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On time-optimal second order discrete control systems

Chao-Tai Li

ON TIME-OPTIMAL SECOND ORDER DISCRETE CONTROL SYSTEMS

BY

CHAO-TAI LI

A

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ABSTRACT

The time optimal control problem in unforced discrete systems is studied in this thesis. Comparison is made between the discrete and the continuous control systems by means of minimum time isochrones. Concerning optimal time, it is shown that using the discrete control system will take at most one more sampling period of time to go to equilibrium.

An investigation is also made for the case when the time constant of the physical plant, G(s), and the control model are different. In such a case, an optimal trajectory can not be obtained. An adaptive process is proposed to adjust the model of the controller to get an almost optimal control.

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CHAPTER I

INTRODUCTION

A. Statement of the Problem

In recent years a considerable amount of work has been devoted to the problem of the optimal control systems, especially for the continuous case. There were also some papers that investigated the use of sampled-data techniques and resulted in certain control systems superior in many respects to unsampled systems. Yet until now few practical applications of sampled-data control systems exist, except in those cases where the input or error signal of the system is inherently sampled. Probably the major reason why sampling techniques have not been more widely used is the complexity and expense of the sampling controller.

Due to the increasing use of digital computers in control systems, the problem which corresponds to the optimal relay servo of the continuous control system can be treated in the discrete systems. The sampled-data type of control problem is very important practically, since the control signal f(t) is frequently computed by means of a digital computer. In which case the introduction of sampling in the system is natural and unavoidable.

This study will be concerned with a second-order saturating sampled-data system, as shown in Fig.1-1, with the following sequence of components in the forward path:

a sampler with period T, a zero-order hold circuit, a linear amplifier with saturation limit of ±1, and a plant with transfer function G(s). And a computer(controller) is set on the feedback loop.

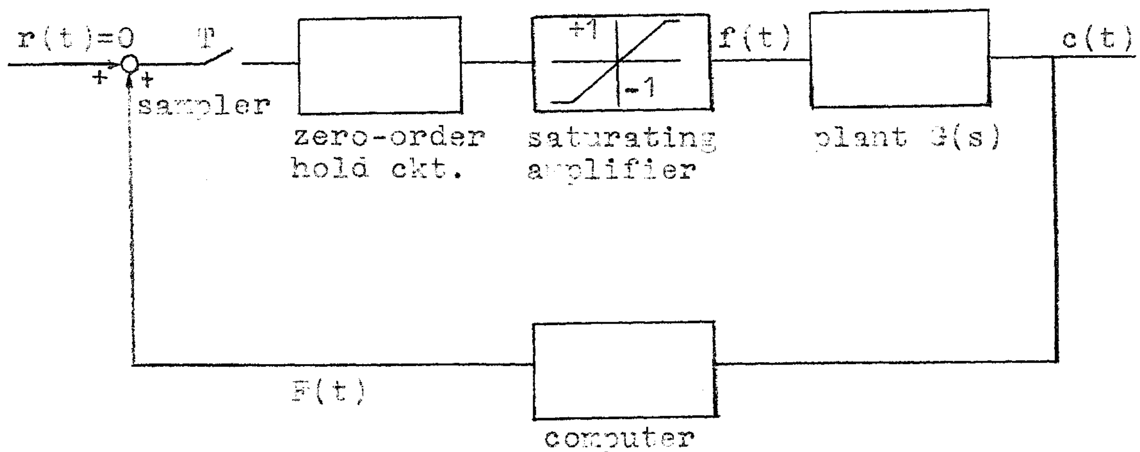


Fig.1-1. Block diagram of the system under consideration.

In chapter II, by following the results of Desoer and Wing[1,2 *], a brief introduction of the optimal strategy in the discrete control system is developed.

Assume that the input is zero for all times, the minimal time control problem for the above system can then be stated as follows: given G(s) with an arbitrary set of initial conditions, find the forcing function F(t) for G(s), and the corresponding computer in the feedback loop which will bring the system to equilibrium in the minimum number of sampling periods. Any such forcing function will be called an optimal strategy.

* All superscript number refer to references in the bibliography.

In chapter III, the minimum time isochrones are constructed both for the discrete and the continuous control systems with the same plant transfer function. The t^* minimum time isochrone is defined as the locus of all points in the state-variable plant with the property that, if a time optimal control policy is used, then all the points on the t^* minimum time isochrone can be forced to the origin in the same minimum time t^*. In order to compare the results of the optimal time and the optimal trajectories between the discrete and the continuous systems, a specific plant with transfer function $\frac{1}{s(s+1)}$ is used in each case.

In chapter IV, the problem of parameter variations and their effect on the optimal strategy is discussed. If the plant has the time constant a, while the control model has the time constant b, for the case $a \neq b$, then the forcing function calculated from the controller can not yield an optimal control. Certain calibrations can be made by the controller to obtain an almost optimal control.

B. Literature Review

In 1957, Kalman[3] presented a paper concerned with the problem of designing an optimal nonlinear controller for a linear dynamic system where input to the linear system is limited by saturation. In contrast to the usual approach to such problems, the output of the controller would be assumed to change only at periodically repeated instants of time. This assumption greatly simplified the analysis of the problem and the design of the controller.

Desoer and Wing[1,2], extending some preliminary results of Kalman, presented an optimal strategy for a saturating sampled-data system. This optimal strategy was concerned with a system described by a linear differential equation with constant coefficients. The control signal of the system is constrained between the saturation limits +1 and -1. The problem is then to determine the forcing function f(t), such that the system is forced to equilibrium in minimum time.

The problem discussed in this thesis is to investigate the discrete optimal control system in comparison with the continuous optimal control system by means of the minimum time isochrone. An investigation is also made for the case that the parameters of the plant and the controller are different.

CHAPTER II

TIME OPTIMAL STRATEGY FOR SECOND ORDER DISCRETE SYSTEM

A. Proposed system

Consider a second-order linear servomechanism as shown in Fig.2-1. The forward path consists of a sampler with period T, a zero-order hold circuit, a linear saturating amplifier and a plant which is described by a second-order transfer function $G(s) = K'/s'(s'+a')$, $(a'>0)$. The feedback loop consists of a computer whose input is c(t) and whose output is F(t). The sampler and the zero-order hold circuit require that f(t) be piecewise constant for any interval $kT \leq t \leq (k+1)T$, where k is an integer. The problem is the following:

Assuming that the input r(t) is zero for all times and given an arbitrary set of initial conditions c(0) and $\dot{c}(0)$, find the forcing function F(t) and the corresponding computer which will bring the system to equilibrium in the minimum number of sampling periods.

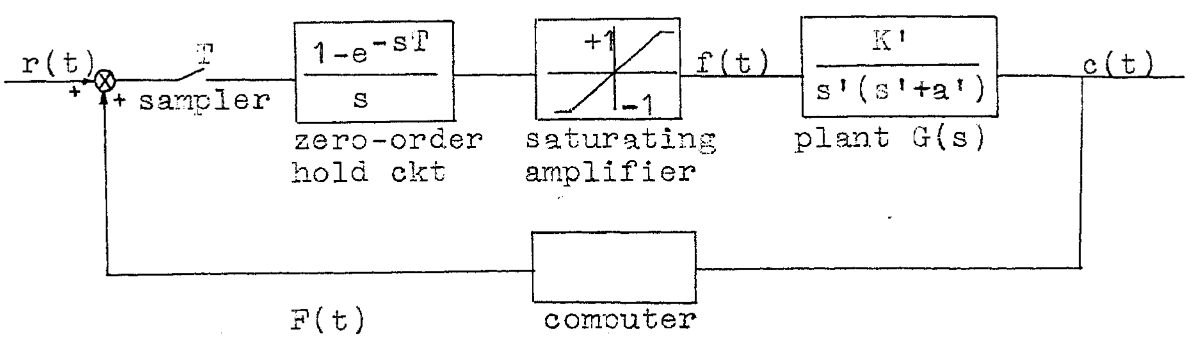


Fig.2-1. Block diagram of the second-order system.

B. Matrix Formulation

By a suitable time normalization, the constant K' of G(s) can be made equal to unity. Expand the time scale by a factor k^{-1}, thus $s=s'/k$, $a=a'/k$, and multiply the transfer function by k. This simultaneous forcing function and time normalization reduces the problem to a two-parameter problem: a, the time constant of the plant G(s), and T, the sampling period. Therefore, the problem is reduced to that shown in Fig.2-2.

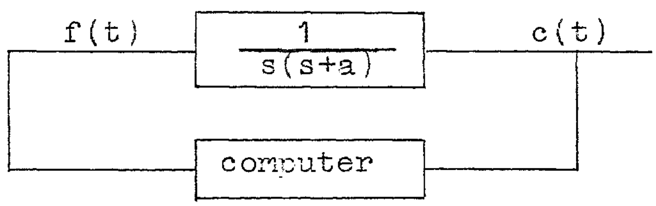


Fig.2-2. System under consideration after normalization.

Here f(t) is a zero-order hold function and $|f(t)| \leqq 1$ at all times.

For $0 \leq t \leq T$

$$\ddot{c}(t) + a\,\dot{c}(t) = f_1 \tag{2-1}$$

The solution for a set of initial conditions, $c(0)$, $\dot{c}(0)$, is

$$\left.\begin{aligned} c(t) &= c(0) + \frac{1-e^{-at}}{a}\,\dot{c}(0) + f_1\,\frac{e^{-at} + at - 1}{a^2} \\ \dot{c}(t) &= e^{-at}\,\dot{c}(0) + f_1\,\frac{1-e^{-at}}{a} \end{aligned}\right\} \tag{2-2}$$

This relation between the initial conditions for the next sampling period, $c(T)$ and $\dot{c}(T)$, is of the form

$$\begin{bmatrix} c(T) \\ \dot{c}(T) \end{bmatrix} = \begin{bmatrix} 1 & (1-e^{-aT})/a \\ 0 & e^{-aT} \end{bmatrix} \begin{bmatrix} c(0) \\ \dot{c}(0) \end{bmatrix} + f_1 \begin{bmatrix} (e^{-aT}+aT-1)/a^2 \\ (1-e^{-aT})/a \end{bmatrix} \qquad (2\text{-}3)$$

Let

$$\underline{A} = \begin{bmatrix} 1 & (1-e^{-aT})/a \\ 0 & e^{-aT} \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} (e^{-aT} + aT - 1)/a^2 \\ (1-e^{-aT})/a \end{bmatrix}$$

Eq.(2-3) can be described by the vector differential equation

$$\underline{c}(T) = \underline{A}\ \underline{c}(0) + f_1\ \underline{u} \qquad (2\text{-}4)$$

The vector $\underline{c}(T)$ can be expressed in terms of the normalized eigenvectors of $\underline{A}$. The eigenvalues of $\underline{A}$ are $\lambda_1 = e^{-aT}$ and $\lambda_2 = 1$. The normalized eigenvectors are

$$\underline{e}_1 = \begin{bmatrix} \frac{-1}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad (2\text{-}5)$$

By change of variable, let

$$\underline{c}(T) = \underline{P}\ \underline{X}(T) \qquad (2\text{-}6)$$

where

$$\underline{P} = [\underline{e}_1 \quad \underline{e}_2], \text{ is a matrix} \qquad (2\text{-}7)$$

$\underline{X}(T)$ is a vector on the (X_1, X_2) plane

The vector on the $(c, \dot{c})$ plane can be transformed to the (X_1, X_2) plane by use of Eq.(2-6). Substitute Eq.(2-6) into Eq.(2-4)

$$\underline{P}\ \underline{X}(T) = \underline{A}\ \underline{P}\ \underline{X}(0) + f_1\ \underline{u}$$

or

$$\underline{X}(T) = \underline{P}^{-1}\underline{A}\ \underline{P}\ \underline{X}(0) + f_1\ \underline{P}^{-1}\underline{u}$$

$$= \underline{\Lambda}\ \underline{X}(0) + f_1\underline{d} \qquad (2\text{-}8)$$

where

$$\underline{\Lambda} = \underline{P}^{-1}\underline{A}\ \underline{P} = \begin{bmatrix} e^{-aT} & 0 \\ 0 & 1 \end{bmatrix} \qquad (2\text{-}9)$$

$$\underline{d} = \underline{P}^{-1}\underline{u} = \begin{bmatrix} (1-e^{-aT})(1+a^2)^{\frac{1}{2}}/a^2 \\ -T/a \end{bmatrix} \tag{2-10}$$

For the general interval $(k-1)T \leq t \leq kT$

$$\underline{X}(kT) = \underline{\Lambda}\ \underline{X}((k-1)T) + f_k\underline{d} \qquad (k=1,2,3,\ldots) \tag{2-11}$$

C. Classification of initial states

The initial conditions $c(0)$, $\dot{c}(0)$ refer to the vector $\underline{X}(0)$ at the point $(X_1(0), X_2(0))$ in the (X_1, X_2) plane. If $\underline{X}(0)$ is such that the system can be brought to equilibrium in N sampling periods, the following relation holds:

$$\underline{X}(NT) = 0 = \underline{\Lambda}^N\underline{X}(0) + f_1\underline{\Lambda}^{N-1}\underline{d} + \ldots + f_{N-1}\underline{\Lambda}\ \underline{d} + f_N\underline{d} \tag{2-12}$$

where $|f_i| \leq 1 \quad (i=1,2,3,\ldots,N)$

Premultiply Eq.(2-12) by $\underline{\Lambda}^{-N}$, then

$$\underline{X}(0) = -f_1\underline{\Lambda}^{-1}\underline{d} - f_2\underline{\Lambda}^{-2}\underline{d} - \ldots - f_{N-1}\underline{\Lambda}^{-N+1}\underline{d} - f_N\underline{\Lambda}^{-N}\underline{d} \tag{2-13}$$

For $k = 1,2,3,\ldots,$ define

$$\underline{r}_k = -\underline{\Lambda}^{-k}\underline{d} = \begin{bmatrix} -e^{kaT}(1-e^{-aT})(1+a^2)^{\frac{1}{2}}/a^2 \\ T/a \end{bmatrix} \tag{2-14}$$

The vectors $\underline{r}_k$ are shown on Fig. 2-3.

Equation (2-13) can be written as

$$\underline{X}(0) = \sum_{i=1}^{N} f_i\underline{r}_i \tag{2-15}$$

Eq.(2-15) is the general representation for the initial states that can be brought into equilibrium in N sampling periods or less.

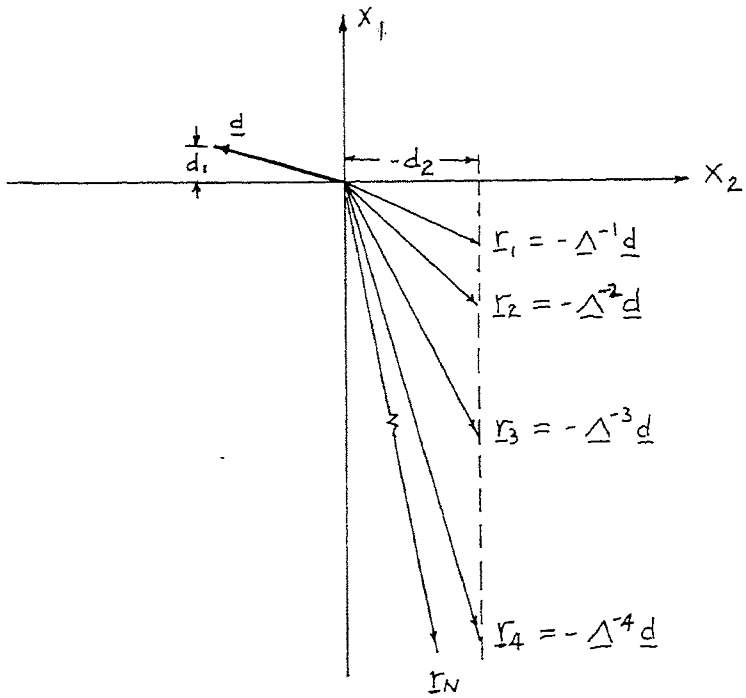


Fig.2-3. Illustration of the vectors $\underline{r}_k$ and $\underline{d}$.

The region R_N' is defined as the set of initial states that can be brought to equilibrium in N sampling periods or less. The properties of R_N':*

1. The region of R_N' is convex. If two initial states represented by the points P_1 and P_2 can be brought to equilibrium in N sampling period or less, the same is true for any initial state on the line segment P_1P_2.

2. R_N' is the closed set whose boundary is the convex polygon which has following 2N vertices:

OP_1, OP_2, ..., OP_N, OP_{-1}, ..., OP_{-N}.

where

* For proof of these properties see reference 1.

$$OP_1 = \underline{r}_1 - \underline{r}_2 - \underline{r}_3 - \ldots - \underline{r}_N$$
$$OP_2 = \underline{r}_1 + \underline{r}_2 - \underline{r}_3 - \ldots - \underline{r}_N$$
......
$$OP_N = \underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \ldots + \underline{r}_N$$
$$OP_{-1} = -\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \ldots + \underline{r}_N$$
......
$$OP_{-N} = -\underline{r}_1 - \underline{r}_2 - \underline{r}_3 - \ldots - \underline{r}_N \qquad (2\text{-}16)$$

where O is the origin of the (X_1, X_2) plane. The convex polygon has the origin as a center of symmetry.

3. $R_N = R'_N - R'_{N-1}$ (2-17)

where R_N is the set of all initial states that can be brought to the origin in N sampling periods and no less.

The regions R'_1, R'_2, R'_3, R'_4 and R_4 with the vertices are shown explicitly on Fig.2-4.

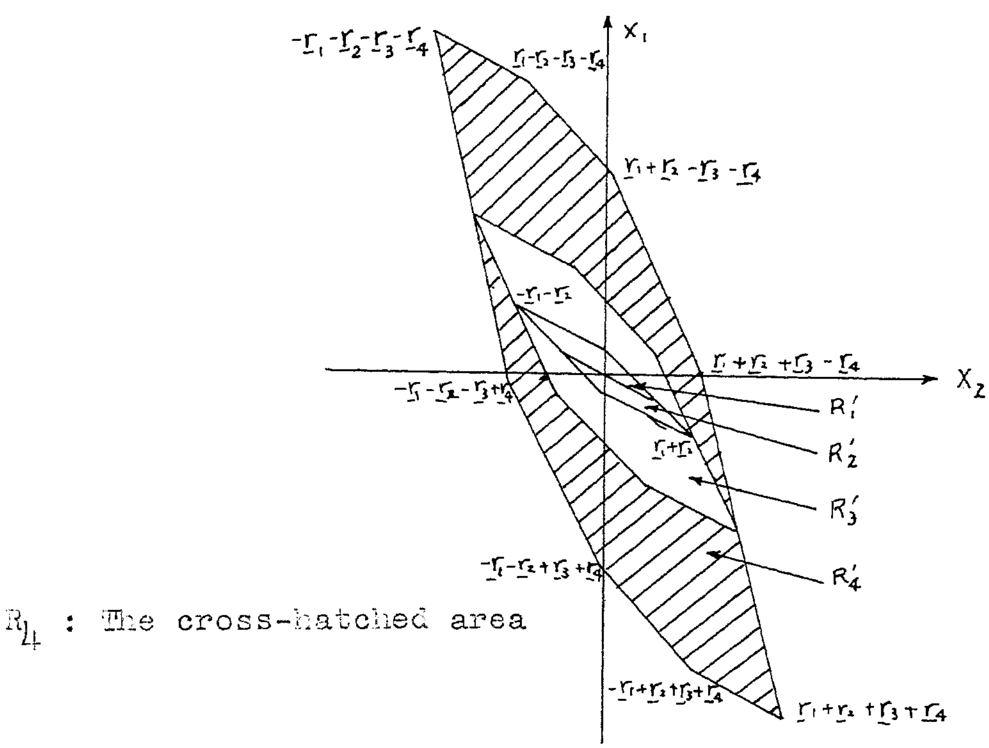


Fig.2-4. The regions R'_1, R'_2, R'_3, R'_4 and R_4.

D. Optimal Strategies

An optimal strategy is defined by the following requirement. Given an $\underline{X}(0)$ that is a point of R_N, an optimal strategy is any effective forcing function f(t), specified by $f_1, f_2, \ldots, f_N$, that brings the point $\underline{X}(0)$ to the origin in exactly N sampling periods.

If $\underline{X}(0)$ belongs to R_1 or R_2, the optimal strategy is unique. If $\underline{X}(0)$ belongs to R_N with $N \geq 3$, there may be more than one solution. For any $\underline{X}(0)$ that is a boundary state of R_N, there is a unique optimal strategy. For any $\underline{X}(0)$ that is a interior state of R_N, (not on the outer boundary of R_N), there exists an infinite number of optimal strategies. In order to prove this statement, start from Eq.(2-8)

$$\underline{X}(0) = \underline{\Delta}^{-1}\underline{X}(T) - f_1\underline{\Delta}^{-1}\underline{d}$$

and using Eq.(2-14)

$$\underline{X}(0) - f_1\underline{r}_1 = \underline{\Delta}^{-1}\underline{X}(T) \qquad (2\text{-}18)$$

If $\underline{X}(0)$ is in R_N, then by the definition of the optimal strategy, an optimal value of f_1 is such that $\underline{X}(T)$ is in R_{N-1}. An equivalent form of Eq.(2-18) expressed in terms of regions R_N and R_{N-1} is

$$R_N - f_1\underline{r}_1 = \underline{\Delta}^{-1}R_{N-1} \qquad (2\text{-}19)$$

Start with R_{N-1}, determine $\underline{\Delta}^{-1}R_{N-1}$, R_N is then generated by translating $\underline{\Delta}^{-1}R_{N-1}$ by all possible $f_1\underline{r}_1$ with $|f_1| \leq 1$. Conversely, if $\underline{X}(0)$ is in R_N, there is an f_1, where $|f_1| \leq 1$, such that $\underline{X}(0) - f_1\underline{r}_1$ is in $\underline{\Delta}^{-1}R_{N-1}$. An optimal value of the forcing function during the first sampling interval is then any such value of f_1.

The region $\underline{\Delta}^{-1}R_{N-1}$ can be determined by the following process, by using Eq.(2-17), get the relation $R_{N-1} = R'_{N-1} - R'_{N-2}$, and therefore

$$\underline{\Delta}^{-1}R_{N-1} = \underline{\Delta}^{-1}R'_{N-1} - \Delta^{-1}R'_{N-2} \qquad (2\text{-}20)$$

Since R'_{N-1} and R'_{N-2} are convex, hence, $\underline{\Delta}^{-1}R'_{N-1}$ and $\underline{\Delta}^{-1}R'_{N-2}$ are convex. To construct the regions of $\underline{\Delta}^{-1}R'_N$, only consider the vertices of $\underline{\Delta}^{-1}R'_N$, which can be derived from the vertices of R'_N. From Eq.(2-14)

$$\Delta^{-1}\underline{r}_k = \underline{r}_{k+1}$$

Table I gives a complete tabulation of the vertices. The crosshatched area in Fig.2-5 is $\underline{\Delta}^{-1}R_3$.

TABLE I

Vertices of Convex Region R'_3	
$\underline{r}_1-\underline{r}_2-\underline{r}_3$	$-\underline{r}_1+\underline{r}_2+\underline{r}_3$
$\underline{r}_1+\underline{r}_2-\underline{r}_3$	$-\underline{r}_1-\underline{r}_2+\underline{r}_3$
$\underline{r}_1+\underline{r}_2+\underline{r}_3$	$-\underline{r}_1-\underline{r}_2-\underline{r}_3$
Vertices of Convex Region R'_2	
$\underline{r}_1-\underline{r}_2$	$-\underline{r}_1+\underline{r}_2$
$\underline{r}_1+\underline{r}_2$	$-\underline{r}_1-\underline{r}_2$
Vertices of Convex Region $\underline{\Delta}^{-1}R'_3$	
$\underline{r}_2-\underline{r}_3-\underline{r}_4$	$-\underline{r}_2+\underline{r}_3+\underline{r}_4$
$\underline{r}_2+\underline{r}_3-\underline{r}_4$	$-\underline{r}_2-\underline{r}_3+\underline{r}_4$
$\underline{r}_2+\underline{r}_3+\underline{r}_4$	$-\underline{r}_2-\underline{r}_3-\underline{r}_4$
Vertices of Convex Region $\Delta^{-1}R'_2$	
$\underline{r}_2-\underline{r}_3$	$-\underline{r}_2+\underline{r}_3$
$\underline{r}_2+\underline{r}_3$	$-\underline{r}_2-\underline{r}_3$

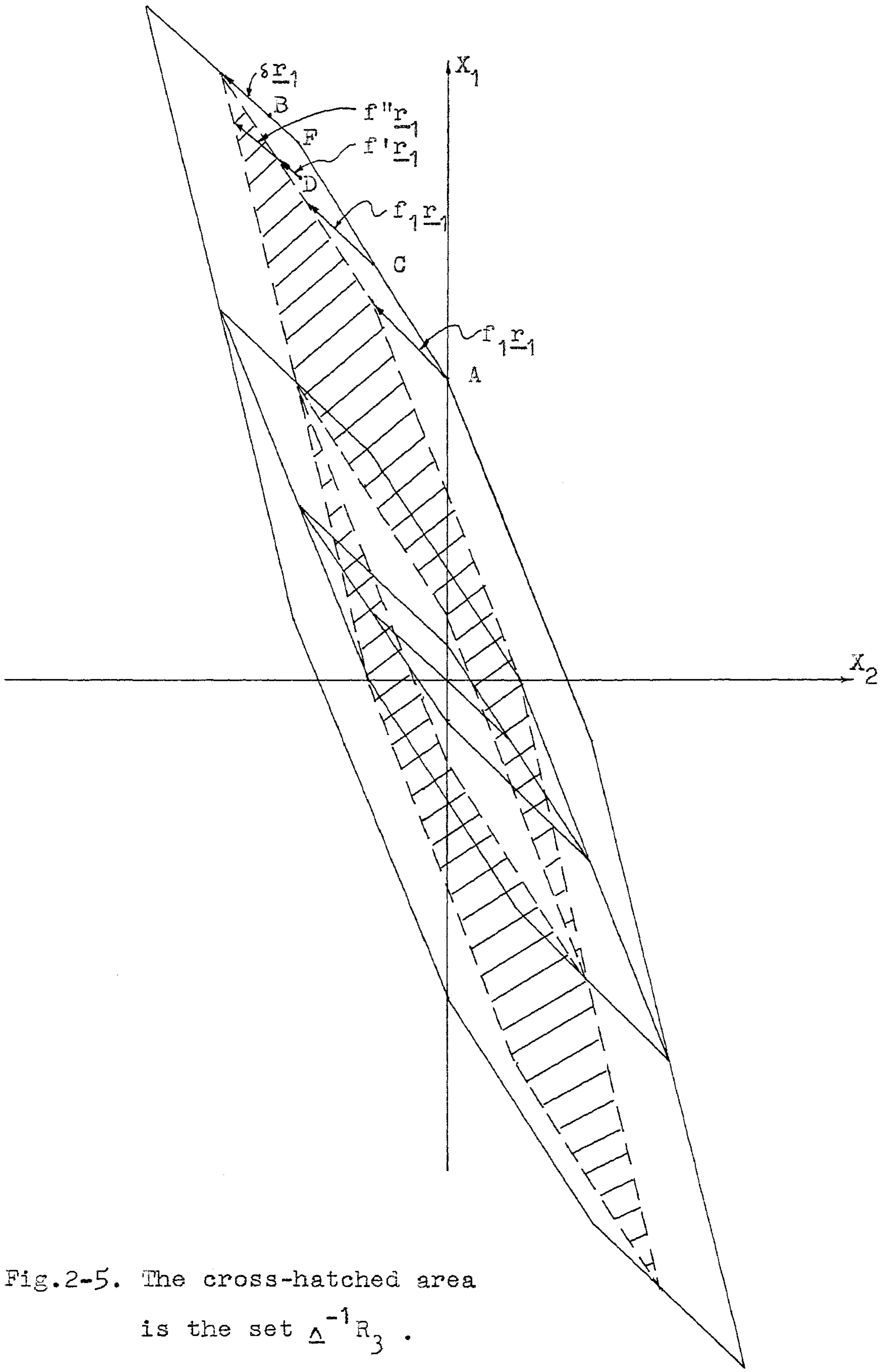


Fig.2-5. The cross-hatched area is the set $\underline{\Delta}^{-1} R_3$.

To illustrate the fact that at the boundary states of R_N ($N \geq 3$), the optimal strategy is unique, while for the interior states of R_N, the optimal strategy is not unique, some examples can be considered from Fig. 2-5. For the boundary states of R_4, such as points A or C, an optimal value of the forcing function during the first sampling interval is +1. For the boundary states such as point B, the optimal value of the forcing function during the first sampling interval is δ($|\delta|<1$). These forcing functions are unique. Otherwise, the initial state can not forced to the region R_3 after one sampling period. But for interior states of R_4, such as point D, the optimal forcing function during the first sampling interval can be f', f" or any f_1 in the range $f' \leq f_1 \leq f''$. There are an infinite number of optimal values of f_1 which force the initial state to the region R_3. Therefore, the optimal strategy of the interior state is not unique.

E. Proposed Optimal Strategy

In order to establish an optimal strategy which is easy to instrument, two particular curves are defined.

(1) The Critical Curve: which is obtained by joining successively the vertices defined by

$$\ldots, -\sum_{i=2}^{N} \underline{r}_i, \ldots, -\underline{r}_2, +r_2, \ldots, +\sum_{i=2}^{N} \underline{r}_i, \ldots$$

(2) The Polygonal Curve K: which is obtained by joining successively the vertices defined by

$$\ldots,\ -\sum_{i=1}^{N} \underline{r}_i,\ldots,-\underline{r}_2-\underline{r}_1,\ -\underline{r}_1,\ +\underline{r}_1,\ \underline{r}_1+\underline{r}_2,\ldots,+\sum_{i=1}^{N}\underline{r}_i,\ldots$$

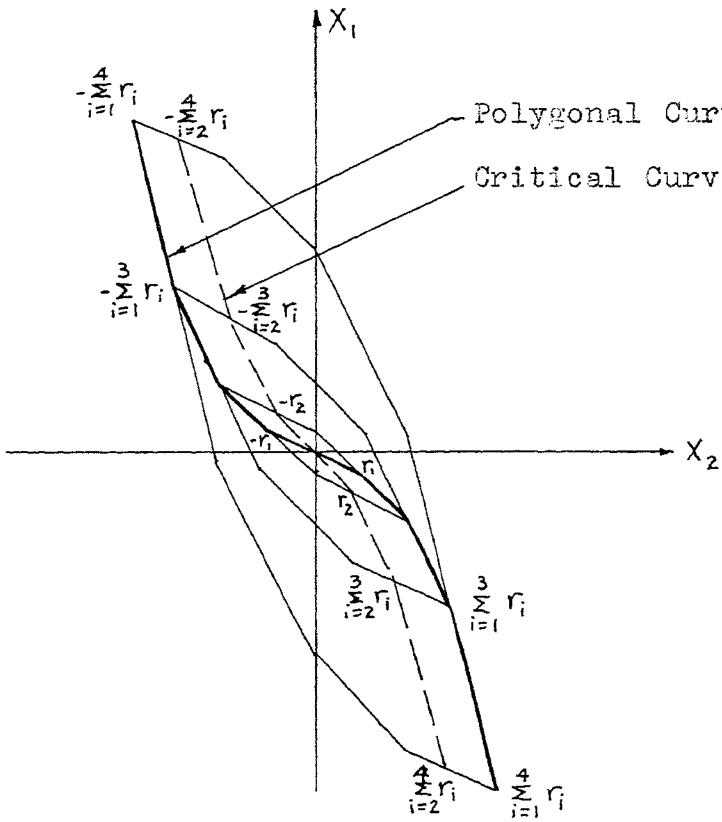


Fig. 2-6 The Critical Curve and the Polygonal Curve, K.

If $\underline{X}(0)$ is in R_N, it can be expressed either as

$$\underline{X}(0) = \underline{r}_1+\underline{r}_2+\ldots+\underline{r}_k+\delta\underline{r}_{k+1}-\underline{r}_{k+2}-\ldots-\underline{r}_{N-1}-\delta_1\underline{r}_N \qquad (2\text{-}21)$$

or as

$$\underline{X}(0) = -\underline{r}_1-\underline{r}_2-\ldots-\underline{r}_k+\delta\underline{r}_{k+1}+\underline{r}_{k+2}+\ldots+\underline{r}_{N-1}+\delta_1\underline{r}_N \qquad (2\text{-}22)$$

where $-1\leq \delta \leq 1$, $0 \leq \delta_1 \leq 1$, $0 \leq k \leq N-1$

The representation in Eq. (2-21) holds for all points to the right of the polygonal curve, K, and Eq. (2-22) holds for those that lie to the left of the polygonal curve, K.

The proposed optimal strategy follows directly from the canonical representations Eq.(2-21) or Eq.(2-22). To get started consider those initial states for which k = 0 in Eq.(2-21) and Eq.(2-22), i.e.

$$\underline{X}(0) = \delta\underline{r}_1-\underline{r}_2-\cdots-\underline{r}_{N-1}-\delta_1\underline{r}_N \tag{2-23}$$

or $$\underline{X}(0) = \delta\underline{r}_1+\underline{r}_2+\cdots+\underline{r}_{N-1}+\delta_1\underline{r}_N \tag{2-24}$$

Take an alternate form

$$\underline{X}(0) - \delta\underline{r}_1 = -\underline{r}_2-\underline{r}_3-\cdots-\underline{r}_{N-1}-\delta_1\underline{r}_N \tag{2-25}$$

$$\underline{X}(0) - \delta\underline{r}_1 = \underline{r}_2+\underline{r}_3+\cdots+\underline{r}_{N-1}+\delta_1\underline{r}_N \tag{2-26}$$

The right hand side of Eq.(2-25) and Eq.(2-26) represent a point on the critical curve. Observe that

(1) If $k > 0$, the optimal strategy implied by Eq.(2-21),(2-22) requires $f_1=\pm1$.(such as point C in Fig. 2-5.)

(2) If k=0 and if $|\delta|=1$, then the optimal strategy implied by Eq.(2-21),(2-22) requires $f_1=\delta$, i.e. $|f_1|=1$. (such as point F in Fig.2-5.)

(3) If k=0 and if $|\delta|<1$, then $f_1=\delta$, where δ is such that $\underline{X}(0)-\delta\underline{r}_1$ be a point on the critical curve.(such as point B in Fig. 2-5.)

This leads to the following rule for determining f_1.

Compute δ such that $\underline{X}(0)-\delta\underline{r}_1$ be a point on the critical curve. If $\delta \geq 1$, take $f_1=1$, where f_1 is the effective forcing function for the first sampling period. If $\delta \leq -1$, take $f_1=-1$. If $-1 < \delta < 1$, take $f_1=\delta$.

At each sampling instant δ must be computed such that $\underline{X}(0)-\delta\underline{r}_1$ be a point on the critical curve. The first step is to transform the critical curve from the (λ_1, λ_2) plane to the $(c, \dot{c})$ plane. The vectors $\underline{s}_1$, $\underline{s}_2, \ldots,$ $\underline{s}_N, \ldots,$ of the $(c, \dot{c})$ plane corresponding to the vectors $\underline{r}_1$, $\underline{r}_2, \ldots,$ $\underline{r}_N, \ldots,$ of the (λ_1, λ_2) plane are given by

$$\underline{s}_k = \underline{P}\,\underline{r}_k \quad (k=1,2,3,\ldots) \tag{2-27}$$

where $\underline{P}$ is the matrix which has the eigenvectors $\underline{e}_1$ and $\underline{e}_2$ as columns. The critical curve can be drawn in the $(c, \dot{c})$ plane. This is done because the proposed optimal strategy requires the determination of δ such that $c(0)-\delta\underline{s}_1$ be a point of the critical curve. However, it is more convenient to rotate the coordinates and use the axes OY_1 and OY_2, shown in Fig. 2-7. where OY_1 is the support of $\underline{s}_1$. Thus, in the (Y_1, Y_2) coordinates, the determination of δ will amount to taking a difference of abscissas. The critical curve in the (Y_1, Y_2) plane is shown in Fig.2-8. Given an initial state $\underline{Y}(0)$, if $f(Y_2(0))$ is the abscissa of the critical curve corresponding to the ordinate $Y_2(0)$ of $\underline{Y}(0)$, then

$$\delta = \frac{f(Y_2(0)) - Y_2(0)}{\underline{s}_1} \tag{2-28}$$

From δ, f_1 can be determined.

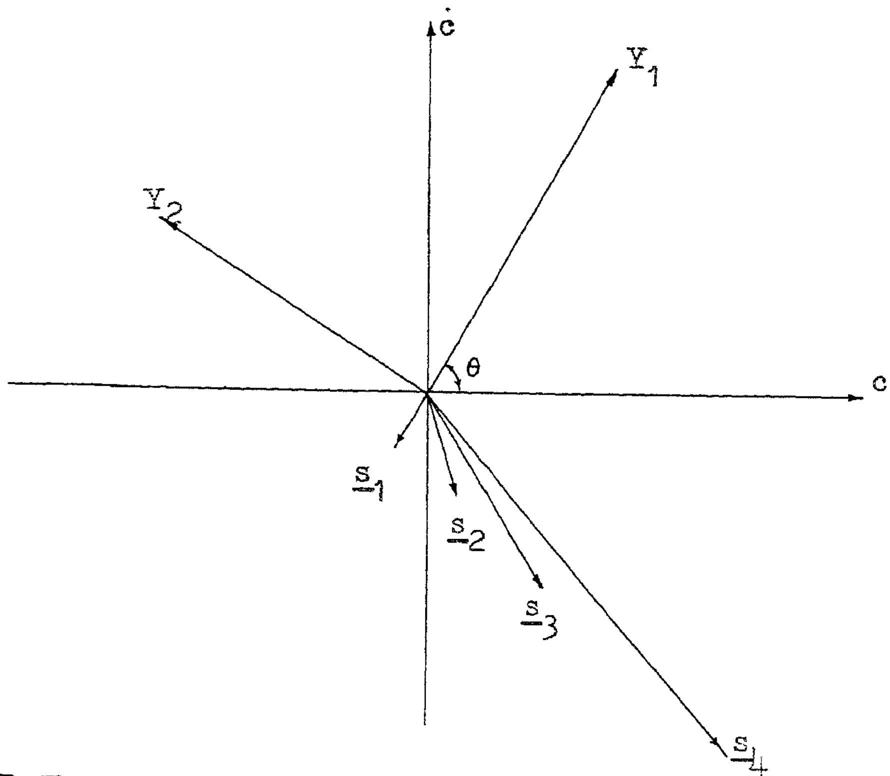


Fig.2-7 The new axes OY_1, OY_2 with respect to the old axes Oc and Oċ.

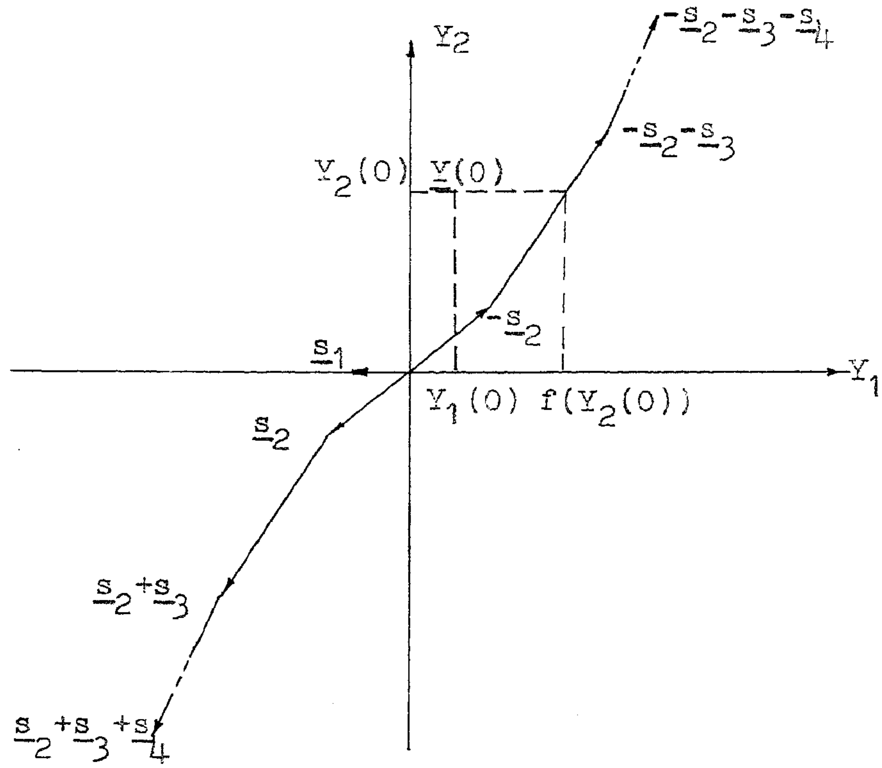


Fig.2-8. The critical curve $Y_1=f(Y_2)$ in the (Y_1,Y_2) plane in terms of the $\underline{s}_i$

CHAPTER III

COMPARISON BETWEEN THE DISCRETE AND THE CONTINUOUS SYSTEMS BY THE MINIMUM TIME ISOCHRONES

The comparison of the optimal time in the continuous control systems and the discrete control systems can be based on comparison of the minimum time isochrones. A t^* minimum isochrone is defined as the locus of all points in the (X_1, X_2) plane with the property that, if a time optimal control policy is used, then all the points on the t^* minimum isochrone can be forced to the origin in the same minimum time t^*.

A. Construction of the isochrones in the continuous system

Consider the plant transfer function $G(s)=\frac{1}{s(s+1)}$. The block diagram of the system is shown in Fig. 3-1.

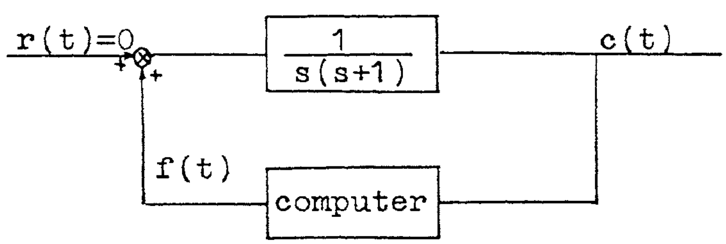


Fig. 3-1. Block diagram of a continuous control system.

Fig. 3-1 implies that

$$\ddot{c}(t) + \dot{c}(t) = f \tag{3-1}$$

For initial condition expressed as $c(0)$ and $\dot{c}(0)$, the solution of Eq.(3-1) is

$$\left.\begin{aligned} c(t) &= c(0) + (1-e^{-t})\dot{c}(0) + f(e^{-t} + t - 1) \\ \dot{c}(t) &= e^{-t}\dot{c}(0) + f(1-e^{-t}) \end{aligned}\right\} \qquad (3\text{-}2)$$

In order to compare with the discrete system, transform the coordinates in the $(c, \dot{c})$ plane to the coordinates (X_1, X_2) plane by the following relations

$$\underline{c}(t) = X_1(t)\,\underline{e}_1 + X_2(t)\,\underline{e}_2 \qquad (3\text{-}3)$$

where

$$\underline{c}(t) = \begin{bmatrix} c(t) \\ \dot{c}(t) \end{bmatrix}$$

$$\underline{e}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

therefore

$$c(t) = -1/\sqrt{2}\;X_1(t) - X_2(t) \qquad (3\text{-}4)$$

$$\dot{c}(t) = 1/\sqrt{2}\;X_1(t) \qquad (3\text{-}5)$$

Letting (ξ_1, ξ_2) be the initial condition in the (X_1, X_2) plane, Eq.(3-2) can be transformed to

$$\left(-\frac{1}{\sqrt{2}}X_1 - X_2\right) = \left(-\frac{1}{\sqrt{2}}\xi_1 - \xi_2\right) + (1-e^{-t})\frac{1}{\sqrt{2}}\xi_1 + f(e^{-t}+t-1) \qquad (3\text{-}6)$$

$$\left(\frac{1}{\sqrt{2}}X_1\right) = \left(\frac{1}{\sqrt{2}}\xi_1\right)e^{-t} + f(1-e^{-t}) \qquad (3\text{-}7)$$

Hereafter, X_1, X_2, represents $X_1(t)$, $X_2(t)$. From Eq.(3-7)

$$X_1 = (\xi_1 - \sqrt{2}f)e^{-t} + \sqrt{2}\,f \qquad (3\text{-}8)$$

Substitute Eq.(3-8) into Eq.(3-6)

$$X_2 = \xi_2 - f\,t \qquad (3\text{-}9)$$

Eliminating t, the trajectory in the (X_1, X_2) plane will be

$$X_1 = (\xi_1 - \sqrt{2}\,f)\exp\left(-\frac{\xi_2 - X_2}{f}\right) + \sqrt{2}\,f \qquad (3\text{-}10)$$

For $f = +1$, $\quad X_1 = (\xi_1 - \sqrt{2})\exp(-\xi_2 + X_2) + \sqrt{2} \qquad (3\text{-}11)$

For $f = -1$, $\quad X_1 = (\xi_1 + \sqrt{2})\exp(\xi_2 - X_2) - \sqrt{2} \qquad (3\text{-}12)$

If the trajectory goes through the origin, thus $\xi_1=0$ and $\xi_2=0$, then from Eq.(3-11) and (3-12), the switching curve can be obtained as

$$S_+(f=+1) \qquad X_1 = \sqrt{2}(1-e^{X_2}) \qquad (3\text{-}13)$$

$$S_-(f=-1) \qquad X_1 = -\sqrt{2}(1-e^{-X_2}) \qquad (3\text{-}14)$$

Eq.(3-13) applies for $X_2 > 0$, and Eq.(3-14) applies for $X_2 < 0$.

Eq.(3-11) generates the trajectories in the (X_1, X_2) plane resulting from f=+1 and starting from the initial states (ξ_1, ξ_2). The trajectories shown in Fig.3-2 are for all (ξ_1, ξ_2) to the right-hand side of the switching curve. The trajectories resulting from f=-1 has the same shape as that of the trajectories of f=+1 except that it is inverted.

The procedure to construct the t^* minimum isochrones is as follows.

(1) Take a finite time t^* which is to be the minimum response time to force (ξ_1, ξ_2) to (0,0).

(2) From Fig.3-2, pick the intersection point of the trajectories and the switching curve S_-, such as point (Z_1, Z_2).

(3) Let t_1 be the time required to force (ξ_1, ξ_2) to (Z_1, Z_2) and t_2 be the time required to force (Z_1, Z_2) to (0,0), then

$$t^* = t_1 + t_2 \qquad (3\text{-}15)$$

From Eq.(3-9), using the above relations with f=+1, then

$$t_2 = -Z_2 \qquad (3\text{-}16)$$

and $$t_1 = \xi_2 - Z_2 \qquad (3\text{-}17)$$

By applying the relations in Eq.(3-8) and (3-9)

$$\xi_1 = (Z_1 - \sqrt{2})\, e^{t_1} + \sqrt{2} \qquad (3\text{-}18)$$

and $$\xi_2 = t_1 + Z_2 \qquad (3\text{-}19)$$

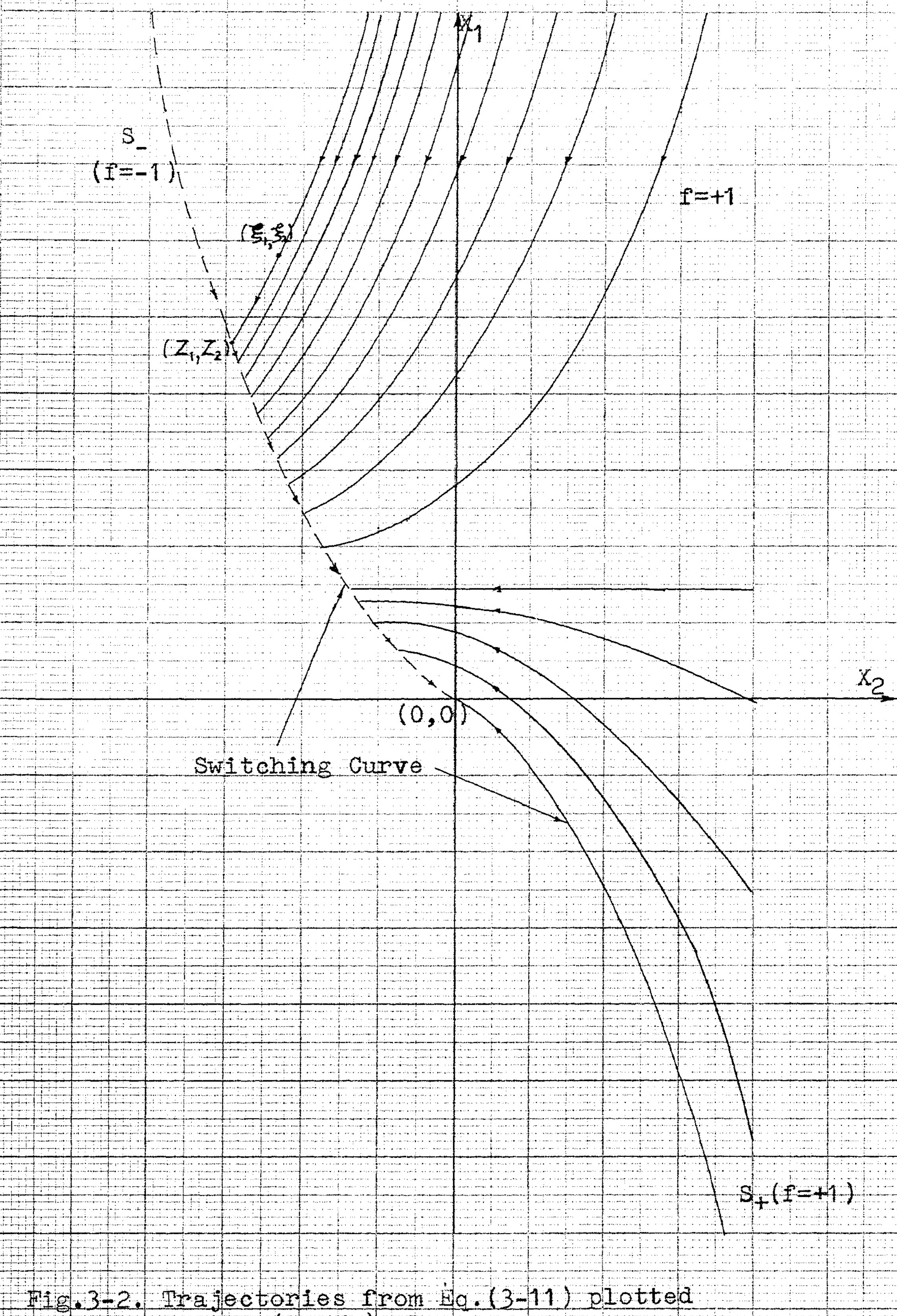


Fig.3-2. Trajectories from Eq.(3-11) plotted on the (X_1, X_2) plane.

(4) For each trajectory in Fig.3-2 the initial state (ξ_1, ξ_2) can be determined by Eq.(3-17') and Eq.(3-18).
Connecting these initial states, the locus of (ξ_1, ξ_2) is the isochrone of t^* in continuous system. Data are listed in Appendix I. The isochrones for $t^* = 0.5$, 1, 1.5, and 2 seconds are plotted on Fig.3-3a and Fig.3-3b.

B. Construction of the isochrone in the discrete system

The block diagram of the discrete control system is shown in Fig.2-1. The plant transfer function $G(s) = \frac{1}{s(s+1)}$ has roots at 0 and -1. From Eq.(2-15), the initial state in the (X_1, X_2) plane can be expressed as

$$\underline{X}(0) = \sum_{k=1}^{N} f_k \underline{r}_k \qquad (3\text{-}19)$$

where $|f_k| \leq 1$

Eq.(3-19) is a general representation for the initial states that can be brought into equilibrium in N sampling periods or less. When $|f_K|=1$ is applied for every sampling instant, a region R_N' can be constructed. The boundary of R_N' is equivalent to an isochrone for N sampling periods.

From Eq.(2-14)

$$\underline{r}_K = -\underline{\Lambda}^{-k}\underline{d} = \begin{bmatrix} -e^{kaT}a^{-2}(1-e^{-aT})(1+a^2)^{\frac{1}{2}} \\ T/a \end{bmatrix} \qquad (3\text{-}20)$$

For control plant with roots at (0, -1)

$$\underline{r}_k = \begin{bmatrix} -e^{kT}(1-e^{-T})\sqrt{2} \\ T \end{bmatrix} \qquad (k=1,2,3,\ldots) \qquad (3\text{-}21)$$

For a sampling period of T=0.5 second,

$$\underline{r}_1 = \begin{bmatrix} -0.94 \\ 0.5 \end{bmatrix} \quad \underline{r}_2 = \begin{bmatrix} -1.51 \\ 0.5 \end{bmatrix} \quad \underline{r}_3 = \begin{bmatrix} -2.48 \\ 0.5 \end{bmatrix} \quad \underline{r}_4 = \begin{bmatrix} -4.08 \\ 0.5 \end{bmatrix}$$

The region R'_N, the isochrone for N sampling periods, can be constructed by the following 2N vertices. These isochrones are shown in Fig.3-3a by dashed lines.

(1) R'_1, the isochrone for $t^* = 0.5$ second, is the straight line segment in the (X_1, X_2) plane.

$-\underline{r}_1 = (0.94, -0.5)$

$\underline{r}_1 = (-0.94, 0.5)$

(2) R'_2, the isochrone for $t^* = 1.0$ second, is the parallelogram with the following 4 vertices in the (X_1, X_2) plane.

$-\underline{r}_1 + \underline{r}_2 = (-0.57, 0)$

$+\underline{r}_1 + \underline{r}_2 = (-2.45, 1)$

$+\underline{r}_1 - \underline{r}_2 = (+0.57, 0)$

$-\underline{r}_1 - \underline{r}_2 = (+2.45, -1)$

(3) R'_3, the isochrone for $t^* = 1.5$ seconds, is the convex polygon with the following 6 vertices in the (X_1, X_2) plane.

$+\underline{r}_1 + \underline{r}_2 + \underline{r}_3 = (-4.93, +1.5)$

$-\underline{r}_1 + \underline{r}_2 + \underline{r}_3 = (-3.05, +0.5)$

$-\underline{r}_1 - \underline{r}_2 + \underline{r}_3 = (-0.03, -0.5)$

$-\underline{r}_1 - \underline{r}_2 - \underline{r}_3 = (+4.93, -1.5)$

$+\underline{r}_1 - \underline{r}_2 - \underline{r}_3 = (+3.05, -0.5)$

$+\underline{r}_1 + \underline{r}_2 - \underline{r}_3 = (+0.03, +0.5)$

(4) R'_4, the isochrone for $t^* = 2.0$ seconds, is the convex polygon with the following 8 vertices in the (X_1, X_2) plane.

$$+\underline{r}_1 \; +\underline{r}_2 \; +\underline{r}_3 \; +\underline{r}_4 = (-9.01,\ +2)$$
$$-\underline{r}_1 \; +\underline{r}_2 \; +\underline{r}_3 \; +\underline{r}_4 = (-7.13,\ +1)$$
$$-\underline{r}_1 \; -\underline{r}_2 \; +\underline{r}_3 \; +\underline{r}_4 = (-4.11,\ 0)$$
$$-\underline{r}_1 \; -\underline{r}_2 \; -\underline{r}_3 \; +\underline{r}_4 = (+0.85,\ -1)$$
$$-\underline{r}_1 \; -\underline{r}_2 \; -\underline{r}_3 \; -\underline{r}_4 = (+9.01,\ -2)$$
$$+\underline{r}_1 \; -\underline{r}_2 \; -\underline{r}_3 \; -\underline{r}_4 = (+7.13,\ -1)$$
$$+\underline{r}_1 \; +\underline{r}_2 \; -\underline{r}_3 \; -\underline{r}_4 = (+4.11,\ 0)$$
$$+\underline{r}_1 \; +\underline{r}_2 \; +\underline{r}_3 \; -\underline{r}_4 = (-0.85,\ +1)$$

As another example, take a sampling period $T = 1$ second, then

$$\underline{r}_1 = \begin{bmatrix} -2.45 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \underline{r}_2 = \begin{bmatrix} -6.56 \\ 1 \end{bmatrix}$$

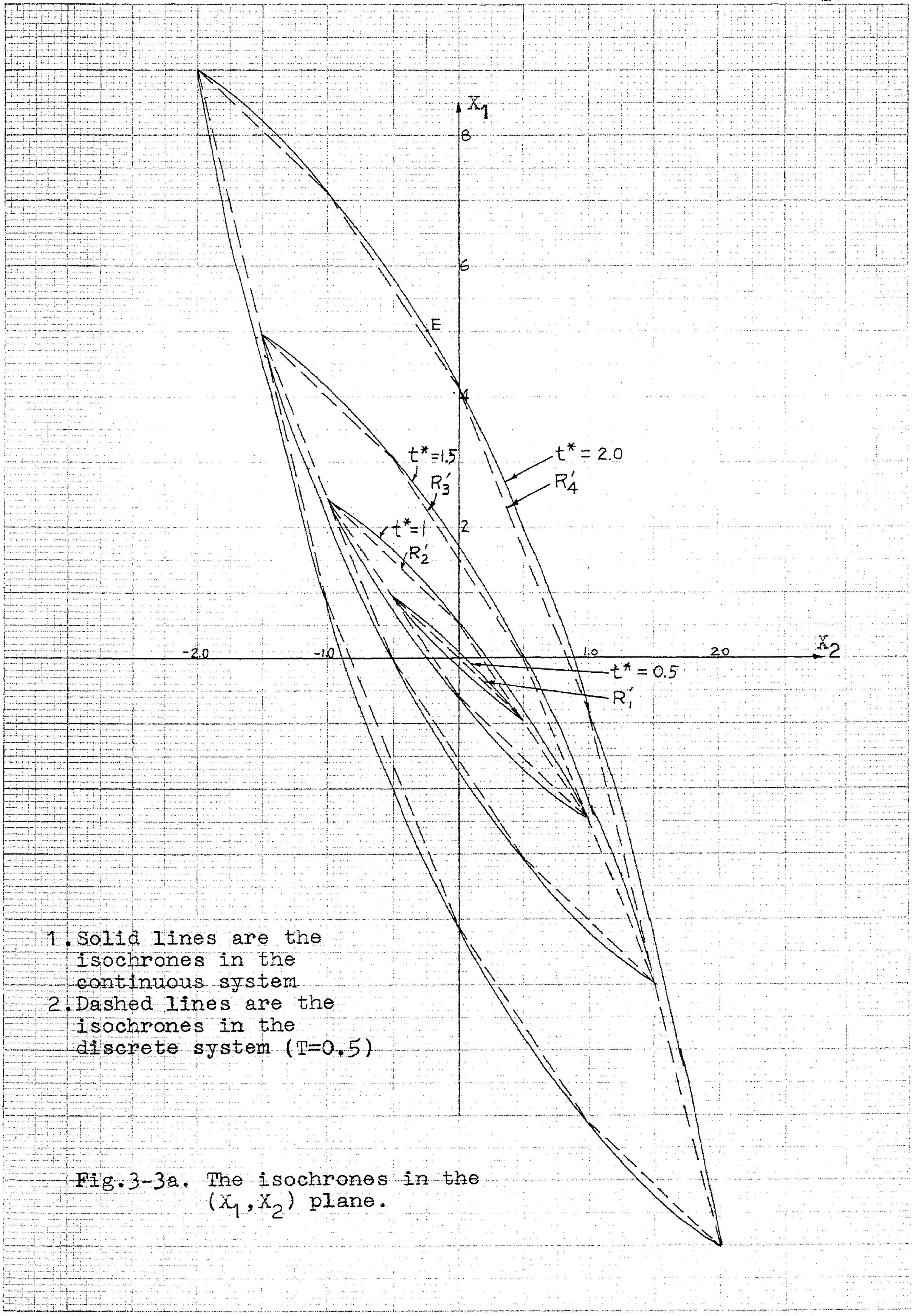
The region R'_N, the isochrone for N sampling periods, can be constructed by the following 2N vertices. These isochrones are shown in Fig.3-3b by dashed lines.

(1) R'_1, the isochrone for $t^* = 1$ second, is the straight line segment in the (X_1, X_2) plane.

$$-\underline{r}_1 = (+2.45,\ -1)$$
$$+\underline{r}_1 = (-2.45,\ +1)$$

(2) R'_2, the isochrone for $t^* = 2$ seconds, is the parallelogram with the following 4 vertices in the (X_1, X_2) plane.

$$-\underline{r}_1 \; +\underline{r}_2 = (+4.11,\ 0) \qquad +\underline{r}_1 \; -\underline{r}_2 = (-4.11,\ 0)$$
$$+\underline{r}_1 \; +\underline{r}_2 = (-9.01,\ 2) \qquad -\underline{r}_1 \; -\underline{r}_2 = (+9.01,\ -2)$$



1. Solid lines are the isochrones in the continuous system
2. Dashed lines are the isochrones in the discrete system (T=0.5)

Fig.3-3a. The isochrones in the (X_1, X_2) plane.

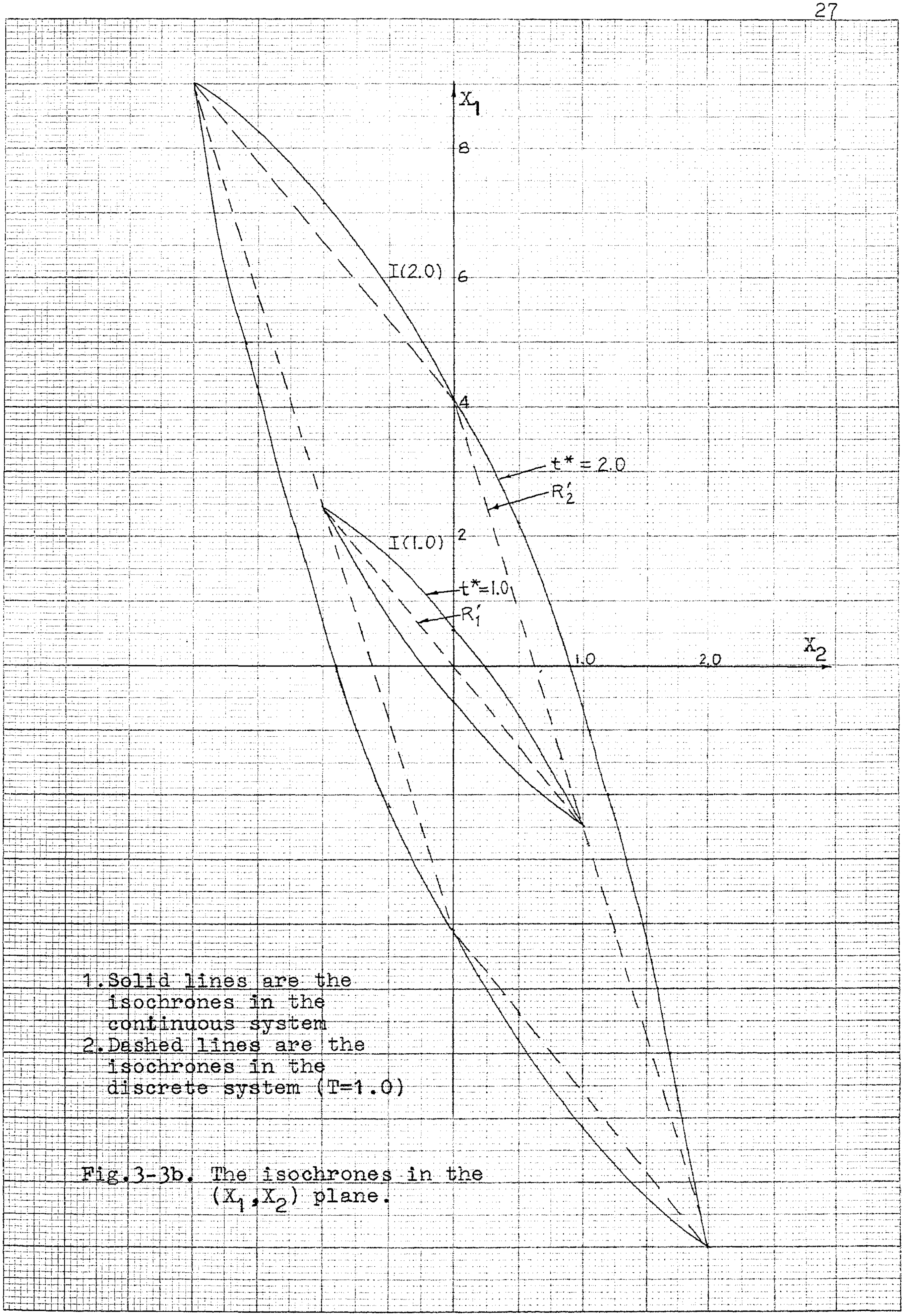


Fig. 3-3b. The isochrones in the (X_1, X_2) plane.

C. Comparison between the discrete and the continuous systems by minimum time isochrones

(1) From Fig.3-3a and Fig.3-3b, the regions for the discrete isochrones and the continuous isochrones are fairly close. The discrete isochrone is a convex polygon, while the continuous isochrone has neither flat part nor corner portion.* If the sampling period, T, is set smaller and smaller, then the discrete isochrone will coincide, in the limit as $T \rightarrow 0$, with that of the continuous isochrone.

(2) The isochrone in the discrete system has smaller area than the isochrone in the continuous case with the same plant transfer function. This means that for some initial states, the optimal control in the discrete system takes a longer time to go to equilibrium. For example point E in Fig.3-3a, the minimum time is different in the two systems. Since point E is on the continuous isochrone of $t^{*}=2.0$, it will take 2 seconds to go to equilibrium using an optimal continuous control. Since point E is outside the discrete isochrone, R_4', it can not go to equilibrium in 4 sampling periods. (In this example, one sampling period equal 0.5 second, 4 sampling periods equal 2 seconds). Point E is inside the discrete isochrone, R_5', and thus it can go to equilibrium in 5 sampling periods. Therefore, using a discrete control system requires one sampling period longer than the continuous control to reach equilibrium.

* For proof of these properties see reference 9.

(3) As illustrated on Fig.3-2, for the continuous control system the optimal strategy is unique. When the initial state (ξ_1, ξ_2) at the right-hand side of the switching curve, takes the forcing function f=+1 to force the initial state (ξ_1, ξ_2) to the switching curve, then applies f=-1, along the switching curve to go to equilibrium. When the initial state (ξ_1, ξ_2) at the left-hand side of the switching curve, the optimal strategy is f=-1, then f=+1. For the discrete control system, the optimal strategy is not unique except on the boundary states of the discrete isochrone. This has been shown on chapter II, section D.

(4) In the continuous optimal control systems, the optimal forcing function always has its absolute value as large as possible, say f=+1 or f=-1. But in the discrete case, the forcing function is not always as large as possible in absolute value. The effective forcing function can be $f_1=+1$, $f_1=-1$ or $f_1=\delta$, where $|\delta|<1$ and δ depends on the distance from the initial state to the critical curve in the $\underline{r}_1$ direction. When the maximum forcing function applies, the initial state moves along the trajectory of the continuous control plane from the region R_N to the region R_{N-1} in one sampling period. This is shown on Fig.3-4 from point A to A', or from point C to C'. When the optimal forcing function is not the maximum value, such as state B_1 or B_2 in Fig.3-4, the initial state no longer moves along the trajectory by f=+1. By using Eq.(3-10) the new trajectory can be constructed for $f=\delta$, where $|\delta|<1$. The initial state, B_1 or B_2, will

move along the trajectory for $f = \delta$, such that B' can be reached in one sampling period.

A few examples of the trajectories of the discrete optimal control are illustrated in Fig.3-4.

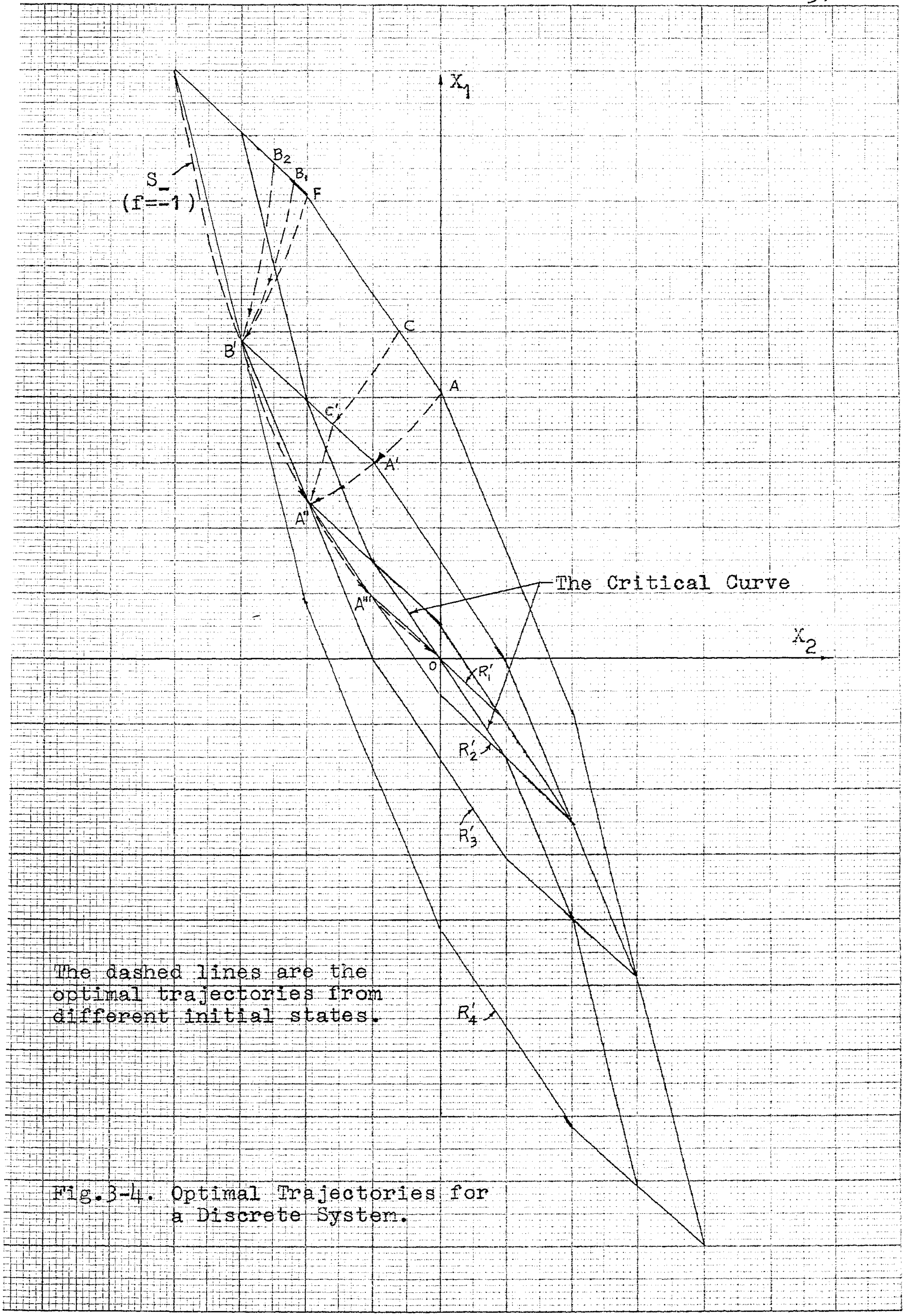
(a) From initial state A to A' to A" to A"' to the origin by the sequence of the forcing functions: +1, +1, -1, -1.

(b) From the initial state C to C' to A" to A"' to the origin by the sequence of the forcing functions: +1, +0.4, -1, -1.

(c) From the initial state B_1 to B' to A" to A"' to the origin by the sequence of the forcing functions: +0.8, -1, -1, -1.

(d) From the initial state B_2 to B' to A" to A"' to the origin by the sequence of the forcing functions: +0.5, -1, -1, -1.

(5) From "The Maximal Principle" of Pontryagin, for nth-order system, the optimal continuous control requires no more then (n-1) switchings from an initial state to go to equilibrium. Thus in the second-order system, there is at most one switching for the continuous optimal control. For the discrete optimal control system, there are at most (n-1) sign variations in the sequence of the forcing functions. This property was expressed in Eq.(2-21) or in Eq.(2-22).



The dashed lines are the optimal trajectories from different initial states.

Fig.3-4. Optimal Trajectories for a Discrete System.

CHAPTER IV

DIFFERENT PARAMETERS IN THE PLANT AND THE CONTROLLER

A. An investigation for different parameters in the plant and the controller

The system under consideration after normalization can be simplified as in Fig.2-2. The plant has transfer function $G(s) = \frac{1}{s(s+a)}$, and the computer(controller) has a fixed model $\frac{1}{s(s+1)}$. The forcing functions are based on the model of the controller. If the parameter "a" in the plant is equal to one, the discrete isochrone and the corresponding control trajectories are shown in Fig.3-4. The case when "a" is not equal to one will be investigated in this section.

From Eq.(2-6), the output of the system can be expressed as

$$c(t) = -\frac{1}{\sqrt{1+a^2}} X_1 - X_2 \qquad (4\text{-}1)$$

$$\dot{c}(t) = \frac{a}{\sqrt{1+a^2}} X_1 \qquad (4\text{-}2)$$

Let (ξ_1, ξ_2) be the initial condition in (X_1, X_2) plane, substitute Eq.(4-1),(4-2) into Eq.(2-2). Then

$$X_1 = \xi_1 e^{-at} + f_1(1-e^{-at})(1+a^2)^{\frac{1}{2}}/a^2 \qquad (4\text{-}3)$$

$$X_2 = \xi_2 - f_1 t/a + \xi_1(a-1)(1-e^{-at})/a(1+a^2)^{\frac{1}{2}} \qquad (4\text{-}4)$$

From Eq.(4-3)

$$t = -\frac{1}{a}\ln \frac{X_1 - f_1(1+a^2)^{\frac{1}{2}}/a^2}{\xi_1 - f_1(1+a^2)^{\frac{1}{2}}/a^2} = -\frac{1}{a}\ln\frac{X_1 - b}{\xi_1 - b} \qquad (4\text{-}5)$$

where $b = f_1(1+a^2)^{\frac{1}{2}}/a^2$

Eliminating t in Eq.(4-3) and (4-4), the trajectory in the (X_1, X_2) plane will be

$$X_2 = \xi_2 + \frac{f_1}{a^2}\ln\frac{X_1 - b}{\xi_1 - b} + \frac{\xi_1(a-1)}{a\sqrt{1+a^2}}\left(1 - \frac{X_1 - b}{\xi_1 - b}\right) \qquad (4\text{-}6)$$

Eq.(4-6) gives the trajectories on the (X_1, X_2) plane between each sampling instants. For a given initial condition (ξ_1, ξ_2), the trajectories on the (X_1, X_2) plane are dependent upon the value of the parameter "a". When a=1, Eq.(4-6) will be identical to Eq.(3-10).

From an arbitrary initial condition (ξ_1, ξ_2), calculate the distance from (ξ_1, ξ_2) to the critical curve in the $\underline{r}_1$ direction, then the forcing function can be determined. If the time constant, a, of the plant G(s) and the sampling period T are given, then by using Eq.(4-3) and (4-4) the state points can be calculated for each of the following sampling instants.

In order to compare the results to optimal trajectories for different values of the parameter "a", a numerical example is given below:

Given an arbitrary initial state (6.000, -0.625) in the (X_1, X_2) plane and a sampling period T=0.5 second. Consider three cases a=1.05, 1.00 and 0.95. Repeated use of Eq.(4-3) and Eq.(4-4) yield the trajectories and are plotted in Fig.4-1. Since the initial state (6, -0.625) is picked from the region R_4, after 4 sampling periods, the trajectory of a=1 goes to equilibrium exactly. But the trajectories of a=1.05 and a=0.95 can not go to equilibrium exactly, though they are very close to the origin.

Using the same initial condition (6, -0.625) and T=0.5 second, the trajectories are plotted in Fig.4-2 for a=0.5, 0.7, 1.5, and 2.0. When $a \neq 1$, none of these trajectories is an optimal trajectory. In comparison with the Fig.4-1, it is shown that for larger value of $|a-1|$, the trajectory will take longer time to approch the origin. Further more, for $a > 1$, the trajectories approch the origin directly, for $a < 1$, the trajectories approch the origin circuitously. A comparison in the time domain is shown in Fig.4-3.

B. Calibration of the parameter in the controller

The situation investigated in the last section is that the plant G(s) has the transfer function $\frac{1}{s(s+a)}$, while the controller(computer) has the fixed model of the transfer function $\frac{1}{s(s+1)}$. When a=1, the control is optimal, but when $a \neq 1$, the control is far from optimal.

If the time constant of the plant varies from the value 1 to the value a', the model of the controller must be calibrated from $\frac{1}{s(s+1)}$ to $\frac{1}{s(s+a')}$. This calibration can be calculated by measuring the state variables in the actual system.

Starting from the initial state (ξ_1, ξ_2), after one sampling period, it will go to the state (λ_1, λ_2) if a=1. Now, from measurment of the state variables, suppose the state is (X_1', X_2'). This means that the plant has a different parameter, a'. From Eq.(4-5)

$$t = -\frac{1}{a}\ln\frac{X_1-b}{\xi_1-b} = -\frac{1}{a'}\ln\frac{X_1'-b'}{\xi_1-b'} \qquad (4\text{-}7)$$

or

$$\frac{a'}{a} = \frac{\ln\left[(X_1'-b')/(\xi_1-b')\right]}{\ln\left[(X_1-b)/(\xi_1-b)\right]} \qquad (4\text{-}8)$$

where

$$b = f_1(1+a^2)^{\frac{1}{2}}/a^2, \qquad b' = f_1(1+a'^2)^{\frac{1}{2}}/a'^2$$

From Eq.(4-8), a suitable model for use in the controller can be determined to get an optimal control.

To illustrate the difference in critical curves for various values of "a", two sets of R_N' with a=1.5 and a=0.5 are plotted on the (X_1, X_2) plane as shown in Fig.4-4.

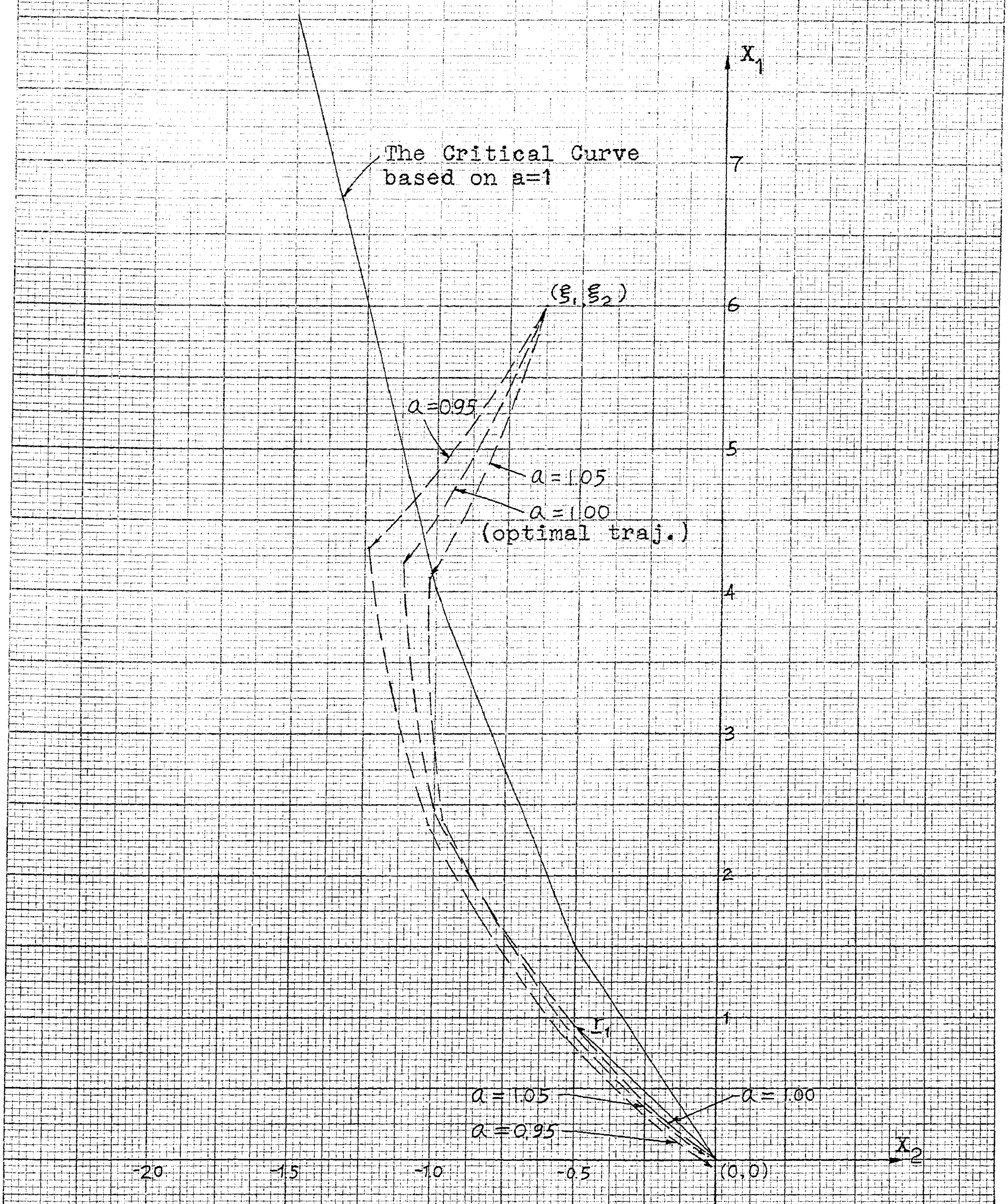


Fig.4-1. Trajectories for a = 1.00, 1.05 and 0.95.

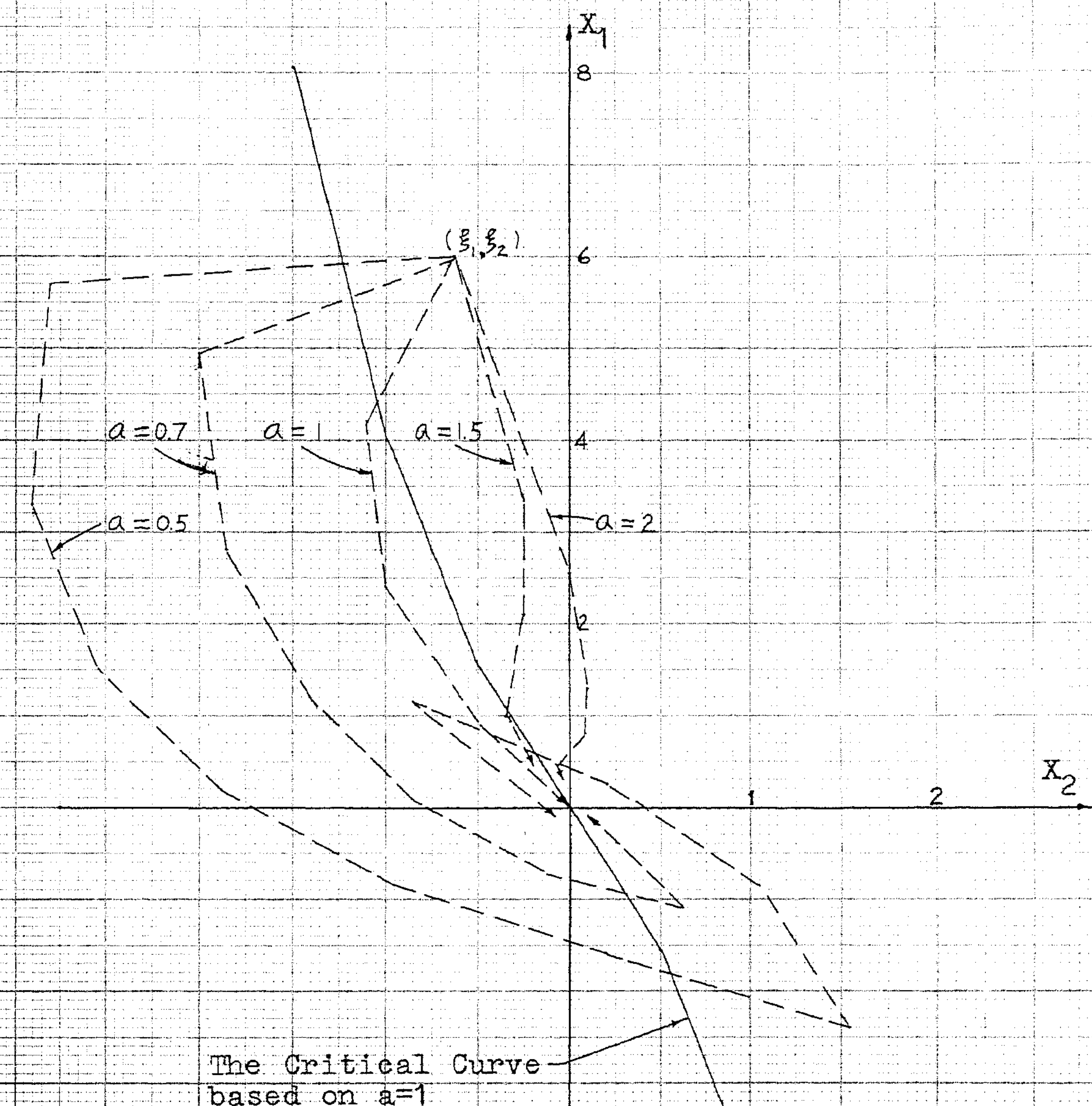


Fig.4-2. Approximate trajectories for

a = 0.5, 0.7, 1.0, 1.5 & 2.0

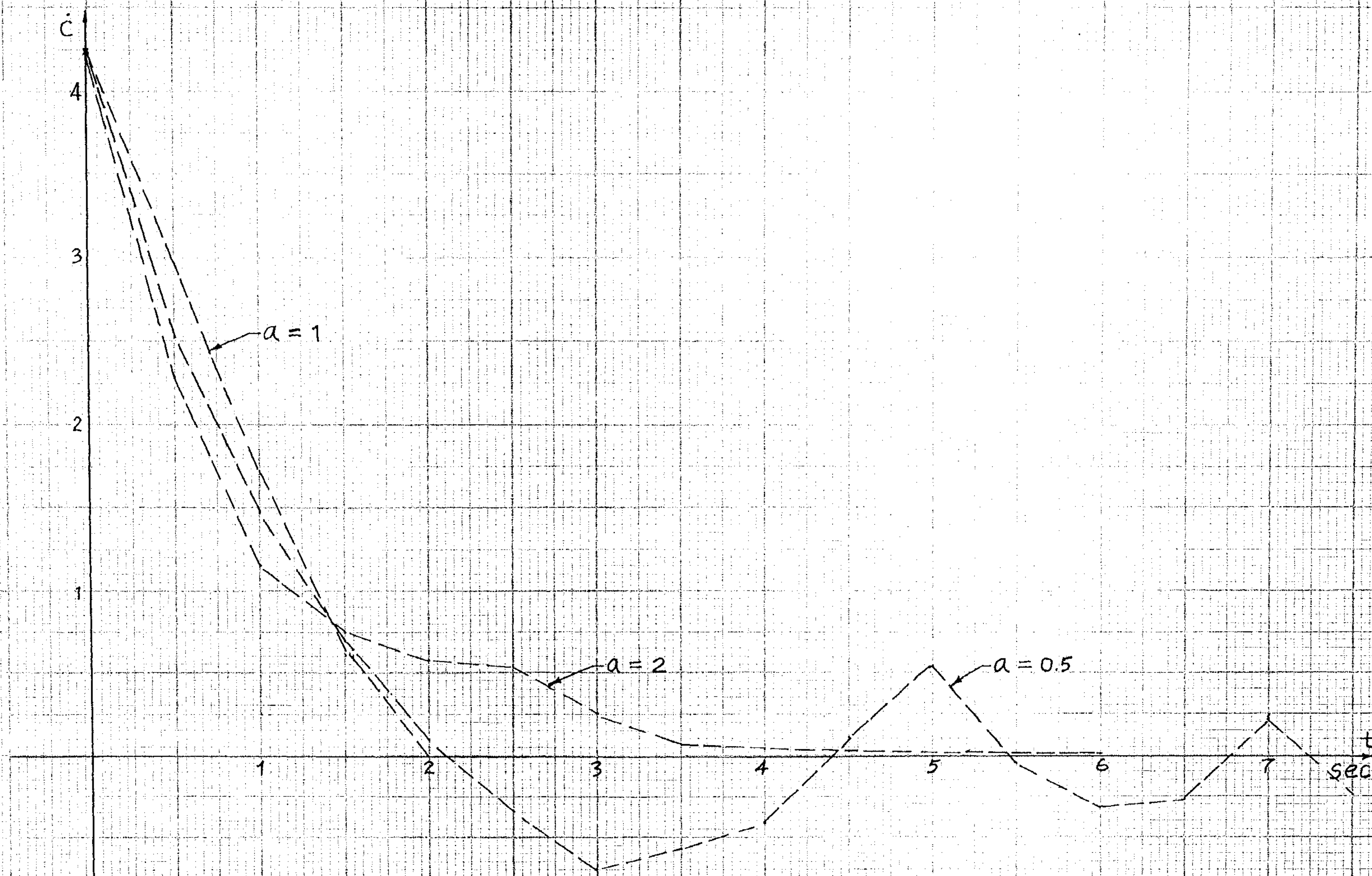


Fig. 4-3 The output response in time domain with different parameters in the plant and the controller.

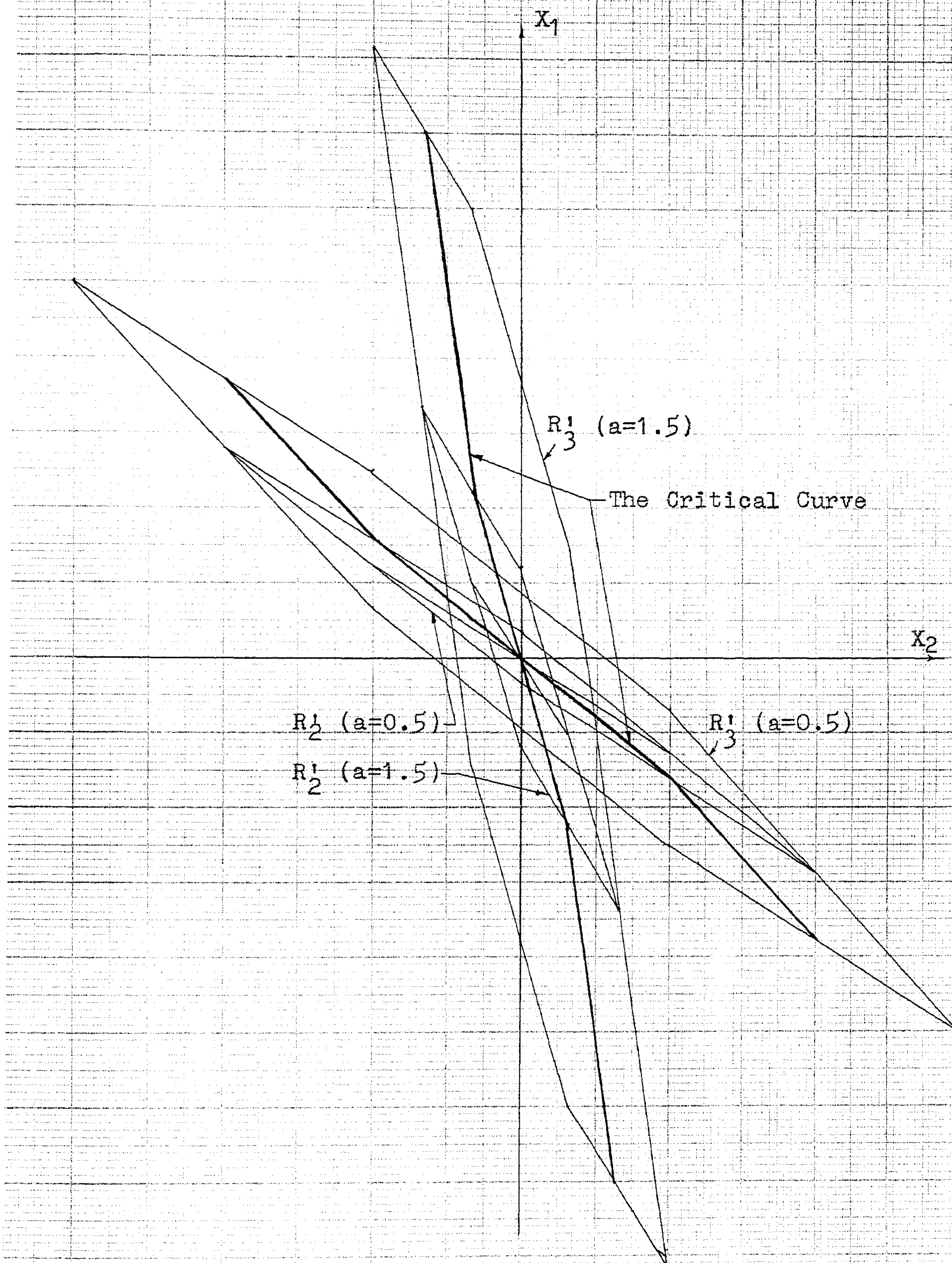


Fig.4-4. Control model for parameter a=0.5 and a=1.5.

CHAPTER V

CONCLUSION

This thesis has studied the problem of the time optimal strategy in the discrete case for a saturating sampled-data control system which has a linear plant $G(s)= \frac{1}{s(s+a)}$. Some investigation have been made in this study:

(1) Comparison of the minimum time isochrones between the discrete and the continuous control systems with the same plant transfer function. The regions for the discrete isochrone and the continuous isochrone are fairly close. In the limit as the sampling period approches zero, the discrete isochrone will coincide with that of the continuous case.

(2) The discrete control will take at most one more sampling period than the continuous system to go to equilibrium.

(3) In the continuous control system, the optimal strategy is unique. But for the discrete control system, the optimal strategy is not unique except on the boundary states of the region R_N.

(4) In the continuous case, the optimal forcing function always has its absolute value as large as possible, say $f=+1$, or $f=-1$. But in the discrete case, the forcing function is not always as large as possible in absolute value. The effective forcing function can be $f=+1$, $f=-1$ or $f=\delta$, where $-1 < \delta < +1$.

(5) From Pontryagin's maximal principle, in the second-order system, only one switching is required to get optimal control. There is at most one sign variation for the discrete optimal

strategy.

(6) If the plant has the time constant "a", while the controller in the feedback loop has the control model $\frac{1}{s(s+b)}$. The optimal control can be obtained only when a=b. If there is some variation of the parameter in the plant, the corresponding calibrations should be made by the controller to get the optimal control.

The study presented in this thesis can be extended in the following respects:

(1) The system under consideration of this study is a second-order system. The plant had a finite pole at the origin, that means the system contained an integrator. It could be extended to the case for the plant has two distinct negative real poles, or a pair of complex poles.

(2) The computer in the feedback loop was considered as an ideal case; it didn't take a finite time to calculate the forcing function. In the practical case, it takes a finite time for computation at the sampling instant. What is the effect of this time on the system response?

(3) The input to the system was assumed to be zero at all time. It could be investigated for the case when a nonzero input is applied to the system.

APPENDIX I

Data for Construction of Isochrones for a Plant, $G(s)=\frac{1}{s(s+1)}$.

For $t^* = 1.0$

t_2	Z_2	Z_1	t_1	ξ_2	ξ_1	$-\xi_2$	$-\xi_1$
0	0	0	1.0	+1.0	-2.45	-1.0	+2.45
0.40	-0.4	0.7	0.6	+0.2	+0.12	-0.2	-0.12
0.55	-0.55	1.05	0.45	-0.1	+0.85	+0.1	-0.85
0.64	-0.64	1.30	0.36	-0.28	+1.25	+0.28	-1.25
0.68	-0.68	1.41	0.32	-0.36	+1.41	+0.36	-1.41
0.86	-0.86	2.0	0.14	-0.72	+2.09	+0.72	-2.09
1.00	-1.0	2.44	0	-1.0	+2.44	+1.0	-2.44

For $t^* = 1.5$

t_2	Z_2	Z_1	t_1	ξ_2	ξ_1	$-\xi_2$	$-\xi_1$
0	0	0	1.5	+1.5	-4.93	-1.5	+4.93
0.4	-0.4	0.70	1.1	+0.7	-0.71	-0.7	+0.71
0.55	-0.55	1.05	0.95	+0.4	+0.47	-0.4	-0.47
0.68	-0.68	1.41	0.82	+0.14	+1.41	-0.14	-1.41
0.86	-0.86	2.00	0.64	-0.22	+2.53	+0.22	-2.53
1.00	-1.00	2.44	0.50	-0.50	+3.05	+0.50	-3.05
1.10	-1.10	2.82	0.40	-0.70	+3.80	+0.70	-3.80
1.22	-1.22	3.45	0.28	-0.94	+4.15	+0.94	-4.15
1.33	-1.33	3.95	0.17	-1.16	+4.40	+1.16	-4.40
1.42	-1.42	4.45	0.08	-1.34	+4.70	+1.33	-4.70
1.46	-1.46	4.70	0.04	-1.42	+4.85	+1.43	-4.85
1.50	-1.50	4.91	0	-1.5	+4.93	+1.50	-4.93

Data for construction of continuous isochrones.

For $t^* = 2.0$

t_2	Z_2	Z_1	t_1	ξ_2	ξ_1	$-\xi_2$	$-\xi_1$
0	0	0	2.0	2.0	-9.01	-2.0	+9.01
0.4	-0.4	0.70	1.6	1.2	-1.90	-1.2	+1.90
0.55	-0.55	1.05	1.45	0.9	-0.10	-0.9	+0.10
0.68	-0.68	1.41	1.32	0.64	1.41	-0.64	-1.41
0.86	-0.86	2.00	1.14	0.28	3.25	-0.28	-3.25
1.00	-1.00	2.44	1.00	0	4.11	0	-4.11
1.10	-1.10	2.82	0.90	-0.2	4.85	+0.2	-4.85
1.16	-1.16	3.15	0.84	-0.32	5.50	+0.32	-5.50
1.22	-1.22	3.45	0.78	-0.44	5.90	+0.44	-5.90
1.28	-1.28	3.70	0.72	-0.56	6.10	+0.56	-6.10
1.33	-1.33	3.95	0.67	-0.66	6.50	+0.66	-6.50
1.38	-1.38	4.20	0.62	-0.76	6.70	+0.76	-6.70
1.42	-1.42	4.45	0.58	-0.84	6.90	+0.84	-6.90
1.46	-1.46	4.70	0.54	-0.92	7.00	+0.92	-7.00
2.00	-2.0	9.01	0	-2.0	9.01	+2.00	-9.01

For $t^* = 0.5$

t_2	Z_2	Z_1	t_1	ξ_2	ξ_1	$-\xi_2$	$-\xi_1$
0	0	0	0.5	+0.5	-0.94	-0.5	+0.94
0.4	-0.4	0.7	0.1	-0.3	+0.63	+0.3	-0.63
0.5	-0.5	0.91	0	-0.5	+0.91	+0.5	-0.91

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VITA

The author was born on march 15, 1937, in Shanghai, China. He received his primary and high school education in Taipei, Taiwan, Republic of China. In September, 1956, he entered The National Taiwan University, and received the Bachelor of Science degree in Electrical Engineering in June 1960. After graduation, he worked in Taiwan Power Company as a junior engineer for two years.

Upon entering the United States in September 1963, he began studying for the Master of Science degree in Electrical Engineering at The Missouri School of Mines and Metallurgy(now The University of Missouri at Rolla).

The author is a member of the Institute of Electrical and Electronics Engineers.