Study of the method of geometric mean distances used in inductance calculations

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STUDY OF THE METHOD OF GEOMETRIC MEAN DISTANCES USED IN INDUCTANCE CALCULATIONS

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A

THESIS

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

INTRODUCTION

The object of this thesis is to study the method of geometric mean distances as applied in the calculation of inductances of transmission line conductors and of multi-circuit transmission lines.

The thesis will consist of: (a) the derivation of some of the basic theorems and equations to be used, (b) the use of these theorems and equations in calculating the geometric mean distances of different transmission line conductors, and (c) practical application of geometric mean distances applied in the calculation of equivalent inductance of multi-circuit transmission lines.

The method of geometric mean distances as applied to individual conductors of single circuit lines is widely used on a well established basis. Methods and rules for application to multicircuit lines of various types are not so well formulated. It is hoped that these latter rules may be better developed in this thesis.

CHAPTER I

DERIVATIONS

The inductance of a transmission line depends upon the material, dimensions, and configurations of the wires themselves, along with the spacing between them; the calculation of the inductance is based on the following fundamental definitions and equations.[1]

1. Definitions. The coefficient of inductance of a circuit of one single turn may be defined with the aid of the fundamental equation of induced voltage,

$$e = L\frac{di}{dt} = \frac{d\lambda}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where e is the voltage of self induction and L is the inductance in henries. The symbol for flux linkages is λ and is defined as the summation of all the elements of flux multiplied by the fraction of the total current linked by each, and (i) is the current in amperes. An ampere is the constant current which, maintained in two straight conductors of infinite length separated by a distance of one meter produces between the conductors a force of 2×10^{-7} Joules

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[1] Throughout this thesis the "rationalized" meter, kilogram, second system or "Giorgi" system of units will be used. In this system space permeability will be equal to $4\pi \times 10^{-7}$ which in turn will make the unit of magnetomative force F simply equal to the ampere-turn instead of ampere-turn divided by 4π.

per meter of length. Solving equation (1) for L and rearranging, the equation becomes

$$L = \frac{d\lambda}{dt}\,\frac{dt}{di} = \frac{d\lambda}{di} \text{ henries}\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

L, the inductance, is defined as the rate of change of flux linkages with current, or when there is no extraneous flux set up by other means than the circuit itself, the inductance in henries is defined as equal to the total number of flux linkages divided by the current.

2. Magnetic Field around a Long Cylindrical Conductor. It may be shown experimentally with iron filings and by use of a compass needle that in a long cylindrical wire of radius (a) carrying a steady current of I amperes, which is assumed uniformly distributed over its cross section[1/], there are set up lines of flux arranged in concentric circles around the wire and also within the wire. Figure 1 represents the concentric lines of flux of thickness dx at a distance x from the center, both inside and outside of the wire. From the above it follows that at any fixed radius from the center of the wire to a point outside the wire the flux density B and magnetizing force H are uniform. Along the elementary flux path outside

[1/] The assumption of uniform current density, although not absolutely correct, is satisfactory for overhead power circuits.

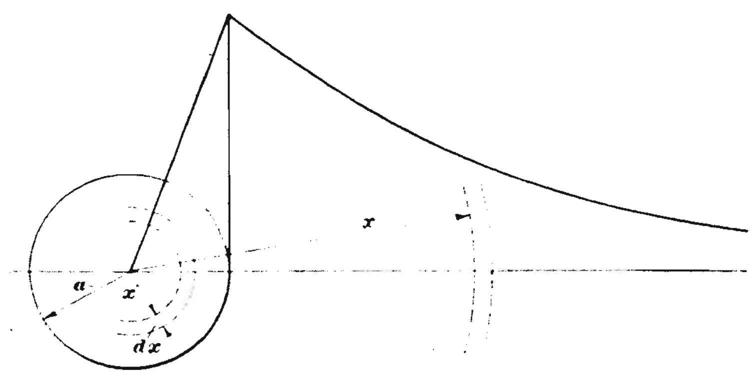


Figure I. Flux Linkages.

of the conductor of radius x meters and thickness dx the magnetic field intensity H is equal to the total magnetomotive force (mmf) F (in ampere-turns) divided by the length of path, which is $2\pi X$.

$$H = \frac{F}{2\pi X} \text{ ampere-turns per meter.}$$

Taking into consideration one turn of flux, F = I ampere-turns. Therefore,

$$H = \frac{I}{2\pi X} \text{ ampere-turns per meter..............(3)}$$

The magnetic flux density B along this path, which is in free space, is equal to the magnetic field intensity times space permeability μ_o. Space permeability μ_o is numerically equal to the ratio between flux density B and field intensity H in free space and is $4\pi 10^{-7} = 1.257 \times 10^{-6}$ 1/. By definition of flux density,

$$B = \mu_o H = 4\pi 10^{-7} \frac{I}{2\pi X}$$

or the magnetic flux density outside the conductor is

$$B = \frac{2 \times 10^{-7} I}{X} \text{ webers per square meter............(4)}$$

1/ See Footnote 1, page 2.

Consider now a tube of flux of infinitesimal thickness dx within the cylindrical conductor as shown in Figure I. The area contained inside this tube is πX^2. The current which flows within this area is I_x and is equal to $I \frac{X^2}{a^2}$ assuming uniform current density [1/] where I is the total current which flows in the total area πa^2 and (a) is the radius of the cylindrical conductor. Therefore, by analogy to equation (3)

$$H = \frac{I_x}{2\pi X} \text{ ampere-turns per meter}$$

and since $I_x = \frac{I X^2}{a^2}$,

therefore, $H = \frac{I X}{2 \pi a^2}$ ampere-turns per meter.

The magnetic flux density in this case is equal to μH where μ is the actual permeability of the conductor and for non-magnetic material is practically constant, but for magnetic materials varies with the type of metal, heat treatment, temperature, etc. In treating the latter analytically it is common to assume an average value of permeability, under average conditions, for the whole wire. Relative permeability is equal to $\frac{\mu}{\mu_0}$ or is the ratio of actual permeability and that of free space. Therefore, for the flux density

$$B = \mu_0 H \frac{\mu}{\mu_0} = \frac{4\pi 10^{-7} I X}{2\pi a^2} \frac{\mu}{\mu_0}$$

[1/] See Footnote 1, page 3.

$$B = \frac{2 \times 10^{-7} IX}{a^2} \quad X \text{ relative permeability}$$

or $$B = \frac{2 \times 10^{-7} IX}{a^2} \text{ webers per square meter} \qquad (5)$$

which takes care of the flux density within the wire itself.

Referring back to Figure 1, the flux outside of the conductor links with all the current flowing in the conductor, or in other words there is total flux linkages or external flux linkages. The flux within the conductor itself does not link with all of the total current; thus this flux forms partial flux linkages or internal flux linkages.

In calculating the inductance it is necessary to determine the external flux linkages and also an equivalent value for the partial flux linkages.

3. Internal Flux Linkages. From equation (5) the flux density due to the flux within the wire is

$$B = \frac{2 \times 10^{-7} IX}{a^2} \text{ webers per square meter.}$$

The flux within the elementary tube of radius x for a unit meter length of the conductor is

$$d\lambda = Bdx = \frac{2 X I}{a^2} \frac{\mu}{\mu_o} x \; 10^{-7} dx$$

Since this flux links with the current only in a portion of the total conductor area $(\frac{X^2}{a^2})$, the equivalent partial flux linkage λ_e in a differential increment of the radius dx is:

$$d\lambda_e = \frac{\mu}{\mu_o} \frac{2 \times 10^{-7}\ IX}{a^2} \times \frac{x^2}{a^2}\ dx$$

$$= \frac{2 \times 10^{-7}\ IX^3}{a^4} \frac{\mu}{\mu_o}\ dx \text{ linkages per meter}$$

The total equivalent partial flux linkages would be the summation or integral of all the flux linkages between the limits $X = 0$ to $X = a$. Thus,

$$\lambda_{et} = \frac{\mu}{\mu_o} \int_0^a \frac{2 \times 10^{-7}\ IX^3\ dx}{a^4} = \frac{\mu}{\mu_o}\ 2 \times 10^{-7}\ I \left[\frac{X^4}{4a^4}\right]_0^a$$

$$= \frac{\mu}{\mu_o} \frac{10^{-7}\ I}{2} \text{ linkages per meter........(6)}$$

4. <u>External Flux Linkages</u>. From equation (4) the magnetic flux density due to the total flux linkages outside the cylindrical conductor of unit relative permeability and radius (a) meters is

$$B = \frac{2 \times 10^{-7}\ I}{X} \text{ webers per square meter}$$

The total external magnetic flux linkages from (a) to a distance (D) outside the conductor per meter length of conductor is

$$\lambda = Bdx = \int_a^D \frac{2 \times 10^{-7}\ I\ dx}{X} = 2 \times 10^{-7} \ln \frac{D}{a} \text{ ...(7)}$$

5. <u>Inductance in Any Cylindrical Wire.</u> Total flux linkages are equal to total equivalent partial flux linkages

plus total external linkages. Therefore, combining equations (6) and (7)

$$\lambda_t = \frac{\mu}{\mu_o} \frac{10^{-7} I}{2} + 2 \times 10^{-7} I \ln \frac{D}{a}$$

$$\lambda_t = 10^{-7} I \left(2 \ln \frac{D}{a} + \frac{1}{2} \frac{\mu}{\mu_o}\right) \quad \ldots\ldots\ldots\ldots\ldots(8)$$

Inductance is equal to flux linkages divided by the total current by our previous definition. Hence

$$L = 10^{-7} \left(2 \ln \frac{D}{a} + \frac{1}{2} \frac{\mu}{\mu_o}\right) \text{ henries per meter..(9)}$$

Looking at equation (9) it is seen that the inductance is the sum of two terms; the term $(2 \ln \frac{D}{a})$ depends upon the size and spacing of the wires, and the second term depends upon the permeability of the wires. In ordinary overhead transmission line conductors, the second term is small compared to the first, except in the case when the conductors are made of iron or steel. Therefore, great importance can be placed on proper spacing and size of wires to obtain the least amount of inductance with minimum cost.

6. <u>Conversion to the CGS System</u>. The classic inductance formula is equation (9) which is

$$L = 10^{-7} \left(2 \ln \frac{D}{a} + \frac{1}{2} \frac{\mu}{\mu_o}\right) \text{ henries per meter}$$

based on the MKS system of units where μ is the actual permeability of the material and μ_o is the space permeability which is equal to $4 \pi 10^{-7}$ in the MKS system of units.[1/] Relative permeability as previously defined is $\frac{\mu}{\mu_o}$. In changing to the CGS system space permeability μ_o is equal to unity; therefore, equation (9) becomes

$$L = 2 \ln \frac{D}{a} + \frac{\mu}{2} \text{ abhenries per cm per conductor}$$

or

$$L = (2 \ln \frac{D}{a} + \frac{\mu}{2}) \; 10^{-8} \text{ henries per cm per conductor}$$

7. Inductance of a Single Wire in Terms of Geometric Mean Radius. Taking the solid cylindrical wire shown in Figure 1 and replacing it by an infinitisimally thin tube so that there are no internal flux linkages, we have a conductor whose radius (a) is known as the Geometric Mean Radius (GMR). This conductor has the same internal inductance, the relative permeability of which will be 10^{-7} in the MKS system or unity in cgs and practical system. Starting with equation (9) in this form

$$L = 10^{-7} (2 \ln D + 2 \ln \frac{1}{a} + \frac{1}{2} \; \frac{\mu}{\mu_o}) \text{henries/meter}$$

1/ See Footnote 1, page 2.

and applying the GMR principle, where by definition the GMR of a conductor is the radius of an infinitisimally thin tube with the same internal inductance as the conductor, the equation becomes

$$L = 10^{-7} \left(2 \ln D + 2 \ln \frac{1}{GMR}\right) \text{ henries per meter}$$

where the first term is the inductance due to the flux from the radius (a) out to a distance D, and the second term is the inductance due to all the flux within the radius (a). The above equation can be written in the following form:

$$L = 2 \times 10^{-7} \left(\ln \frac{D}{GMR}\right) \text{ henries per meter} \ldots\ldots\ldots (11)$$

Equation (11) is important because assuming uniform current density,[1] it can be used for calculating the inductance of any type of an overhead conductor, such as a stranded conductor, merely by using the proper value of GMR.

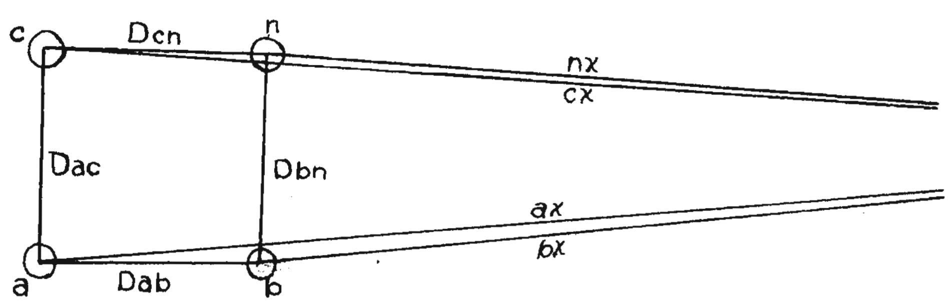


Figure 2. N-Parallel Wires Carrying Current.

[1] See Footnote 1, page 3.

8. Flux Linkages of N-Parallel Wires. In Figure 2 a group of N-parallel wires is represented so that they carry all of the current of the complete circuit. X is any point whose distance from a is greater than the distance from a to any other conductor in this system. Let it be required to develop an equation for the total flux linkages of N-parallel wires.

Calling the current I_a in conductor (a), and with the aid of GMR, the flux linkage about conductor (a) due to its own current I_a is from equation (11)

$$\lambda_a = 2 \times 10^{-7} \ln \frac{Dax}{GMR} I_a \text{ linkages per meter} \qquad (12)$$

where Dax is the distance from conductor (a) to point X.

Now the flux linkages in conductor (a), due to the current in any other one (k) of the remaining wires b, c, and d and produced by lines of flux between a and x, are from equation (7) for external flux linkages

$$\lambda_{ak} = \int_{D_{ka}}^{D_{kX}} \frac{2 \times 10^{-7} I_k \, dx}{X}$$

$$= 2 \times 10^{-7} I_k \ln \frac{D_{kx}}{D_{ka}} \qquad (13)$$

where k stands for conductors b, c, d, ...N.

This equation assumes that the diameters of the conductors are very small compared with the spacing of the wires in the circuit since the equation ignores the partial flux linkage effect. There is obviously very little error introduced by this assumption.

The total flux linkages about conductor (a) are therefore

$$\lambda_{at} = 2 \times 10^{-7} \left(I_a \ln \frac{Dax}{GMRa} + I_b \ln \frac{Dbx}{Dba} + I_c \ln \frac{Dcx}{Dca} + \cdots\cdots + I_n \ln \frac{Dnx}{Dna} \right) \ldots\ldots\ldots\ldots (14)$$

Since it was assumed that the N conductors carry all of the current,

$$I_N = -I_a - I_b - I_c \cdots\cdots - I_{n-1} \ldots\ldots\ldots\ldots (15)$$

Substituting the value for I_N in equation 14 and combining the logarithms the equation becomes

$$\lambda_{at} = 2 \times 10^{-7} \left(I_a \ln \frac{Dna}{GMRa} \cdot \frac{Dax}{Dnx} + I_b \ln \frac{Dna}{Dba} \cdot \frac{Dbx}{Dnx} + \cdots\cdots + I_{n-1} \ln \frac{Dna}{D(n-1)a} \cdot \frac{Dnx}{Dnx} \right)$$

Now let X approach infinity so that DaX, dbX and DnX will approach infinity, thus the fractions involving X will then approach unity as a limit. Therefore the actual total number of linkages about conductor (a) is equal to

$$\lambda_{at} = 2 \times 10^{-7} \left(I_a \ln \frac{Dna}{GMRa} + I_b \ln \frac{Dnax}{Dba} + \cdots + I_{n-1} \ln \frac{Dna}{D(n-1)a} \right)$$

or,

$$\lambda_{at} = 2 \times 10^{-7}\left[I_a\left(\ln\frac{1}{GMRa} - \ln\frac{1}{Dna}\right) + I_b\left(\ln\frac{1}{Dba} - \ln\frac{1}{Dna}\right) + \cdots\cdots + I_{n-1}\left(\ln\frac{1}{D(n-1)a} - \ln\frac{1}{Dna}\right)\right]$$

and then applying equation (15) based on Kirchkoff's Law, the above equation becomes finally

$$\lambda_{at} = 2 \times 10^{-7}\left(I_a \ln\frac{1}{GMRa} + I_b \ln\frac{1}{Dba} + I_c \ln\frac{1}{Dca} + \cdots\cdots + I_n \ln\frac{1}{Dna}\right) \text{ linkages per meter} \quad (16)$$

The above equation is an important one because it forms the basis from which it is possible to determine the total inductance of any system of parallel conductors and is the important step in the development of the widely used method of inductance calculation by Geometric Mean Distances.

9. Geometric Mean Distances. Figure 3 represents an

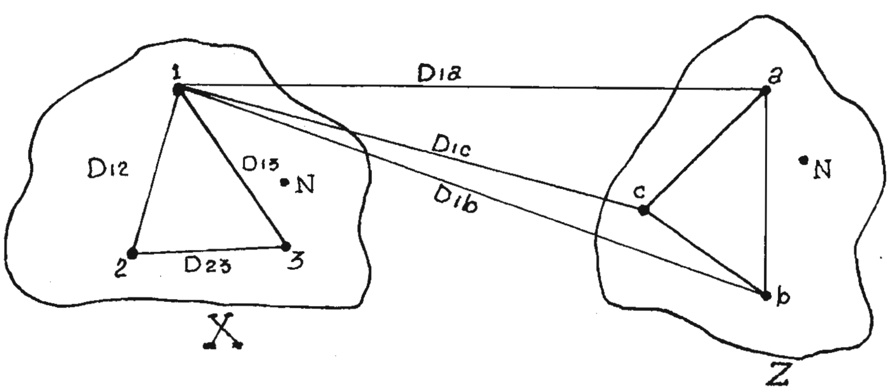


Figure 3. Two Cylindrical Non-Magnetic Parallel Conductors of Irregular Sections.

irregular cross section of cylindrical non-magnetic conductors X and Z. Let the cross sections of both X and Z be divided into an infinite number of infinitisimally small parts respectively so that each part carries equal current and, assuming uniform current density, is equal in size. Let the current in X be I and the current in Z be -I so that X and Z conductors make up a complete circuit. The current in each small element of X will be $\frac{I}{N}$ and of Z will be $\frac{-I}{N}$. The elements in X will be numbered 1, 2, 3,............N.

Now applying equation (16) just as if the various elements were cross sections of independent conductors, the number of linkages about conductor 1 is

$$\lambda_1 = 2 \times 10^{-7} \frac{I}{N}\left(\ln\frac{1}{GMR} + \ln\frac{1}{D_{21}} + \ln\frac{1}{D_{31}} + \cdots\cdots \right.$$

$$\left. \cdots\cdots + \ln\frac{1}{D_n} \right) - \frac{I}{N}\left(\ln\frac{1}{D_{a1}} + \ln\frac{1}{D_{b1}} + \cdots + \ln\frac{1}{D_{n1}} \right)$$

The first part of the above term includes the contribution of the elements in X and the second set of terms is from all the elements in conductor Z. The above equation by combining terms can be written

$$\lambda_{t1} = 2 \times 10^{-7} \frac{I}{N} \ln \sqrt[N]{\frac{D_{a1} \cdot D_{b1} \cdot D_{c1} \cdots\cdots D_n}{GMR_1 \cdot D_{21} \cdot D_{31}}}$$

Similar expressions may be written for the flux linkages about elements 2, 3, 4,..........N of irregular conductors or

$$\lambda_{tk} = 2 \times 10^{-7} \frac{I}{N} \ln \frac{\sqrt[N]{D_{aK} \cdot D_{bK} \cdot D_{cK} \cdots D_{nK}}}{\sqrt[N]{GMR_K \cdot D_{1K} \cdot D_{2K}}}$$

where K is any one of the elements in conductor X.

Now the sum of all the linkages about all the elements from 1 to N of conductor X will be equal to the number of linkages about the entire conductor. However, it must be remembered that one linkage about one of N equal elements contributes only $\frac{1}{N}$ of the linkage around the entire conductor. Therefore, summing up all of the equations for N linkages and dividing that sum by the number of elements (N), the equation becomes

$$\lambda_{tave} = 2 \times 10^{-7} \frac{I}{N} \left[\ln \frac{\sqrt[N]{D_{a1} \cdot D_{b1} \cdot D_{c1} \cdots D_{n1}}}{\sqrt[N]{GMR_1 \cdot D_{21} \cdot D_{31}}} + \ln \frac{\sqrt[N]{D_{a2} \cdot D_{b2} \cdot D_{c2} \cdots D_{n2}}}{\sqrt[N]{GMR_2 \cdot D_{12} \cdot D_{32}}} + \cdots\cdots + \ln \frac{\sqrt[N]{D_{an} \cdot D_{bn} \cdot D_{cn}}}{\sqrt[N]{GMR_n \cdot D_{1n} \cdot D_{2n}}} \right] \ldots\ldots\ldots\ldots (17)$$

which can be written in the form

$$\lambda_{tave} = 2 \times 10^{-7} \; I \left[\ln \frac{\sqrt[N^2]{(D_{a1} \cdot D_{b1} \cdots D_{n1})(D_{a2} \cdot D_{b2} \cdots D_{n2})(D_{an} \cdot D_{bn} \cdots D_{nn})}}{\sqrt[N^2]{GMR_{1 \cdot 2 \cdots n} \; D_{21} \, D_{31} \, D_{n1} \cdots D_{12} \, D_{32} \cdots D_{n2}}} \right]$$

Now let the number of elements into which each conductor is divided approach infinity, which in turn lets the numerator of the equation approach the geometric mean distance from one conductor to another. The denominator approaches the geometric mean distance of conductor X to itself or therefore equation (17) can be written

$$\lambda = 2 \times 10^{-7} I \ln_e \frac{D_m}{D_s} \text{ linkages/meter........(18)}$$

where D_m equals to the geometric mean distance between conductors and where geometric mean is the n^{th} root of an n-fold product. D_s equals the geometric mean radius or self GMD of a single conductor or a group of parallel conductors such as stranded conductors.

Solving equation (18) for L

$$L = \frac{\lambda}{I} = 2 \times 10^{-7} \ln_e \frac{D_m}{D_s} \text{ henries/meter....(19)}$$

or since there are 1609.4 meters in a mile, the equation can be written as

$$L = 0.000322 \ln_e \frac{D_m}{D_s} \text{ henries per mile........(20)}$$

or,

$$L = 0.000741 \log_{10} \frac{D_m}{D_s} \text{ henries per mile........(20a)}$$

Now, $X_L = 2\pi fL$, where X_L is the reactance in ohms, and f is the frequency. Since the frequency for most power transmission lines is 60 cycles per second, equation (20a) becomes in terms of the inductive reactance

$$X_L = 2\pi f \; 0.000741 \log_{10} \frac{D_m}{D_s}$$

$$= 0.004657 f \quad \log_{10} \frac{D_m}{D_s} \text{ ohms/mile per phase} \ldots (21)$$

Thus, at 60 cycles

$$X_L = 0.2794 \log_{10} \frac{D_m}{D_s} \text{ ohms/mile per phase} \ldots\ldots\ldots (21a)$$

CHAPTER II

SELF GEOMETRIC MEAN DISTANCE OF COMMON TRANSMISSION LINE CONDUCTORS

1. Theorems to be Used.[1/] The following well established theorems will be used in the calculation of the self geometric mean distance of solid stranded conductors.

I. Self geometric mean distances (D_s) of a circular area is $re^{-\frac{1}{4}}$ where r is the radius and e is 2.718.

$$D_s = re^{-\frac{1}{4}} \quad \text{..(22)}$$

II. Geometric mean distance of a circular line to any point, line or area wholly enclosed by the circular line is equal to the radius of the circular line.

$$D_m = r \quad \text{..(23)}$$

III. Geometric mean distance between two circular areas external to each other is equal to the distance between their centers.

$$D_m = D \quad \text{..(24)}$$

IV. If a circular line of radius r has on its periphery n equally spaced points the geometric mean distance between them is $r^{n-1}\sqrt{n}$. This is Guye's Theorem.

1/ See the appendix of this thesis for proofs of this theorem.

$$D_m = r^{n-1}\sqrt{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The two following theorems will be used for annular area conductors.

V. The self geometric mean distance of an annular area has its natural logarithm as follows:

$$\ln D_s = \ln r_1 - \frac{\frac{r_1^4}{4} - r_1^2 r_2^2 + r_2^4\left(\frac{3}{4} + \ln\frac{r_1}{r_2}\right)}{(r_1^2 - r_2^2)^2} \quad \ldots\ldots\ldots\ldots (26)$$

where r_1 and r_2 are the outer and inner radius respectively.

VI. The geometric mean distance of any point, line or area wholly within the annular area has for its natural logarithm the following expression:

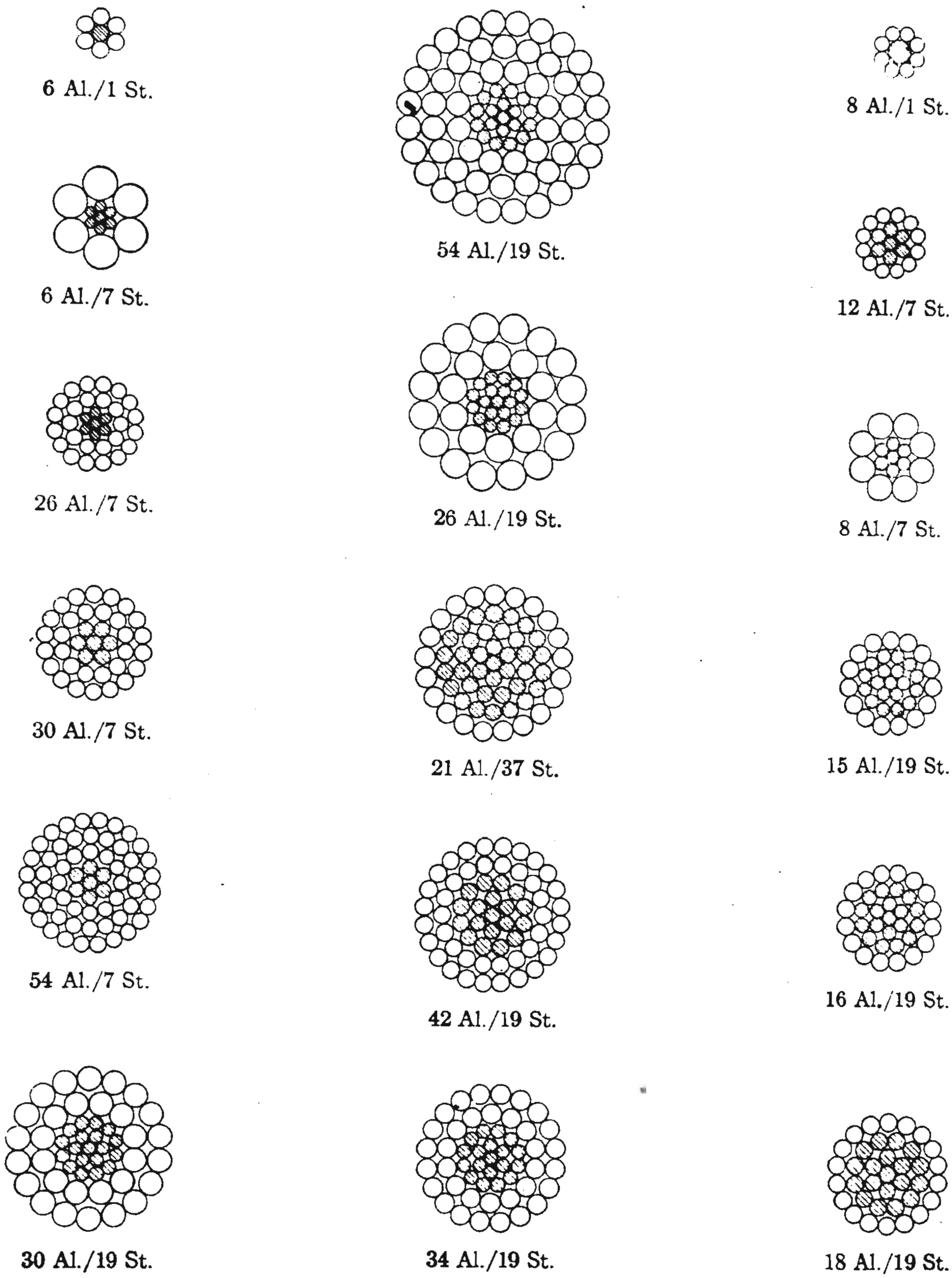
$$\ln D_m = \frac{r_1^2 \ln r_1 - r_2^2 \ln r_2}{r_1^2 - r_2^2} - \frac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots (27)$$

where r_1 and r_2 are the outer and inner radius respectively.

VII. The self geometric mean distance of a rectangular area of width X and length Y is

$$D_s = 0.2235\ (X + Y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

2. <u>Common Conductors</u>. The most common conductors used for high-voltage power transmission lines are stranded copper conductors, stranded aluminum cable - steel reinforced commonly called A.C.S.R. (see Figure 4) and hollow copper



ALUMINUM CABLE — STEEL REINFORCED

Some Types of A. C. S. R. Stranding

Figure 4.

conductors. Copper covered steel (known as Copperweld), Copperweld copper, and steel conductors, are occasionally used in transmission and distribution lines.

The stranded cables are made of concentric strands with all strands usually made of the same size as shown in Figure 5. Successive layers are spiraled in opposite directions to prevent one layer tending to settle into the interstices of the one underneath.

Figure 5. A Typical Stranded Conductor (bare copper)

For the first layer of strands around a central straight strand six strands would be required to fill the annular space. A second layer would require twelve more strands; a third eighteen more; a fourth would require 4 x 6 or 24 more; and so on, adding n x 6 for each increase in the number of layers. In other words the total number of strands in such cables whether homogeneous or not would be 7, 37, 61, 91, 127, etc.

Since most common transmission lines are built with these stranded conductors, it is necessary to develop methods of handling the calculation of inductance of such conductors. This is done by replacing the actual conductors by an equivalent cylindrical wire of equal geometric mean radius. Stranded conductors are made up of a number of such parallel wires. Then it is only necessary to obtain the proper geometric mean radius or distance for the whole conductor.

The self geometric mean distance will be called d_s when referring to the one conductor, and later in calculating inductance of multicircuit lines, the self geometric mean distance of the entire phase will be called D_s. The self geometric mean distance d_s is calculated in terms of the outside radius of the conductor or in terms of the area of the cable in circular mils where the area is equal to $n\ (2a)^2$ where n is the number of strands and a is the radius of the individual strands. The first case will be used here.

3. <u>Geometric Mean Distance of Different Stranded Homogeneous Conductors</u>. The most common of the homogeneous stranded conductors are made of copper, but occasionally homogeneous aluminum or steel cables are used.

(a) Concentric Cable of Seven Strands. Referring to Figure 6, there is a conductor made up of seven strands, each strand of radius (a), or a total radius for the whole conductor of (3a). Assuming uniform current density there

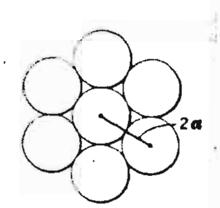


Figure 6. Seven-Strand Conductor.
(a in inches)

are seven equal currents in the seven strands and remembering that by definition the geometric mean is the $n^{\underline{th}}$ root of an n-fold product, the self geometric mean distance of the entire conductor is the $49^{\underline{th}}$ root of the 49 individual geometric mean distances among the seven strands, or

$$d_s = \sqrt[N^2]{(r_1\ r_2 \ldots r_n)(D_{21}\ D_{31} \ldots D_{n1})(D_{1n}\ D_{2n} \ldots)}$$

Listing the actual terms involved in this case by application of the theorems which have previously been proved,

(a) $ae^{-\frac{1}{4}} = 0.7788a$ ....7 terms

(b) $2a$ .... 12 terms

(c) $2a\sqrt[5]{6}$ .... 6 x 5 terms

49 terms

In the above expressions (a) is the self geometric mean distance of each of the seven strands as stated by theorem I. The term (b) represents the mutual distance between pairs of strands each one of the outer row with the center making a pair. The mutual distance is equal to their distance between centers or 2a by theorem III. There are twelve such terms because both directions must be taken into consideration. The last term is the geometric mean distance among the six outer strands as given by Guye's theorem or theorem IV. There are 6 x 5 such terms as each strand in the outer layer must be taken with respect to the other five. The product of them is equal to the $30^{\underline{th}}$ power of the geometric. Putting these terms in an equation for the self geometric mean distance

$$d_s = \sqrt[49]{(0.7788a)^7 \; (2a)^{12} \; 2a \; \sqrt[5]{6}^{\,6\times5}}$$

$$d_s = 2.180a$$

In terms of the outer radius R which in this case is equal to 3a

$$d_s = 0.726R \text{ inches} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

The self geometric mean distance in terms of the area A in circular measure, where A is equal to $7\ (2a)^2 = 28a^2$, is

$$d_s = .4114\ \sqrt{A} \text{ inches} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29a)$$

b. Homogeneous Concentric Cable of 37 Strands. All strands are the same size and carry the same current as they are all made of the same material whether copper, aluminum or steel. The strands are arranged in circular layers of 6, 12, and 18 strands about the central strand. Letting (a) be the radius of each individual strand, the radii of circles drawn through the centers of successive layers will be 2a, 4a, and 6a respectively. The self geometric mean distance of the entire homogeneous area to itself will include the following terms:

(a)	$0.7788a$	..........................37 terms
(b)	$2a$	.... 2 x 6 terms
(c)	$4a$	..................12 x 2 x 7 terms
(d)	$6a$	.... 18 x 2 x 19 terms
(e)	$2a\sqrt[5]{6}$	.... 6 x 5 terms
(f)	$4a\sqrt[11]{12}$	.... 12 x 11 terms
(g)	$6a\sqrt[17]{18}$	.... 18 x 17 terms
		Total $(37)^2 =$ 1369 terms

In this case (a) is the self geometric mean distance of each of the 37 strands as shown by theorem I. The parts (b), (c), and (d) are all based upon theorem III, which states that the geometric mean distance between two circular areas external to each other is equal to the distance between their centers. The $(4a)^7$ in part (c) is the geometric mean distance of the twelve outer strands of radius 4a to the seven strands within this radius. The $(6a)^{19}$ in part (d) is the geometric mean distance of the 18 outer strands of radius 6a to the 19 strands within this radius. The two above terms are based on theorem II.

The parts (e), (f), and (g) come from IV, which takes each strand in the layers of radius 2a, 4a and 6a with the remaining strands of the same layer. Hence,

$$d_s = \sqrt[37^2]{\left\{(0.7788)^{37}(2a)^{12}(4a)^{12\times 2\times 7}(6a)^{36\times 19} (2a\sqrt[5]{6})^{6\times 5}(4a\sqrt[11]{12})^{11\times 12}(6a\sqrt[17]{18})^{18\times 17}\right\}}$$

$$d_s = (5.375)(a)$$

in terms of the outer radius R, which is equal to 7a

$$d_s = 0.7679\ R \text{ inches} \quad \dots\dots\dots\dots\dots\dots\dots\dots (30)$$

or in terms of $\sqrt{A}$ where $A = 37\ (2a)^2$

$$d_s = .4419\sqrt{A} \text{ inches} \quad \dots\dots\dots\dots\dots\dots\dots\dots (30a)$$

where A is in circular inches.

(c) Homogeneous Cable of Three Strands. Assume that a homogeneous cable is made up of three equal strands placed in such a manner that they are tangent to each other externally and that lines drawn connecting their centers form an equilateral triangle. Obviously, there is no center strand and the cable cannot be considered as a concentric cable. It is, however, easily seen that value of d_s is

$$d_s = \sqrt[3^2]{(2a)^6\ (0.7788a)^3}$$

simplified,

$$d_s = \sqrt[3]{(2a)^2\ (0.7788)}\ a \text{ inches} \quad \dots\dots\dots\dots\dots (31)$$

where 2a is the distance between centers taken in both directions and (0.7788a) is the self geometric mean distance of the strands.

4. Self Geometric Mean Distance of Anaconda Hollow Conductors. Anaconda hollow conductors are made up of strands of copper wire wound on a twisted copper I-beam as a core, the I-beam being twisted in a direction opposite to that of the inner layer of strands (see Figure 7). Hollow conduc-

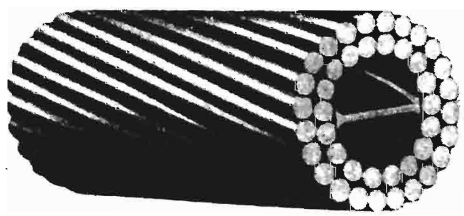


Figure 7. A Typical Anaconda Hollow Copper Conductor

tors have been developed for the purpose of reducing skin effects, corona formations, and inductance. The first layer of strands is spiraled in the opposite direction of the twist in the center I-beam, the second layer is twisted opposite to the first layer. This same pattern is carried throughout the different layers. There are several standard designs in use today with different size I-beams and different layers of strands. For example, design No. 378 is made up of two layers of strands wound around an I-beam which has a width equal to the diameter of three strands. The first layer wound on this beam consists of 12 strands, and the layer has an equivalent radius equal to 4a where a is the radius of each strand. The second layer is made up of 18 strands and has a radius equal to 6a. Design No. 378[1] has an outside diameter of 0.742 inches and an area of 0.2796 square inches. Each strand has a diameter of 0.1060 inches. The entire conductor, I-beam and strands, is made of copper. The geometric distance of this conductor is calculated as follows:

Area of the 30 strands is $30 \frac{\pi}{4} (.1060)^2 = 0.26476$ sq. in.

Total area = 0.2796 sq. in.

Area of core = .0148 sq. in.

Area of one strand = $(.1060)^2 \frac{\pi}{4} = .00883$ sq. in.

Core is therefore equal to 0.0148/.00883 = 1.6761 strands

The width of the core is the diameter of three conductors or 0.318 inches and assuming the core to be a rectangle the thickness would be the area/width or 0.0469 inches, which is equal to the area of 1.6761 strands. The total area is the equivalent of 31.676 strands. The value of self geometric mean distance is composed of the following terms:

(a) $(0.7788\ 6a)^{30} = (0.24766)$ ....30 terms

(b) $(0.2235\ \sqrt{x+y})^{(1.676)^2} = 0.08156$ ..... 2.809 terms

(c) $(6a)^{18 \times 2 \times 13.676} = 0.318$ .... 492.336 terms

(d) $(4a)^{12 \times 2 \times 1.676} = 0.212$ .... 40.224 terms

(e) $(6a \sqrt[17]{18})^{17 \times 18} = 0.3769$ .... 306.0 terms

(f) $(4a \sqrt[11]{12})^{11 \times 12} = 0.26574$ .... 132.0 terms

Total terms = $(31.676)^2$ = 1003.369 terms

[1] Woodruff, L. F., __Electric Power Transmission__, Second Ed., Table VI, p. 16 (1938).

The parts in order are: (a) the self geometric mean distance of the 30 strands to themselves; (b) the geometric mean distance from the core to itself based upon theorem VII for rectangular conductors; (c) the geometric mean distance from the outer layer to the inner layer and core, and return based on theorems II and III; (d) the geometric mean distance from the inner layer to the core, and return based on theorems II and III; (e) the geometric mean distance among the outer layer; (f) the geometric mean distance among the inner layer. Parts (e) and (f) are based upon Guye's theorem. The self geometric mean distance d_s will be equal to the $(31.676)^2$ root of the product of the above six parts which gives a self geometric mean distance equal to 0.31199 inches compared to 0.310 inches as given in the table.1/

5. Self Geometric Mean Distance of a General Cable Type H H Conductor. Hollow tubular conductors are being used increasingly for transmission line conductors. They have the advantage of small skin effect-resistance ratio, diminished inductance and lower corona loss due to a decreased dielectric gradient as compared to solid conductors of the same area of metal. Tubular conductors have a better current distribution than any other shape of conductor of similar cross-sectional area, but have a relatively small

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1/ See Footnote 1, page 29.

surface area for dissipating heat losses. Tubular conductors are usually made of copper but sometimes aluminum tubes are used.

For sake of illustration of calculation of tubular conductors, the self geometric mean distance of a General Cable type H H hollow copper conductor will be calculated (see Figure 8), taking a standard H H copper conductor[1/]

Figure 8. A Typical General Cable Type H H Conductor

of area 400,000 circular mils, with an outside diameter of 1.103 inches and uniform thickness of 0.100 inches. In the cases where the thickness is not uniform, an average value for the thickness should be taken. This conductor has an approximate current capacity of 838 amperes. The geometric mean radius is given as 0.0428 feet found by experiment. It can be calculated from theorem V, equation (26) which states that the self geometric distance of an annular area has as its natural logarithm

$$\ln d_s = \ln r_1 - \frac{\frac{r_1^4}{4} - r_1^2 r_2^2 + r_2^4\left(\frac{3}{4} + \ln\frac{r_1}{r_2}\right)}{(r_1^2 - r_2^2)^2}$$

1/ Westinghouse Electric Company, Electrical Transmission and Distribution Reference Book, Third Ed., Table 3-B, page 33, (1944) where r_1 and r_2 are the outer and inner radii respectively. The outside diameter is equal to 1.103 inches, the inside diameter is equal to 0.903 inches. Therefore,

$$\ln d_s = \ln 0.5515 - \frac{\frac{(0.5515)^4}{4} - (0.5515)^2(0.4515)^2 + (0.4515)^4\left(0.75 + \ln\frac{0.5515}{0.4515}\right)}{\left[(0.5515)^2 - (0.4515)^2\right]^2}$$

$$= 9.40488 - 10 - 0.0602778$$

$$= 9.34460 - 10$$

$$d_s = 0.5192 \text{ inches} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$

The experimental geometric mean distance of this conductor was previously given as 0.0428 feet or 0.5136 inches. The difference between the measured value and the calculated value is .0056 inches or an error of a little less than 1.1%. Since the experimental value is obtained by measurement of the inductance when the conductor has skin effect, corona loss, etc., this error is within expected limits.

6. Self Geometric Distance of Aluminum Cable-Steel Reinforced Conductors. On account of the relatively low tensile strength of all-aluminum conductors, it is necessary to use a composite cable combining the electrical conductance of aluminum with the tensile strength of steel. (See Figure 9.) Aluminum cable-steel reinforced (A.C.S.R.) is a concentric cable consisting of a central core (of one or

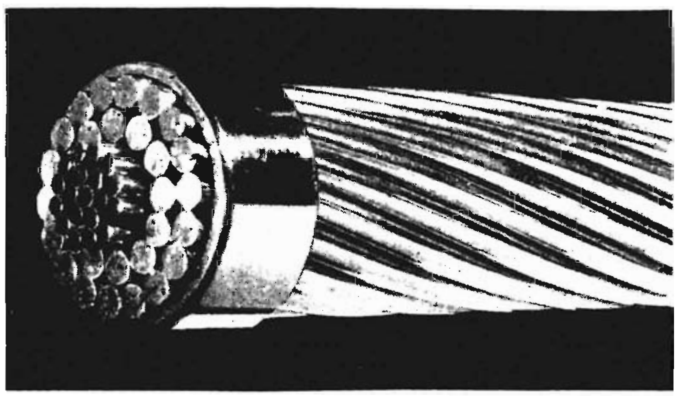

Figure 9. A Typical A.C.S.R. Cable

more galvanized high-strength steel wires) around which one or more layers of hard-drawn aluminum wires are stranded. A.C.S.R. is made with different proportions of steel depending upon various loading requirements.

The inductance of A.C.S.R. is reduced by the presence of a steel core because the current does not flow readily in the latter due to the poor conductivity of steel compared to aluminum; thereby giving the conductor the effect of a tube. With one layer of conducting strands, the solenoid action slightly more than counterbalances the tube effect. With two or more layers, however, the inductance is less than that for a solid non-magnetic conductor of the same overall diameter.

Consider a standard 30 x 7 strand A.C.S.R. cable which is made up of two layers of aluminum 12 and 18 strands respectively wound or spiraled on a steel core of seven strands; calculation of the self geometric mean distance would involve the following parts:

(a)	$0.7788a$	30 terms
(b)	$6a$	18 x 12 x 2 terms
(c)	$4a\ \sqrt[11]{12}$	11 x 12 terms
(d)	$6a\ \sqrt[17]{18}$	18 x 17 terms
	Total	$(30)^2 = 900$ terms

Since the major part of the total current flows in the aluminum, both because of its larger area and its greater conductivity, it is permissible to calculate the inductance from the geometric mean distance of this part alone and to apply to the effect of the steel reinforcement as a small correction of the inductance later. However, it is common practice to neglect the correction due to the steel core since it is very small, as will be shown later.

Part (a) above is the self geometric mean distance of the 30 individual strands of radius a. Part (b) is the geometric mean distance between the strands of the first and second layers taken both directions based upon theorems II and III. Part (c) is the geometric mean distance between the strands in the outside layer and part (d) is the geometric mean distance between strands in the inside layer, both parts being based upon Guye's theorem. In the above parts, a is the radius of the individual strands.

Applying the definition for geometric mean distance,

$$d_s = a \sqrt[(30)^2]{(0.7788)^{30}(6)^{432}(5.014)^{132}(7.1113)^{306}}$$

$$d_s = 5.7845a$$

or in terms of the outer radius R = 6a

$$d_s = 0.964093R \text{ inches} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

or in terms of area in circular measure

$$d_s \quad 0.52805\sqrt{A} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33a)$$

where A in this design is $30\ (2a)^2$

7. <u>Effect of the Steel Core in A.C.S.R. upon the Inductance</u>. The effect of the steel core is to increase

the number of linkages over what the cable would have if it were merely hollow, both because of linkages in the core itself, and because the shifting of part of the current toward the center results in greater flux density throughout the aluminum portion. The boundary condition controlling the determination of the relative currents in the aluminum and steel portions is that at the surface of contact the current densities of the two metals are in direct proportion to their conductivities. At regular power transmission frequency, skin effect is very small and can be neglected as far as the inductance calculations are concerned. At 60 cycles per second the skin effect-resistance ratio is 1.007 at 10 amperes and 1.013 at 20 amperes for Siemens Martin steel, which is commonly used as the core in A.C.S.R. Obviously, this indicates that there is, for all practical purposes, uniform current density throughout the steel. We can then say the currents in the two metals will be inversely proportional to their d-c resistances. The resistivity of steel is about 110 ohms per mil foot compared to about 17.0 ohms per mil foot for aluminum, both measured at 20 degrees centigrade. Therefore, the aluminum of a 30 + 7 A.C.S.R. cable will carry $\frac{30}{7} \times \frac{110}{17.0} = 27.7$ times as much current as steel. In other words, the steel would carry uniformly about $\frac{1}{27.7+1.0} \times 100$ or 3.48% of the current. This would have the effect of removing 3.48% of the current in the aluminum and its flow in the steel would obviously have no effect upon the external magnetic flux of the cable. It will, however, increase the flux density in the aluminum region by an amount which varies from zero at the outside to a maximum at the point of contact between the aluminum and the steel. Also, the value of flux density in the steel is increased, because previously the core current was assumed to be zero. For example, a standard 556,500 circular mil, 30 + 7 strand, A.C.S.R. cable will be considered. If the aluminum is considered as a tube with inside diameter of 6a or .4086 inch and outside diameter of 14a or 0.9534 inches, (a) being the radius of the individual strands or equal to 0.06455 inches, the increase in flux linkages per meter length due to the increased flux in the aluminum is equal to

$$10^{-7} I \left\{ \int_{0.2043}^{0.4767} \frac{4\pi}{2\pi x} \left[0.0348 + 0.9652 \frac{x^2 - (0.2043)^2}{(0.4767)^2 - (0.2043)^2} \right]^2 dx - \int_{0.2043}^{0.4767} \frac{4\pi}{2\pi x} \left[\frac{x^2 - (0.2043)^2}{(0.4767)^2 - (0.2043)^2} \right]^2 dx \right\}$$

or

$$\lambda_{a1} = 10^{-7} I (0.376 - 0.358)$$

$$= 2.0 \times 10^{-9} I \text{ linkages per meter}$$

where I is expressed in amperes.[1] Since there are 1609.4

[1] Woodruff, L. F., "Inductance of Aluminum Cable Steel Reinforced," AIEE Transactions, Vol. 54, page 299 (1935).

meters per mile, the increase of inductance per mile would be

$$L = \frac{\lambda}{I} = 1609.4 \times 2 \times 10^{-9} = 3.219 \times 10^{-6} \text{ henries/mile}$$

or .00322 millihenries per mile of one conductor. Also, in addition to the increase in inductance due to the aluminum we have an increase owing to the linkages between the steel core current and core flux. Experimental results on seven strand steel cores of this size and type indicate an internal inductance that varies slightly with the current, but an average value of 1.50 millihenries may be used. In order to make use of this datum for A.C.S.R., it must be multiplied by the square of that fraction of the current carried by the steel core or $(.0348)^2 \times 1.50 = 00.00182$ millihenries per mile per conductor. Therefore, the total contribution to the inductance due to the presence of the seven strand steel core in the center instead of a non-conductor would be the sum of the 0.00322 millihenry due to the effect of the core on the aluminum and the 00.00182 millihenry due to the core itself or 0.00504 millihenry per mile of conductor. The same increase would apply to other A.C.S.R. cables of different size provided they are geometrically similar; that is, composed of 30 + 7 strands of aluminum and steel. Obviously these corrections are negligible but by applying them to the inductance as calculated from the d_s found by neglecting the core the calculated value and the value obtained experimentally as explained in the succeeding paragraph, will give for all practical purposes the same value.

8. Calculation of Inductance from Self Geometric Mean Radius and Vice Versa. Going back to equations (20), (20a) and (21a) which were derived in chapter 1, it is seen that the inductance and the inductive reactance can be calculated if the self geometric mean distance d_s of the conductor is known, along with the mutual geometric mean distance or equivalent spacing. Starting with equation (20a),

$$L = 0.000741 \log_{10} \frac{D_m}{d_s} \text{ henries/mile}$$

and letting the equivalent spacing be one foot, a table could be made for the inductance as

$$L = 0.000741 \log_{10} \frac{\text{one foot spacing}}{d_s \text{ in feet}} \text{ henries/mile} \quad \ldots\ldots\ldots\ldots\ldots(34)$$

or from equation (21a) at 60 cycles the inductive reactance is

$$X_L = 2\pi fL = 0.2794 \log_{10} \frac{\text{one foot spacing}}{d_s \text{ in feet}} \text{ ohms/mile} \quad \ldots\ldots(35)$$

A table could be made in terms of the 60 cycle reactance.

The self geometric mean distance from a tabulated or experimental reactance may be obtained from the following equation:

$$d_s = \frac{1}{\text{antilog}_{10} \dfrac{\text{reactance with one foot spacing 60 cycles}}{0.2794}} \text{ in ft.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(36)$$

If the reactance is now known to a one foot spacing but is a value corresponding to a spacing equal to the conductor diameter it is commonly called the internal reactance and

$$d_s = \frac{\text{physical radius in feet}}{\text{antilog}_{10} \dfrac{\text{internal reactance (60 cycles)}}{0.2794}} \text{ in ft.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(37)$$

Now going back to the 556,500 circular-mil 30 + 7 strand A.S.C.R. cable used in paragraph 8 and actually comparing the difference between the calculated value and the experimental value of inductance or reactance at one foot spacings, the inductance, neglecting the steel core from our calculation of d_s is as follows:

$$d_s = 0.5280 \sqrt{A} = 0.5280 \sqrt{556,500} \times 10^{-3} = 0.394 \text{ in.}$$

$$= \frac{0.394}{12} = 0.0328 \text{ ft.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38)$$

From equation (34)

$$L = 0.000741 \log_{10} \frac{\text{one foot spacing}}{\text{GMR}} = 0.000741 \log_{10} \frac{12}{0.394}$$

$$= 1.0994 \text{ mh/mile} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

Adding to this value the 0.00504 increase due to the steel 7 strand core we have 1.1044 mh/mile. It can readily be seen that the effect of the steel core on the inductance is very small, or .00504 mh compared to 1.0994 found when neglecting the core. For the sake of comparing the calculated value with the experimental value(which is given as 0.415 ohms at 60 cycles at one foot spacings, by Aluminum Company of America's tables on Characteristics of A.C.S.R.)[1/] the reactance for one foot spacings is calculated as follows:

$$L = 1.1044 \text{ mh/mile}$$

$$X_L = 2\pi fL = 0.001104 \times 2\pi \times 60 \text{ ohms per mile}$$

$$X_L = 0.4161 \text{ ohms/mile} \qquad \text{..................(40)}$$

The results are well within the accuracy of calculation or measurement.

Since the experimental values also include skin effect and proximity effect it can be assumed, as has been previously stated, that these effects are negligible.

[1/] Westinghouse Electric Company, Electrical Transmission and Distribution Reference Book, 3rd ed., Table 2, page 32, (1944)

CHAPTER III

APPLICATION OF GEOMETRIC MEAN DISTANCE IN THE CALCULATION OF EQUIVALENT INDUCTANCE OF MULTICIRCUIT LINES

1. General Inductance Formula. In sections 8 and 9, chapter 1, the equation for the total flux linkages between two conductors made up of N-parallel wires was found to be from equation (18)

$$\lambda = 2 \times 10^{-7} I \ln_e \frac{D_m}{d_s} \text{ linkages per meter.......(41)}$$

where D_m was equal to the geometric mean distance between conductors and d_s was equal to the geometric mean radius of a group of N-parallel wires of one conductor. From this, the above equation for the inductance by definition is

$$L = \frac{\lambda}{I} = 2 \times 10^{-7} \ln_e \frac{D_m}{d_s} \text{ henries/meter.....(42)}$$

or at 60 cycles, the inductive reactance is

$$X_L = 0.2794 \log_{10} \frac{D_m}{d_s} \text{ ohms/mile..............(42a)}$$

Polyphase transmission lines can be considered as merely a special case of the general N-conductor line and can be treated as such by using the formula as

$$L = 2 \times 10^{-7} \ln_e \frac{D_m}{d_s} \text{ ....(43)}$$

where D_m here represents the mutual geometric mean distance between all of the conductors of one phase with all of the conductors of the other phases. D_s is the self geometric mean distance of one phase, which can be made up of any number of individual conductors, each of which may or may not be stranded but has a self geometric mean distance d_s. D_s then takes into account the individual self geometric mean distance d_s of each conductor of a phase along with the mutual geometric mean distance of these conductors making up this phase. Obviously, equation (43) can be used to calculate the inductance of any type of transmission line made up of any number of phases and any number of conductors per phase as long as the proper values of D_m and D_s are used, providing the line is properly transposed.

Theorem III or equation (24), which states the geometric mean distance between two circular areas external to each other and in a common plane is equal to the distance between their centers, is used to find the D_m and D_s in equation (43). This theorem was proved in the appendix and will be used throughout the rest of this chapter in finding D_m and D_s

A limitation to equation (43) is that conditions must remain so that uniform current density over all of each conductor is maintained (or else non-uniformity within a single conductor is taken into account by the use of the correct value of equivalent self geometric mean distance d_s). This is taken care of by transposition, which will be fully explained later.

2. <u>Inductance of a Single Phase Line</u>. Taking a single phase line made up of two conductors of geometric mean radius d_{sa} and d_{sb} separated from each other by a distance D and calculating the inductance,

$$L = 2 \times 10^{-7} \ln_e \frac{D^2}{d_{sa}\ d_{sb}} \text{ henries per meter}$$

The terms d_{sa} and d_{sb} are the self geometric mean distances of conductors a and b calculated as was shown in chapter 2, where conductor a carries I current and b carries the return current or -I. If the two conductors making up the complete circuit were the same size and of the same material, which is generally the case, the equation would reduce to

$$L = 2 \times 10^{-7} \ln_e \frac{D_m}{D_s} \text{ henries per meter......(44)}$$

where D_m is the mutual geometric mean distance based on theorem III or equation(24).

3. <u>Symmetrical Three-Phase Single Circuit Line</u>. The simplest three-phase arrangement, and the only one which does not require transposition to balance the inductive reactance drop per phase is the equilateral triangle arrangement. Here is found one conductor for each phase placed at each vertix of an equilateral triangle at some distance D from each other or where D is the length of the sides of the equilateral triangle. Using the same type and size of

conductor for each phase, the inductance would be

$$L = 2 \times 10^{-7} \ln_e \frac{\sqrt[3]{D^3}}{\sqrt[3]{d_{s1}\ d_{s2}\ d_{s3}}}$$

or which reduces to

$$L = 2 \times 10^{-7} \ln \frac{D}{D_s} \text{ henries per meter........(45)}$$

or at 60 cycles, the inductive reactance is

$$X_L = 0.2794 \log_{10} \frac{D}{D_s} \text{ ohms per mile per phase...(45a)}$$

It is seen from the above equation that with the same spacing D and the same conductor size that the inductance per unit length of a three-phase line is the same as that of a single phase line which is given by equation (42).

4. Symmetrical Single-Phase Double-Circuit Line. From Figure 10 it can be seen that in this case conductors 1 and

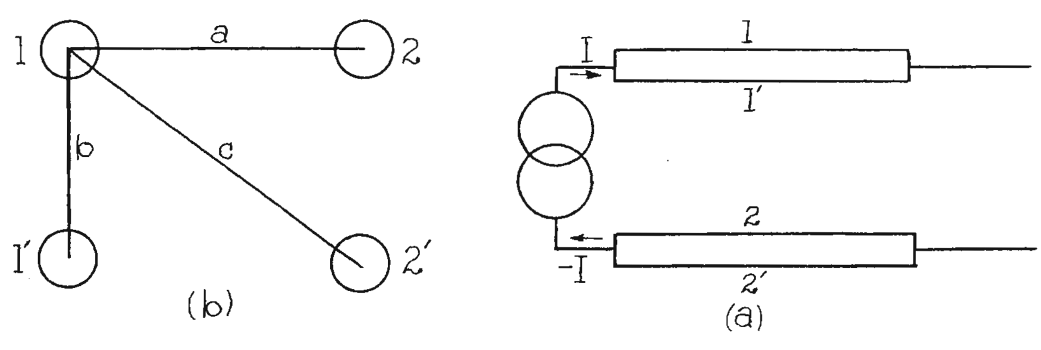

Figure 10. Symmetrical Single-Phase Double-Circuit Line.

1' are in parallel and carry together the line current I, and conductors 2 and 2' in parallel together carry the return current -I. It will be assumed that all of the conductors are similar in size, material and design or in other words all conductors have the same d_s and will carry equal currents uniformly distributed. It can be seen from the symmetry of the arrangement that there will be no unbalanced voltages introduced in conductor 1 and 1' due to currents in conductors 2 and 2'. Therefore, the currents in 1 and 1' will be equal. Applying the fundamental inductance formula we have, letting X represent lines 1 and 1' and Y represent lines 2 and 2',

$$L = 2 \times 10^{-7} \ln_e \frac{D_m}{D_s} \text{ henries/meter}$$

D_m is the geometric mean distance from lines X to Y and would be equal to by definition the fourth root of the lengths of the four lines joining the centers of conductors 1 and 1' with the centers of 2 and 2' or D_m would equal to the fourth root of A^2C^2, or $D_m = \sqrt{A\ C}$. D_s is the self geometric mean distance of line X or line Y, as previously stated, and is equal to the square root of the product of the distances between the two conductors making up lines X and Y, times the self geometric mean distance d_s of the conductor, there fore,

$$D_s = \sqrt{b\ d_s}$$

Thus,

$$L = 2 \times 10^{-7} \ln_e \frac{\sqrt{A\ C}}{\sqrt{b\ d_s}} \text{ henries per meter....(46)}$$

In looking at this equation for this type of line and realizing that a minimum of inductance is desirable in order to improve voltage regulations, increase the power limit, and improve the power factor, etc., it is seen that best results would be obtained by keeping the individual conductors of a phase as far apart as practical while at the same time keeping the distance between lines X and Y as small as practical. Therefore, in designing a line of this type, the above considerations along with cost should be taken into account.

5. Unbalanced Single-Conductor Three-Phase Line. So far in this chapter all of the lines taken into consideration have been balanced and did not need transposition. In the case of an unbalanced line (and by this is meant a line designed so that without transposition unequal inductance occurs in the different phases because of the effect of any one phase inducing a field on a second phase but not on a third) all phases do not affect each other equally. This in turn often causes unequal currents to flow in the individual conductors making up that phase. This condition is taken care of by transposition. Take for example a three-phase line with one conductor carrying the current in each phase. Calling the three conductors a, b, and c, assume

they are arranged in a triangle such as is shown in Figure II.

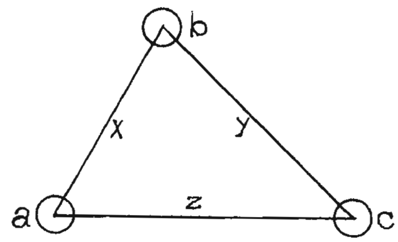

Figure II. Cross Section of a Three-Phase Unsymmetrical Line.

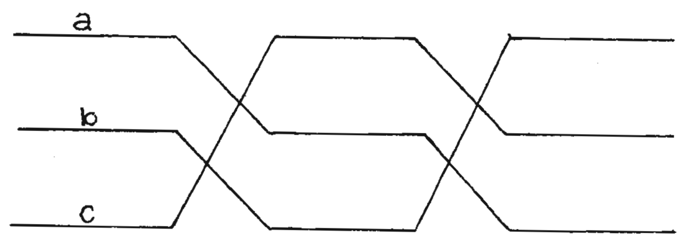

Figure IIa. Three-Phase Line Transposed.

The inductance would be

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_s} = 2 \times 10^{-7} \ln \frac{\sqrt[6]{x^2 y^2 z^2}}{\sqrt[3]{d_s{}^3}} =$$

$$2 \times 10^{-7} \ln \frac{\sqrt[3]{x\ y\ z}}{d_s} \text{ henries/meter..........(47)}$$

Offhand, it might seem that this design is impractical in the sense that if only one conductor per phase was used the conductors would be placed in an equilateral triangle arrangement as was shown in paragraph 3. This is not true since unsymmetrical lines are often used because of more convenient mounting on poles or towers, or for the purpose of keeping the average height of the conductors above the ground as low as practical to prevent as much as possible lightning hazards. Sometimes because of inductance interference effect of parallel communication circuits, symmetrical lines become unbalanced; this would be eliminated by transposition in the absence of zero sequence currents.

6. <u>Unsymmetrical Single Phase Multi-Circuit Line</u>. Take for an example the case where the conductors of the single-phase line in Figure 10 are arranged to form a line as shown in Figure 12a. Without transposition unequal currents

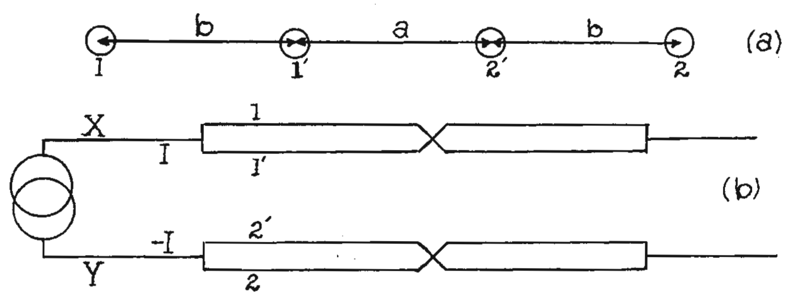

Figure 12. Unsymmetrical Double-Circuit Single-Phase Line and Transposition Cycle.

would flow in the two individual conductors of line X and line Y, because the currents of line Y would cause more linkages in line 1'of line X than in the outer conductor 1, and vice versa. This effect is eliminated by proper transposition as shown in Figure 12b. The mutual geometric mean distance (D_m) in this case would be

$$D_m = \sqrt[4]{(a)\ (a+b)^2\ (a+2b)}$$

and

$$D_s = \sqrt[4]{b^2\ d_s^2} = \sqrt{b\ d_s}$$

Therefore, the inductance would be

$$L = 2 \times 10^{-7} \ln \frac{\sqrt{a\ (a+b)^2\ (a+2b)}}{\sqrt{b}\ d_s}$$

henries per meter..............(48)

7. <u>Three-Phase Double-Circuit Symmetrical Line</u>. Assume a three-phase line composed of six equal conductors arranged so that lines drawn connecting the centers of the adjacent conductors would form a regular hexagon with each side equal to a distance D. For this three-phase symmetrical arrangement there are two conductors per phase. In order to increase the self geometric mean distance, thereby decreasing the inductance, the two conductors of each phase are placed as far apart as possible (as shown in Figure 13). Obviously, it

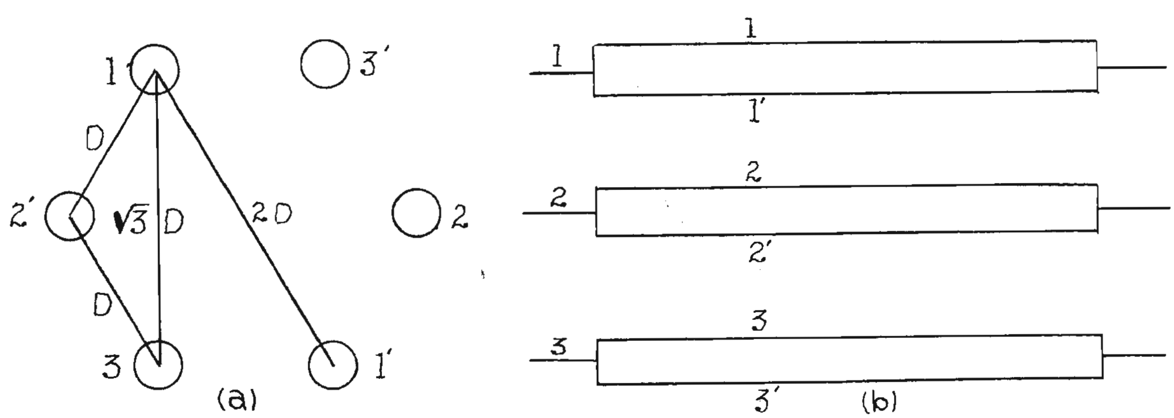

Figure 13. A Three-Phase Double-Circuit Symmetrical Line.

will not be necessary to transpose this line owing to the fact that the flux linkages about each of the two conductors of a phase, produced by the currents of the other two phases, will be equal. The current density over the entire cross-section of each phase made up of two conductors will be uniform (or else, if there is non-uniformity within a single conductor, it is taken into account by using the correct equivalent self geometric mean distance d_s as was shown in chapter 2.)

The mutual geometric mean distance D_m in this case then is applying equation (24) from theorem III,

$$D_m = \sqrt[24]{(D\sqrt{3}\ D\sqrt{3}\ D\ D)^6} = \sqrt[24]{D^{12}\ (\sqrt{3}\ D)^{12}}$$

$$= \sqrt[12]{D^6\ (\sqrt{3}\ D)^6} = D\ \sqrt[12]{3^3} = D\sqrt[4]{3}$$

In finding D_m it is necessary actually to take the distance from the center of each conductor of each phase to the center of the four other conductors making up the other two phases. In other words, the first term in the radical $(D\sqrt{3}\ D\sqrt{3}\ D\ D)$ is the product of the distance from conductor 1 of phase 1 to conductors 2', 3, 2 and 3' respectively and there will be five more such products for the other five conductors. The mutual geometric mean distance is then the 24th root of all such products from six conductors.

The self geometric mean distance of the entire phase will be

$$D_s = \sqrt[4]{(2D)^2\ d_s^2} = \sqrt{2D\ d_s}$$

where 2D is the distance between the centers of the two conductors making up a phase and d_s is the self geometric mean distance of each individual conductor calculated as was shown in chapter 2 or else found from tables. Therefore,

$$L = 2 \times 10^{-7} \ln \frac{D_M}{D_s} = 2 \times 10^{-7} \ln \frac{\sqrt[4]{3}\ D}{\sqrt{2D\ d_s}}$$

$$= 10^{-7} \ln \frac{\sqrt{3}\ D}{2d_s} \text{ henries/meter/phase} \ldots\ldots\ldots (49)$$

or at 60 cycles, the inductive reactance is

$$X_L = 0.1347 \log_{10} \frac{\sqrt{3}\ D}{2d_s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49a)$$

8. Three-Phase Double-Circuit Unsymmetrical Line. Sometimes it is necessary to use a line in which the conductors are not arranged symmetrically and have to be transposed, as has been previously explained. Reasons for this are as follows: (a) to bring about more convenient mounting on poles, (b) to keep the average height above the ground as low as practicable in order to minimize hazards due to lightning, (c) to reduce the effect of parallel communication circuits. Take for an example a line similar to the one discussed in paragraph 7, except in this case the conductors are arranged as shown in Figure 14.

The mutual geometric mean distance D_m is composed of the following terms, taking each conductor with the four other conductors of the other two phases in a counter-clockwise direction:

Conductor 1 ... a e g d .... 4 terms

Conductor 2 ... b i g a .... 4 terms

Conductor 3 ... c i e b .... 4 terms

Conductor 1'... b e i c .... 4 terms

Conductor 2'... a g i b .... 4 terms

Conductor 3'... d g e a .... 4 terms

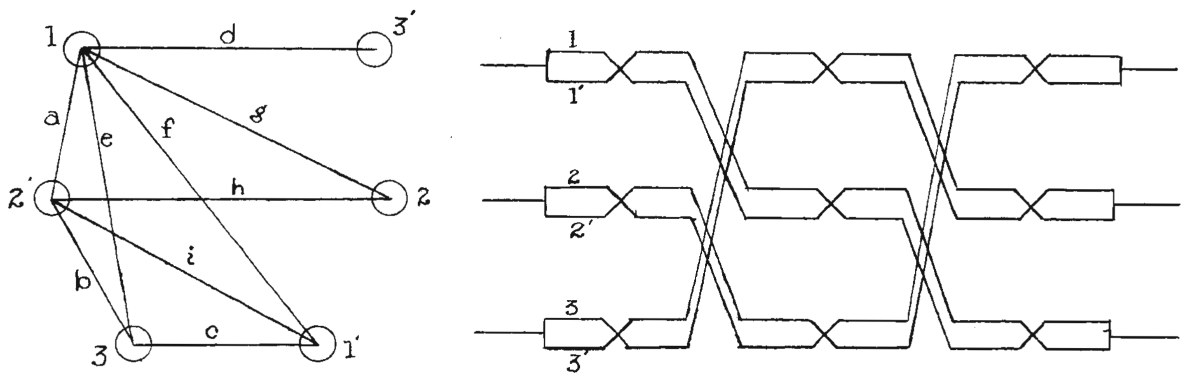

Figure 14. Unsymmetrical Three Phase Double Circuit Line.

Applying the definition for geometric mean distances, it is seen that the mutual geometric mean distance D_m is equal to the 24th root of the product of the above terms, or collecting terms,

$$D_m = \sqrt[24]{a^4\ b^4\ c^2\ d^2\ e^4\ g^4\ i^4}$$

$$= \sqrt[12]{a^2\ b^2\ c\ \ d\ \ e^2\ g^2\ i^2} \quad \text{.................(50)}$$

similarly,

$$D_s = \sqrt[12]{d_s\ f^4\ h^2} = \sqrt[6]{d_s^{\ 3}\ f^2\ h} \text{ ....(51)}$$

Notice that all of the distances are from center to center of the conductors, which applies to equation (24) based on theorem III which is proved in the appendix. From the fundamental inductance formula the inductance of this unsymmetrical line is

$$L = 2 \times 10^{-7} \ln \frac{\text{equation (50)}}{\text{equation (51)}} \text{ henries/meter.....(52)}$$

It can be seen from the above equation that in order to keep the inductance as low as possible it is best to put the two conductors of each phase as far apart as possible, as in this case f and h are both greater than any of the distances in the numerator.

One method of transposition is shown in Figure 14b. Instead of transposing the two wires of each phase in the middle of each third of the line as shown in Figure 14b, the two phase conductors could be reversed at the beginning of each new cycle of transposition but this effect amounts to doubling the length of the cycle. Some of the benefits are lost if the cycle is too long because of variation of current and voltage from section to section. In general, lengths of a cycle range from 12 to 40 miles, depending upon inductance interference due to the exposure of parallel communication circuits.

9. Three-Phase Multi-Circuit Line Flat Spaced. It is common practice to place the conductors in an unsymmetrical arrangement, because of convenience, low cost, and other reasons previously mentioned. A common arrangement used is that of placing the conductors in horizontal planes, and is sometimes called "flat spacing".

Consider the 18-conductor, 6-circuit, 3-phase line shown in Figure 15a.

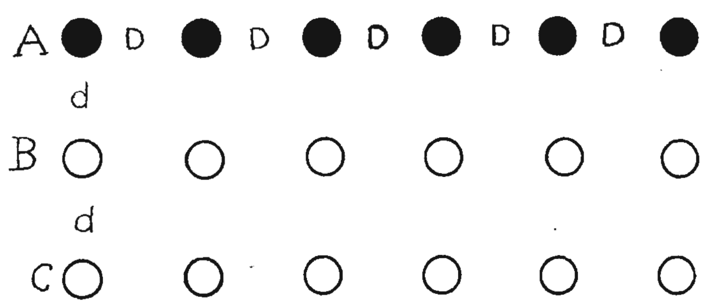

Figure 15a. Six-Circuit, Three-Phase Line
(Phase 1 shown in solid)

Obviously, it will be necessary to transpose the line but in this case it will be assumed that transposition has been taken care of. There are several different possibilities which could be used in placing the six conductors of the three phases and still not change the arrangement of the circuit as a whole. First, let the arrangement be that all the conductors in each horizontal plane be one phase. In this case the self geometric mean distance D_s of the entire phase is composed of the following terms found by taking the distance from each conductor to the other five and multiplying that by the self geometric mean distance d_s of each individual conductor.

Conductor 1 ... d_s D 2D 3D 4D 5D .... 6 distances

Conductor 2 ... d_s D^2 2D 3D 4D .... 6 distances

Conductor 3 ... d_s D^2 $(2d)^2$ 3D .... 6 distances

Conductor 4 ... d_s D^2 $(2d)^2$ 3D .... 6 distances

Conductor 5 ... d_s D^2 2D 3D 4D ....6 distances

Conductor 6 ... d_s d 2D 3D 4D 5D .... 6 distances

D_s then would be the 36th root of the 36 above distances, which when simplified is

$$D_s = \sqrt[36]{d_s{}^6\ D^{30}\ 1^{10}\ 2^8\ 3^6\ 4^4\ 5^2}$$

$$= \sqrt[18]{d_s{}^3\ D^{15}\ 1^5\ 2^4\ 3^3\ 4^2\ 5^1}$$

or

$$D_s = 1.790\ \sqrt[6]{d_s\ D^5} \qquad \text{.........................(53)}$$

In calculating the mutual geometric mean distance D_m each of the six individual conductors, making up one phase, must be taken with respect to the 12 conductors making up the other two phases. It is necessary to take into account the fact that due to transposition each phase will occupy in succession all three positions. (See Figure 15a)

The mutual geometric mean distance from one phase to the other two phases for the transposed line will contain the following terms:

First Phase in the A Position

Conductor 1 with respect to the 12 conductors of the other phases:

$$d \quad 2d \quad (d^2+D^2)^{\frac{1}{2}} \quad (d^2 + 4D^2)^{\frac{1}{2}} \quad (4d^2 + D^2)^{\frac{1}{2}} \quad (4d^2 + 4D^2)^{\frac{1}{2}}$$
$$(d^2 + 9D^2)^{\frac{1}{2}} \quad (4d^2 + 9D^2)^{\frac{1}{2}} \quad (d^2 + 16D^2)^{\frac{1}{2}} \quad (4d^2 + 16D^2)^{\frac{1}{2}}$$
$$(d^2 + 25D^2)^{\frac{1}{2}} \quad (4d^2 + 25D^2)^{\frac{1}{2}}$$ ....12 distances

Conductor 2:

$$(d^2 + D^2)^{\frac{2}{2}} \quad (4d^2 + D^2)^{\frac{2}{2}} \quad d \quad 2d \quad (d^2 + 4D^2)^{\frac{1}{2}} \quad (4d^2 + 4D^2)^{\frac{1}{2}}$$
$$(d^2 + 9D^2)^{\frac{1}{2}} \quad (4d^2 + 9D^2)^{\frac{1}{2}} \quad (d^2 + 16D^2)^{\frac{1}{2}} \quad (4d^2 + 16D^2)^{\frac{1}{2}}$$
.............12 distances

Conductor 3:

$$d \quad 2d \quad (d^2 + D^2)^{\frac{2}{2}} \quad (4d^2 + D^2)^{\frac{2}{2}} \quad (d^2 + 4D^2)^{\frac{2}{2}} \quad (4d^2 + 4D^2)^{\frac{2}{2}}$$
$$(d^2 + 9D^2)^{\frac{1}{2}} \quad (4d^2 + 9D^2)^{\frac{1}{2}}$$ ....12 distances

Conductor 4:

The same terms as conductor 3 ....12 distances

Conductor 5:

The same terms as conductor 2 ....12 distances

Conductor 6:

The same terms as conductor 1 ....12 distances

First Phase in the B Position

Conductor 1 with respect to the other 12:

$$d^2 \ (d^2 + D^2)^{\frac{2}{2}} \ (d^2 + 4D^2)^{\frac{2}{2}} \ (d^2 + 9D^2)^{\frac{2}{2}} \ (d^2 + 16D^2)^{\frac{2}{2}} \ (d^2 + 25D^2)^{\frac{2}{2}}$$ ....12 distances

Conductor 2:

$$d^2 \ (d^2 + D^2)^{\frac{4}{2}} \ (d^2 + 9D^2)^{\frac{2}{2}} \ (d^2 + 16D^2)^{\frac{2}{2}} (d^2 + 4D^2)^{\frac{2}{2}}$$ ....12 distances

Conductor 3:

$$d^2 \ (d^2 + D^2)^{\frac{4}{2}} \ (d^2 + 4D^2)^{\frac{4}{2}} \ (d^2 + 9D^2)^{\frac{2}{2}}$$ ....12 distances

Conductor 4:

The same terms as conductor 3 above......12 distances

Conductor 5:

The same terms as Conductor 2 above....12 distances

Conductor 6:

The same terms as Conductor 1 above....12 distances

The first phase transposed in the C position will give the same terms as the first phase in the A position.

There will be a total of 3 x 72 or 216 distances measured from center to center. By definition, the mutual geometric mean distance D_m will be the 216th root of the above 216 terms. Therefore,

$$D_m = \sqrt[216]{\left\{d^{24} \ (d^2+D^2)^{\frac{40}{2}} \ (d^2+4D^2)^{\frac{32}{2}} \ (d^2+9D^2)^{\frac{24}{2}} \ (d^2+16D^2)^{\frac{16}{2}} \ (d^2+25D^2)^{\frac{8}{2}} \ (2d)^{12} \ (4d^2+D^2)^{\frac{20}{2}} \ (4d^2+4D^2)^{\frac{16}{2}} \ (4d^2+9D^2)^{\frac{12}{2}} \ (4d^2+16D^2)^{\frac{8}{2}} \ (4d^2+25D^2)^{\frac{4}{2}}\right\}}$$

and simplifying,

$$D_m = \sqrt[108]{\left\{d^{12} \ (d^2+D^2)^{10} \ (d^2+4D^2)^{8} \ (d^2+9D^2)^{6} \ (d^2+16D^2)^{4} \ (d^2+25D^2)^{2} \ (2d)^{6} \ (4d^2+D^2)^{5} \ (4d^2+4D^2)^{4} \ (4d^2+9D^2)^{3} \ (4d^2+16D^2)^{2} \ (4d^2+25D^2)^{1}\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(54)$$

The inductance in this case is

$$L = 2 \times 10^{-7} \ln_e \frac{\text{equation (54)}}{\text{equation (53)}} \text{ henries per meter...(55)}$$

Another possible arrangement of the phases is to have the first and fourth vertical columns of conductors comprise one phase; the second and fifth another phase and the third and sixth columns the remaining phase. (See Figure 15b.)

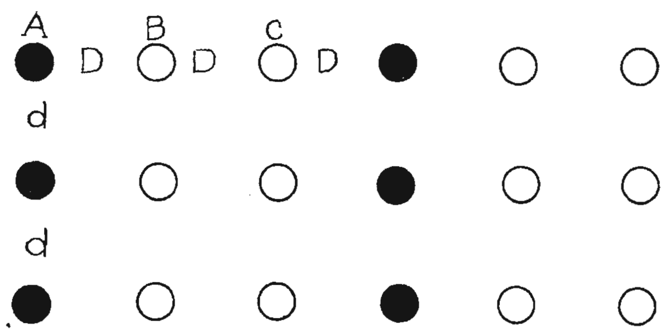

Figure 15b. Six-Circuit, Three-Phase Line
(Phase 1 shown in solid)

Obviously, transposition is again necessary, and taking this into account the self geometric mean distance of the entire phase is made up of the following terms, found by taking the distance from each conductor of a phase to the other five making up a complete phase and multiplying that by the self geometric mean distance d_s of each individual conductor.

Conductor 1....d_s d 2d 3D $(d^2+9D^2)^{\frac{1}{2}}$ $(4d^2+9D^2)^{\frac{1}{2}}$...6 distances

Conductor 2....d_s d^2 $(d^2+9D^2)^{\frac{1}{2}}$ 3D...............6 distances

Conductor 3....same distances as 1 above.......6 distances

Conductor 4....same distances as 1 above.......6 distances

Conductor 5....same distances as 2 above.......6 distances

Conductor 6....same distances as 1 above.......6 distances

D_s is the 36th root of the product of the 36 above distances, or,

$$D_s = \sqrt[36]{d_s^6 \quad d^8 \quad (2d)^4 \quad (3D)^6 \quad (d^2 + 9D^2)^{\frac{8}{2}} \quad (4d^2 + 9D^2)^{\frac{4}{2}}}$$

which when simplified is

$$D_s = \sqrt[18]{d_s^3 \quad 4d^6 \quad 27D^3 \quad (d^2 + 9D^2)^2 \quad (4d^2 + 9D^2)^1} \text{......(56)}$$

The value of D_s is the same for each transposition section assuming that the phases are interchanged at one-third and two-thirds the distance along a transposition section. The effective value of D_s for the entire line will be the same as the value of D_s for each transposition cycle.

The value of mutual geometric mean distance D_m from one phase to the other two will be different in the different transposition sections. To obtain the effective value of D_m the phases must be taken in the different transposition sections. The mutual geometric mean distance from one phase to the other two phases for the transposed line will contain the following terms:

First Phase in the A Position

Conductor 1 with respect to the other 12 conductors:

$$D \;\; 2D \;\; 4D \;\; 5D \;\; (d^2 + D^2)^{\frac{1}{2}} \;\; (d^2 + 4D^2)^{\frac{1}{2}} \;\; (d^2 + 16D^2)^{\frac{1}{2}}$$
$$(d^2 + 25D^2)^{\frac{1}{2}} \;\; (4d^2 + D^2)^{\frac{1}{2}} \;\; (4d^2 + 4D^2)^{\frac{1}{2}} \;\; (4d^2 + 16D^2)^{\frac{1}{2}}$$
$$(4d^2 + 25D^2)^{\frac{1}{2}} \text{.............................12 distances}$$

Conductor 2:

$D\ 2D\ 4D\ 5D\ (d^2 + D^2)^{\frac{2}{2}}\ (d^2 + 4D^2)^{\frac{2}{2}}\ (d^2 + 16D^2)^{\frac{2}{2}}$
$(d^2 + 25D^2)^{\frac{2}{2}}$..............................12 distances

Conductor 3:

The same terms as Conductor 1 above......12 distances

Conductor 4:

$D^2\ (2D)^2\ (d^2 + D^2)^{\frac{2}{2}}\ (d^2 + 4D^2)^{\frac{2}{2}}\ (4d^2 + D^2)^{\frac{2}{2}}$
$(4d^2 + 4D^2)^{\frac{2}{2}}$..............................12 distances

Conductor 5:

$D^2\ (2D)^2\ (d^2 + D^2)^{\frac{4}{2}}\ (d^2 + 4D^2)^{\frac{4}{2}}$ ....12 distances

Conductor 6:

The same terms as Conductor 4 above......12 distances

First Phase in the B Position

Conductor 1:

$D^2\ 2D\ 4D\ (d^2 + D^2)^{\frac{2}{2}}\ (d^2 + 4D^2)^{\frac{1}{2}}\ (d^2 + 16D^2)^{\frac{1}{2}}\ (4d^2 + D^2)^{\frac{2}{2}}$
$(4d^2 + 4D^2)^{\frac{1}{2}}\ (4d^2 + 16D^2)^{\frac{1}{2}}$.................12 distances

Conductor 2:

$D^2\ 2D\ 4D\ (d^2 + D^2)^{\frac{4}{2}}\ (d^2 + 4D^2)^{\frac{2}{2}}\ (d^2 + 16D^2)^{\frac{2}{2}}$.. 12 distances

Conductor 3:

The same terms as Conductor 1 above......12 distances

Conductor 4:

The same terms as Conductor 1 above......12 distances

Conductor 5:

The same terms as Conductor 2 above......12 distances

Conductor 6:

The same terms as Conductor 1 above......12 distances

First phase in the C position will give the same terms as the first phase in the A position but in a different order. The mutual geometric mean distance D_m will be the 216th root of the product of the 216 distances, or

$$D_m = \sqrt[216]{\left\{D^{30}\ (2D)^{24}\ (4D)^{12}\ (5D)^{6}\ (d^2+D^2)^{\frac{40}{2}}\ (d^2+4D^2)^{\frac{32}{2}}\ (d^2+16D^2)^{\frac{16}{2}}\ (d^2+25D^2)^{\frac{8}{2}}\ (4d^2+D^2)^{\frac{20}{2}}\ (4d^2+4D^2)^{\frac{16}{2}}\ (4d^2+16D^2)^{\frac{8}{2}}\ (4d^2+25D^2)^{\frac{4}{2}}\right\}}$$

and simplifying,

$$D_m = \sqrt[108]{\left\{D^{15}\ (2D)^{12}\ (4D)^{6}\ (5D)^{3}\ (d^2+D^2)^{10}\ (d^2+4D^2)^{8}\ (d^2+16D^2)^{4}\ (d^2+25D^2)^{2}\ (4d^2+D^2)^{5}\ (4d^2+4D^2)^{4}\ (4d^2+16D^2)^{2}\ (4d^2+25D^2)\right\}} \quad \ldots\ldots\ldots\ldots(57)$$

The inductance for this arrangement can be calculated from the following equation:

$$L = 2 \times 10^{-7} \ln_e \frac{\text{equation (57)}}{\text{equation (56)}} \quad \ldots\ldots\ldots\ldots(58)$$

Another possible arrangement of the phases is to have the conductor of each phase always separated from one another by diagonal distance; thus the first phase might comprise the top conductor of the first and fourth columns, the middle conductors of the second and fifth columns, and the bottom conductors of the third and sixth columns. The other two phases will be arranged similarly. (See Figure 15c.)

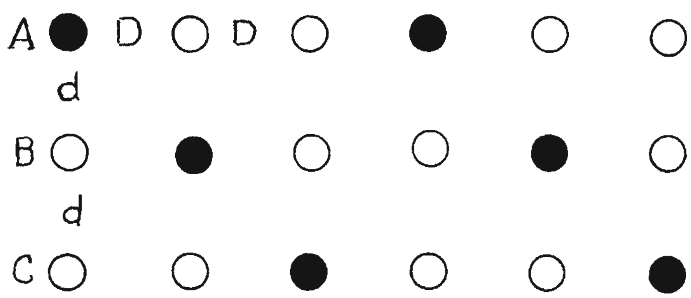

Figure 15c. Six-Circuit, Three-Phase Line
(Phase 1 shown in solid)

With this arrangement, transposition is again necessary, and with this taken care of the equation for the self geometric mean distance of each phase contains the following terms:

Conductor 1:

$$d_s\ (d^2+D^2)^{\frac{1}{2}}\ (4d^2+4D^2)^{\frac{1}{2}}\ 3D\ (d^2+16D^2)^{\frac{1}{2}}$$
$$(4d^2+25D^2)^{\frac{1}{2}}$$ ....6 distances

Conductor 2:

$$d_s\ (d^2+D^2)^{\frac{2}{2}}\ (d^2+4D^2)^{\frac{1}{2}}\ 3D\ (d^2+16D^2)^{\frac{1}{2}}$$ ...6 distances

Conductor 3:

$$d_s\ (d^2+D^2)^{\frac{1}{2}}\ (4d^2+4D^2)^{\frac{1}{2}}\ (4d^2+D^2)^{\frac{1}{2}}\ (d^2+4D^2)^{\frac{1}{2}}$$
$$(3D)$$ ....6 distances

Conductor 4:

$$d_s\ (3D)\ (d^2+4D^2)^{\frac{1}{2}}\ (4d^2+D^2)^{\frac{1}{2}}\ (d^2+D^2)^{\frac{1}{2}}$$
$$(4d^2+4D^2)^{\frac{1}{2}} \text{.............................6 distances}$$

Conductor 5:

The same terms as Conductor 2 above......6 distances

Conductor 6:

The same terms as Conductor 1 above......6 distances

D_s is the 36th root of the 36 above distances, or

$$D_s = \sqrt[36]{\left\{d_s^6\ \ (3D)^6\,(d^2+D^2)^{\frac{8}{2}}\ (d^2+16D^2)^{\frac{4}{2}}\ (4d^2+4D^2)^{\frac{4}{2}}\ (4d^2+25D^2)^{\frac{2}{2}}\ (d^2+4D^2)^{\frac{4}{2}}\ 4(d^2+D^2)^{\frac{2}{2}}\right\}}$$

and simplifying,

$$D_s = \sqrt[18]{\left\{d_s^3\ \ 27D^3\ (d^2+D^2)^2\ (d^2+4D^2)^1\ (d^2+16D^2)^1\ (4d^2+D^2)^{\frac{1}{2}}\ (4d^2+4D^2)^1\ (4d^2+25D^2)^{\frac{1}{2}}\right\}} \text{........(59)}$$

The mutual geometric mean distance D_m taking transposition into account is found by taking the 216th root of the following 216 terms:

First Phase as Shown in Figure 15c.

Conductor 1 with respect to the other 12 conductors:

$$d\ 2d\ D\ (4d^2+D^2)^{\frac{1}{2}}\ 2D\ (d^2+4D^2)^{\frac{1}{2}}\ (d^2+9D^2)^{\frac{1}{2}}\ (4d^2+9D^2)^{\frac{1}{2}}$$
$$4D\ (4d^2+16D^2)^{\frac{1}{2}}\ (5D)\ (d^2+25D^2)^{\frac{1}{2}} \text{ ....12 distances}$$

Conductor 2 with respect to the other 12 conductors:

$D^2 \quad (d^2+D^2)^{\frac{2}{2}} \quad d^2 \quad 2D \quad (d^2+4D^2)^{\frac{1}{2}} \quad (d^2+9D^2)^{\frac{2}{2}} \quad (d^2+16D^2)^{\frac{1}{2}}$

$4D$ ....12 distances

Conductor 3:

$(d^2+4D^2)^{\frac{1}{2}} \quad (2D)^2 \quad D^2 \quad (4d^2+D^2)^{\frac{1}{2}} \quad d \quad 2d \quad (d^2+D^2)^{\frac{1}{2}}$

$(4d^2+4D^2)^{\frac{1}{2}} \quad (4d^2+9D^2)^{\frac{1}{2}} \quad (d^2+9D^2)^{\frac{1}{2}}$ ......12 distances

Conductor 4:

The same terms as Conductor 3 above......12 distances

Conductor 5:

The same terms as Conductor 2 above......12 distances

Conductor 6:

The same terms as Conductor 1 above......12 distances

Phase One Transposed in Phase Two Position

Conductor 1:

$d^2 \quad D \quad (d^2+D^2)^{\frac{1}{2}} \quad 2D \quad (d^2+4D^2)^{\frac{1}{2}} \quad (d^2+9D^2)^{\frac{2}{2}} \quad 4D$

$(d^2+16D^2)^{\frac{1}{2}} \quad 5D \quad (d^2+25D^2)^{\frac{1}{2}}$ ....12 distances

Conductor 2:

$(4d^2+D^2)^{\frac{1}{2}} \quad D^2 \quad d \quad 2d \quad (d^2+D^2)^{\frac{1}{2}} \quad 2D \quad (4d^2+4D^2)^{\frac{1}{2}}$

$(d^2+9D^2)^{\frac{1}{2}} \quad (4d^2+9D^2)^{\frac{1}{2}} \quad 4D \quad (d^2+16D^2)^{\frac{1}{2}}$.....12 distances

Conductor 3:

$$D^2 (2D)^2 (4d^2 + 4D^2)^{\frac{1}{2}} (d^2 + D^2)^{\frac{1}{2}} \; d \; 2d \; (d^2 + 4D^2)^{\frac{1}{2}}$$
$$(4d^2 + D^2)^{\frac{1}{2}} (d^2 + 9D^2)^{\frac{1}{2}} (4d^2 + 9D^2)^{\frac{1}{2}} \; \ldots\ldots\text{12 distances}$$

Conductor 4:

$$D^2 (2D)^2 \; d^2 \; (d^2 + 9D^2)^{\frac{2}{2}} (d^2 + 4D^2)^{\frac{2}{2}} (d^2 + D^2)^{\frac{2}{2}}$$
$$\ldots\ldots\ldots\text{12 distances}$$

Conductor 5:

The same terms as Conductor 2 above......12 distances

Conductor 6:

$$D \; 2D \; 4D \; 5D \; d \; 2d \; (d^2 + D^2)^{\frac{1}{2}} (4d^2 + 4D^2)^{\frac{1}{2}} (d^2 + 9D^2)^{\frac{1}{2}}$$
$$(4d^2 + 9D^2)^{\frac{1}{2}} (d^2 + 16D^2)^{\frac{1}{2}} (4d^2 + 25D^2)^{\frac{1}{2}} \; \ldots\text{12 distances}$$

Phase 1 transposed in phase 3 position will give the same distance as in the above case with phase 1 in phase 2 position. Therefore, for this arrangement D_m is

$$D_m = \sqrt[216]{\Big\{d^{24} \; D^{30} (2d)^{16}} \; (2D)^{20} (4D)^{12} (5D)^6 (d^2 + D^2)^{\frac{20}{2}}$$
$$(d^2 + 4D^2)^{\frac{14}{2}} (d^2 + 9D^2)^{\frac{24}{2}} (d^2 + 16D^2)^{\frac{10}{2}} (d^2 + 25D^2)^{\frac{4}{2}}$$
$$(4d^2 + D^2)^{\frac{10}{2}} (4d^2 + 4D^2)^{\frac{10}{2}} (4d^2 + 9D^2)^{\frac{12}{2}} (4d^2 + 16D^2)^{\frac{2}{2}}$$
$$(4d^2 + 25D^{2})^{\frac{2}{2}}\Big\}$$

which when simplified is

$$D_m = \sqrt[216]{\Big\{d^{24} \; D^{30} (2d)^{16}} \; (2D)^{20} (4D)^{12} (5D)^6 (d^2 + D^2)^{10} (d^2 + 4D^2)^7$$
$$(d^2 + 9D^2)^{12} (d^2 + 16D^2)^5 (d^2 + 25D^2)^2 (4d^2 + D^2)^5$$
$$(4d^2 + 4D^2)^5 (4d^2 + 9D^2)^6 (4d^2 + 16D^2)^1 (4d^2 + 25D^2)^1\Big\}$$
$$\ldots\ldots\ldots\ldots\ldots(60)$$

The inductance for this arrangement is

$$L = 2 \times 10^{-7} \ln_e \frac{\text{equation (60)}}{\text{equation (59)}} \ldots\ldots\ldots\ldots\ldots\ldots(61)$$

Assuming that d and D are each equal to four feet in Figure 15 and also that the conductors in this arrangement are all the standard 30 x 7 strand, 556,000 cir-mil, A.C.S.R. conductor (for which the self geometric mean distance d_s was calculated in equation (38), chapter II) the inductance will be found for comparison for each of the three phase arrangements shown in Figures 15a, 15b and 15c. The self geometric mean distance d_s of each of the conductors is 0.394 inches.

Arrangement Shown in Figure 15a

From equation (53)

$$\begin{aligned} D_s &= 1.790 \sqrt[6]{d_s\ D^5} \\ &= 1.790\ (48)^{\frac{5}{6}}\ (0.394)^{\frac{1}{6}} \\ &= 1.700 \times 25.1 \times 0.856 = 38.3 \text{ inches} \ldots\ldots\ldots\ldots(53a) \end{aligned}$$

From equation (54), and since d equals D equals 4 feet

$$D_m = 48 \sqrt[108]{(1^{12}\ 2^{10}\ 5^{8}\ 10^{6}\ 17^{4}\ 26^{2}\ 2^{6}\ 5^{5}\ 8^{4}\ 13^{3}\ 20^{2}\ 29^{1})}$$

$$\doteq 48 \times 2.280 = 109.4 \text{ inches} \ldots\ldots\ldots\ldots\ldots\ldots(54a)$$

From equation (55)

$$L = 2 \times 10^{-7} \ln_e \frac{109.4}{38.3} \text{ henries per meter}$$

From equation (23a), chapter I, the inductive reactance drop at 60 cycles is

$$X_{60} = 0.2794 \log_{10} \frac{D_m}{D_s} = 0.2794 \log_{10} \frac{109.4}{38.3}$$

$$= 0.2794 \times 0.455 = 0.1271 \text{ ohm per mile per phase} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(55a)$$

With the spacings and conductors the same as above, the inductance and reactance drop for the arrangement of phases as shown in Figure 15b is calculated as follows:

$$D_s = (48)^{\frac{5}{6}} \times 0.394^{\frac{1}{6}} \sqrt[18]{4 \times 27 \cdot 10^2 \times 13}$$

$$= 25.1 \times 0.856 \times 1.932 = 41.5 \text{ inches} \ldots\ldots\ldots(56a)$$

from equation (56) with d = D = 4 feet. From equation (57),

$$D_m = 48 \times \sqrt[108]{1^{15}\ 2^{12}\ 4^{6}\ 5^{3}\ 2^{10}\ 5^{8}\ 17^{4}\ 26^{2}\ 5^{5}\ 8^{4}\ 20^{2}\ 29^{1}}$$

$$= 48 \times 2.191 = 105.0 \text{ inches} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(57a)$$

The inductive reactance at 60 cycles is

$$X_{60} = 0.2794 \times \log_{10} \frac{D_m}{D_s} = 0.2794 \times \log_{10} \frac{105.0}{41.5}$$

$$= 0.2794 \times 0.403 = 0.1086 \text{ ohms per mile per phase} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(58)$$

The inductance and reactance drop for the phase arrangement as shown in Figure 15c is calculated as follows:

From equation (57)

$$D_s = 48^{\frac{5}{6}}\ 0.394^{\frac{1}{6}}\ \sqrt[18]{27 \times 2^2 \times 5 \times 17 \times \sqrt{5} \times 8 \times \sqrt{29}}$$

$$= 25.1 \times 0.856 \times 2.139 = 45.8 \text{ inches} \ldots\ldots\ldots (59a)$$

The mutual geometric mean distance from equation (60) is

$$D_m = 48\ \sqrt[216]{(1^{54}\ 2^{36}\ 4^{12}\ 5^{6}\ 2^{10}\ 5^{7}\ 10^{12}\ 17^{5}\ 26^{2}\ 5^{5}\ 8^{5}\ 13^{6}\ 20^{1}\ 29^{1})}$$

$$= 48 \times 2.0777 = 99.73 \text{ inches} \ldots\ldots\ldots\ldots\ldots\ldots (60a)$$

The 60 cycle inductance reactance drop in this case is

$$X_{60} = 0.2794 \log_{10} \frac{99.73}{45.8}$$

$$= 0.2794 \times 0.337 = 0.0942 \text{ ohms per mile per phase} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61a)$$

Obviously, the best arrangement of phases is the one shown in Figure 15c; the next best of the three arrangements is the one shown in Figure 15b, since the reactance is approximately 26 and 14 per cent lower in the above cases respectively, than in the first case. This is as was expected, since the distances between conductors of each phase were increased in the arrangement of Figure 15b compared to

Figure 15a; and this is also true for the arrangement of Figure 15c as compared to Figure 15b. The mutual geometric mean distance decreased from Figure 15a to Figure 15c respectively. Therefore, as has been previously stated, the best arrangement is obtained by placing the conductors of one phase as far apart as practicable and the distances between conductors of one phase with respect to another as close as practicable, keeping in mind the expense as compared to the advantages obtained by the above.

SUMMARY

A thorough study has been made in this thesis of the application of geometric mean distances to inductance calculations.

The classic induction formula, based upon the "rationalized" M.K.S. system of units is,

$$L = 2 \times 10^{-7} \ln_e \frac{D_m}{D_s} \text{ henries per meter}$$

where D_m is the mutual geometric mean distance of one phase to the remaining phases and D_s is the self geometric mean distance between conductors of a phase. This formula was derived in Chapter I, starting with the fundamental equation of induced voltage, equation (1), and carrying it on through to its final form. It was derived from the definition of inductance and from a consideration of internal and external flux linkages of any conductor carrying a current. The formula is based upon the assumption of uniform current density (or non-uniformity was taken into account by the self geometric mean distance d_s of the individual conductor as in the case of A.C.S.R.). This formula, in its present form, applies to non-magnetic conductors, and A.C.S.R. and copperweld conductors where aluminum or copper carry practically all of the current.

The method of calculating the self geometric mean distance d_s of standard conductors was shown in Chapter III, and is

based upon some well established theorems[1/].

The application of the basic inductance formula to the calculation of inductance on multicircuit lines was shown in chapter III. Here, the method of calculating the mutual geometric mean distance D_m of one phase with respect to the other phases was developed, along with methods of calculating the self geometric mean distance of the entire phase.

D_m and D_s were calculated for several different arrangements of the conductors making up a phase along with different transposition cycles.

This thesis gives the complete method of calculating the inductance of different types of lines composed of different conductors by means of geometric mean distances.

[1/] See the appendix of this thesis.

APPENDIX

MATHEMATICAL PROOFS OF THREE BASIC GMD THEOREMS

If three terms are in a geometric progression the middle term is said to be the geometric mean of the other two and equal to the square root of their product, for if a, b, and c are in geometric progression then

$$\frac{b}{a} = \frac{c}{b} \quad \text{or} \quad b = \sqrt{ac}\,.$$

The geometric mean may also be considered the exponential of the average of the logarithm, or since

$$b = a^{\frac{1}{2}} c^{\frac{1}{2}}$$

$$\ln b = \frac{1}{2}(\ln a + \ln c)$$

therefore

$$e^{\frac{\ln a + \ln c}{2}} = \sqrt{ac}$$

Which gives the geometric mean of a function as the exponential of the average of the logarithm of the function.

For example, consider a circle of radius r, and a point, P, whose distance from the center, O, is b (as shown in Figure a). By the law of cosines, the distance from point P to the extremity of r radius, d, making an angle θ

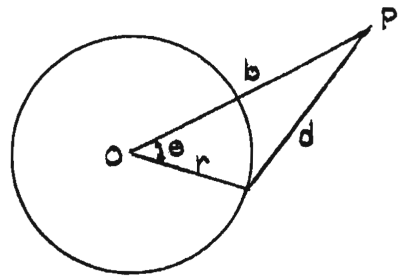

Figure a

with OP is :

$$d = \sqrt{r^2 + b^2 - 2rb\cos\theta}$$

Taking this as a function of the arc length on the circle, and applying the geometric mean, as just defined, over the whole circle, d will become the geometric mean distance, D, from point P to the circumference. Since the average is unchanged by a change in scale, it can be taken with respect to θ, and since it is an even function, the average may be taken from 0 to π. Thus,

$$\ln D = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\ln(r^2 + b^2 - 2rb\cos\theta)\,d\theta.$$

This integral is a continuous function of r for all values, therefore differentiating with respect to r, which will hold except when r equals b, the equation becomes,

$$\frac{d\ln D}{dr} = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\,\frac{2r - 2b\cos\theta}{r^2 + b^2 - 2rb\cos\theta}\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{r - b\cos\theta}{r^2 + b^2 - 2rb\cos\theta}\,d\theta$$

multiplying by $\frac{2r}{2r}$ and adding and subtracting b^2,

$$\frac{r - b\cos\theta}{r^2 + b^2 - 2rb\cos\theta} \times \frac{2r}{2r} = \frac{r^2 + r^2 - 2rb\cos\theta + b^2 - b^2}{2r(r^2 + b^2 - 2rb\cos\theta)}$$

reducing,

$$\frac{r^2 - 2rb\cos\theta + b^2}{2r(r^2 + b^2 - 2rb\cos\theta)} + \frac{r^2 - b^2}{2r(r^2 + b^2 - 2rb\cos\theta}$$

the final equation becomes,

$$\frac{d \ln D}{dr} = \frac{1}{\pi}\int_0^{\pi}\left[\frac{1}{2r} + \frac{1}{2r}\,\frac{r^2 - b^2}{r^2 + b^2 - 2rb\cos\theta}\right] d\theta$$

Rearranging the denominator of the last term so it can be more readily integrated and applying trigometric identity, $\cos^2 x + \sin^2 x = 1$, and $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$, it becomes

$$r^2 + b^2 - 2rb\cos\theta = (r^2+b^2)\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} -$$

$$2rb\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)$$

$$= (r+b)^2 \sin^2\frac{\theta}{2} + (r-b)^2\cos^2\frac{\theta}{2}$$

Now multiplying and dividing by $\cos^2\frac{\theta}{2}$

$$\left[(r+b)^2 \sin^2\frac{\theta}{2} + (r-b)^2\cos^2\frac{\theta}{2}\right]\frac{\cos^2\theta}{\cos^2\theta} =$$

$$\left[(r+b)^2 \tan^2\theta + (r-b)^2\right]\cos^2\theta$$

Since $\sec^2\theta = \frac{1}{\cos^2\theta}$ the equation becomes

$$\frac{r^2 - b^2}{2r}\,\frac{1}{\pi}\int_0^{\pi}\frac{1}{r^2 + b^2 - 2rb\cos\theta}\, d\theta =$$

$$\frac{r^2 - b^2}{2r}\,\frac{1}{\pi}\int_0^{\pi}\frac{\sec^2\theta}{(r+b)^2\tan^2\theta + (r-b)^2}\, d\theta$$

or

$$= \frac{r^2 - b^2}{2r}\,\frac{1}{\pi}\,\frac{1}{2(r+b)^2}\int\frac{2\sec^2\theta}{\tan^2\theta + \frac{(r-b)^2}{(r+b)^2}}\, d\theta$$

Which is now of the form $\int \frac{du}{u^2+a^2} = \frac{1}{a} \text{arc tan} \frac{u}{a}$

therefore,

$$\frac{r^2 - b^2}{2r} \frac{1}{\pi} \int_0^{\pi} \frac{1}{r^2 + b^2 - 2rb\cos\theta} d\theta =$$

$$\frac{r^2 - b^2}{2r} \frac{1}{\pi} \frac{2}{(r+b)(r-b)} \left[\text{arc tan} \left(\frac{\tan\frac{\theta}{2}}{\frac{r-b}{r+b}} \right) \right]_0^{\pi}.$$

Since arc tan $\infty = \frac{\pi}{2}$

$$= \frac{r^2 - b^2}{2r} \frac{1}{\pi} \frac{2}{|r^2 - b^2|} \frac{\pi}{2} = \frac{1}{2r} \frac{r^2 - b^2}{|r^2 - b^2|}$$

Therefore the final equation becomes

$$\frac{d \ln D}{dr} = \frac{1}{\pi} \frac{\theta}{2r} \Big]_0^{\pi} + \frac{1}{2r} \frac{r^2 - b^2}{|r^2 - b^2|}$$

$$= \frac{1}{2r} + \frac{1}{2r} \frac{r^2 - b^2}{|r^2 - b^2|}$$

Since $|r^2 - b^2|$ is an absolute value and will always be positive, therefore when r is greater than b

$$\frac{d \ln D}{dr} = \frac{1}{2r} + \frac{1}{2r} = \frac{1}{r}$$

and when r is less than b

$$\frac{d \ln D}{dr} = \frac{1}{2r} - \frac{1}{2r} = 0.$$

Referring back to the original integral for the ln D it is seen that when r = 0

$$\ln D = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2} \ln b^2 \, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \ln b \, d\theta = \frac{\ln b}{\pi}\,\theta\Big]_0^{\pi}$$

$$= \ln b.$$

Therefore when r = 0, D = b. If r is greater than b, from the above equation

$$\frac{d \ln D}{dr} = \frac{1}{r} \quad \text{or,} \quad d \ln D = \frac{dr}{r}$$

Taking the integral of both sides the equation becomes

$$\int d \ln D = \int \frac{dr}{r}$$

$$\ln D = \ln r$$

therefore D = r when r is greater than b. Now if r is less than b

$$d \ln D = 0$$

$$\therefore \quad \ln D = c,$$

but when r = 0 ln D = ln b which must hold for all values of b, therefore when r is less than b.

$$c = \ln b$$

$$\text{hence} \quad \ln D = \ln b$$

$$\therefore \quad D = b$$

These results are consistent with the fact that the integral is symmetrical in r and b.

Thus, the gmd from a point to a circumference is equal to its distance from the center (Theorem III, Chapter II) since it was proved that when $b > r$, $D = b$.

When $b < r$, $D = r$. This proves Theorem II (Chapter II) which states that the gmd from a circular line of radius r to any point, line, or area in the same plane and wholly enclosed by the circular line is equal to its radius.

THE SELF GMD OF A CIRCULAR AREA

The GMD from an area is defined by

$$\ln D = \frac{1}{A}\frac{1}{A'}\int_A\int_{A'} \ln \delta_{AA'}\, dA\, dA'$$

Given a circle of radius a, to find its self GMD

$$\delta^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$$

(Law of Cosines)

$$\ln\delta = \frac{1}{2}\ln(r_1^2 + r_2^2 - 2r_1r_2\cos\theta)$$

$$A = A' = \pi a^2$$

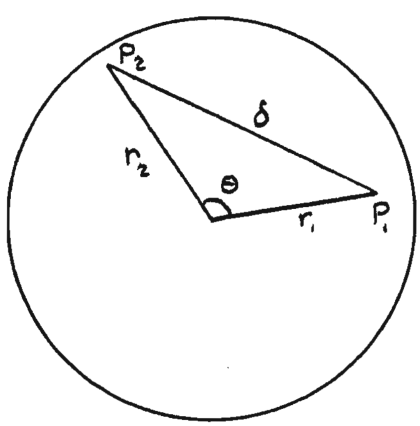

Figure b

Since the distance from P_1 to P_2 equals the distance from P_2 to P_1 over the circle the log of the GMD is:

$$\ln D = \frac{2}{\pi^2 a^4}\int_A\int_{A'} \ln\delta\, dA\, dA'$$

It is necessary to consider the length from P_1

to P_2 as well as the length from P_2 to P_1. Since the integral form includes this distance only once, the total expression must be multiplied by 2.

$$dA = r_1\,dr_1\,d\theta \qquad dA' = r_2\,dr_2\,d\theta$$

therefore

$$\ln D = \frac{2}{\pi^2 a^4}\int_0^a\int_0^{r_2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}\ln(r_1^2+r_2^2-2r_1r_2\cos\theta)\,d\theta\,d\theta\,r_1\,dr_1\,dr_2\,r_2$$

and since the integral is an even function

$$\ln D = \frac{2}{\pi^2 a^4}\int_0^a\int_0^{r_2}\int_0^{2\pi}\int_0^{\pi}\ln(r_1^2+r_2^2-2r_1r_2\cos\theta)\,d\theta\,d\theta\,r_1\,dr_1\,r_2\,dr_2$$

Let $I = \int_0^{\pi}\ln(r_1^2+r_2^2-2r_1r_2\cos\theta)\,d\theta$

$$\frac{dI}{dr_2} = \int_0^{\pi}\frac{2r_2-2r_1\cos\theta}{r_1^2+r_2^2-2r_1r_2\cos\theta}\,d\theta$$

$$= \int_0^{\pi}\frac{-2r_1\cos\theta+2r_2}{-2r_1r_2\cos\theta+r_1^2+r_2^2}\,d\theta$$

adding and subtracting $\frac{r_1^2+r_2^2}{r_2}$

$$\frac{dI}{dr_2} = \int_0^{\pi}\frac{1}{r_2}+\frac{2r_2-\frac{r_1^2+r_2^2}{r_2}}{r_1^2+r_2^2-2r_1r_2\cos\theta}\,d\theta$$

$$\frac{dI}{dr_2} = \frac{1}{r_2}\int_0^{\pi} d\theta + \frac{r_2^2 - r_1^2}{r_2}\int_0^{\pi} \frac{d\theta}{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}$$

$$= \frac{\pi}{r_2} + \frac{r_2^2 - r_1^2}{r_2}\int_0^{\pi} \frac{d\theta}{r_1^2 + r_2^2 - r_1r_2\cos\theta}$$

but $\int \frac{d\theta}{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}$ is of form $\int \frac{d\theta}{a - b\cos\theta}$

$$\therefore \int_0^{\pi} \frac{d\theta}{a - b\cos\theta} = \int_0^{\infty} \frac{\frac{2dz}{1+z^2}}{a - b\frac{1-z^2}{1+z^2}} = \int_0^{\infty} \frac{2dz}{(a-b)+(a+b)z^2}$$

Let $z = \tan\frac{\theta}{2}$

$$= \frac{2}{\sqrt{a+b}} \cdot \frac{1}{\sqrt{a-b}} \text{ arc } \tan\sqrt{\frac{a+b}{a-b}}\,z\,\Big]_0^{\infty}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{a^2-b^2}} = \frac{\pi}{\sqrt{(r_1^2+r_2^2) - 4r_1^2r_2^2}}$$

$$= \frac{\pi}{r_2^2 - r_1^2}$$

therefore

$$\frac{dI}{dr_2} = \frac{\pi}{r_2} + \frac{r_2^2 - r_1^2}{r_2} \cdot \frac{\pi}{r_2^2 - r_1^2} = \frac{2\pi}{r_2}$$

$$I = \int \frac{2\pi}{r_2}\,dr_2 = 2\pi \ln r_2 + C$$

when $r_1 = 0$ $\quad I = 2\pi \ln r_2 + C$

$$I = \int_0^{\pi} \ln r_2^2 \, d\theta = 2\pi \ln r_2$$

therefore

$$C = 0 \qquad I = 2\pi \ln r_2$$

Substituting I in the original equation

$$\ln D = \frac{2}{\pi^2 a^4} \int_0^a \int_0^{r_2} \int_0^{2\pi} 2\pi \ln r_2 \, d\theta \, r_1 \, dr_1 \, r_2 \, dr_2$$

$$= \frac{4}{\pi^2 a^4} \int_0^a \int_0^{r_2} \theta \Big]_0^{2\pi} \ln r_2 \, r_1 \, dr_1 \, r_2 \, dr_2$$

$$= \frac{8}{a^4} \int_0^a \int_0^{r_2} \ln r_2 \, r_1 \, dr_1 \, r_2 \, dr_2$$

$$= \frac{8}{a^4} \int_0^a \frac{r_1^2}{2} \Big]_0^{r_2} \ln r_2 \, r_2 \, dr_2$$

$$= \frac{4}{a^4} \int_0^a r_2^3 \ln r_2 \, dr_2$$

which is of the form $\int x^p (ax) \, dx$

$$\therefore \ln D = \frac{4}{a^4} \left[\frac{r_2^4}{4} \ln r_2 - \frac{r_2^4}{16} \right]_0^a$$

$$\ln D = \frac{4}{a^4}\left[\frac{a^4}{4}\ln a - \frac{a^4}{16}\right]$$

$$= \ln a - \frac{1}{4}$$

$$= \ln a - \ln e^{0.25} = \frac{a}{e^{0.25}}$$

therefore

$$D = ae^{-\frac{1}{4}} \quad \text{(Theorem I, Chapter II)}$$

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