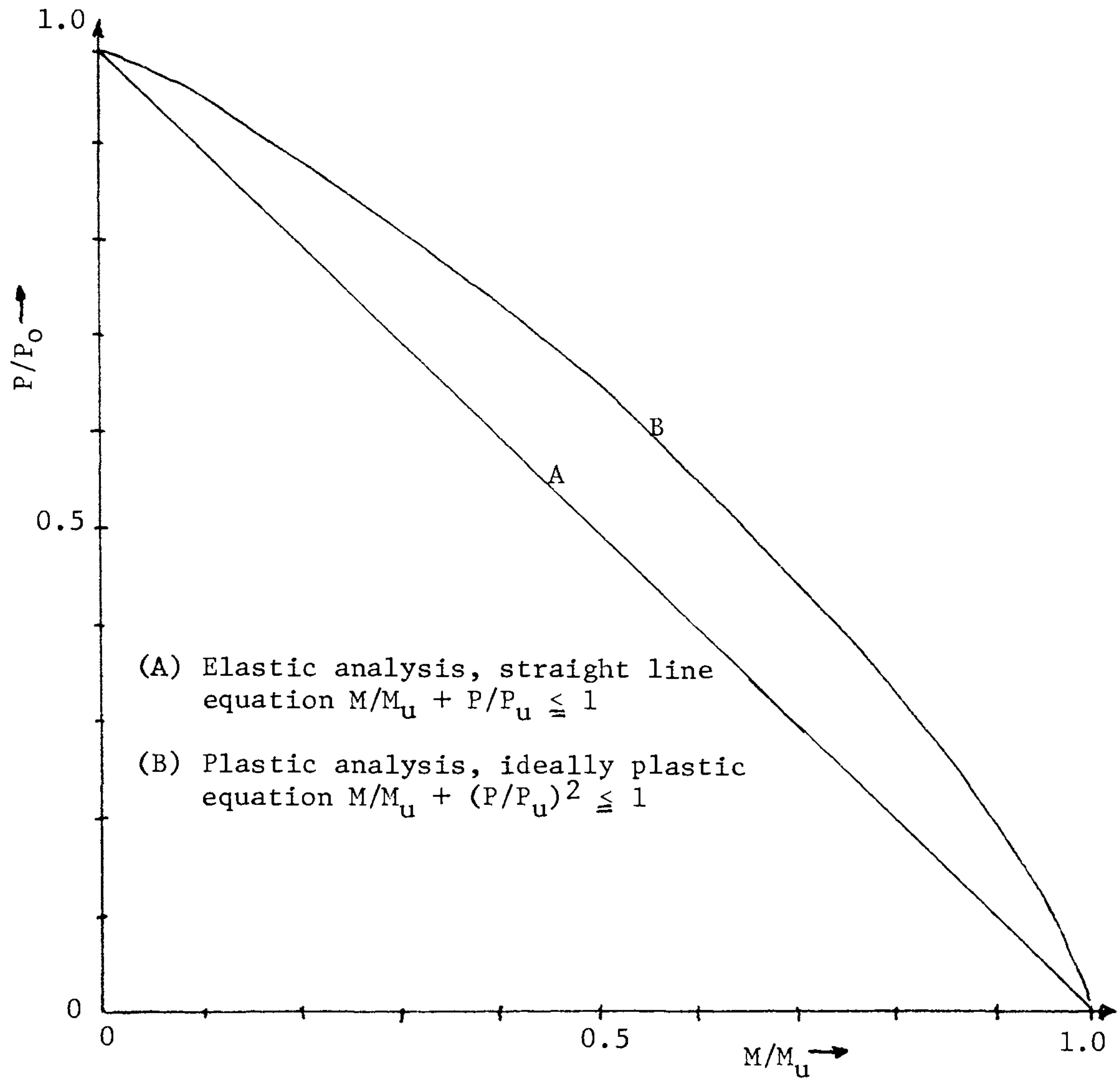


Figure 5. Interaction curves, bending and axial loads

II. COMBINED BENDING AND AXIAL LOAD

(1) Influence of Axial Load

In the range of elastic behavior the stress distribution of a beam-column subjected to an axial load P and a bending moment M can be resolved into two parts, as shown in Figure 6. When the extreme fiber reaches the yield stress, f_y, plastic flow begins. As the loading is increased, yielding first begins in the flange on the compression side, then yielding will occur on the tension side until the fully plastic condition is reached. The progress of the plastic penetrations are illustrated in Figure 6. In the fully plastic case the existance of axial load tends to reduce the plastic moment capacity of the member. If the axial load were great enough to produce compression yield stress over the entire cross-section, there would be no resistance bending moment. However, the fully plastic case occurs only in short compressive blocks. For long columns, failure may occur by buckling before the yield point is reached.

Most practical column failures occur in the inelastic range before the fully plastic condition is reached. It is this case that is to be investigated in this paper.

The general equations leading to the development of interaction curves have been derived for a rectangular cross-section in Chapter I. For the case of wide-flange shapes, it was difficult to find a direct theoretical expression because of irregularities in the cross-section, and an attempt to derive an equation for wide-flange shapes similar to that for a rectangular section failed due to the unwieldy mathematics involved. However, expressions for axial load, moment and curvature were obtained.

It is most important, in this type of column behavior, to find the relationship between the applied moment, axial load and resultant curvature for the region in the plastic range.

(2) Plastic Deformation of Wide-Flange Sections

The relationship between the angle change ϕ per unit length and the bending moment M at any section of the member is

$$\phi = M/EI \qquad (2.1a)$$

as long as the structural member remains in the elastic range. If the stress is above the elastic limit, a linear relationship between M and ϕ is no longer existent. For these cases the behavior of ϕ will be described systematically below.

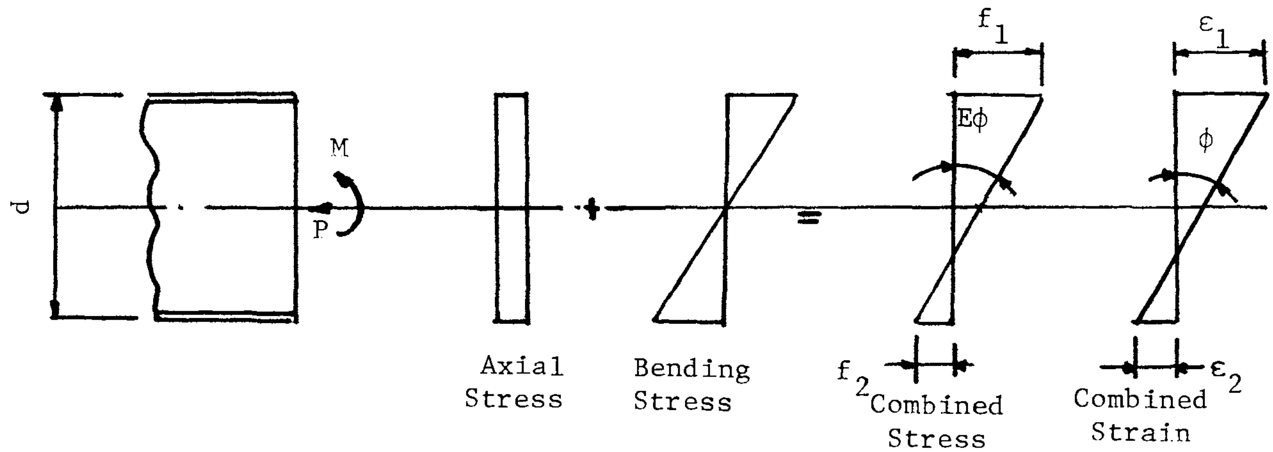
When the loads are applied to the member within the elastic limit by combined bending and axial load, stress and strain diagrams are shown in Figure 6 where the angle change is:

$$\phi = (\varepsilon_1 - \varepsilon_2)/d = (f_1 - f_2)/Ed \qquad (2.1b)$$

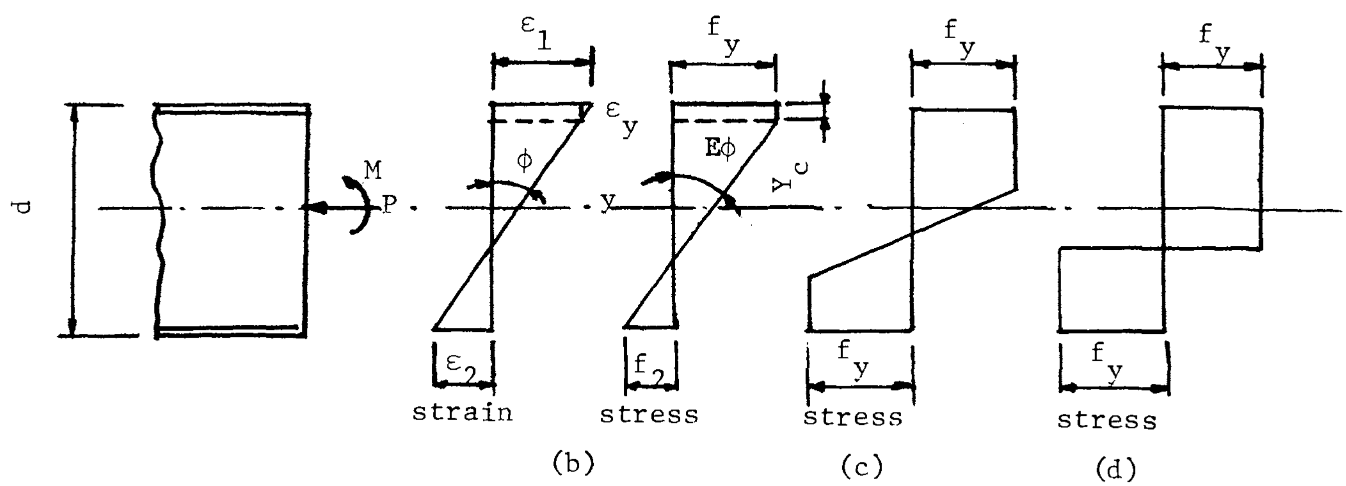
where ε_1 and ε_2 are unit strains, f_1 and f_2 are fiber stresses, and d is the depth of the cross-section. Since ϕ is very small, $\tan\phi \doteq \phi$. For members loaded beyond the elastic limit, the stress diagrams are shown in Figures 6(b) and 6(c). The determination of ϕ in the plastic range requires consideration of that part of the section which remains elastic, thus

$$\phi = (\varepsilon_y - \varepsilon_2)/(d - Y_c) = (f_y - f_2)/(d - Y_c) \qquad (2.1c)$$

where ε_y is the strain corresponding to the initial yield point stress and Y_c represents the distance of compressive yield stress penetration.



(a) Elastic range



(b) and (c) are partially plastic

(d) fully plastic

Figure 6. Elastic and plastic stress and strain distribution

In Equations (2.1b) and (2.1c), the sign convention considered for compressive strain and stress is positive, and tension is negative.

From a stress diagram it is possible to determine both end moments and the applied load acting on the section. The values of P and M correspond to stress distributions obtained by applying the basic equilibrium equations to the forces on the member as shown in Figure 6, in which

$$P = \int f \cdot dA \tag{2.2}$$

and $$M = \int f \cdot Y \cdot dA \tag{2.3}$$

Derivation of values M and P for each stress pattern, will be given in the following section.

(3) Derivation of Basic Equations

Using basic equations (2.1), (2.2) and (2.3), the values of the axial load, bending moment and curvature can be determined for any given stress distribution diagram owing to the shape of the wide-flange cross-section and the partially plastic to fully plastic region penetrations. There are three cases to be considered in the derivation[1*],

Case 1. Yield stress at one extreme fiber.

Case 2. Plastic-flow penetrates the cross-section on one side while other side remains elastic.

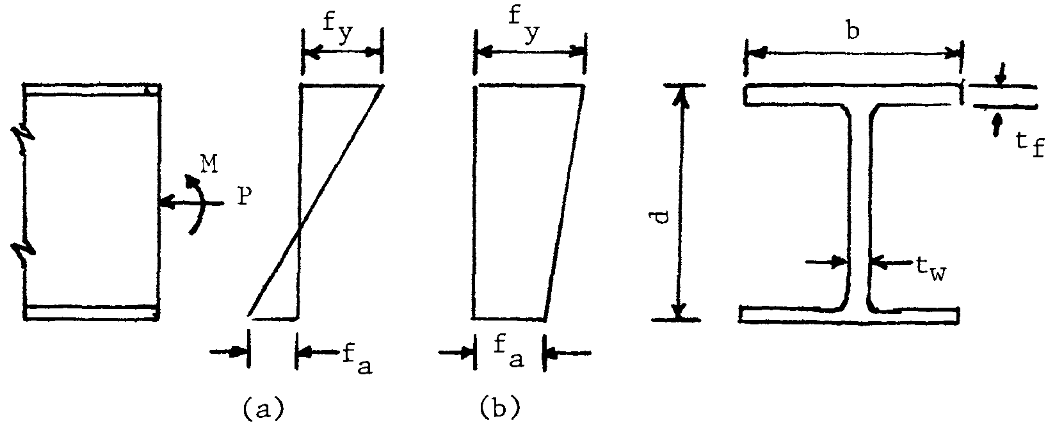
Case 3. Plastic-flow penetrates through both sides of the cross-section.

Diagrams for the three cases are shown in Figure 7. The following equations take both stress ranges into consideration:

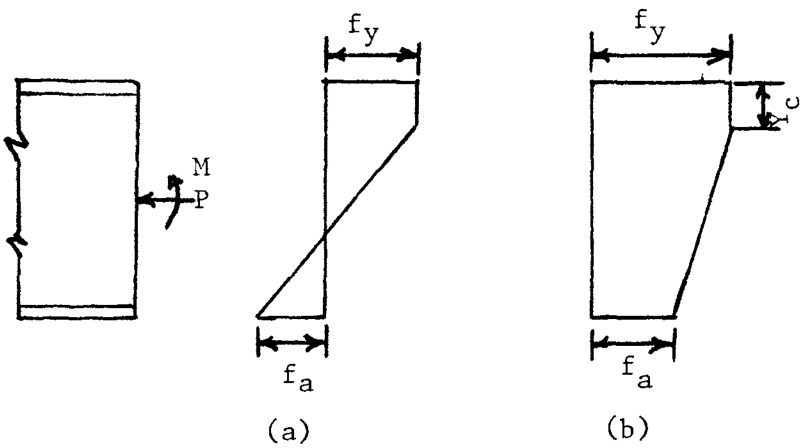
(a) Compressive stress on one side and tension on the other side

---

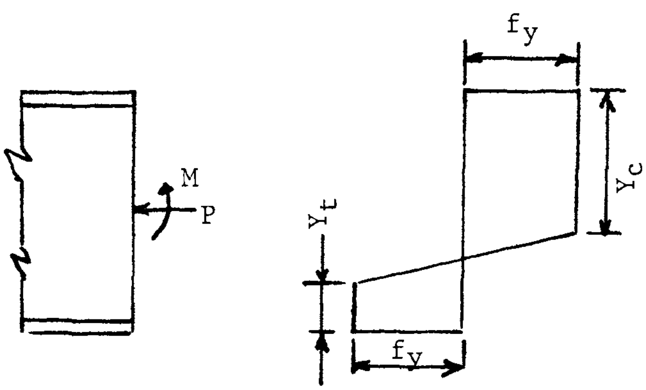
* *Reference applies to the numbered item in the list of references (page 50).*



Case 1. First yield stress distribution



Case 2. Yielding on one side of the cross-section



Case 3. Yielding on both sides of the cross-section

Figure 7. Three cases of stress distribution

(b) Compressive stress on both sides.

Case 1.

Axial load

$$P = \frac{fy + fa}{2} A \tag{2.4a}$$

If in terms of A of wide-flange cross-section

$$P = \frac{fy + fa}{2} \left[2bt_f + t_w(d-2t_f)\right] \tag{2.4b}$$

Where b is the width of flange, t_f is the thickness of flange and t_w denotes the thickness of web.

Bending moment

$$M_e = f \frac{I}{c} = f \cdot S = \frac{fy - fa}{2} S \tag{2.5a}$$

Where c is the distance from neutral axis to the extreme fiber, and S represents the section modulus of the cross section.

If in terms of S of wide-flange section

$$M_e = \frac{fy - fa}{12d} \left[t_w(d-2t_f)^3 + 2bt_f(3d^2 - 6dt_f + 4t_f^2)\right] \tag{2.5b}$$

Angle change

$$\phi_e = \frac{fy - fa}{Ed} \tag{2.6}$$

Case 2.

Axial load

$$P = 2fy\ bt_f + fy\ tw(d - 2t_f) - \frac{bt_f}{2} \left[(fy - fa) + \frac{fy - fa}{d - Y_c}(d - Y_c - t_f)\right] - \frac{t_w}{2} \frac{fy - fa}{d - Y_c}(d - Y_c - t_f)^2$$

$$= fy\ (t_w d + 2bt_f - 2t_w t_f) - \frac{bt_f}{2} \left[2(fy - fa)\right.$$

$$- \frac{t_f(fy - fa)}{d - Y_c}\Bigg] - \frac{t_w}{2}\frac{fy - fa}{d - Y_c}\left[(d - Y_c)^2 - 2(d - Y_c)t_f \quad t_f^2\right]$$

Let $Y_c/d = \alpha$

$$P = fy(t_w d + 2bt_f - 2t_w t_f) - bt_f(fy - fa) + \frac{bt_f^2(fy - fa)}{2d(1 - \alpha)}$$

$$- \frac{t_w\,(fy - fa)}{2d(1 - \alpha)}\left[d\,(1 - \alpha)^2 - 2d(1 - \alpha)t_f + t_f^2\right]$$

$$P = fy\,t_w d + t_f(fy + fa)(b - t_w) + \frac{fy - fa}{2d(1 - \alpha)}$$

$$\left[bt_f^2 - t_w d^2(1 - \alpha)^2 - t_w t_f^2\right] \quad (2.7)$$

<u>Bending moment</u>

$$M = \frac{bt_f(fy - fa)}{2}\left(\frac{d}{2} - \frac{t_f}{3}\right) + \frac{bt_f(fy - fa)(d - Y_c - t_f)}{2(d - Y_c)}\left(\frac{d}{2} - \frac{2}{3}t_f\right)$$

$$+ \frac{t_w(fy - fa)}{2(d - Y_c)}(d - Y_c - t_f)^2\left[\frac{d}{2} - t_f - \frac{1}{3}(d - Y_c - t_f)\right]$$

$$M = \frac{bt_f(fy - fa)}{2}\left(\frac{d}{2} - \frac{t_f}{3}\right) + \frac{bt_f(fy - fa)(d - Y_c - t_f)}{2(d - Y_c)}\left(\frac{d}{2} - \frac{2}{3}t_f\right)$$

$$+ \frac{t_w(fy - fa)}{2(d - Y_c)}(d - Y_c - t_f)^2\left[\frac{1}{6}(d + 2Y_c - 4t_f)\right]$$

Let $Y_c/d = \alpha$

$$M = \frac{bt_f(fy - fa)}{12}(3d - 2t_f) + \frac{bt_f(fy - fa)}{12d}\left[d - \frac{t_f}{(1 - \alpha)}\right](3d - 4t_f)$$

$$+ \frac{t_w(fy - fa)}{12d(1 - \alpha)}\left[d\left(1 - \alpha - \frac{t_f}{d}\right)\right]^2\left[d(1 + 2\alpha) - 4t_f\right]$$

$$M = \frac{(fy - fa)}{12d(1 - \alpha)}\left\{bt_f\left[6d(1 - \alpha)(d - t_f) - 3dt_f + 4t_f^2\right]\right.$$

$$\left. + t_w\left[d\left(1 - \alpha - \frac{t_f}{\alpha}\right)\right]^2\left[d(1 + 2\alpha) - 4t_f\right]\right\} \quad (2.8)$$

<u>Angle change</u>

$$\phi = \frac{(fy - fa)}{E(d - Y_c)} = \frac{(fy - fa)}{Ed(1 - \alpha)} \quad (2.9)$$

Case 3

Axial load

$P = fy \cdot t_w (Y_c - Y_t) \quad Y_c < Y_t < t_f$

Let $\alpha = Y_c/d \quad \beta = Y_t/d$

where Y_t is the distance representing tensile yield stress penetration

$$P = fy \cdot t_w d(\alpha - \beta) \tag{2.10}$$

Bending moment

$$M = fybt_f(d-t_f) + fyt_w(Y_t-t_f)(d-Y_t-t_f)$$

$$+ \frac{2fyt_w}{2} (d-Y_c-Y_t)(\frac{d}{6} - \frac{2}{3} Y_t + \frac{Y_c}{3}$$

$$M = fybt_f(d-t_f) + \frac{fyt_w\alpha^2}{6} \left[6(\beta- \frac{t_f}{d})(1-\beta- \frac{t_f}{d}) + (1-\alpha-\beta)(1+2\alpha-4\beta)\right] \tag{2.11}$$

Angle change

$$\phi = \frac{2fy}{E(d-Y_t-Y_c)} = \frac{2fy}{Ed(1-\alpha-\beta)} \tag{2.12}$$

(4) Load Deformation Behavior of Beam-Columns

The computation of the ultimate load of a beam-column failing by inelastic instability is usually done by establishing sufficient points on the M-Δ curve, since a direct analytical expression is not available in the inelastic range. For a given axial load, a typical M-Δ curve is shown in Figure 8, where the bending moment applied at the ends of a pinned-end column is plotted against the deflection at the center-section. The slope of this curve consists of two parts; one is the ascending slope which means the member is stable, the other is the descending slope, which represents instability. At the peak where the slope is zero, the ultimate load is reached. The methods of construction of M-Δ curves will be discussed in Chapter III.

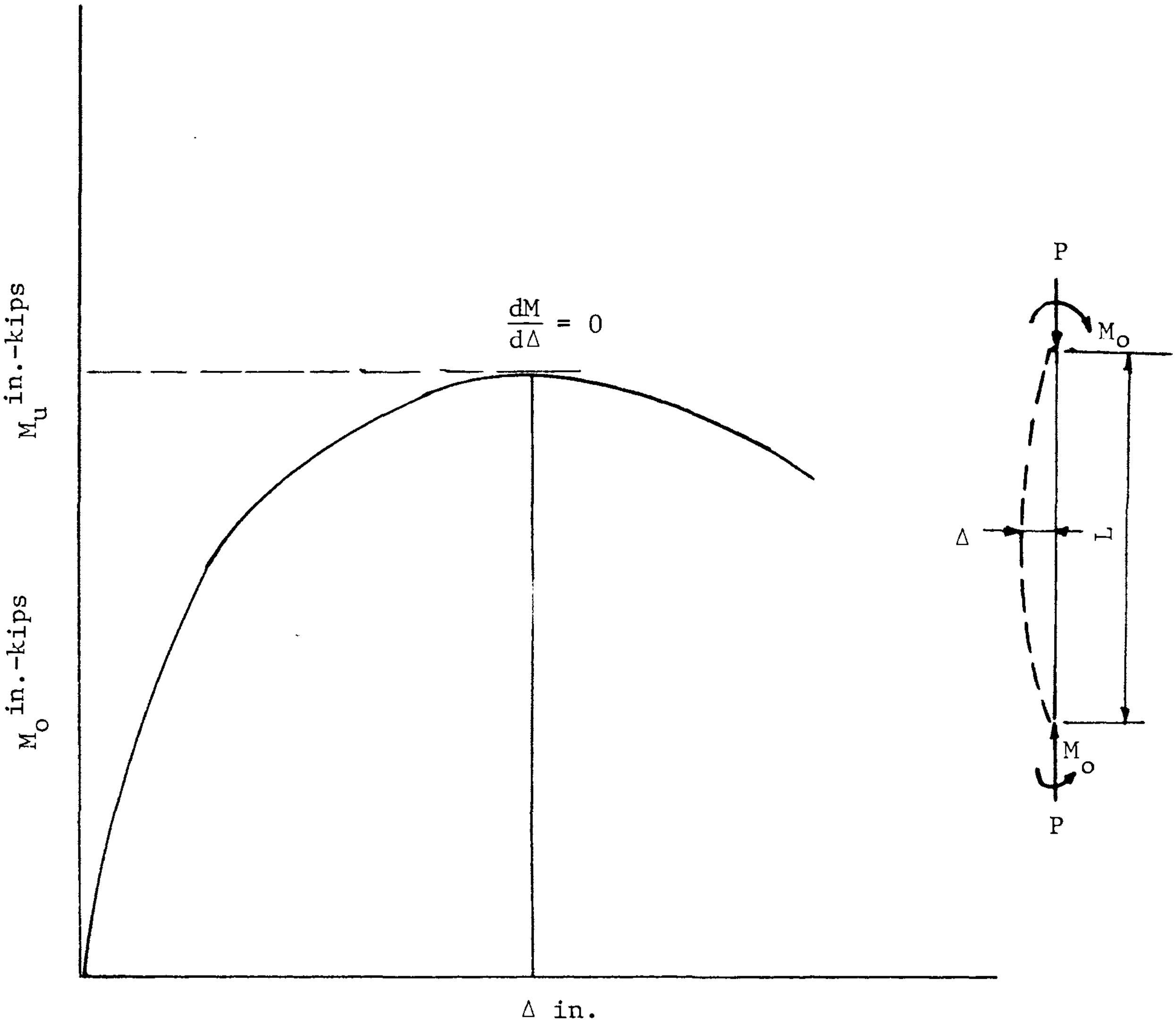


Figure 8. Typical load-deformation diagram

III. ANALYSIS OF ULTIMATE CAPACITY OF BEAM-COLUMNS

(1) Direct Numerical Method

The deflected column as shown in Figure 9 may be obtained by the superposition of a non-sine curve caused by end moments and a sine curve due to PY. Since the PY effect becomes more predominant as ultimate load is approached, a sine curve combination is assumed. The following equations based on this sine-curve deformation are developed for determining the end moment. Deflection Y at any point having a distance X from the end of the column is:

$$Y = \Delta \sin \frac{\pi x}{L} , \tag{3.1}$$

where Δ is maximum deflection at mid-height of the column, and the curvature at mid-height is,

$$\phi = \left. \frac{d^2y}{dx^2} \right|_{x=\frac{1}{2}L} = \Delta \frac{\pi^2}{L^2} \tag{3.2}$$

The moment at this same section for the column subjected to end moments M_o and axial load P is given by

$$M = M_o + P \cdot \Delta \tag{3.3}$$

$$\text{or} \quad M_o = M - P \cdot \Delta .$$

From Equations (3.2), (3.3) and (2.4) to (2.12), the ultimate end moment can be determined for any given axial load P and column length L by plotting the moment-deflection curve.

The procedure of calculation can be outlined as follows:

A. Using Equation (2.4a), find f_a.

B. Substituting f_a into Equation (2.5a), find M_e.

C. Substituting f_a into Equation (2.6), find ϕ_e.

D. Using Equation (3.2), compute Δ_e.

E. Using Equation (3.3), compute M_{oe}.

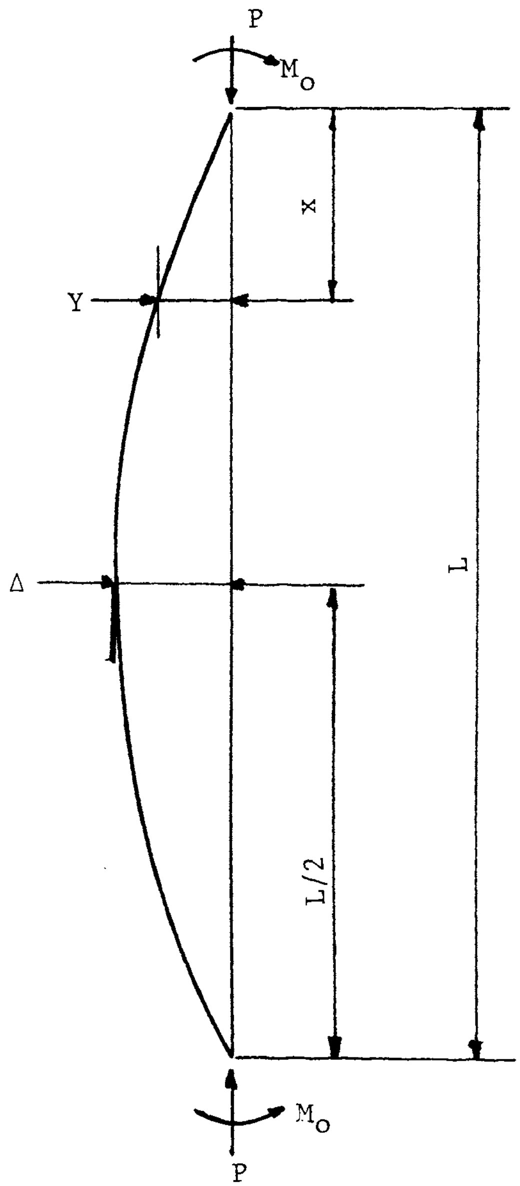


Figure 9. Deflection of a beam-column under combined bending and axial load

Denote Δ_e and M_{oe} instead of Δ and M_o to signify the first yield value. Steps A through E determine the first yield values.

F. Start assuming $Y_c = 0.5$ in using Equation (2.7) find f_a.

G. Substituting f_a into Equation (2.8), find M.

H. Substituting f_a into Equation (2.9), find ϕ.

I. Using Equation (3.2), compute Δ.

J. Using Equation (3.3), compute M_o.

Repeat steps F through J at Y_c = 1, 1.5, 2.0, 2.5 etc. in 0.5 increments, in order to find Δ and M_o when -36 ksi $\leqq f_a \leqq 36$ ksi and $t_f \leqq Y_c \leqq d-t_f$

K. If $f_a < -36$ ksi, use Equation (2.10) to find Y_t.

L. Substituting Y_t into Equation (2.11) find M.

M. Substituting Y_t into Equation (2.12) find ϕ.

N. Using Equation (3.2), compute Δ.

O. Using Equation (3.3), compute M_o

Increase Y_c in 0.5 increments, repeat steps K through O to find M_o and Δ.

For each set of L/r and P/Py ratios, ($Py = fy \cdot A$) steps A through O should be followed in analyzing the end moment and deformation. The tedious series of calculations can be checked by computer program.

Fortunately, the variation between the M-ϕ-P curves, and thus the interaction curves, is almost negligible for different WF - shapes, and therefore the interaction curves developed for the 8WF31 shape can also be used for other sections, because it has one of the lowest shape factors of all rolled steel sections (1.10 compared to the average shape factor 1.14). It has been shown that the curves are slightly on the conservative side for other WF - shapes.[5] For other 8WF31 data see page 22.

Before beginning the calculations, it is better to know the first yield values for the beam-column. By following steps A to E, the results shown in Table I were obtained. Table II to Table X contain the computed ultimate and end moments for

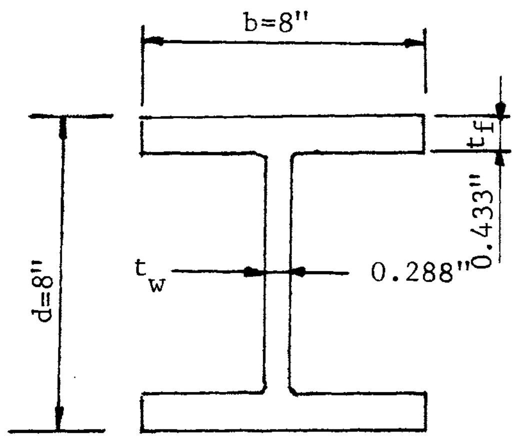
$L/r = 40$ $P/Py = 0.2, 0.4, 0.6,$ and 0.8

$L/r = 60$ $P/Py = 0.2, 0.4, 0.6,$ and 0.8

$L/r = 80$ $P/Py = 0.2, 0.4, 0.6,$ and 0.8

$L/r = 100$ $P/Py = 0.2, 0.4, 0.6,$ and 0.8

$L/r = 120$ $P/Py = 0.2, 0.4, 0.6,$ and 0.8



$A = 9.12$ in.^{2}

$I = 109.7$ in.^{4}

$S = 27.4$ in.^{3}

$r = 3.47$ in.

$fy = 36$ ksi

$E = 30 \times 10^6$ psi

Data for 8 WF 31

$Py = fy \cdot A = 36 \times 9.12 = 328.3^{kips}$

$M_y = fy \cdot S = 36 \times 27.4 = 986.4^{in\text{-}kips}$

$M_p = fy \cdot z = 36 \times 29.876 = 1075.6$ in.-kips

$\phi_y = \frac{My}{EI} = 986.4/30 \times 10^6 \times 109.7 = 0.000299$ rad.

TABLE I

FIRST YIELD VALUES FOR BEAM-COLUMN, 8WF31

(See Appendix for the detailed computer data)

L/r	P/Py	fa (ksi)	M (in.-kips)	ϕ (rad)	Δ (in.)	M_o (in.-kips)
40	.2	-21.60	789.12	.00024	.4685	758.36
40	.4	-7.20	591.84	.00018	.3514	545.70
40	.6	7.20	394.56	.00012	.2342	348.42
40	.8	21.60	197.28	.00006	.1171	166.52
80	.2	-21.60	789.12	.00024	1.8739	666.07
80	.4	-7.20	591.84	.00018	1.4054	407.27
80	.6	7.20	394.56	.00012	.9370	309.99
80	.8	21.60	197.28	.00006	.4685	74.23
120	.2	-21.60	789.12	.00024	4.2163	512.26
120	.4	-7.20	591.84	.00018	3.1622	176.55
120	.6	7.20	394.56	.00012	2.1081	-20.73
120	.8	21.60	197.28	.00006	1.0541	-79.58

TABLE II

8WF31 BEAM-COLUMN IN PLASTIC REGION

$L/r = 40$ and $P/P_y = 0.2$

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	f_a (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	-21.60	789.12	0.00024000	.4685	758.36
2	0.5	-	-23.61	831.31	0.00026496	.5172	797.35
2	1.0	-	-24.72	849.24	0.00028916	.5644	812.18
2	1.5	-	-25.89	866.34	0.00031738	.6195	825.66
2	2.0	-	-27.12	882.57	0.00035067	.6845	837.62
2	2.5	-	-28.43	897.87	0.00039048	.7622	847.82
2	3.0	-	-29.83	912.21	0.00043887	.8567	855.96
2	3.5	-	-31.35	925.53	0.00049886	.9738	861.59
2	4.0	-	-33.01	937.79	0.00057506	1.1225	864.08
2	4.5	-	-34.86	948.92	0.00067488	1.3174	862.42
2	5.0	-	-37.00	958.88	0.00081110	1.5833	854.91
	5.5	-					
	6.0	-					
	6.5	-					
	7.0	-					

NOTE: The computed values below the horizontal line do not apply for case 2.

TABLE III

8WF31 BEAM-COLUMN IN PLASTIC REGION

L/r = 40 and P/Py = 0.4

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	fa (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	-7.20	591.84	.00018000	.3514	545.70
2	0.5	-	-8.43	619.49	.00019744	.3854	568.88
2	1.0	-	-9.25	632.86	.00021548	.4206	577.62
2	1.5	-	-10.12	645.60	.00023651	.4617	584.97
2	2.0	-	-11.04	657.69	.00026132	.5101	590.70
2	2.5	-	-12.01	669.10	.00029099	.5680	594.50
2	3.0	-	-13.06	679.78	.00032705	.6384	595.94
2	3.5	-	-14.19	689.71	.00037175	.7257	594.41
2	4.0	-	-15.42	698.84	.00042853	.8365	588.99
2	4.5	-	-16.81	707.14	.00050292	.9817	518.21
2	5.0	-	-18.40	714.56	.00060443	1.1798	559.16
2	5.5	-	-20.32	721.07	.00075089	1.4657	528.58
2	6.0	-	-22.80	726.64	.00098008	1.9131	475.40
2	6.5	-	-26.49	731.32	.00138862	2.7106	375.34
2	7.0	-	-33.57	735.34	.00231910	4.5269	140.84
2	7.5	-	-61.71	741.19	.00651427	12.7158	-928.75

TABLE IV

8 WF31 BEAM-COLUMN IN PLASTIC REGION

L/r = 40 and P/Py = 0.6

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	fa (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	7.20	394.56	0.00012000	0.2342	348.42
2	0.5	-	6.76	407.68	0.00012939	0.2536	357.71
2	1.0	-	6.22	416.47	0.00014180	0.2768	361.94
2	1.5	-	5.65	424.86	0.00015564	0.3038	365.01
2	2.0	-	5.03	432.81	0.00017197	0.3357	366.69
2	2.5	-	4.40	440.32	0.00019149	0.3738	366.68
2	3.0	-	3.72	447.35	0.00021522	0.4201	364.59
2	3.5	-	2.97	453.88	0.00024464	0.4775	359.81
2	4.0	-	2.16	459.89	0.00028201	0.5505	351.45
2	4.5	-	1.25	465.35	0.00033096	0.6460	338.09
2	5.0	-	0.20	470.24	0.00039777	0.7764	317.28
2	5.5	-	-1.06	474.52	0.00049414	0.9646	284.51
2	6.0	-	-2.70	478.19	0.00064497	1.2590	230.18
2	6.5	-	-5.12	481.27	0.00091382	1.7838	129.88
2	7.0	-	-9.78	483.91	0.00152616	2.9790	-102.93
2	7.5	-	-28.30	487.76	0.00428691	8.3680	-1160.67

TABLE V

8WF31 BEAM-COLUMN IN PLASTIC REGION

$L/r = 40$ and $P/Py = 0.8$

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	f_a (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	21.60	197.28	0.00006000	0.1171	166.52
2	0.5	-	21.95	195.86	0.00006242	0.1219	163.85
2	1.0	-	21.69	200.08	0.00006813	0.1330	165.16
2	1.5	-	21.42	204.11	0.00007478	0.1460	165.78
2	2.0	-	21.13	207.94	0.00008262	0.1613	165.58
2	2.5	-	20.82	211.54	0.00009200	0.1796	164.37
2	3.0	-	20.49	214.92	0.00010340	0.2018	161.91
2	3.5	-	20.13	218.06	0.00011753	0.2294	157.80
2	4.0	-	19.74	220.95	0.00013549	0.2645	151.48
2	4.5	-	19.30	223.57	0.00015901	0.3104	142.05
2	5.0	-	18.80	225.92	0.00019110	0.3730	127.94
2	5.5	-	18.19	227.97	0.00023740	0.4634	106.26
2	6.0	-	17.41	229.74	0.00030986	0.6048	70.87
2	6.5	-	16.24	231.21	0.00043903	0.8570	6.12
2	7.0	-	14.00	232.49	0.00073321	1.4312	-143.43
2	7.5	-	5.11	234.33	0.00205956	4.0202	-821.61

TABLE VI

8WF31 BEAM-COLUMN IN PLASTIC REGION

L/r = 80 and P/Py = 0.2

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	fa (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	-21.60	789.12	0.00024000	1.8739	666.07
2	0.5	-	-23.61	831.31	0.00026496	2.0688	695.46
2	1.0	-	-24.72	849.24	0.00028916	2.2577	700.99
2	1.5	-	-25.89	866.34	0.00031738	2.4781	703.62
2	2.0	-	-27.12	882.57	0.00035067	2.7380	702.78
2	2.5	-	-28.43	897.87	0.00039048	3.0489	697.67
2	3.0	-	-29.83	912.21	0.00043887	3.4267	687.20
2	3.5	-	-31.35	925.53	0.00049886	3.8951	669.77
2	4.0	-	-33.01	937.79	0.00057506	4.4900	642.95
2	4.5	-	-34.86	948.92	0.00067488	5.2695	602.91
2	5.0	-	-37.00	958.88	0.00081110	6.3300	543.02

TABLE VII

8WF 31 BEAM-COLUMN IN PLASTIC REGION

L/r = 80 and P/Py = 0.4

(See Appendix for the detailed computer data

Case	Y_c (in.)	Y_t (in.)	fa (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	-7.20	591.84	0.00018000	1.4054	407.27
2	0.5	-	-8.43	619.49	0.00019744	1.5416	417.03
2	1.0	-	-9.25	632.86	0.00021548	1.6825	411.90
2	1.5	-	-10.12	645.60	0.00023651	1.8467	403.08
2	2.0	-	-11.04	657.69	0.00026132	2.0404	389.73
2	2.5	-	-12.01	669.10	0.00029099	2.2720	344.42
2	3.0	-	-13.06	679.78	0.00032705	2.5536	344.42
2	3.5	-	-14.19	689.71	0.00037175	2.9026	308.51
2	4.0	-	-15.42	698.84	0.00042853	3.3460	259.42
2	4.5	-	-16.81	707.14	0.00050292	3.9268	191.44
2	5.0	-	-18.40	714.56	0.00060443	4.7194	94.77
2	5.5	-	-20.32	721.07	0.00075089	5.8629	-48.89

TABLE VIII

8 W 31 BEAM-COLUMN IN PLASTIC REGION

L/r = 80 and P/Py = 0.6

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	fa (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	7.20	394.56	0.00012000	0.9370	309.99
2	0.5	-	6.76	407.68	0.00012993	1.0145	207.82
2	1.0	-	6.22	416.47	0.00014180	1.1072	198.36
2	1.5	-	5.65	424.86	0.00015564	1.2153	185.46
2	2.0	-	5.05	432.81	0.00017197	1.3427	168.31
2	2.5	-	4.40	440.32	0.00019149	1.4952	145.78
2	3.0	-	3.72	447.35	0.00021522	1.6805	116.31
2	3.5	-	2.97	453.88	0.00024464	1.9102	77.60
2	4.0	-	2.16	459.89	0.00028201	2.2019	26.13
2	4.5	-	1.25	465.35	0.00033096	2.5842	-43.71

TABLE IX

8WF31 BEAM-COLUMN IN PLASTIC REGION

L/r = 120 and P/Py = 0.2

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	fa (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	-21.60	789.12	0.00024000	4.2163	512.26
2	0.5	-	-23.61	831.31	0.00026496	4.6547	525.66
2	1.0	-	-24.72	849.24	0.00028916	5.0799	515.68
2	1.5	-	-25.89	866.34	0.00031738	5.5757	500.22
2	2.0	-	-27.12	882.57	0.00035067	6.1605	478.04
2	2.5	-	-28.43	897.87	0.00039048	6.8600	447.42
2	3.0	-	-29.83	912.21	0.00043887	7.7101	405.93
2	3.5	-	-31.35	925.53	0.00049886	8.7639	350.06
2	4.0	-	-33.01	937.79	0.00057506	10.1025	274.42
2	4.5	-	-34.86	948.92	0.00067488	11.8563	170.39
2	5.0	-	-37.00	958.88	0.00081110	14.2493	23.21

TABLE X

8WF31 BEAM-COLUMN IN PLASTIC REGION

L/r = 120 and P/Py = 0.4

(See Appendix for the detailed computer data)

Case	Y_c (in.)	Y_t (in.)	fa (ksi)	M (in.-kips)	ϕ (rad.)	Δ (in.)	M_o (in.-kips)
1	0.0	-	-7.20	591.84	0.00018000	3.1622	176.55
2	0.5	-	-8.43	619.49	0.00019744	3.4687	163.96
2	1.0	-	-9.25	632.86	0.00021548	3.7855	135.71
2	1.5	-	-10.12	645.60	0.00023651	4.1550	99.93
2	2.0	-	-11.04	657.69	0.00026132	4.5908	54.78
2	2.5	-	-12.01	669.10	0.00029099	5.1121	-2.26

L/r = 120 P/Py = 0.6

1	0.0	-	+7.20	394.56	0.00012000	2.1081	-20.73

L/r = 120 P/Py = 0.8

1	0.0	-	21.60	197.28	0.00006000	1.0542	-92.19

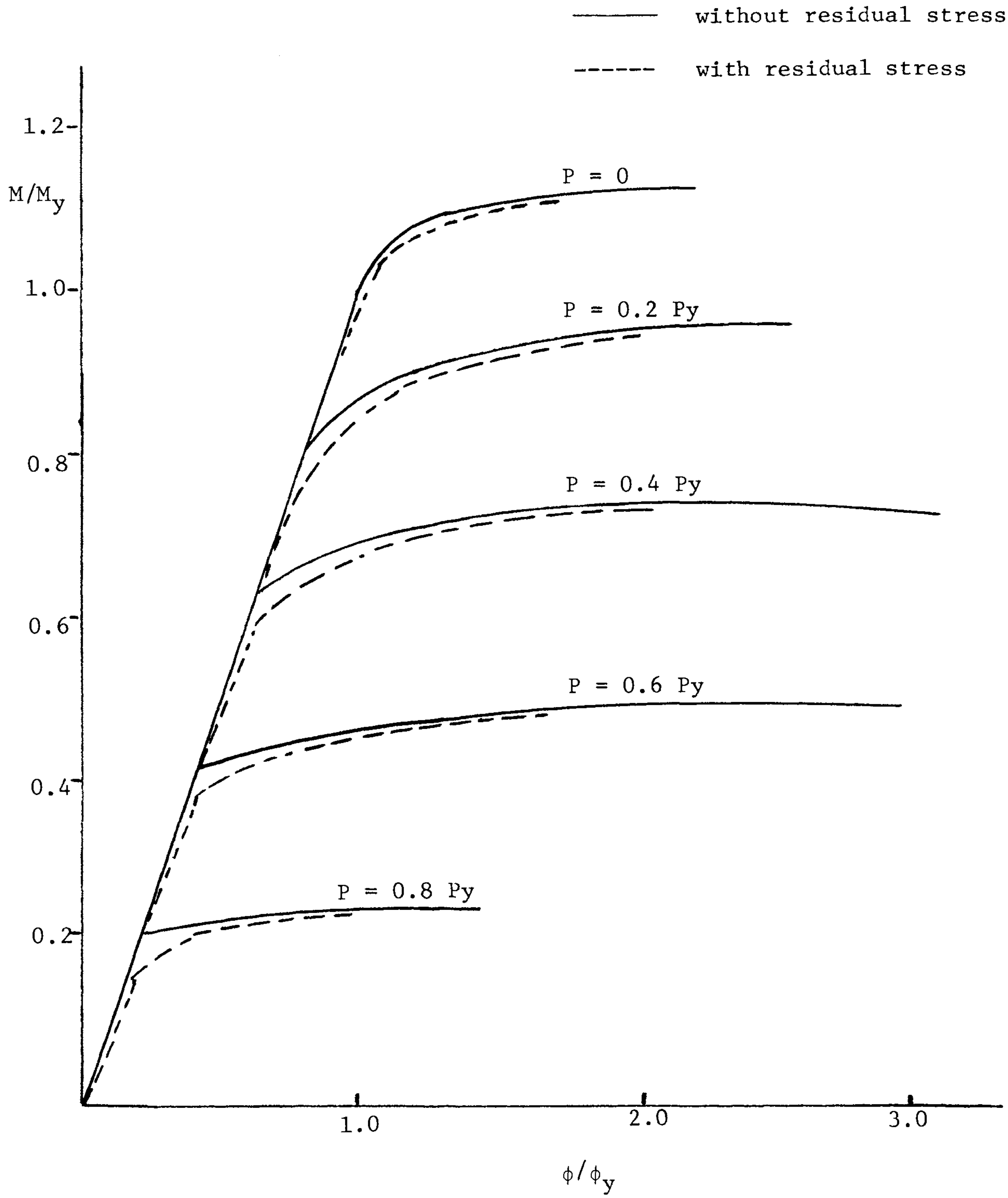


Figure 10. Moment-curvature diagram for 8WF31[10]

The above results from the I.B.M. computer which is available in the computer center of the University of Missouri-Rolla are given in detail in the Appendix.

(2) Moment-Curvature Method

The relationship between moment and curvature is important because the data allows for the determination of the maximum moments and angle changes that a member can substain. Using equations in Chapter II, moment-curvature graphs can be constructed for a given constant axial load P. One set of curves is shown in Figure 10 where the non-linear curves, M-ϕ are shown for P=0, 0.2Py, 0.4Py, 0.6Py and 0.8Py. For P=0, M/My ratio reaches the maximum limit 1.10 which is the shape factor of the 8WF31 wide-flange section.

The M-ϕ curve method is based on a direct numerical method. The process is shown as follows:

A. Assume a center-curvature ϕ.

B. Using Equation (3.2) compute Δ.

C. Entering M-ϕ curve (Figure 10) with the value of ϕ/ϕ_y ratio, pick up the corresponding M/My ratio.

D. From the moment in step C compute M_o from Equation (3.3)

For example:

Given L/r = 40 P/Py = 0.2

L = 138.8 in. $P = 65.7^k$

A. Assuming ϕ = 0.00025 rad. which is greater than first yield angle change ϕ_e.

B. Using Equation (3.2) compute Δ.

The maximum deflection at middle height is

$$\Delta = \frac{\phi L^2}{\pi^2} = \frac{0.00025 \times 138.8^2}{9.86} = 0.489 \text{ in.}$$

C. From $\phi/\phi_y = \frac{0.00025}{0.000299} = 0.835$

Using (Figure 10) $M = 0.815 \times 986.4 = 805''^{k}$

D. Using Equation (3.3)

$M_o = M - P\cdot\Delta = 805 - 65.7 \times 0.489 = 762.9''^{k}$ (end moment)

The following results in Table XI is a set of M_o for $L/r = 40$ and $P/Py = 0.4$.

TABLE XI

COMPUTED MOMENTS FOR BEAM-COLUMNS BASED ON FIGURE 10

L/r = 40 P/Py = 0.4 My = 986.4 in.-kips P = 131.32 kips

ϕ	Δ	$\phi/\phi y$	M/My	M (in.-kips)	$P \cdot \Delta$ (in.-kips)	M_o (in.-kips)
0.00018	0.352	0.600	0.600	591.9	46.2	545.70
0.00022	0.430	0.734	0.635	626.0	56.5	569.50
0.00026	0.507	0.867	0.668	652.8	66.5	586.30
0.00030	0.586	1.000	0.683	672.0	77.0	595.00
0.00034	0.664	1.130	0.691	681.9	87.4	594.60
0.00038	0.742	1.265	0.698	689.8	97.5	592.40
0.00042	0.820	1.400	0.706	698.0	108.0	590.00
0.00046	0.899	1.532	0.715	704.7	118.0	586.70
0.00 050	0.976	1.670	0.718	708.3	128.4	579.90

(3) Numerical Integration Method[6]

Based on the M-ϕ curve, a numerical integration method can be applied to find the maximum moment capacity of the beam-column when the value of M_o and configuration of deflected structure are assumed. Several cycles of calculation are needed before the correct deflected shape is obtained. However, for each set of given values of P and L, the above process must be repeated until enough values of moment have been obtained to construct the M-ϕ curve for determining the critical value of bending moment. The calculation is time consuming and tedious.

The procedure is as follows:

A. Divide the beam-column into equal segments (Roughly λ = 3r or 4r)

B. Try a set of deflection values at each section.

C. Compute the moment due to axial load.

D. Assume an end moment M_o, find total moment.

E. Find the deflection.

F. Compare the deflection value to assumed value, if two values are identical, the deflection is the correct one. If not, repeat above procedures as required.

Example:

Determine the ultimate strength of the following beam-column.

Material: A 36 steel, F_y = 36 ksi, E = 30,000 ksi

Section: 8WF31 $P_y = AF_y = 9.12 \times 36$

$$M_y = S \cdot F_y = 27.4 \times 36 = 986.4 \text{ in.-kips}$$

$$\phi_y = \frac{2F_y}{dE} = \frac{2 \times 36}{8 \times 30{,}000} = 0.000299 \text{ rad.}$$

$L/r = 40$ $\qquad$ $L = 138.8$ in.

$P/Py = 0.4$ $\qquad$ $P = 65.7^{kips}$

Loading condition: Equal end moments.

Results are shown in Table XII (pages 39 and 40).

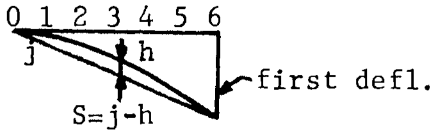
It should be noted that for the assumed end moments of $600^{in.-kips}$ (page 40), the computed moment to yield moment ratio (M/M_y) exceeds the maximum value given in Figure 10 for $P = 0.4\ P_y$. For this reason, the deflection will not converge to its true shape. Thus, the ultimate moment computed on the basis of $M_o = 600^{in.-kips}$ is larger than the maximum bending capacity of the member.

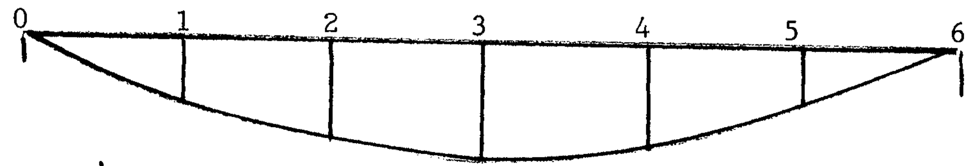
TABLE XII

DATA USED FOR THE EXAMPLE

L/r = 40   L = 138.8 in.   P/Py = 0.4 $P = 131.32^k$

ϕy = 0.000299 rad. M_y = 986.4in.-kips λ= 23.13 in.





a	Assumed end moment	Common factor	546	546	546	546	546	546	546
b	Assumed deflection		0	0.20	0.28	0.35	0.28	0.20	0
c	Moment due to P		0	26.3	36.8	46	36.8	26.3	0
d	Total moment		546	572.3	582.8	592	582.8	572.3	546
e	M/My		0.554	0.58	0.59	0.60	0.59	0.58	0.554
f	curvature	ϕ_y	0.554	0.58	0.59	0.60	0.59	0.58	0.554
g	Slope	$\lambda\phi_y$	0	0.554	1.134	1.724	2.324	2.914	3.494
h	First deflection	$\lambda^2\phi_y$	0	0.554	1.688	3.412	5.736	8.650	12.144
i	Liner correction deflection	$\lambda^2\phi_y$	0	2.02	4.04	6.06	8.08	10.12	12.144
j	Final deflection in (h - i)	$\lambda^2\phi_y$	0	1.47	2.35	2.65	2.35	1.47	0
k	Final deflection in inches		0	0.235	0.376	0.424	0.376	0.235	0
	Final trial deflection	$\lambda^2\phi_y$	0	0.237	0.384	0.432	0.384	0.237	0
	Final deflection in inches		0	0.238	0.384	0.432	0.384	0.238	0

TABLE XII

(continued)

$M_{ult} = M_o + P \cdot \Delta = 546 + 131.32 \times 0.432 = 597.8$ in.-kips

Try end moment = 570 in.-kips	0	1	2	3	4	5	6
Final try deflection	0	0.271	0.444	0.500	0.443	0.271	0
Final deflection in inches	0	0.272	0.444	0.500	0.444	0.272	0

$M_{ult} = 570 + 131.31 \times 0.500 = 615.7$ in.-kips

Try end moment = 590 in.-kips							
Final try deflection	0	0.320	0.527	0.600	0.527	0.325	0
Final deflection in inches	0	0.320	0.530	0.602	0.530	0.320	0

$M_{ult} = 590 + 131.32 \times 0.602 = 669$ in.-kips

Try end moment = 600in.-kips

Deflection is divergent (see discussion on page 38)

(4) Development of Interaction Curves

A. Ultimate strength interaction curves

The results of the ultimate strength calculation for beam-columns are presented best in the form of P-M-L interaction curves. A set of these is given in Figure 11. These particular interaction curves are for the case of two equal end moments causing single-curvature deformation about the strong axis of an 8WF31 member.

Each curve in Figure 11 shows the relationship between P/Py and M/Mp for given slenderness ratio L/r.

The following observations can be made about these interaction curves:

a. When P=0, the member is a beam and can support a moment equal to Mp.

b. When M=0, the member is a column which is able to carry a load equal to its own critical load.

c. Except P=0, M=0, extreme cases, between these extremes, beam-column action takes place.

d. For a given value of P, the member for which $L/r_x=0$ can carry considerably more moment than the member with $L/r_x=120$. Thus, short members are stronger than the long members.

e. Up to $L/r_x = 60$, the interaction curves are nearly straight lines.

For higher slenderness ratios the curve sags downward, thus showing the larger influence of secondary moments due to deflection.

Fortunately, the variation between M-ϕ-P curves, and thus the interaction curves, is almost negligible for different WF - shapes, and therefore the interaction curves developed for the 8WF31 shape can be used for other sections also. It has been shown that the curves are slightly on the conservative side for all other WF-shapes.

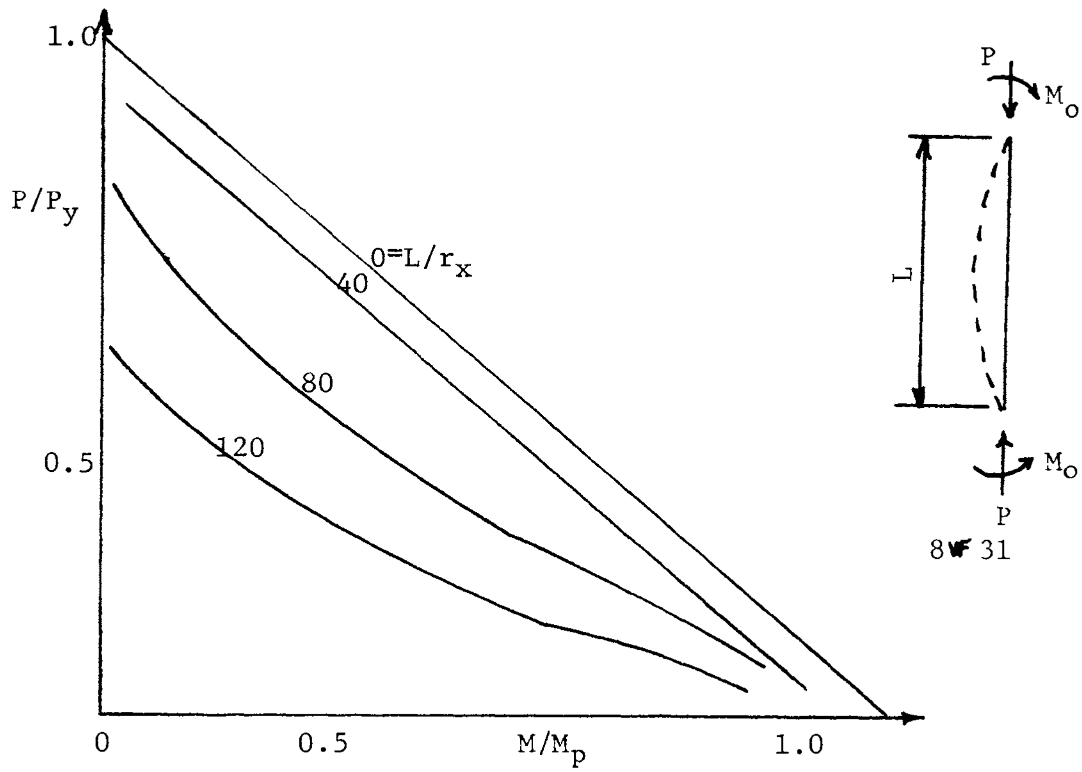


Figure 11. Interaction curves[5]

B. Interaction equations

The relatively simple, empirically determined beam-column interactions have proven themselves more popular with specification writers than other methods because the interaction equations are easily adaptable to a multitude of situations. Most significantly, these formulas permit ready inclusion of provisions for lateral-torsional buckling and biaxial bending.

In the following discussion an ultimate strength interaction equation will be derived from the analytically determined ultimate strength interaction curves such as those shown in Figure 12. The dashed line represents the analytically developed curve, and the solid line corresponds to the approximation.

$M/Mp = 1.18\ [1 - P/P_y]$ ———

Analytical solution --------

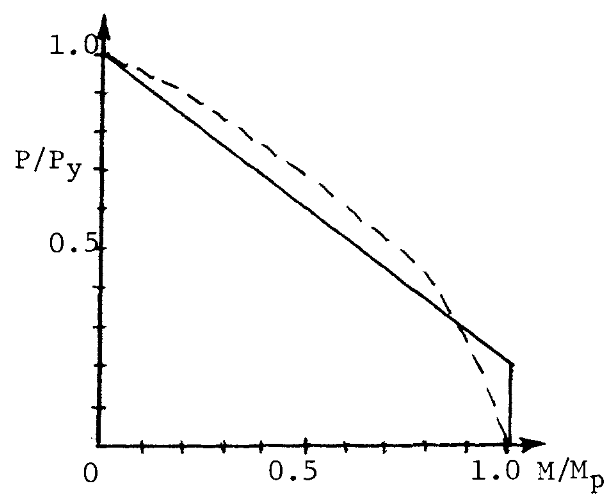


Figure 12. Interaction curve for "Zero Length" member[6]

C. The straight-line interaction formula

$P/P_o + M/M_p = 1.0$ when $L/r_x \leq 40$

$P/P_o + M/M_p \left(\frac{1}{1 - P/P_e}\right) = 1.0$ when $40 < L/r_x < 120$

where:

$$\frac{1}{1 - P/P_e} = \text{Amplification factor}$$

Where P_e = The elastic buckling load of the member in the plane of bending or in nondimensional form

$$P_e/P_y = \frac{\pi^2 E}{F_y} \left(\frac{1}{L/r_x}\right)^2$$

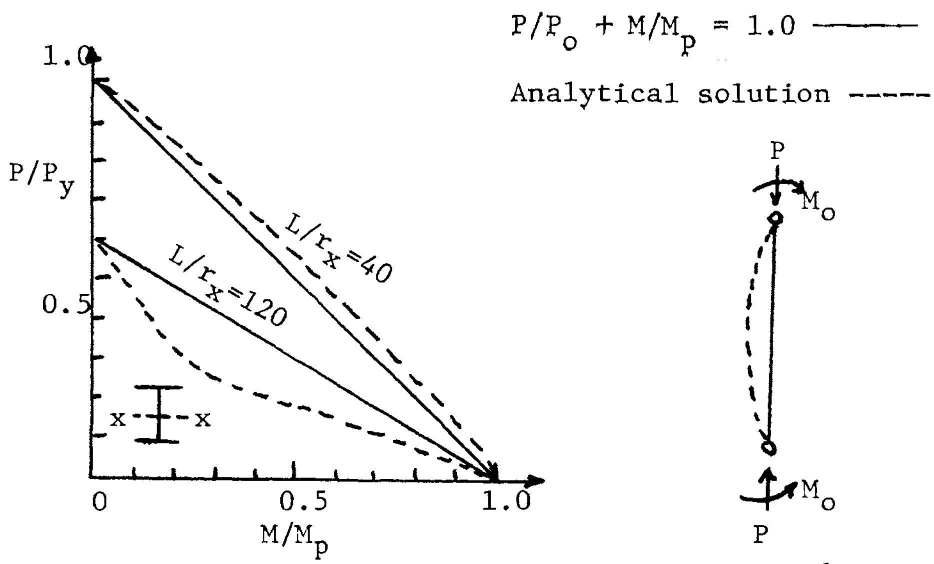


Figure 13. Straight-line interaction formula[6]

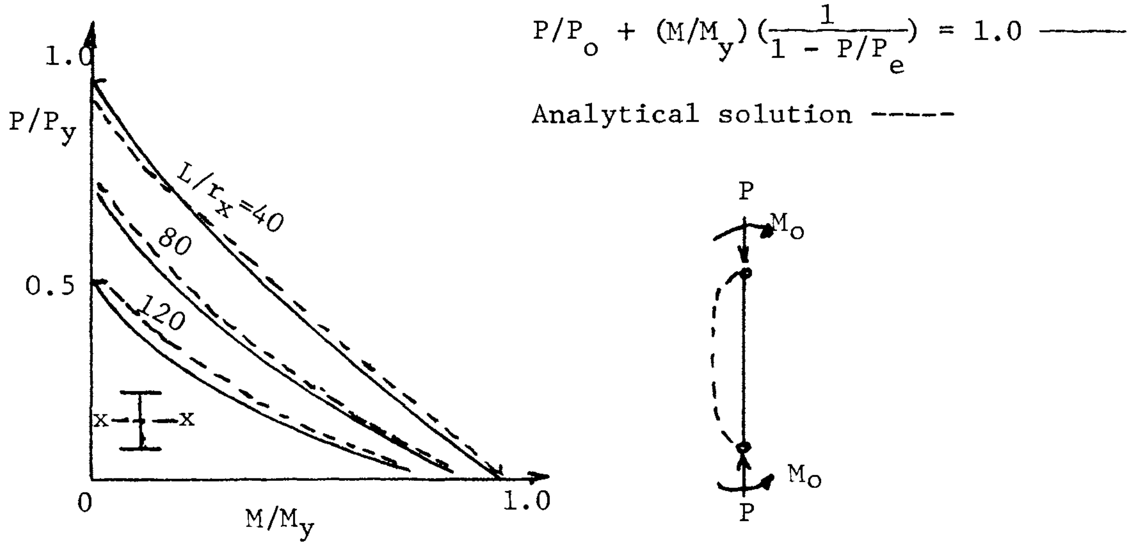


Figure 14. The interaction formula with amplification factor[6]

D. AISC Specification[9]

Case (1) for columns bent in single curvature

$$M_o/M_p = 1.0 - H(P/P_y) - J(P/Py)^2 \quad L/r < 120 \tag{3.4}$$

H and J are numerical coefficient values which depend on L/r ratio and steel yielding stress.

(5) Discussion and Comparison of Results

The calculations in this Chapter were based on the assumption that the deflected shape of the beam-column was a single sine curve. Since the deflection at mid-height is small compared to the column length, satisfactory results were obtained. The sine curve assumption yields good results for a column subjected to equal end moments with opposite direction. For columns loaded with either equal end moment in same direction or with end moment at one end only, the assumption is no longer applicable.

There is no problem to determine the ultimate moment of beam-column by direct numerical method or numerical integration method. However, the relatively simple, empirically determined beam-column interaction equations have proven themselves more popular with specification because the interaction equations are easily adaptable and conservative. In practice, most engineers are interested in actually applied moment at ends instead of the ultimate capacity of beam-column at mid height.

The results of an example obtained by different methods as presented in this Chapter may be summarized as follows:

A. Direct numerical method (page 25)

M_{ult} = 679.78 in.-kips

M_o = 595.94 in.-kips

B. Numerical integration method

M_{ult} = 669.0 in.-kips

M_o = 590.0 in.-kips

C. Straight-line interaction formula (without correction by an amplification factor)

$$M_o = M_p\ (1 - P/P_y)$$
$$= 1075.6 \times (1 - 0.4)$$
$$= 645.4 \text{ in.-kips}$$

D. A.I.S.C. formula

$M_o/M_p = 1.0 - H(P/Py) - J(P/Py)^2$

where, H = 1.020 and J = 0.154 for L/r = 40

M_p = 1075.6 in.-kips, P/Py = 0.4

substituting H, J, M and P/Py into the above equation, solve for M_o,

$M_o = 1075.6\ (1.0 - 1.020 \times 0.4 - 0.154 \times 0.4^2)$

$= 1075.6 \times 0.5674$

= 610.3 in.-kips

The above comparison indicates that the end moment computed by the straight-line interaction formula without correction by an amplification factor is about 8% higher than those computed by the direct numerical method and the numerical integration method. The end moment using AISC formula would provide closer agreement between the computed values.

(6) Influence of Residual Stresses[7]

Residual stresses are formed in a structural member as a result of plastic deformations. They are stresses which exist in the cross section even before the application of an external load. These plastic deformations may be due to cooling after hot-rolling or welding, or due to fabrication operations such as cold-bending or cambering. In rolled shapes, these deformations always occur during the process of cooling from the rolling temperature to air temperature; the plastic deformations result from the fact that some parts of the shape cool much more rapidly than others, causing inelastic deformations in the slower cooling portions. However, the magnitude and distribution of residual stresses depends on the shape of the cross section, rolling or welding temperature, cooling condition, and material properties. Typical cooling residual stresses for WF-shapes result in average compressive residual stresses at the flange tip of about 13 ksi for structural carbon steel A-7, A-36 and also A-242.

Due to the presence of residual stresses, yielding commences as the sum of the applied compressive stress and the pre-existing maximum compressive residual stress becomes equal to the yield stress

$$f_y = f_{rc} + f_c \tag{3.5a}$$

or interaction equation defining this limit is,

$$P/P_y + \psi \frac{M_o}{M_y} = 1 - \frac{f_{rc}}{f_y} \tag{3.5b}$$

where, f_{rc} is the compression residual stress and ψ is a factor by which the end moments are modified to obtain the maximum moment.

Because of the increased amount of yielding with each increment in M, the stiffness of the member is reduced progressively until finally the overall member stiffness becomes zero and no additional moment can be supported at the peak of the curve.

Figure 15 illustrates two conditions. The solid lines represent cases where residual stresses were neglected. Dotted lines denote cases where residual stresses were included. The series of plotted points represent Equation (3.4).

The method (Equations 3.5a and 3.5b) can be applied to members containing residual stresses after a distribution pattern for such stress has been assumed. Then the idealized moment curvature diagram as shown in Figure 10 should be corrected as indicated by dotted lines which represent the influence of residual stresses.

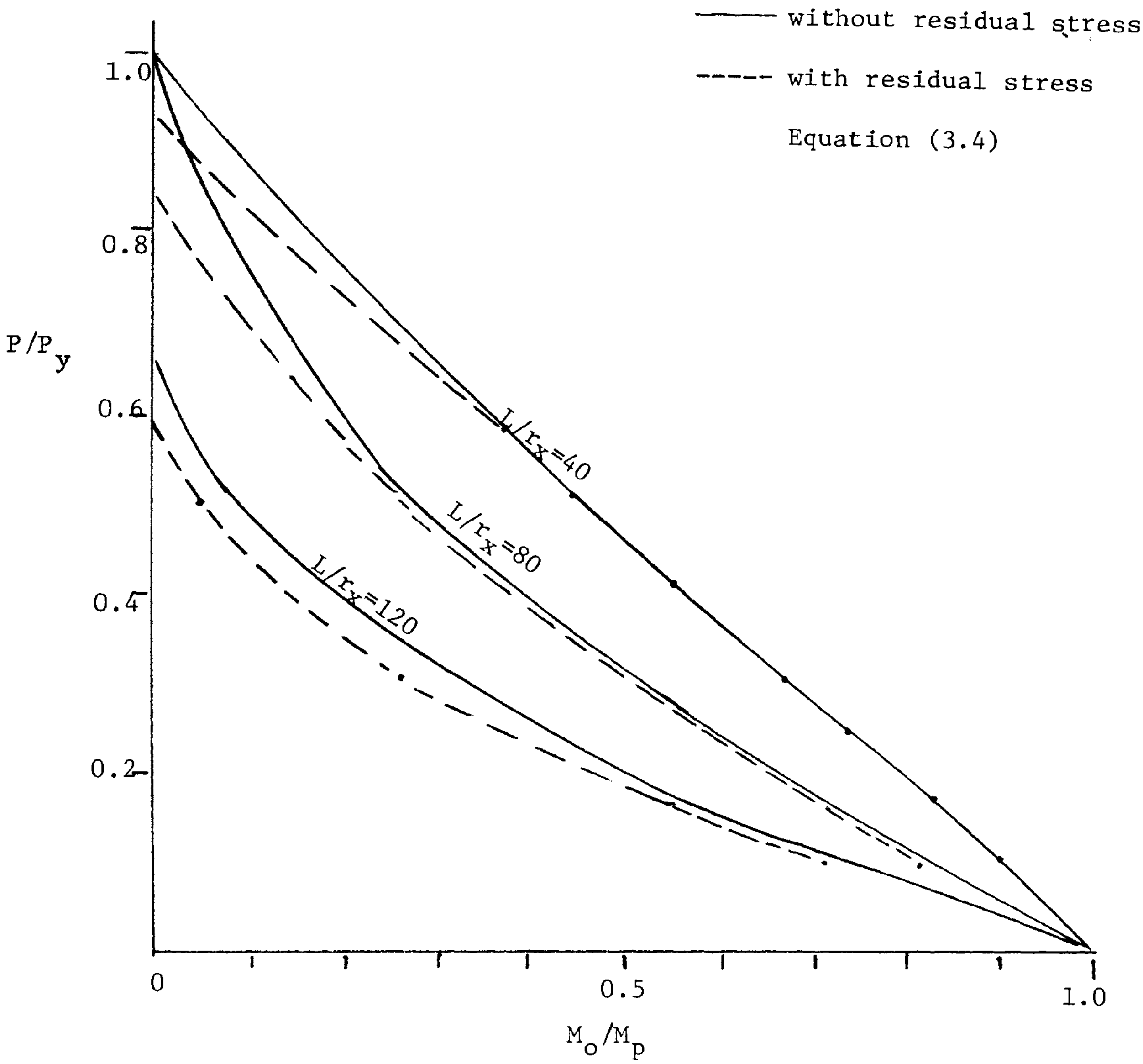


Figure 15. Interaction curves with and without residual stresses[7]

IV. CONCLUSIONS

Three methods were presented in this thesis for determining the maximum strength of beam-columns subjected to a thrust and equal end moments with opposite direction. Failure due to lateral torsional or local buckling was not considered.

In this presentation, the influence of an axial load on M-ϕ relationship in the plastic range, considering the combination of thrust, moment and curvature were discussed in Chapter III. In addition, approximate interaction equations were adopted from other references in order to compare the results with the analytical study presented herein.

This investigation indicates that (1) the ultimate moment of beam-columns can be predicted by the direct numerical method or numerical integration method, (2) satisfactory results can be obtained by assuming a single sine curve for the deflected shape, and (3) the AISC formula provides a satisfactory result for end bending moments.

V. REFERENCES

1. Robert L. Ketter and Edmund L. Kaminsky, "Plastic Deformation of Wide-Flange Beam-Columns," Transactions of ASCE, Vol. 120, p. 1028, 1955.

2. F. R. Shanley, "Applied Column Theory," Transaction of ASCE, Vol. 115, p. 698, 1950.

3. Frank Baron and Harold S. Davis, "Pressure Line and Inelastic Buckling of Column," Transaction of ASCE, Vol. 124, 1959.

4. Robert L. Ketter, "Further Studies of the Strength of Beam-Columns," Transaction of ASCE, Vol. 127, 1962.

5. Lynn S. Beedle, "Plastic Design of Steel Frame," John Wiley and Sons, Inc., New York, 1958.

6. Beedle, Blackmon, Copper, Driscoll, Eney, Errera, Estuar, Fisher, Galambos, Hansell, Lay, Levi, Lu, Ostapenko, Reemsnyder, Rumpf, Slutter, Tall, Yen, "Structural Steel Design," Chapter 11, Combined Bending and Compression, The Ronald Press Company, 1964, pp. 344-412.

7. Boris Bresler and T. Y. Lin, "Design of Steel Structures," Compression Member, John Wiley and Sons, Inc., New York, 4th Printing, 1964, pp. 280-363.

8. Bowin H. Gaylord, Jr. and Charles N. Gaylord, "Design of Steel Structures," Chapter 4, Compression Members, McGraw-Hill Book Company, Inc., New York, 1957, pp. 91-136.

9. American Institute of Steel Construction, "Manual of Steel Construction," 6th Edition, 1964.

10. Theodore V. Galambos, "Structural Members and Frames," Prentice-Hall, 1968.

VI. APPENDIX - COMPUTER PROGRAMS

APPENDIX

The Appendix contains two computer programs used for preparation of Tables I to X in the text.

The first computer program (page 53) was used to compute first yield values for L/r = 40, 80 and 120 combined with different P/Py ratios ranging from 0.2 to 0.8 for each specific L/r ratio.

The second computer program (pages 54 to 61) was used to calculate ultimate and end moments based on various L/r ratios (40, 80 and 120). The P/Py ratios used in this investigation ranged from 0.4 to 0.8.

COMPUTER DATA FOR TABLE I

```
      READ 100,A,S,R,FY,E,D
  100 FORMAT (6F12.2)
      PRINT 100,A,S,R,FY,E,D
      PY=FY*A
      DO 200 J=40,120,40
      B=J
      X=B*R
      DO 200 I=1,4,1
      C=I
      CC=C/5.
      P=CC*PY
      FA=2.*P/A-FY
      EM=(FY-FA)/2.*S
      RAD=(FY-FA)/(E*D)
      DE=(X**2*RAD)/(3.1416**2)
      EOM=EM-P*DE
  200 PRINT 300,J,CC,FA,EM,RAD,DE,EOM
  300 FORMAT (I5,F10.1,F10.2,F10.2,F15.8,F10.4,F10.2)
      STOP
      END
        9.12        27.40         3.47        36.00    30000.00        8.00

C***12868CEX026  N C TSAO              02/10/66 FORMO         00FORMO 02/10/66-
        9.12        27.40         3.47        36.00    30000.00        8.00
```

40	.2	-21.60	789.12	.00024000	.4685	758.36
40	.4	-7.20	591.84	.00018000	.3514	545.70
40	.6	7.20	394.56	.00012000	.2342	348.42
40	.8	21.60	197.28	.00006000	.1171	166.52
80	.2	-21.60	789.12	.00024000	1.8739	666.07
80	.4	-7.20	591.84	.00018000	1.4054	407.27
80	.6	7.20	394.56	.00012000	.9370	209.99
80	.8	21.60	197.28	.00006000	.4685	74.23
120	.2	-21.60	789.12	.00024000	4.2163	512.26
120	.4	-7.20	591.84	.00018000	3.1622	176.55
120	.6	7.20	394.56	.00012000	2.1081	-20.73
120	.8	21.60	197.28	.00006000	1.0541	-79.58

COMPUTER DATA FOR TABLE II

```
      READ 100,D,B,TF,TW,FY,X,A
  100 FORMAT (7F10.3)
      E=30000.0
      N=40
      PY=FY*A
      DO 10 J=1,4
      Z=J
      ZZ=Z/5.
      P=ZZ*PY
      DO 10 I=1,15
      Y=I
      YY=0.5*Y
      W=YY/8.
      F1=FY*TW*D+TF*FY*(B-TW)-P
      F2=FY*(B*TF**2-TW*D**2*(1.-W)**2-TW*TF**2)/(2.*D*(1.-W))
      F3=F2/FY-TF*(B-TW)
      FA=(F1+F2)/F3
      E1=B*TF*(6.*D*(1.-W)*(D-TF)-3.*D*TF+4.*TF**2)
      E2=TW*((D*(1.-W-TW/D))**2*(D*(1.+2.*W)-4.*TF))
      EM=(FY-FA)*(E1+E2)/(12.*D*(1.-W))
      RAD=(FY-FA)/(E*D*(1.-W))
      DE=(X**2*RAD)/(3.1416**2)
      EOM=EM-P*DE
   10 PRINT 200,N,ZZ,YY,FA,EM,RAD,DE,EOM
  200 FORMAT (I5,F5.1,F5.1,F10.2,F10.2,F15.8,F10.4,F10.2)
      STOP
      END
```

40	.2	.5	-23.61	831.31	.00026496	.5172	797.35
40	.2	1.0	-24.72	849.24	.00028916	.5644	812.18
40	.2	1.5	-25.89	866.34	.00031738	.6195	825.66
40	.2	2.0	-27.12	882.57	.00035067	.6845	837.62
40	.2	2.5	-28.43	897.87	.00039048	.7622	847.82
40	.2	3.0	-29.83	912.21	.00043887	.8567	855.96
40	.2	3.5	-31.35	925.53	.00049886	.9738	861.59
40	.2	4.0	-33.01	937.79	.00057506	1.1225	864.08

40	.2	4.5	-34.86	948.92	.00067488	1.3174	862.42
40	.2	5.0	-37.00	958.88	.00081110	1.5833	854.91
40	.2	5.5	-39.57	967.61	.00100763	1.9669	838.46
40	.2	6.0	-42.91	975.10	.00131519	2.5672	806.52
40	.2	6.5	-47.85	981.37	.00186341	3.6374	742.52
40	.2	7.0	-57.36	986.77	.00311205	6.0747	587.88
40	.2	7.5	-95.12	994.61	.00874162	17.0635	-125.85
40	.4	.5	-8.43	619.49	.00019744	.3854	568.88
40	.4	1.0	-9.25	632.86	.00021548	.4206	577.62
40	.4	1.5	-10.12	645.60	.00023651	.4617	584.97
40	.4	2.0	-11.04	657.69	.00026132	.5101	590.70
40	.4	2.5	-12.01	669.10	.00029099	.5680	594.50
40	.4	3.0	-13.06	679.78	.00032705	.6384	595.94
40	.4	3.5	-14.19	689.71	.00037175	.7257	594.41
40	.4	4.0	-15.42	698.84	.00042853	.8365	588.99
40	.4	4.5	-16.81	707.14	.00050292	.9817	578.21
40	.4	5.0	-18.40	714.56	.00060443	1.1798	559.61
40	.4	5.5	-20.32	721.07	.00075089	1.4657	528.58
40	.4	6.0	-22.80	726.64	.00098008	1.9131	475.40
40	.4	6.5	-26.49	731.32	.00138862	2.7106	375.34
40	.4	7.0	-33.57	735.34	.00231910	4.5269	140.84
40	.4	7.5	-61.71	741.19	.00651427	12.7158	-928.75
40	.6	.5	6.76	407.68	.00012993	.2536	357.71
40	.6	1.0	6.22	416.47	.00014180	.2768	361.94
40	.6	1.5	5.65	424.86	.00015564	.3038	365.01
40	.6	2.0	5.05	432.81	.00017197	.3357	366.69
40	.6	2.5	4.40	440.32	.00019149	.3738	366.68
40	.6	3.0	3.72	447.35	.00021522	.4201	364.59
40	.6	3.5	2.97	453.88	.00024464	.4775	359.81
40	.6	4.0	2.16	459.89	.00028201	.5505	351.45
40	.6	4.5	1.25	465.35	.00033096	.6460	338.09
40	.6	5.0	.20	470.24	.00039777	.7764	317.28
40	.6	5.5	-1.06	474.52	.00049414	.9646	284.51
40	.6	6.0	-2.70	478.19	.00064497	1.2590	230.18

40	.6	6.5	-5.12	481.27	.00091382	1.7838	129.88
40	.6	7.0	-9.78	483.91	.00152616	2.9790	-102.93
40	.6	7.5	-28.30	487.76	.00428691	8.3680	-1160.67
40	.8	.5	21.95	195.86	.00006242	.1219	163.85
40	.8	1.0	21.69	200.08	.00006813	.1330	165.16
40	.8	1.5	21.42	204.11	.00007478	.1460	165.78
40	.8	2.0	21.13	207.94	.00008262	.1613	165.58
40	.8	2.5	20.82	211.54	.00009200	.1796	164.37
40	.8	3.0	20.49	214.92	.00010340	.2018	161.91
40	.8	3.5	20.13	218.06	.00011753	.2294	157.80
40	.8	4.0	19.74	220.95	.00013549	.2645	151.48
40	.8	4.5	19.30	223.57	.00015901	.3104	142.05
40	.8	5.0	18.80	225.92	.00019110	.3730	127.94
40	.8	5.5	18.19	227.97	.00023740	.4634	106.26
40	.8	6.0	17.41	229.74	.00030986	.6048	70.87
40	.8	6.5	16.24	231.21	.00043903	.8570	6.12
40	.8	7.0	14.00	232.49	.00073321	1.4312	-143.43
40	.8	7.5	5.11	234.33	.00205956	4.0202	-821.61

```
      READ 100,D,B,TF,TW,FY,X,A
  100 FORMAT (7F10.3)
      E=30000.0
      N=80
      PY=FY*A
      DO 10 J=1,4
      Z=J
      ZZ=Z/5.
      P=ZZ*PY
      DO 10 I=1,15
      Y=I
      YY=0.5*Y
      W=YY/8.
      F1=FY*TW*D+TF*FY*(B-TW)-P
      F2=FY*(B*TF**2-TW*D**2*(1.-W)**2-TW*TF**2)/(2.*D*(1.-W))
      F3=F2/FY-TF*(B-TW)
      FA=(F1+F2)/F3
      E1=B*TF*(6.*D*(1.-W)*(D-TF)-3.*D*TF+4.*TF**2)
```

```
      E2=TW*((D*(1.-W-TW/D))**2*(D*(1.+2.*W)-4.*TF))
      EM=(FY-FA)*(E1+E2)/(12.*D*(1.-W))
      RAD=(FY-FA)/(E*D*(1.-W))
      DE=(X**2*RAD)/(3.1416**2)
      EOM=EM-P*DE
   10 PRINT 200,N,ZZ,YY,FA,EM,RAD,DE,EOM
  200 FORMAT (I5,F5.1,F5.1,F10.2,F10.2,F15.8,F10.4,F10.2)
      STOP
      END
```

80	.2	.5	-23.61	831.31	.00026496	2.0688	695.46
80	.2	1.0	-24.72	849.24	.00028916	2.2577	700.99
80	.2	1.5	-25.89	866.34	.00031738	2.4781	703.62
80	.2	2.0	-27.12	882.57	.00035067	2.7380	702.78
80	.2	2.5	-28.43	897.87	.00039048	3.0489	697.67
80	.2	3.0	-29.83	912.21	.00043887	3.4267	687.20
80	.2	3.5	-31.35	925.53	.00049886	3.8951	669.77
80	.2	4.0	-33.01	937.79	.00057506	4.4900	642.95
80	.2	4.5	-34.86	948.92	.00067488	5.2695	602.91
80	.2	5.0	-37.00	958.88	.00081110	6.3330	543.02
80	.2	5.5	-39.57	967.61	.00100763	7.8675	451.00
80	.2	6.0	-42.91	975.10	.00131519	10.2689	300.80
80	.2	6.5	-47.85	981.37	.00186341	14.5494	25.99
80	.2	7.0	-57.36	986.77	.00311205	24.2987	-608.79
80	.2	7.5	-95.12	994.61	.00874162	68.2542	-3487.23
80	.4	.5	-8.43	619.49	.00019744	1.5416	417.03
80	.4	1.0	-9.25	632.86	.00021548	1.6825	411.90
80	.4	1.5	-10.12	645.60	.00023651	1.8467	403.08
80	.4	2.0	-11.04	657.69	.00026132	2.0404	389.73
80	.4	2.5	-12.01	669.10	.00029099	2.2720	370.71
80	.4	3.0	-13.06	679.78	.00032705	2.5536	344.42
80	.4	3.5	-14.19	689.71	.00037175	2.9026	308.51
80	.4	4.0	-15.42	698.84	.00042853	3.3460	259.42
80	.4	4.5	-16.81	707.14	.00050292	3.9268	191.44

COMPUTER DATA FOR TABLES VII AND VIII

80	.4	5.0	-13.40	714.56	.00060443	4.7194	94.77
80	.4	5.5	-20.32	721.07	.00075089	5.8629	-48.89
80	.4	6.0	-22.80	726.64	.00098008	7.6524	-278.33
80	.4	6.5	-26.49	731.32	.00138862	10.8423	-692.58
80	.4	7.0	-33.57	735.34	.00231910	18.1074	-1642.68
80	.4	7.5	-61.71	741.19	.00651427	50.8631	-5938.56
80	.6	.5	6.76	407.68	.00012993	1.0145	207.82
80	.6	1.0	6.22	416.47	.00014180	1.1072	198.36
80	.6	1.5	5.65	424.86	.00015564	1.2153	185.46
80	.6	2.0	5.05	432.81	.00017197	1.3427	168.31
80	.6	2.5	4.40	440.32	.00019149	1.4952	145.78
80	.6	3.0	3.72	447.35	.00021522	1.6805	116.31
80	.6	3.5	2.97	453.88	.00024464	1.9102	77.60
80	.6	4.0	2.16	459.89	.00028201	2.2019	26.13
80	.6	4.5	1.25	465.35	.00033096	2.5842	-43.71
80	.6	5.0	.20	470.24	.00039777	3.1057	-141.57
80	.6	5.5	-1.06	474.52	.00049414	3.8582	-285.52
80	.6	6.0	-2.70	478.19	.00064497	5.0359	-513.84
80	.6	6.5	-5.12	481.27	.00091382	7.1351	-924.29
80	.6	7.0	-9.78	483.91	.00152616	11.9162	-1863.48
80	.6	7.5	-28.30	487.76	.00428691	33.4720	-6105.96
80	.8	.5	21.95	195.86	.00006242	.4874	67.84
80	.8	1.0	21.69	200.08	.00006813	.5319	60.37
80	.8	1.5	21.42	204.11	.00007478	.5838	50.76
80	.8	2.0	21.13	207.94	.00008262	.6451	38.50
80	.8	2.5	20.82	211.54	.00009200	.7183	22.87
80	.8	3.0	20.49	214.92	.00010340	.8073	2.87
80	.8	3.5	20.13	218.06	.00011753	.9177	-22.98
80	.8	4.0	19.74	220.95	.00013549	1.0579	-56.91
80	.8	4.5	19.30	223.57	.00015901	1.2415	-102.52
80	.8	5.0	18.80	225.92	.00019110	1.4921	-165.99
80	.8	5.5	18.19	227.97	.00023740	1.8536	-258.89
80	.8	6.0	17.41	229.74	.00030986	2.4194	-405.73
80	.8	6.5	16.24	231.21	.00043903	3.4279	-669.15
80	.8	7.0	14.00	232.49	.00073321	5.7249	-1271.19
80	.8	7.5	5.11	234.33	.00205956	16.0809	-3[illegible]

```
      READ 100,D,B,TF,TW,FY,X,A
  100 FORMAT (7F10.3)
      E=30000.0
      N=120
      PY=FY*A
      DO 10 J=1,4
      Z=J
      ZZ=Z/5.
      P=ZZ*PY
      DO 10 I=1,15
      Y=I
      YY=0.5*Y
      W=YY/8.
      F1=FY*TW*D+TF*FY*(B-TW)-P
      F2=FY*(B*TF**2-TW*D**2*(1.-W)**2-TW*TF**2)/(2.*D*(1.-W))
      F3=F2/FY-TF*(B-TW)
      FA=(F1+F2)/F3
      E1=B*TF*(6.*D*(1.-W)*(D-TF)-3.*D*TF+4.*TF**2)
      E2=TW*((D*(1.-W-TW/D))**2*(D*(1.+2.*W)-4.*TF))
      EM=(FY-FA)*(E1+E2)/(12.*D*(1.-W))
      RAD=(FY-FA)/(E*D*(1.-W))
      DE=(X**2*RAD)/(3.1416**2)
      EOM=EM-P*DE
   10 PRINT 200,N,ZZ,YY,FA,EM,RAD,DE,EOM
  200 FORMAT (I5,F5.1,F5.1,F10.2,F10.2,F15.8,F10.4,F10.2)
      STOP
      END
```

120	.2	.5	-23.61	831.31	.00026496	4.6547	525.66
120	.2	1.0	-24.72	849.24	.00028916	5.0799	515.68
120	.2	1.5	-25.89	866.34	.00031738	5.5757	500.22
120	.2	2.0	-27.12	882.57	.00035067	6.1605	478.04
120	.2	2.5	-28.43	897.87	.00039048	6.8600	447.42
120	.2	3.0	-29.83	912.21	.00043887	7.7101	405.93
120	.2	3.5	-31.35	925.53	.00049886	8.7639	350.06
120	.2	4.0	-33.01	937.79	.00057806	10.1025	274.42

COMPUTER DATA FOR TABLE IX AND X

120	.2	4.5	-34.86	948.92	.00067488	11.8563	170.39
120	.2	5.0	-37.00	958.88	.00081110	14.2493	23.21
120	.2	5.5	-39.57	967.61	.00100763	17.7019	-194.77
120	.2	6.0	-42.91	975.10	.00131519	23.1050	-542.07
120	.2	6.5	-47.85	981.37	.00186341	32.7362	-1168.23
120	.2	7.0	-57.36	986.77	.00311205	54.6722	-2603.23
120	.2	7.5	-95.12	994.61	.00874162	153.5719	-9089.53
120	.4	.5	-8.43	619.49	.00019744	3.4687	163.96
120	.4	1.0	-9.25	632.86	.00021548	3.7855	135.71
120	.4	1.5	-10.12	645.60	.00023651	4.1550	99.93
120	.4	2.0	-11.04	657.69	.00026132	4.5908	54.78
120	.4	2.5	-12.01	669.10	.00029099	5.1121	-2.26
120	.4	3.0	-13.06	679.78	.00032705	5.7456	-74.77
120	.4	3.5	-14.19	689.71	.00037175	6.5309	-167.98
120	.4	4.0	-15.42	698.84	.00042853	7.5284	-289.85
120	.4	4.5	-16.81	707.14	.00050292	8.8353	-453.19
120	.4	5.0	-18.40	714.56	.00060443	10.6186	-679.97
120	.4	5.5	-20.32	721.07	.00075089	13.1915	-1011.34
120	.4	6.0	-22.80	726.64	.00098008	17.2179	-1534.55
120	.4	6.5	-26.49	731.32	.00138862	24.3951	-2472.44
120	.4	7.0	-33.57	735.34	.00231910	40.7418	-4615.19
120	.4	7.5	-61.71	741.19	.00651427	114.4420	-14288.25
120	.6	.5	6.76	407.68	.00012993	2.2827	-41.99
120	.6	1.0	6.22	416.47	.00014180	2.4912	-74.27
120	.6	1.5	5.65	424.86	.00015564	2.7343	-113.78
120	.6	2.0	5.05	432.81	.00017197	3.0211	-162.33
120	.6	2.5	4.40	440.32	.00019149	3.3641	-222.39
120	.6	3.0	3.72	447.35	.00021522	3.7810	-297.49
120	.6	3.5	2.97	453.88	.00024464	4.2979	-392.76
120	.6	4.0	2.16	459.89	.00028201	4.9543	-516.06
120	.6	4.5	1.25	465.35	.00033096	5.8144	-680.03
120	.6	5.0	.20	470.24	.00039777	6.9879	-906.33
120	.6	5.5	-1.06	474.52	.00049414	8.6811	-1235.58
120	.6	6.0	-2.70	478.19	.00064497	11.3308	-1753.88

COMPUTER DATA FOR TABLE X

120	.6	6.5	-5.12	481.27	.00091382	16.0539	-2681.23
120	.6	7.0	-9.78	483.91	.00152616	26.8114	-4797.71
120	.6	7.5	-28.30	487.76	.00428691	75.3121	-14348.11
120	.8	.5	21.95	195.86	.00006242	1.0967	-92.19
120	.8	1.0	21.69	200.08	.00006813	1.1968	-114.27
120	.8	1.5	21.42	204.11	.00007478	1.3136	-140.92
120	.8	2.0	21.13	207.94	.00008262	1.4514	-173.30
120	.8	2.5	20.82	211.54	.00009200	1.6162	-212.97
120	.8	3.0	20.49	214.92	.00010340	1.8165	-262.20
120	.8	3.5	20.13	218.06	.00011753	2.0648	-324.28
120	.8	4.0	19.74	220.95	.00013549	2.3802	-404.23
120	.8	4.5	19.30	223.57	.00015901	2.7934	-510.13
120	.8	5.0	18.80	225.92	.00019110	3.3572	-655.87
120	.8	5.5	18.19	227.97	.00023740	4.1706	-867.47
120	.8	6.0	17.41	229.74	.00030986	5.4436	-1200.07
120	.8	6.5	16.24	231.21	.00043903	7.7128	-1794.60
120	.8	7.0	14.00	232.49	.00073321	12.8810	-3150.78
120	.8	7.5	5.11	234.33	.00205956	36.1821	-9269.12

VII. VITA

Ning-Cheng Tsao was born in China on December 30, 1933. He attended high school at Provincial Pingtung High School. He received his Bachelor of Science Degree in Civil Engineering in June of 1958 from Provincial Cheng Kung University, Tainan, Taiwan, China. From August, 1958 to March, 1960, he served in the Chinese Army as a Second Lieutenant. From 1960 to 1963, he worked on the Shehmem Dam Project, Taiwan, China, as a junior engineer. In September 1964 he entered the graduate school of the University of Missouri-Rolla, Rolla Missouri, for advanced study in structural engineering leading to the degree of Master of Science in Civil Engineering.