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Multiport synthesis with multi-layered distributed parameter RC networks

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MULTIPORT SYNTHESIS WITH MULTI-LAYERED DISTRIBUTED PARAMETER RC NETWORKS

by

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ABSTRACT

Synthesis and design techniques for multiport, multilayered, distributed parameter RC networks are investigated in this dissertation. The basic building block is a network section of uniform width consisting of alternate layers of resistive and dielectric materials on top of a perfectly conducting plane. The resulting network is composed of a cascade of these sections with all access terminals at one end of the cascaded network. The terminals at the other end are either open circuited or short circuited. A method is discussed for synthesizing a network from a prescribed network admittance matrix. An approximation technique is given for obtaining a realizable network immittance matrix from Bode plots of the short circuit admittance parameters for the four-layered case. Since this approximation technique may be difficult to implement in practical engineering problems, high and low frequency asymptotic relationships are given which are useful in practical design of networks. Design suggestions are given for notch, low pass, and high pass networks.

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LIST OF SYMBOLS

A line under a quantity denotes the quantity as a matrix.

a - A parameter which is a function of the components of the RC product matrix $\underline{T}$

a_j - A constant in the lumped LC rational fraction immittance representation

A_j - A constant in the general distributed realizable admittance matrix

b - A parameter which is a function of the components of the RC product matrix $\underline{T}$

b_j - A constant in the lumped LC rational fraction immittance representation

B_j - A constant in the general distributed admittance matrix

c - Ratio of the dielectric constants of a four-layered network

c_α - Capacitance per unit length in the α^{th} dielectric layer of a distributed network

$C_{\alpha i}$ - Total capacitance due to the α^{th} dielectric layer in the i^{th} section of a cascaded network

d - A parameter which is a function of the components of the RC product matrix $\underline{T}$

F_i - The functional form of a realizable admittance matrix for a four-layered cascaded network

G_{ij} - Open circuit voltage transfer ratio V_j/V_i

h_{ϵ_α} - Thickness of the α^{th} dielectric layer

h_{ρ_α} - Thickness of the α^{th} resistivity layer

H - Admittance level scaling parameter in the hyperbolic form of the general realizable admittance matrix

$\hat{H}$ - Admittance level scaling parameter in the exponential form of the general realizable admittance matrix

I - Electrical current

$\underline{I}$ - Unit matrix

k_i -Constant obtained as a direct result of the synthesis procedure

l - Length of each section of the cascaded network

L - Ratio of the eigenvalues of the RC product matrix for the four-layered network

Lm - Log magnitude

$\underline{M}$ - Matrix used to diagonalize a realizable admittance matrix when evaluated at infinite frequency

$\underline{\underline{M}}$ - $\underline{M} \times \sqrt{\underline{T}}^{-1}$

$\hat{P}_i$ - Functional form of a general i section cascaded distributed network in the $\sqrt{s}$ plane

r - Ratio of the resistivities of a four-layered distributed network

r_α - Ratio of the resistivity of the α^{th} resistive layer to the a^{th} layer of a general multi-layered network

$R_{\alpha i}$ - Total resistance due to the α^{th} resistive layer in the i^{th} section of the cascaded network

s - Laplace transform variable

T - Indicates the transpose of a matrix if used as a superscript

V - Electrical voltage

w - Scalar transformation $\tanh \sqrt{s\underline{T}}$

w_i - Width of the i^{th} section of the cascaded network

$\hat{w}_i$ - Ratio of the width of the i^{th} section to the p^{th} section of the cascaded network

$\underline{W}$ - Matrix transformation $\tanh \sqrt{s\underline{T}}$

y_{ij} - Short circuit admittance parameter

$\underline{Y}_i$ - Admittance of a cascaded distributed network looking into the i^{th} section of the cascade. A lower case y indicates the scalar admittance of the two-layered network.

$\underline{Y}_{i_{LC}}$ - Admittance in the $\sqrt{s}$ plane. The equation form is similar to that of a lumped LC network.

$\hat{\underline{Y}}_i$ - The diagonalized admittance matrix in the $\underline{W}$ plane

$\underline{Z}$ - Impedance matrix

Zo - Constituent idempotent

ϵ_α - Permittivity of the α^{th} dielectric layer

$\hat{\epsilon}_\alpha$ - Capacitance of the α^{th} dielectric layer per unit width of the distributed network

η - Normalized frequency

λ - Eigenvalue of the RC product matrix $\underline{T}$

ρ - Resistivity of the α^{th} dielectric layer

ρ - Resistivity of the α^{th} resistive layer per unit width of the distributed network

τ - Product of resistance and capacitance

$\underline{T}$ - Product of resistance and capacitance matrices

Φ_i - Phase angle of F_i evaluated at $j\omega\lambda_i$

ω - Radian frequency

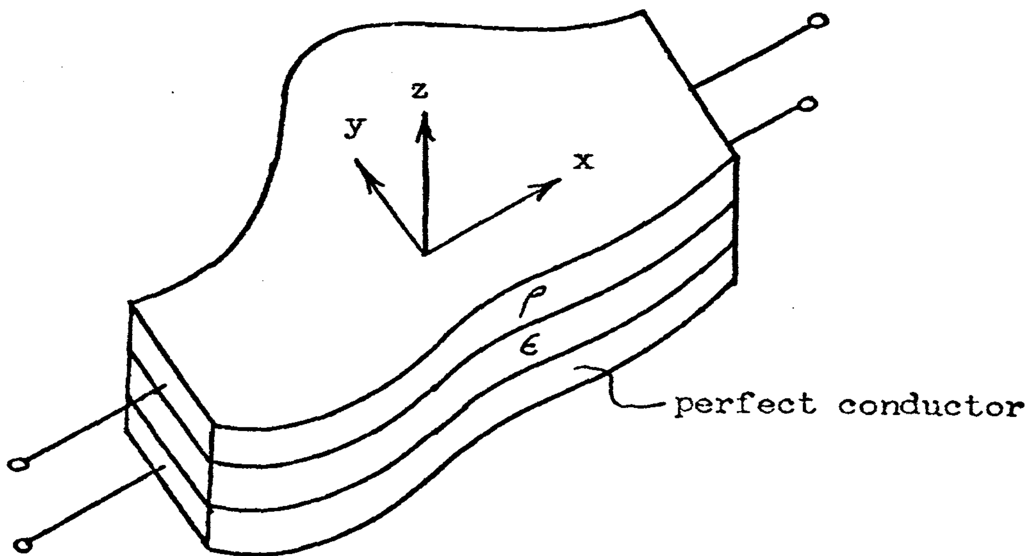
1.0 INTRODUCTION TO DISTRIBUTED $\overline{RC}$ NETWORKS

1.1 Purpose of Distributed $\overline{RC}$ Networks

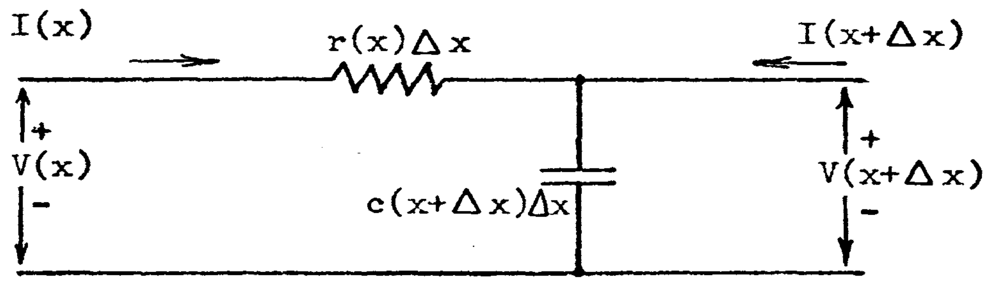
With the advent of microelectronic technology, devices which can be constructed using thin film or monolithic techniques are much in demand. Among these devices, distributed parameter $\overline{RC}$ networks are becoming popular as a means of overcoming the difficulty of constructing microminiature inductors (Ref. 3, 23) because they can be used to construct low pass, high pass, and notch networks. A notch network placed in the feedback loop of an active device can produce a bandpass network (Ref. 31).

1.2 Definition of Distributed Parameter $\overline{RC}$ Network

A distributed parameter $\overline{RC}$ network is an electrical network composed of alternate layers of resistive and dielectric films. Figure 1.1 shows an example of a distributed $\overline{RC}$ network. In this dissertation it will be assumed that these layers are deposited on a perfectly conducting film or ground plane and that the dielectric layers are non-conductive. Further it will be assumed that each layer is electrically homogeneous and of uniform thickness. Thus the resistivity ρ and permittivity ϵ of Figure 1-1 are not functions of the spacial coordinates, and, since the width of the network varies, the resistance and capacitance per unit length are functions of x only. A convention used in this dissertation is to call a circuit an n-layered device if the number of resistive layers plus the number of capacitive layers equals n. The perfectly conducting layer is not counted. By this convention the device of Figure 1.1 has two layers. If in discribing a network a bar appears



(a).



(b).

Figure 1-1. (a). Two-layered distributed parameter $\overline{RC}$ network.
(b). An incremental model.

above the letters R and C, specifically $\overline{RC}$, a distributed parameter network is to be understood.

1.3 Review of Previous Work

The analysis of distributed $\overline{RC}$ networks is based on transmission line theory. The network differential equations in the s domain, sometimes called the telegrapher's equations, can be written as the matrix equation (Ref. 1)

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = -\begin{bmatrix} 0 & r(x) \\ sc(x) & 0 \end{bmatrix}\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} \qquad (1.1)$$

in which r and c are respectively the resistance and capacitance per unit length of the network. Elimination of either V or I from these equations results in

$$V''(x,s) - \frac{r'(x)}{r(x)} V'(x,s) - r(x)\ c(x)\ sV(x,s) = 0 \qquad (1.2a)$$

$$I''(x,s) - \frac{c'(x)}{c(x)} I'(x,s) - r(x)\ c(x)\ sI(x,s) = 0 \qquad (1.2b)$$

where the prime denotes differentiation with respect to x. In early analytic work, geometries were chosen such that equations (1.2) could be solved in closed form. Some of the tapers studied for the two layered network were the uniform (Ref. 14), exponential (Ref. 31,14), trigonometric (Ref. 29), and linear (Ref. 14) tapers. A solution to the completely general taper was obtained in the form of an infinite series of multiple integrals by Protonotarios and Wing (Ref. 23). By solving equation (1.1) as a matrix differential equation, Bertnolli (Ref. 1) showed this solution to be the matrizant. Googe and Su (Ref. 9) investigated the four-layered case of the uniform distributed RC network, paying special attention to the two-port obtained by connecting leads between the two ends of the top resistive layer and the conductive plane. Bertnolli

(Ref. 1) pointed out that the multilayered case may be obtained from the two layered case by replacing all scalar electric quantities by corresponding matrix quantities.

Some design work has been done by referring to a catalog of known immittance and transfer functions of previously studied network morphologies (Ref. 11, 13). Heizer (Ref. 16) and Hesselberth (Ref. 17) succeeded in realizing a finite number of poles along the negative real axis of the s-plane by assuming special nonhomogenous distributions of either the permittivity or the resistivity of the network layers. The positive real transformation $W = \tanh\sqrt{rcs}$ enabled Wyndrum (Ref. 32) to develop a synthesis procedure using two-layered uniform sections of varying widths as the basic network building blocks. Network functions of distributed networks in the W-plane resemble those of discrete component L-C networks in the s-plane and can be realized exactly by the use of Wyndrum's procedure. The W-plane function is obtained by fitting log magnitude plots of certain cataloged factors to a given admittance curve. O'Shea (Ref. 22) used the transformation $p = \cosh\sqrt{rcs}$ and realized transmission zeros in the p-plane. By using gyrators and transformers, Newcomb (Ref. 20,21) described an n-port synthesis procedure for lossless transmission lines.

1.4 Scope of this Dissertation

The motivation of this work stems from the lack of distributed multiport synthesis procedures in the literature. Wyndrum's work is extended to include multilayered distributed uniform networks. Chapter Two describes an exact n-port synthesis procedure provided the network function to be synthesized is in a form that is realizable. A method of calculating some of the network parameters from the low and high asymptotic frequency response of short circuit admittance parameters and open circuit voltage transfer functions is given in Chapter Three. Chapter Four shows how a realizable function may be derived from the short circuit admittance parameters by curve fitting in a transformation plane. Finally in Chapter Five a design procedure is discussed which uses digital computation to optimize selected performance characteristics of four-layered two-port networks. Several interesting sample circuits are given along with their log magnitude and phase plots.

2.0 MULTIPORT SYNTHESIS

2.1 Introduction

In his Ph.D. dissertation (Ref. 32) Ralph W. Wyndrum described a one port synthesis procedure and an approximation procedure using two-layered distributed parameter $\overline{RC}$ networks of uniform width as elements of a cascaded network. In this chapter Wyndrum's synthesis procedure will be extended to multiport synthesis in which multi-layered $\overline{RC}$ network sections similar to the four-layered network of Figure 2-1 are elements. First the uniform multilayered $\overline{RC}$ network is analized, then the synthesis procedure is developed, and finally examples are given which demonstrate the procedure.

2.2 Analysis of the Multi-Layered Distributed $\overline{RC}$ Network

To begin the analysis of the uniform $\overline{RC}$ network consider the incremental model of an n-layered circuit shown in Figure 2-2 in which the quantities r_a, c_a, etc. are the resistances and capacitances per unit length of the uniform section. In equation (2.1) the voltage and current relations are written for this model.

$$
\begin{aligned}
V_a(x) &= r_a \Delta x I_a(x) + V_a(x+\Delta x) \\
V_b(x) &= r_b \Delta x I_b(x) + V_b(x+\Delta x) \\
&\cdots\cdots\cdots\cdots\cdots \\
V_n(x) &= r_n \Delta x I_n(x) + V_n(x+\Delta x) \\
I_a(x) &= sc_a \Delta x V_a(x+\Delta x) - sc_b \Delta x V_b(x+\Delta x) + I_a(x+\Delta x) \qquad (2.1) \\
I_b(x) &= -sc_a \Delta x V_a(x+\Delta x) + s(c_a+c_b)\Delta x V_b(x+\Delta x) \\
&\quad -sc_b \Delta x V_c(x+\Delta x) + I_b(x+\Delta x) \\
&\cdots\cdots\cdots\cdots\cdots \\
I_n(x) &= -sc_{n-1} \Delta x V_{n-1}(x+\Delta x) + s(c_{n-1}+c_n) \Delta x V_n(x+\Delta x) + I_n(x+\Delta x)
\end{aligned}
$$

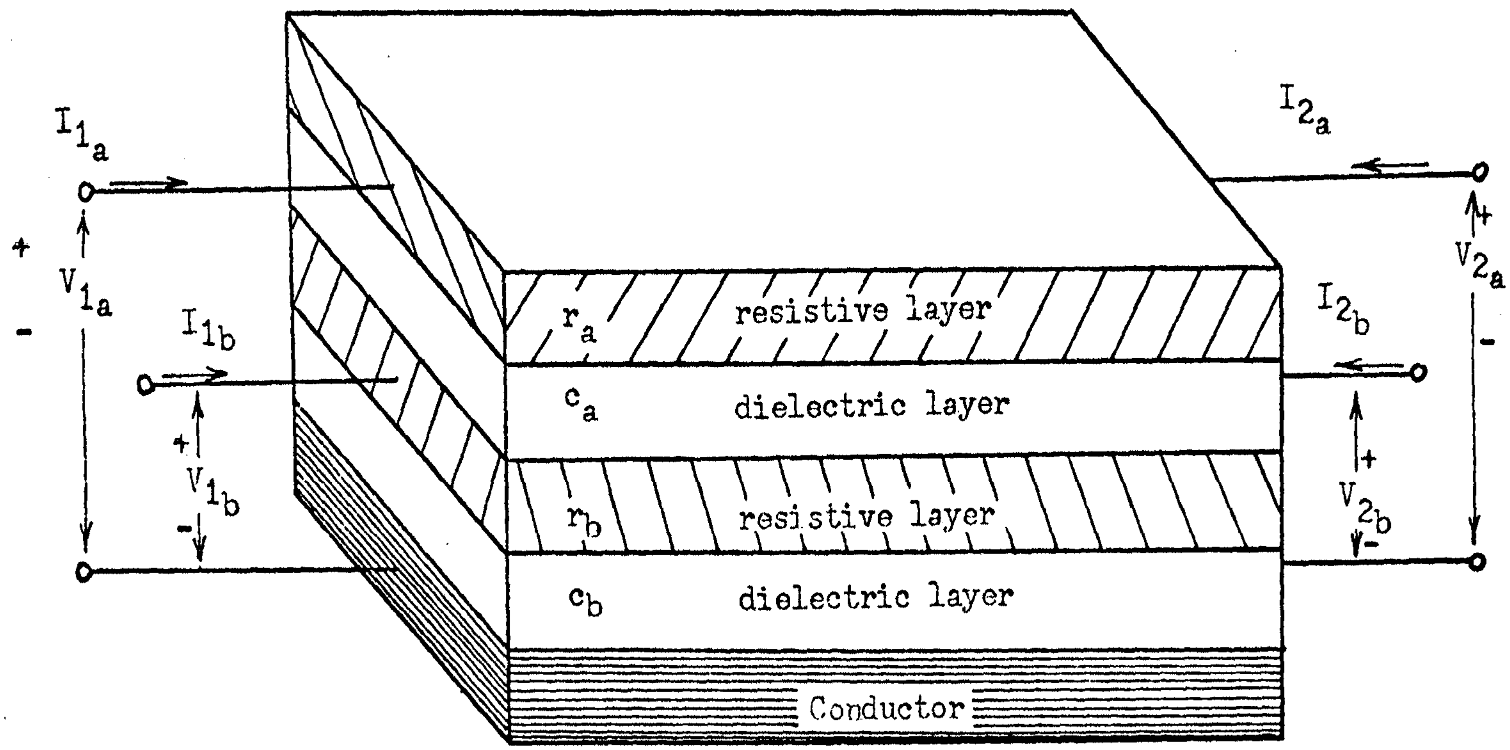


Figure 2-1. Four-layered distributed $\overline{RC}$ network.

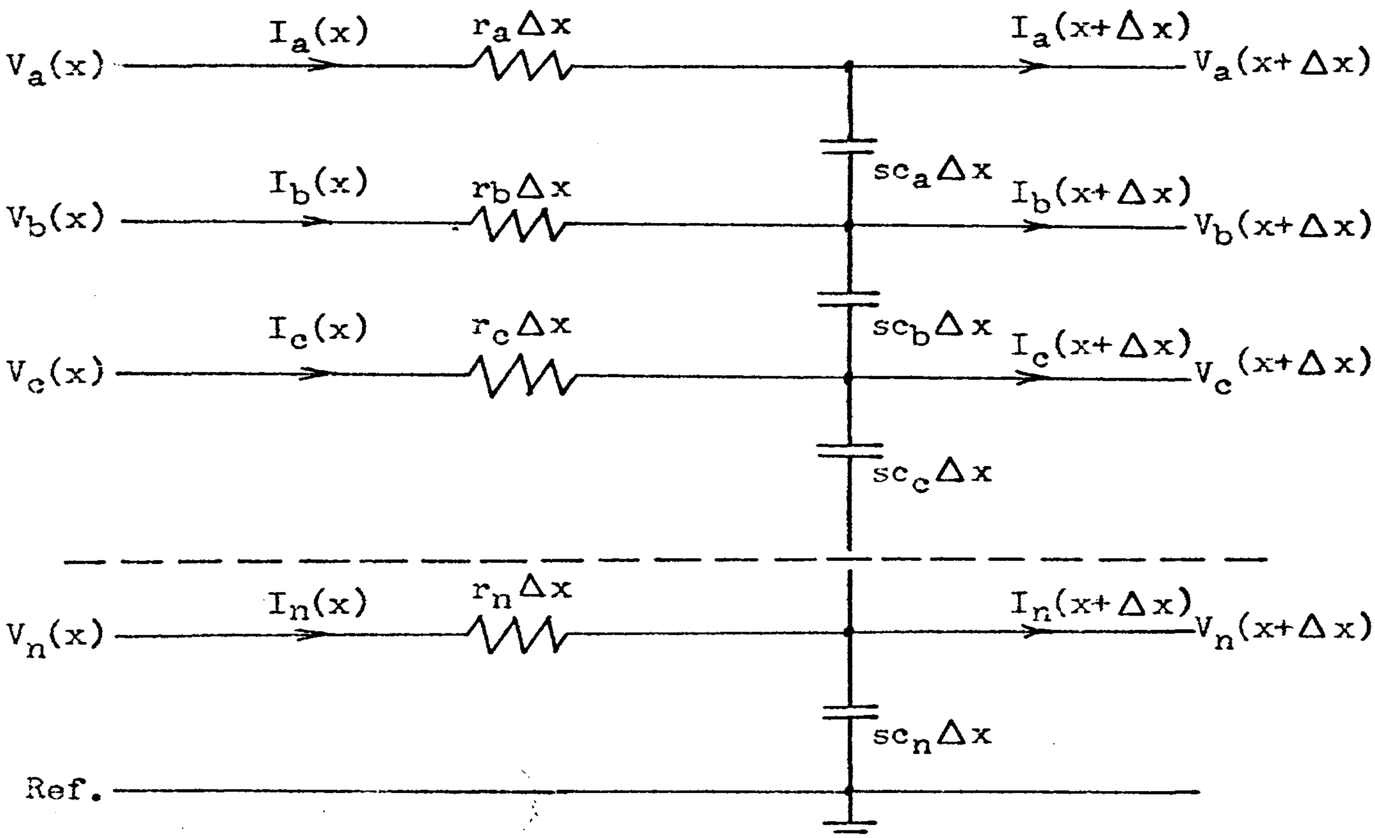


Figure 2-2. Equivalent circuit of an incremental length (Δx) of a 2n-layered distributed RC network.

If l is the total length of the section and R_a, C_a, etc. are the total resistance and capacitance of the respective layers of the section, then $r_a = \frac{R_a}{l}$, $c_a = \frac{C_a}{l}$, etc. When Δx approaches zero in equation (2.1), the differential equations of the circuit result.

$$\frac{\partial V_a(x)}{\partial x} = -\frac{R_a}{l} I_a(x)$$

$$\frac{\partial V_b(x)}{\partial x} = -\frac{R_b}{l} I_b(x)$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\partial V_n(x)}{\partial x} = -\frac{R_n}{l} I_n(x)$$

$$\frac{\partial I_a(x)}{\partial x} = -s\frac{C_a}{l} V_a(x) + x\frac{C_a}{l} V_b(x) \qquad (2.2)$$

$$\frac{\partial I_b(x)}{\partial x} = s\frac{C_a}{l} V_a(x) - s\left(\frac{C_a}{l} + \frac{C_b}{l}\right) V_b(x) + s\frac{C_b}{l} V_c(x)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial I_n(x)}{\partial x} = s\frac{C_{n-1}}{l} V_{n-1}(x) - s\left(\frac{C_{n-1}}{l} + \frac{C_n}{l}\right) V_n(x)$$

These equations can be written in the matrix form

$$\frac{\partial}{\partial x}\begin{bmatrix}\underline{V}\\ \underline{I}\end{bmatrix} = -\frac{1}{l}\begin{bmatrix}\underline{0} & \underline{R}\\ \underline{C}s & \underline{0}\end{bmatrix}\begin{bmatrix}\underline{V}\\ \underline{I}\end{bmatrix} \qquad (2.3)$$

where $\underline{V} = \begin{bmatrix}V_a\\ V_b\\ \cdot\\ \cdot\\ V_n\end{bmatrix}$, $\underline{I} = \begin{bmatrix}I_a\\ I_b\\ \cdot\\ \cdot\\ I_n\end{bmatrix}$,

$$\underline{R} = \begin{bmatrix} R_a & 0 & \cdots & 0 \\ 0 & R_b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix}, \tag{2.4}$$

and

$$\underline{C} = \begin{bmatrix} C_a & -C_a & 0 & 0 & \cdots & 0 \\ -C_a & C_a+C_b & -C_b & 0 & \cdots & 0 \\ 0 & -C_b & C_b+C_c & -C_c & \cdots & 0 \\ 0 & 0 & -C_c & C_c+C_d & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_{n-1}+C_n \end{bmatrix}. \tag{2.5}$$

Note that R and C are symmetric. If the reference directions for voltage and current of Figure 2-1 are used, the solution of equation (2.3) as given by Bertnolli (Ref. 1) in terms of terminal voltages and currents is

$$\begin{bmatrix} \underline{V}_1 \\ \underline{I}_1 \end{bmatrix} = \begin{bmatrix} \cosh\sqrt{s\underline{RC}} & (s\underline{C})^{-1}\sqrt{s\underline{CR}}\,\sinh\sqrt{s\underline{CR}} \\ \underline{R}^{-1}\sqrt{s\underline{RC}}\,\sinh\sqrt{s\underline{RC}} & \cosh\sqrt{s\underline{CR}} \end{bmatrix} \begin{bmatrix} \underline{V}_2 \\ -\underline{I}_2 \end{bmatrix} \tag{2.6}$$

where $\underline{V}_i = \begin{bmatrix} V_{ia} \\ V_{ib} \\ \vdots \\ V_{in} \end{bmatrix}$, $\underline{I}_i = \begin{bmatrix} I_{ia} \\ I_{ib} \\ \vdots \\ I_{in} \end{bmatrix}$, $i = 1, 2$, and the square root and hyperbolic functions of the matrices are as defined in Appendix A. It is shown in Appendix B that equation (2.6) may be written in either the form

$$\underline{V} = \underline{Z}\,\underline{I} \tag{2.7}$$

or

$$\underline{I} = \underline{Y}\,\underline{V} \tag{2.8}$$

where $\underline{V} = \begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \end{bmatrix}$, $\underline{I} = \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$, and $\underline{Z}$ and $\underline{Y}$ are the 2nx2n matrices

$$\underline{Z} = \begin{bmatrix} \underline{Z}_{11} & \underline{Z}_{12} \\ \underline{Z}_{21} & \underline{Z}_{22} \end{bmatrix} = \begin{bmatrix} \coth\sqrt{s\underline{RC}} & \operatorname{csch}\sqrt{s\underline{RC}} \\ \operatorname{csch}\sqrt{s\underline{RC}} & \coth\sqrt{s\underline{RC}} \end{bmatrix} \begin{bmatrix} \sqrt{s\underline{RC}}^{-1}\,\underline{R} & \underline{0} \\ \underline{0} & \sqrt{s\underline{RC}}^{-1}\,\underline{R} \end{bmatrix} \tag{2.9}$$

and
$$\underline{Y} = \begin{bmatrix} \underline{Y}_{11} & \underline{Y}_{12} \\ \underline{Y}_{21} & \underline{Y}_{22} \end{bmatrix} = \begin{bmatrix} \underline{R}^{-1}\sqrt{s\underline{RC}} & \underline{0} \\ \underline{0} & \underline{R}^{-1}\sqrt{s\underline{RC}} \end{bmatrix} \begin{bmatrix} \coth\sqrt{s\underline{RC}} & -\operatorname{csch}\sqrt{s\underline{RC}} \\ -\operatorname{csch}\sqrt{s\underline{RC}} & \coth\sqrt{s\underline{RC}} \end{bmatrix}. \quad (2.10)$$

Equations (2.9) and (2.10) are the desired forms of the solution of the uniform multilayered distributed RC network.

2.3 Open and Short Circuited Driving Point Immittance Matrices

Observe that if the outputs of the $\overline{RC}$ section are open circuited, i.e. $\underline{I}_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then

$$\underline{V}_1 = \underline{Z}_{11}\,\underline{I}_1 = \coth\sqrt{s\underline{RC}}\,\sqrt{s\underline{RC}}^{-1}\,\underline{R}\,\underline{I}_1\,. \quad (2.11)$$

and if the outputs are short circuited, i.e. $\underline{V}_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

$$\underline{I}_1 = \underline{Y}_{11}\,\underline{V}_1 = \underline{R}^{-1}\sqrt{s\underline{RC}}\,\coth\sqrt{s\underline{RC}}\,\underline{V}_1. \quad (2.12)$$

These open circuit and short circuit relations of equations (2.11) and (2.12) will be useful later.

2.4 Cascade Synthesis of Multi-Layered $\overline{RC}$ Networks

Let several distributed $\overline{RC}$ networks be cascaded as depicted in Figure 2-3. To ensure uniform current flow through each individual network or section in the cascade assume a narrow conductive film across the resistive films at the ends of each section of the cascaded network as in the four-layered three section network of Figure 2-4. For the i^{th} network in the cascade let $\underline{\mathcal{T}}_i$ represent the matrix

$$\underline{\mathcal{T}}_i = \underline{R}_i\underline{C}_i = \begin{bmatrix} \mathcal{T}_{11} & -\mathcal{T}_{11} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\mathcal{T}_{21} & \mathcal{T}_{21}+\mathcal{T}_{22} & -\mathcal{T}_{22} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -\mathcal{T}_{32} & \mathcal{T}_{32}+\mathcal{T}_{33} & -\mathcal{T}_{33} & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & -\mathcal{T}_{43} & \mathcal{T}_{43}+\mathcal{T}_{44} & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \mathcal{T}_{n,n-1}+\mathcal{T}_{nn} \end{bmatrix} \quad (2.13)$$

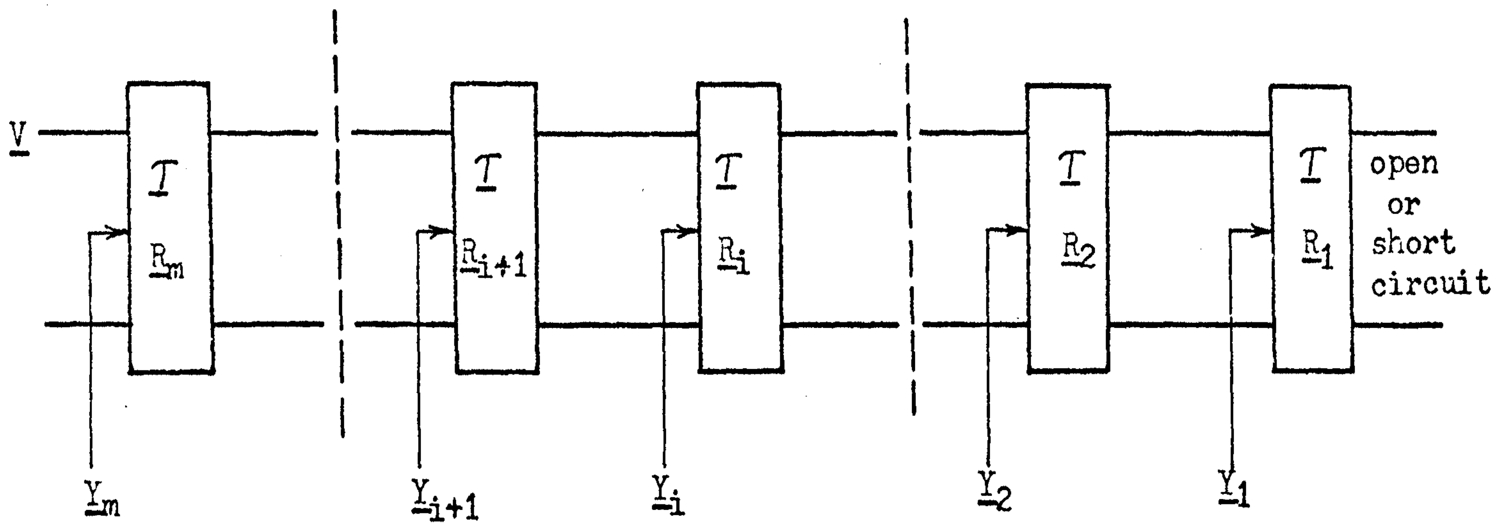


Figure 2-3. m cascaded sections of a multi-layered $\overline{RC}$ network.

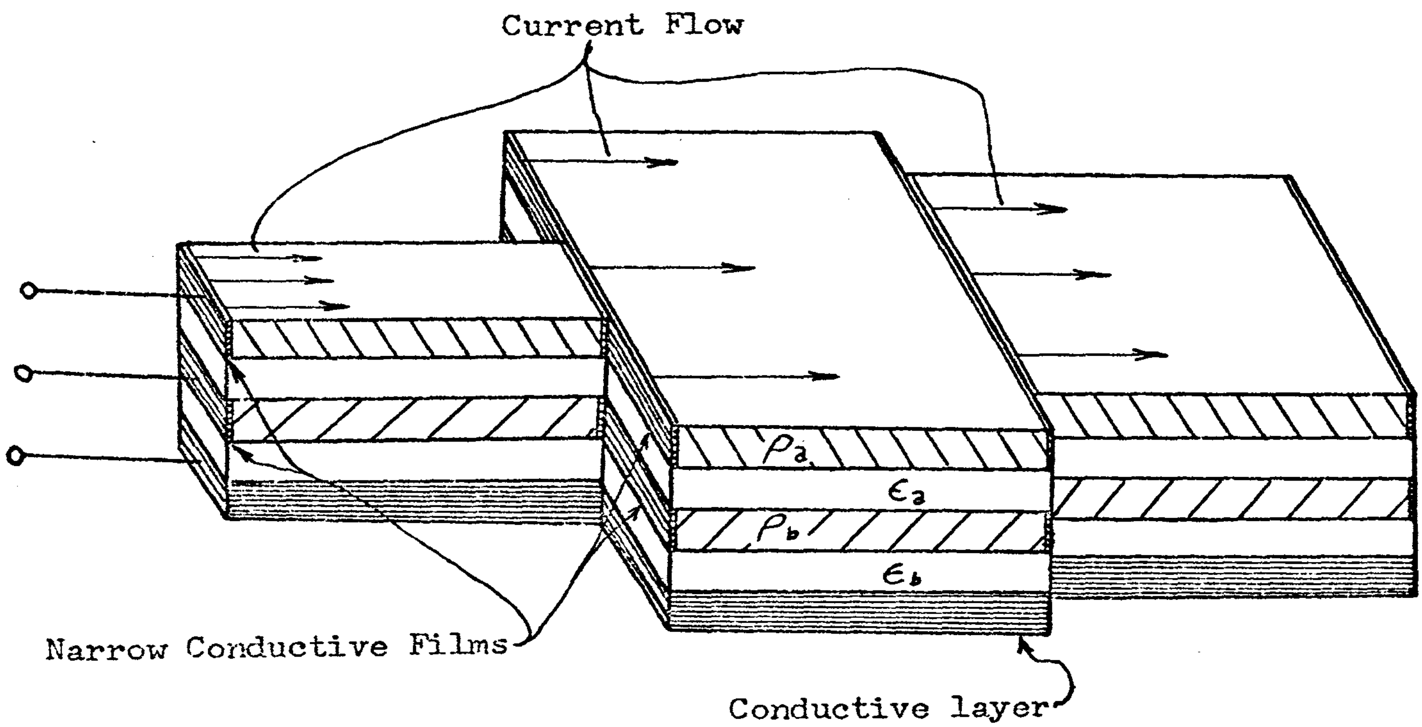


Figure 2-4. Four-layered three section cascaded network showing narrow conductive films which ensure uniform current flow through each section.

where $\mathcal{T}_{11} = R_{a_1}C_{a_1}$, $\mathcal{T}_{21} = R_{b_1}C_{a_1}$, $\mathcal{T}_{22} = R_{b_1}C_{b_1}$, etc. The first subscript on R and C denotes the layer index while the second subscript denotes which section in the cascade is under consideration. Later, a change of variable will be made involving $\tanh\sqrt{s\underline{\tau}_i}$. Since the variable will have to be independent of the particular section under consideration, $\tanh\sqrt{s\underline{\tau}_i}$ must be the same for each section in the cascade. This requires that $\underline{\tau}_i=\underline{\tau}_j=\underline{\tau}$ which implies that corresponding RC products of all sections are equal. Now write the equation for the input admittance matrix of the ith section in terms of the input admittance matrix and the parameters of the i+1th section. The details are involved and are left to Appendix B, the result is given in equation (2.14).

$$\underline{Y}_i(s) = \underline{R}_{i+1}^{-1}\sqrt{s\underline{\tau}}\,(\underline{R}_{i+1}^{-1}\sqrt{s\underline{\tau}} - \underline{Y}_{i+1}\tanh\sqrt{s\underline{\tau}})^{-1} \times (\underline{Y}_{i+1} - \underline{R}_{i+1}^{-1}\sqrt{s\underline{\tau}}\tanh\sqrt{s\underline{\tau}}) \tag{2.14}$$

If $Y_i(s)$ is multiplied by $1/\sqrt{s}$ an equation results which describes a corresponding LC circuit in the $\sqrt{s}$ plane.

$$\underline{Y}_i^{LC}(\sqrt{s}) = (1/\sqrt{s})\underline{Y}_i(s) = \underline{R}_{i+1}^{-1}\sqrt{\underline{\tau}}\left[\underline{R}_{i+1}^{-1}\sqrt{\underline{\tau}} - \underline{Y}_{i+1}^{LC}(\sqrt{s})\tanh\sqrt{s\underline{\tau}}\right]^{-1} \times \left[\underline{Y}_{i+1}^{LC}(\sqrt{s}) - \underline{R}_{i+1}^{-1}\sqrt{\underline{\tau}}\tanh\sqrt{s\underline{\tau}}\right] \tag{2.15}$$

Now make a change of variables similar to one first suggested by Richards (Ref. 28). Let

$$\underline{W} = \tanh\sqrt{s\underline{\tau}} = (e^{2\sqrt{s\underline{\tau}}} - \underline{I})\,(e^{2\sqrt{s\underline{\tau}}} + \underline{I})^{-1} \tag{2.16a}$$

or

$$e^{2\sqrt{s\underline{\tau}}} = (\underline{I} + \underline{W})\,(\underline{I} - \underline{W})^{-1} \tag{2.16b}$$

$$\underline{Y}_{i_{LC}}(\underline{W}) = \underline{R}_{i+1}^{-1}\sqrt{\underline{\tau}}\left[\underline{R}_{i+1}^{-1}\sqrt{\underline{\tau}} - \underline{Y}_{i+1_{LC}}(\underline{W})\,\underline{W}\right]^{-1}\left[\underline{Y}_{i+1_{LC}}(\underline{W}) - \underline{R}_{i+1}^{-1}\sqrt{\underline{\tau}}\,\underline{W}\right] \tag{2.17}$$

Note that now the variable is the matrix $\underline{W}$ rather than the scalar s.

One form of Richard's Theorem (Ref. 28) for the scalar case states that if

$$y_i(s) = y_{i+1}(1)\,\frac{y_{i+1}(s) - s y_{i+1}(1)}{y_{i+1}(1) - s y_{i+1}(s)}\,, \tag{2.18}$$

$y_i(s)$ is positive real if $y_{i+1}(s)$ is positive real and is of lower degree than $y_{i+1}(s)$. Wyndrum (Ref. 32) described a synthesis procedure for cascaded sections of the two-layered distributed RC network using equation (2.18). He obtained an equation for the driving point impedance of a portion of the network in terms of the driving point impedance and parameters of the preceding section. If Figure 2-3 is used as a diagram, Wyndrum's equation is

$$y_i = \frac{\sqrt{s\tau}}{R_{i+1}} \; \frac{y_{i+1} - \frac{\sqrt{s\tau}}{R_{i+1}} \tanh\sqrt{s\tau}}{\frac{\sqrt{s\tau}}{R_{i+1}} - y_{i+1} \tanh\sqrt{s\tau}} \tag{2.19}$$

in which R_i and τ are scalars. After multiplying by $1/\sqrt{s}$ and making the substitution $w = \tanh\sqrt{s\tau}$, this equation becomes

$$y_{i_{LC}}(w) = \frac{\sqrt{\tau}}{R_{i+1}} \; \frac{y_{i+1}(w) - \frac{\sqrt{\tau}}{R_{i+1}} w}{\frac{\sqrt{\tau}}{R_{i+1}} - y_{i+1}(w)\, w} \; . \tag{2.20}$$

Equation (2.20) is in the form of a lumped LC network in the w plane. Thus it could be represented by a positive real ratio of polynomials which have the characteristics of lumped LC networks. Wyndrum observed that if

$$y_{i+1_{LC}}(1) = \frac{\sqrt{\tau}}{R_{i+1}}$$

equation (2.20) is in the form of Richard's Theorem, equation (2.18). Inspection of equation (2.17) reveals that in form it is identical to equation (2.20), the difference being that in the former all quantities (including the variable $\underline{W}$) are matrices while in the latter all quantities are scalars. This leads one to suspect that a synthesis procedure similar to Wyndrum's could be applied to multi-layered networks. Observe that in the scalar case

$$w(s)\Big|_{s\to\infty} = \lim_{s\to\infty} \tanh\sqrt{s\tau} = \lim_{s\to\infty} \frac{e^{2\sqrt{s\tau}}-1}{e^{2\sqrt{s\tau}}+1} = 1$$

and in the matrix case application of Sylvester's Theorem leads to

$$\underline{W}(s)\Big|_{s\to\infty} = \lim_{s\to\infty} \tanh\sqrt{s\underline{T}} = \sum_{i=1}^{n} \lim_{s\to\infty} \tanh\sqrt{s\lambda_i}\ \underline{Z}o(\lambda_i)$$

$$= \sum_{i=1}^{n} 1 \cdot \underline{Z}o(\lambda_i)$$

where the λ_i's are the eigenvalues of the matrix $\underline{T}$ and the $\underline{Z}o(\lambda_i)$'s are the constituent idempotents of $\underline{T}$ as defined in equation (A.10) of Appendix A. The term $\underline{Z}o(\lambda_i)$ has the properties

$$\sum_{i=1}^{n} \underline{Z}o(\lambda_i) = \underline{I}$$

so that

$$\underline{W}(s)\Big|_{s} = \underline{I}.$$

Since equation (2.20) takes the form of a lumped LC network,

$$y_{\substack{i\\LC}}(w) = w^{\pm 1} \frac{\prod_j (w^2+a_j)}{\prod_j (w^2+b_j)} \quad .$$

assume that $\underline{Y}_{\substack{i\\LC}}(\underline{W})$ of equation (2.17) can be written in an analogous fashion,

$$\underline{Y}_{\substack{i\\LC}}(\underline{W}) = \underline{W}^{\pm 1} \prod_j (\underline{W}^2 + a_j\underline{I}) \prod_j (\underline{W}^2 + b_j\,\underline{I})^{-1} \equiv \underline{\hat{P}}_i(\underline{W}), \qquad (2.21)$$

and follow Wyndrum's lead by identifying $\underline{R}_{i+1}^{-1}\sqrt{\underline{T}}$ with $\underline{Y}_{\substack{i+1\\LC}}(\underline{I})$. Unfortunately it can be seen from equation (2.21) that

$$\underline{Y}_{\substack{i+1\\LC}}(\underline{I}) = \underline{\hat{P}}_{i+1}(\underline{I}) = k_{i+1}\ \underline{I} \neq \underline{R}_{i+1}^{-1}\sqrt{\underline{T}}$$

where k_i is a scalar. $\underline{Y}_{\substack{i+1\\LC}}(\underline{I})$ is a scalar times the unit matrix; $\underline{R}_{i+1}^{-1}\sqrt{\underline{T}}$ is not even a diagonal matrix, much less a scalar matrix! This trouble may be avoided by assuming a slightly different polynomial form for $\underline{Y}_{\substack{i\\LC}}(\underline{W})$. Let

$$\underline{Y}_{i_{Lc}}(\underline{W}) = \underline{M}\,\hat{\underline{P}}_i(\underline{W}) \tag{2.22}$$

where $\underline{M}$ is some yet to be determined constant matrix independent of i. Then define another Y matrix as

$$\hat{\underline{Y}}_i(\underline{W}) \equiv \underline{M}^{-1}\,\underline{Y}_{i_{Lc}}(\underline{W}) = \hat{\underline{P}}_i(\underline{W}) \tag{2.23}$$

Thus

$$\hat{\underline{Y}}_i(\underline{W}) = \underline{M}^{-1}\left\{\underline{R}_{i+1}^{-1}\sqrt{\underline{T}}\left[\underline{R}_{i+1}^{-1}\sqrt{\underline{T}} - \underline{Y}_{i+1_{Lc}}\,\underline{W}\right]^{-1}\underline{M}\,\underline{M}^{-1}\times\left[\underline{Y}_{i+1_{Lc}} - \underline{R}_{i+1}^{-1}\sqrt{\underline{T}}\,\underline{W}\right]\right\}$$

$$= (\underline{R}_{i+1}\underline{M})^{-1}\sqrt{\underline{T}}\left[(\underline{R}_{i+1}^{-1}\underline{M})^{-1}\sqrt{\underline{T}} - \hat{\underline{Y}}_{i+1}\underline{W}\right]^{-1}\times\left[\hat{\underline{Y}}_{i+1} - (\underline{R}_{i+1}\underline{M})^{-1}\sqrt{\underline{T}}\,\underline{W}\right] \tag{2.24}$$

Now if $(\underline{R}_i\underline{M})^{-1}\sqrt{\underline{T}}$ is identified with $\hat{\underline{Y}}_i(\underline{I})$ one obtains

$$(\underline{R}_i\underline{M})^{-1}\sqrt{\underline{T}} = \hat{\underline{Y}}_i(\underline{I}) = \hat{\underline{P}}_i(\underline{I}) = k_i\underline{I}. \tag{2.25}$$

The matrix $\underline{M}$ must be chosen so as to diagonalize the left side of equation (2.25). Thus solving for $\underline{M}$ one arrives at

$$\underline{M} = \frac{1}{k_i}\,\underline{R}_i^{-1}\sqrt{\underline{T}} \tag{2.26a}$$

or

$$k_i\underline{M} = \underline{R}_i^{-1}\sqrt{\underline{T}} \tag{2.26b}$$

There is considerable freedom in the choice of $\underline{M}$, but it must be chosen so that it is independent of the index i. For convenience choose $\underline{M}$ so that $k_i = 1/R_{a_i}$. Then

$$\underline{M} = R_{a_i}\,\underline{R}_i^{-1}\sqrt{\underline{T}} = R_{a_i}\begin{bmatrix} 1/R_{a_i} & 0 & \cdots & 0 \\ 0 & 1/R_{b_i} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/R_{n_i} \end{bmatrix}\sqrt{\underline{T}}. \tag{2.27a}$$

Let $r_b = R_{b_i}/R_{a_i}$, $r_c = R_{c_i}/R_{a_i}$, etc. so that

$$\underline{M} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1/r_b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/r_n \end{bmatrix}\sqrt{\underline{T}} \equiv \underline{\mathcal{M}}\sqrt{\underline{T}} \tag{2.27b}$$

By substituting the matrices $\underline{R}$ and $\underline{C}$ given by equations (2.4) and (2.5) into the equation

$$\underline{R}_i \, \underline{C}_i = \underline{R}_j \, \underline{C}_j = \underline{T}$$

and equating corresponding components, it can be shown that the r_b, r_c, etc. are the same for each network section, i.e. $r_{b_i} = r_{b_j}$, $r_{c_i} = r_{c_j}$, etc. Thus $\underline{M}$ is independent of the section index as required and equation (2.25) becomes

$$\hat{\underline{Y}}_i(\underline{\Xi}) = k_i \, \underline{\Xi} = \frac{1}{R_{a_i}} \, \underline{\Xi} \, . \tag{2.28}$$

Now equation (2.24) can be written in the form

$$\hat{\underline{Y}}_i(\underline{W}) = \frac{1}{R_{a_{i+1}}} \left[\frac{1}{R_{a_{i+1}}} \, \underline{\Xi} - \hat{\underline{Y}}_{i+1} \, \underline{W} \right]^{-1} \left[\hat{\underline{Y}}_{i+1} - \frac{1}{R_{a_{i+1}}} \, \underline{W} \right] . \tag{2.29}$$

2.5 The Synthesis Procedure

This synthesis procedure will yield the width of each section to within an arbitrary constant. The starting point is a prescribed matrix function of the form

$$\underline{Y}_p(s) = H \, \underline{M} \sqrt{s\underline{T}} \left\{ \tanh \sqrt{s\underline{T}} \, \prod \left[(\tanh \sqrt{s\underline{T}})^2 + a_j \underline{\Xi} \right] \times \prod \left[(\tanh \sqrt{s\underline{T}})^2 + b_j \underline{\Xi} \right]^{-1} \right\}^{\pm 1} \tag{2.30}$$

where H, b_j, a_j are positive constants, $b_i < a_i < b_{i+1}$, n=m or n=m+1, and p the number of sections in the cascade is equal to n+m+1. First pre-multiply $\underline{Y}_p(s)$ by $(1/\sqrt{s})\underline{M}^{-1}$.

$$\hat{\underline{Y}}_p(\tanh \sqrt{s\underline{T}}) = \frac{1}{\sqrt{s}} \underline{M}^{-1} \, \underline{Y}_p(s) = \sqrt{s\underline{T}}^{-1} \, \underline{M}^{-1} \, \underline{Y}_p(s)$$

$$= H \left\{ \tanh \sqrt{s\underline{T}} \, \prod_{j=1}^{m} \left[(\tanh \sqrt{s\underline{T}})^2 + a_j \underline{\Xi} \right] \prod_{j=1}^{n} \left[(\tanh \sqrt{s\underline{T}})^2 + b_j \underline{\Xi} \right]^{-1} \right\}^{\pm 1}$$

Next make the change of variables indicated in equation (2.16a) obtaining

$$\hat{\underline{Y}}_p(\underline{W}) = H \left[\underline{W} \prod_{j=1}^{m} (\underline{W}^2 + a_j \underline{\Xi}) \prod_{j=1}^{n} (\underline{W}^2 + b_j \underline{\Xi})^{-1} \right]^{\pm 1} \tag{2.31}$$

This is the form of the admittance matrix that will be synthesized. Note that since the only matrices equation (2.31) contains are the unit matrix and $\underline{W}$, all of the factors commute with respect to multiplication. Also the functional form of (2.31) is isomorphic to a positive real quotient of rational polynomials. Richard's Theorem, equation (2.18), can be written in the matrix form

$$\underline{Y}_i(\underline{W}) = \underline{Y}_{i+1}(\underline{I}) \left[\underline{Y}_{i+1}(\underline{W}) - \underline{W}\, \underline{Y}_{i+1}(\underline{I})\right] \left[\underline{Y}_{i+1}(\underline{I}) - \underline{W}\, \underline{Y}_{i+1}(\underline{W})\right]^{-1} \tag{2.32}$$

Equation (2.32) is not to be confused with the matrix forms of Richard's equation usually found in the literature (Ref. 15,20) which use the Laplace transform variable s as the independent variable. The variable of equation (2.32) is the matrix $\underline{W}$. Richard's Theorem assumes that $\hat{\underline{Y}}_{p-1}(\underline{W})$ will be of lower degree than $\hat{\underline{Y}}_p(\underline{W})$ if equation (2.32) is applied to (2.31) to find $\hat{\underline{Y}}_{p-1}(\underline{W})$. Successive applications of (2.32) to (2.31) will eventually lead to an equation of either the form

$$\hat{Y}_1(\underline{W}) = k_1\, \underline{W} \tag{2.33}$$

or the form

$$\hat{\underline{Y}}_1(\underline{W}) = k_1\, \underline{W}^{-1}. \tag{2.34}$$

Equation (2.33) may be identified with the open circuited $\overline{RC}$ section of equation (2.11) while (2.34) is the short circuited section given by (2.12). That this identification is valid may be shown by applying the transformations used to obtain $\hat{\underline{Y}}_n(\underline{W})$ to equations (2.11) and (2.12). Thus

$$\begin{aligned} \hat{\underline{Y}}_{1_{oc}}(\underline{W}) &= \frac{1}{\sqrt{s}} M^{-1} \left(\coth \sqrt{s\underline{\tau}}\, \sqrt{s\underline{\tau}}^{-1}\, \underline{R}_1\right)^{-1} \Bigg|_{\tanh \sqrt{s\underline{\tau}} = \underline{W}} \\ &= \frac{1}{R_{a1}} \sqrt{s\underline{\tau}}^{-1}\, \underline{R}_1\, \underline{R}_1^{-1} \sqrt{s\underline{\tau}} \tanh \sqrt{s\underline{\tau}} \\ &= \frac{1}{R_{a_1}} \underline{W} \end{aligned} \tag{2.35}$$

and $$\hat{\underline{Y}}_{1sc}(\underline{W}) = \frac{1}{\sqrt{s}} \underline{M}^{-1} \underline{R}_1^{-1} \sqrt{s\underline{\tau}} \coth \sqrt{s\underline{\tau}}$$

$$= \frac{1}{R_{a_1}} \underline{W}^{-1} \tag{2.36}$$

The next step is to calculate the R_{a_i}'s from equation (2.28), i.e. $R_{a_i} = 1/k_i$. To complete the synthesis procedure observe from Figure 2-5 that

$$R_{a_i} = \frac{\rho_a \, l}{h_{\rho a} \, w_i} \, . \tag{2.37}$$

and consequently

$$w_i = \left(\frac{\rho_a \, l}{h_{\rho a}}\right) k_i \, . \tag{2.38}$$

For ease of construction the thickness h and the conductivity ρ or permittivity ϵ of a given layer should be the same for each section. The restriction that $\tau_{\alpha\beta}$ be identical for each section has previously been stated. Therefore

$$\tau_{\alpha\beta} = R_{\alpha_i} C_{\beta_i} = R_{\alpha_j} C_{\beta_j}$$

$$\frac{\rho_\alpha l_i}{h_{\rho\alpha} \, w_i} \times \frac{\epsilon_\beta l_i w_i}{h_{\epsilon\beta}} = \frac{\rho_\alpha \, l_j}{h_{\rho\alpha} \, w_j} \times \frac{\epsilon_\beta l_j w_j}{h_{\epsilon\beta}}$$

or $$l_i = l_j$$

showing that the length of each section is also constant. From this it can be seen that the term $\rho_a l/h_{\rho a}$ in equation (2.38) is a constant and may be arbitrarily selected.

2.6 Additional Comments

Application of this synthesis procedure assumes that equation (2.30) is the starting point. It applies to any such 2n layered network with n greater than zero. In the next chapter a method will be discussed for

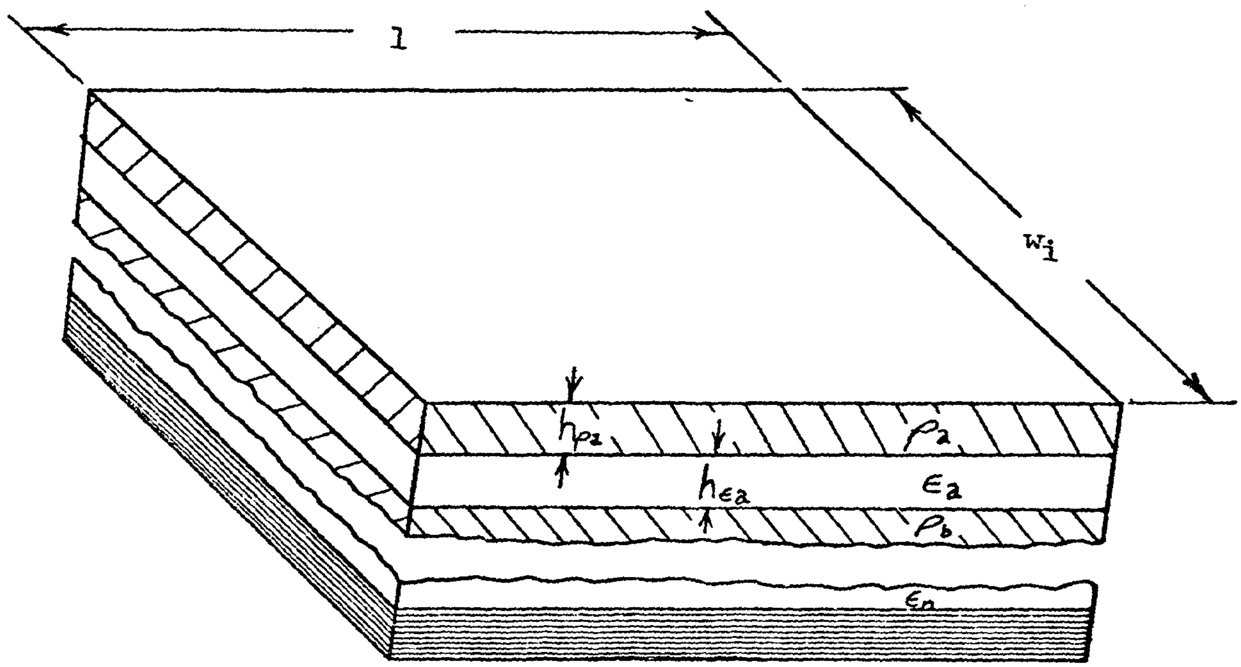


Figure 2-5. The i^{th} section of an n-section cascaded $\overline{RC}$ network.

$$R_{a_i} = \frac{\rho_a l}{h_{\rho_a} w_i}, \quad C_{a_i} = \frac{\epsilon_a l w_i}{h_{\epsilon_a}}.$$

calculating the components of the matrix $\underline{T}$ from the high and low frequency asymptotes of either the desired short circuit admittance parameters or the voltage transfer functions.

One more analogy may be made to Wyndrum's work. Recall that

$$\tanh \sqrt{s\underline{\tau}} = (e^{2\sqrt{s\underline{\tau}}} - \underline{I})\ (e^{2\sqrt{s\underline{\tau}}} + \underline{I})^{-1}. \tag{2.39}$$

If this is substituted into equation (2.30), one obtains after simplification an equation of the form

$$\underline{Y}_p(s) = \hat{H}\,\underline{H}\sqrt{s\underline{\tau}}\ \Big\{ \Big[(e^{2\sqrt{s\underline{\tau}}} - \underline{I})\prod_{j=1}^{m}(e^{4\sqrt{s\underline{\tau}}} + A_j e^{2\sqrt{s\underline{\tau}}} + \underline{I})\Big] \Big[(e^{2\sqrt{s\underline{\tau}}} + \underline{I})^{2(m-n)+1} \prod_{j=1}^{n}(e^{4\sqrt{s\underline{\tau}}} + B_j e^{2\sqrt{s\underline{\tau}}} + \underline{I})\Big]^{-1}\Big\}^{\pm 1} \tag{2.40}$$

in which $\hat{H}$, A_j, and B_j are scalar constants, $n=m$ or $n=m+1$, $p=n+m+1$, and A_j and B_j are subject to the conditions

$$-2 < B_j < A_j < B_{j+1} < 2. \tag{2.41}$$

The derivation of relation (2.41) is straight forward and comes from the restriction in equation (2.30) that $0 < b_j < a_j < b_{j+1} < \infty$. As this is completely analogous to the case where all quantities in (2.40) are one-by-one matrices (scalars) the reader is referred to Theorem VI, page 47 of reference 32 for the proof.

The network resulting from this synthesis technique is of the type shown in Figure 2-4, with all connection leads located on the left side. The right hand ports are either open circuited or short circuited depending on whether the final application of Richard's equation leads to equation (2.33) or equation (2.34) and, as a result, on whether the ± 1 exponent of the starting equation (relation (2.30) or (2.40)) is positive or negative respectively.

2.7 Examples

Example 2.1: Synthesize the network function

$$\underline{Y}_3(s) = 3\underline{M}\sqrt{s\tau}\tanh\sqrt{s\tau}(\tanh^2\sqrt{s\tau} + 4\underline{I})(\tanh^2\sqrt{s\tau} + 2\underline{I})^{-1}.$$

Solution: First obtain the $\hat{\underline{Y}}$ admittance matrix by premultiplying by $1/\sqrt{s}\,\underline{M}^{-1}$ and making the change of variables of equation (2.16a).

$$\hat{\underline{Y}}_3(\underline{W}) = \frac{1}{\sqrt{s}}\underline{M}^{-1}\underline{Y}_3(s)\Big|_{\tanh\sqrt{s\tau} = \underline{W}} = \sqrt{s\tau}\,\underline{M}^{-1}\underline{Y}_3(s)\Big|_{\tanh\sqrt{s\tau} = \underline{W}}$$

$$= 3\underline{W}(\underline{W}^2 + 4\underline{I})(\underline{W}^2 + 2\underline{I})^{-1}.$$

Now $\hat{\underline{Y}}_3$ must be evaluated at $\underline{W} = \underline{I}$ and Richard's equation in the form of relation (2.32) should be applied.

$$\hat{\underline{Y}}_3(\underline{I}) = 3(\underline{I} + 4\underline{I})(\underline{I} + 2\underline{I})^{-1} = 5\underline{I} \quad \text{and } k_3 = 5$$

$$\hat{\underline{Y}}_2(\underline{W}) = 5\underline{I}\left[3\underline{W}(\underline{W}^2 + 4\underline{I})(\underline{W}^2 + 2\underline{I})^{-1} - \underline{W}\cdot 5\underline{I}\right]$$

$$\times\left[5\underline{I} - \underline{W}\cdot 3\underline{W}(\underline{W}^2 + 4\underline{I})(\underline{W}^2 + 2\underline{I})^{-1}\right]^{-1}$$

$$= 10\underline{W}(-\underline{W}^2 + \underline{I})(3\underline{W}^2 + 10\underline{I})^{-1}(-\underline{W}^2 + \underline{I})^{-1}$$

$$= 10\underline{W}(3\underline{W}^2 + 10\underline{I})^{-1}$$

Repeat this process until the variable $\underline{W}$ is of degree one.

$$\hat{\underline{Y}}_2(\underline{I}) = 10(3\underline{I} + 10\underline{I})^{-1} = \frac{10}{13}\underline{I} \quad \text{and} \quad k_2 = \frac{10}{13}$$

$$\hat{\underline{Y}}_1(\underline{W}) = \frac{10}{13}\underline{I}\left[10\underline{W}(3\underline{W}^2 + 10\,\underline{I})^{-1} - \underline{W}\cdot\frac{10}{13}\underline{I}\right]$$

$$\times\left[\frac{10}{13}\underline{I} - \underline{W}\cdot 10\underline{W}(3\underline{W}^2 + 10\underline{I})^{-1}\right]^{-1}$$

$$= \frac{3}{13}\underline{W} \tag{2.42}$$

$$\hat{\underline{Y}}_1(\underline{I}) = \frac{3}{13}\underline{I} \quad \text{and} \quad k_1 = \frac{3}{13}$$

By equation (2.38)

$$w_3 = 5 \frac{\rho_a 1}{h\rho_a}$$

$$w_2 = \frac{10}{13} \frac{\rho_a 1}{h\rho_a}$$

$$w_1 = \frac{3}{13} \frac{\rho_a 1}{h\rho_a}$$

and comparing equation (2.42) with (2.33) it can be seen that the final section is open circuited.

Example 2.2: Start with equation (2.40) and for a three section cascaded network with the end section open circuited, find the valves of the A_j's and B_j's in terms of the widths of the sections.

Solution: This is to be a three section cascaded network so n=m=1 and there is only one A and one B. The open circuit specification requires that ± 1 in equation (2.40) be positive. The network equation then is

$$\underline{Y}_3(s) = \hat{H} \mp \sqrt{s\underline{\tau}} \; (e^{2\sqrt{s\underline{\tau}}} - \underline{I})\; (e^{4\sqrt{s\underline{\tau}}} + A\; e^{2\sqrt{s\underline{\tau}}} + \underline{I})$$
$$\times (e^{2\sqrt{s\underline{\tau}}} + \underline{I})\; (e^{4\sqrt{s\underline{\tau}}} + B\; e^{2\sqrt{s\underline{\tau}}} + \underline{I})^{-1} \qquad (2.43)$$

To get this into the form of equation (2.31) substitute (2.16b) into (2.43) and premultiply by $(1/\sqrt{s})\underline{M}^{-1}$.

$$\hat{\underline{Y}}_3(\underline{W}) = \frac{1}{\sqrt{s}} \underline{M}^{-1} \hat{H}\, \underline{M} \sqrt{s\underline{\tau}}\, \underline{W} \left[(\underline{I}+\underline{W})^2 (\underline{I}-\underline{W})^{-2} + A(\underline{I}+\underline{W})(\underline{I}-\underline{W})^{-1} + \underline{I}\right]$$
$$\times \left[(\underline{I}+\underline{W})^2 (\underline{I}-\underline{W})^{-2} + B(\underline{I}+\underline{W})(\underline{I}-\underline{W})^{-1} + \underline{I}\right]$$
$$= \hat{H}\, \underline{W} \left[(2-A)\underline{W}^2 + (2+A)\underline{I}\right] \left[(2-B)\underline{W}^2 + (2+B)\underline{I}\right]^{-1} \qquad (2.44)$$

Repeated application of Richard's equation, i.e. equation (2.32), will lead to

$$\hat{\underline{Y}}_3(\underline{I}) = \hat{H}\, \underline{I}$$

$$\hat{\underline{Y}}_2(\underline{\Xi}) = \frac{\hat{H}(A-B)}{4-(A-B)}\,\underline{\Xi}$$

$$\hat{\underline{Y}}_1(\underline{\Xi}) = \frac{\hat{H}(A-B)}{4-(A-B)}\,\frac{2-A}{2+B}\,\underline{\Xi}.$$

Therefore

$$k_3 = \hat{H} \tag{2.45a}$$

$$k_2 = \frac{\hat{H}(A-B)}{4-(A-B} = k_3\,\frac{A-B}{4-(A-B)} \tag{2.45b}$$

$$k_1 = \frac{\hat{H}(A-B)}{4-(A-B)}\,\frac{2-A}{2+B} = k_2\,\frac{2-A}{2+B}\,. \tag{2.45c}$$

The simultaneous solution of equations (2.45b) and (2.45c) for A and B yields

$$A = \frac{2\left[k_3(k_2-k_1) + k_2(k_2+k_1)\right]}{(k_1+k_2)\;(k_2+k_3)} \tag{2.46a}$$

$$B = \frac{2\left[k_3(k_2-k_1) - k_2(k_2+k_1)\right]}{(k_1+k_2)\;(k_2+k_3)}\,. \tag{2.46b}$$

According to equation (2.38),

$$w_i = \frac{\rho_a\,1}{h_{\rho_a}}\;k_i\;,$$

but if this is substituted into equations (2.46) the constant term $(\rho_a\,1)/h_{\rho_a}$ cancels, yielding the required results.

$$A = \frac{2\left[w_3(w_2-w_1) + w_2(w_2+w_1)\right]}{(w_1+w_2)\;(w_2+w_3)} \tag{2.47a}$$

$$B = \frac{2\left[w_3(w_2-w_1) - w_2(w_2+w_1)\right]}{(w_1+w_2)\;(w_2+w_3)} \tag{2.47b}$$

These equations will be useful in the circuit design procedures of Chapter 5.

3.0 ASYMPTOTIC RELATIONSHIPS

3.1 Introduction

In the last chapter a procedure was developed to synthesize cascaded multi-layered $\overline{RC}$ networks from a prescribed network admittance matrix. The information actually obtained from this procedure is the width of each section of the cascaded network. To make this a useful procedure, more must be known about the relation between the prescribed network admittance matrix and the physical network. Because of the complexity of the mathematics for multi-layered distributed $\overline{RC}$ networks, this paper will deal exclusively with the four-layered case from this point on. Figure 3-1 shows a four-layered network with the voltage, current, and admittance notation that is used in the remainder of this dissertation. Assuming a prescribed network matrix, the equations for low and high frequency asymptotic properties of the short circuit admittance parameters and open circuit voltage transfer ratios are developed in this chapter. From these properties, the values of the components of the $\underline{\mathcal{T}}$ matrix can be calculated.

For cascaded four-layered networks, the $\underline{\mathcal{T}}$ matrix is

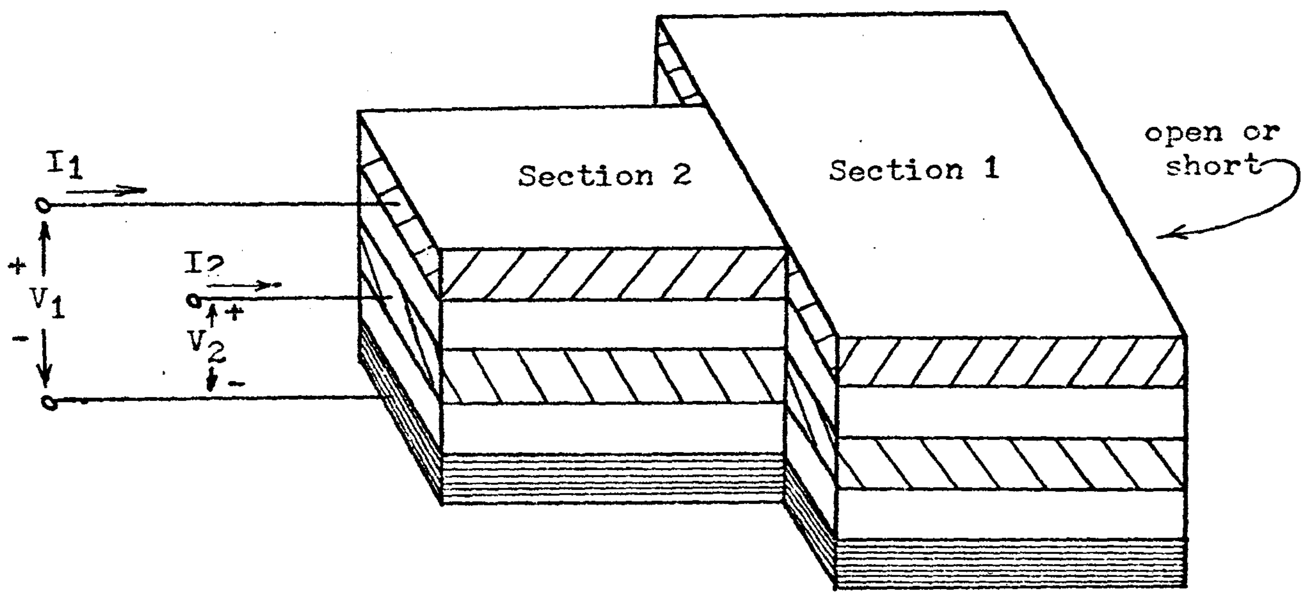
$$\underline{\mathcal{T}} = \begin{bmatrix} \mathcal{T}_{11} & -\mathcal{T}_{11} \\ -\mathcal{T}_{21} & \mathcal{T}_{21}+\mathcal{T}_{22} \end{bmatrix} \tag{3.1}$$

and it's eigenvalues, as calculated in Appendix C, are

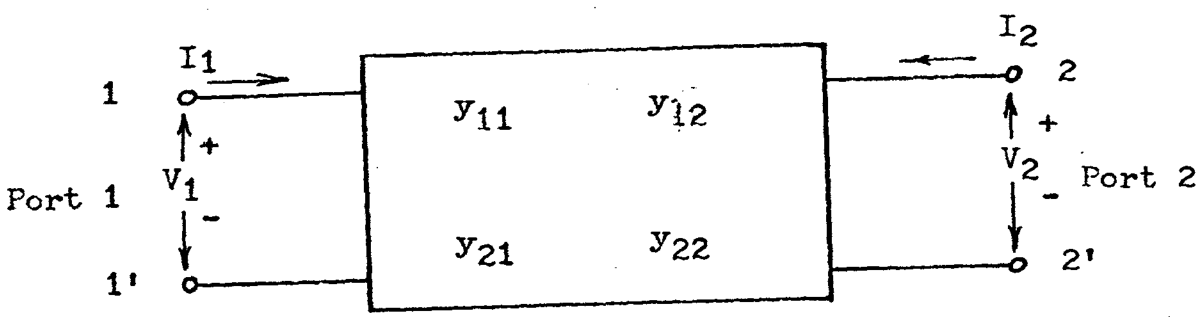
$$\lambda_1 = \frac{1}{2}\,(\mathcal{T}_1-\mathcal{T}_0) \tag{3.2a}$$

$$\lambda_2 = \frac{1}{2}\,(\mathcal{T}_1+\mathcal{T}_0) \tag{3.2b}$$

where $$\mathcal{T}_1 \equiv \mathcal{T}_{11}+\mathcal{T}_{21}+\mathcal{T}_{22} \tag{3.2c}$$



(a).



(b).

Figure 3-1. (a). Four-layered distributed $\overline{RC}$ network. (b). Black box approach.

and $\mathcal{T}_0 \equiv \sqrt{\mathcal{T}_1^2 - 4\mathcal{T}_{11}\mathcal{T}_{22}}$. (3.2d)

Now apply Sylvester's Theorem to either equation (2.30) or (2.40). Define the function of λ_1 and λ_2

$$F_i = F(s\lambda_i) = H\sqrt{s\lambda_i}\left[\tanh\sqrt{s\lambda_i}\,\frac{\prod_{j=1}^{m}(\tanh^2\sqrt{s\lambda_i}+a_j)}{\prod_{j=1}^{n}(\tanh^2\sqrt{s\lambda_i}+b_j)}\right]^{\pm 1}$$

$$= \hat{H}\sqrt{s\lambda_i}\left[\frac{(e^{2\sqrt{s\lambda_i}}-1)\prod_{j=1}^{m}(e^{4\sqrt{s\lambda_i}}+A_j\,e^{2\sqrt{s\lambda_i}}+1)}{(e^{2\sqrt{s\lambda_i}}+1)^{2(m-n)+1}\prod_{j=1}^{n}(e^{4\sqrt{s\lambda_i}}+B_j\,e^{2\sqrt{s\lambda_i}}+1)}\right]^{\pm 1} \qquad (3.3)$$

where i = 1 or 2. Then the admittance matrix of a network composed of m+n+1 cascaded four-layered network sections may be written

$$\underline{Y}(s) = \frac{1}{\mathcal{T}_0}\begin{bmatrix}1 & 0\\ 0 & \frac{1}{r}\end{bmatrix}\left[(\lambda_2\underline{\underline{I}} - \underline{\mathcal{T}})\,F(s\lambda_1) - (\lambda_1\underline{\underline{I}} - \underline{\mathcal{T}})\,F(s\lambda_2)\right]$$

$$= \begin{bmatrix}-aF_1+bF_2 & -d(F_2-F_1)\\ -d(F_2-F_1) & \frac{1}{r}(bF_1-aF_2)\end{bmatrix} \qquad (3.4)$$

where the constants a, b, d, and r are defined in Appendix C. Appendix C derives several relationships among the constants a, b, d, and r, as well as their relationship to the components and eigenvalues of $\mathcal{T}$.

3.2 High Frequency Asymptotes

As a first step toward studying the high frequency asymptotic behavior consider the function $F(s\lambda_i)$ of equation (3.3). Either form could be used, but the equations to be derived will be simpler if the second form is used. As $|s|$ becomes extremely large compared with $1/\lambda_i$,

$$F_{i\infty} \equiv F(s\lambda_i)\Big|_{s\to\infty} = \hat{H}\sqrt{s\lambda_i} \ . \qquad (3.5)$$

Relation (3.5) holds whether the end section (section one or the section on the right in Figure 3-1) is open or short circuited. Let $L = \lambda_2/\lambda_1$, i.e. the ratio of the eigenvalues. Due to the prominence of the term

$e^{2\sqrt{s\lambda_1}}$ in equation (3.3), normalize frequency ω such that $\left|e^{2\sqrt{j\omega\lambda_1}}\right| = e^{\sqrt{\eta}}$, or

$$\eta = 2\omega\lambda_1 = 2\omega\lambda_2/L. \tag{3.6}$$

Then substitution of equation (3.5) into (3.4) gives the high frequency asymptotes for the y_{11}, y_{12}, and y_{22} parameters.

$$y_{11\infty}(j\eta) = -a\,\hat{H}\sqrt{j\frac{\eta}{2}} + b\,\hat{H}\sqrt{j\frac{\eta L}{2}}$$

$$= \frac{\hat{H}}{\sqrt{2}}\,(b\sqrt{L}-a)\sqrt{j\eta} \tag{3.7a}$$

$$y_{12\infty}(j\eta) = y_{21}(j\eta) = -d\left(\hat{H}\sqrt{j\frac{\eta L}{2}} - H\sqrt{j\frac{\eta}{2}}\right)$$

$$= -\frac{d\hat{H}}{\sqrt{2}}\,(\sqrt{L}-1)\sqrt{j\eta} \tag{3.7b}$$

$$y_{22\infty}(j\eta) = \frac{1}{r}\left(b\,\hat{H}\sqrt{\frac{j\eta}{2}} - a\,\hat{H}\sqrt{j\,\frac{\eta L}{2}}\right)$$

$$= \frac{\hat{H}}{r\sqrt{2}}\,(b-a\sqrt{L})\sqrt{j\eta} \tag{3.7c}$$

Define G as the voltage gain such that

$$G_{12} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{-y_{12}}{y_{22}} \tag{3.8a}$$

and

$$G_{21} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{-y_{12}}{y_{11}} \tag{3.8b}$$

Substitution of equation (3.7) together with the values of a, b, d, r, and L given in Appendix C yields the high frequency asymptotes for gain.

$$G_{12\infty}(j\eta) = d\;r\,\frac{\sqrt{L}-1}{b-a\sqrt{L}} = \frac{\tau_{21}}{\tau_{21}+\tau_{22}+\sqrt{\tau_{11}\tau_{22}}} \tag{3.9a}$$

$$G_{21\infty}(j\eta) = \frac{d(\sqrt{L}-1)}{b\sqrt{L}-a} = \frac{1}{1+\sqrt{\tau_{22}/\tau_{11}}} \tag{3.9b}$$

Equations (3.7) and (3.9) show that as frequency becomes large the log magnitude of the short circuit admittance parameters increases at the rate of 10 db per decade causing both voltage gains to approach a constant at high frequency. This behavior is independent of the termination that might be applied to the right hand terminals of the cascade.

3.3 Low Frequency Asymptotes

Two separate cases must be considered when finding the low frequency asymptotes of the short circuit admittance parameters. The case where the right end section of the network of Figure 2-1 is open circuited will be considered first, and after that, the case where this section is short circuited.

3.3.1 Right End Section Open Circuited

For the circuit configuration with the right end section open circuited the plus one exponent in equation (3.3) is retained so that

$$F_{i_o} = F(s\lambda_i)\Big|_{s\to \text{small}} = H\sqrt{s\lambda_i}\left[\frac{(e^{2\sqrt{s\lambda_i}} - 1)}{(e^{2\sqrt{s\lambda_i}} + 1)^{2(m-n)+1}} \times \frac{\prod_{j=1}^{m}(e^{4\sqrt{s\lambda_i}} + A_j\, e^{2\sqrt{s\lambda_i}} + 1)}{\prod_{j=1}^{n}(e^{4\sqrt{s\lambda_i}} + B_j\, e^{2\sqrt{s\lambda_i}} + 1)}\right]_{s\to \text{small}}$$

$$\cong H\sqrt{s\lambda_i}\;\frac{\left(1+2\sqrt{s\lambda_i} + \frac{(2+\sqrt{s\lambda_i})^2}{2!} + \ldots - 1\right)\prod_{j=1}^{m}(2+A_j)}{2^{2(m-n)+1}\prod_{j+1}^{n}(2+B_j)}\Bigg|_{s\to \text{small}}$$

$$\cong \frac{4^{(n-m)}\prod_{j=1}^{m}(2+A_j)}{\prod_{j=1}^{n}(2+B_j)}\,(s\lambda_i) \tag{3.10}$$

The desired asymptotes may be found using equation 3.4. The results follow:

$$y_{11_o}(j\eta) = \frac{\hat{H}\,4^{(n-m)}}{2} \frac{\prod_{j=1}^{m}(2+A_j)}{\prod_{j=1}^{n}(2+B_j)} (bL-a)\,(j\eta) \tag{3.11a}$$

$$y_{12_o}(j\eta) = -\frac{\hat{H}\,4^{(n-m)}}{2} \frac{\prod_{j=1}^{m}(2+A_j)}{\prod_{j=1}^{n}(2+B_j)} d(L-1)\,(j\eta)$$

$$= -y_{11_o}(j\eta) \tag{3.11b}$$

$$y_{22_0}(j\eta) = \frac{\hat{H}\,4^{(n-m)}}{2} \frac{\prod_{j=1}^{m}(2+A_j)}{\prod_{j=1}^{n}(2+B_j)} \frac{(b-aL)}{r}\,(j\eta) \tag{3.11c}$$

$$G_{12_o} = \frac{d(L-1)}{b-aL}\,r = \frac{\tau_{21}}{\tau_{21}+\tau_{22}} \tag{3.12a}$$

$$G_{21_o} = 1 \tag{3.12b}$$

Thus it is seen that as frequency becomes very small y_{11} and y_{12} become equal in magnitude, the log magnitude plots of the short circuit admittance parameters have a positive slope of 20 db per decade, and the log magnitude of the voltage gains have zero slope.

3.3.2 Right End Section Short Circuited

In the short circuit case, $F_i(s \to 0)$ becomes

$$F_{i_o} = F(s\lambda_i)\Big|_{s\to 0} = \left.\frac{\hat{H}\sqrt{s\lambda_i}\;2^{2(m-n)+1}\prod_{j=1}^{n}(2+B_j)}{\left(1+2\sqrt{s\lambda_i}+\frac{(2\sqrt{s\lambda_i})^2}{2!}+\ .\ .\ .\ -1\right)\prod_{j=1}^{m}(2+A_j)}\right|_{s\to 0}$$

$$= \frac{\hat{H}\prod_{j=1}^{n}(2+B_j)}{4^{n-m}\prod_{j=1}^{m}(2+A_j)}\ . \tag{3.13}$$

Applying this relation to equation (3.4) one obtains for the short circuit driving point admittances

$$y_{11_o} = \frac{\hat{H}}{4^{n-m}} \frac{\prod_{j=1}^{n}(2+B_j)}{\prod_{j=1}^{m}(2+A_j)} \tag{3.14a}$$

$$y_{22o} = \frac{y_{11o}}{r} \tag{3.14b}$$

Because equation (3.13) indicates that F_{1_o} and F_{2_o} both approach the same constant at low frequency it must be used with caution. It may be used to find the low-frequency asymptotes of a term of the form $pF_1(\omega \to 0) + q\,F_2(\omega \to 0)$ so long as this term does not vanish. Since in the case under consideration y_{12} is identically zero if (3.13) is used, a different approach is required for this parameter. Suppose the distributed network is approximated by the lumped element pi section shown in Figure 3-2. Since C_3 and C_4 are short circuited they do not enter into the calculations. The node voltage equations may be written by inspection as

$$I_1 = V_1(C_1 s + G_1) - V_2 C_1 s$$

$$I_2 = -V_1\ C_1\ s + V_2(C_1 s + C_2 s + G_2).$$

The transfer y parameters then are

$$y_{12} = y_{21} = -C_1 s$$

or with $\quad s = j\omega = j\eta/(2\lambda_1)$

$$y_{12} = -j\eta \times \text{constant},$$

i.e. the magnitude of y_{12} is proportional to frequency. If one begins with the equation

$$y_{12} = -d(F_2 - F_1)$$

in which F_1 is the exact form (equation (3.3) using the negative one exponent) and replaces the exponentials by the first few terms of their respective Maclaurin's series expansion, the result is

$$y_{12_o} = \frac{-\hat{H}}{4^{n-m}}\left[\frac{d(L-1)}{2}\ \frac{\prod_{j=1}^{n}(2+B_j)}{\prod_{j=1}^{m}(2+A_j)}\right]\left[\sum_{j=1}^{n}\frac{2-B_j}{2+B_j} - \sum\frac{2-A_j}{2+A_j} + \frac{1}{3}\right](j\eta) \tag{3.14c}$$

The derivation of equation (3.14c), though straight forward, is a long drawn out process, and therefore the details are omitted from this discussion.

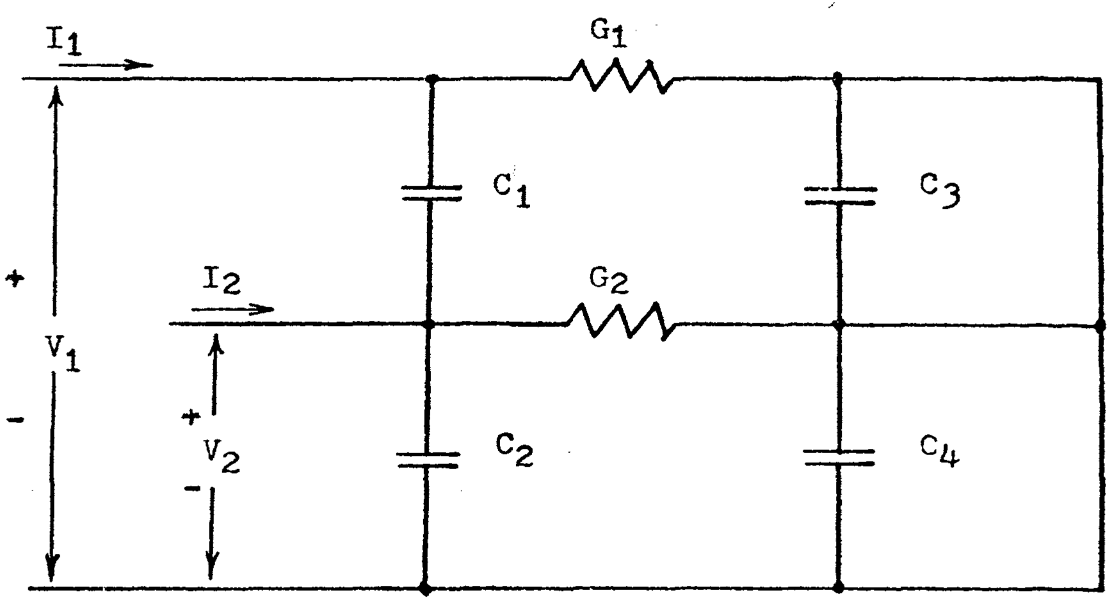


Figure 3-2. Short circuited distributed network approximated as a lumped pi network.

The open circuit voltage gains are

$$G_{12_o} = \frac{rd}{2}(L-1)\left[\sum_{j=1}^{n} \frac{2-B_j}{2+B_j} - \sum_{j=1}^{m} \frac{2-A_j}{2+A_j} + \frac{1}{3}\right](j\eta) \qquad (3.15a)$$

$$G_{21_o} = \frac{G_{12_o}}{r} \qquad (3.15b)$$

where $$\frac{rd(L-1)}{2} = \frac{\mathcal{T}_{21}\left[(\mathcal{T}_{11}+\mathcal{T}_{21}+\mathcal{T}_{22}) + \sqrt{(\mathcal{T}_{11}+\mathcal{T}_{21}+\mathcal{T}_{22})^2 - 4\mathcal{T}_{11}\mathcal{T}_{22}}\right]}{4\mathcal{T}_{11}\mathcal{T}_{22}} .$$

Equations (3.14) and (3.15) show that at very low frequencies y_{11} and y_{22} approach a constant value while y_{12} and the voltage gains are directly proportional to frequency.

3.4 Calculation of the Matrix $\underline{\mathcal{T}}$ and the Constant $\hat{H}$

The results of the previous sections may be used to calculate $\mathcal{T}_{11}, \mathcal{T}_{21}, \mathcal{T}_{22}$, and $\hat{H}$ if the asymptotic values of either the voltage gains or the short circuit admittance parameters are specified. Since this is a passive, resistive network, one would expect the asymptotic voltage gains to be less than or equal to unity, and therefore

$$|y_{11}| \geq |y_{12}| \leq |y_{22}| .$$

Equations (3.9) and (3.12) are especially useful because they are simple functions of $\mathcal{T}_{11}$, $\mathcal{T}_{21}$, and $\mathcal{T}_{22}$. Since equations (3.14a) and (3.14b) contain the constants A_j and B_j they are of little value for the present discussion.

There are four constants that are to be determined $\mathcal{T}_{11}$, $\mathcal{T}_{21}$, $\mathcal{T}_{22}$, and $\hat{H}$, and thus the number of quantities that can be chosen arbitrarily is equal to four minus the number of quantities specified. Since the normalized frequency η equals $2\omega\lambda_1$ the eigenvalue λ_1, as will be shown in Chapter 5, can be used as a frequency scaling constant and therefore should, if possible, be selected as one of the arbitrary constants. Because the constant $\hat{H}$ does not appear in the gain equations it is arbitrary if only asymptotic gains are specified. A look at the high frequency asymptotic short circuit admittance (y_{ij}) equations will show that $\hat{H}$ together with the normalized frequency η determine the admittance level. If any of the y_{ij} parameters are specified, the normalized frequency for that value of y_{ij} must also be specified. However, this value of η is not one of the degrees of freedom because knowledge of it does not give the value of the admittance level product $\hat{H}\eta$.

3.5 Summary of Asymptotic Parameter Values

Table 3-1 summerizes the asymptotic parameter values developed in this chpater.

3.6 Examples

As it is impractical to attempt to write explicit equations yielding the $\mathcal{T}_{ij}$'s and $\hat{H}$ for every possible set of specifications, three representative examples are given here.

Example 3.1: Find the components of the $\underline{\mathcal{T}}$ matrix and $\hat{H}$ if the gain G_{21} at large frequency and the frequency scaling factor λ_1 are specified.

Solution: Since two quantities are specified, two may be selected arbitrarily. Let $\mathcal{T}_0$ be selected arbitrarily. Then from (3.2a)

$$\mathcal{T}_1 = 2\lambda_1 + \mathcal{T}_0$$

and by (3.2d)

$$\mathcal{T}_{11}\,\mathcal{T}_{22} = \frac{\mathcal{T}_1^2 - \mathcal{T}_0^2}{4} \quad (3.16)$$

	Low Frequency Asymptotes		High Frequency Asymptotes
	Right End Section Open Circuited	Right End Section Short Circuited	
$y_{11}(j\eta)$	$\frac{\hat{H}}{2} 4^{n-m} \frac{\prod_{j=1}^{m}(2+A_j)}{\prod_{j=1}^{n}(2+B_j)} (bL-a)(j\eta)$	$\frac{\hat{H}}{4^{n-m}} \frac{\prod_{j=1}^{n}(2+B_j)}{\prod_{j=1}^{m}(2+A_j)}$	$\frac{\hat{H}}{2}(b\sqrt{L}-a)\sqrt{j\eta}$
$y_{12}(j\eta)$	$-y_{11}(j\eta)$	$\frac{-\hat{H}}{4^{n-m}} \frac{d(L-1)}{2} \frac{\prod_{j=1}^{n}(2+B_j)}{\prod_{j=1}^{m}(2+A_j)} \left[\sum_{j=1}^{n}\frac{2-B_j}{2+B_j} - \sum_{j=1}^{m}\frac{2-A_j}{2+A_j} + \frac{1}{3}\right](j\eta)$	$-\frac{d\hat{H}}{2}(\sqrt{L}-1)\sqrt{j\eta}$
$y_{22}(j\eta)$	$\frac{\hat{H}}{2} 4^{n-m} \frac{\prod_{j=1}^{m}(2+A_j)}{\prod_{j=1}^{n}(2+B_j)} \frac{(b-aL)}{r}(j\eta)$	$\frac{y_{11}}{r}$	$\frac{\hat{H}}{r\sqrt{2}}(b-a\sqrt{L})\sqrt{j\eta}$
$G_{12}(j\eta)$	$\frac{\tau_{21}}{\tau_{21}+\tau_{22}}$	$\frac{rd(L-1)}{2}\left[\sum_{j=1}^{n}\frac{2-B_j}{2+B_j} - \sum_{j=1}^{m}\frac{2-A_j}{2+A_j} + \frac{1}{3}\right](j\eta)$	$\frac{\tau_{21}}{\tau_{21}+\tau_{22}+\sqrt{\tau_{11}\tau_{22}}}$
$G_{21}(j\eta)$	1	G_{12}/r	$1/(1+\sqrt{\tau_{22}/\tau_{11}})$
$a = (\tau_{11}-\tau_{21}-\tau_{22}-\tau_o)/(2\tau_o)$ $b = (\tau_{11}-\tau_{21}-\tau_{22}+\tau_o)/(2\tau_o)$	$d = \tau_{11}/\tau_o$ $r = \tau_{21}/\tau_{11}$	$\tau_o = \sqrt{(\tau_{11}+\tau_{21}+\tau_{22})^2 - 4\,\tau_{11}\tau_{22}}$	

Table 3-1. Low and high frequency asymptotic expressions for y_{11}, y_{12}, y_{22}, G_{12}, and G_{21}.

From Table 3-1

$$\frac{\mathcal{T}_{22}}{\mathcal{T}_{11}} = \left(\frac{1 - G_{21\infty}}{G_{21\infty}}\right)^2. \tag{3.17}$$

The square root of the product of these last two equations yields $\mathcal{T}_{22}$ while the square root of their quotient yields $\mathcal{T}_{11}$. Then from equation (3.2c)

$$\mathcal{T}_{21} = \mathcal{T}_1 - \mathcal{T}_{11} - \mathcal{T}_{22}$$

and the components of $\underline{\mathcal{T}}$ are determined. Since none of the y_{ij}'s were specified, $\hat{H}$ is arbitrary.

Example 3.2: Given the asymptotic gain G_{12} at both zero and infinity calculate the $\underline{\mathcal{T}}$ matrix and $\hat{H}$. Use the configuration where the right end section is open circuited.

Solution: By Table 3-1

$$\frac{\mathcal{T}_{22}}{\mathcal{T}_{21}} = c = \frac{1-G_{12_o}}{G_{12_o}}$$

and substituting this into the equation for $G_{12\infty}$ gives

$$\frac{\mathcal{T}_{11}}{\mathcal{T}_{21}} = \frac{1}{c}\left[\frac{1}{G_{12\infty}} - \frac{1}{G_{12_o}}\right]^2 = \frac{1}{r}.$$

Choose λ_1 arbitrarily. Then with the aid of (3.2a)

$$\mathcal{T}_{21} = \frac{\lambda_1}{\frac{1}{2}\left[\frac{\mathcal{T}_{11}+\mathcal{T}_{21}+\mathcal{T}_{22}}{\mathcal{T}_{21}} - \sqrt{\left(\frac{\mathcal{T}_1}{\mathcal{T}_{21}}\right)^2 - \frac{4\mathcal{T}_{11}\mathcal{T}_{22}}{\mathcal{T}_{21}^2}}\right]}$$

$$= \frac{2\lambda_1}{\frac{1}{r} + 1 + c - \sqrt{\left(\frac{1}{r} + 1 + c\right)^2 - 4\frac{c}{r}}}$$

where c and r have just previously been calculated.

Finally $\mathcal{T}_{11} = \frac{1}{r}\mathcal{T}_{21}$

$$\mathcal{T}_{22} = c\,\mathcal{T}_{21}$$

and again $\hat{H}$ may be selected arbitrarily.

Example 3.3: If $|y_{11}|$, $|y_{12}|$, and $|y_{22}|$ are specified at some large normalized frequency η_∞, find $\hat{H}$ and the $\underline{T}$ matrix.

Solution: First find the two gains for large frequency from equations (3.8a) and (3.8b). Then by Table 3-1

$$\frac{T_{22}}{T_{11}} = \left(\frac{1-G_{21}}{G_{21}}\right)^2 = rc$$

and

$$\frac{T_{21}}{T_{11}} = \frac{G_{12\infty}\left(\frac{T_{22}}{T_{11}} + \sqrt{\frac{T_{22}}{T_{11}}}\right)}{1 - G_{12\infty}} = r \,.$$

Choose λ_1 and calculate T_{11} with the aid of (3.2a).

$$T_{11} = \frac{\lambda_1}{\frac{1}{2}\left[\frac{T_1}{T_{11}} - \sqrt{\left(\frac{T_1}{T_{11}}\right)^2 - \frac{4T_{11}T_{22}}{T_{11}^2}}\right]}$$

$$= \frac{2\lambda_1}{1+r+rc-\sqrt{(1+r+rc)^2 - 4rc}}$$

Then $\quad T_{21} = rT_{11}$

and $\quad T_{22} = rcT_{11}$

Finally the constants a, b, and L may be determined from equations (C.10), (C.11), and (C.23) of Appendix C and η is specified, so by (3.7a)

$$\hat{H} = \frac{|y_{11}|\sqrt{2}}{(b\sqrt{L}-a)\sqrt{\eta_\infty}} \,.$$

4.0 SYNTHESIS FROM A GRAPHICAL MAGNITUDE SPECIFICATION

4.1 Introduction

Chapter 2 described a procedure for finding the widths of the individual sections of a cascaded $\overline{RC}$ network from a prescribed admittance matrix the form of equation (2.40). Chapter 3 explained how to calculate the $\underline{T}$ matrix and the constant $\hat{H}$ of equation (2.40) for a four-layered network if the high or low frequency asymptotes of either the open circuit voltage gain or short circuit admittance parameters are specified. To make use of this synthesis procedure a method is needed to obtain the matrix function of equation (2.40) from specifications. This chapter attempts to deal with that problem.

4.2 Discussion of the Procedure

In most synthesis problems the starting point is a graphical response specification, often in the form of a log magnitude plot. The question to be answered in this chapter is "How can an admittance matrix that is realizable as a cascade of sections of a uniform multi-layered distributed $\overline{RC}$ network, i.e. an equation of the form of equation (2.40), be obtained from the log magnitude plots of a prescribed set of short circuit admittance parameters." Because the bulk of the mathematics rapidly becomes prohibitive as the number of resistance and dielectric layers increases, this discussion will again be limited to the exemplary four-layered $\overline{RC}$ network cascade of the type shown in Figure 2-4.

Since the circuit under consideration is a series of cascaded sections with the right end section either open or short circuited, the left end driving point admittance matrix is that of a two port network. Suppose log magnitude versus frequency plots are given for the short circuit admittance parameters of some network that is to be synthesized. The first step in synthesis is to obtain a realizable equation that represents the given plots to within a required degree of accuracy. Since 20 log ab = 20 log a + 20 log b, the usual procedure (Chapter 9 of reference 30) is to match a combination of log magnitude curves from a catalogue of the factors allowed by the class of realizable functions under consideration to the given curve or curves. For example if a given curve is to be realized as a lumped LC network the allowed factors are $s^{\pm 1}$ and $(s^2+a_i^2)^{\pm 1}$. The desired equation is obtained by matching a combination of the curves of $\pm$ 20 log $|j\omega|$ and $\pm$ 20 log $|-\omega^2+a_i^2|$ to the given curve. In the case of this dissertation it is known that the realizable equation belongs to the class of functions given by equation (2.40), but the factors of this equation are matrices, not scalars. While it is possible to define by Sylvester's Theorem the magnitude of a matrix such that

$$20 \log |\underline{AB}| = 20 \log |\underline{A}| + 20 \log |\underline{B}|$$

where $\underline{A}$ and $\underline{B}$ are matrix factors, this definition does not result in the log magnitude of each factor of each component of the original matrix (the y_{ij} parameters in this case). Since it has been supposed that the log magnitudes of the short circuit admittance parameters (y_{ij}) are specified, the usual procedure for the derivation of a realizable equation must be modified.

Equation (2.40) will be used as the starting point for deriving a procedure to obtain a realizable equation from the log magnitude plots of the short circuit admittance parameters. The procedure develops equations that can be used to obtain a plot of the log magnitude of the scalar function $F(s\lambda_1)$, defined in equation (3.3), from a plot of the log magnitudes of the y_{ij} parameters. From this plot of $F(s\lambda_1)$ an equation for this function may be obtained in the usual manner, that is by fitting a sum of log magnitude plots of certain allowed factors, specifically $\sqrt{s\lambda_1}$, $(e^{2\sqrt{s\lambda_1}} \pm 1)$, and $(e^{4\sqrt{s\lambda_1}} + Ae^{2\sqrt{s\lambda_1}} + 1)$, to the graph of $F(j\omega\lambda_1)$. Once the scalar equation for $F(s\lambda_1)$ is obtained, the matrix equation corresponding to (2.40) is obtained merely by replacing the scalar λ_1 by the matrix $\underline{T}$.

4.3 Development of the Procedure

In Chapter 3 it was shown that

$$\underline{Y}(s) = \begin{bmatrix} -aF(s\lambda_1)+bF(s\lambda_2) & -d\left[F(s\lambda_2)-F(s\lambda_1)\right] \\ -d\left[F(s\lambda_2)-F(s\lambda_1)\right] & \frac{1}{r}\left[bF(s\lambda_1)-aF(s\lambda_2)\right] \end{bmatrix} \tag{4.1}$$

in which the constants a, b, d, and r and the eigenvalues λ_1 and λ_2 are defined in Appendix C. Define $Lm\,\underline{Y}(j\omega)$ to be the log magnitude of each of the components of $\underline{Y}(j\omega)$, i.e.

$$Lm\underline{Y}(j\omega) \equiv \begin{bmatrix} 20\log\left|y_{11}(j\omega)\right| & 20\log\left|y_{12}(j\omega)\right| \\ 20\log\left|y_{21}(j\omega)\right| & 20\log\left|y_{22}(j\omega)\right| \end{bmatrix}. \tag{4.2}$$

$$\text{Let } F_i = F(j\omega\lambda_i) = \left|F_i\right| e^{j\Phi_i} \text{ where } \Phi_i = \tan^{-1}\frac{Im\,F_i}{Re\,F_i}. \tag{4.3}$$

Then from equation (4.1),

$$\begin{aligned}\left|y_{11}\right| &= \left|-aF_1 + bF_2\right| = \left|-a\left|F_1\right|(\cos\Phi_1 + j\sin\Phi_1) + b\left|F_2\right|(\cos\Phi_2 + j\sin\Phi_2)\right| \\ &= \left[(-a\left|F_1\right|\cos\Phi_1 + b\left|F_2\right|\cos\Phi_2)^2 + (-a\left|F_1\right|\sin\Phi_1 + b\left|F_2\right|\sin\Phi_2)^2\right]^{\frac{1}{2}} \\ &= \left[a^2\left|F_1\right|^2 + b^2\left|F_2\right|^2 - 2ab\left|F_1\right|\left|F_2\right|\cos(\Phi_2 - \Phi_1)\right]^{\frac{1}{2}} \end{aligned} \tag{4.4a}$$

$$\left|y_{12}\right| = \left|-d(F_2 - F_1)\right| = d\left[\left|F_1\right|^2 + \left|F_2\right|^2 - 2\left|F_1\right|\left|F_2\right|\cos(\Phi_2 - \Phi_1)\right]^{\frac{1}{2}} \tag{4.4b}$$

$$\left|y_{22}\right| = \left|\frac{1}{r}(bF_1 - aF_2)\right| = \frac{1}{r}\left[b^2\left|F_1\right|^2 + a^2\left|F_2\right|^2 - 2ab\left|F_1\right|\left|F_2\right|\cos(\Phi_2 - \Phi_1)\right]^{\frac{1}{2}} \tag{4.4c}$$

If these equations are solved for $|F_1|$ and $|F_2|$ — the details are left for Appendix C — equations (4.5) result.

$$\left|F_1\right|^2 = \frac{d^2(a\left|y_{11}\right|^2 + br^2\left|y_{22}\right|^2) - ab(b+a)\left|y_{12}\right|^2}{d^2(b+a)} \tag{4.5a}$$

$$|F_2|^2 = \frac{d^2(b|y_{11}|^2 + a\ r^2|y_{22}|^2) - ab(b+a)|y_{12}|^2}{d^2(b+a)} \tag{4.5b}$$

$$|F_1|\ |F_2| \cos(\Phi_2 - \Phi_1) = \frac{d^2(|y_{11}|^2 + r^2|y_{22}|^2) - (b^2+a^2)|y_{12}|^2}{2d^2} \tag{4.5c}$$

Equation (4.5a) is the equation to be used to obtain the log magnitude plot of $F(s\lambda_1)$. Since $F_1 = F(j\omega\lambda_1)$

$$\text{Lm } F(s\lambda_1) = 20 \log |F_1|$$

$$= 10 \log \left[\frac{d^2(a\, 10^{\text{Lm } y_{11}/10} + br^2 10^{\text{Lm } y_{22}/10}) - ab(b+a)10^{\text{Lm } y_{12}/10}}{d^2(b+a)} \right] \tag{4.6}$$

Equations (4.5b) and (4.5c) are useful in deriving certain restrictions as will now be shown.

Recall that $F_i = F(j\omega\lambda_i)$ and choose some frequency ω_i. Let

$$\omega_{i-1} = \omega_i \frac{\lambda_1}{\lambda_2}\ . \tag{4.7}$$

Then $F_2\Big|_{s=j\omega_{i-1}} = F(j\omega_{i-1}\lambda_2) = F(j\omega_i \frac{\lambda_1}{\lambda_2} \lambda_2) = F(j\omega_i\lambda_1) = F_1\Big|_{s=j\omega_i}$

or $\qquad |F_2(s=j\omega_{i-1})|^2 - |F_1(s=j\omega_i)|^2 = 0.$

From this relation and equations (4.5a) and (4.5b) one obtains

$$d^2 \left[a|y_{11}(j\omega_i)|^2 - b|y_{11}(j\omega_{i-1})|^2\right] + d^2 r^2 \left[b|y_{22}(j\omega_i)|^2 - a|y_{22}(j\omega_{i-1})|^2\right.$$
$$\left. - ab(b+a)\left[|y_{12}(j\omega_i)|^2 - |y_{12}(j\omega_{i-1})|^2\right] = 0 \tag{4.8}$$

Since for any angle θ

$$-1 \le \cos\theta \le 1$$

equation (4.5c) suggests the inequality

$$-|F_1|\ |F_2| \le \frac{d^2(|y_{11}|^2 + r^2|y_{22}|^2) - (b^2+a^2)|y_{12}|^2}{2d^2} \le |F_1|\ |F_2|$$

or $$0 \le \left[\frac{d^2(|y_{11}|^2 + r^2|y_{22}|^2) - (b^2+a^2)|y_{12}|^2}{2d^2}\right]^2 \le |F_1|^2|F_2|^2. \tag{4.9}$$

Substitution of $|F_1|^2$ and $|F_2|^2$ from equations (4.5a) and (4.5b) into (4.9) leads to

$$4d^2(b+a)^2\, r^2 |y_{22}|^2 |y_{12}|^2 - \left[d^2(|y_{11}|^2 - r^2|y_{22}|^2) - (b+a)^2|y_{12}|^2\right]^2 \geq 0. \tag{4.10}$$

Equations (4.8) and (4.10) place resitrictions on the relative magnitudes of the specified parameters y_{11}, y_{12}, and y_{22}. If only two of the y_{ij}'s are given equation (4.8) could be used to find the third.

The method of deriving the matrix function $F(s\underline{T})$ described here is theoretically correct but of questionable use in practice. The author attempted to synthesize a notch filter by this method and discovered that slight errors in constructing certain portions of the Lm $F(s\lambda_1)$ curve could cause large variations in the y_{ij} parameters. Perhaps a certain class of functions such as a notch filter or low-pass filter could be synthesized by this method and then optimization techniques could be used to improve the initial results. Certainly this procedure requires further study before it will become a useful tool to the practicing engineer. For this reason Chapter 5 is devoted to design techniques and a study of several specific types of network.

5.0 DESIGN OF FOUR-LAYERED CASCADED DISTRIBUTED $\overline{RC}$ NETWORKS

5.1 Introduction - Discussion of the Procedure

It was stated at the end of the last chapter that the approximation technique presented there is difficult to apply to a practical engineering problem. For that reason Chapter 5 is devoted to design procedures which can be applied to practical engineering problems. The circuit used is the cascaded configuration of Figures 2-4 and 3-1. Appendix E discusses possibilities of selecting the circuit parameters. It shows that the parameter $\hat{H}$ of equation (2-40) changes the admittance level of the network while the eigenvalue λ_1 of $\mathcal{T}$ may be used to shift responses along the frequency axis. Also $\hat{\rho}_a$, the constant determining the resistivity of the top layer per unit width, is arbitrary and can be chosen to give convenient values of resistivity of the top layer, length of the individual sections, and the thickness of the top layer. Since the designer will want to put practical limits or "stops" on the ratios of the resistivities and permittivities of the layers as well as on the ratios of the widths of the different sections, it is convenient to choose r, c, and $\hat{w}_i$ as the network parameters where r and c are defined in equations (C.13) and (C.14) and

$$\hat{w}_i \equiv \frac{w_i}{w_p} .$$

Specification of either the high or low frequency voltage gain may remove a degree of freedom, leaving only one of the quantities r and c as an independent parameter. If both the low and high frequency voltage gains are specified, as perhaps in a low or high pass filter, both r and c are determined, restricting further the freedom of the designer.

The remainder of this chapter discusses in order notch filters, low pass networks, and finally high pass networks.

5.2 Notch Filter

The approach to the design of a notch filter will be to look at the high and low frequency asymptotic requirements, select the quantities to be used as parameters and using convenient starting values calculate an initial response curve. If the log magnitude versus frequency response curve has no minimum or "dip," new starting values must be selected. If there is a dip in the response curve, no matter how slight, an iterative process may be used to optimize the network by varying each undetermined parameter in turn until either further perturbation of that parameter results in no improvement in the response curve or the value of the parameter reaches one of the limits or "stops" predetermined by practical engineering considerations. A digital computer is an invaluable aid for these iterative computations.

A passive notch filter requires a gain of as near unity as possible at both low and high frequencies and a gain much less than one at some intermediate frequency. Table 3-1 shows that the low frequency asymptotic gain of the cascaded configuration with the right end section short circuited approaches zero as frequency approaches zero and thus this configuration is not useful as a notch filter. On the other hand if the right end section is open circuited, Table 3-1 shows that the gain at low frequency is unity if the gain is taken as the ratio V_1/V_2 with $I_1=0$, i.e. port 2 of Figure 3-1b is taken as the input and port 1 is considered the output. This would be the logical configuration to use for a notch filter. In this case specification of the low frequency gain as unity does not remove a degree of freedom since G_{21_o} is unity and thus independent of all parameters. Equation (3.9b) gives the high frequency gain for this configuration as a function of $\mathcal{T}_{11}$ and $\mathcal{T}_{22}$, and from the definitions for r and c it can be seen that this is equivalent to

$$G_{21} = \frac{1}{1+\sqrt{rc}} \cdot \qquad (5.1)$$

If r is selected as an independent variable parameter, equation (5.1) may be solved for c to obtain the permittivity ratio. A computer program was written which would read the value of $G_{21\infty}$, the starting values for r and the width ratios $\hat{w}_i$, and the stops for r, c, and $\hat{w}_i$. It was decided that 0.01 and 100 would be physically feasible minimum and maximum limits for r, c, and the width ratios (Ref. 3). For a three section filter the constants A and B of equations (2.40) were calculated in terms of the widths in Example 2.2 of Chapter 2. To get A and B in terms of the width ratios, simply divide the numerators and denominators of equations (2.47a) and (2.47b) by w_3. Figure 5-1 gives an abbreviated flow chart of the computer program used. First the minimum gain was found for the initial values, then one of the widths was varied until the minimum possible gain was achieved. Next the other width was varied, the first width optimized again and the results tested for any improvement. This process was repeated until both widths (for the three section case) produced optimum results. Finally, the resistance ratio r was optimized in a similar fashion. In the vicinity of the dip in the log magnitude response curve double precision (16 digit) accuracy was necessary. One, two, and three section networks with high freqency asymptotic gains from

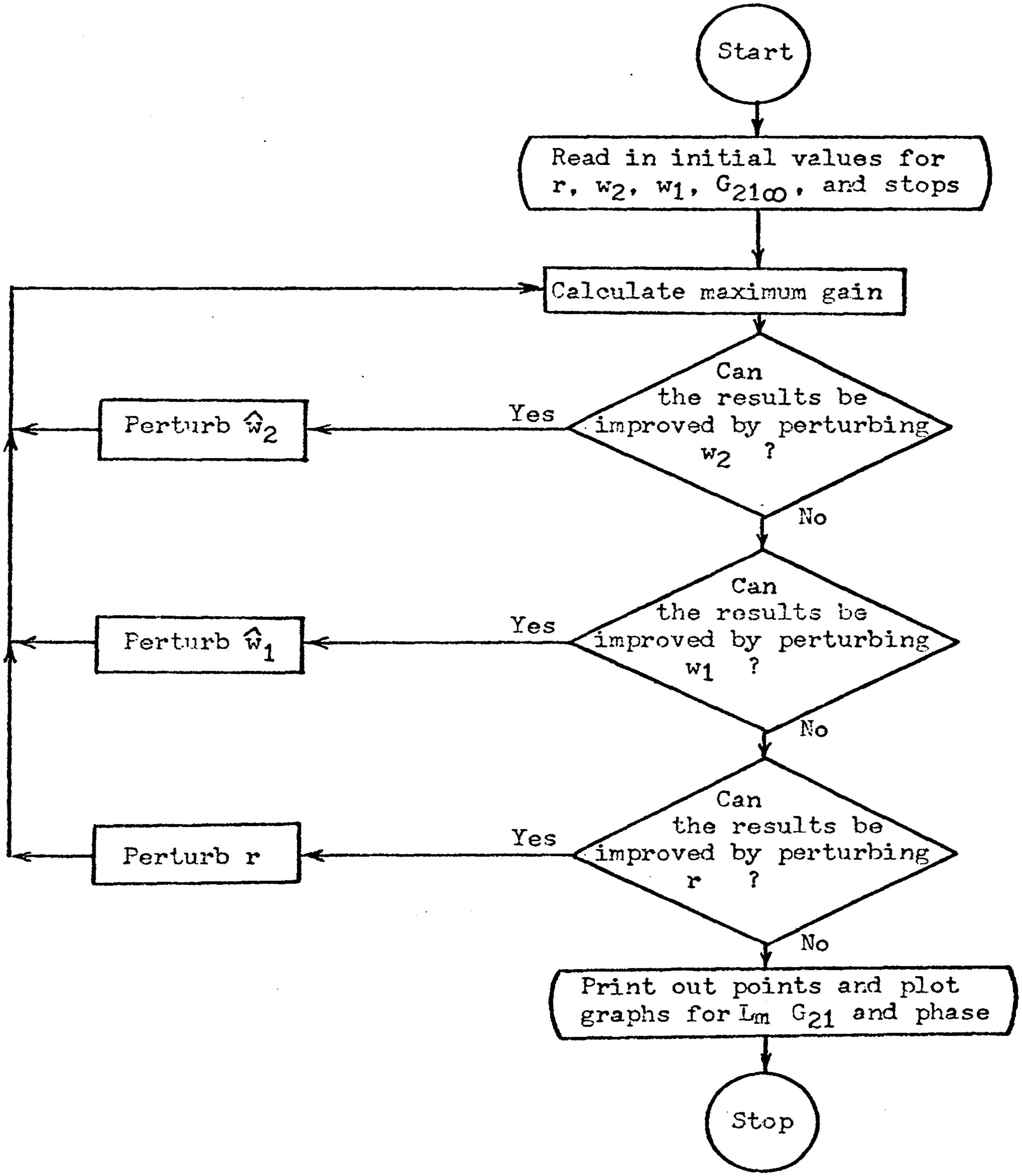


Figure 5-1. Flow chart for the computer program used to design a three section distributed $\overline{RC}$ notch filter.

-6 to -1 decibels and under no-load conditions were investigated in this manner.

The results for one, two, and three section notch networks are shown in the log magnitude and phase versus normalized frequency plots of Figures 5-2 through 5-7. If these networks are used in the feedback loop of an active device, bandpass characteristics result. It can be seen from Figures 5-2, 5-4, and 5-6 that the one section network is useless as a notch filter, the two section network has a minimum gain of about -42 db if -6 db gain at high frequencies can be tolerated, and the corresponding minimum for the three section network is almost 6 db below that. If the equation for circuit Q of a symmetrical gain characteristic (Ref. 33) is used, i.e.

$$Q = \frac{\eta_o}{BW}$$

where η is the normalized resonant frequency and BW the bandwidth, curve 1 of the three section network of Figure 5-6 has a Q of about 10 if the network is used in a bandpass configuration. For all three networks investigated, best results occurred when the resistance ratio was set as low as possible and the width ratios as high as possible.

Figures 5-8 and 5-9 show the log magnitude and phase plots of a two section notch network with $G_{21\infty}$ equal to -6 db under various loads. The notch characteristics are improved slightly by a load, but the low frequency end of the curve is severely degraded. Instead of having a low frequency gain of one (or zero db) the loaded network has a low frequency gain with a 20 db per decade slope.

5.3 Low Pass Network

The response characteristic of a low pass network requires that

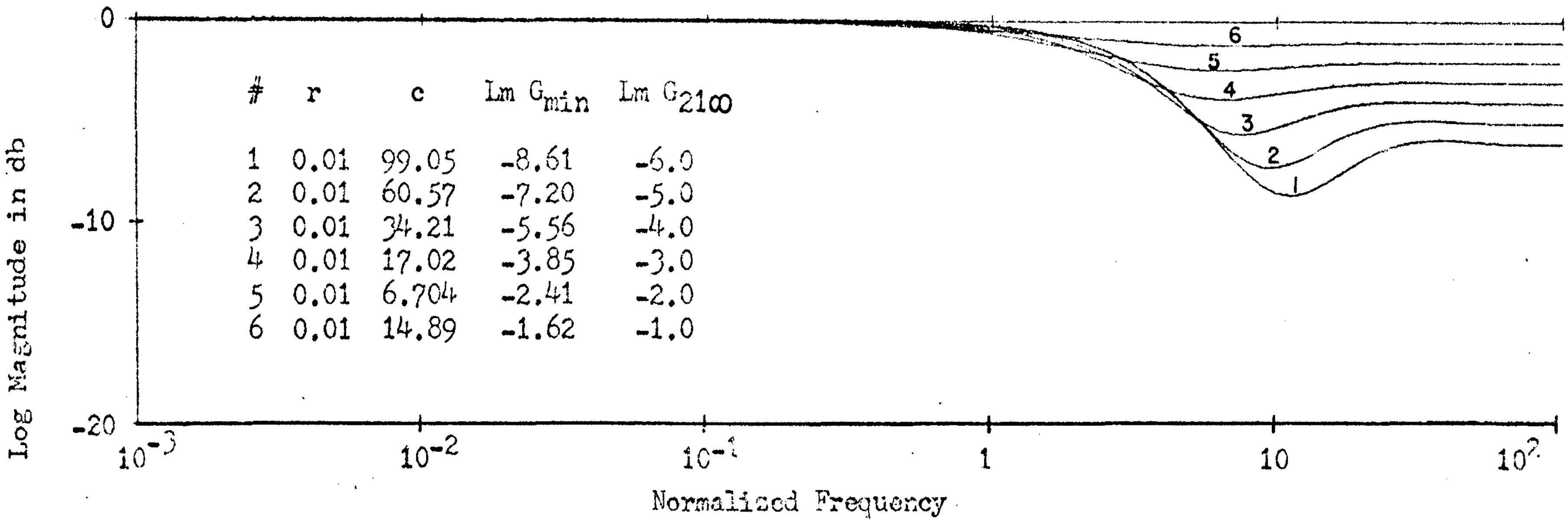


Figure 5-2. Gain characteristics of a one section distributed $\overline{RC}$ notch network with zero load. V_2= input voltage, V_1= output voltage.

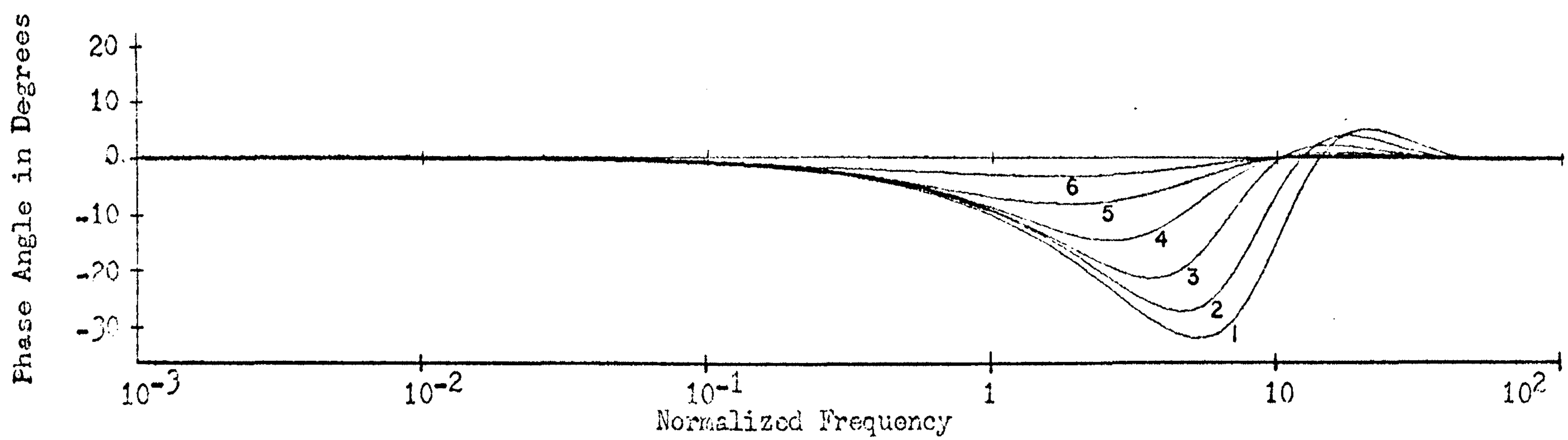


Figure 5-3. Phase characteristics of a one section distributed $\overline{RC}$ notch network with zero load. V_2= input voltage, V_1= output voltage.

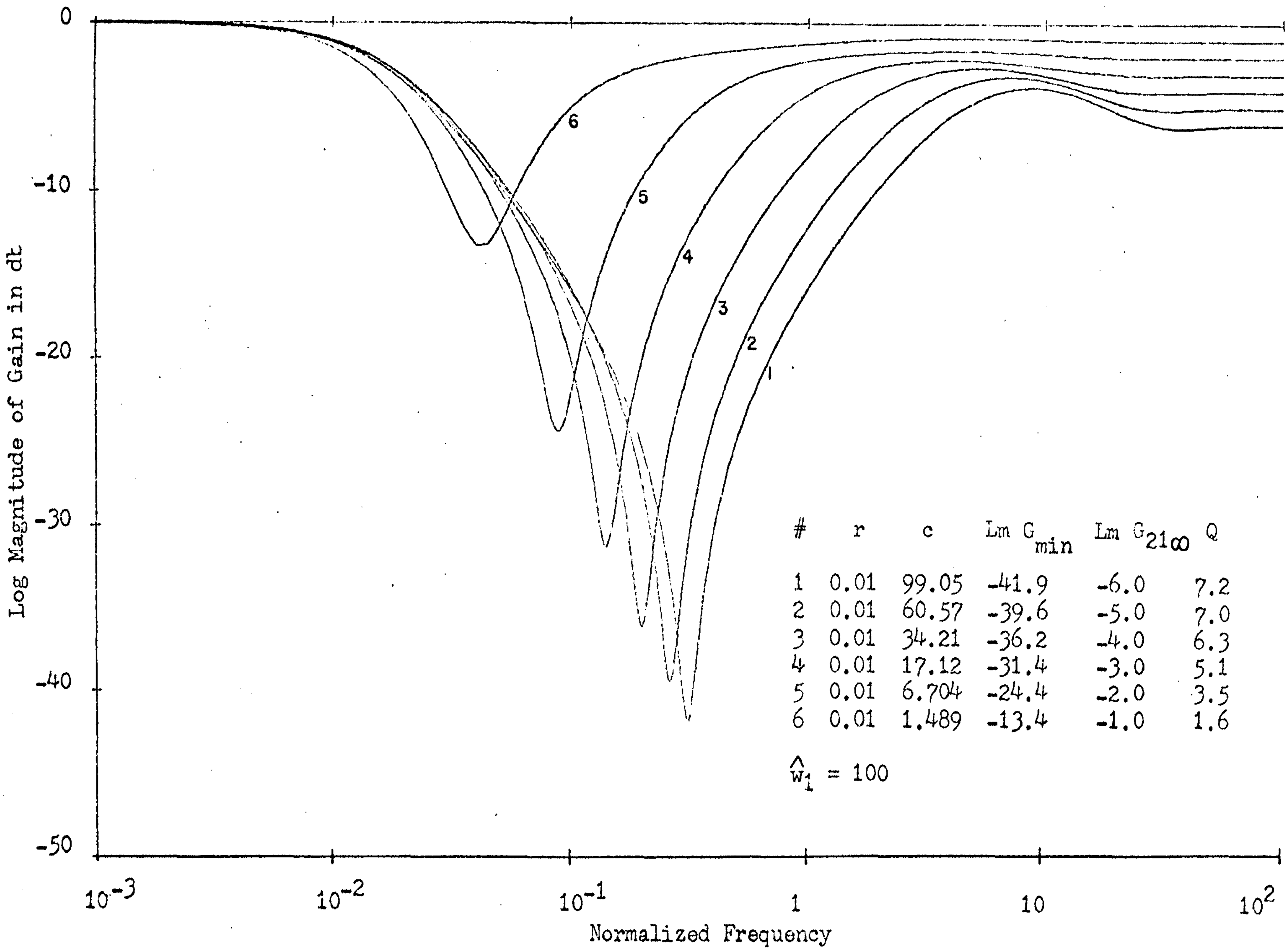


Figure 5-4. Gain characteristics of a two section distributed $\overline{RC}$ notch network with zero load. V_2= input voltage, V_1= output voltage. Q applies only if the network is used in a bandpass configuration.

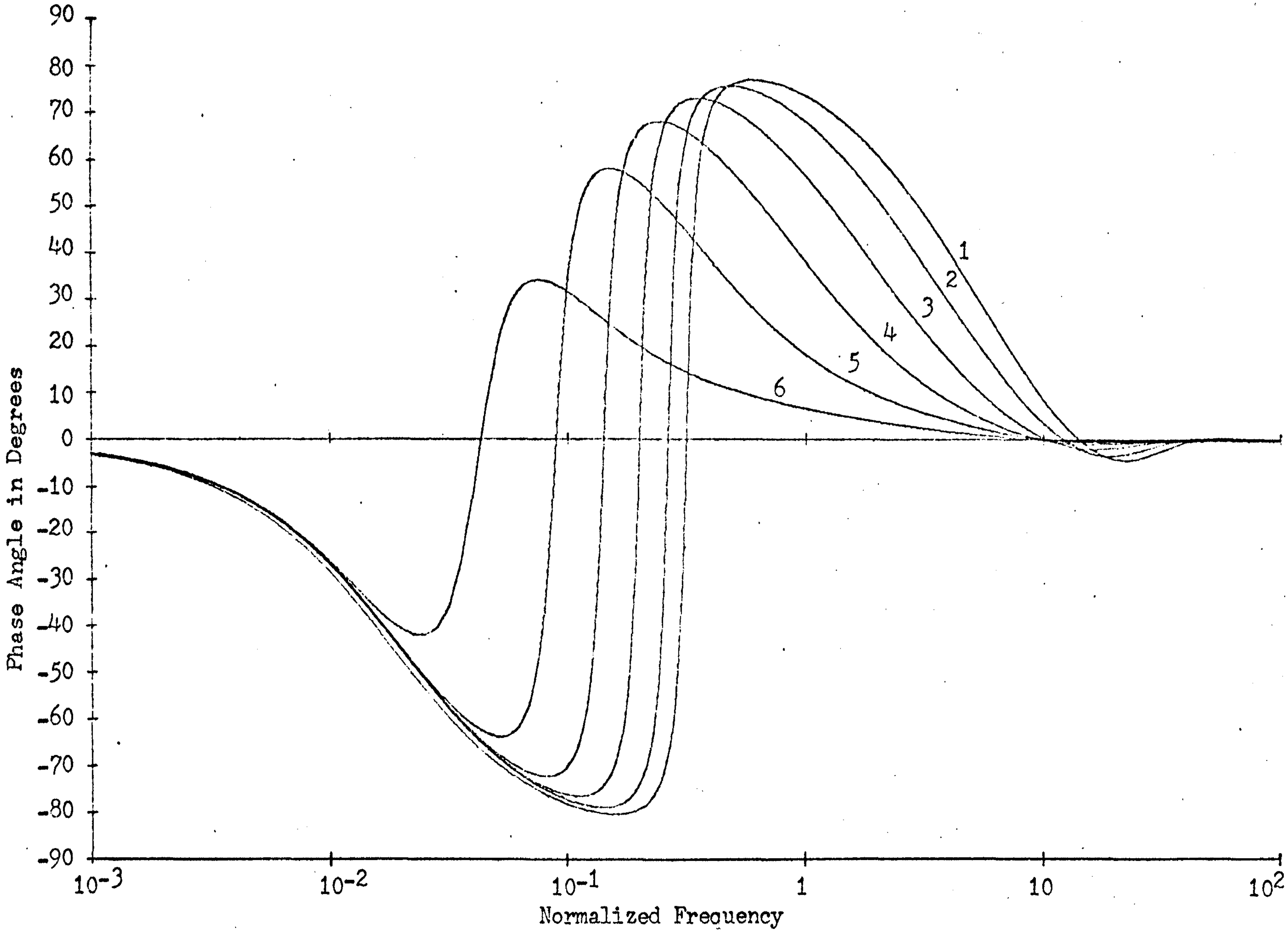


Figure 5-5. Phase characteristics of a two section distributed $\overline{RC}$ notch network with zero load. V_2= input voltage, V_1= output voltage.

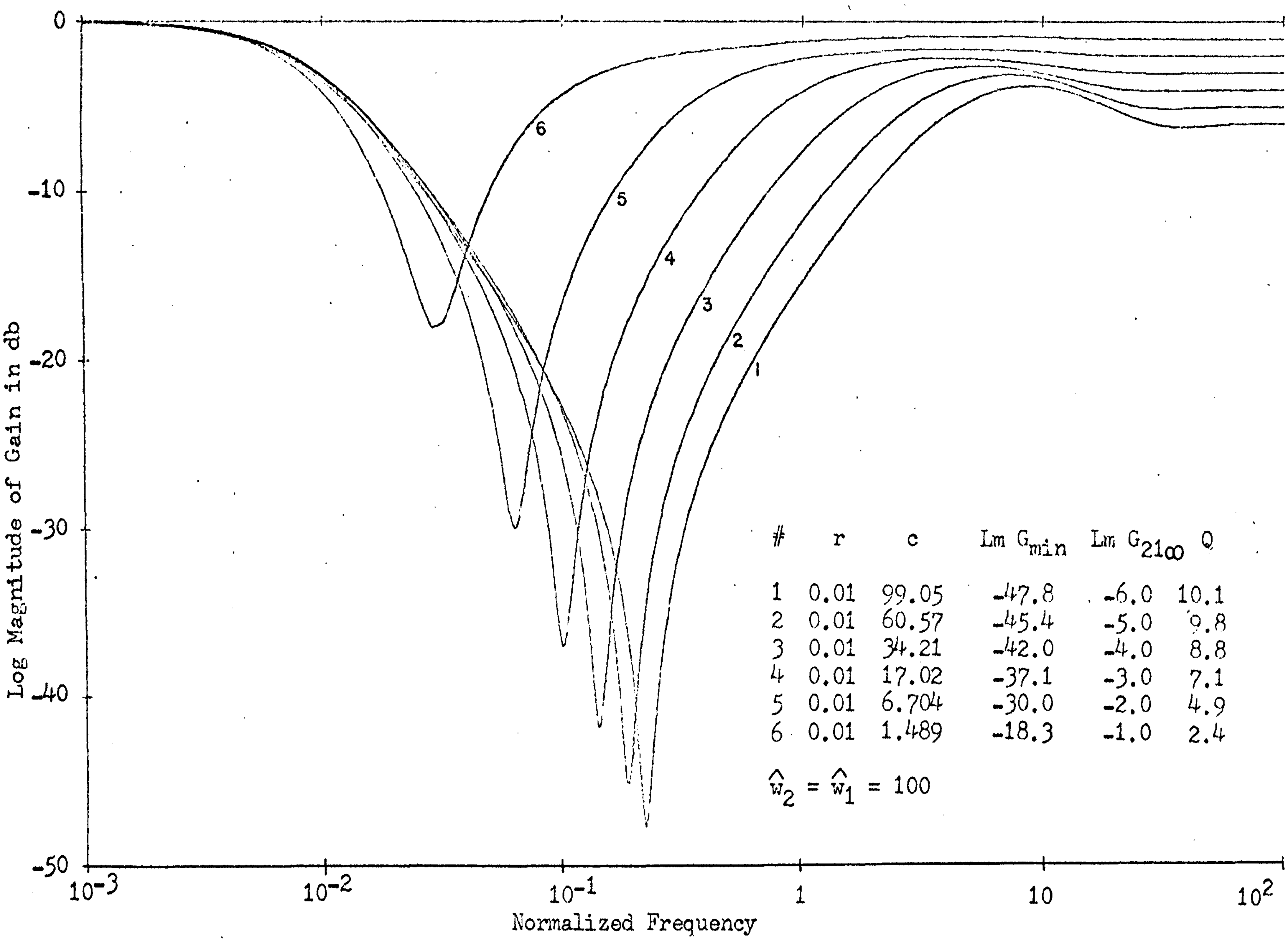


Figure 5-6. Gain characteristics of a three section distributed $\overline{RC}$ notch network with zero load. V_2= input voltage, V_1= output voltage. Q applies only if the network is used in a bandpass configuration.

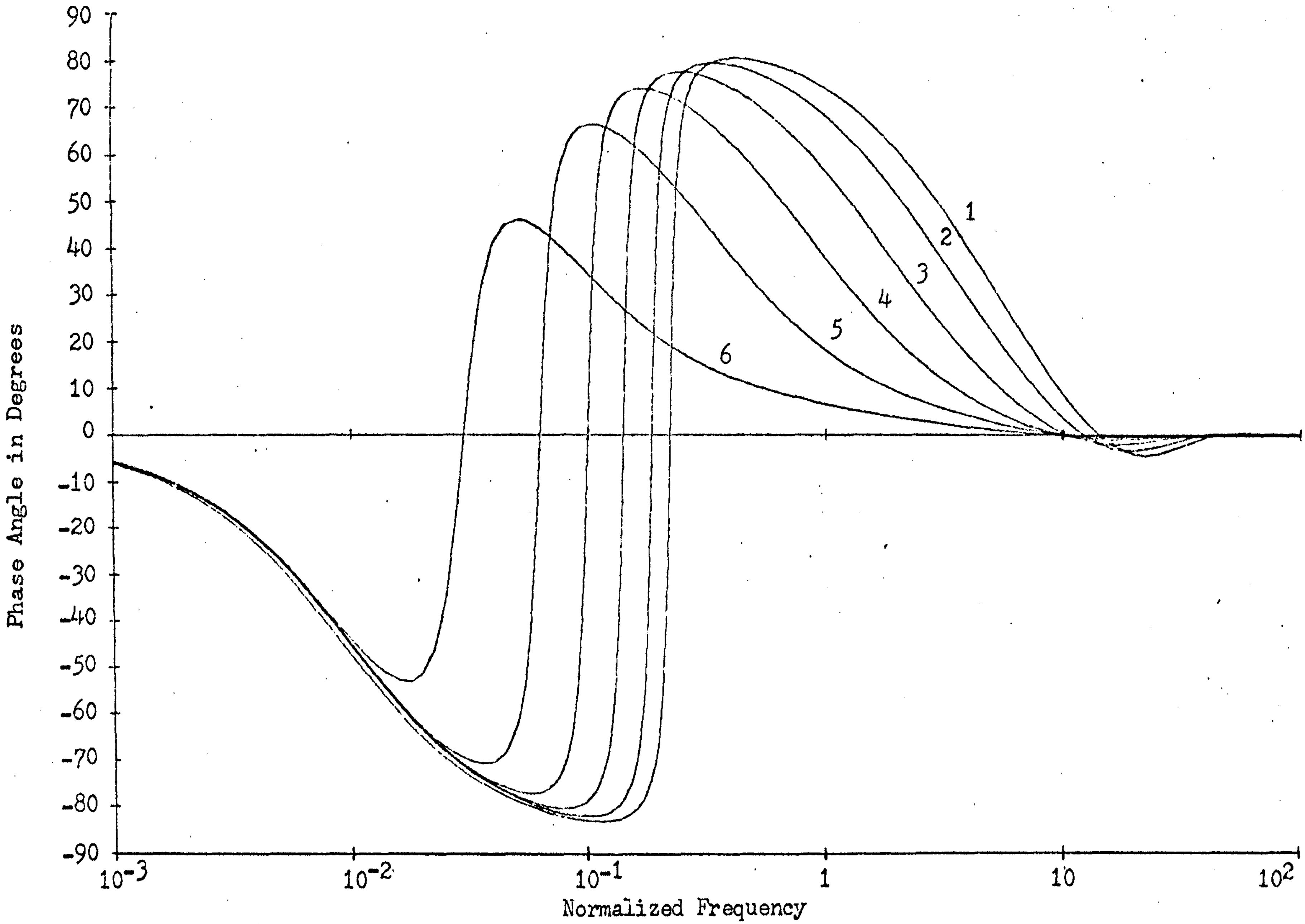


Figure 5-7. Phase characteristics of a three section distributed $\overline{RC}$ notch network with zero load. V_2= input voltage, V_1= output voltage.

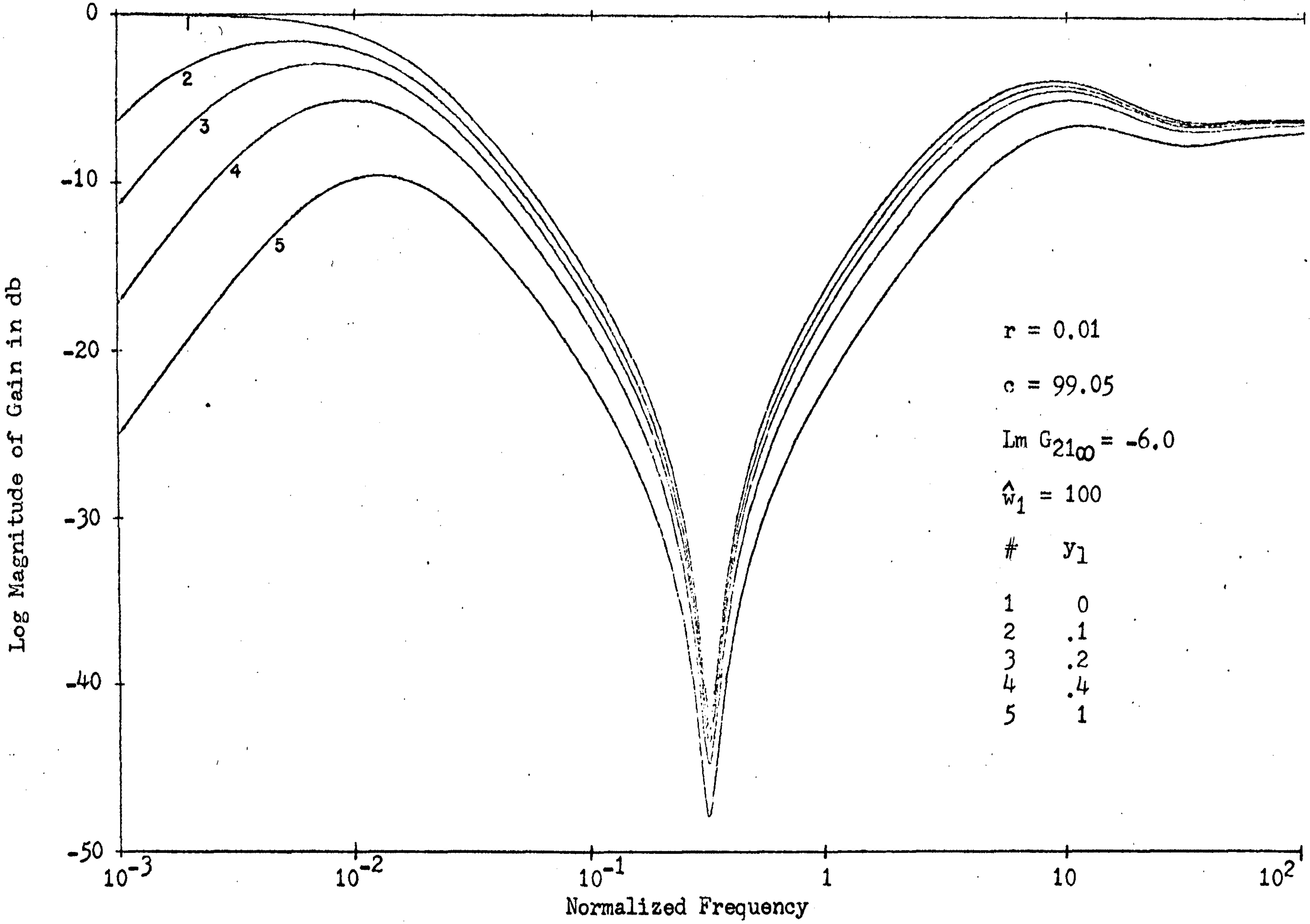


Figure 5-8. Gain characteristics of a two section distributed $\overline{RC}$ notch network with finite load. V_2= input voltage, V_1= output voltage.

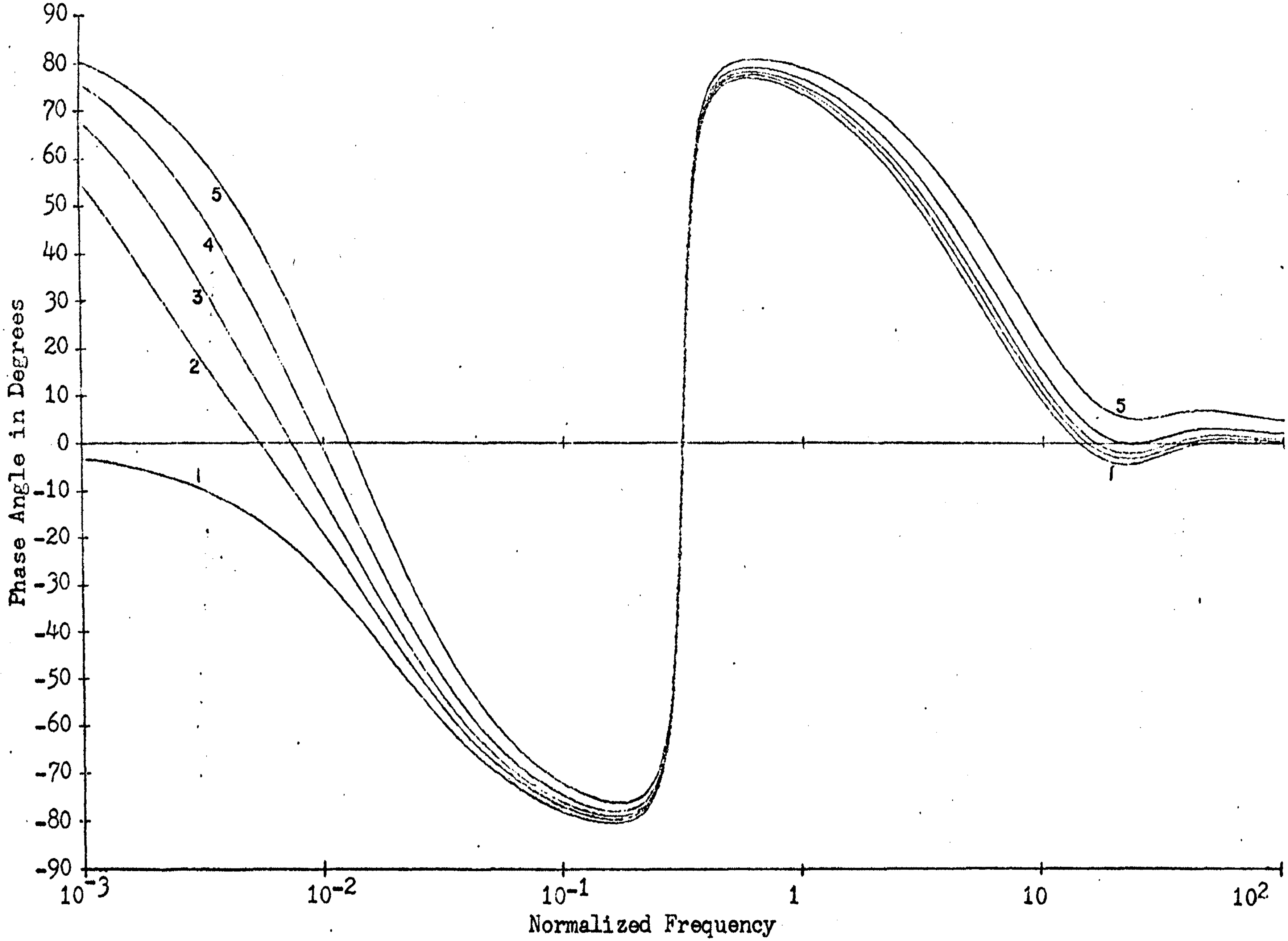


Figure 5-9. Phase characteristics of a two section distributed $\overline{RC}$ notch network with finite load. V_2= input voltage, V_1= output voltage.

the voltage gain be near unity at low frequency and much less than unity at high frequency. Looking at the asymptotic gain relationships of Table 3-1 on page 36, one can see that the cascaded distributed network with an end section open circuited would make the best low pass network. Also port 2 and port 1 of Figure 3-1 (page 27) should be used as input and output respectively. The equations of Table 3-1 which apply to this case show that this configuration has a voltage gain of unity at low frequency and $1/(1+\sqrt{rc})$ at high frequencies. Note that this is the same configuration used for the notch network. The low pass network was investigated for one, two, and three section cascaded networks while keeping the width, resistance, and capacitance ratios within the same bounds as before, that is, between 0.01 and 100.

Figures 5-10 through 5-17 show the results. These plots were made by a digital computer, plotter combination using frequency increments of $\Delta \log \eta = 0.1$. The sharp changes of slope in the immediate vicinity of the minimums or dips of the curves of Figures 5-12 and 5-16 are not network characteristics, but are caused rather by the size of the frequency increments chosen for plotting and the fact that the plotter drew straight lines between plotted points. In all of the Figures four different sets of values of r and c were chosen which would give a high frequency attenuation of 20 db. These sets of r and c were spaced from minimum to maximum r (or maximum to minimum c respectively) keeping both r and c within the bounds of 0.01 and 100. A fifth r and c set was chosen which would make the high frequency gain as small as possible within the permitted range of r and c. This asymptotic gain turned out to be about -40 db. In an attempt to make the magnitude of the slope of the log magnitude re-

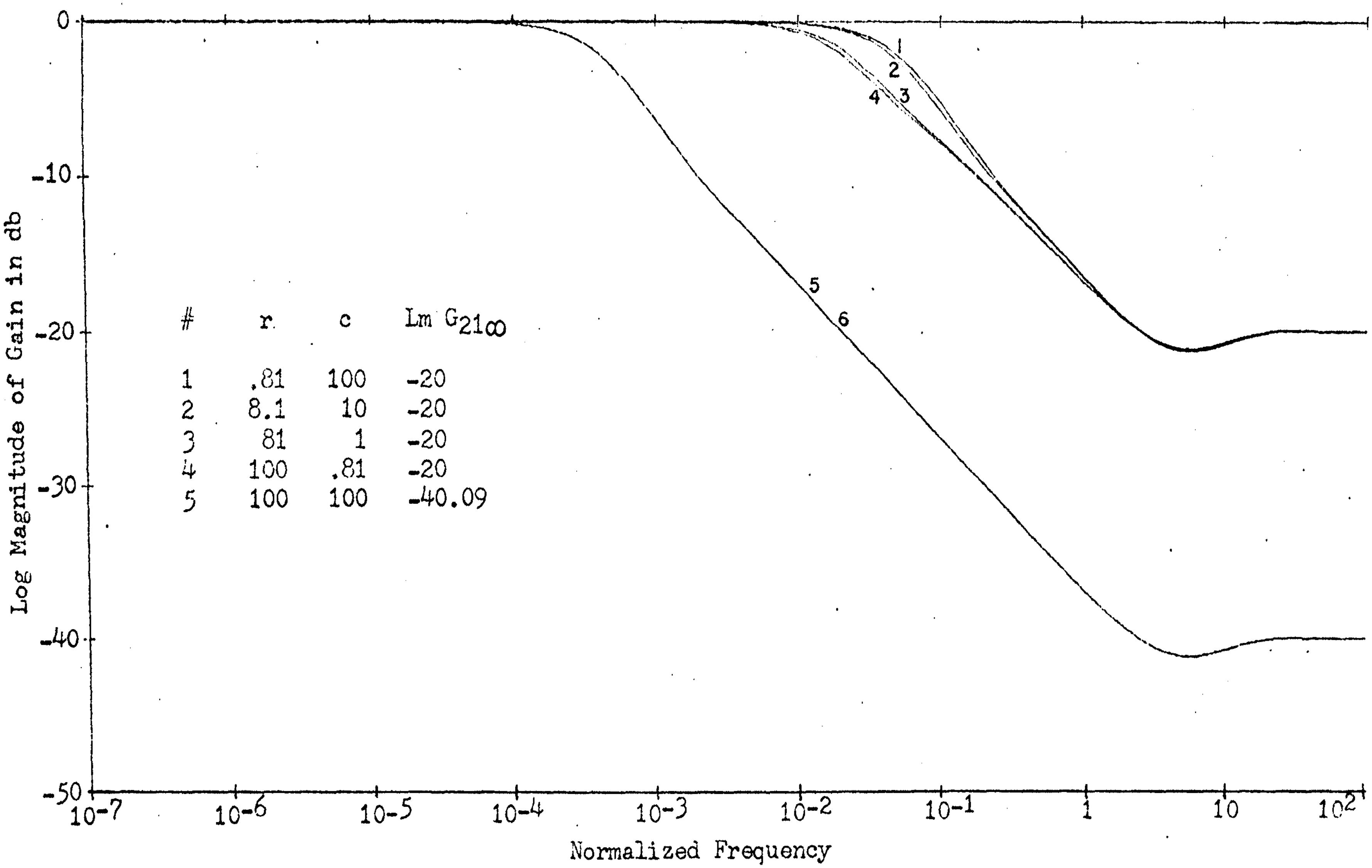


Figure 5-10. Gain characteristics of a one section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage.

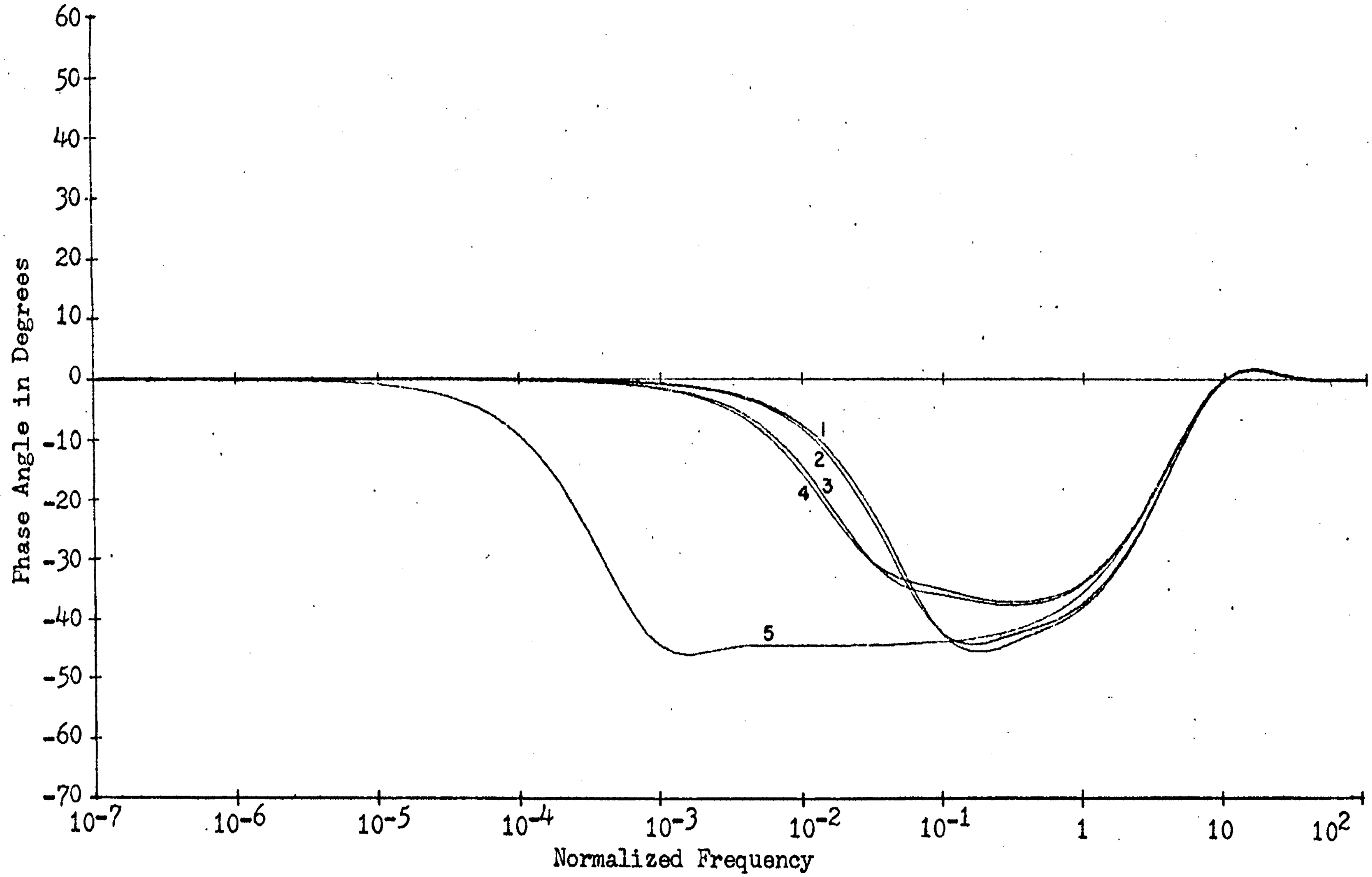


Figure 5-11. Phase characteristics of a one section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage.

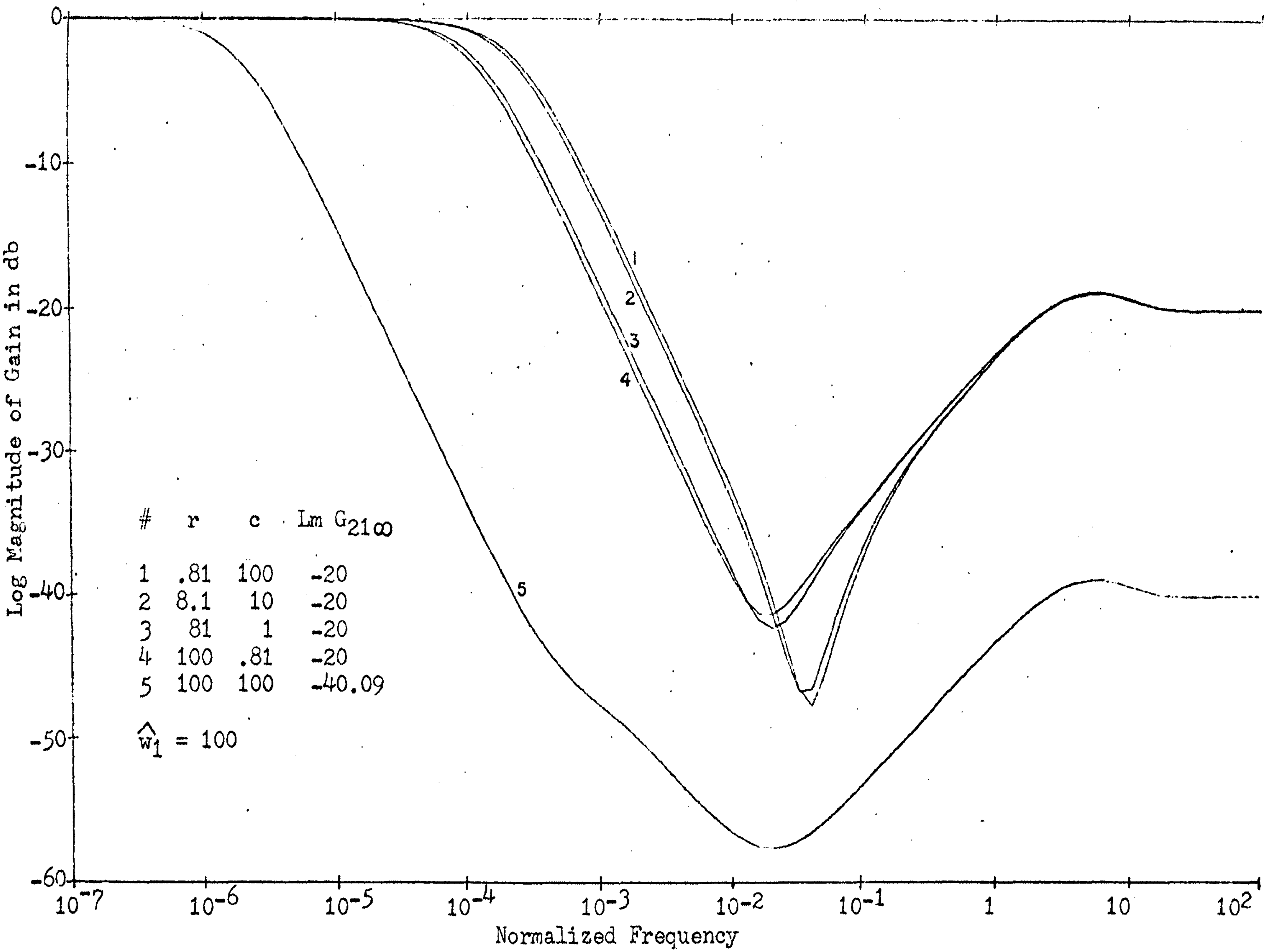


Figure 5-12. Gain characteristics of a two section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage, w_1 = 100.

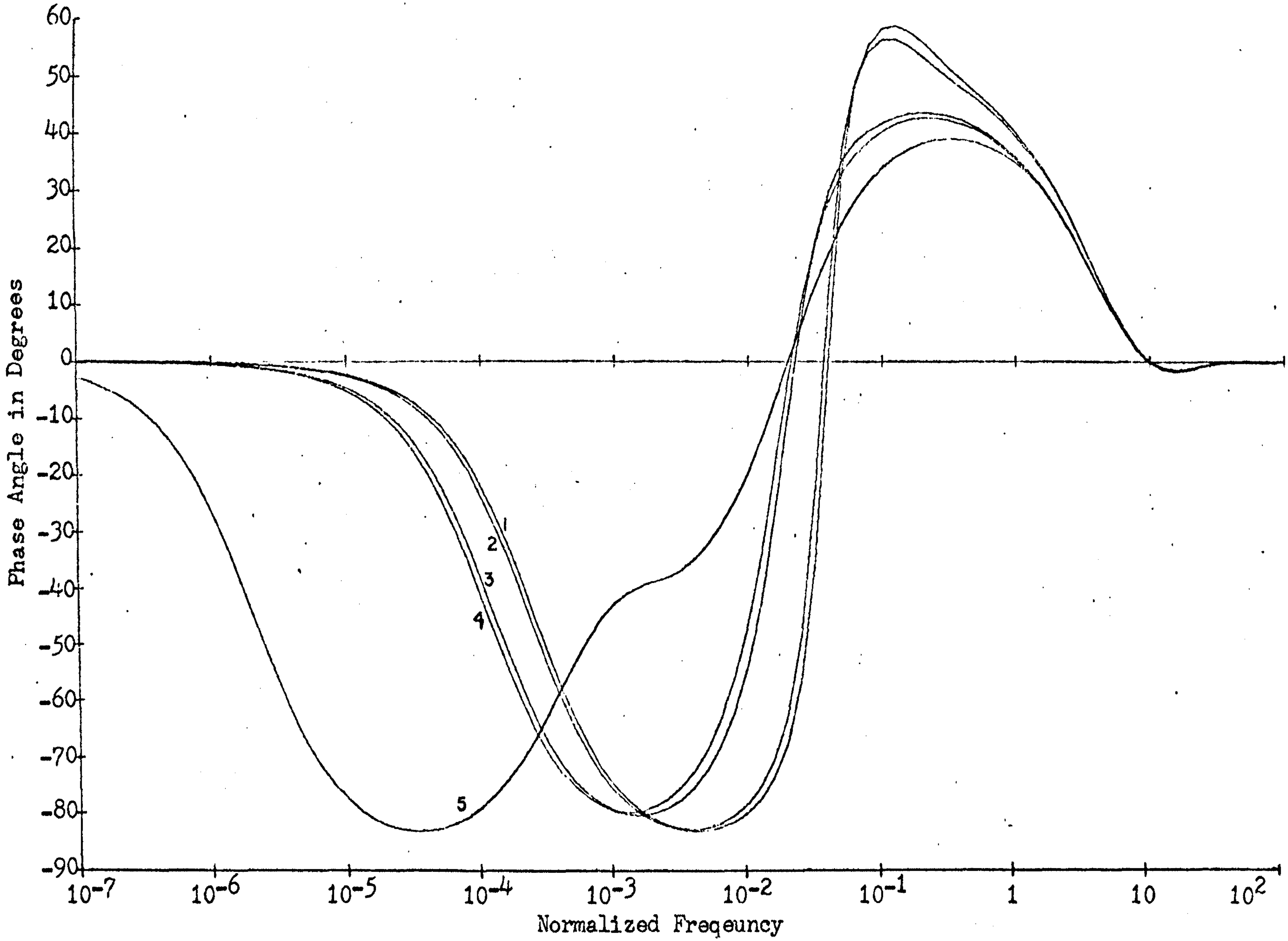


Figure 5-13. Phase characteristics of a two section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage, w_1 = 100.

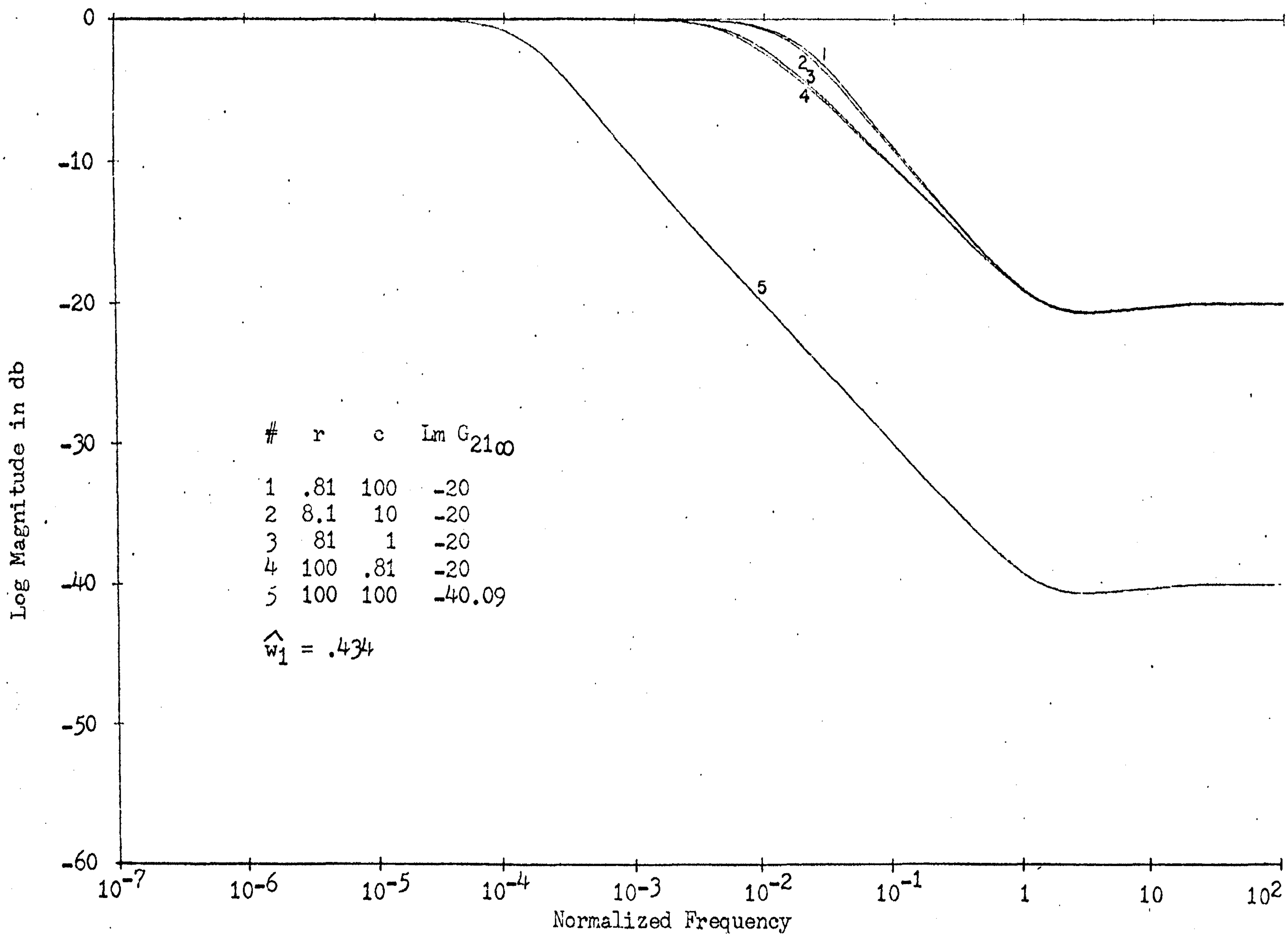


Figure 5-14. Gain characteristics of a two section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage, w_1 = 0.434.

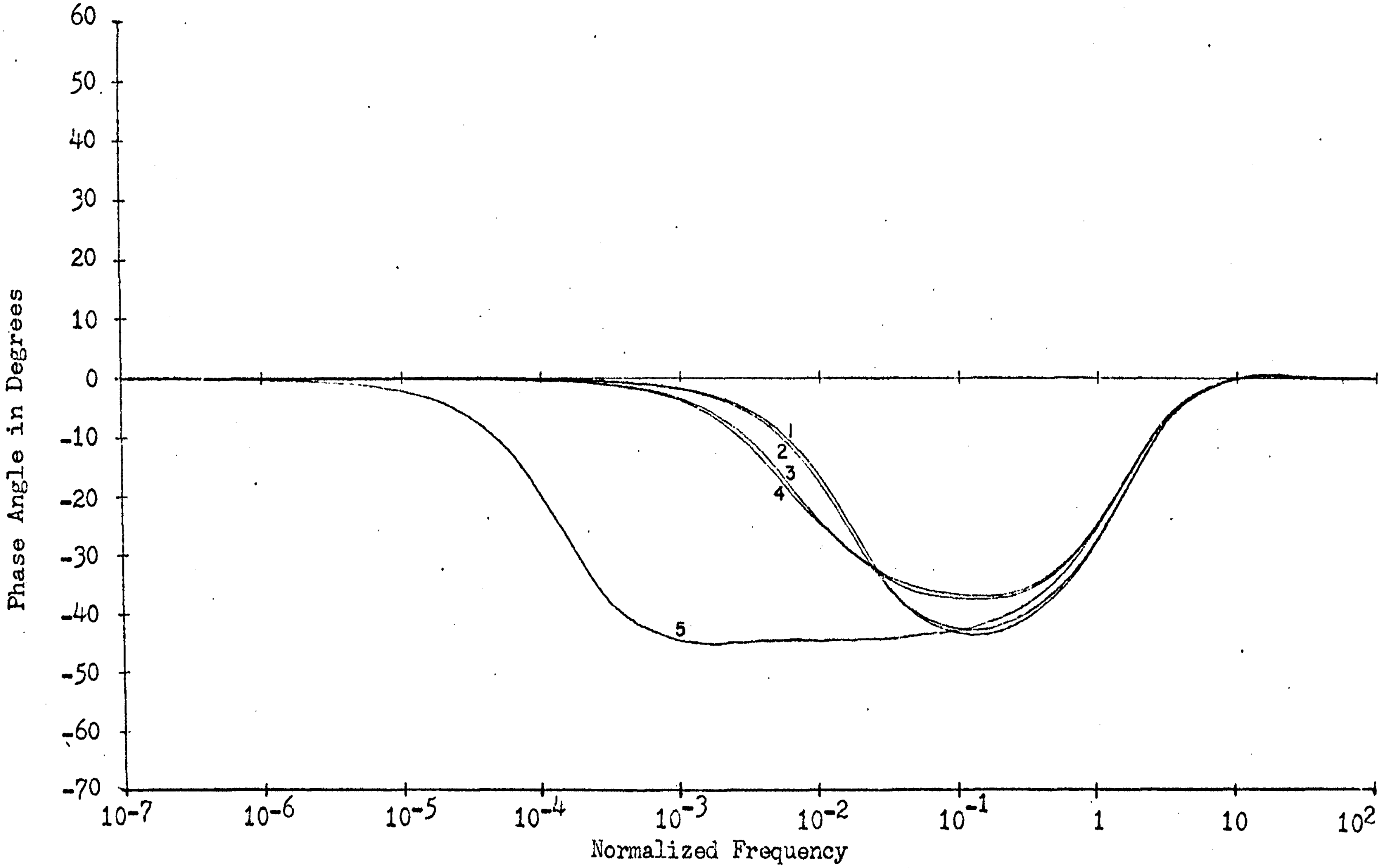


Figure 5-15. Phase characteristics of a two section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage, w_1 = 0.434.

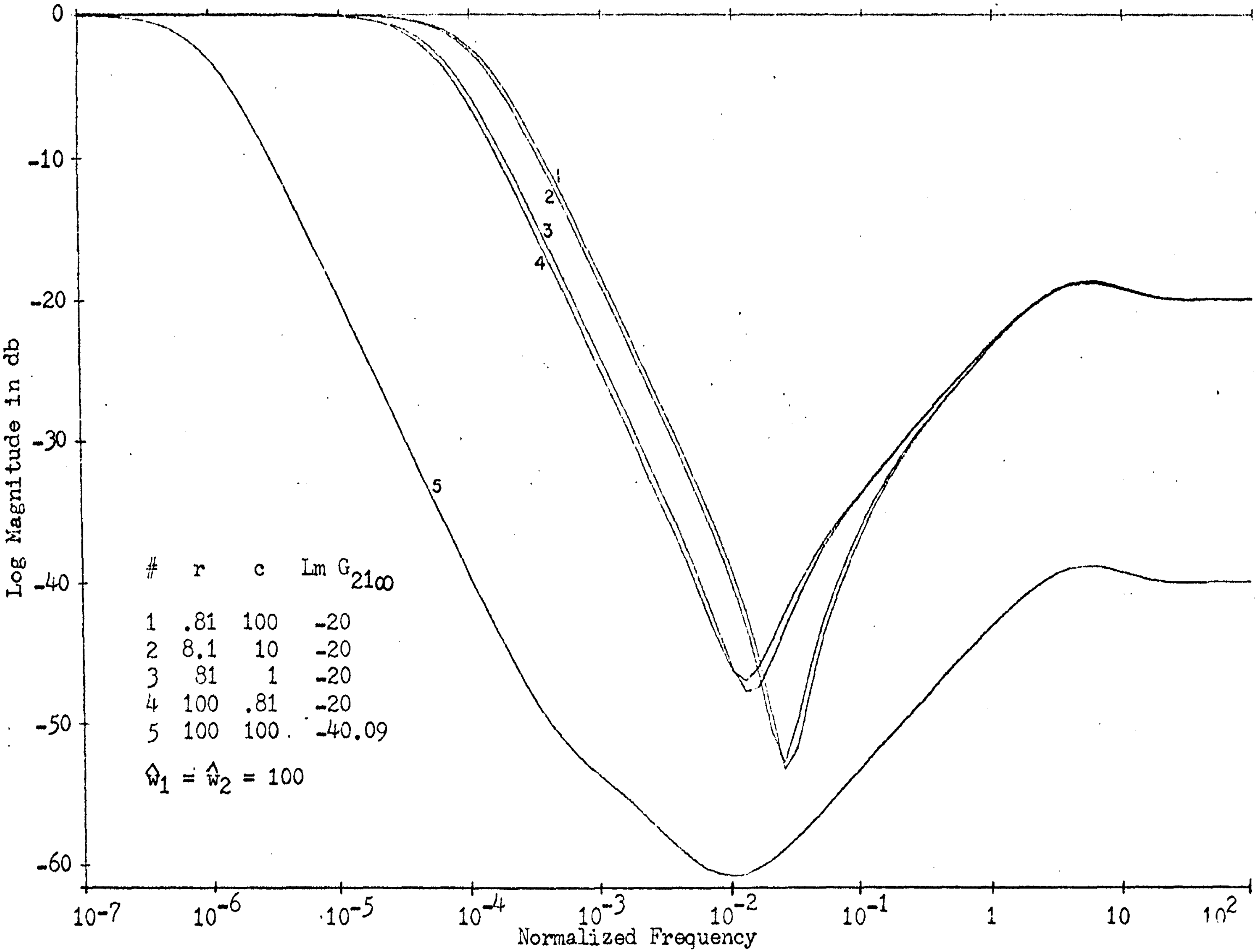


Figure 5-16. Gain characteristics of a three section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage.

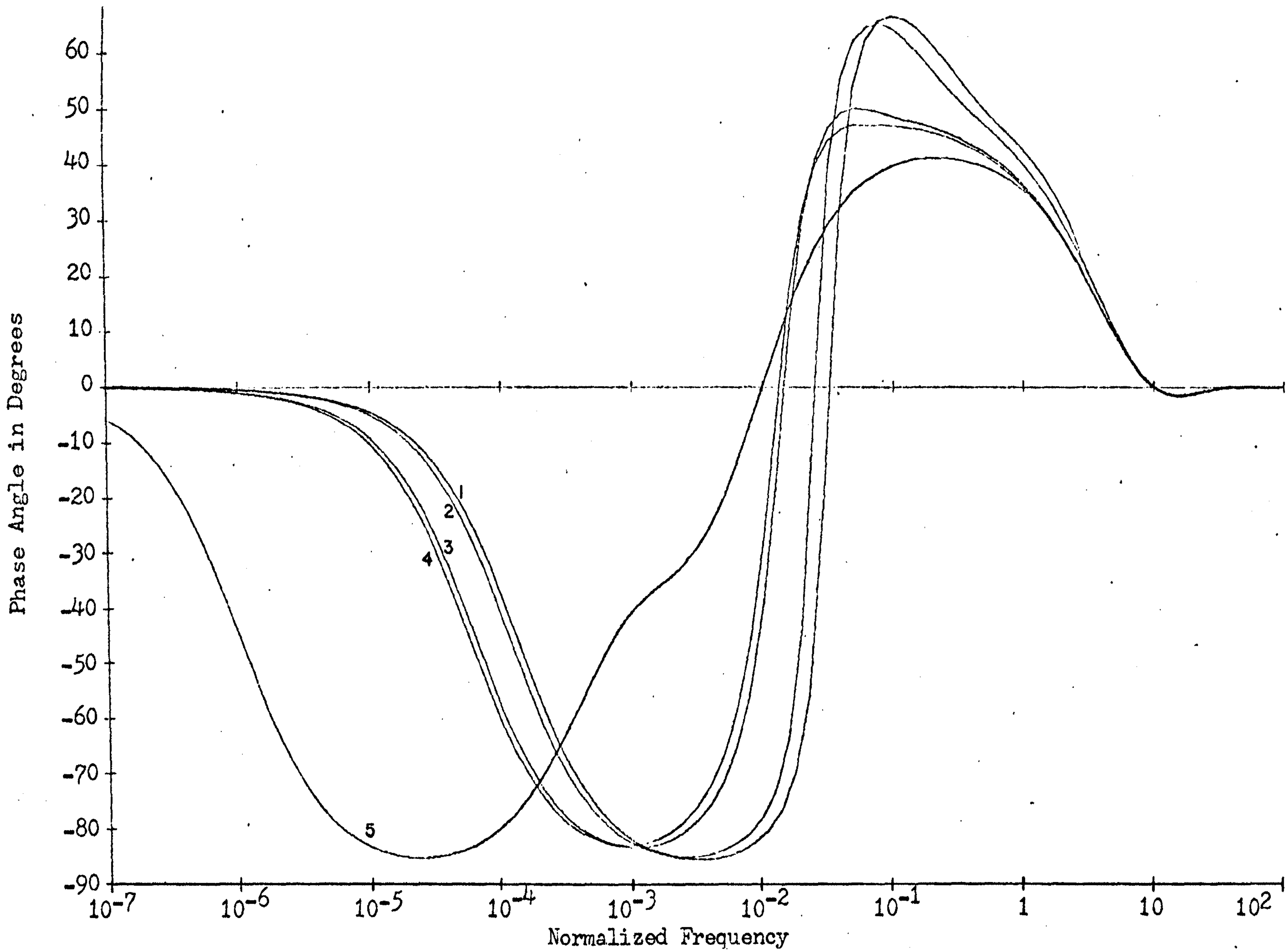


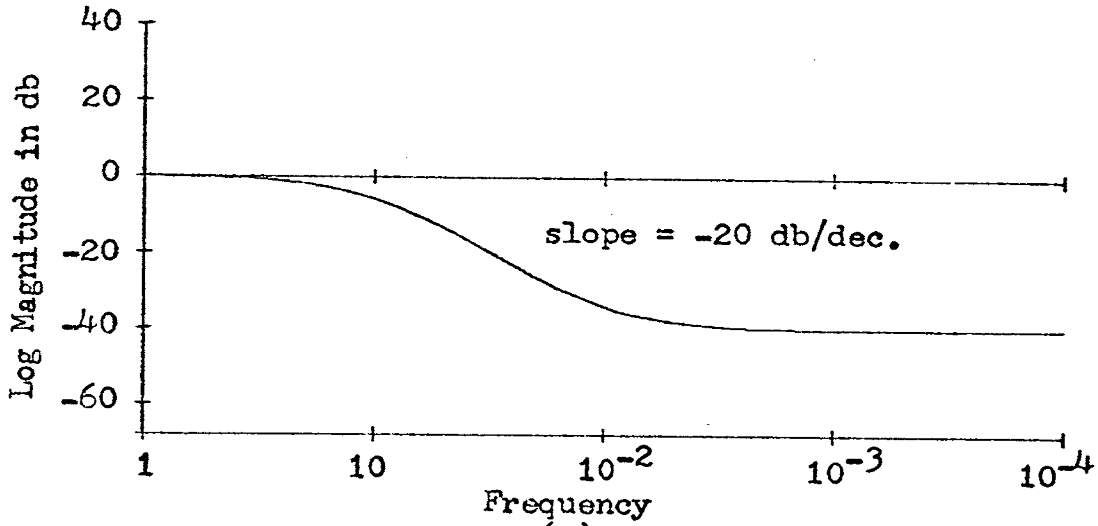
Figure 5-17. Phase characteristics of a three section distributed $\overline{RC}$ low pass network with zero load. V_2= input voltage, V_1= output voltage.

sponse curves as great as possible at intermediate frequencies, the width ratio of the two and three section networks was chosen to be the same that produced the best notch filter of Section 5.2. This width ratio, $\hat{w}_1=\hat{w}_2=100$, produced a slope of -20 db per decade (Figures 5-12 and 5-16). As radical changes in phase occurred at the point on these curves where the gain dips are extreme (Figures 5-13 and 5-17) a two section low pass network was investigated in which the width ratio was $\hat{w}_1=0.434$. Figures 5-14 and 5-15 show that this caused a more moderate phase characteristic, but the slope of the gain curve at intermediate frequencies was reduced to -10 db per decade.

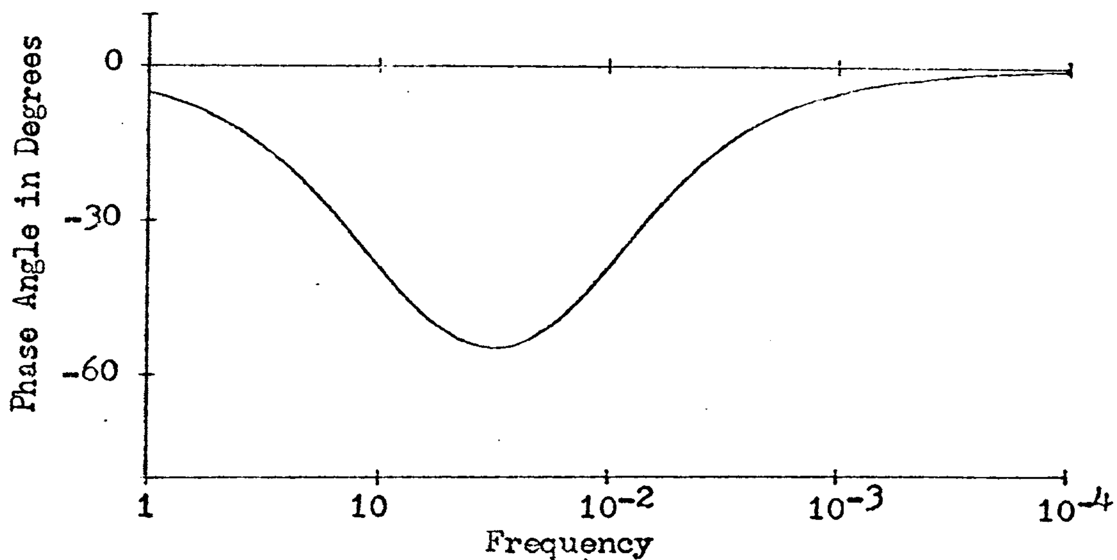
The networks of Figures 5-10 and 5-14 are also suitable for use as phase lag compensators. The response characteristics for the typical lumped element phase lag compensator with voltage gain $G_c(s) = (1+s\tau)/(1+s\alpha\tau)$, $\alpha>1$, are shown in Figure 5-18 (Ref. 5, pp 295 and 314). If G_u is the uncompensated open loop voltage gain of a unity feedback system, a properly designed lag componsator has the effect of rotating the phasor of the polar plot of $G_u(j\omega)$ clockwise in the vicinity of the -1.0 point and reducing the gain at large frequencies while leaving the low frequency end of the plot unchanged. Because all of the curves of Figures 15-10 and 15-14 meet these requirements, the networks they represent can be used as lag compensators.

5.4 High Pass Network

A passive high pass network should have a gain much less than unity at low frequencies and as near unity as possible at high frequencies. A search of the asymptotic gain equations of Table 3-1 on page 36 shows that for the case of the cascaded networks with the right end



(a).



(b).

Figure 5-18. Typical log magnitude and phase plot of a lumped element RC lag compensator. (a). Log magnitude plot. (b). Phase plot.

short circuited the gain at low frequency is directly proportional to frequency regardless of which port is used as input or output. Thus r and c should be selected so that the high frequency asymptotes are near unity. Several possible combinations of r and c were investigated for the one section case first using port 1 as the input, then using port 2 as the input, with the remaining port open circuited in both cases. The results are shown in Figures 5-19 through 5-22. As expected, the low frequency portion of each curve has a positive slope of 20 db per decade. Using port 1 as input Figure 5-19 shows that the best results are obtained with r=100 and c=0.01. With port 2 as input the best results occur for r=c=.01. For these two cases the attenuation at high frequencies is less than two tenths of a decibel.

Another type of high pass network, especially suited for use as a coupling network can be constructed by adding a resistive load to an all pass network. The design procedure for the all pass network is almost identical to that of the notch filter, the only difference being that the computer program should minimize instead of maximize the notch. The same network configuration can be used for the all pass coupling network as for the notch filter, i.e. the right end section of the cascaded network is open circuited and terminals 2-2' of Figure 3-1 on page 27 are used as the input. A one section and a two section network were investigated with high frequency gains G_{21_∞} of -0.5 to -0.1 decibels. Since the results were best for small c and improved as r became lower, the case with r and c both at the lower stop limits (0.01) was also investigated. These results are shown in the graphs of Figures 5-23 through 5-26. Because the

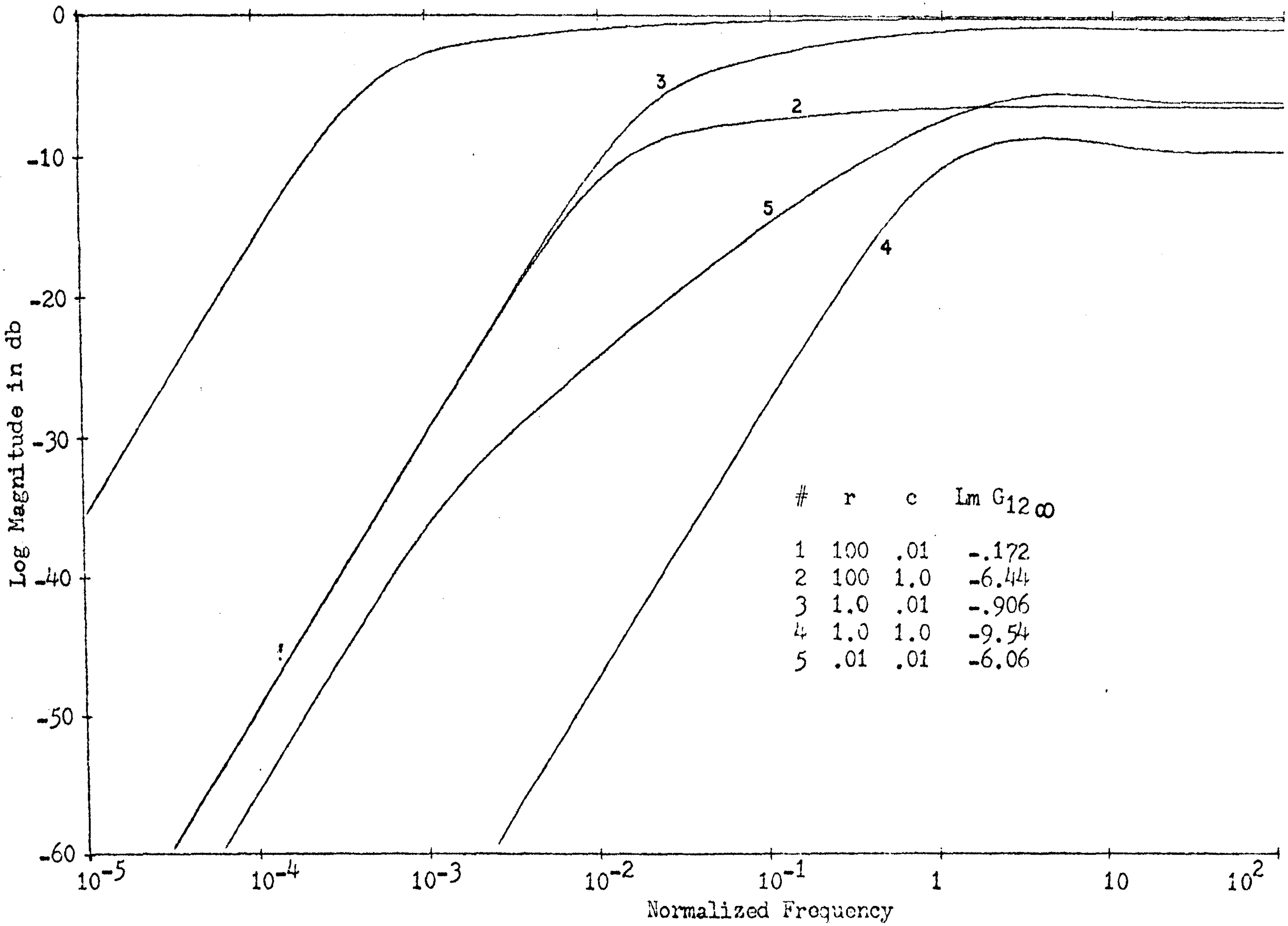


Figure 5-19. Gain characteristics of a one section distributed $\overline{RC}$ high pass network with zero load. V_1= input voltage, V_2= output voltage.

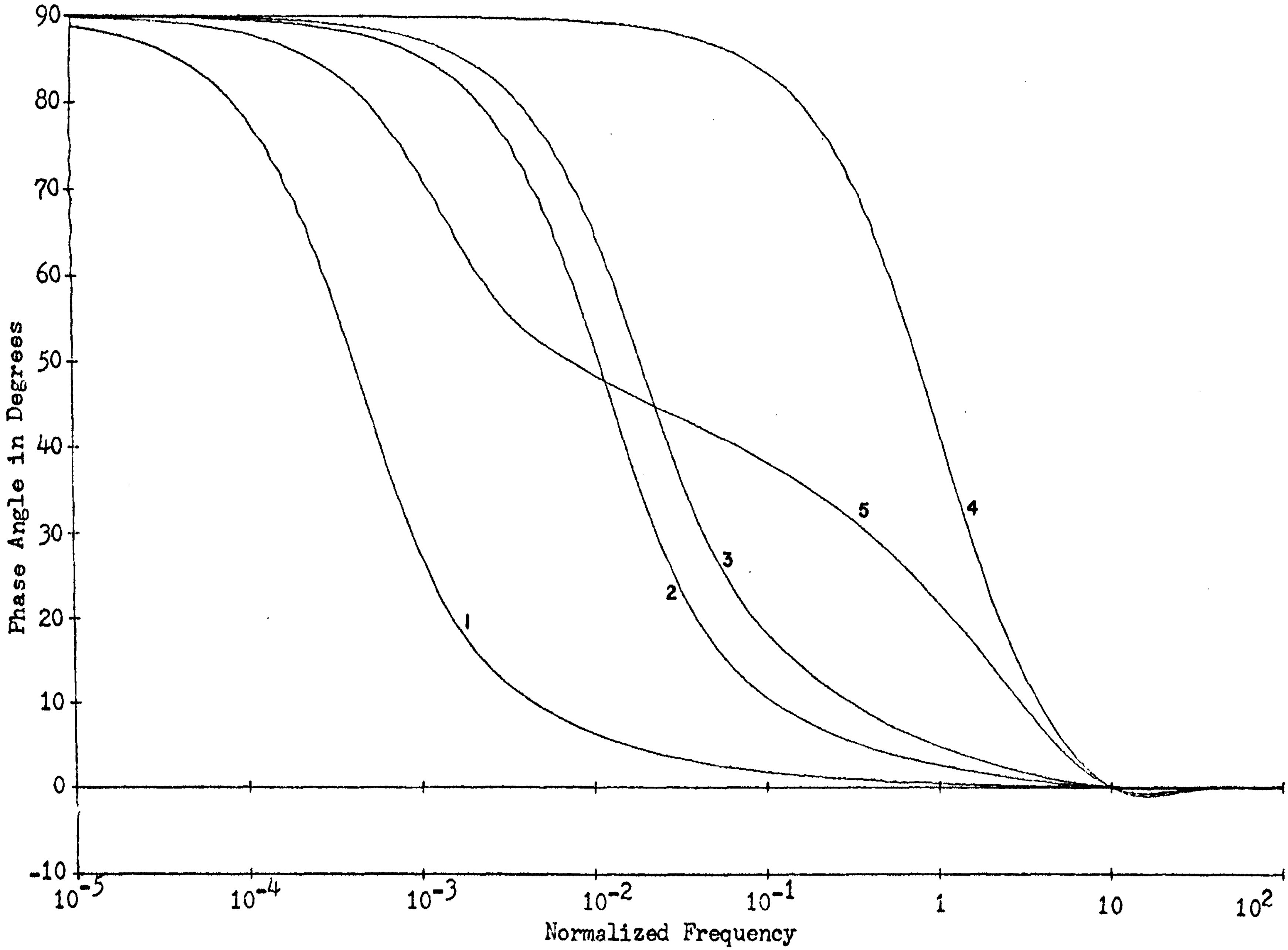


Figure 5-20. Phase characteristics of a one section distributed $\overline{RC}$ high pass network with zero load. V_1= input voltage, V_2= output voltage.

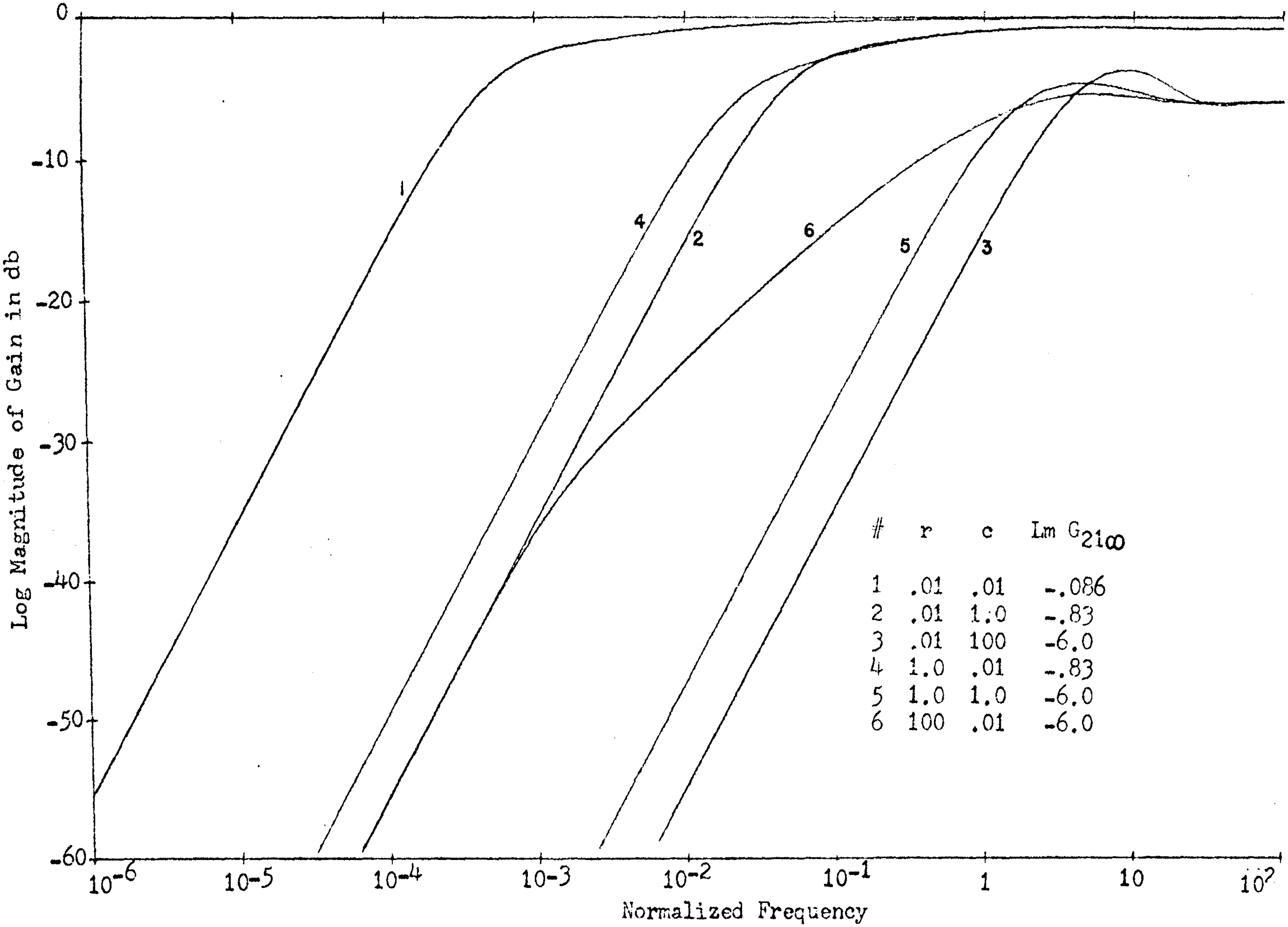


Figure 5-21. Gain characteristics of a one section distributed $\overline{RC}$ high pass network with zero load. V_2= input voltage, V_1= output voltage.

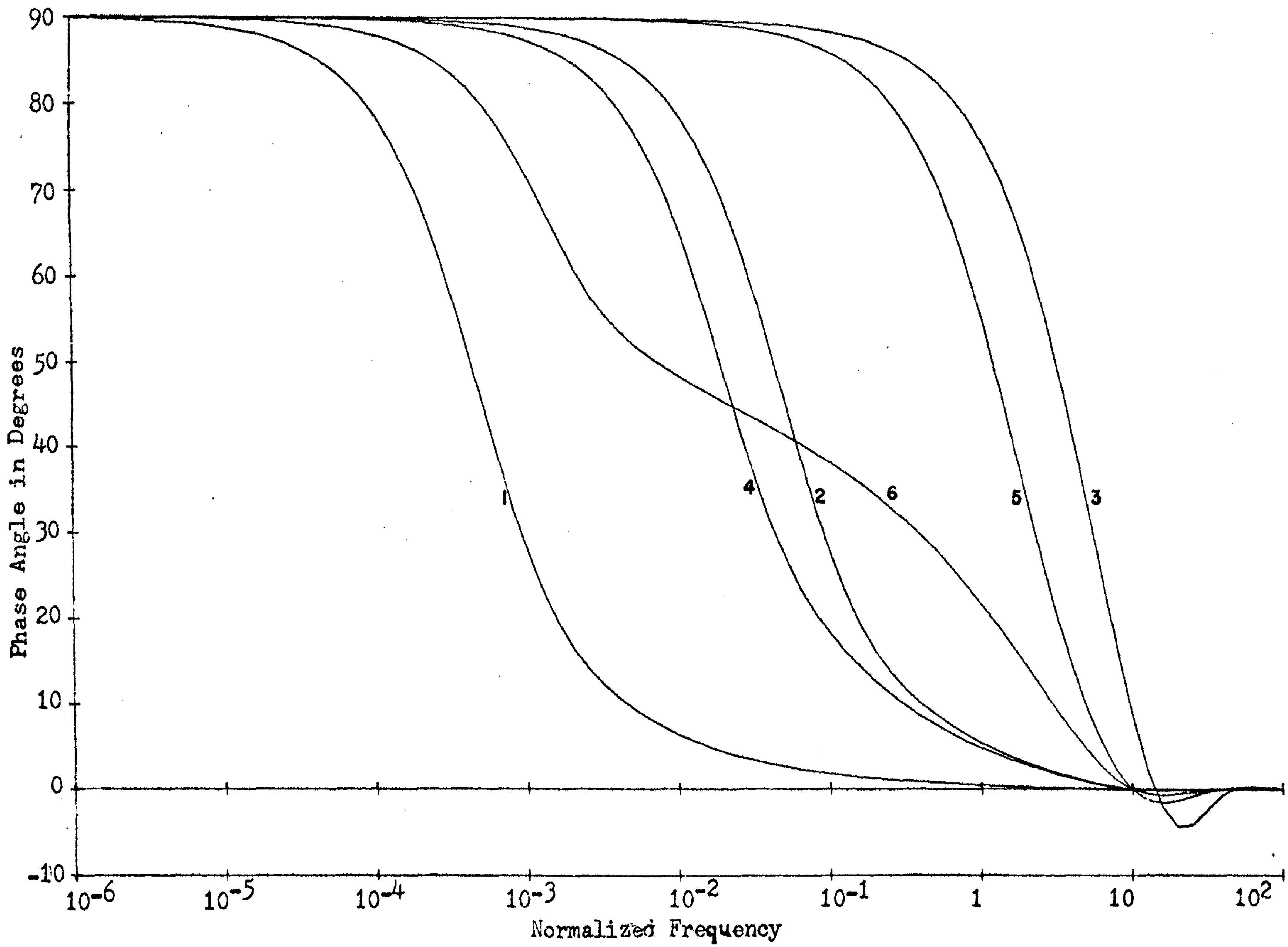


Figure 5-22. Phase characteristics of a one section distributed $\overline{RC}$ high pass network with zero load. V_2= input voltage, V_1= output voltage.

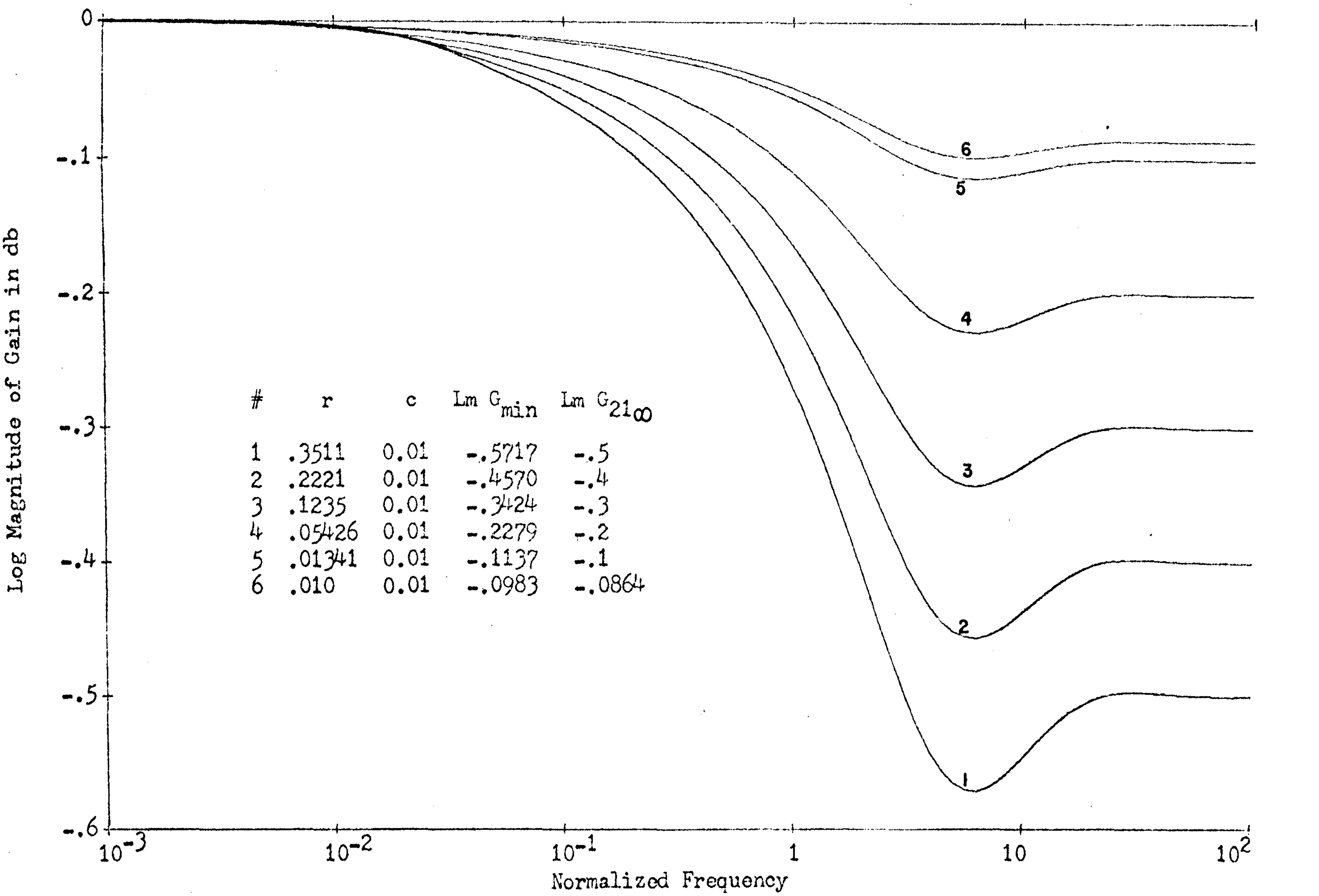


Figure 5-23. Gain characteristics of a one section distributed $\overline{RC}$ all pass network with zero load. V_2= input voltage, V_1= output voltage.

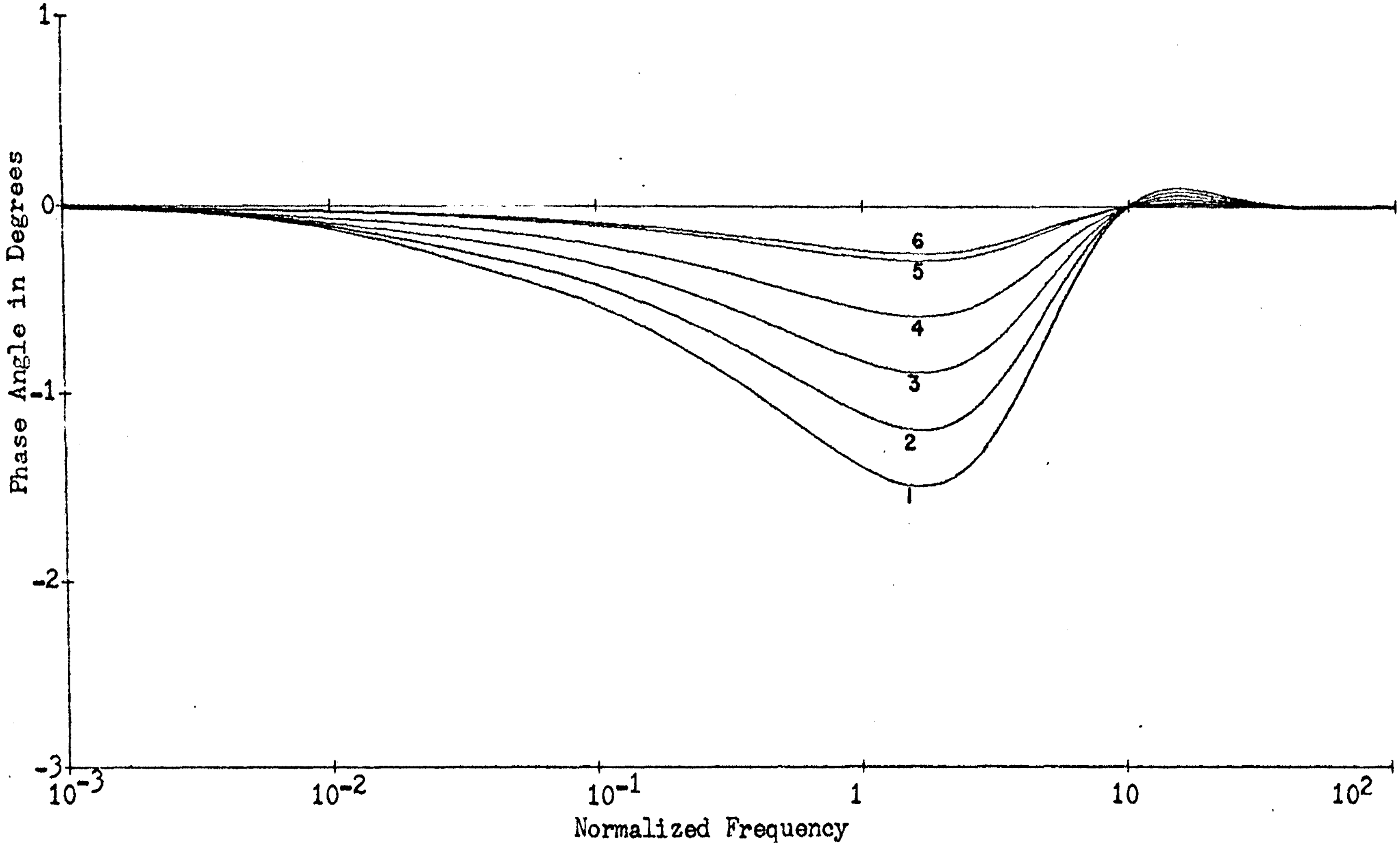


Figure 5-24. Phase characteristics of a one section distributed $\overline{RC}$ all pass network with zero load. V_2= input voltage, V_1= output voltage.

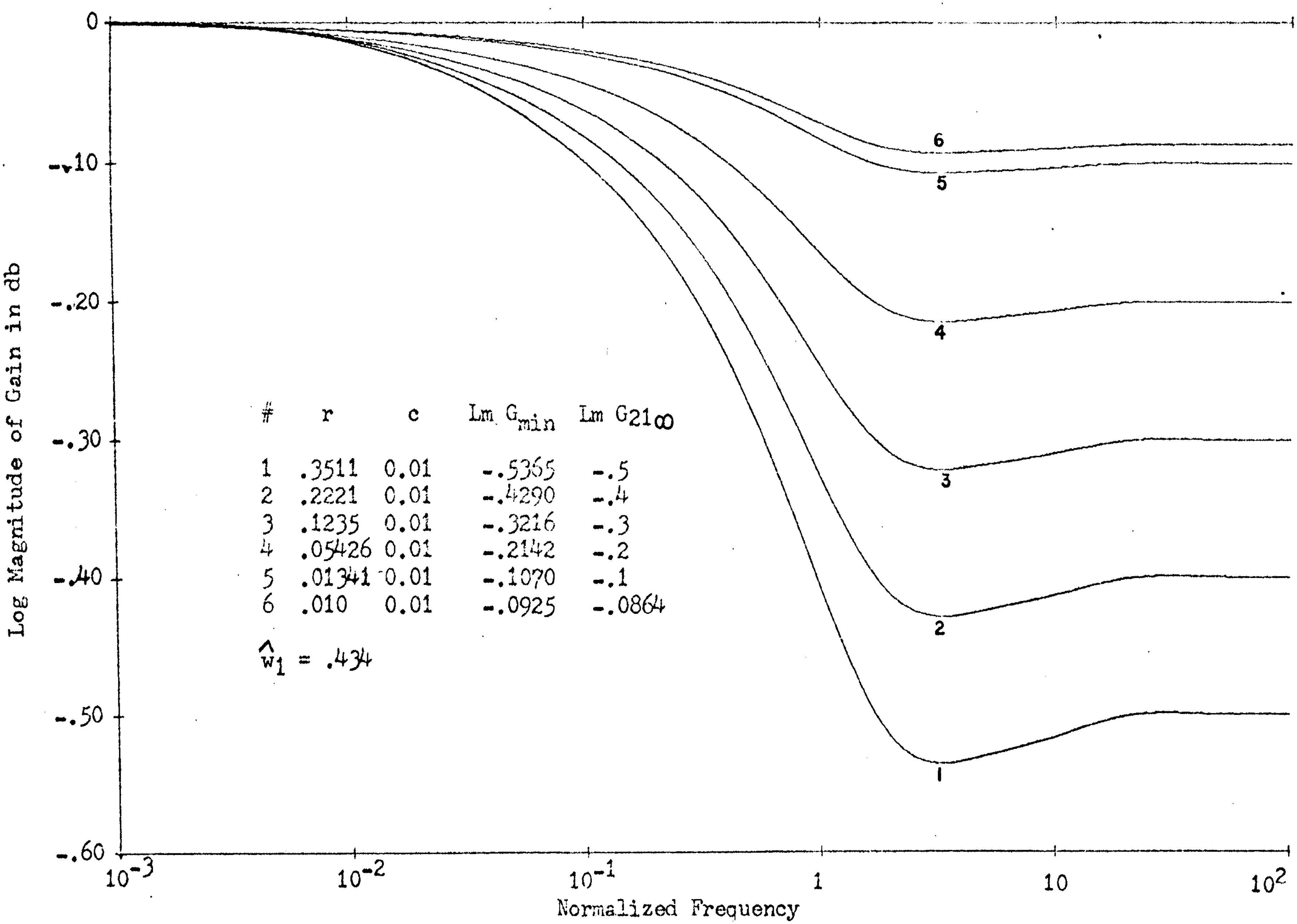


Figure 5-25. Gain characteristics of a two section distributed $\overline{RC}$ all pass network with zero load. V_2= input voltage, V_1= output voltage.

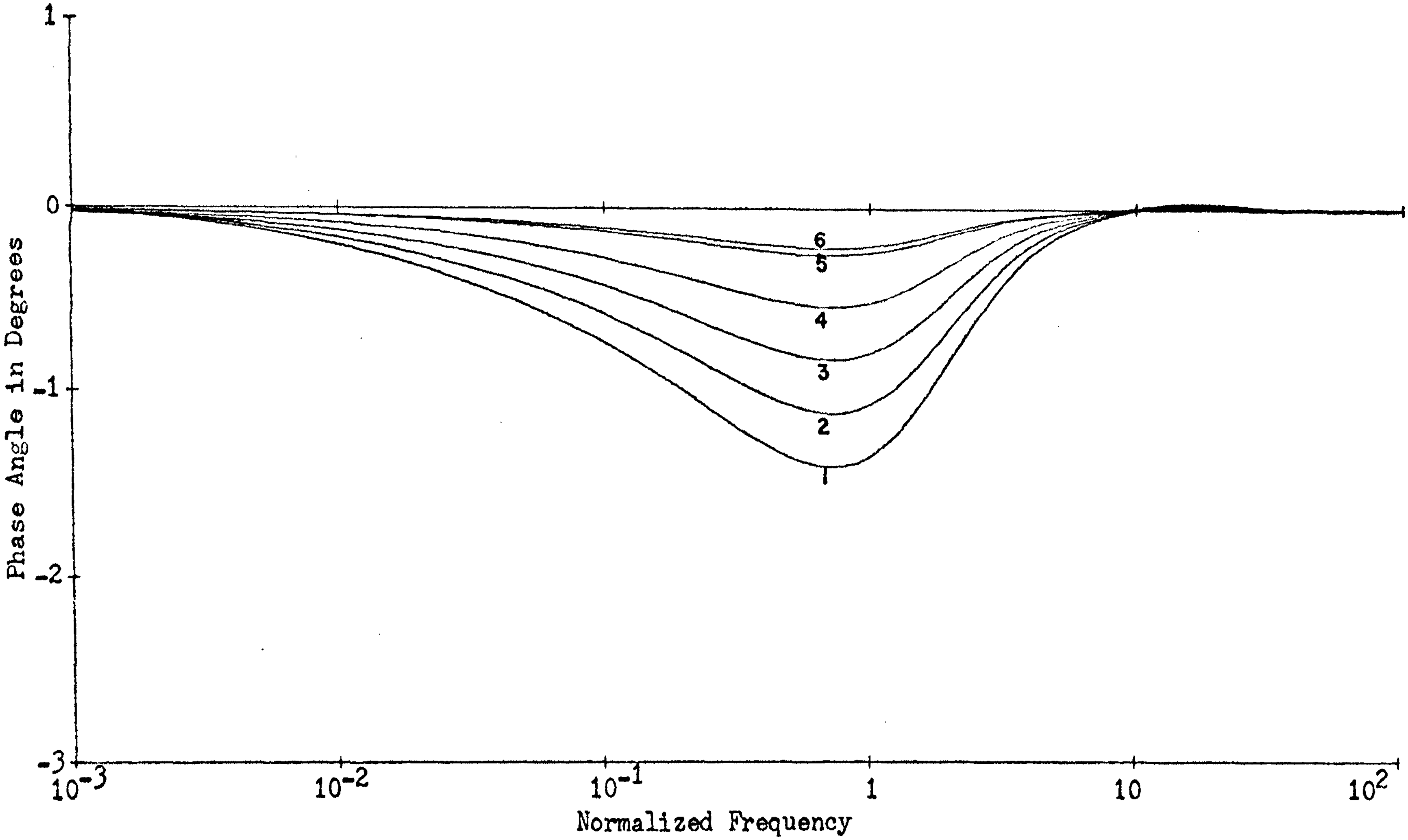


Figure 5-26. Phase characteristics of a two section distributed $\overline{RC}$ all pass network with zero load. V_2= input voltage, V_1= output voltage.

results of the two section network were not significantly better than those of the one section (0.0058 db improvement), a three section network was not investigated. The unloaded one section coupling network could be made to have a minimum gain of -0.0983 db with phase distortion less than 0.3 degrees. Figures 5-27 and 5-28 show the log magnitude and phase plots respectively for the best of the one section networks of Figures 5-23 and 5-24, but under loaded conditions. Curve #1 of these Figures corresponds to the no load case. Under loaded conditions the gain equation becomes

$$G_{21} = \frac{-y_{12}}{y_{11}+y_{l}} \qquad (5.2)$$

where y_l is the load admittance. The equations in the first column of Table 3-1 show that at low frequency y_{12} and y_{11} are both proportional to frequency. Therefore if y_l is a purely resistive load, the demoninator of equation (5.2) will be approximately equal to y_l at low frequency making G_{21} proportional to frequency. Thus as is shown by Figure 5-27, if the all pass network works into a finite load it becomes a high pass network.

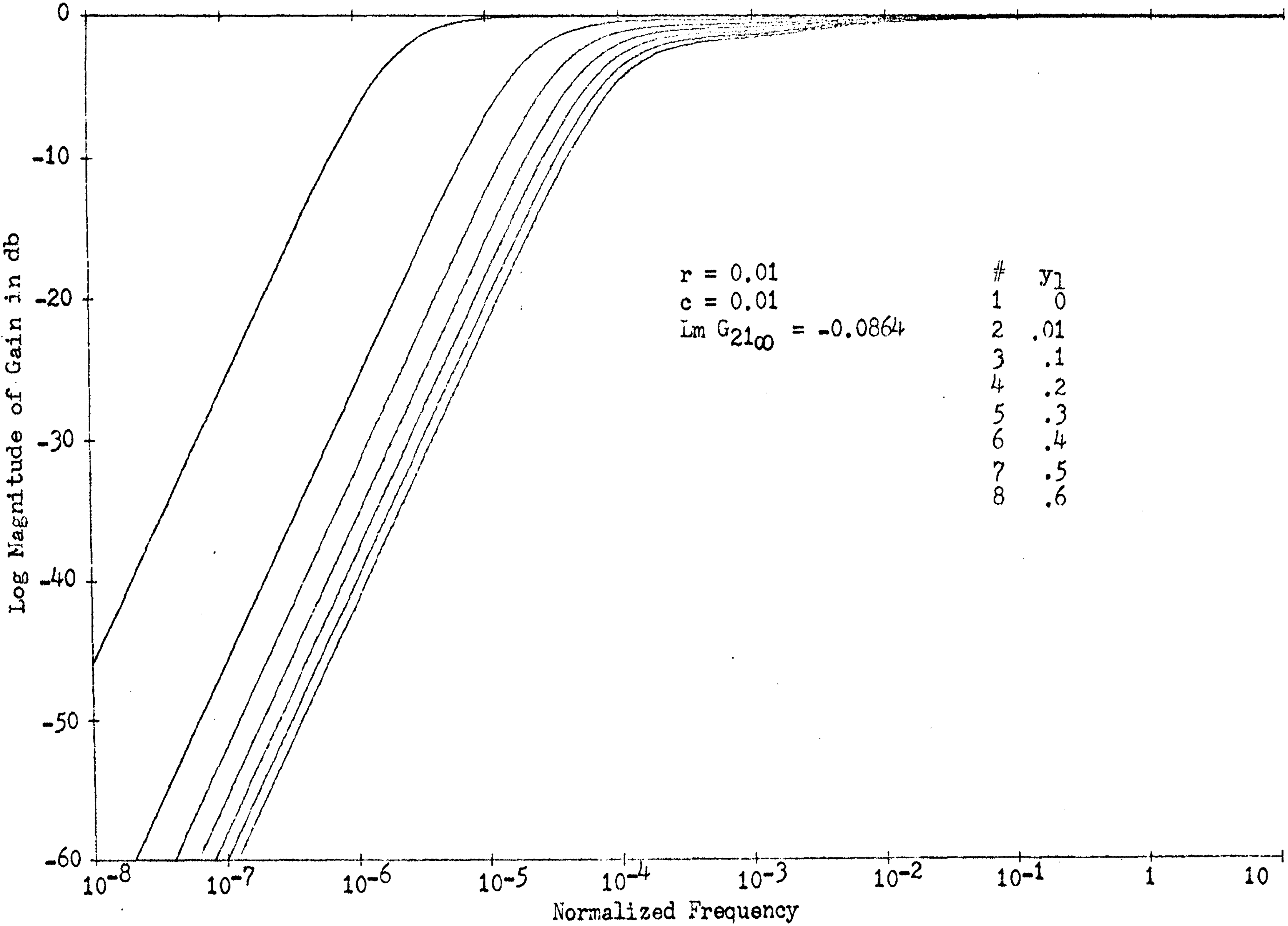


Figure 5-27. Gain characteristics of a loaded one section distributed $\overline{RC}$ coupling network. V_2= input voltage, V_1= output voltage.

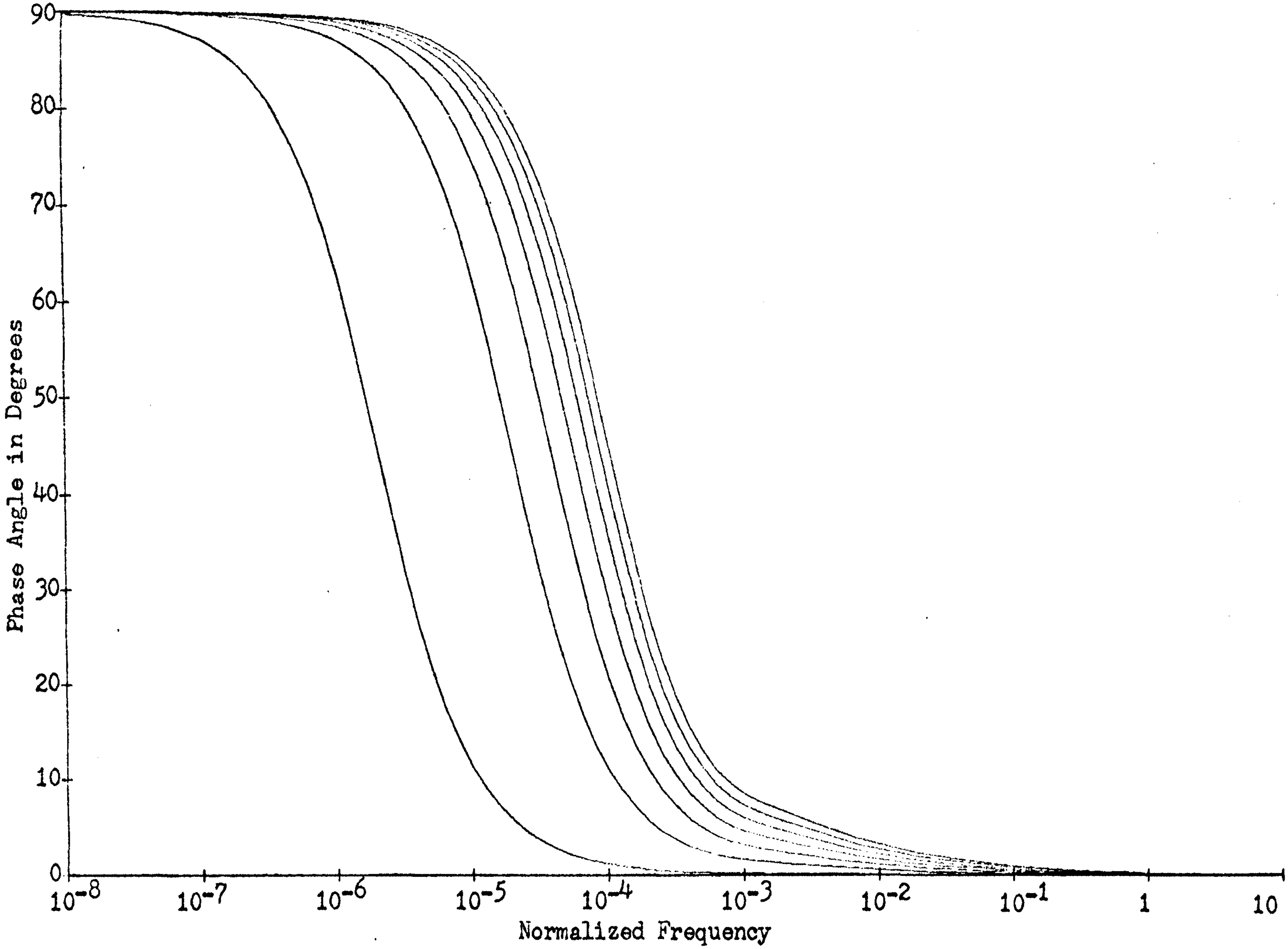


Figure 5-28. Phase characteristics of a loaded one section distributed $\overline{RC}$ coupling network. V_2= input voltage, V_1= output voltage.

6.0 SUGGESTIONS FOR FURTHER RESEARCH

The results of this dissertation suggest several areas for further research on synthesis problems. First recall that the $\overline{RC}$ product matrix was chosen from the high and low frequency asymptotic requirements of prescribed short circuit admittance parameters. Are there other ways of choosing the components of this matrix? Are there methods of graphically obtaining a realizable matrix equation, equation (2.30) or (2.40), from prescribed short circuit admittance parameters that are more accurate than the method presented in Chapter 4? Note that the high frequency asymptotic equations, third column of Table 3-1 on page 36, are independent of A_j and B_j and thus are also independent of the widths of the individual sections of the cascade. Do these equations also apply to any four-layered distributed parameter $\overline{RC}$ network regardless of the geometric taper of the widths? Only left end connections of the cascaded network were investigated leaving all the terminals of the right end section either open circuited or short circuited. What kind of network responses could be obtained if some but not all layers are short circuited? Can a synthesis procedure be developed using connections at both ends of the cascaded network as ports? It was assumed in this dissertation that the $\overline{RC}$ product matrix $\mathcal{T}$ was the same for each section in the cascade, and thus that each section is the same length. A synthesis procedure which does not require this restriction would allow more freedom of design. This leads naturally into the problem of synthesis using some morphology other than the cascade of uniform sections, or perhaps even synthesis of completely arbitrarily tapered networks.

Several analysis problems also remain to be solved. One dimensional current flow was assumed in each section of the cascaded network. What happens to the network responses if current flows in two dimensions as it would if the narrow conductive strips of Figure 2-4 on page 13 were removed. In the transmission line equations in Chapter 2, inductance was assumed negligible compared to resistance. Would this type of network be useful if inductance became a significant parameter? Could it have some use at frequencies so high that the device begins to act as a lossy waveguide?

Construction of distributed parameter $\overline{RC}$ networks suggests a few more interesting problems. Suppose a device of this type is constructed using monolithic techniques with the resistive layers being composed of either n-type or p-type materials and the dielectric is formed by the depletion layer between adjacent n-type and p-type materials. If a d-c reverse bias is applied to the p-n junction, the width of the depletion layer changes, changing the capacitance. Can a change in the d-c bias alter the resonant frequency of a notch filter or, perhaps, change the notch filter to an entirely different type of circuit such as a low pass filter? The capacitance inherent at a back-biased p-n junction is the same principle used for varactors. Perhaps some rather exotic properties could be obtained by using monolithic distributed $\overline{RC}$ networks of various morphologies as varactors in parametric amplifier or multiplier configurations.

REFERENCES

1. Bertnolli, Edward C. "Exact and Numerical Analyses of Distributed Parameter RC Networks," Kansas State University Bulletin, Special Report Number 57, Kansas Engineering Experiment Station, Manhattan, Kansas, Vol. 49, #9, Sept., 1965.

2. Bertnolli, Edward C. "Distributed Parameter RC Network Analysis," _1966 IEEE International Convention Record_, Part 7, pp. 243-249.

3. Blank, J. M., I. A. Lesk, J. J. Suran. "Practical Considerations in Thin Circuit Microelectronics," _Proceedings of the National Electronics Conference_, 1960, Vol. 16, pp. 438-443.

4. Castro, P. S. and W. W. Happ. "Distributed Parameter Circuits and Microsystem Electronics," _Proceedings of the National Electronics Conference_, 1960, Vol. 16, pp. 448-460.

5. D'Azzo, John J., and Constantine H. Houpis. _Feedback Control System Analysis and Synthesis_. New York: McGraw-Hill, 1960.

6. Dell, R. A., Jr. and S. L. Hakimi. "Limitations of Distributed RC Filters," _Proceedings of the National Electronics Conference_, Vol. 20, 1964, pp. 765-769.

7. Frazer, R. A., W. J. Duncan, and A. R. Collar. _Elementary Matrices_, Cambridge: University Press, 1938.

8. Fuller, W. D. "Titanium Thin Film Circuits," _Proceedings of the National Electronics Conference_, 1961, Vol. 17, pp. 32-43.

9. Googe, J. M. and K. L. Su, "The Double-Kelvin Transmission Line and Some Applications," _IEEE Transactions on Circuit Theory_, Vol. CT-11, No. 3, Sept., 1964, pp. 372-377.

10. Hager, Charles K. "Network Design of Microcircuits." _Electronics_, Vol. 32, No. 36, Sept. 4, 1959, pp. 44-49.

11. Happ, W. W. and P. S. Castro. "Distributed Parameter Circuit Design Techniques," _Proceedings of the National Electronics Conference_, Vol. 17, Oct., 1961, pp. 45-70.

12. Happ, W. W., P. S. Castro, and W. D. Fuller. "Synthesis of Solid State Distributed Parameter Functions." _IRE International Convention Record_, Vol. 10, pt. 6, March, 1962, pp. 262-278.

13. Happ, W. W. and W. D. Fuller. "Design Procedures for Film Type Distributed Parameter Circuits," _Proceedings of the National Electronics Conference_, Vol. 17, Oct., 1961, pp. 597-610.

14. Kaufman, W. M. and S. J. Garrett. "Tapered Distributed Filters."

IRE Transactions on Circuit Theory, Vol. CT-9, No. 4, Dec., 1962, pp. 329-336.

15. Hazony, Dov. Elements of Network Synthesis, New York: Reinhold, 1963.

16. Heizer, K. W. "Distributed RC Networks with Rational Transfer Functions," IRE Transactions on Circuit Theory, Vol. CT-9, No. 4, Dec., 1962, pp. 356-362.

17. Hesselberth, C. A. "Synthesis of Some Distributed RC Networks," Ph.D. Thesis, University of Illinois, Department of Electrical Engineering, Urbana, Illinois, 1963.

18. Hildebrand, Francis B. Methods of Applied Mathematics, 2nd edition, Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1965.

19. Kelly, John J. and Mohammed S. Ghausi. "Network Properties of Distributed RC Networks with Arbitrary Geometric Shapes." Laboratory for Electroscience Research, Technical Report 400-107, AFCRL-65-312, Dept. of Electrical Engineering, School of Engineering and Science, New York University, New York, March, 1965.

20. Newcomb, Rovert W. Linear Multiport Synthesis, New York: McGraw-Hill, 1966.

21. Newcomb, Robert W. "Nonreciprocal Transmission-Line n-Port Synthesis," Proceedings of the Institution of Radio and Electronics Engineers of Australia, Vol. 26, No. 4, Apr., 1965, pp. 135-142.

22. O'Shea, Richard P. "Synthesis Using Distributed RC Networks," 1965 IEEE International Convention Record, Part 7, pp. 18-29.

23. Protonotarios, E. N. and O. Wing. "Delay and Rise Time of Arbitrarily Tapered RC Transmission Lines," 1965 IEEE International Convention Record, 1965, pt. 7, pp. 1-6.

24. Protonotarios, E. N. and O. Wing. "Distribution of Zeros of ABCD Parameters of Arbitrary RC Transmission Lines," Proceedings of the Third Annual Allerton Conference on Circuit and System Theory, University of Illinois, Oct., 1965, pp. 162-171.

25. Protonotarios, E. N. and O. Wing. "Theory of Nonuniform RC Lines. Part I: Analytic Properties and Realizability Conditions in the Frequency Domain," IEEE Transactions on Circuit Theory, Vol. CT-14, No. 1, March, 1967, pp. 2-12.

26. Rao, T. N., R. W. Newcomb. "Synthesis of Lumped-Distributed RC n-Ports," IEEE Transactions on Circuit Theory, Vol. CT-13, No. 4, Dec., 1966, pp. 458-460.

27. Rice, S. O. "Steady State Solutions of Transmission Line Equa-

tions," *The Bell System Technical Journal*, Vol. 20, No. 2, April, 1941, pp. 131-178.

28. Richards, P. I. "Resistor-Tranmission-Line Circuits," *Proceedings of the IRE*, Vol. 36, Feb., 1948, pp. 217=220.

29. Su, Kendall L. "The Trigonometric RC Transmission Lines, *1963 IEEE International Convention Record*, Part 2, pp. 43-55.

30. VanValkenburg, M. E. *Introduction to Modern Network Synthesis*, New York: John Wiley and Sons, 1960.

31. Woo, Barry B. and James M. Barlemay. "Characteristics and Applications of a Tapered, Thin Film Distributed Parameter Structure." *1963 IEEE International Convention Record*, Part 2, pp. 56-75.

32. Wyndrum, Ralph W., Jr. "The Exact Synthesis of Distributed RC Networks," New York University, College of Engineering, May, 1963, Technical Report 400-76.

33. Seely, S. *Radio Electronics*, New York: Mcgraw-Hill, 1956, p. 216.

VITA

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The author is a member of Eta Kappa Nu, Tau Beta Pi, and Phi Kappa Phi and a student member of IEEE. He is married to the former Edith Genelle Trexler and has one son.

APPENDIX A. MATRIX FUNCTIONS AND IDENTITIES

While books on matrix theory are plentiful, they generally either deal with theory useful mostly for numerical analysis or else with the very sophisticated mathematical peculiarities of matrices in regard to set theory. Therefore the author feels that definitions of certain matri functions much used in this dissertation are in order. This appendix provides these definitions as well as some matrix identities that are useful in the proofs of Chapter 2 and Appendix B.

The literature (Ref. 7, 18) defines a quantity $e^{\underline{A}}$, in which $\underline{A}$ is a matrix, by the same infinite series expansion used to define the scalar quantity e^{a}, i.e.

$$e^{\underline{A}} \equiv \underline{I} + \underline{A} + \frac{1}{2!}\underline{A}^2 + \frac{1}{3!}\underline{A}^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!}\underline{A}^i. \tag{A.1}$$

where $\underline{I}$ is the identity matrix. If and only if the matrices $\underline{A}$ and $\underline{B}$ comm with respect to multiplication

$$e^{\underline{A}}\ e^{\underline{B}} = e^{\underline{B}}\ e^{\underline{A}} = e^{\underline{A}+\underline{B}}\ . \tag{A.2}$$

This can be shown by multiplying the series expansions of the two terms on the left of equation (A.2) and identifying the product term by term with the series on the right. Thus

$$e^{\underline{A}}\ e^{-\underline{A}} = \underline{I}.$$

The matrix hyperbolic functions sinh $\underline{A}$ and cosh $\underline{A}$ may be defined by serie expansions analogous to the power series expansions of the corresponding scalar functions sinh a and cosh a; however, it is more convenient to use exponential definitions. So the definitions

$$\sinh \underline{A} \equiv \tfrac{1}{2}(e^{\underline{A}} - e^{-\underline{A}}) \tag{A.3}$$

$$\cosh \underline{A} \equiv \tfrac{1}{2}(e^{\underline{A}} + e^{-\underline{A}}) \tag{A.4}$$

are chosen. In general the product of two matrix functions of $\underline{A}$, e.g. P(and Q($\underline{A}$), commute since each is expressible as a polynomial (or series) in $\underline{A}$. Thus the remaining matrix hyperbolic functions may be defined in terms of sinh $\underline{A}$ and cosh $\underline{A}$.

$$\operatorname{sech} \underline{A} \equiv (\coth \underline{A})^{-1} \tag{A.5}$$

$$\operatorname{csch} \underline{A} \equiv (\sinh \underline{A})^{-1} \tag{A.6}$$

$$\tanh \underline{A} \equiv \sinh \underline{A} \operatorname{sech} \underline{A} = \operatorname{sech} \underline{A} \sinh \underline{A} \tag{A.7}$$

$$\coth \underline{A} \equiv \cosh \underline{A} \operatorname{csch} \underline{A} = \operatorname{csch} \underline{A} \cosh \underline{A} \tag{A.8}$$

The definitions given by equations (A.3) to (A.8) can be shown to be compatible with Sylvester's Theorem.

Sylvester's Theorem, a very powerful tool of matrix theory, states that if the n eigenvalues λ_i of a matrix $\underline{A}$ are all distinct and if $P(\underline{A})$ is any polynomial in $\underline{A}$ (including infinite series), then

$$P(\underline{A}) = \sum_{i=1}^{n} P(\lambda_i)\, \underline{Z}o(\lambda_i) \tag{A.9}$$

where

$$\underline{Z}o(\lambda_i) \equiv \prod_{j\neq i}^{n} (\lambda_j \underline{I} - \underline{A}) \Big/ \prod_{j\neq i}^{n} (\lambda_j - \lambda_i). \tag{A.10}$$

The quantity $\underline{Z}o$ has the properties

$$\underline{Z}o(\lambda_i)\underline{Z}o(\lambda_j) = \underline{Z}o(\lambda_i)\delta_{ij} \tag{A.11}$$

and

$$\sum_{i=1}^{n} \underline{Z}o(\lambda_i) = \underline{I} \tag{A.12}$$

where δ_{ij} is the well known Kronecker delta.

Another function that is used throughout this dissertation is the square root of a matrix. This function is defined as a matrix $\underline{A}^{\frac{1}{2}}$ such that $\underline{A}^{\frac{1}{2}}\, \underline{A}^{\frac{1}{2}} = \underline{A}$ and is most simply expressed by Sylvester's Theorem. Thus

$$\sqrt{\underline{A}} = \sum_{i=1} \pm\sqrt{\lambda_i}\ \prod_{j\neq i}^{n} (\lambda_i \underline{I} - \underline{A}) \Big/ \prod_{j\neq i}^{n} (\lambda_j - \lambda_i) \tag{A.13}$$

in which the $\pm$ signs of the various $\sqrt{\lambda_i}$'s are independent of each other.

Several matrix identities are very useful throughout this dissertation and will now be given in the form of a theorem.

Theorem 1. Given (1.) $\underline{A}$, $\underline{B}$, and $\underline{C}$ are $n \times n$ matrices

(2.) $\underline{A} = \underline{A}^T$ and $\underline{B} = \underline{B}^T$, i.e. $\underline{A}$ and $\underline{B}$ are symmetric

(3.) $\underline{C} = \underline{A}\,\underline{B}$

(4.) Sylvester's Theorem applies to the function $P(\underline{C})$

Then: 1.) $\underline{A}\, P(\underline{C}^T) = P(\underline{C})\, \underline{A}$ or $P(\underline{C})\, \underline{A}$ is symmetric (A.14)

2.) $\underline{B}^{-1}\, P(\underline{C}^T) = P(\underline{C})\, \underline{B}^{-1}$ or $P(\underline{C})\, \underline{B}^{-1}$ is symmetric (A.15)

3.) $\underline{B}^{-1}\sqrt{\underline{C}^T} = \sqrt{\underline{C}}^{-1}\, \underline{A}$ (A.16)

Proof: For the case where $P(\underline{C})$ is a polynomial in $\underline{C}$ parts 1 and 2 are obvious by inspection. The proofs given will apply also to functions which can be expressed by Sylvester's Theorem but not by a polynomial, e.g. $\sqrt{\underline{C}}$.

Part 1.) Since the determinate of a matrix and its transpose are identical, the eigenvalues of a matrix and its transpose are also identical. From Sylvester's Theorem

$$\underline{A}\, P(\underline{C}^T) = \sum_{i=1}^{n} P(\lambda_i)\, \underline{A} \prod_{j \neq i}^{n} (\lambda_j \underline{I} - \underline{C}^T) / \prod_{j \neq i}^{n} (\lambda_j - \lambda_i).$$

Consider the term $\underline{A}\,(\lambda_1 \underline{I} - \underline{C}^T)$.

$$\underline{A}\,(\lambda_1 \underline{I} - \underline{C}^T) = (\lambda_1 \underline{A} - \underline{A}\,\underline{B}\,\underline{A}) = (\lambda_1 \underline{I} - \underline{C})\,\underline{A}$$

Reversing the order of $\underline{A}$ and $(\lambda_i \underline{I} - \underline{C}^T)$ successively to each term in the product $\underline{A} \prod (\lambda_j \underline{I} - \underline{C}^T)$ leads to $\left[\prod (\lambda_j \underline{I} - \underline{C})\right] \underline{A}$ or

$$\underline{A}\, P(\underline{C}^T) = \left[\sum_{i=1}^{n} P(\lambda_i) \prod_{j \neq i}^{n} (\lambda_j \underline{I} - \underline{C}) / \prod_{j \neq i}^{n} (\lambda_j - \lambda_i)\right] \underline{A} = P(\underline{C})\, \underline{A} \qquad \text{Q.E.D.}$$

Part 2.) As before apply Sylvester's Theorem.

$$\underline{B}^{-1}\, P(\underline{C}^T) = \sum_{i=1}^{n} P(\lambda_i)\, \underline{B}^{-1} \prod_{j \neq i}^{n} (\lambda_j \underline{I} - \underline{C}^T) / \prod_{j \neq i}^{n} (\lambda_j - \lambda_i)$$

But $\underline{B}^{-1}\,(\lambda_j \underline{I} - \underline{C}^T) = (\lambda_j \underline{B}^{-1} - \underline{B}^{-1}\underline{B}\,\underline{A}) = (\lambda_j \underline{I} - \underline{A}\,\underline{B})\,\underline{B}^{-1}$

so $$\underline{B}^{-1} \prod_{j \neq i}^{n} (\lambda_j \underline{I} - \underline{C}^T) = \left[\prod_{j \neq i}^{n} (\lambda_j \underline{I} - \underline{C}^T) \right] \underline{B}^{-1}$$

and $$\underline{B}^{-1} \, P(\underline{C}^T) = \left[\sum_{i=1}^{n} P(\lambda_i) \prod_{j \neq i}^{n} (\lambda_j \underline{I} - \underline{C}) / \prod_{j \neq i}^{n} (\lambda_j - \lambda_i) \right] \underline{B}^{-1} = P(\underline{C}) \, \underline{B}^{-1}$$

Q.E.D.

Part 3.) By equation (A.15)

$$\underline{B}^{-1} \sqrt{\underline{C}^T} = \sqrt{\underline{C}} \, \underline{B}^{-1} = \sqrt{\underline{C}} \, \underline{C}^{-1} \, \underline{C} \, \underline{B}^{-1} = \sqrt{\underline{C}}^{-1} \, \underline{A} \, \underline{B} \, \underline{B}^{-1} = \sqrt{\underline{C}}^{-1} \, \underline{A}$$ Q.E.D.

APPENDIX B. DERIVATION OF EQUATIONS (2.9), (2.10), AND (2.14)

B.1 Derivation of Equation (2.9)

If
$$\begin{bmatrix} \underline{V}_1 \\ \underline{I}_1 \end{bmatrix} = \begin{bmatrix} \coth \sqrt{s\underline{RC}} & (s\underline{C})^{-1} \sqrt{s\underline{CR}} \sinh \sqrt{s\underline{CR}} \\ \underline{R}^{-1} \sqrt{s\underline{RC}} \sinh \sqrt{s\underline{RC}} & \cosh \sqrt{s\underline{CR}} \end{bmatrix} \begin{bmatrix} \underline{V}_2 \\ -\underline{I}_2 \end{bmatrix} \tag{B.1}$$
find matrices $\underline{Z}$ and $\underline{Y}$ and that

$$\underline{V} = \underline{Z} \; \underline{I}$$

and
$$\underline{I} = \underline{Y} \; \underline{V}$$

where $\underline{V} = \begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \end{bmatrix}$ and $\underline{I} = \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$. Equation (B.1) may be written as the two matrix equations

$$\underline{V}_1 = \cosh \sqrt{s\underline{RC}} \; \underline{V}_2 - (s\underline{C})^{-1} \sqrt{s\underline{CR}} \sinh \sqrt{s\underline{CR}} \; \underline{I}_2 \tag{B.2}$$

$$\underline{I}_1 = \underline{R}^{-1} \sqrt{s\underline{RC}} \sinh \sqrt{s\underline{RC}} \; \underline{V}_2 - \cosh \sqrt{s\underline{CR}} \; \underline{I}_2. \tag{B.3}$$

If equation (B.3) is solved for $\underline{V}_2$ one obtains

$$\underline{V}_2 = \operatorname{csch} \sqrt{s\underline{RC}} \sqrt{s\underline{RC}}^{-1} \underline{R} \left[\underline{I}_1 + \cosh \sqrt{s\underline{CR}} \; \underline{I}_2 \right]. \tag{B.4}$$

If this is substituted into equation (B.2) there results

$$\underline{V}_1 = \coth \sqrt{s\underline{RC}} \sqrt{s\underline{RC}}^{-1} \underline{R} \; \underline{I}_1 + \left[\coth \sqrt{s\underline{RC}} \sqrt{s\underline{RC}}^{-1} \underline{R} \cosh \sqrt{s\underline{CR}} - (s\underline{C})^{-1} \sqrt{s\underline{CR}} \sinh \sqrt{s\underline{CR}} \right] I_2 \tag{B.5}$$

These two equations can be written

$$\begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \end{bmatrix} = \begin{bmatrix} \coth \sqrt{s\underline{RC}} \sqrt{s\underline{RC}}^{-1} \underline{R} & \coth \sqrt{sRC} \sqrt{sRC}^{-1} R \cosh \sqrt{sCR} - (s\underline{C})^{-1} \sqrt{s\underline{CR}} \sinh \sqrt{s\underline{CR}} \\ \operatorname{csch} \sqrt{s\underline{RC}} \sqrt{s\underline{RC}}^{-1} \underline{R} & \operatorname{csch} \sqrt{s\underline{RC}} \sqrt{s\underline{RC}}^{-1} \underline{R} \cosh \sqrt{s\underline{CR}} \end{bmatrix} \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix} \tag{B.6}$$

which is of the form $\underline{V} = \underline{Z} \; \underline{I}$. Since the geometry of the network under consideration is symmetric, it would be natural to expect that $\underline{Z}_{11} = \underline{Z}_{22}$ and $\underline{Z}_{12} = \underline{Z}_{21}$. That this is in fact true will now be shown. Consider the

$\underline{Z}_{22}$ term of equation (B.6).

$$\underline{Z}_{22} = \operatorname{csch}\sqrt{s\underline{RC}}\,\sqrt{s\underline{RC}}^{-1}\,\underline{R}\,\cosh\sqrt{s\underline{CR}}$$

If $\underline{R}$ is identified with the matrix $\underline{A}$ and $\underline{C}$ with the matrix $\underline{B}$ of equation (A.15), then

$$\underline{Z}_{22} = \operatorname{csch}\sqrt{s\underline{RC}}\,\sqrt{s\underline{RC}}^{-1}\cosh\sqrt{s\underline{RC}}\,\underline{R}.$$

Matrices which are functions of a common matrix—in this case $\underline{RC}$—commute with respect to multiplication so that

$$\underline{Z}_{22} = \operatorname{csch}\sqrt{s\underline{RC}}\,\cosh\sqrt{s\underline{RC}}\,\sqrt{s\underline{RC}}^{-1}\,\underline{R}$$

$$= \coth\sqrt{s\underline{RC}}\,\sqrt{s\underline{RC}}^{-1}\,\underline{R} = \underline{Z}_{11}.$$

The $\underline{Z}_{12}$ term can be similarly reduced.

$$\underline{Z}_{12} = \coth\sqrt{s\underline{RC}}\,\sqrt{s\underline{RC}}^{-1}\,\underline{R}\,\cosh\sqrt{s\underline{CR}} - (s\underline{C})^{-1}\sqrt{s\underline{CR}}\,\sinh\sqrt{s\underline{CR}}$$

Application of equation (A.14) and (A.16) to $\underline{R}\cosh\sqrt{s\underline{CR}}$ and $(s\underline{C})^{-1}\sqrt{s\underline{CR}}$ respectively leads to

$$\underline{Z}_{12} = \coth\sqrt{s\underline{RC}}\;\sqrt{s\underline{RC}}^{-1}\cosh\sqrt{s\underline{RC}}\,\underline{R} - \sqrt{s\underline{RC}}^{-1}\,\underline{R}\,\sinh\sqrt{s\underline{CR}}.$$

Again using (A.14), this time to $\underline{R}\sinh\sqrt{s\underline{CR}}$ yields

$$\underline{Z}_{12} = \coth\sqrt{s\underline{RC}}\,\cosh\sqrt{s\underline{RC}}\quad s\underline{RC}^{-1}\,R - \quad s\underline{RC}^{-1}\,\sinh\quad s\underline{RC}\,\underline{R}$$

$$= (\cosh^2\sqrt{s\underline{RC}} - \sinh^2\sqrt{s\underline{RC}})\,(\sinh\sqrt{s\underline{RC}})^{-1}\;\sqrt{s\underline{RC}}^{-1}\,\underline{R}.$$

An appeal to the exponential definitions of $\cosh\sqrt{s\underline{RC}}$ and $\sinh\sqrt{s\underline{RC}}$ shows that the scalar hyperbolic identity also applies to the corresponding matrix case, i.e.

$$\cosh^2\underline{A} - \sinh^2\underline{A} = \underline{I}.$$

Thus $\underline{Z}_{12} = \operatorname{csch}\sqrt{s\underline{RC}}\;\sqrt{s\underline{RC}}^{-1}\,\underline{R} = \underline{Z}_{21}$

and equation (B.6) becomes

$$\begin{bmatrix}\underline{V}_1\\ \underline{V}_2\end{bmatrix} = \begin{bmatrix}\coth\sqrt{s\underline{RC}} & \operatorname{csch}\sqrt{s\underline{RC}}\\ \operatorname{csch}\sqrt{s\underline{RC}} & \coth\sqrt{s\underline{RC}}\end{bmatrix}\begin{bmatrix}\sqrt{s\underline{RC}}^{-1}\,\underline{R} & \underline{0}\\ \underline{0} & \sqrt{s\underline{RC}}^{-1}\,R\end{bmatrix}\begin{bmatrix}\underline{I}_1\\ \underline{I}_2\end{bmatrix}$$

Q.E.D.

(B.7)

B.2 Derivation of Equation (2.10)

In order to get the form $\underline{I} = \underline{Y}\ \underline{V}$ one could either invert $\underline{Z}$ of equation (B.7) or go back to equations (B.2) and (B.3). Choosing the latter approach and solving (B.2) for $\underline{I}_2$ results in

$$\underline{I}_2 = \text{csch}\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\left[-V_1 + \cosh\sqrt{s\underline{RC}}\ \underline{V}_2\right]. \tag{B.8}$$

Substitute this into (B.3).

$$\underline{I}_1 = \coth\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\underline{V}_1 - \Big[\coth\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\cosh\sqrt{s\underline{RC}} - \underline{R}^{-1}\sqrt{s\underline{RC}}\sinh\sqrt{s\underline{RC}}\Big]\ \underline{V}_2 \tag{B.9}$$

Equations (B.9) and (B.8) can be written as the single matrix equation

$$\begin{bmatrix}\underline{I}_1\\ \underline{I}_2\end{bmatrix} = \begin{bmatrix}\coth\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C} & -\coth\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\cosh\sqrt{s\underline{RC}} \\ & -\underline{R}^{-1}\sqrt{s\underline{RC}}\sinh\sqrt{s\underline{RC}} \\ -\text{csch}\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C} & \text{csch}\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\cosh\sqrt{s\underline{RC}}\end{bmatrix}\begin{bmatrix}\underline{V}_1\\ \underline{V}_2\end{bmatrix} \tag{B.10}$$

This equation is of the form $\underline{I} = \underline{Y}\ \underline{V}$, but again it should be expected from symmetry that $\underline{Y}_{11} = \underline{Y}_{22}$ and $\underline{Y}_{12} = \underline{Y}_{21}$. Apply equation (A.14) to $\underline{Y}_{22}$.

$$\begin{aligned}\underline{Y}_{22} &= \text{csch}\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\cosh\sqrt{s\underline{RC}}\\ &= \text{csch}\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\cosh\sqrt{s\underline{CR}}\ s\underline{C}\\ &= \coth\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C} = \underline{Y}_{11}\end{aligned}$$

Repeated application of equation (A.14) and use of (A.16) on $\underline{Y}_{12}$ leads to

$$\begin{aligned}-\underline{Y}_{12} &= \coth\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ sC\cosh\sqrt{s\underline{RC}} - \underline{R}^{-1}\sqrt{s\underline{RC}}\sinh\sqrt{s\underline{RC}}\\ &= \coth\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\cosh\sqrt{s\underline{CR}}\ s\underline{C} - \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\sinh\sqrt{s\underline{RC}}\\ &= (\cosh^2\sqrt{s\underline{CR}} - \sinh^2\sqrt{s\underline{CR}})\ \text{csch}\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\\ &= \text{csch}\sqrt{s\underline{CR}}\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C} = -\underline{Y}_{21}\end{aligned}$$

Thus equation (B.10) takes the form

$$\begin{bmatrix}\underline{I}_1\\ \underline{I}_2\end{bmatrix} = \begin{bmatrix}\coth\sqrt{s\underline{CR}} & -\text{csch}\sqrt{s\underline{CR}}\\ -\text{csch}\sqrt{s\underline{CR}} & \coth\sqrt{s\underline{CR}}\end{bmatrix}\begin{bmatrix}\sqrt{s\underline{CR}}^{-1}\ s\underline{C} & 0\\ 0 & \sqrt{s\underline{CR}}^{-1}\ s\underline{C}\end{bmatrix}\begin{bmatrix}\underline{V}_1\\ \underline{V}_2\end{bmatrix} \tag{B.11}$$

Each of the components of this equation fits the form $P(s\underline{CR})\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C}$.
By equation (A.16) and (A.15)

$$P(s\underline{CR})\ \sqrt{s\underline{CR}}^{-1}\ s\underline{C} = P(s\underline{CR})\ \underline{R}^{-1}\ \sqrt{s\underline{RC}} = \underline{R}^{-1}\ \sqrt{s\underline{RC}}\ P(s\underline{RC}).$$

Thus equation (B.11) can be written

$$\begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix} = \begin{bmatrix} \underline{R}^{-1}\sqrt{s\underline{RC}} & \underline{0} \\ \underline{0} & \underline{R}^{-1}\sqrt{s\underline{RC}} \end{bmatrix} \begin{bmatrix} \coth\sqrt{s\underline{RC}} & -\text{csch}\sqrt{s\underline{RC}} \\ -\text{csch}\sqrt{s\underline{RC}} & \coth\sqrt{s\underline{RC}} \end{bmatrix} \begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \end{bmatrix} \tag{B.12}$$

If one multiplies $\underline{Y}$ by $\underline{Z}$ the expected $2n \times 2n$ identity matrix results.

$$\underline{Z}\ \underline{Y} = \underline{Y}\ \underline{Z} = \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{I} \end{bmatrix}$$

B.3 Derivation of Equation (2.14)

Derive the driving point admittance matrix of the i^{th} section of the cascaded $\overline{RC}$ network of Figure 2-3 in terms of the driving point admittance matrix and the parameters of the $i+1^{th}$ section, i.e. derive equation (2.14),

$$\underline{Y}_i(s) = \underline{R}_{i+1}^{-1}\ \sqrt{s\underline{T}}\ (\underline{R}_{i+1}^{-1}\ \sqrt{s\underline{T}} - \underline{Y}_{i+1}\ \tanh\sqrt{s\underline{T}})^{-1}\ (\underline{Y}_{i+1} - \underline{R}_{i+1}^{-1}\sqrt{s\underline{T}}\ \tanh\sqrt{s\underline{T}}). \tag{B.13}$$

Equation (B.13) can be derived more easily if the following theorem is first proved.

Theorem 2. If $\underline{Y}_i$ is the driving point admittance matrix of the i^{th} section of a cascaded multi-layered distributed $\overline{RC}$ network, $P(\underline{T})$ is a matrix polynomial in $\underline{T}$, $\underline{T} = \underline{RC}$, and $\underline{R}$ and $\underline{C}$ are symmetrical matrices, then

$$P(\underline{T}^T)\underline{Y}_i = \underline{Y}_i\ P(\underline{T}). \tag{B.14}$$

The proof of Theorem 2 will be proof by induction. First it will be shown that if (B.14) holds for i=j, it also holds for i=j+1. Then it will be shown that (B.14) holds for i=1.

Proof: Assume $P(\underline{T}^T)\underline{Y}_j = \underline{Y}_j P(\underline{T})$, then find $\underline{Y}_{j+1}$ in terms of $\underline{Y}_j$. Reference to Figure B-1 shows that

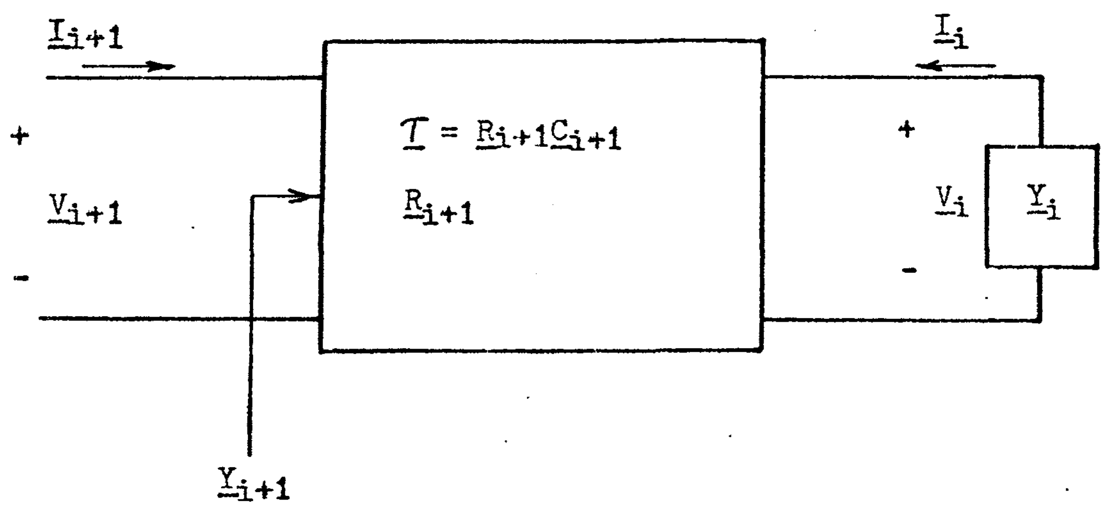

Figure B-1. A block diagram of the i+1th section of a cascaded $\overline{RC}$ network.

$$\underline{I}_{j+1} = \underline{Y}_{j+1}\,\underline{V}_{j+1} \tag{B.15}$$

$$-\underline{I}_j = \underline{Y}_j\,\underline{V}_j. \tag{B.16}$$

With Figures 2-3 and B-1 in mind equation 2.10 can be written

$$\begin{bmatrix}\underline{I}_{j+1}\\ -\underline{I}_j\end{bmatrix} = \begin{bmatrix}R^{-1}\sqrt{s\underline{T}} & \underline{0}\\ \underline{0} & R^{-1}\sqrt{s\underline{T}}\end{bmatrix}\begin{bmatrix}\coth\sqrt{s\underline{T}} & -\operatorname{csch}\sqrt{s\underline{T}}\\ -\operatorname{csch}\sqrt{s\underline{T}} & \coth\sqrt{s\underline{T}}\end{bmatrix}\begin{bmatrix}\underline{V}_{i+1}\\ \underline{V}_j\end{bmatrix}. \tag{B.17}$$

By substitution of (B.16) into (B.17) one obtains

$$\underline{I}_{j+1} = \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\,(\coth\sqrt{s\underline{T}}\,\underline{V}_{j+1} - \operatorname{csch}\sqrt{s\underline{T}}\,\underline{V}_j)$$

$$-\underline{Y}_j\underline{V}_j = \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\,(-\operatorname{csch}\sqrt{s\underline{T}}\,\underline{V}_{j+1} + \cosh\sqrt{s\underline{T}}\,\underline{V}_j).$$

Elimination of $\underline{V}_j$ from these two equations gives

$$\underline{I}_{j+1} = \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\Big[\coth\sqrt{s\underline{T}} - \operatorname{csch}\sqrt{s\underline{T}}\,(\underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\coth\sqrt{s\underline{T}} + \underline{Y}_j)^{-1}$$

$$\times\underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\operatorname{csch}\sqrt{s\underline{T}}\Big]\underline{V}_{j+1}$$

$$= \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\Big[\coth\sqrt{s\underline{T}} - \operatorname{csch}\sqrt{s\underline{T}}\operatorname{sech}\sqrt{s\underline{T}}(\underline{I}$$

$$+ \sinh\sqrt{s\underline{T}}\,\sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,\underline{Y}_j\operatorname{sech}\sqrt{s\underline{T}})^{-1}\Big]\,\underline{V}_{j+1}$$

or $\quad \underline{Y}_{j+1} = \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}(\tanh\sqrt{s\underline{T}} + \cosh\sqrt{s\underline{T}}\,\sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,\underline{Y}_j\operatorname{sech}\sqrt{s\underline{T}})$

$$\times(\underline{I} + \sinh\sqrt{s\underline{T}}\,\sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,\underline{Y}_j\operatorname{sech}\sqrt{s\underline{T}})^{-1}$$

If equation (B.14) holds for i=j then

$$\underline{R}_{j+1}\,\underline{Y}_j\operatorname{sech}\sqrt{s\underline{T}} = \underline{R}_{j+1}\operatorname{sech}\sqrt{s\underline{T}}^T\underline{Y}_j = \operatorname{sech}\sqrt{s\underline{T}}\,\underline{R}_{j+1}\,\underline{Y}_j\,,$$

the last equality coming from equation (A.14). Thus

$$\underline{Y}_{j+1} = \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\,(\tanh\sqrt{s\underline{T}} + \sqrt{s\underline{T}}^{-1}\,\underline{R}_{i+1}\,\underline{Y}_j)\,(\underline{I} + \tanh\sqrt{s\underline{T}}\sqrt{s\underline{T}}^{-1}\underline{R}_{j+1}\underline{Y}_j)^{-1} \tag{B.18}$$

and $\quad P(\underline{T}^T)\underline{Y}_{j+1} = P(\underline{T}^T)\underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}(\tanh\sqrt{s\underline{T}} + \sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,\underline{Y}_j)$

$$\times(\underline{I} + \tanh\sqrt{s\underline{T}}\,\sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,\underline{Y}_j)^{-1}$$

$$= \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\Big[\tanh\sqrt{s\underline{T}}\,P(\underline{T}) + \sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,P(\underline{T}^T)\underline{Y}_j\Big]$$

$$\times\Big[\underline{I} + \tanh\sqrt{s\underline{T}}\,\sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,\underline{Y}_j\Big]^{-1}$$

$$= \underline{R}_{j+1}^{-1}\sqrt{s\underline{T}}\Big[\tanh\sqrt{s\underline{T}} + \sqrt{s\underline{T}}^{-1}\,\underline{R}_{j+1}\,\underline{Y}_j\Big]$$

$$\times\Big[\underline{I} + \tanh\sqrt{s\underline{T}}\,\sqrt{s\underline{T}}^{-1}\,\underline{R}_{i+1}\,\underline{Y}_j\Big]^{-1}P(\underline{T})$$

or $P(\underline{T}^T)\ \underline{Y}_{j+1} = \underline{Y}_{j+1}\ P(\underline{T})$.

Thus if (B.14) holds for i=j it also holds for i=j+1. Now if it holds for i=1 the theorem will be proved. If the output of the i=1 section is open circuited, its driving point admittance is given by equation (2.11), or

$$\underline{Y}_1 = (\underline{Z}_{11})^{-1} = \underline{R}_1^{-1} \sqrt{s\underline{T}} \tanh \sqrt{s\underline{T}}.$$

Then $P(\underline{T}^T)\ \underline{Y}_1 = P(\underline{T}^T)\ \underline{R}_1^{-1} \sqrt{s\underline{T}} \tanh \sqrt{s\underline{T}} = \underline{R}^{-1} \sqrt{s\underline{T}} \tanh \sqrt{s\underline{T}}\ P(\underline{T}) = \underline{Y}_1\ P(\underline{T})$

If the output of the i=1 section is short circuited, its driving point admittance is given by equation (2.12), or

$$\underline{Y}_1 = \underline{Y}_{11} = \underline{R}_1^{-1} \sqrt{s\underline{T}} \coth \sqrt{s\underline{T}}$$

Then $P(\underline{T}^T)\ Y_1 = P(\underline{T}^T)\ \underline{R}_1^{-1} \sqrt{s\underline{T}} \coth \sqrt{s\underline{T}} = \underline{R}^{-1} \sqrt{s\underline{T}} \coth \sqrt{s\underline{T}}\ P(\underline{T}) = \underline{Y}_1\ P(\underline{T})$

and the theorem is proved.

Equation (B.13) can now be derived from equation (B.15), (B.16), and (B.17) with the help of Theorem 2; however, since the theorem is true, equation (B.18) holds for any value of j and offers a shorter derivation. Solving (B.18) for $\underline{Y}_i$ leads to

$$\underline{Y}_{i+1} + \underline{Y}_{i+1} \tanh \sqrt{s\underline{T}}\ \sqrt{s\underline{T}}^{-1}\ \underline{R}_{i+1}\ \underline{Y}_i = \underline{R}_{i+1}^{-1} \sqrt{s\underline{T}} \tanh \sqrt{s\underline{T}} + \underline{Y}_i$$

$$\underline{Y}_1 = (\underline{I} - \underline{Y}_{i+1} \tanh \sqrt{s\underline{T}}\ \sqrt{s\underline{T}}^{-1}\ \underline{R}_{i+1})^{-1}(\underline{Y}_{i+1} - \underline{R}_{i+1}^{-1} \sqrt{s\underline{T}} \tanh \sqrt{s\underline{T}})$$

or $$\underline{Y}_i = \underline{R}_{i+1}^{-1} \sqrt{s\underline{T}}(\underline{R}_{i+1}^{-1} \sqrt{s\underline{T}} - \underline{Y}_{i+1} \tanh \sqrt{s\underline{T}})^{-1}(\underline{Y}_{i+1} - \underline{R}_{i+1}^{-1} \sqrt{s\underline{T}} \tanh\sqrt{s\underline{T}})$$

Q.E.D.

APPENDIX C. DERIVATION OF EIGENVALUES AND THE CONSTANTS a, b, d, and r.

C.1 Derivation and Discussion of Eigenvalues

For the four-layered distributed $\overline{RC}$ network of Chapter 3 the matrix $\underline{\mathcal{T}}$ is defined as

$$\underline{\mathcal{T}} \equiv \begin{bmatrix} \mathcal{T}_{11} & -\mathcal{T}_{11} \\ -\mathcal{T}_{21} & \mathcal{T}_{21}+\mathcal{T}_{22} \end{bmatrix}$$

where $\mathcal{T}_{\alpha\beta} = R_{\alpha i}\, C_{\beta i} = R_{\alpha j}\, C_{\beta j}$ and $\alpha, \beta = a, b$ and each of the $\mathcal{T}_{\alpha\beta}$'s are real and positive. Let

$$\mathcal{T}_1 = \mathcal{T}_{11} + \mathcal{T}_{21} + \mathcal{T}_{22} \tag{C.1}$$

and

$$\hat{\mathcal{T}}_1 = \mathcal{T}_{11} - \mathcal{T}_{21} - \mathcal{T}_{22}. \tag{C.2}$$

The eigenvalues are obtained from the determinant equation

$$\left\|\lambda \underline{\underline{I}} - \underline{\mathcal{T}}\right\| = \lambda^2 - \lambda\mathcal{T}_1 + \mathcal{T}_{11}\,\mathcal{T}_{22} = 0$$

or

$$\lambda = \frac{\mathcal{T}_1 \pm \sqrt{\mathcal{T}_1^{\,2} - 4\,\mathcal{T}_{11}\,\mathcal{T}_{22}}}{2}.$$

If one lets

$$\mathcal{T}_o = \sqrt{\mathcal{T}_1^{\,2} - 4\,\mathcal{T}_{11}\,\mathcal{T}_{22}} = \sqrt{\hat{\mathcal{T}}_1^{\,2} + 4\,\mathcal{T}_{11}\,\mathcal{T}_{21}} \tag{C.3}$$

then

$$\lambda_1 = \tfrac{1}{2}(\mathcal{T}_1 - \mathcal{T}_o), \tag{C.4}$$

$$\lambda_2 = \tfrac{1}{2}(\mathcal{T}_1 + \mathcal{T}_o), \tag{C.5}$$

and

$$\lambda_2 - \lambda_1 = \tfrac{1}{2}(\mathcal{T}_1 + \mathcal{T}_o) - \tfrac{1}{2}(\mathcal{T}_1 - \mathcal{T}_o) = \mathcal{T}_o. \tag{C.6}$$

From (C.3) it can be seen that $\mathcal{T}_o$ is always real and greater than zero, i.e.

$$\mathcal{T}_o = \sqrt{\hat{\mathcal{T}}_1^{\,2} + 4\,\mathcal{T}_{11}\,\mathcal{T}_{21}} > 0. \tag{C.7}$$

Equation (C-3) suggests that

$$\mathcal{T}_1^{\,2} - 4\,\mathcal{T}_{11}\,\mathcal{T}_{22} = \mathcal{T}_o^{\,2} = \hat{\mathcal{T}}_1^{\,2} + 4\,\mathcal{T}_{11}\,\mathcal{T}_{21}$$

or

$$\mathcal{T}_1^{\,2} > \mathcal{T}_o^{\,2} > \hat{\mathcal{T}}_1^{\,2},$$

and since $\mathcal{T}_o$ is greater than zero

$$\mathcal{T}_1 > \mathcal{T}_o > \left|\hat{\mathcal{T}}_1\right| \geq 0 \tag{C.8}$$

With inequality (C.8) in mind, inspection of equations (C.4) and (C.5) shows that

$$0 < \lambda_1 < \frac{\mathcal{T}_1}{2} < \lambda_2 < \mathcal{T}_1. \qquad (C.9)$$

Hence both of the eigenvalues are positive and they are distinct.

C.2 Definition and Discussion of the Constants a, b, d, r, and c

The quantities a, b, d, and r, used for convenience starting in Chapter 3, are defined as

$$a \equiv -\frac{\lambda_2 - \mathcal{T}_{11}}{\mathcal{T}_o} \qquad (C.10)$$

$$b \equiv -\frac{\lambda_1 - \mathcal{T}_{11}}{\mathcal{T}_o} \qquad (C.11)$$

$$d \equiv \frac{\mathcal{T}_{11}}{\mathcal{T}_o} \qquad (C.12)$$

$$r \equiv \frac{R_b}{R_a} = \frac{R_b C_a}{R_a C_a} = \frac{\mathcal{T}_{21}}{\mathcal{T}_{11}} \quad . \qquad (C.13)$$

Another useful definition is

$$c \equiv \frac{C_b}{C_a} = \frac{C_b R_b}{C_a R_b} = \frac{\mathcal{T}_{22}}{\mathcal{T}_{21}} \quad . \qquad (C.14)$$

In addition to (C.10) "a" may also be written

$$a = \frac{-\lambda_2 + \mathcal{T}_{11}}{\mathcal{T}_o} = \frac{1}{2\mathcal{T}_o}(\mathcal{T}_{11} - \mathcal{T}_{21} - \mathcal{T}_{22} - \mathcal{T}_o)$$

$$= \frac{1}{2\mathcal{T}_o}(\mathcal{T}_1 - \mathcal{T}_o - 2\mathcal{T}_{21} - 2\mathcal{T}_{11}) = \frac{\lambda_1 - (\mathcal{T}_{21} + \mathcal{T}_{22})}{\mathcal{T}_o} \quad . \qquad (C.15)$$

Similarly it can be shown that

$$b = \frac{\lambda_2 - (\mathcal{T}_{21} + \mathcal{T}_{22})}{\mathcal{T}_o} \quad . \qquad (C.16)$$

Relationships among the quantities a, b, d, and r are useful in the manipulation of many of the equations of Chapter 3. To begin with, subtract equation (C.11) from (C.10).

$$b-a = -\frac{\lambda_1 - \tau_{11}}{\tau_o} + \frac{\lambda_2 - \tau_{11}}{\tau_o} = \frac{\lambda_2 - \lambda_1}{\tau_o} = 1 \tag{C.17}$$

Substituting (C.12) into (C.11) and (C.10) and solving for the eigenvalues leads to

$$\lambda_1 = (d-b)\tau_o \tag{C.18}$$

and

$$\lambda_2 = (d-a)\tau_o \ . \tag{C.19}$$

With the aid of inequality (C.8), bounds may be put on "a".

$$a = \frac{-\lambda_2 + \tau_{11}}{\tau_o} = \frac{-\tau_1 - \tau_o + 2\tau_{11}}{2\tau_o} = \frac{\hat{\tau}_1 - \tau_o}{2\tau_o} \leq \frac{|\hat{\tau}_1| - \tau_o}{2\tau_o}$$

$$< \frac{|\hat{\tau}_1| - |\hat{\tau}_1|}{2\tau_o} = 0$$

Also

$$a = \frac{\hat{\tau}_1 - \tau_o}{2\tau_o} \geq \frac{-|\hat{\tau}_1| - \tau_o}{2\tau_o} > \frac{-\tau_o - \tau_o}{2\tau_o} = -1.$$

Therefore

$$-1 < a < 0. \tag{C.20}$$

Since b = 1 + a relation (C.20) may be written

$$-1+1 < a+1 < 0+1$$

or

$$0 < b < 1 \tag{C.21}$$

From equation (C.18) and inequality (C.9)

$$0 < \lambda_1 = (d-b)\tau_o$$

or

$$d > b. \tag{C.22}$$

One more relation that is useful can be obtained by letting $L = \lambda_2/\lambda_1$ and solving the ratio of equation (C.19) to (C.18) for d.

$$L \equiv \frac{\lambda_2}{\lambda_1} = \frac{d-a}{d-b} \qquad d = \frac{bL-a}{L-1} \tag{C.23}$$

APPENDIX D. DERIVATION OF EQUATION (4.5)

Starting with equations (4.4) derive equations (4.5). From equation (4.4c),

$$2|F_1|\,|F_2|\cos(\Phi_2-\Phi_1) = |F_1|^2 + |F_2|^2 - \frac{|y_{12}|^2}{d^2}\,. \qquad \text{(D.1)}$$

Substitute this into equations (4.4a) and (4.4b) and remember that b-a = 1.

$$|y_{11}|^2 = a^2|F_1|^2 + b^2|F_2|^2 - ab\left(|F_1|^2 + |F_2|^2 - \frac{|y_{12}|^2}{d^2}\right)$$

$$= -a|F_1|^2 + b|F_2|^2 + \frac{ab}{d^2}|y_{12}|^2 \qquad \text{(D.2)}$$

$$r^2|y_{22}|^2 = b^2|F_1|^2 + a^2|F_2|^2 - ab\left(|F_1|^2 + |F_2|^2 - \frac{|y_{12}|^2}{d^2}\right)$$

$$= b|F_1|^2 - a|F_2|^2 + \frac{ab}{d^2}|y_{12}|^2 \qquad \text{(D.3)}$$

From equation (D.2)

$$|F_1|^2 = \frac{b|F_2|^2 + \frac{ab}{d^2}|y_{12}|^2 - |y_{11}|^2}{a} \qquad \text{(D.4)}$$

Substitute this into (D.3) and solve for $|F_2|^2$.

$$r^2|y_{22}|^2 = \frac{b}{a}\left[b|F_2|^2 + \frac{ab}{d^2}|y_{12}|^2 - |y_{11}|^2\right] - a|F_2|^2 + \frac{ab}{d^2}|y_{12}|^2$$

$$|F_2|^2 = \frac{d^2\,(b|y_{11}|^2 + a\,r^2|y_{22}|^2) - ab(b+a)|y_{12}|^2}{d^2\,(b+a)} \qquad \text{(D.5)}$$

Substituting (D.5) into (D.4) results in

$$|F_1|^2 = \frac{b}{a}\left[\frac{d^2(b|y_{11}|^2 + a\,r^2|y_{22}|^2) - ab(b+a)|y_{12}|^2}{d^2(b+a)}\right]$$

$$+ \frac{b}{d^2}|y_{12}|^2 - \frac{|y_{11}|^2}{a}$$

$$= \frac{d^2(a|y_{11}|^2 + b\, r^2|y_{22}|^2) - ab(b+a)|y_{12}|^2}{d^2(b+a)} \tag{D.6}$$

Substitution of (D.6) and (D.5) into (D.3) leads to

$$|F_1|\,|F_2|\,\cos(\Phi_2 - \Phi_1) = \frac{d^2(|y_{11}|^2 + r^2|y_{22}|^2) - (b^2+a^2)|y_{12}|^2}{2d^2} \tag{D.7}$$

Equations (D.6), (D.5), and (D.7) are equations (4.5a, b, and c) respectively.

APPENDIX E. DISCUSSION OF NETWORK PARAMETERS

In the design of any network the engineer must decide what are the parameters, how many independent parameters or degrees of freedom he has to work with, and which parameters are then dependent, and which if any have no effect whatsoever on the network response. This appendix discusses these questions as they apply to the design procedure of Chapter 5.

The general circuit equation for a distributed $\overline{RC}$ network with $p = n+m+1$ cascaded sections, equation (2.40), i.e.

$$\underline{Y}_p(s) = \hat{H}\,\underline{M}\,\sqrt{s\underline{T}}\left[(e^{2\sqrt{s\underline{T}}} - \underline{I})\prod_{j=1}^{m}(e^{4\sqrt{s\underline{T}}} + A_j e^{2\sqrt{s\underline{T}}} + \underline{I})\right]^{\pm 1} \times\left[(e^{2\sqrt{s\underline{T}}} + \underline{I})^{2(m-n)+1}\prod_{j=1}^{n}(e^{4\sqrt{s\underline{T}}} + B_j e^{2\sqrt{s\underline{T}}} + \underline{I})\right]^{\mp 1} \tag{E.1}$$

has p+3 constants or parameters that require evaluating, T_{11}, T_{21}, T_{22}, $\hat{H}$, A_j $(j=1,2,\cdots,m)$, and B_j $(j=1,2,\cdots,n)$ for p+3 degrees of freedom of the equation. Let

$$\hat{\rho} = \frac{\rho_\alpha l}{h_{\rho\alpha}} \tag{E.2}$$

and

$$\hat{\epsilon} = \frac{\epsilon_\alpha l}{h_{\epsilon\alpha}}, \quad \alpha = a,\ b. \tag{E.3}$$

Then, looking at the physical device the degrees of freedom are $\hat{\rho}_a$, $\hat{\rho}_b$, $\hat{\epsilon}_a$, $\hat{\epsilon}_b$, and the p widths for a total of p+4. Since the physical device has one more degree of freedom than the equation, one of the physical parameters may be chosen completely arbitrarily. Let this be $\hat{\rho}_a$ so that the resistivity, length, and thickness (ρ_a, l, and $h_{\rho a}$ respectively) may be selected by the engineer within the physical realizable range. Observe from equation (2.40) or (3.3) and (3.4) that the constant $\hat{H}$ is a multiplicative factor of each of the short circuit admittance parameters (y_{ij}'s). Because of this, the network may be designed by initially ignoring $\hat{H}$ or setting it equal to unity before evaluating the other network constants. After the basic design, which consists of the selection of the other circuit constants, is complete $\hat{H}$ may be adjusted to give the desired admittance level without altering the network performance in any other way.

The normalized frequency $\eta = 2\omega\lambda_1$ will now be discussed. By refering to the definition of λ_1, $\mathcal{T}_1$, and $\mathcal{T}_o$ of equation (C.4), (C.1), and (C.3) it can be seen that multiplying λ_1 by some factor q is the same as multiplying $\mathcal{T}_{11}$, $\mathcal{T}_{21}$, $\mathcal{T}_{22}$, and consequently $\mathcal{T}_1$ and $\mathcal{T}_o$ by the same factor. Further, reference to the defining equations for a, b, and d, equations (C.10), (C.11), and (C.12), shows that multiplying λ_1 by q leaves these unchanged, and consequently by equation (3.4) the short circuit admittance parameters as functions of η are also unchanged. Thus multiplication of λ_1 by the constant q leaves the plot of the network response unchanged in shape but shifted along the frequency axis by a factor 1/q. Therefore in the design procedure the value of λ_1 should be selected at the same time $\hat{H}$ is chosen, after the basic design is complete.

If equation (2.39) of Chapter 2 is solved for $e^{2\sqrt{s\underline{\tau}}}$ and the result substituted into (2.40) it can be seen that

$$\hat{\underline{Y}}_p(\underline{\Xi}) = \hat{H}\,\underline{\Xi}\ ,$$

and by (2.28)

$$\hat{\underline{Y}}_p(\underline{\Xi}) = k_p\,\underline{\Xi} = \hat{H}\,\underline{\Xi}\ .$$

Thus $\hat{H} = k_p$ and recalling from Chapter 2 equation (2.38) that

$$w_i = \frac{\rho_a 1}{h_{\rho a}}\ k_i = \hat{\rho}_a k_i$$

it can be seen that

$$w_p = \hat{\rho}_a\ k_p = \hat{\rho}_a\ \hat{H} \tag{E.4}$$

so that the actual value of the widths of the p^{th} section is determined by $\hat{\rho}_a$ and $\hat{H}$. In other words the selection of $\hat{H}$ as a parameter replaces w_p

as a degree of freedom.

It has just been shown that $\hat{\rho}_a$, $\hat{H}$, and λ_1 do not affect the shape of th response plot and are therefore to be selected after the basic design is completed. Substracting these quantities from the original p+4 degrees of freedom leaves p+1 parameters to be discussed. In design work the engine is concerned with practical considerations such as how great a width ratio can be obtained between two sections and what are the maximum practical limits to the ratios of permittivity or resistivity between layers. Since w_p is determined by $\hat{H}$ and $\hat{\rho}_a$, it is convenient to use w_ρ as a reference and divide the width of each section by w_p and choose these ratios as parameters, giving p-1 more parameters. Therefore let

$$\hat{w}_i \equiv \frac{w_i}{w_p}, \; i = 1, 2, \cdots, p-1. \tag{E.5}$$

Since for the four layered case

$$r \equiv \frac{R_{bi}}{R_{ai}} = \frac{\hat{\rho}_b/w_i}{\hat{\rho}_a/w_i} = \frac{\hat{\rho}_b}{\hat{\rho}_a} \tag{E.6}$$

and

$$c \equiv \frac{C_{bi}}{C_{ai}} = \frac{\hat{\epsilon}_b w_i}{\hat{\epsilon}_a w_i} = \frac{\hat{\epsilon}_b}{\hat{\epsilon}_a}, \tag{E.7}$$

r and c would be convenient choices as the remaining two degrees of freedom. If an asymptotic gain is specified such as $G_{12\infty}$ or $G_{21\infty}$ of equations (3.9a and (3.9b), only one of the quantities r and c remains an independent parameter.

After all of the $\hat{w}_i$'s, r, c, $\hat{H}$, λ_1, and $\hat{\rho}_a$ are selected, the physical parameters may be calculated. The widths are calculated from equations (E.4) and (E.5).

$$w_p = \hat{\rho}_a \hat{H} \tag{E.8}$$

and

$$w_i = \hat{w}_i \cdot w_p, \; i = 1,2, \cdots, p-1. \tag{E.9}$$

Also, from (E.6) $\hat{\rho}_b = r\hat{\rho}_a$. (E.10)

From equations (C.1), (C.3), and (C.4)

$$\lambda_1 = \frac{1}{2}\left[(T_{11}+T_{21}+T_{22}) - \sqrt{(T_{11}+T_{21}+T_{22})^2 - 4T_{11}T_{22}}\right].$$

Division of each side of this equation by T_{11} leads to

$$T_{11} = \frac{2\lambda_1}{(1+r+rc) - \sqrt{(1+r+rc)^2 - 4rc}}\ , \tag{E.11}$$

and since $T_{11} = R_{ai}\ C_{ai} = \hat{\rho}_a\ \hat{\epsilon}_a$,

$$\hat{\epsilon}_a = T_{11}/\hat{\rho}_a \tag{E.12}$$

and

$$\hat{\epsilon}_b = c\ \hat{\epsilon}_a \tag{E.13}$$

Equations (E.8) through (E.13) constitute the final design phase, that of determining the magnitudes of the constants of the real, physical network.