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Volterra difference equations

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VOLTERRA DIFFERENCE EQUATIONS

by

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DEDICATION

This dissertation is dedicated to my loving mother

NoorJahan Begum

and

my beloved son

Mahir Rayan

PUBLICATION DISSERTATION OPTION

This dissertation contains the following five articles: (i). Subexponential solutions of linear Volterra difference equations, pages 25–49, (ii). Rate of convergence of solutions of linear Volterra difference equations, pages 50–62, (iii). Subexponential solutions of linear Volterra delay difference equations, pages 63–83, (iv). Bounded solutions of a Volterra difference equation, pages 84–98, and (v). Asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations, pages 99–113. The article (i) is submitted. The articles (ii), (iii), and (iv) will be submitted. The article (v) is already published in Opuscula Mathematica.

ABSTRACT

This dissertation consists of five papers in which discrete Volterra equations of different types and orders are studied and results regarding the behavior of their solutions are established. The first paper presents some fundamental results about subexponential sequences. It also illustrates the subexponential solutions of scalar linear Volterra sum-difference equations are asymptotically stable. The exact value of the rate of convergence of asymptotically stable solutions is found by determining the asymptotic behavior of the transient renewal equations. The study of subexponential solutions is also continued in the second and third articles. The second paper investigates the same equation using the same process as considered in the first paper. The discussion focuses on a positive lower bound of the rate of convergence of the asymptotically stable solutions. The third paper addresses the rate of convergence of the solutions of scalar linear Volterra sum-difference equations with delay. The result is proved by developing the rate of convergence of transient renewal delay difference equations. The fourth paper discusses the existence of bounded solutions on an unbounded domain of more general nonlinear Volterra sum-difference equations using the Schaefer fixed point theorem and the Lyapunov direct method. The fifth paper examines the asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations and establishes some new criteria based on so-called time scales, which unifies and extends both discrete and continuous mathematical analysis. Beside these five research papers that focus on discrete Volterra equations, this dissertation also contains an introduction, a section on difference calculus, a section on time scales calculus, and a conclusion.

ACKNOWLEDGMENTS

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NOMENCLATURE

$\mathbb{T}$ A Time Scale

$\mathbb{R}$ Set of Real Numbers

$\mathbb{N}$ Set of Natural Numbers

$\mathbb{N}_0$ $\mathbb{N} \cup \{0\}$

$\mathbb{N}_0^2$ The Set $\{0, 1, 4, 9, 16, \ldots\}$

$h\mathbb{Z}$ The Set $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{C}$ Set of Complex Numbers

$\overline{q^{\mathbb{Z}}}$ The Set $\{\ldots, q^{-2}, q^{-1}, 1, q, q^2 \ldots\}$ for $q > 1$.

$q^{\mathbb{N}_0}$ The Set $\{1, q, q^2, q^3 \ldots\}$ for $q > 1$.

σ Forward Jump Operator

ρ Backward Jump Operator

μ Graininess Function

Δ Delta Derivative Operator

Δ Forward Difference Operator

1. INTRODUCTION

Volterra integral equations are a special type of integral equation. These equations were introduced by the Italian mathematician and physicist Vito Volterra. Volterra is known for his contributions to mathematical biology, integral equations, and the foundation of functional analysis. Traian Lalescu studied Volterra's integral equations in his thesis, *Sueles Équations de Volterra,* written under the direction of Émile Picard, in 1908. Lalescu wrote the first book on integral equations in 1911. Two types of Volterra integral equations exist, namely the first and second kind. The linear equations of the form (i) $f(t) = \int_a^t k(t, s)x(s)ds$ and (ii) $x(t) = f(t) + \int_a^t k(t, s)x(s)ds$ are known as the Volterra integral equation of the first and second kind, respectively, where f is a given function and x is unknown. The function k in the integral is often known as the kernel. The linear equations of the form (iii) $x(t) = f(t) + \int_a^t k(t - s)x(s)ds$ are known as Volterra convolution integral equations. The application of Volterra integral equations can be found in demography, viscoelastic material, and insurance mathematics.

While working on population growth models, Volterra also studied hereditary influence. This research led to a special type of equation in which both the integral and the differential operator appeared together in the same equation. The general form of this special type of equation is

$$x^{(n)}(t) = f(t) + \lambda \int_0^t K(t, s)x(s)ds, \quad \text{where} \quad x^n(t) = \frac{d^n x}{dt^n},$$

which was termed Volterra integro-differential equation in the *Theory of Functionals of Integrals and Integro-Differential Equations by V. Volterra.* The initial conditions $x(0), x'(0), \ldots, x^{(n-1)}(0)$ must be defined before the solution x can be identified. They are characterized by the existence of one or more of the derivatives $x'(t), x''(t), \ldots$ outside the integral sign. The Volterra integro-differential equation may be observed

at the time of conversion of an initial value problem to an integral equation by using the Leibnitz rule. This type of equation appears in many physical applications such as heat transfer, the neutron diffusion process, the glass-forming process, wind ripple, reactor dynamics, viscoelasticity, and coexistence of biological species together with increasing and decreasing generating rate.

Several variables are allowed to take any value on some interval of the real line during mathematical representation of a real-life problem. These variables are known as continuous variables. These variables cannot be continuous in other real-life problems (e.g., an investment with compound interest). Thus, choosing the variables to be discrete would be more appropriate. The variables would, therefore, be evaluated at discrete times. These values would remain unchanged throughout each time period and then jump to another period as time moves forward (e.g., time moves from one time period to the next). Each variable of interest in this framework is measured once at each time period, and the number of measurements between any two time periods is finite. The discrete variables are measured sequentially. As a result, such variables can be represented in terms of either other values or their own prior values. This process is known as a *recurrence relation* (or *difference equation*). Difference equations have applications in almost every area of study, such as in probability theory, a stochastic time series, number theory, combinatorics, electrical networks, genetics, psychology, sociology, and economics. Although difference equations were discovered much earlier than differential equations, little research has been done on them compared to differential equations. Difference equations have recently come to light due to advancements in computer programming because solving a differential equation with the aid of a computer requires a formulation using their approximate difference equation. Saber Elaydi in [13] introduced the difference equations of the Volterra type that are the analogue to the differential equations. Many works have been devoted to these types of equations during the last few years.

Discrete Volterra equations, meaning Volterra equations with discrete time, of different types and orders are studied here. These equations primarily arise from the mathematical modeling of some real phenomena and from the application of numerical methods to Volterra integral equations. Therefore, studying such equations is important. Developing their quantitative and, notably, their qualitative behavior is essential as well. Relatively few works have addressed discrete Volterra equations and their asymptotic behavior. A number of researchers (e.g., Medina [23], Győri and Horvath [15], Song and Baker [29, 30], Kolmanovskii et al. [21, 22], Migda [24], Győri and Reynolds [16], and Appleby et al. [6]) recently investigated the asymptotic properties (stability, oscillation) of Volterra difference equations and discrete Volterra systems.

Much of the general qualitative theory and asymptotic properties of discrete Volterra equation solutions need to be developed. The research discussed here was focused on Volterra sum-difference equations, which are analogous to Volterra integro-differential equations. In contrast, the individual discrete and continuous models may not clearly model some physical phenomena. Discrete-continuous hybrid models are more meaningful in this situation to describe these physical phenomena. The time scales theory not only unifies discrete and continuous approaches but also extends the entire theory to a much more general framework. In this framework, the discrete and continuous counterparts are only the special cases. Hence, this theory is used to model the asymptotic behaviors of higher order integro-dynamic equations.

This dissertation consists of five papers focusing on discrete Volterra equations. The first three are concentrated on subexponential solutions of scaler linear Volterra sum-difference equations of the second kind. The fourth focuses on bounded solutions of more general scaler nonlinear Volterra sum-difference equations of the second kind. The final paper in this dissertation examines asymptotic behavior of higher-order Volterra integro-dynamic equations of the first kind on time scales.

By using subexponential properties, the asymptotic behavior of solutions of a scalar linear Volterra sum-difference equation is discussed in the first paper. Some fundamental results regarding subexponential sequences are developed first. Both the subexponential solutions and the rate of convergence of transient renewal equations are studied next. We show the solution of the considered scalar convolution sum-difference equation is unique and asymptotically stable. Finally, we find the rate of convergence of its solution assuming the kernel to be a positive subexponential sequence and using the asymptotic behavior of the transient renewal equation. We also give an example of subexponential sequences including verification.

The second paper is based on the same problem that was examined in the first paper. Here, instead of assuming the kernel is positive subexponential, we assume the kernel is positive, summable, and its first order difference converges to zero. Instead of an exact value, we obtain a positive lower bound for the rate of convergence of asymptotically stable solutions. Initially, we find a positive lower bound for the rate of convergence of solutions of the transient renewal equation. This bound is used to determine the positive lower bound for the rate of convergence of asymptotically stable solutions of the considered difference equation.

The third paper is focused on delay difference equations. We consider a scalar linear Volterra sum-difference equation with delay and assume that the kernel is a positive, summable, and subexponential sequence. By the subexponential kernel, we mean the ratio of the kernel to a positive subexponential sequence that converges to a positive limit. Some important results on delay difference equations are established first. The rate of convergence of solutions of delay transient renewal equations are determined next. The rate of convergence of asymptotically stable solutions of the considered delay equations is observed by applying the results for the rate of convergence of solutions of delay transient renewal equations.

The existence of bounded solutions of more general nonlinear Volterra sum-difference equations is discussed in the fourth paper. Schaefer's fixed point theorem

is applied, under some hypothesis, to show the existence of a bounded solution on an unbounded domain. To find the *a priori* bound for the Schaefer's fixed point theorem, we use Lyapunov's direct method. An upper bound is found to exist for all solutions. Moreover, we give examples of discrete nonlinear Volterra sum-difference equations including a bounded oscillatory solution.

The fifth and last paper is based on so-called time scales, which unifies and extends discrete and continuous mathematics. Using some assumptions and established results, we develop some new criteria for the asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations under various restrictions on constants which are the ratios of positive odd integers. Finally, we present several remarks and extensions of the obtained results.

2. INTRODUCTION TO DIFFERENCE CALCULUS

We usually do not begin with an explicit formula for terms of a sequence because we may only know some relationship between various terms. An equation which defines a value of a sequence as a function of other terms of the sequence is called a *difference equation* or *recurrence equation.* These types of equations may appear in many different settings and forms, both in mathematics itself and in its applications to various fields such as statistics, economics, biology, dynamical systems, and electrical circuit analysis.

Difference calculus is a collection of mathematical tools which is quite similar to differential calculus. It is used to simplify many calculations involved in solving and analyzing difference equations. For basic concepts, we refer to $[1, 2, 13, 20]$.

2.1. DIFFERENCE OPERATOR

The basic component of calculations of finite differences in difference calculus is the difference operator. Its role is similar to the differential operator in differential calculus.

Definition 2.1. Let $x(t)$ be a sequence of real or complex numbers. The difference operator Δ is defined by

$$\Delta x(t) = x(t+1) - x(t).$$

We consider $t \in \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Note, there is no loss of generality by using step size 1. Consider a difference operator with step size $h > 0$, say $y(s+h) - y(s)$ and let $x(t) = y(th)$. Then

$$y(s+h) - y(s) = y(th+h) - y(th) = y((t+1)h) - y(th) = x(t+1) - x(t) = \Delta x(t).$$

Higher order differences are iterations of the basic difference operator. For example, the second order difference is

$$
\begin{aligned}
\Delta^2 x(t) &= \Delta(\Delta x(t)) \\
&= \Delta(x(t+1) - x(t)) \\
&= (x(t+2) - x(t+1)) - (x(t+1) - x(t)) \\
&= x(t+2) - 2x(t+1) + x(t).
\end{aligned}
$$

Definition 2.2. For any $n \in \mathbb{N}$,

$$
\Delta^n x(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x(t+n-k). \tag{2.1}
$$

Other useful operators are the shift operator and the identity operator.

Definition 2.3. The shift operator E is defined by

$$
Ex(t) = x(t+1).
$$

Definition 2.4. The identity operator I is defined by

$$
Ix(t) = x(t).
$$

The composition of I and E is the same as multiplication of numbers. Clearly, $\Delta = E - I$, therefore (2.1) is in fact similar to the binomial theorem:

$$
\begin{aligned}
\Delta^n x(t) &= (E - I)^n x(t) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-I)^k E^{n-k} x(t) \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x(t+n-k).
\end{aligned}
$$

Similarly, we have

$$E^n x(t) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} x(t).$$

The fundamental properties of Δ are given in the following theorem:

Theorem 2.5. *For any $m, n \in \mathbb{N}$ and any $c \in \mathbb{R}$:*

(a) $\Delta^m(\Delta^n x(t)) = \Delta^{m+n} x(t).$

(b) $\Delta(x(t) + y(t)) = \Delta x(t) + \Delta y(t).$

(c) $\Delta(cx(t)) = c\Delta x(t).$

(d) $\Delta(x(t)y(t)) = x(t)\Delta y(t) + Ey(t)\Delta x(t).$

(e) $\Delta\left(\frac{x(t)}{y(t)}\right) = \frac{y(t)\Delta x(t) - x(t)\Delta y(t)}{y(t)Ey(t)}.$

Proof. Parts (a)–(c) are easily proven using the previous definitions. The calculation

$$
\begin{aligned}
\Delta(x(t)y(t)) &= x(t+1)y(t+1) - x(t)y(t) \\
&= x(t+1)y(t+1) - x(t)y(t+1) + x(t)y(t+1) - x(t)y(t) \\
&= y(t+1)(x(t+1) - x(t)) + x(t)(y(t+1) - y(t)) \\
&= x(t)\Delta y(t) + Ey(t)\Delta x(t)
\end{aligned}
$$

proves (d), and (e) can be shown in a similar way. □

The next theorem states the difference formulas for some basic functions.

Theorem 2.6. *Let a be any constant. Then*

(a) $\Delta a^t = (a - 1)a^t.$

(b) $\Delta \sin at = 2 \sin \frac{a}{2} \cos a(t + \frac{1}{2}).$

(c) $\Delta \cos at = -2 \sin \frac{a}{2} \sin a(t + \frac{1}{2}).$

(d) $\Delta \log at = \log(1 + \frac{1}{t})$.

(e) $\Delta \log \Gamma(t) = \log(t)$.

Remark 2.7. All formulas in Theorem 2.6 remain true for a constant shift k of t, i.e., when t is replaced by $t + k$.

Combinations of Theorems 2.5 and 2.6 can be used to find the differences of more complicated expressions. In many cases, it may be easier to use the given definitions to find these differences.

There are some functions whose derivatives are simple but differences are complicated. There are other functions that are not often studied in calculus but whose differences are easily examined. For example, in differential calculus one of the basic formulas is the power rule

$$d(t^n) = nt^{n-1}.$$

Unfortunately, in difference calculus, the difference of a power is complicated and therefore is not very useful:

$$
\begin{aligned}
\Delta t^n &= (t+1)^n - t^n \\
&= \sum_{k=0}^{n} \binom{n}{k} t^k - t^n \\
&= \sum_{k=0}^{n-1} \binom{n}{k} t^k.
\end{aligned}
$$

2.2. SUMMATION

In order to apply the difference operator effectively, we introduce an antidifference operator. This is known as the right inverse operator and is sometimes called the indefinite sum.

Definition 2.8. An indefinite sum of $x(t)$, denoted by $\sum x(t)$, is any function such that

$$\Delta\left(\sum x(t)\right) = x(t)$$

for all t in the domain of x.

As in integration, summation requires a summation constant, which might not always be constant.

Theorem 2.9. *If $y(t)$ is an indefinite sum of $x(t)$, then every indefinite sum of $x(t)$ can be expressed as*

$$\sum x(t) = y(t) + C(t),$$

where C has the same domain as x and $\Delta C(t) = 0$.

If the domain of x is the set of real numbers, then $\Delta C(t) = 0$ implies $C(t+1) = C(t)$, which means that C is a periodic function of period one.

Corollary 2.10. *If the domain of $x(t)$ is the set of the form $\{a, a + 1, a + 2, ...\}$, where a is any real number, and $y(t)$ is an indefinite sum of $x(t)$, then every indefinite sum of $x(t)$ has the form*

$$\sum x(t) = y(t) + C,$$

where C is an arbitrary constant.

The summation of some basic functions following from Theorems 2.5 and 2.6 are given in the next theorem.

Theorem 2.11. *Let a be a constant. Then for $\Delta C(t) = 0$,*

(a) $\sum a^t = \frac{a^t}{(a-1)} + C(t)$, $a \neq 1$.

(b) $\sum \sin at = \frac{\cos a(t+\frac{1}{2})}{2\sin\frac{a}{2}} + C(t)$, $a \neq 2n\pi$.

(c) $\sum \cos at = \frac{\sin a(t+\frac{1}{2})}{-2\sin\frac{a}{2}} + C(t)$, $a \neq 2n\pi$.

(d) $\sum \log t = \log \Gamma(t) + C(t)$, $t > 0$.

Many general properties of the indefinite sum can be derived from Theorem 2.5.

Theorem 2.12. *For any constant c*

(a) $\sum(x(t) + y(t)) = \sum x(t) + \sum y(t)$.

(b) $\sum(cx(t)) = c\sum x(t)$.

(c) $\sum(x(t)\Delta y(t)) = x(t)y(t) - \sum Ey(t)\Delta x(t)$.

(d) $\sum(Ex(t)\Delta y(t)) = x(t)y(t) - \sum y(t)\Delta x(t)$.

Remark 2.13. Parts (c) and (d) of Theorem 2.12 are known as "summation by parts" formulas.

Proof. Parts (a) and (b) follow directly from Theorem 2.5. From part (d) of Theorem 2.5, we have

$$\Delta(x(t)y(t)) = x(t)\Delta y(t) + Ey(t)\Delta x(t).$$

By Theorem 2.9, we have

$$\sum(x(t)\Delta y(t) + Ey(t)\Delta x(t)) = x(t)y(t) + C(t).$$

Then (c) follows from (a) with some rearrangement. Finally, (d) is simply a rearrangement and relabeling of (c). $\square$

The summation by parts formulas can be used to compute certain indefinite sums similar to how the integration by parts formula is used to compute integrals. Moreover, these formulas turn out to have fundamental importance in the analysis of

difference equations. Now, for $m < n$, Corollary 2.10 gives us

$$\sum y_n = \sum_{k=m}^{n-1} y_k + C \tag{2.2}$$

for some constant C and, alternatively, for $n \leq p$,

$$\sum y_n = -\sum_{k=n}^{p} y_k + C \tag{2.3}$$

for some constant C. Equations (2.2) and (2.3) give us a way of relating indefinite sums to definite sums.

The following theorem is used to compute definite sums analogous to the fundamental theorem of calculus.

Theorem 2.14 (Fundamental Theorem of Difference Calculus)**.** *If y_n is an indefinite sum of x_n, then for $m < n$*

$$\sum_{k=m}^{n-1} x_k = [y_k]_m^n = y_n - y_m.$$

The next theorem gives a version of the summation by parts method for definite sums.

Theorem 2.15. *If $m < n$, then*

$$\sum_{k=m}^{n-1} a_k \Delta b_k = [a_k b_k]_m^n - \sum_{k=m}^{n-1} (\Delta a_k) b_{k+1}.$$

Proof. Let $x_n = a_n$ and $y_n = b_n$, then by part (c) of Theorem 2.12, we have

$$\sum a_n \Delta b_n = a_n b_n - \sum (\Delta a_n) b_{n+1}.$$

By (2.2),
$$\sum_{k=m}^{n-1} a_k \Delta b_k = [a_k b_k]_m^n - \sum_{k=m}^{n-1} (\Delta a_k) b_{k+1} + C.$$

Substituting $n = m + 1$ in the above equation, we get

$$\sum a_m \Delta b_m = a_{m+1} b_{m+1} - (\Delta a_m) b_{m+1} + C.$$

This implies $C = -a_m b_m$, and hence, the proof is complete. □

Remark 2.16. An equivalent form of Theorem 2.15 is Abel's summation formula

$$\sum_{k=m}^{n-1} a_k b_k = b_n \sum_{k=m}^{n-1} a_k - \sum_{k=m}^{n-1} \left(\sum_{i=m}^{k} a_i \right) \Delta b_k, \quad (n > m)$$

and, alternatively,

$$\sum_{k=n}^{p} a_k b_k = b_{n-1} \sum_{k=n}^{p} a_k + \sum_{k=n}^{p} \left(\sum_{i=k}^{p} a_i \right) \Delta b_{k-1}, \quad (p \geq n).$$

We now turn to linear difference equations. There will be many parallels between the difference equation theory and the corresponding differential equation theory. Assume the domain is a discrete set, $t = a, a + 1, a + 2, \cdots$, and that there exists a function $p(t)$ with $p(t) \neq 0$ for all t. Then the linear homogeneous first-order difference equation is

$$u(t + 1) = p(t)u(t). \tag{2.4}$$

The solution is obtained by iteration

$$
\begin{aligned}
u(a + 1) &= p(a)u(a) \\
u(a + 2) &= p(a + 1)p(a)u(a) \\
&\vdots \\
u(a + n) &= u(a) \prod_{k=0}^{n-1} p(a + k).
\end{aligned}
$$

This can be written as

$$u(t) = u(a) \prod_{s=a}^{t-1} p(s), \quad (t = a, a+1, \cdots),$$

where $\prod_{s=a}^{a-1} p(s) \equiv 1$ and, for $t \geq a+1$, the product is taken over $a, a+1, \cdots, t-1$.

Consider the linear nonhomogeneous difference equation

$$x(t+1) - p(t)x(t) = r(t). \qquad (2.5)$$

This can be solved by using a similar technique as in differential equations such as reduction of order or variation of parameters. Let $x(t) = u(t)v(t)$, where $u(t)$ is the solution of the corresponding homogeneous difference equation, i.e., any nontrivial solution of (2.4) and $v(t)$ is to be determined:

$$u(t+1)v(t+1) - p(t)u(t)v(t) = r(t).$$

Using (2.4), we get

$$u(t+1)\Delta v(t) = r(t),$$

i.e.,

$$\Delta v(t) = \frac{r(t)}{Eu(t)},$$

i.e.,

$$v(t) = \sum \frac{r(t)}{Eu(t)} + C,$$

where C is any arbitrary constant. Therefore,

$$x(t) = u(t)\left[\sum \frac{r(t)}{Eu(t)} + C\right] = u(a)\prod_{s=a}^{t-1} p(s)\left[\sum \frac{r(t)}{Eu(t)} + C\right].$$

The above results are summarized in the following theorem.

Theorem 2.17. *Let $p(t) \neq 0$ and $r(t)$ be a given function defined on $t = a, a+1, a+2, \cdots$. Then (2.4) has solutions of the form*

$$u(t) = u(a) \prod_{s=a}^{t-1} p(s), \quad (t = a, a+1, \cdots),$$

and all solutions of (2.5) are given by

$$x(t) = u(t) \left[\sum \frac{r(t)}{Eu(t)} + C \right] = u(a) \prod_{s=a}^{t-1} p(s) \left[\sum \frac{r(t)}{Eu(t)} + C \right],$$

where C is any arbitrary constant and $u(t)$ is any nonzero solution of (2.4).

3. INTRODUCTION TO TIME SCALE CALCULUS

Time scales calculus is the unification of the theory of difference calculus with differential and integral calculus. This new school of thought was introduced by German mathematician Stefan Hilger in 1988. The key concept of the study of dynamic equations on time scales is a way of unifying and extending continuous and discrete mathematical analysis, which allows us to generalize a process to deal with both continuous and discrete cases and any combination. Time scales have applications in any field where modeling requires both continuous and discrete data simultaneously. Because of its hybrid formalism, in the past few years, this area of mathematics has received considerable attention. Application of time scales can be found in various fields such as population dynamics, economics, electrical circuits, heat transfer, etc. We refer to [8,9] for in-depth study of time scales.

3.1. BASIC DEFINITIONS

Definition 3.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

Some common examples of time scales are given below.

Example 3.2. (i) $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$;

(ii) $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$ for $h > 0$;

(iii) $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ for $q > 1$;

(iv) $\mathbb{T} = 2^{\mathbb{N}_0} := \{2^k : k \in \mathbb{N}_0\}$;

(vi) $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$;

(vii) the Cantor set.

The combination of any of the above sets is also a time scale and called a hybrid time scale. On the other hand, the sets $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{C}$, and (a, b) with $a, b \in \mathbb{R}$ are not time scales.

Now, we define the forward jump and backward operator.

Definition 3.3. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, for the empty set $\emptyset$, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t).

Definition 3.4. Let $t \in \mathbb{T}$. Then t is said to be

   (i) right-scattered, if $\sigma(t) > t$,

  (ii) left-scattered, if $\rho(t) < t$,

 (iii) isolated, if t is right-scattered and left-scattered,

 (iv) right-dense, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$,

  (v) left-dense, if $t > \inf \mathbb{T}$ and $\rho(t) = t$,

 (vi) dense, if t is right-dense and left-dense.

Table 3.1 summarizes the classification of points defined in Definition 3.4.

Definition 3.5. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Table 3.1. Classification of points

t right-scattered	$t < \sigma(t)$
t left-scattered	$t > \rho(t)$
t isolated	$\rho(t) < t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-dense	$t = \rho(t)$
t dense	$\rho(t) = t = \sigma(t)$

Remark 3.6. Due to our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$, both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$.

In Table 3.2, the forward jump operator, the backward jump operator, and the graininess of some common time scales are given.

Table 3.2. Some common time scales and their corresponding forward and backward jump operators and graininess

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$h\mathbb{Z}$	$t+h$	$t-h$	h
$q^{\mathbb{N}_0}$	qt	$\frac{t}{q}$	$(q-1)t$
$2^{\mathbb{N}_0}$	$2t$	$\frac{t}{2}$	t
$\mathbb{N}_0^2$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$	$(2\sqrt{t}+1)$

Definition 3.7. If $\mathbb{T}$ is a time scale with left-scattered maximum m, then the set $\mathbb{T}^\kappa$ is defined by $\mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 3.8. For any function $f : \mathbb{T} \to \mathbb{R}$, the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined as

$$f^{\sigma}(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T};$$

i.e., $f^{\sigma} = f \circ \sigma$.

3.2. DIFFERENTIATION

In this section, we give the definition of delta (or Hilger) derivative and some of its useful properties.

Definition 3.9. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. The delta derivative $f^{\Delta}(t)$ is the number (if it exists) such that given any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U.$$

Properties of the delta derivative are stated in the next two theorems.

Theorem 3.10 (See [8, Theorem 1.16, page 5]). *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then, we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(*iv*) *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

Note the following examples.

Example 3.11. (*a*) When $\mathbb{T} = \mathbb{R}$, then we can deduce from Theorem 3.10 (*iii*) that

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t),$$

provided that the limit exists, i.e., $f : \mathbb{R} \to \mathbb{R}$ is differentiable at t in the usual sense.

(*b*) When $\mathbb{T} = \mathbb{Z}$, then Theorem 3.10 (*iv*) yields that

$$f^{\Delta}(t) = f(t + 1) - f(t) = \Delta f(t),$$

where Δ is the well-known difference operator.

(*c*) When $\mathbb{T} = q^{\mathbb{N}_0}$, then Theorem 3.10 (*iv*) yields

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q - 1)t}.$$

The next theorem allows us to find derivatives of sums, products, and quotients of differentiable functions.

Theorem 3.12 (See [8, Theorem 1.20, page 8]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are functions and let $t \in \mathbb{T}^{\kappa}$. Then:*

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) *If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

3.3. INTEGRATION

Consider integrable functions on an arbitrary time scale. The following two concepts are important in order to describe the class of integrable functions.

Definition 3.13. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be regulated if its left-sided and right-sided limits exist at all left-dense and right-dense points in $\mathbb{T}$, respectively.

Definition 3.14. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The

set of rd-continuous functions is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Some results regarding regulated and rd-continuous functions are stated in the following theorem.

Theorem 3.15. *Let* $f : \mathbb{T} \to \mathbb{R}$.

   (i) *If* f *is continuous, then* f *is rd-continuous.*

  (ii) *If* f *is rd-continuous, then* f *is regulated.*

 (iii) *The jump operator* σ *is rd-continuous.*

 (iv) *If* f *is regulated or rd-continuous, then so is* f^{σ}.

  (v) *Assume* f *is continuous. If* $g : \mathbb{T} \to \mathbb{R}$ *is regulated or rd-continuous, then so is*
    $f \circ g$.

Definition 3.16. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if

$$F^{\Delta}(t) = f(t) \text{ for all } t \in \mathbb{T}^{\kappa}.$$

Definition 3.17. Let $F : \mathbb{T} \to \mathbb{R}$ be an antiderivative of $f : \mathbb{T} \to \mathbb{R}$. Then the Cauchy integral of f is given by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for all } a, b \in \mathbb{T}.$$

Note the following example.

Example 3.18. Let $a, b \in \mathbb{T}$ and f be rd-continuous.

 (a) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(b) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t), & \text{if } a < b \\ 0, & \text{if } a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t), & \text{if } a > b. \end{cases}$$

(c) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a < b \\ 0, & \text{if } a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h, & \text{if } a > b. \end{cases}$$

(d) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t), & \text{if } a < b \\ 0, & \text{if } a = b \\ -\sum_{t=b}^{a-1} f(t), & \text{if } a > b. \end{cases}$$

Theorem 3.19 (Existence of Antiderivatives, see [8, Theorem 1.74, page 27]). *Every rd-continuous function possesses an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) = \int_{t_0}^t f(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}$$

is an antiderivative of f.

Some basic properties of integration on time scale are listed in the next theorem.

Theorem 3.20. *For $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f \in C_{\mathrm{rd}}$*

(i) $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b (\alpha f(t))\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^a f(t)\Delta t = 0$;

(vi) $\int_a^b f^\sigma(t) g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(t)\Delta t$;

(vii) $\int_a^b f(t) g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g^\sigma(t)\Delta t$.

Note that the formulas in Theorem 3.20 (*vi*) and (*vii*) are called "integration by parts" formulas. Also note that all of the formulas given in Theorem 3.20 also hold for the case that f and g are only regulated functions.

Finally, we consider a generalized form of the Leibniz rule.

Theorem 3.21 (See [8, Theorem 1.117, page 46]). *Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$, and assume $f : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^\kappa$ with $t > a$. Also assume that $f^\Delta(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a, \sigma(t)]$, such that*

$$|f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \ \text{ for all } \ s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

(i) $g(t) := \int_a^t f(t, \tau)\Delta\tau$ *implies* $g^\Delta(t) = \int_a^t f^\Delta(t, \tau)\Delta\tau + f(\sigma(t), t)$;

(ii) $h(t) := \int_t^b f(t, \tau)\Delta\tau$ *implies* $h^\Delta(t) = \int_t^b f^\Delta(t, \tau)\Delta\tau - f(\sigma(t), t)$.

PAPER

I. SUBEXPONENTIAL SOLUTIONS OF LINEAR VOLTERRA DIFFERENCE EQUATIONS

ABSTRACT

We study the asymptotic behavior of the solutions of a scalar convolution sum-difference equation. The rate of convergence of the solution is found by determining the asymptotic behavior of the solution of the transient renewal equation.

1. INTRODUCTION

We consider the discrete equation

$$\Delta x(t) = -ax(t) + \sum_{s=0}^{t-1} k(t-1-s)x(s), \quad t \in \mathbb{N}_0, \tag{1.1}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We suppose that $a \in (0, 1)$ and $k : \mathbb{N}_0 \to (0, \infty)$. We show that if

$$\sum_{s=0}^{\infty} k(s) < a,$$

then all solutions x of (1.1) satisfy $x(t) \to 0$ as $t \to \infty$ with the rate of convergence

$$\lim_{t \to \infty} \frac{x(t)}{k(t)} = \frac{x(0)}{(a - \sum_{s=0}^{\infty} k(s))^2}$$

provided k is in a class of subexponential sequences. The result is proved by determining the asymptotic behavior of the solution of the transient renewal equation

$$r(t) = h(t) + \sum_{s=0}^{t-1} h(t-1-s)r(s), \qquad \text{where} \qquad \sum_{s=0}^{\infty} h(s) < 1. \tag{1.2}$$

If h is subexponential, then the solution r of (1.2) satisfies

$$\lim_{t \to \infty} \frac{r(t)}{h(t)} = \frac{1}{(1 - \sum_{s=0}^{\infty} h(s))^2}.$$

The same problem has been studied in [2] for corresponding linear integro-differential equations. For basic properties and formulas concerning difference equations, we refer the reader to [1,4,6]. We also refer to [3,5] for basic results on the existence of solutions of scalar linear Volterra equations and Lyapunov functionals in the continuous case.

2. PRELIMINARIES

For $f, g : \mathbb{N}_0 \to \mathbb{R}$, we define the convolution of f and g (see [4, Section 3.10])
by

$$(f * g)(t) = \sum_{s=0}^{t-1} f(t - 1 - s)g(s), \quad t \in \mathbb{N}_0.$$

The n-fold convolution f^{*n} is given by $f^{*1} = f$ and $f^{*(n+1)} = f * f^{*n}$ for $n \in \mathbb{N}$.

Definition 2.1. A *subexponential sequence* is a discrete mapping $h : \mathbb{N}_0 \to (0, \infty)$
with $\sum_{s=0}^{\infty} h(s) < \infty$ and

(S$_1$) $\lim_{t \to \infty} \frac{h^{*2}(t)}{h(t)} = 2 \sum_{s=0}^{\infty} h(s)$;

(S$_2$) $\lim_{t \to \infty} \frac{h(t-s)}{h(t)} = 1$ for each fixed $s \in \mathbb{N}_0$.

The class of subexponential sequences is denoted by $\mathcal{U}$.

Remark 2.2. Condition (S$_2$) is equivalent to

(S$_{2a}$) $\lim_{t \to \infty} \frac{h(t+1)}{h(t)} = 1$,

and it is also equivalent to

(S$_{2b}$) $\lim_{t \to \infty} \sup_{0 \le s \le T} \left| \frac{h(t-s)}{h(t)} - 1 \right| = 0$ for each $T \in \mathbb{N}_0$.

We show that (S$_2$) and (S$_{2a}$) are equivalent: If h satisfies (S$_2$), then (take $s = 1$)

$$\lim_{t \to \infty} \frac{h(t - 1)}{h(t)} = 1, \quad \text{so} \quad \lim_{t \to \infty} \frac{h(t)}{h(t + 1)} = 1,$$

and hence (S$_{2a}$) holds. If h satisfies (S$_{2a}$), then

$$\lim_{t \to \infty} \frac{h(t + v + 1)}{h(t + v)} = 1 \quad \text{for any} \quad v \in \mathbb{Z},$$

and thus

$$\frac{h(t)}{h(t-s)} = \prod_{v=-s}^{-1} \frac{h(t+v+1)}{h(t+v)} \to 1 \text{ as } t \to \infty \text{ for all } s \in \mathbb{N}_0,$$

and hence (S_2) holds.

Remark 2.3. The terminology "subexponential" is justified by the following: If $k : \mathbb{N}_0 \to (0, \infty)$ satisfies (S_{2a}), then

$$\lim_{t \to \infty} \frac{(1-a)^t}{k(t)} = 0 \text{ for all } a \in (0, 1). \tag{2.1}$$

We prove (2.1): Define

$$K(t) := \frac{k(t)}{(1-a)^t}, \quad t \in \mathbb{N}_0.$$

Then,

$$\frac{K(t+1)}{K(t)} = \frac{k(t+1)}{(1-a)k(t)} \to \frac{1}{1-a} \text{ as } t \to \infty. \tag{2.2}$$

Hence, there exists $T \in \mathbb{N}_0$ so that

$$\frac{K(t+1)}{K(t)} > \frac{1}{1-a} - \frac{a}{2(1-a)} = \frac{2-a}{2(1-a)} > 1 \text{ for all } t \geq T.$$

Thus,

$$K(t+1) > K(t) \text{ for all } t \geq T.$$

Hence, K is eventually increasing, so either

$$\lim_{t \to \infty} K(t) = \infty \tag{2.3}$$

or otherwise,

$$\lim_{t \to \infty} K(t) =: K^* \in (0, \infty). \tag{2.4}$$

Since (2.4) implies $\lim_{t\to\infty} \frac{K(t+1)}{K(t)} = 1$, a contradiction to (2.2), we must have (2.3). Thus, (2.1) holds.

Remark 2.4. If $h \in \mathcal{U}$, then there exists $M_h \in (0, \infty)$ such that

$$\sup_{t\in\mathbb{N}_0} \frac{h^{*2}(t)}{h(t)} = M_h. \tag{2.5}$$

Lemma 2.5. *Let $h \in \mathcal{U}$, $n \in \mathbb{N}$, and*

$$\mu = \sum_{s=0}^{\infty} h(s). \tag{2.6}$$

Then

$$\sum_{s=0}^{\infty} h^{*n}(s) = \mu^n. \tag{2.7}$$

Proof. For $n = 2$, we have

$$\sum_{s=0}^{t-1} h^{*2}(s) = \sum_{s=0}^{t-1} \sum_{u=0}^{s-1} h(s-1-u)h(u)$$
$$= \sum_{u=0}^{t-2} \sum_{s=u+1}^{t-1} h(s-1-u)h(u)$$
$$= \sum_{u=0}^{t-2} \sum_{v=0}^{t-2-u} h(v)h(u).$$

As $t \to \infty$, we obtain

$$\sum_{s=0}^{\infty} h^{*2}(s) = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} h(v)h(u) = \mu^2.$$

Hence, (2.7) holds for $n = 2$. Now, assume (2.7) holds for some $n \in \mathbb{N} \setminus \{1\}$. Then

$$\sum_{s=0}^{t-1} h^{*(n+1)}(s) = \sum_{s=0}^{t-1} \sum_{u=0}^{s-1} h^{*n}(s-1-u)h(u).$$

Following the same calculation as above, we get

$$\sum_{s=0}^{\infty} h^{*(n+1)}(s) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} h^{*n}(v)h(u) = \mu^{n+1}.$$

By the principle of mathematical induction, the proof is complete. $\qquad\square$

Lemma 2.6. *Let $h \in \mathcal{U}$ and assume $\mu < 1$, where μ is defined as in (2.6). Let $\varepsilon \in (0,1)$ satisfy $(1+4\varepsilon)\mu < 1$. Then there exists $C_0 \in \mathbb{N}$ and $\lambda \geq 1$ such that*

$$\sup_{t \geq C_0} \frac{h^{*n}(t)}{h(t)} \leq \max\{\lambda, 1+\varepsilon\}\mu^{n-1}\sum_{k=0}^{n-2}(1+4\varepsilon)^k + 2(1+\varepsilon)\mu^{n-1}(1+4\varepsilon)^{n-2} \quad (2.8)$$

for all $n \in \mathbb{N} \setminus \{1\}$.

Proof. For a given $\varepsilon > 0$, due to $h > 0$, (2.6), and (S$_1$), we can choose $C_0 \in \mathbb{N}$ such that for all $t \geq C_0$,

(a) $\sum_{s=0}^{t-1} h(s) > (1-\varepsilon)\mu$,

(b) $\frac{h^{*2}(t)}{h(t)} \leq 2\mu(1+\varepsilon)$,

and also due to $h > 0$ and (S$_2$), we can choose an integer $T_0 > C_0$ such that for all $t \geq T_0$,

(c) $\left|\frac{h(t-s)}{h(t)} - 1\right| < \varepsilon$ for all $0 \leq s \leq C_0$.

Let

$$\lambda = \lambda(C_0, T_0) := \max\left\{\frac{h(t-1-s)}{h(t)} : 0 \leq s \leq t-1 \ \text{and} \ C_0 \leq t \leq T_0 - 1\right\}.$$

In order to prove (2.8), we will use the method of induction. Clearly, for $n = 2$, (2.8) holds by (b). Assume that (2.8) is true for some $n \in \mathbb{N} \setminus \{1\}$. Therefore, if $t \geq T_0$, then using (c) and the induction hypothesis, we have

$$\frac{h^{*(n+1)}(t)}{h(t)} = \sum_{s=0}^{C_0-1} h^{*n}(s)\left(\frac{h(t-1-s)}{h(t)} - 1\right) + \sum_{s=0}^{C_0-1} h^{*n}(s)$$

$$+ \sum_{s=C_0}^{t-1} \frac{h^{*n}(s)}{h(s)} \frac{h(s)h(t-1-s)}{h(t)}$$

$$\leq (1+\varepsilon) \sum_{s=0}^{C_0-1} h^{*n}(s) + \left(\max\{\lambda, 1+\varepsilon\} \mu^{n-1} \sum_{k=0}^{n-2} (1+4\varepsilon)^k \right.$$

$$\left. +2(1+\varepsilon)\mu^{n-1} (1+4\varepsilon)^{n-2} \right) \sum_{s=C_0}^{t-1} \frac{h(s)h(t-1-s)}{h(t)}. \tag{2.9}$$

Now, if $t \geq T_0$, then using (c) and (a), we obtain

$$\sum_{s=C_0}^{t-1} \frac{h(s)h(t-1-s)}{h(t)} = \sum_{s=0}^{t-1} \frac{h(s)h(t-1-s)}{h(t)} - \sum_{s=0}^{C_0-1} \frac{h(s)h(t-1-s)}{h(t)}$$

$$= \frac{h^{*2}(t)}{h(t)} - \left(\sum_{s=0}^{C_0-1} h(s) \left(\frac{h(t-1-s)}{h(t)} - 1 \right) + \sum_{s=0}^{C_0-1} h(s) \right)$$

$$\leq 2\mu(1+\varepsilon) - \left(-\varepsilon \sum_{s=0}^{C_0-1} h(s) + \sum_{s=0}^{C_0-1} h(s) \right)$$

$$= 2\mu(1+\varepsilon) - (1-\varepsilon) \sum_{s=0}^{C_0-1} h(s)$$

$$\leq 2\mu(1+\varepsilon) - (1-\varepsilon)^2 \mu$$

$$= (2 + 2\varepsilon - 1 + 2\varepsilon - \varepsilon^2)\mu$$

$$\leq (1+4\varepsilon)\mu. \tag{2.10}$$

Using (2.10) and Lemma 2.5 in (2.9), if $t \geq T_0$, then we get

$$\frac{h^{*(n+1)}(t)}{h(t)} \leq (1+\varepsilon)\mu^n + \left(\max\{\lambda, 1+\varepsilon\} \mu^{n-1} \sum_{k=0}^{n-2} (1+4\varepsilon)^k \right.$$

$$\left. +2(1+\varepsilon)\mu^{n-1} (1+4\varepsilon)^{n-2} \right) (1+4\varepsilon)\mu$$

$$= (1+\varepsilon)\mu^n + \max\{\lambda, 1+\varepsilon\} \mu^n \sum_{k=1}^{n-1} (1+4\varepsilon)^k$$

$$+ 2(1+\varepsilon)\mu^n (1+4\varepsilon)^{n-1}. \tag{2.11}$$

Now, suppose $C_0 \leq t \leq T_0 - 1$. Then

$$\frac{h^{*(n+1)}(t)}{h(t)} = \sum_{s=0}^{t-1} h^{*n}(s) \frac{h(t-1-s)}{h(t)} \leq \lambda \sum_{s=0}^{t-1} h^{*n}(s) \leq \lambda \mu^n. \tag{2.12}$$

Therefore, if $t \geq C_0$, then (2.11) and (2.12) imply

$$\begin{aligned}
\frac{h^{*(n+1)}(t)}{h(t)} &\leq \max\{\lambda, 1+\varepsilon\}\mu^n \\
&\quad + \max\{\lambda, 1+\varepsilon\}\mu^n \sum_{k=1}^{n-1}(1+4\varepsilon)^k + 2(1+\varepsilon)\mu^n (1+4\varepsilon)^{n-1} \\
&= \max\{\lambda, 1+\varepsilon\}\mu^n \sum_{k=0}^{n-1}(1+4\varepsilon)^k + 2(1+\varepsilon)\mu^n (1+4\varepsilon)^{n-1}.
\end{aligned}$$

By the principle of mathematical induction, the proof is complete. $\qquad\square$

Lemma 2.7. *Let $f, g : \mathbb{N}_0 \to (0, \infty)$. Suppose further that g is summable, satisfies (S_2), and*

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} =: \lambda > 0, \qquad \lim_{t\to\infty} \frac{(f*g)(t)}{g(t)} =: \nu.$$

Then f is summable, satisfies (S_2), and

$$\lim_{t\to\infty} \frac{(f*f)(t)}{f(t)} = \nu + \sum_{s=0}^{\infty}(f(s) - \lambda g(s)).$$

Proof. The fact that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lambda > 0$ implies that f is summable. Since g also satisfies (S_2),

$$\frac{f(t-s)}{f(t)} = \frac{f(t-s)}{g(t-s)} \cdot \frac{g(t-s)}{g(t)} \cdot \frac{g(t)}{f(t)}$$

implies that f must satisfy (S_2). Now, note that, for all $t \in \mathbb{N}$, we have

$$\frac{(f*f)(t)}{f(t)} - \lambda \frac{(f*g)(t)}{f(t)} - \sum_{s=0}^{t-1}(f(s)-\lambda g(s)) = \sum_{s=0}^{t-1}\left(\frac{f(t-1-s)}{f(t)} - 1\right)(f(s)-\lambda g(s)). \tag{2.13}$$

To establish the remainder of the assertions, we show that the right-hand side of (2.13) tends to 0 as $t \to \infty$. Let $\varepsilon > 0$. Then there exists $T \in \mathbb{N}$ such that

$$\left| \frac{f(t)}{g(t)} - \lambda \right| < \varepsilon \quad \text{for all } t \geq T.$$

Hence, for $t \geq T$

$$\left| \sum_{s=T}^{t-1} \left(\frac{f(t-1-s)}{f(t)} - 1 \right) (f(s) - \lambda g(s)) \right|$$

$$= \left| \sum_{s=T}^{t-1} \left(\frac{f(t-1-s)}{f(t)} - 1 \right) \left(\frac{f(s)}{g(s)} - \lambda \right) g(s) \right|$$

$$\leq \varepsilon \sum_{s=T}^{t-1} \left| \frac{f(t-1-s)g(s)}{f(t)} - g(s) \right|$$

$$\leq \varepsilon \left(\frac{(f * g)(t)}{f(t)} + \sum_{s=0}^{t-1} g(s) \right)$$

$$\to \varepsilon \left(\frac{\nu}{\lambda} + \sum_{s=0}^{\infty} g(s) \right) \quad \text{as } t \to \infty.$$

Also, it follows from

$$\frac{f(t-1-s)}{f(t)} - 1 = \frac{f(t-1-s)/g(t-1-s)}{f(t)/g(t)} \left(\frac{g(t-1-s)}{g(t)} - 1 \right)$$
$$+ \left(\frac{f(t-1-s)/g(t-1-s)}{f(t)/g(t)} - 1 \right)$$

that f satisfies (S_{2b}). Hence,

$$\left| \sum_{s=0}^{T-1} \left(\frac{f(t-1-s)}{f(t)} - 1 \right) (f(s) - \lambda g(s)) \right| \leq \sup_{0 \leq s \leq T} \left| \frac{f(t-s)}{f(t)} - 1 \right| \sum_{s=0}^{T-1} (f(s) - \lambda g(s)),$$

which tends to 0 as $t \to \infty$.

Therefore, it has been proved that

$$\limsup_{t\to\infty} \left| \sum_{s=0}^{t-1} \left(\frac{f(t-1-s)}{f(t)} - 1 \right) (f(s) - \lambda g(s)) \right| \le \varepsilon \left(\frac{\nu}{\lambda} + \sum_{s=0}^{\infty} g(s) \right).$$

Since $\varepsilon > 0$ is arbitrary small, the proof is complete. $\qquad\square$

Definition 2.8. Let $h \in \mathcal{U}$. Then $\mathrm{BC}_h(\mathbb{N}_0)$ is defined to be the space of sequences f on $\mathbb{N}_0$ such that $f = \phi h$ for some bounded sequence ϕ on $\mathbb{N}_0$. We will use BC_h in short in stead of $\mathrm{BC}_h(\mathbb{N}_0)$. It is a Banach space equipped with the norm

$$||f||_h = M_h \sup_{t\in\mathbb{N}_0} \left| \frac{f(t)}{h(t)} \right|,$$

where M_h is defined in (2.5). We denote by BC_h^l the closed subspace of sequences in BC_h for which

$$L_h f := \lim_{t\to\infty} \frac{f(t)}{h(t)}$$

exists. The operator $L : \mathrm{BC}_h^l \to \mathbb{R}$ is a bounded linear operator on BC_h^l. BC_h^0 is defined to be the closed subspace of sequences in BC_h^l for which $L_h f = 0$.

Theorem 2.9. *Suppose that $h \in \mathcal{U}$. Then BC_h is a commutative Banach algebra with the convolution as product, and BC_h^l and BC_h^0 are subalgebras. If $f, g \in \mathrm{BC}_h^l$, then*

$$L_h(f * g) = L_h f \sum_{s=0}^{\infty} g(s) + L_h g \sum_{s=0}^{\infty} f(s). \tag{2.14}$$

Proof. Let $f, g \in \mathrm{BC}_h$. Then

$$M_h \frac{|(f*g)(t)|}{h(t)} \le M_h \sum_{s=0}^{t-1} \left| \frac{f(t-1-s)}{h(t-1-s)} \right| \cdot \left| \frac{g(s)}{h(s)} \right| \cdot \frac{h(t-1-s)h(s)}{h(t)} \le ||f||_h ||g||_h \frac{1}{M_h} \frac{h^{*2}(t)}{h(t)}.$$

Hence, (2.5) implies that $||f * g||_h \leq ||f||_h ||g||_h$. Now, let $g \in \mathrm{BC}_h^0$. Then clearly g is summable. Let $\varepsilon > 0$. Then there exists $B > 0$ such that

$$\left| \frac{g(t)}{h(t)} \right| < \varepsilon \ \text{ for all } \ t \geq B \ \text{ and } \ \sum_{s=B}^{\infty} |g(s)| < \varepsilon.$$

Suppose that $t \geq B$. Since

$$\frac{(g * h)(t)}{h(t)} - \sum_{s=0}^{\infty} g(s) = \sum_{s=0}^{B-1} \left(\frac{h(t-1-s)}{h(t)} - 1 \right) g(s)$$

$$+ \frac{1}{h(t)} \sum_{s=B}^{t-1} \frac{g(s)}{h(s)} h(s) h(t-1-s) - \sum_{s=B}^{\infty} g(s),$$

we have

$$\left| \frac{(g * h)(t)}{h(t)} - \sum_{s=0}^{\infty} g(s) \right| \leq \sup_{0 \leq s \leq B} \left| \frac{h(t-s)}{h(t)} - 1 \right| \sum_{s=0}^{B-1} g(s)$$

$$+ \sup_{B \leq s \leq t-1} \left| \frac{g(s)}{h(s)} \right| \frac{1}{h(t)} \sum_{s=0}^{t-1} h(s) h(t-1-s) + \varepsilon$$

$$\leq \sup_{0 \leq s \leq B} \left| \frac{h(t-s)}{h(t)} - 1 \right| \sum_{s=0}^{B-1} g(s) + \frac{\varepsilon h^{*2}(t)}{h(t)} + \varepsilon.$$

By taking the limit superior on both sides as $t \to \infty$ and then using (S$_1$) and (S$_{2b}$), we get

$$\limsup_{t \to \infty} \left| \frac{(g * h)(t)}{h(t)} - \sum_{s=0}^{\infty} g(s) \right| \leq 2\varepsilon \sum_{s=0}^{\infty} h(s) + \varepsilon.$$

Thus, we have shown that

$$L_h(g * h) = \sum_{s=0}^{\infty} g(s) \ \text{ for all } \ g \in \mathrm{BC}_h^0. \tag{2.15}$$

Also, we observe that if $t \geq 2B$, then

$$\left| \frac{1}{h(t)} \sum_{s=0}^{t-1} g(t-1-s) g(s) \right|$$

$$\leq \left| \sum_{s=0}^{B-1} \left(\frac{g(t-1-s)}{h(t-1-s)} \left(\frac{h(t-1-s)}{h(t)} - 1 \right) + \frac{g(t-1-s)}{h(t-1-s)} \right) g(s) \right|$$

$$+ \frac{1}{h(t)} \sum_{s=B}^{t-1} \left| \frac{g(s)}{h(s)} \right| |g(t-1-s)| h(s)$$

$$\leq \left(1 + \sup_{0 \leq s \leq B} \left| \frac{h(t-s)}{h(t)} - 1 \right| \right) \sum_{s=0}^{B-1} \frac{|g(t-1-s)|}{h(t-1-s)} |g(s)|$$

$$+ \frac{\varepsilon}{h(t)} \sum_{s=B}^{t-1} |g(t-1-s)| h(s)$$

$$\leq \varepsilon \left(1 + \sup_{0 \leq s \leq B} \left| \frac{h(t-s)}{h(t)} - 1 \right| \right) \sum_{s=0}^{\infty} |g(s)| + \varepsilon \frac{(|g| * h)(t)}{h(t)}.$$

Taking limit superior on both sides as $t \to \infty$ and then using (S_{2b}) and (2.15) implies

$$\limsup_{t \to \infty} \left| \frac{1}{h(t)} \sum_{s=0}^{t-1} g(t-1-s) g(s) \right| \leq 2\varepsilon \sum_{s=0}^{\infty} |g(s)|.$$

Thus, we have shown

$$L_h(g * g) = 0 \quad \text{for all} \quad g \in \mathrm{BC}_h^0. \tag{2.16}$$

Now, let $f \in \mathrm{BC}_h^l$. Then $\tilde{f} := f - (L_h f) h \in \mathrm{BC}_h^0$. Therefore, the linearity of L, (S_1), and (2.15) imply that

$$L_h(f * h) = (L_h f) L_h(h * h) + L_h(\tilde{f} * h)$$

$$= 2 L_h f \sum_{s=0}^{\infty} h(s) + \sum_{s=0}^{\infty} \tilde{f}(s)$$

$$= 2 L_h f \sum_{s=0}^{\infty} h(s) + \sum_{s=0}^{\infty} f(s) - L_h f \sum_{s=0}^{\infty} h(s)$$

$$= L_h f \sum_{s=0}^{\infty} h(s) + \sum_{s=0}^{\infty} f(s). \tag{2.17}$$

Now, if $f, g \in \mathrm{BC}_h^0$, then by (2.16), we get $L_h((f+g) * (f+g)) = 0$, which implies $L_h(g * g) = 0$. Hence, BC_h^0 is a subalgebra. This fact and (2.17) implies that BC_h^l is

also a subalgebra. For the remaining part of the proof, let $f, g \in \mathrm{BC}_h^l$ and put

$$\tilde{f} := f - (L_h f)h, \quad \tilde{g} := g - (L_h g)h.$$

Using (2.16), (S_1), and (2.15), we obtain

$$
\begin{aligned}
L_h(f * g) &= L_h(((L_h f)h + \tilde{f}) * ((L_h g)h + \tilde{g})) \\
&= L_h f L_h g L_h(h * h) + L_h f L_h(\tilde{g} * h) + L_h g L_h(\tilde{f} * h) + L_h(\tilde{f} * \tilde{g}) \\
&= 2 L_h f L_h g \sum_{s=0}^{\infty} h(s) + L_h f \sum_{s=0}^{\infty} \tilde{g}(s) + L_h g \sum_{s=0}^{\infty} \tilde{f}(s) \\
&= 2 L_h f L_h g \sum_{s=0}^{\infty} h(s) + L_h f \sum_{s=0}^{\infty} (g(s) - (L_h g)h(s)) \\
&\quad + L_h g \sum_{s=0}^{\infty} (f(s) - (L_h f)h(s)) \\
&= L_h f \sum_{s=0}^{\infty} g(s) + L_h g \sum_{s=0}^{\infty} f(s).
\end{aligned}
$$

Thus, (2.14) holds and the proof is complete. □

Corollary 2.10. *Let $h \in \mathcal{U}$. Then for every $n \in \mathbb{N} \setminus \{1\}$, $h^{*n} \in \mathrm{BC}_h^l$ and $L_h(h^{*n}) = n\mu^{n-1}$, where μ is defined as in (2.6).*

Proof. An induction argument using Theorem 2.9 with Lemma 2.5 establishes the claim. □

Lemma 2.11. *If $f \in \mathrm{BC}_h^l$ and $L_h f \neq 0$, then $f \in \mathcal{U}$.*

Proof. Since $f \in \mathrm{BC}_h^l$, by Theorem 2.9, we have

$$L_h(f * f) = 2 L_h f \sum_{s=0}^{\infty} f(s).$$

Also, since
$$\frac{(f * f)(t)}{f(t)} = \frac{(f * f)(t)}{h(t)} \cdot \frac{h(t)}{f(t)}$$

and $L_h f \neq 0$, f satisfies (S$_1$). Since

$$\frac{f(t-s)}{f(t)} = \frac{f(t-s)}{h(t-s)} \cdot \frac{h(t-s)}{h(t)} \cdot \frac{h(t)}{f(t)},$$

$L_h f \neq 0$, and h satisfies (S$_2$), f satisfies (S$_2$). Hence, $f \in \mathcal{U}$. $\qquad\square$

3. TRANSIENT RENEWAL EQUATIONS

Consider the solution r of the linear scalar convolution equation

$$r(t) = h(t) + \sum_{s=0}^{t-1} h(t-1-s)r(s), \quad t \in \mathbb{N}_0 \tag{3.1}$$

with $h \in \mathcal{U}$ and

$$\mu = \sum_{s=0}^{\infty} h(s) < 1. \tag{3.2}$$

Then r is positive on $\mathbb{N}$. Summing (3.1) from $s = 0$ to $t - 1$, we get

$$\sum_{s=0}^{t-1} r(s) = \sum_{s=0}^{t-1} h(s) + \sum_{s=0}^{t-1}\sum_{u=0}^{s-1} h(s-1-u)r(u)$$

$$= \sum_{s=0}^{t-1} h(s) + \sum_{u=0}^{t-2}\sum_{s=u+1}^{t-1} h(s-1-u)r(u)$$

$$= \sum_{s=0}^{t-1} h(s) + \sum_{u=0}^{t-2}\sum_{v=0}^{t-u-2} h(v)r(u).$$

Now, as $t \to \infty$, we have

$$\sum_{s=0}^{\infty} r(s) = \sum_{s=0}^{\infty} h(s) + \sum_{u=0}^{\infty} r(u) \sum_{v=0}^{\infty} h(v) = \mu + \mu \sum_{s=0}^{\infty} r(s).$$

Using (3.2), we obtain

$$\sum_{s=0}^{\infty} r(s) = \frac{\mu}{1-\mu}. \tag{3.3}$$

It can also be represented, since $r = h + (h * r)$ by (3.1), by the Neumann series

$$r = \sum_{n=1}^{\infty} h^{*n}. \tag{3.4}$$

r is called the *resolvent* of h, since every solution of

$$y(t) = f(t) + \sum_{s=0}^{t-1} y(t-1-s)h(s), \quad t \in \mathbb{N}_0, \tag{3.5}$$

can be represented as

$$y(t) = f(t) + \sum_{s=0}^{t-1} r(t-1-s)f(s). \tag{3.6}$$

Theorem 3.1. *Let $h \in \mathcal{U}$ satisfy (3.2). Then the resolvent r defined by (3.1) is in* BC_h^l *and*

$$L_h r = \frac{1}{(1-\mu)^2}. \tag{3.7}$$

Also, $r \in \mathcal{U}$.

Proof. By the representation (3.4) for r, Corollary 2.10, and the uniform convergence implied by Lemma 2.6,

$$L_h r = \lim_{t \to \infty} \frac{r(t)}{h(t)} = \lim_{t \to \infty} \sum_{n=1}^{\infty} \frac{h^{*n}(t)}{h(t)} = \sum_{n=1}^{\infty} \lim_{t \to \infty} \frac{h^{*n}(t)}{h(t)} = \sum_{n=1}^{\infty} L_h(h^{*n}(t))$$

$$= \sum_{n=1}^{\infty} n\mu^{n-1} = \frac{1}{(1-\mu)^2}.$$

Since $L_h r > 0$, it follows that $r \in \mathrm{BC}_h^l$, and also by Lemma 2.11, $r \in \mathcal{U}$. $\qquad \square$

4. LINEAR SUM-DIFFERENCE EQUATIONS

Consider the asymptotic stability of the scalar linear Volterra sum-difference equation

$$\Delta x(t) = -ax(t) + \sum_{s=0}^{t-1} k(t-1-s)x(s) + f(t), \quad x(0) = x_0, \tag{4.1}$$

under the assumptions that $f, k > 0$ on $\mathbb{N}_0$ and $\sum_{s=0}^{\infty} k(s) < \infty$. It is convenient to introduce the *difference resolvent* z, which is the solution of

$$\Delta z(t) = -az(t) + \sum_{s=0}^{t-1} k(t-1-s)z(s), \quad z(0) = 1. \tag{4.2}$$

Theorem 4.1. *If $a \in (0,1)$ and $k > 0$ on $\mathbb{N}_0$, then all solutions z of (4.2) with $z(0) > 0$ are positive. If moreover,*

$$\sum_{s=0}^{\infty} k(s) < a,$$

then all solutions z of (4.2) satisfy $\sum_{s=0}^{\infty} z(s) < \infty$ and $\lim_{t \to \infty} z(t) = 0$.

Proof. To show $z(t) > 0$ for all $t \in \mathbb{N}_0$, we use the method of induction. First, $z(0) > 0$ holds by assumption. If for some $T \in \mathbb{N}_0$, $z(t) > 0$ for all $0 \le t \le T$, then from (4.2) and using the assumptions, we get

$$z(T+1) = (1-a)z(T) + \sum_{s=0}^{T-1} k(T-1-s)z(s) > 0.$$

Now, we use a Lyapunov functional to show $\sum_{s=0}^{\infty} z(s) < \infty$, and then $\lim_{t \to \infty} z(t) = 0$ follows from the property of convergent series. Let

$$A := a - \sum_{s=0}^{\infty} k(s) > 0.$$

Since $z, k > 0$ on $\mathbb{N}_0$, we have that

$$V(t) := z(t) + \sum_{s=0}^{t-1} \sum_{u=t}^{\infty} k(u-1-s)z(s) \geq 0 \quad \text{for all} \ \ t \in \mathbb{N}_0.$$

Taking the difference of $V(t)$ and substituting $\Delta z(t)$ from (4.2), we get

$$\Delta V(t) = -az(t) + \sum_{s=0}^{t-1} k(t-1-s)z(s) + \sum_{s=0}^{t-1} \left[\sum_{u=t+1}^{\infty} k(u-1-s)z(s) \right.$$

$$\left. - \sum_{u=t}^{\infty} k(u-1-s)z(s) \right] + \sum_{u=t+1}^{\infty} k(u-1-t)z(t)$$

$$= -az(t) + \sum_{s=0}^{t-1} k(t-1-s)z(s) - \sum_{s=0}^{t-1} k(t-1-s)z(s)$$

$$+ \sum_{u=t+1}^{\infty} k(u-1-t)z(t)$$

$$= \left(-a + \sum_{u=t+1}^{\infty} k(u-1-t) \right) z(t)$$

$$= \left(-a + \sum_{s=0}^{\infty} k(s) \right) z(t)$$

$$= -Az(t).$$

Now, taking sums on both sides, we have

$$0 \leq V(t) = V(0) - A \sum_{s=0}^{t-1} z(s).$$

Thus, $V(t) + A \sum_{s=0}^{t-1} z(s) = V(0)$. Hence $\sum_{s=0}^{\infty} z(s) < \infty$. $\qquad \square$

The following lemma shows the significance of the difference resolvent.

Lemma 4.2. *If z is the difference resolvent defined by (4.2), then x defined by*

$$x(t) = z(t)x_0 + \sum_{s=0}^{t-1} z(t-1-s)f(s), \quad t \in \mathbb{N}_0 \tag{4.3}$$

solves (4.1).

Proof. Taking differences in (4.3) and using (4.2), we get

$$
\begin{aligned}
\Delta x(t) =& \Delta z(t)x_0 + \sum_{s=0}^{t-1} \Delta z(t-1-s)f(s) + z(0)f(t) \\
=& (-az(t) + (k*z)(t))x_0 + ((-az + (k*z))*f)(t) + f(t) \\
=& -az(t)x_0 + (k*z)(t)x_0 - a(z*f)(t) + ((k*z*f)(t) + f(t) \\
=& -ax(t) + (k*(zx_0 + (z*f)))(t) + f(t) \\
=& -ax(t) + (k*x)(t) + f(t)
\end{aligned}
$$

and $x(0) = z(0)x_0 = x_0$. $\qquad\square$

Lemma 4.3. *Let k be any sequence. The unique solution of (4.2) satisfies*

$$z = e_{-a} + e_{-a} * r, \tag{4.4}$$

where $e_{-a}(t) = (1-a)^t$ for $a \in (0,1)$, $t \geq 0$, $h = e_{-a} * k$ and r is the resolvent given by (3.1).

Proof. Taking difference on (4.4), we obtain

$$
\begin{aligned}
\Delta z(t) =& \Delta(1-a)^t + \Delta \sum_{s=0}^{t-1} (1-a)^{t-1-s} r(s) \\
=& -a(1-a)^t + \sum_{s=0}^{t-1} \Delta(1-a)^{t-s} r(s) + r(t)
\end{aligned}
$$

$$= -a(1-a)^t - a \sum_{s=0}^{t-1} (1-a)^{t-1-s} r(s) + r(t)$$

$$= -az(t) + r(t).$$

Now, it remains to show that $r = k * z$. From (3.1), using $h = e_{-a} * k$ and (4.4), we get

$$r = h + (h * r) = (k * e_{-a}) + ((k * e_{-a}) * r) = (k * (e_{-a} + e_{-a} * r)) = (k * z).$$

Thus, z given by (4.4) solves (4.2) $\qquad\square$

Theorem 4.4. *Let $k \in \mathcal{U}$. Suppose that*

$$\sum_{s=0}^{\infty} k(s) < a.$$

Then the difference resolvent z given by (4.2) satisfies

$$\lim_{t\to\infty} \frac{z(t)}{k(t)} = \frac{1}{(a - \sum_{s=0}^{\infty} k(s))^2}, \quad \lim_{t\to\infty} \frac{z(t+1)}{z(t)} = 1, \tag{4.5}$$

and $z \in \mathrm{BC}_k^l$. Moreover, $z \in \mathcal{U}$.

Proof. Define $h := e_{-a} * k$, where e_{-a} is as in Lemma 4.3. First, we prove that $h \in \mathcal{U}$ so that we can apply the results for subexponential functions that have already been established. Clearly, $h > 0$ on $\mathbb{N}$, and (3.2) holds due to

$$\mu := \sum_{s=0}^{\infty} h(s)$$

$$= \sum_{s=0}^{\infty} \sum_{u=0}^{s-1} (1-a)^{s-1-u} k(u)$$

$$= \sum_{u=0}^{\infty} \sum_{s=u+1}^{\infty} (1-a)^{s-1-u} k(u)$$

$$\tag{4.6}$$

$$= \sum_{u=0}^{\infty} k(u) \sum_{s=0}^{\infty} (1-a)^s$$

$$= \frac{1}{a} \sum_{s=0}^{\infty} k(s) < 1. \tag{4.7}$$

Since $k \in \mathcal{U}$, Remark 2.3 yields

$$L_k e_{-a} = \lim_{t \to \infty} \frac{e_{-a}(t)}{k(t)} = 0. \tag{4.8}$$

So, $e_{-a} \in \mathrm{BC}_k^0$. Again, by using the fact that k is subexponential, $h = e_{-a} * k \in \mathrm{BC}_k^l$. By Theorem 2.9, (4.8), and $L_k k = 1$, we have

$$L_k h = L_k(e_{-a} * k) = L_k e_{-a} \sum_{s=0}^{\infty} k(s) + L_k k \sum_{s=0}^{\infty} e_{-a}(s) = \frac{1}{a} (\neq 0). \tag{4.9}$$

Therefore, by Lemma 2.11, $h \in \mathcal{U}$. Now, using (4.4), (4.8), (4.9), (2.14), (3.7), $\sum_{s=0}^{\infty} e_{-a}(s) = \frac{1}{a}$, $L_h e_{-a} = 0$, and (3.3), we obtain

$$L_k z = L_k e_{-a} + L_k(r * e_{-a})$$

$$= (L_k h) L_h(r * e_{-a})$$

$$= \frac{1}{a} \left(L_h r \sum_{s=0}^{\infty} e_{-a}(s) + L_h e_{-a} \sum_{s=0}^{\infty} r(s) \right)$$

$$= \frac{1}{a^2(1-\mu)^2} > 0. \tag{4.10}$$

Hence, $z \in \mathrm{BC}_k^l$. Using (4.6) in (4.10), we get

$$\lim_{t \to \infty} \frac{z(t)}{k(t)} = \frac{1}{(a - \sum_{s=0}^{\infty} k(s))^2}.$$

Now, summing (4.2) from 0 to $t-1$ and then using $z(t) \to 0$ as $t \to \infty$ from Theorem 4.1, by (4.6), we obtain

$$\sum_{s=0}^{\infty} z(s) = \frac{1}{a(1-\mu)}. \tag{4.11}$$

Using (2.14), (4.10), (4.6), and (4.11), we get

$$L_k(k * z) = L_k z \sum_{s=0}^{\infty} k(s) + L_k k \sum_{s=0}^{\infty} z(s) = \frac{1}{a(1-\mu)^2}. \tag{4.12}$$

Next, using (4.2), (4.12), and (4.10), we obtain

$$
\begin{aligned}
\lim_{t \to \infty} \frac{z(t+1)}{z(t)} &= \lim_{t \to \infty} \frac{(1-a)z(t) + (k * z)(t)}{z(t)} \\
&= (1-a) + \lim_{t \to \infty} \left[\frac{(k * z)(t)}{k(t)} \cdot \frac{k(t)}{z(t)} \right] \\
&= (1-a) + L_k(k * z) \cdot \frac{1}{L_k z} \\
&= 1.
\end{aligned}
$$

Finally, since $z \in \mathrm{BC}_k^l$ and $L_k z > 0$, Lemma 2.11 implies that $z \in \mathcal{U}$. $\qquad \square$

Corollary 4.5. *Let $k \in \mathcal{U}$. If*

$$\sum_{s=0}^{\infty} k(s) < a,$$

then for every $f \in \mathrm{BC}_k^l$, the solution x of (4.1) satisfies $x \in \mathrm{BC}_k^l$ and

$$L_k x = \frac{1}{\left(a - \sum_{s=0}^{t-1} k(s)\right)^2} \left(x_0 + \sum_{s=0}^{\infty} f(s) \right) + \frac{L_k f}{\left(a - \sum_{s=0}^{\infty} k(s)\right)}. \tag{4.13}$$

Proof. Since $f \in \mathrm{BC}_k^l$ and $z \in \mathrm{BC}_k^l$ (by Theorem 4.4), we get by Theorem 2.9 that $z * f \in \mathrm{BC}_k^l$. Hence, from (4.3), we obtain $x \in \mathrm{BC}_k^l$. Applying (2.14) to (4.3) and then using (4.10) and (4.11), the proof of (4.13) is complete. $\qquad \square$

Theorem 4.6. *Let* $k : \mathbb{N}_0 \to (0, \infty)$ *and* z *the difference resolvent defined in* (4.2).
If

$$\sum_{s=0}^{\infty} k(s) < a,$$

then the two conditions

    (i) $k \in \mathcal{U}$,

    (ii) z *satisfies the conditions* (4.5)

are equivalent.

Proof. If $k \in \mathcal{U}$, then by Theorem 4.4, z satisfies the conditions (4.5). Suppose now
that z satisfies the conditions (4.5). Applying the second expression of (4.5) to (4.2),
we get

$$\lim_{t \to \infty} \frac{(k * z)(t)}{z(t)} = \lim_{t \to \infty} \frac{z(t+1) - (1-a)z(t)}{z(t)} = a. \tag{4.14}$$

Since z is summable and satisfies (by the second condition of (4.5)) (S$_{2a}$), z satisfies
(by Remark 2.2) (S$_2$), and we may apply Lemma 2.7 with $f = k$ and $g = z$. Using
(4.14) and (4.11), we have

$$\lim_{t \to \infty} \frac{(k * k)(t)}{k(t)} = a + a\mu - \frac{a^2(1-\mu)^2}{a(1-\mu)} = 2a\mu = 2\sum_{s=0}^{\infty} k(s).$$

Thus, k satisfies the subexponential property (S$_1$). From the subexponential property
(S$_2$) of z and $L_k z > 0$ from (4.10), it follows that

$$\lim_{t \to \infty} \frac{k(t-s)}{k(t)} = \lim_{t \to \infty} \frac{\frac{k(t-s)}{z(t-s)} \frac{z(t-s)}{z(t)}}{\frac{k(t)}{z(t)}} = 1$$

holds, i.e., k satisfies the subexponential property (S$_2$). Hence, $k \in \mathcal{U}$. $\qquad\square$

Example 4.7. Consider $h(t) = \frac{1}{(t+b)^n}$, where $t \in \mathbb{N}_0$, $b \in \mathbb{N} \setminus \{1\}$ and $n \in \mathbb{N} \setminus \{1\}$.
Then $h \in \mathcal{U}$.

Proof. We prove the statement in the case $n = 2$ and $b = 2$. Therefore, $h(t) = \frac{1}{(t+2)^2}$, $t \in \mathbb{N}_0$ clearly satisfies (S$_2$), (S$_{2a}$), and $\sum_{s=0}^{\infty} h(s) < 1$. To see that h satisfies (S$_1$), we calculate

$$
\begin{aligned}
\frac{h^{*2}(t)}{h(t)} &= \sum_{s=0}^{t-1} \frac{h(s)h(t-1-s)}{h(t)} \\
&= \sum_{s=0}^{t-1} \left(\frac{t+2}{(s+2)(t+1-s)} \right)^2 \\
&= \sum_{s=0}^{t-1} \left(\frac{1}{s+2} + \frac{1}{t+1-s} \right)^2 \left(\frac{t+2}{t+3} \right)^2 \\
&= \left(\frac{t+2}{t+3} \right)^2 \left(\sum_{s=0}^{t-1} \frac{1}{(s+2)^2} + \sum_{s=0}^{t-1} \frac{2}{(s+2)(t+1-s)} + \sum_{\sigma=0}^{t-1} \frac{1}{(\sigma+2)^2} \right) \\
&= \left(\frac{t+2}{t+3} \right)^2 \left(2\sum_{s=0}^{t-1} \frac{1}{(s+2)^2} + 2\sum_{s=0}^{t-1} \frac{1}{(s+2)(t+1-s)} \right).
\end{aligned}
$$

Now, it is enough to show $\sum_{s=0}^{t-1} \frac{2}{(s+2)(t+1-s)} \to 0$ as $t \to \infty$:

$$
\begin{aligned}
0 \leq \sum_{s=0}^{t-1} \frac{1}{(s+2)(t+1-s)} &= \frac{1}{t+3} \sum_{s=0}^{t-1} \left[\frac{1}{s+2} + \frac{1}{t+1-s} \right] \\
&= \frac{2}{t+3} \sum_{s=0}^{t-1} \frac{1}{s+2} \\
&\leq \frac{2}{t+3} \int_1^{t+1} \frac{dx}{x} \\
&= \frac{2\ln(t+1)}{t+3} \quad \to 0 \text{ as } t \to \infty.
\end{aligned}
$$

Thus, h satisfies (S$_1$). $\qquad\square$

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II. RATE OF CONVERGENCE OF SOLUTIONS OF LINEAR VOLTERRA DIFFERENCE EQUATIONS

ABSTRACT

We study the asymptotic behavior of solutions of a scalar linear Volterra sum-difference equation. A positive lower bound for the rate of convergence of asymptotically stable solutions is found by assuming the kernel is a positive and summable sequence such that $k(t+1)/k(t) \to 1$ as $t \to \infty$.

1. INTRODUCTION

We consider the discrete equation

$$\Delta x(t) = -ax(t) + \sum_{s=0}^{t-1} k(t-1-s)x(s), \quad t \in \mathbb{N}_0, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We suppose that $a \in (0, 1)$ and

$$k : \mathbb{N}_0 \to (0, \infty) \quad \text{satisfies} \quad \sum_{s=0}^{\infty} k(s) < \infty. \tag{1.3}$$

By using elementary analysis, we show that if all solutions x of (1.1) satisfy $x(t) \to 0$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{k(t+1)}{k(t)} = 1, \tag{1.4}$$

then a positive lower bound of the rate of convergence is

$$\liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \geq \frac{|x(0)|}{a\left(a - \sum_{s=0}^{\infty} k(s)\right)}.$$

The result is proved by determining the asymptotic behavior of the solution of the transient renewal equation

$$r(t) = h(t) + \sum_{s=0}^{t-1} h(t-1-s)r(s) \quad \text{with} \quad \sum_{s=0}^{\infty} h(s) < 1,$$

where the asymptotic behavior of $r(t)/h(t)$ as $t \to \infty$ is

$$\liminf_{t\to\infty} \frac{r(t)}{h(t)} \geq \frac{1}{\left(1 - \sum_{s=0}^{\infty} h(s)\right)}.$$

It is shown in [6, Theorem 4.4] that

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{x(0)}{(a - \sum_{s=0}^{\infty} k(s))^2}$$

if (1.3),

$$\sum_{s=0}^{\infty} k(s) < a, \quad a \in (0,1) \tag{1.5}$$

and

$$\lim_{t\to\infty} \frac{\sum_{s=0}^{t-1} k(t-1-s)k(s)}{k(t)} = 2 \sum_{s=0}^{\infty} k(s) \quad \text{and} \quad \lim_{t\to\infty} \frac{k(t+1)}{k(t)} = 1 \tag{1.6}$$

hold. Thus, in that case, the exact value of the rate of convergence of the solution x of (1.1) is known. The relationship between these hypotheses and the present work is that (1.4) is the second condition of (1.6) (see also [6, Remark 2.2]). In [6], in order to prove the rate of convergence of the solution of (1.1), we also used the rate of convergence of the solution of the transient renewal equation

$$\lim_{t\to\infty} \frac{r(t)}{h(t)} = \frac{1}{(a - \sum_{s=0}^{\infty} h(s))^2}.$$

Note that the results obtained in [6] are discrete analogues of the work in [3] for the continuous case. The continuous case of the problem that we are studying in the present paper has been investigated in [2]. For basic properties and formulas concerning difference equations, we refer the reader to [1, 5, 8]. We also refer to [4, 7] for basic results on the existence of solutions of scalar linear Volterra equations and Lyapunov functionals in the continuous case.

2. PRELIMINARIES

Definition 2.1. For $f, g : \mathbb{N}_0 \to \mathbb{R}$, we define the convolution of f and g (see [5, Section 3.10]) by

$$(f * g)(t) = \sum_{s=0}^{t-1} f(t - 1 - s)g(s), \quad t \in \mathbb{N}_0.$$

The n-fold convolution f^{*n} is given by $f^{*1} = f$ and $f^{*(n+1)} = f * f^{*n}$ for $n \in \mathbb{N}$.

Consider the solution r of the linear scalar convolution equation

$$r(t) = h(t) + \sum_{s=0}^{t-1} h(t - 1 - s)r(s) = h(t) + (h * r)(t), \quad t \in \mathbb{N}_0 \tag{2.1}$$

with $h(0) = 0$, $h(t) > 0$ for all $t \in \mathbb{N}$, $\lim_{t \to \infty} \frac{h(t+1)}{h(t)} = 1$, and

$$\mu := \sum_{s=0}^{\infty} h(s) < 1. \tag{2.2}$$

Then clearly (2.1) has a solution r with $r(t) > 0$ for all $t \in \mathbb{N}$, and r is summable. Here r is called the *resolvent* of h. It can also be represented, since $r = h + (h * r)$ by (2.1), by the Neumann series

$$r = \sum_{n=1}^{\infty} h^{*n}. \tag{2.3}$$

Theorem 2.2. *Let $h(t) > 0$ for all $t \in \mathbb{N}$, $\lim_{t \to \infty} \frac{h(t+1)}{h(t)} = 1$, and assume (2.2) is satisfied. Then the lower bound for the rate of convergence of the solution r of (2.1) is*

$$\liminf_{t \to \infty} \frac{r(t)}{h(t)} \geq \frac{1}{1 - \mu}. \tag{2.4}$$

Proof. Since $h(t) > 0$ for all $t \in \mathbb{N}$, (2.3) shows that $r(t) \geq \sum_{n=1}^{N} h^{*n}(t)$ for each $N \in \mathbb{N}$. Hence, for $N \in \mathbb{N}$, we have

$$\liminf_{t \to \infty} \frac{r(t)}{h(t)} \geq \liminf_{t \to \infty} \sum_{n=1}^{N} \frac{h^{*n}(t)}{h(t)} \geq \sum_{n=1}^{N} \liminf_{t \to \infty} \frac{h^{*n}(t)}{h(t)}. \tag{2.5}$$

For any $0 < T \leq t$, we obtain

$$\begin{aligned}
\frac{h^{*n}(t)}{h(t)} &= \frac{\sum_{s=0}^{t-1} h(t-1-s) h^{*(n-1)}(s)}{h(t)} \\
&= \sum_{s=0}^{T-1} \frac{h(t-1-s)}{h(t)} h^{*(n-1)}(s) + \sum_{s=T}^{t-1} \frac{h(t-1-s)}{h(t)} h^{*(n-1)}(s) \\
&\geq \sum_{s=0}^{T-1} \frac{h(t-1-s)}{h(t)} h^{*(n-1)}(s) \\
&= \sum_{s=0}^{T-1} \left(\frac{h(t-1-s)}{h(t)} - 1 \right) h^{*(n-1)}(s) + \sum_{s=0}^{T-1} h^{*(n-1)}(s).
\end{aligned}$$

Using

$$\sum_{s=0}^{\infty} h^{*(n-1)}(s) = \mu^{n-1} \tag{2.6}$$

(see [6, Lemma 2.5] and note that [6, (S_1)] was not used in this proof), where μ is defined as in (2.2), yields

$$\frac{h^{*n}(t)}{h(t)} - \mu^{n-1} \geq \sum_{s=0}^{T-1} \left(\frac{h(t-1-s)}{h(t)} - 1 \right) h^{*(n-1)}(s) - \sum_{s=T}^{\infty} h^{*(n-1)}(s). \tag{2.7}$$

Now, we show that the first sum on the right-hand side of (2.7) tends to 0 as $t \to \infty$. Since $\lim_{t \to \infty} \frac{h(t+1)}{h(t)} = 1$, we have for each fixed $s \in \mathbb{N}$,

$$\frac{h(t-1-s)}{h(t)} = \prod_{v=-s-1}^{-1} \frac{h(t+v)}{h(t+v+s)} \to 1 \quad \text{as } t \to \infty.$$

This yields

$$\left| \sum_{s=0}^{T-1} \left(\frac{h(t-1-s)}{h(t)} - 1 \right) h^{*(n-1)}(s) \right| \leq \sup_{0 \leq s \leq T} \left| \frac{h(t-s)}{h(t)} - 1 \right| \sum_{s=0}^{T-1} h^{*(n-1)}(s)$$

$$\to 0 \quad \text{as} \quad t \to \infty.$$

Therefore, (2.7) implies

$$\liminf_{t \to \infty} \frac{h^{*n}(t)}{h(t)} \geq \mu^{n-1} - \sum_{s=T}^{\infty} h^{*(n-1)}(s) \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad T \in \mathbb{N} \tag{2.8}$$

and thus by letting $T \to \infty$ in (2.8),

$$\liminf_{t \to \infty} \frac{h^{*n}(t)}{h(t)} \geq \mu^{n-1} \quad \text{for} \quad n \in \mathbb{N}. \tag{2.9}$$

Using (2.9) in (2.5), we get

$$\liminf_{t \to \infty} \frac{r(t)}{h(t)} \geq \sum_{n=1}^{N} \mu^{n-1}. \tag{2.10}$$

By letting $N \to \infty$ in (2.10), we obtain (2.4). $\qquad \square$

3. RESULTS

Theorem 3.1. *Let $k : \mathbb{N}_0 \to [0, \infty)$ be summable. Suppose that x is a summable solution of* (1.1). *Then there exists $\beta > 0$ such that*

$$\sum_{s=t}^{\infty} |x(s)| \geq \beta |x_0| \sum_{s=t}^{\infty} k(s), \quad t \in \mathbb{N}_0. \tag{3.1}$$

Moreover, if $\sum_{s=t}^{\infty} k(s) > 0$ for all large $t \in \mathbb{N}$, then

$$\liminf_{t \to \infty} \frac{\sum_{s=t}^{\infty} |x(s)|}{\sum_{s=t}^{\infty} k(s)} \geq \frac{|x_0|}{a\left(a - \sum_{s=0}^{\infty} k(s)\right)}. \tag{3.2}$$

Proof. Let x be a nonzero solution of (1.1). Since $-x$ is also a solution of (1.1), we may assume $x(0) = x_0 > 0$. From (1.1), we have

$$x(t+1) = (1-a)x(t) + \sum_{s=0}^{t-1} k(t-1-s)x(s),$$

from which, due to $a \in (0,1)$, we infer by induction that $x(t) > 0$ for all $t \in \mathbb{N}_0$. Summing (1.1) from t to ∞ and using the assumption $x(t) \to 0$ as $t \to \infty$ yields

$$
\begin{aligned}
a \sum_{s=t}^{\infty} x(s) &= x(t) + \sum_{s=t}^{\infty} \sum_{u=0}^{s-1} k(s-1-u)x(u) \\
&> \sum_{s=t}^{\infty} \sum_{u=0}^{s-1} k(s-1-u)x(u) \\
&= \sum_{s=t}^{\infty} \sum_{u=0}^{t-1} k(s-1-u)x(u) + \sum_{s=t}^{\infty} \sum_{u=t}^{s-1} k(s-1-u)x(u) \\
&\geq \sum_{u=0}^{t-1} \sum_{s=t}^{\infty} k(s-1-u)x(u)
\end{aligned}
$$

$$= \sum_{u=0}^{t-1} \sum_{v=t-1-u}^{\infty} k(v)x(u)$$

$$\geq \sum_{u=0}^{t-1} x(u) \sum_{v=t}^{\infty} k(v), \quad \text{for } t \in \mathbb{N}_0. \tag{3.3}$$

Now, using the additional assumption $\sum_{s=t}^{\infty} k(s) > 0$ for large t, we get from (3.3)

$$\liminf_{t\to\infty} \frac{\sum_{s=t}^{\infty} x(s)}{\sum_{s=t}^{\infty} k(s)} \geq \frac{1}{a} \sum_{s=0}^{\infty} x(s). \tag{3.4}$$

Again, summing (1.1) from 0 to t and using the assumption that the solution x is summable, we obtain

$$x(t) - x(0) = -a \sum_{s=0}^{t-1} x(s) + \sum_{s=0}^{t-1} \sum_{u=0}^{s-1} k(s-1-u)x(u)$$

$$= -a \sum_{s=0}^{t-1} x(s) + \sum_{u=0}^{t-2} \sum_{s=u+1}^{t-1} k(s-1-u)x(u)$$

$$= -a \sum_{s=0}^{t-1} x(s) + \sum_{u=0}^{t-2} \sum_{v=0}^{t-2-u} k(v)x(u)$$

and thus

$$x_0 = \left(\sum_{s=0}^{\infty} x(s) \right) \left(a - \sum_{s=0}^{\infty} k(s) \right) > 0.$$

Hence,

$$\sum_{s=0}^{\infty} x(s) = \frac{x_0}{a - \sum_{s=0}^{\infty} k(s)} > 0. \tag{3.5}$$

Thus, estimate (3.2) for $x_0 > 0$ follows from (3.4) and (3.5). Estimate (3.1) follows from (3.3) and (3.5) with $\beta = \frac{1}{a\left(a - \sum_{s=0}^{\infty} k(s)\right)} > 0$. $\qquad \square$

Remark 3.2. Under the hypothesis on k in Theorem 3.1, it is clear from (3.5) that x is summable if and only if $a > \sum_{s=0}^{\infty} k(s)$.

Recall that by [6, Theorem 4.4], (1.3), (1.5), and (1.6) imply

$$\lim_{t \to \infty} \frac{|x(t)|}{k(t)} = \frac{x(0)}{\left(a - \sum_{s=0}^{\infty} k(s)\right)^2}.$$

Then by the discrete version of L'Hôpital's rule [1, Corollary 1.8.8] for $u(t) = \sum_{s=t}^{\infty} |x(s)|$ and $v(t) = \sum_{s=t}^{\infty} k(s)$, we get

$$\lim_{t \to \infty} \frac{\sum_{s=t}^{\infty} |x(s)|}{\sum_{s=t}^{\infty} k(s)} = \lim_{t \to \infty} \frac{\Delta u(t)}{\Delta v(t)} = \lim_{t \to \infty} \frac{|x(t)|}{k(t)} = \frac{x(0)}{\left(a - \sum_{s=0}^{\infty} k(s)\right)^2}.$$

Thus, the exact value of the right-hand side of (3.2) is known in that case.

In the remainder of this paper, instead of (1.3), (1.5), and (1.6), we only assume (1.3), (1.5), and (1.4).

Corollary 3.3. *Suppose k satisfies (1.3), (1.5), and (1.4). If $x_0 \neq 0$, then*

$$\liminf_{t \to \infty} \frac{|x(t)|}{e_{-b}(t)} = \infty \quad \text{for all} \quad b \in (0, 1), \tag{3.6}$$

where $e_{-b}(t) := (1 - b)^t$.

Proof. By [6, Remark 2.3], (1.4) implies

$$\lim_{t \to \infty} \frac{k(t)}{e_{-b}(t)} = \infty, \tag{3.7}$$

and so, using Theorem 3.1,

$$
\begin{aligned}
\liminf_{t \to \infty} \frac{|x(t)|}{e_{-b}(t)} &= \liminf_{t \to \infty} \frac{|x(t)|}{k(t)} \frac{k(t)}{e_{-b}(t)} \\
&\geq \liminf_{t \to \infty} \frac{|x(t)|}{k(t)} \cdot \liminf_{t \to \infty} \frac{k(t)}{e_{-b}(t)} = \infty
\end{aligned}
$$

for every $b \in (0, 1)$, provided $x_0 \neq 0$. $\qquad \square$

Theorem 3.4. *Let k satisfy* (1.3), (1.5), *and* (1.4). *Suppose that x is a solution of* (1.1) *satisfying $x(t) \to 0$ as $t \to \infty$. Then there exists $\alpha > 0$ such that*

$$|x(t)| \geq \alpha |x_0| k(t), \quad t \in \mathbb{N}_0. \tag{3.8}$$

Moreover, if $x_0 \neq 0$, then

$$\liminf_{t \to \infty} \frac{|x(t)|}{k(t)} \geq \frac{|x_0|}{a\left(a - \sum_{s=0}^{\infty} k(s)\right)}. \tag{3.9}$$

Proof. Define

$$h := e_{-a} * k, \tag{3.10}$$

where $e_{-a}(t) = (1-a)^t$ for $t \in \mathbb{N}_0$. Then (see [6, Lemma 4.3]) the unique solution of (1.1) and (1.2) is

$$x = (e_{-a} + (e_{-a} * r)) x_0, \tag{3.11}$$

where r is the unique solution of (2.1). Note that the function h defined in (3.10) is summable with $h(0) = 0$ and $h(t) > 0$ for all $t \in \mathbb{N}$. Also, in [6, Theorem 4.1], it has been shown that a necessary condition for $\lim_{t\to\infty} x(t) = 0$ for all solutions x of (1.1) is that (1.5) holds. Using this result, we obtain (see [6, (4.6)])

$$\mu := \sum_{s=0}^{\infty} h(s) = \frac{1}{a} \sum_{s=0}^{\infty} k(s) < 1. \tag{3.12}$$

By (3.12), $h(t) \to 0$ as $t \to \infty$. Using the discrete L'Hôpital rule [1, Theorem 1.8.9] for $u(t) = \sum_{s=0}^{t-1}(1-a)^{-s}k(s)$ and $v(t) = (1-a)^{1-t}k(t)$ together with (1.4) and (3.7), we have

$$\lim_{t\to\infty} \frac{h(t)}{k(t)} = \lim_{t\to\infty} \frac{\sum_{s=0}^{t-1}(1-a)^{-s}k(s)}{(1-a)^{1-t}k(t)}$$

$$= \lim_{t \to \infty} \frac{u(t)}{v(t)} = \lim_{t \to \infty} \frac{\Delta u(t)}{\Delta v(t)}$$

$$= \lim_{t \to \infty} \frac{(1-a)^{-t}k(t)}{(1-a)^{-t}k(t+1) - (1-a)^{1-t}k(t)}$$

$$= \lim_{t \to \infty} \frac{1}{\frac{k(t+1)}{k(t)} - (1-a)}$$

$$= \frac{1}{a}. \tag{3.13}$$

Since from (3.10), $h(t) = \sum_{s=0}^{t-1}(1-a)^{t-1-s}k(s)$, we get

$$\Delta h(t) = \sum_{s=0}^{t}(1-a)^{t-s}k(s) - \sum_{s=0}^{t-1}(1-a)^{t-1-s}k(s)$$

$$= k(t) - a\sum_{s=0}^{t-1}(1-a)^{t-1-s}k(s)$$

$$= k(t) - ah(t),$$

i.e.,

$$h(t+1) = k(t) + (1-a)h(t). \tag{3.14}$$

Hence, using (3.14) and (3.13), we have

$$\lim_{t \to \infty} \frac{h(t+1)}{h(t)} = \lim_{t \to \infty} \frac{k(t)}{h(t)} + 1 - a = 1. \tag{3.15}$$

Since $r(t) > 0$ for all $t \in \mathbb{N}$, (3.11) implies that

$$\frac{|x(t)|}{|x_0|} = (1-a)^t + \sum_{s=0}^{t-1}(1-a)^{t-1-s}r(s) = \frac{1}{(1-a)^{-t}} + \frac{\sum_{s=0}^{t-1}(1-a)^{-s}r(s)}{(1-a)^{1-t}}.$$

Now, diving both sides by $k(t) > 0$, we get

$$\frac{|x(t)|}{|x_0|k(t)} = \frac{1}{k(t)(1-a)^{-t}} + \frac{\sum_{s=0}^{t-1}(1-a)^{-s}r(s)}{k(t)(1-a)^{1-t}} > \frac{\sum_{s=0}^{t-1}(1-a)^{-s}r(s)}{k(t)(1-a)^{1-t}}. \tag{3.16}$$

Let $0 < \varepsilon < 1$. Due to $h(t) > 0$ for all $t \in \mathbb{N}$, (3.12), and (3.15), we may apply Theorem 2.2. Thus, there exists $T \in \mathbb{N}$ such that

$$r(t) > \frac{(1 - \varepsilon)h(t)}{1 - \mu} \quad \text{for all } t \geq T.$$

Therefore, for $t \geq T$,

$$\frac{\sum_{s=0}^{t-1}(1-a)^{-s}r(s)}{k(t)(1-a)^{1-t}} \geq \frac{\sum_{s=T}^{t-1}(1-a)^{-s}r(s)}{k(t)(1-a)^{1-t}} > \left(\frac{1-\varepsilon}{1-\mu}\right)\frac{\sum_{s=T}^{t-1}(1-a)^{-s}h(s)}{k(t)(1-a)^{1-t}}. \quad (3.17)$$

It follows from (3.7), the discrete L'Hôpital rule [1, Theorem 1.8.9] for $u(t) = \sum_{s=T}^{t-1}(1-a)^{-s}h(s)$ and $v(t) = (1-a)^{1-t}k(t)$, (1.4), and (3.13)

$$
\begin{aligned}
\lim_{t\to\infty} \frac{\sum_{s=T}^{t-1}(1-a)^{-s}h(s)}{(1-a)^{1-t}k(t)} &= \lim_{t\to\infty} \frac{u(t)}{v(t)} = \lim_{t\to\infty} \frac{\Delta u(t)}{\Delta v(t)} \\
&= \lim_{t\to\infty} \frac{(1-a)^{-t}h(t)}{(1-a)^{-t}k(t+1) - (1-a)^{1-t}k(t)} \\
&= \lim_{t\to\infty} \frac{\frac{h(t)}{k(t)}}{\frac{k(t+1)}{k(t)} - (1-a)} \\
&= \frac{1}{a^2}. \quad (3.18)
\end{aligned}
$$

Using (3.18) in (3.17) and employing (3.12) yields

$$\liminf_{t\to\infty} \frac{\sum_{s=T}^{t-1}(1-a)^{-s}r(s)}{(1-a)^{1-t}k(t)} \geq \frac{1-\varepsilon}{a^2(1-\mu)} = \frac{1-\varepsilon}{a\left(a - \sum_{s=0}^{\infty}k(s)\right)}.$$

Therefore, by taking the $\liminf$ of both sides of (3.16) as $t \to \infty$ and then letting $\varepsilon \to 0$, we obtain (3.9). (3.8) follows from (3.16) and (3.9). $\qquad\square$

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III. SUBEXPONENTIAL SOLUTIONS OF LINEAR VOLTERRA DELAY DIFFERENCE EQUATIONS

ABSTRACT

We study the asymptotic behavior of the solutions of a scalar linear Volterra sum-difference equation with delay. The rate of convergence of the solution is found by assuming the kernel k is a nonnegative, summable, and subexponential sequence in the sense that $k(t)/h(t)$ converges to a positive number as $t \to \infty$, where h is a positive subexponential sequence.

1. INTRODUCTION

We consider the discrete equation with delay

$$\Delta x(t) = -\sum_{i=1}^{n} a_i x(t - \tau_i) + \sum_{s=0}^{t-1} k(t - 1 - s) x(s), \quad t \in \mathbb{N}_0, \tag{1.1}$$

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0, \tag{1.2}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\tau_i \in \mathbb{N}_0$ and $\tau = \max_{1 \leq i \leq n} \tau_i$. In this problem, we suppose that $a_i > 0$ with $\sum_{i=1}^{n} a_i < 1$, $k : \mathbb{N}_0 \to (0, \infty)$ is summable, and $\phi : \{-\tau, -\tau + 1, \ldots, 0\} \to \mathbb{R}$. We show that if k satisfies

$$\lim_{t \to \infty} \frac{k(t)}{h(t)} > 0$$

for some positive subexponential sequence h, then the rate of convergence of solutions (1.1) and (1.2) is

$$\lim_{t \to \infty} \frac{x(t)}{k(t)} = \frac{\phi(0) - \sum_{i=1}^{n} a_i \sum_{s=-\tau_i}^{0} \phi(s)}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s)\right)^2}.$$

The result is proved by determining the asymptotic behavior of the difference resolvent r of (1.1), which is the solution of the delay difference equation

$$\Delta r(t) = -\sum_{i=1}^{n} a_i r(t - \tau_i) + \sum_{s=0}^{t-1} k(t - 1 - s) r(s), \quad t \in \mathbb{N}_0$$

with

$$r(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \leq t < 0, \end{cases}$$

where the asymptotic behavior of $r(t)/k(t)$ as $t \to \infty$ is

$$\lim_{t \to \infty} \frac{r(t)}{k(t)} = \left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)^{-2}.$$

The same problem that we have studied here has been studied in [2] for corresponding linear integro-differential equations. We refer to [2–4, 7, 8] for related studies in the continuous and discrete cases. For basic properties and formulas concerning difference equations, we refer the reader to [1, 6, 11]. We also refer to [5, 9] for basic results on the existence of solutions of scalar linear Volterra equations in the continuous case.

2. PRELIMINARIES

Definition 2.1. For $f, g : \mathbb{N}_0 \to \mathbb{R}$, we define the convolution of f and g (see [11, Section 3.10]) by

$$(f * g)(t) = \sum_{s=0}^{t-1} f(t-1-s)g(s), \quad t \in \mathbb{N}_0.$$

The n-fold convolution f^{*n} is given by $f^{*1} = f$ and $f^{*(n+1)} = f * f^{*n}$ for $n \in \mathbb{N}$.

Definition 2.2. (See [8, Definition 2.1]) A *positive subexponential sequence* is a discrete mapping $h : \mathbb{N}_0 \to (0, \infty)$ with $\sum_{s=0}^{\infty} h(s) < \infty$ and

(S$_1$) $\lim_{t \to \infty} \dfrac{h^{*2}(t)}{h(t)} = 2 \sum_{s=0}^{\infty} h(s)$;

(S$_2$) $\lim_{t \to \infty} \dfrac{h(t-s)}{h(t)} = 1$ for each fixed $s \in \mathbb{N}_0$.

If h is a positive subexponential sequence and f is a sequence on $\mathbb{N}$ such that $\lim_{t \to \infty} \frac{f(t)}{h(t)}$ exists, then we define

$$L_h f := \lim_{t \to \infty} \frac{f(t)}{h(t)}.$$

By [8, Remark 2.3], (S$_2$) implies

$$\lim_{t \to \infty} \frac{h(t)}{e_{-b}(t)} = \infty, \tag{2.1}$$

where $e_{-b}(t) = (1-b)^t$ for $b \in (0,1)$.

We will use the following results from [8].

Lemma 2.3 (See [8, Theorem 2.9]). *Suppose that h is a positive subexponential sequence. Let f and g be summable sequences on $\mathbb{N}$ such that $L_h f$ and $L_h g$ both exist.*

Then $L_h(f * g)$ exists and

$$L_h(f * g) = L_h f \sum_{s=0}^{\infty} g(s) + L_h g \sum_{s=0}^{\infty} f(s). \qquad (2.2)$$

3. ASSUMPTIONS AND AUXILIARY RESULTS

Assume

(A$_1$) $\tau_i \geq 0$, $a_i \geq 0$, $\sum_{i=1}^{n} a_i < 1$ and $\tau = \max_{1 \leq i \leq n} \tau_i$;

(A$_2$) the characteristic equation

$$\lambda = \sum_{i=1}^{n} \frac{a_i}{e_{-\lambda}(\tau_i)},$$

where $e_{-\lambda}(\tau_i)$ is defined as in (2.1), has a real root $\lambda \in (0,1)$;

(A$_3$) $k : \mathbb{N}_0 \to [0, \infty)$ is nontrivial and summable;

(A$_4$) $\sum_{s=0}^{\infty} k(s) < \sum_{i=1}^{n} a_i$;

(A$_5$) $\lim_{t \to \infty} \dfrac{k(t)}{h(t)} > 0$ for some positive subexponential sequence h.

It is convenient to introduce the *difference resolvent* r of (1.1), which is the solution of the delay difference equation

$$\Delta r(t) = -\sum_{i=1}^{n} a_i r(t - \tau_i) + \sum_{s=0}^{t-1} k(t-1-s)r(s), \quad t \in \mathbb{N}_0 \tag{3.1}$$

with

$$r(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \leq t < 0. \end{cases} \tag{3.2}$$

The following lemma shows the significance of the difference resolvent r.

Lemma 3.1. *If x solves (1.1) and (1.2), then*

$$x(t) = \phi(0)r(t) + \sum_{s=0}^{t-1} r(t-1-s)\tilde{\phi}(s), \quad t \in \mathbb{N}_0, \tag{3.3}$$

where

$$\tilde{\phi}(t) = -\sum_{i=1}^{n} a_i \phi(t - \tau_i) \chi_{[0,\tau_i)}(t), \quad t \in \mathbb{N}_0 \tag{3.4}$$

and χ_I denotes the indicator function of a set I.

Proof. Suppose x solves (1.1) and (1.2). We claim that (3.3) holds for all $t \in \mathbb{N}_0$. Since

$$x(0) = \phi(0) = \phi(0)r(0) + (r * \tilde{\phi})(0),$$

(3.3) holds for $t = 0$. Now, assume (3.3) holds for all $0 \le t \le T$, where $T \in \mathbb{N}_0$. Then using assumption and (1.1), we get

$$x(T+1) = x(T) - \sum_{i=1}^{n} a_i x(T - \tau_i) + (k * x)(T)$$

$$= \phi(0)r(T) + (r * \tilde{\phi})(T) - \sum_{i=1}^{n} a_i x(T - \tau_i)$$

$$+ \phi(0)(k * r)(T) + (k * r * \tilde{\phi})(T).$$

Since (3.1) and (3.2), we get

$$(r * \tilde{\phi})(T+1) = \sum_{s=0}^{T} r(T - s)\tilde{\phi}(s)$$

$$= \sum_{s=0}^{T-1} \left(\Delta r(T - 1 - s) + r(T - 1 - s) \right) \tilde{\phi}(s) + \tilde{\phi}(T)$$

$$= \left(\Delta r * \tilde{\phi} \right)(T) + (r * \tilde{\phi})(T) + \tilde{\phi}(T)$$

$$= -\sum_{i=1}^{n} a_i (r * \tilde{\phi})(T - \tau_i) + (k * r * \tilde{\phi})(T)$$

$$+ (r * \tilde{\phi})(T) + \tilde{\phi}(T).$$

This implies,

$$x(T+1) = \phi(0)\left(r(T+1) + \sum_{i=1}^{n} a_i r(T - \tau_i)\right) - \sum_{i=1}^{n} a_i x(T - \tau_i)$$

$$+ (r * \tilde{\phi})(T+1) - \tilde{\phi}(T) + \sum_{i=1}^{n} a_i (r * \tilde{\phi})(T - \tau_i)$$

$$= \phi(0)r(T+1) + (r * \tilde{\phi})(T+1) + \sum_{i=1}^{n} a_i \left(\phi(0)r(T - \tau_i)\right.$$

$$+ (r * \tilde{\phi})(T - \tau_i) - x(T - \tau_i) + \phi(T - \tau_i)\chi_{[0,\tau_i)}(T)\right)$$

$$= \phi(0)r(T+1) + (r * \tilde{\phi})(T+1)$$

since the terms in the last sum are all zero due to the induction hypothesis, (1.2), and (3.4). □

Now, we introduce the *difference resolvent* z associated with the purely point delay part of (1.1), defined as the solution of

$$\Delta z(t) = -\sum_{i=1}^{n} a_i z(t - \tau_i), \quad t \in \mathbb{N}_0 \tag{3.5}$$

which satisfies

$$z(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \le t < 0. \end{cases} \tag{3.6}$$

The next theorem states the existence, uniqueness, and positivity of the solution of (3.5) and (3.6). Note also that in [10, Theorem 2.1], the positivity of the solution of delay differential systems has been studied.

Theorem 3.2. *Suppose that (A_1)–(A_2) hold. Then there exists a unique solution to (3.5) subject to (3.6). Moreover, $z(t) > 0$ for all $t \in \mathbb{N}_0$.*

Proof. An elementary induction proof shows that, for each positive integer m, there exists one and only one real function $z = z_m(t)$ with domain $\{-\tau, -\tau+1, \ldots, m\}$ and satisfying

$$\Delta z(t) = -\sum_{i=1}^{n} a_i z(t - \tau_i), \quad 0 \leq t \leq m - 1$$

with

$$z(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \leq t < 0. \end{cases}$$

Hence, there exists a unique solution to (3.5) subject to (3.6).

Now, we will show $z(t) > 0$ for all $t \in \mathbb{N}_0$. If $\sum_{i=1}^{n} a_i \tau_i = 0$, then (3.5) reduces to

$$\Delta z(t) = -\sum_{j=1}^{m} b_j z(t), \quad m \leq n, \quad b_j = a_{i_j}, \quad \tau_{i_j} = 0, \quad t \in \mathbb{N}_0.$$

Then the solution of (3.5) is $z(t) = \left(1 - \sum_{j=1}^{m} b_j\right)^t z(0)$ for all $t \in \mathbb{N}_0$. Hence, by assumption (A$_1$) and (3.6), $z(t) > 0$ for all $t \in \mathbb{N}_0$.

Next, we assume $\sum_{i=1}^{n} a_i \tau_i > 0$. Let $\lambda_0 \in (0, 1)$ be a real solution to the characteristic equation guaranteed by assumption (A$_2$). Then $\lambda_0 = \sum_{i=1}^{n} \frac{a_i}{(1-\lambda_0)^{\tau_i}} > \sum_{i=1}^{n} a_i$ and thus by (3.6), we have

$$\left| -\sum_{i=1}^{n} a_i z(-\tau_i) \right| \leq \sum_{i=1}^{n} a_i < \lambda_0.$$

Now, suppose there exists $t_1 \in \mathbb{N}$ such that

$$z(t) > 0 \text{ for all } 0 \leq t < t_1 \text{ and } z(t_1) \leq 0. \tag{3.7}$$

Let

$$a(t) = \left| \frac{-\sum_{i=1}^{n} a_i z(t - \tau_i)}{z(t)} \right| \quad \text{for all } 0 \leq t < t_1.$$

We will show that

$$a(t) < \lambda_0 \quad \text{for all } 0 \leq t < t_1. \tag{3.8}$$

Otherwise, there exists an integer t_2 where $0 < t_2 < t_1$ such that

$$a(t) < \lambda_0 \quad \text{for all } 0 \leq t < t_2 \quad \text{and} \quad a(t_2) \geq \lambda_0. \tag{3.9}$$

Relation (3.5) yields

$$\Delta z(t) = \left(\frac{-\sum_{i=1}^{n} a_i z(t - \tau_i)}{z(t)} \right) z(t), \quad 0 \leq t < t_1, \tag{3.10}$$

and $z(0) = 1$. Set $b(t) = \frac{-\sum_{i=1}^{n} a_i z(t-\tau_i)}{z(t)}$ for $0 \leq t < t_1$, and $y(v) = \frac{z(t-v)}{z(t)}$ for $0 \leq v \leq t < t_1$. Then (3.10) yields

$$\Delta y(v) = -b(t - v - 1)y(v + 1), \quad 0 \leq v \leq t < t_1. \tag{3.11}$$

Then the unique solution of (3.11) with $y(0) = 1$ is

$$y(v) = \left(\prod_{s_1=1}^{v} \frac{1}{1 + b(t - s_1)} \right) = \prod_{s=t-v}^{t-1} \frac{1}{1 + b(s)}, \quad 0 \leq v \leq t < t_1.$$

Therefore, from (3.9), the above equation yields

$$y(t - u) = \prod_{s=u}^{t-1} \frac{1}{1 + b(s)} < \prod_{s=u}^{t-1} \frac{1}{1 - \lambda_0} = \frac{1}{(1 - \lambda_0)^{t-u}}, \quad 0 \leq u \leq t < t_2.$$

Thus, by the definition of $y(t)$, we obtain

$$|y(t-u)| = \left|\frac{z(u)}{z(t)}\right| < \frac{1}{(1-\lambda_0)^{t-u}}, \quad 0 \le u \le t < t_2. \tag{3.12}$$

Denote $[t-\tau_i]_+ = \max\{t-\tau_i, 0\}$. Then for $0 \le t < t_1$ and because of (3.6), we have

$$\left|\frac{-a_i z(t-\tau_i)}{z(t)}\right| = \left|\frac{-a_i z(t-\tau_i)z([t-\tau_i]_+)}{z([t-\tau_i]_+)z(t)}\right|$$

$$= |-a_i|\left|\frac{z(t-\tau_i)}{z([t-\tau_i]_+)}\right|\left|\frac{z([t-\tau_i]_+)}{z(t)}\right|$$

$$\le |-a_i|\left|\frac{z([t-\tau_i]_+)}{z(t)}\right|.$$

Hence, using (3.12), we get for $0 \le t < t_2$

$$\sum_{i=1}^{n}\left|\frac{-a_i z(t-\tau_i)}{z(t)}\right| \le \sum_{i=1}^{n}|-a_i|\left|\frac{z([t-\tau_i]_+)}{z(t)}\right|$$

$$< \sum_{i=1}^{n}|-a_i|\frac{1}{(1-\lambda_0)^{t-[t-\tau_i]_+}}$$

$$\le \sum_{i=1}^{n}|-a_i|\frac{1}{(1-\lambda_0)^{\tau_i}}.$$

Thus, using (A_2)

$$a(t_2) = \left|\frac{-\sum_{i=1}^{n}a_i z(t_2-\tau_i)}{z(t_2)}\right| < \sum_{i=1}^{n}|-a_i|\frac{1}{(1-\lambda_0)^{\tau_i}} = \lambda_0,$$

which contradicts (3.9). Therefore, (3.8) holds.

From (3.10), we have for $0 \le t < t_1$ that $y(t) = \frac{1}{z(t)}$ satisfies the equation

$$\Delta y(t) = -y(t+1)\frac{\sum_{i=1}^{n}-a_i z(t-\tau_i)}{z(t)}, \quad 0 \le t < t_1. \tag{3.13}$$

The unique solution of (3.13) with $y(0) = \frac{1}{z(0)} = 1$ is

$$y(t) = \prod_{s=0}^{t-1} \left(\frac{1}{1 + \frac{\sum_{i=1}^{n} -a_i z(s-\tau_i)}{z(s)}} \right) = \prod_{s=0}^{t-1} \frac{1}{1 + b(s)}, \quad 0 \leq t \leq t_1.$$

Using (3.8), we have for $0 \leq t \leq t_1$,

$$0 < y(t) < \prod_{s=0}^{t-1} \frac{1}{1 - \lambda_0} = \frac{1}{(1 - \lambda_0)^t} \leq \frac{1}{(1 - \lambda_0)^{t_1}}.$$

Now, if $z(t_1) < 0$, then $y(t_1) = \frac{1}{z(t_1)} < 0$. But since $a(t) = |b(t)| < \lambda_0 < 1$ implies $1 + b(t) > 0$ on $0 \leq t \leq t_1$, we obtain

$$y(t_1) = \prod_{s=0}^{t_1-1} \frac{1}{1 + b(s)} > 0,$$

which is a contradiction. Therefore, $z(t_1) = 0$ so $y(t_1) = \frac{1}{z(t_1)}$ is undefined. However,

$$0 < \prod_{s=0}^{t_1-1} \frac{1}{1 + b(s)} = y(t_1) \leq \frac{1}{(1 - \lambda_0)^{t_1}} < \infty,$$

which is again a contradiction.

Consequently, the assumption (3.7) is incorrect. Hence, $z(t) > 0$ for all $t \in \mathbb{N}_0$. $\qquad\square$

Lemma 3.3. *Suppose that (A_1)–(A_2) hold. Then*

$$z(t) \to 0 \quad \text{exponentially as } t \to \infty \tag{3.14}$$

and

$$\sum_{s=0}^{\infty} z(s) = \frac{1}{\sum_{i=1}^{n} a_i}. \tag{3.15}$$

Moreover, if h is a positive subexponential sequence, then

$$L_h z = 0. \tag{3.16}$$

Proof. Since by Theorem 3.2, $z(t) > 0$ for all $t \in \mathbb{N}_0$ and $a_i \geq 0$, we observe that (3.5) implies that z is decreasing, and hence (3.5) yields

$$\Delta z(t) \leq -\sum_{i=1}^{n} a_i z(t).$$

Using [6, Theorem 6.1], (3.6), and (2.1), we get

$$0 < z(t) \leq z(0)e_{-\sum_{i=1}^{n} a_i}(t) = \left(1 - \sum_{i=1}^{n} a_i\right)^t. \tag{3.17}$$

Due to (A_1), (3.17) yields (3.14). Summing (3.5) from 0 to $t-1$, then letting $t \to \infty$ and then using (3.14), we obtain

$$\sum_{i=1}^{n} a_i \sum_{u=-\tau_i}^{\infty} z(u) = 1,$$

and hence applying (3.6), we get (3.15). Since h is a positive subexponential sequence, from (2.1) and (3.14), we have (3.16). $\qquad\square$

Now, consider the general delay difference equation

$$\Delta y(t) = -\sum_{i=1}^{n} a_i y(t - \tau_i) + f(t), \quad t \in \mathbb{N}_0, \tag{3.18}$$

subject to

$$y(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \leq t < 0. \end{cases} \tag{3.19}$$

Lemma 3.4. *Suppose z solves* (3.5) *and* (3.6) *and define y by*

$$y = z + (z * f). \tag{3.20}$$

Then y solves (3.18) *and* (3.19)

Proof. Taking difference in (3.20) and then using (3.5) and (3.6), we get for $t \in \mathbb{N}_0$

$$
\begin{aligned}
\Delta y(t) =& \Delta z(t) + \sum_{s=0}^{t-1} \left(\Delta z(t-1-s) \right) f(s) + z(0) f(t) \\
=& -\sum_{i=1}^{n} a_i z(t - \tau_i) - \sum_{i=1}^{n} a_i \sum_{s=0}^{t-1} z(t-1-s-\tau_i) f(s) + f(t) \\
=& -\sum_{i=1}^{n} a_i z(t - \tau_i) - \sum_{i=1}^{n} a_i \left(\sum_{s=0}^{t-\tau_i-1} z(t - \tau_i - 1 - s) f(s) \right. \\
& \left. + \sum_{s=t-\tau_i}^{t-1} z(t - \tau_i - 1 - s) f(s) \right) + f(t) \\
=& -\sum_{i=1}^{n} a_i z(t - \tau_i) - \sum_{i=1}^{n} a_i \sum_{s=0}^{t-\tau_i-1} z(t - \tau_i - 1 - s) f(s) + f(t) \\
=& -\sum_{i=1}^{n} a_i \left(z(t - \tau_i) + \sum_{s=0}^{t-\tau_i-1} z(t - \tau_i - 1 - s) f(s) \right) + f(t) \\
=& -\sum_{i=1}^{n} a_i y(t - \tau_i) + f(t),
\end{aligned}
$$

i.e., (3.18) holds. Using (3.6) in (3.20), we obtain (3.19). $\qquad \square$

4. MAIN RESULTS

Theorem 4.1. *Suppose that (A_1)–(A_4) hold. Then the resolvent r, defined by (3.1) and (3.2), is positive, summable, and satisfies $r(t) \to 0$ as $t \to \infty$. In addition, if (A_5) holds, then*

$$\lim_{t\to\infty} \frac{r(t)}{k(t)} = \left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)^{-2}, \qquad \lim_{t\to\infty} \frac{\Delta r(t)}{k(t)} = 0. \tag{4.1}$$

Moreover,

$$\lim_{t\to\infty} \frac{\sum_{s=0}^{t-1} r(t-1-s)r(s)}{r(t)} = 2 \sum_{s=0}^{\infty} r(s). \tag{4.2}$$

Proof. Applying Lemma 3.4 to the resolvent r of (1.1), which satisfies (3.1) and (3.2), we obtain

$$r = z + (z * (k * r)) = z + (h * r), \tag{4.3}$$

where

$$h = z * k. \tag{4.4}$$

Due to (A_3) and Theorem 3.2, we get $h \geq 0$. Now, since for $t = 0$, we have $r(0) = 1 > 0$. Let us assume $r(t) > 0$ for all $0 \leq t \leq T$ for some $T \in \mathbb{N}_0$. Then from (4.3), using the assumption and Theorem 3.2, we obtain $r(T+1) = z(T+1) + \sum_{s=0}^{T} h(T-s)r(s) > 0$. Thus, $r(t) > 0$ for all $t \in \mathbb{N}_0$. Summing (4.4) from 0 to $t-1$, then taking $t \to \infty$ and then using (3.15) and (A_4), we obtain

$$0 \leq \sum_{s=0}^{\infty} h(s) = \sum_{s=0}^{\infty} z(s) \sum_{s=0}^{\infty} k(s) = \frac{\sum_{s=0}^{\infty} k(s)}{\sum_{i=1}^{n} a_i} < 1. \tag{4.5}$$

By taking the convolution of each term in (4.3) with k, we get

$$\rho = h + (h * \rho), \tag{4.6}$$

where $\rho = k * r$. Then $\rho > 0$. Summing (4.6) from 0 to $t - 1$, we get for all $t \in \mathbb{N}_0$

$$
\begin{aligned}
0 \leq \sum_{s=0}^{t-1} \rho(s) &= \sum_{s=0}^{t-1} h(s) + \sum_{s=0}^{t-1} \sum_{u=0}^{s-1} h(s - 1 - u)\rho(s) \\
&= \sum_{s=0}^{t-1} h(s) + \sum_{u=0}^{t-2} \rho(u) \sum_{s=0}^{t-u-2} h(s) \\
&\leq \sum_{s=0}^{\infty} h(s) + \sum_{u=0}^{t-1} \rho(u) \sum_{s=0}^{\infty} h(s) \\
&\leq \frac{\sum_{s=0}^{\infty} h(s)}{(1 - \sum_{s=0}^{\infty} h(s))}.
\end{aligned}
$$

and thus

$$0 \leq \sum_{s=0}^{\infty} \rho(s) \leq \frac{\sum_{s=0}^{\infty} h(s)}{(1 - \sum_{s=0}^{\infty} h(s))}. \tag{4.7}$$

Therefore, ρ is summable. It is then an immediate consequence of Lemma 3.3 and

$$r = z + \rho * z,$$

that r is summable. Hence, $r(t) \to 0$ as $t \to \infty$ follows automatically from the property of convergent series. Now, summing (3.1) from 0 to $t - 1$ and using (3.2) gives

$$
\begin{aligned}
-1 &= - \sum_{i=1}^{n} a_i \sum_{s=0}^{\infty} r(s - \tau_i) + \sum_{s=0}^{\infty} r(s) \sum_{s=0}^{\infty} k(s) \\
&= - \sum_{i=1}^{n} a_i \sum_{u=-\tau_i}^{\infty} r(u) + \sum_{s=0}^{\infty} r(s) \sum_{s=0}^{\infty} k(s) \\
&= - \sum_{i=1}^{n} a_i \sum_{u=0}^{\infty} r(u) + \sum_{s=0}^{\infty} r(s) \sum_{s=0}^{\infty} k(s),
\end{aligned}
$$

which implies

$$\sum_{s=0}^{\infty} r(s) = \frac{1}{\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s)}. \tag{4.8}$$

Applying Lemma 2.3 in (4.4) and by (3.16) and (3.15), it deduce

$$L_k h = L_k(z * k) = L_k z \sum_{s=0}^{\infty} k(s) + L_k k \sum_{s=0}^{\infty} z(s) = \frac{1}{\sum_{i=1}^{n} a_i}. \tag{4.9}$$

Under the additional assumption $k(t) > 0$ for all $t \in \mathbb{N}_0$, it follows that $h(t) > 0$ for all $t \in \mathbb{N}$. Then by [8, Lemma 2.11], h is a subexponential sequence. By applying [8, Theorem 3.1] to (4.3), we conclude that $L_h r$ exists and hence $L_k r$ exists. Since $L_k r$ exits, we can infer from Lemma 2.3, (3.16), (4.9), (4.5), (4.8), and the above formulas that

$$
\begin{aligned}
L_k r &= L_k z + L_k h \sum_{s=0}^{\infty} r(s) + L_k r \sum_{s=0}^{\infty} h(s) \\
&= \frac{1}{\sum_{i=1}^{n} a_i \left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)} + L_k r \frac{1}{\sum_{i=1}^{n} a_i} \sum_{s=0}^{\infty} k(s) \\
&= \frac{1}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)^2}.
\end{aligned}
$$

Note that (S$_2$) implies $L_k r(. - \tau_i) = L_k r$, so by applying Lemma 2.3 to (3.1), using $L_k r$, and (4.8), we get

$$L_k \Delta r = -L_k r \left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right) + L_k k \sum_{s=0}^{\infty} r(s) = 0.$$

Therefore, (4.2) simply implies from Lemma 2.3

$$L_k(r * r) = 2 L_k r \sum_{s=0}^{\infty} r(s).$$

The proof is complete. $\qquad\square$

Theorem 4.2. *Suppose (A_1)–(A_4) hold and that $k(t) > 0$ for all $t \in \mathbb{N}_0$. If the resolvent r satisfies (4.1), then k is a positive subexponential sequence and (4.2) is true.*

Proof. Dividing (3.1) by $k(t)$ and then taking $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{\Delta r(t)}{k(t)} = -\sum_{i=1}^{n} a_i \lim_{t\to\infty} \frac{r(t-\tau_i)}{k(t)} + \lim_{t\to\infty} \frac{(k*r)(t)}{k(t)}.$$

By using the second expression of (4.1), we get

$$\lim_{t\to\infty} \frac{(k*r)(t)}{k(t)} = \frac{\sum_{i=1}^{n} a_i}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s)\right)^2}.$$

Hence, using the first part of (4.1),

$$\lim_{t\to\infty} \frac{(k*r)}{r(t)} = \lim_{t\to\infty} \frac{(k*r)(t)}{k(t)} \cdot \frac{k(t)}{r(t)} = \sum_{i=1}^{n} a_i.$$

Since r and k are positive, we have from (4.1), $\lim_{t\to\infty} \frac{k(t)}{r(t)} > 0$, so, [8, Lemma 2.7] applies with $f = k$ and $g = r$. Therefore, we can conclude from this that k satisfies (S_2). Now, using (4.8), we obtain

$$\lim_{t\to\infty} \frac{(k*k)(t)}{k(t)} = \sum_{i=1}^{n} a_i + \sum_{s=0}^{\infty} k(s) - \left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s)\right)^2 \sum_{s=0}^{\infty} r(s) = 2\sum_{s=0}^{\infty} k(s).$$

Thus, k satisfies (S_1). The proof is complete. $\qquad\square$

Theorem 4.3. *Suppose that (A_1)–(A_5) hold. Then the solution of (1.1) and (1.2) satisfies*

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{\phi(0) - \sum_{i=1}^{n} a_i \sum_{-\tau_i}^{0} \phi(s)}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s)\right)^2}, \tag{4.10}$$

$$\lim_{t\to\infty} \frac{\Delta x(t)}{k(t)} = 0. \tag{4.11}$$

Proof. Since $\tilde{\phi} \to 0$ as $t \to \infty$ for all $t > \tau_i$, we observe from (3.4) that $L_k\tilde{\phi} = 0$ and also summing (3.4) from 0 to ∞, we get

$$
\begin{aligned}
\sum_{s=0}^{\infty} \tilde{\phi}(s) &= -\sum_{i=1}^{n} a_i \sum_{s=0}^{\infty} \phi(s - \tau_i)\chi_{[0,\tau_i)}(s) \\
&= -\sum_{i=1}^{n} a_i \sum_{s=0}^{\tau_i} \phi(s - \tau_i) \\
&= -\sum_{i=1}^{n} a_i \sum_{s=-\tau_i}^{0} \phi(s).
\end{aligned}
\tag{4.12}
$$

By applying L_k to (3.3), using Lemma 2.3, (4.12), $L_k\tilde{\phi} = 0$, and (4.1), we obtain

$$
\begin{aligned}
L_k x &= \phi(0)L_k r + L_k r \sum_{s=0}^{\infty} \tilde{\phi} + L_k\tilde{\phi} \sum_{s=0}^{\infty} r(s) \\
&= L_k r \left(\phi(0) - \sum_{i=1}^{n} a_i \sum_{s=-\tau_i}^{0} \phi(s) \right) \\
&= \frac{\phi(0) - \sum_{i=1}^{n} a_i \sum_{s=-\tau_i}^{0} \phi(s)}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)^2}.
\end{aligned}
$$

Summing (3.3) from 0 to ∞ and then using (4.8) and (4.12), it follows that

$$
\sum_{s=0}^{\infty} x(s) = \frac{\phi(0) - \sum_{i=1}^{n} a_i \sum_{s=-\tau_i}^{0} \phi(s)}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)}.
\tag{4.13}
$$

By applying L_k to (1.1), Lemma 2.3, (4.10), and (4.13), we then see that

$$
\begin{aligned}
L_k \Delta x &= -\sum_{i=1}^{n} a_i L_k x(. - \tau_i) + L_k x \sum_{s=0}^{\infty} k(s) + L_k k \sum_{s=0}^{\infty} x(s) \\
&= \left(\sum_{s=0}^{\infty} k(s) - \sum_{i=1}^{n} a_i \right) L_k x + \sum_{s=0}^{\infty} x(s) \\
&= -\frac{\phi(0) - \sum_{i=1}^{n} a_i \sum_{s=-\tau_i}^{0} \phi(s)}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)} + \frac{\phi(0) - \sum_{i=1}^{n} a_i \sum_{s=-\tau_i}^{0} \phi(s)}{\left(\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s) \right)} \\
&= 0.
\end{aligned}
$$

The proof is complete. □

From (4.10) and (4.13), we can make the following remark

Remark 4.4. The decay rate given in (4.10) can also be expressed as

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{\sum_{s=0}^{\infty} x(s)}{\sum_{i=1}^{n} a_i - \sum_{s=0}^{\infty} k(s)}.$$

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IV. BOUNDED SOLUTIONS OF A VOLTERRA DIFFERENCE EQUATION

ABSTRACT

In this article, we study the existence of a bounded solution to a nonlinear Volterra difference equation. The technique and the tools employed in the analysis are Schaefer's fixed point theorem and Lyapunov's direct method.

1. INTRODUCTION

We consider the discrete equation

$$\Delta x(t) = a(t) - b(t)f(x(t+1)) + \sum_{s=0}^{t-1} k(t,s)g(s, x(s+1)), \quad x(0) = 0, \quad t \in \mathbb{N}_0, \quad (1.1)$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $a, b : \mathbb{N}_0 \to \mathbb{R}$ are real sequences, where $\mathbb{R} = (-\infty, \infty)$, the set of all real numbers, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Also, $k(t,s)$ is continuous on $0 \le s \le t < \infty$.

In this paper, we study the existence of a bounded solution of (1.1) on $\mathbb{N}_0$, an unbounded domain. We employ Schaefer's fixed point theorem, stated below, as the main mathematical tool in the analysis. The mapping H in the equation $x = \lambda H x$ of Schaefer's fixed point theorem needs to be completely continuous, i.e., it is continuous as a mapping and it maps bounded sets into relatively compact sets. The relative compactness property is normally obtained employing the Arzelà–Ascoli theorem when the domain of the problem is bounded. Since the domain of (1.1) is unbounded, Arzelà–Ascoli's theorem does not apply. To overcome this problem, we resort to a theorem, Theorem 2.1, which can be found in (see [2, Theorem 4.3.1, page 150]).

Schaefer's fixed point theorem requires an *a priori* bound on all solutions of an auxiliary equation. We employ a variant of Lyapunov's direct method to obtain the *a priori* bound. In particular, we show in Theorem 2.4 that (1.1) has a bounded solution on $\mathbb{N}_0$ provided all such solutions of an auxiliary equation, (2.10), has an *a priori* bound for all λ, $0 < \lambda \le 1$. Then in Theorem 2.6, we obtained such an *a priori* bound on solutions of (2.10) applying Lyapunov's method. We refer the readers to Burton [4] for basic results on the existence of solutions of continuous case of (1.1). For basic properties and formulas for difference equations, we refer to [1–3,5], and [6].

For convenience, we state Schaefer's fixed point theorem below.

Theorem 1.1 (Schaefer's fixed point theorem [7]). *Let $(S, \| \cdot \|)$ be a normed space and let H be a completely continuous mapping of S into S. Then either*

(i) the equation $x = \lambda H x$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Note that a mapping is completely continuous if it is continuous and maps bounded sets into relatively compact sets. A set B is relatively compact if the closure of B is compact.

Theorem 1.2. *The sequence x satisfies the IVP*

$$\Delta x(t) = F(t, x(t)), \quad x(t_0) = x_0 \tag{1.2}$$

if and only if it satisfies the following sum equation

$$x(t) = x_0 + \sum_{s=t_0}^{t-1} F(s, x(s)). \tag{1.3}$$

Proof. Suppose x satisfies (1.2). Summing both sides of (1.2), we get

$$x(t) - x(t_0) = \sum_{s=t_0}^{t-1} F(s, x(s))$$

and then substituting $x(t_0) = x_0$ yields (1.3). Conversely, suppose x satisfies (1.3). Taking difference to (1.3), we get $\Delta x(t) = F(t, x(t))$ and evaluating (1.3) at $t = t_0$, we get $x(t_0) = x_0$. $\square$

2. EXISTENCE OF BOUNDED SOLUTIONS

Assume

(A$_1$) (i) $|a(t)| \leq \bar{a} < \infty$,

(ii) $|b(t)| \leq \bar{b} < \infty$ for all $t \in \mathbb{N}_0$;

(A$_2$) there exist constants $\bar{f}$ and $\bar{g}$ such that

(i) $|f(x) - f(y)| \leq \bar{f}|x - y|$,

(ii) $|g(t, x) - g(t, y)| \leq \bar{g}|x - y|$;

(A$_3$) (i) $|f(0)| \leq f^* < \infty$,

(ii) $|g(t, 0)| \leq g^* < \infty$;

(A$_4$) $\displaystyle\sup_{t \in \mathbb{N}_0} \sum_{s=0}^{t-1} |k(t, s)| \leq k^* < \infty$.

By adding $x(t + 1)$, multiplying by 2^t, and then taking sum and finally, multiplying by 2^{-t} on both sides of equation (1.1), we obtain

$$x(t) = \sum_{s=0}^{t-1} 2^{-(t-s)} x(s+1) + \sum_{s=0}^{t-1} 2^{-(t-s)} a(s)$$
$$- \sum_{s=0}^{t-1} 2^{-(t-s)} b(s) f(x(s+1)) + \sum_{s=0}^{t-1} 2^{-(t-s)} \sum_{u=0}^{s-1} k(s, u) g(u, x(u+1)). \quad (2.1)$$

Let $\mathbb{B}$ be the space of bounded functions $\phi : \mathbb{N}_0 \to \mathbb{R}$ with the supremum norm. Define a mapping H by the right-hand side of (2.1)

$$H\phi(t) = \sum_{s=0}^{t-1} 2^{-(t-s)} \phi(s+1) + \sum_{s=0}^{t-1} 2^{-(t-s)} a(s)$$
$$- \sum_{s=0}^{t-1} 2^{-(t-s)} b(s) f(\phi(s+1)) + \sum_{s=0}^{t-1} 2^{-(t-s)} \sum_{u=0}^{s-1} k(s, u) g(u, \phi(u+1)). \quad (2.2)$$

For any $\phi \in \mathbb{B}$ with $|\phi(t)| \leq \bar{\phi} < \infty$, it follows from (2.2) and assumptions (A_1)–(A_4)

$$|(H\phi)(t)| \leq (\bar{a} + \bar{\phi}) + \bar{b}(\bar{f}\bar{\phi} + f^*) + k^*(\bar{g}\bar{\phi} + g^*) := M < \infty, \qquad (2.3)$$

which shows that $(H\phi)(t)$ is bounded on $\mathbb{N}_0$. Therefore, H maps from $\mathbb{B}$ into $\mathbb{B}$.

Now, to apply Schaefer's theorem, we need to show that H is completely continuous, i.e., H is a continuous mapping and that H maps bounded sets into relatively compact sets. To show the continuity of the mapping H, let $\phi, \psi \in \mathbb{B}$. Then (2.2) and assumptions (A_1)–(A_4) yield

$$|(H\phi)(t) - (H\psi)(t)| \leq (1 + \bar{b}\bar{f} + k^*\bar{g})||\phi - \psi||.$$

Therefore, H is continuous as a mapping.

Now, we will show that H maps bounded sets into relatively compact sets. To show this, we use the following result [2] for the relative compactness.

Theorem 2.1 (see [2, Theorem 4.3.1, page 150]). *Let M be the space of all bounded continuous real-valued functions on $[0, \infty)$ and $S \subset M$. Then S is relatively compact in M, if the following conditions hold.*

(i) S is bounded in M,

(ii) the functions in S are equicontinuous on any compact interval of $[0, \infty)$,

(iii) the functions in S are equiconvergent, i.e., given $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $|\phi(t) - \phi(\infty)| < \varepsilon$ for all $t > T$ and for all $\phi \in S$.

Let

$$K = \left\{ \phi \in \mathbb{B} : ||\phi|| \leq m, \ \phi(0) = 0, \ \lim_{t \to \infty} \phi(t) = \theta \right\},$$

where $m > 0$ and θ is an arbitrary but fixed real number.

Additional assumptions:

(A_5) (i) $\lim_{t\to\infty} a(t) = \hat{a} < \infty,$

   (ii) $\lim_{t\to\infty} b(t) = \hat{b} < \infty;$

(A_6) $\lim_{t\to\infty} g(t, x) = \hat{g} < \infty$ uniformly with respect to $x \in K;$

(A_7) $\lim_{t\to\infty} k(t, s) = 0$ for each $s \in \mathbb{N}_0;$

(A_8) $\lim_{t\to\infty} \sum_{s=0}^{t-1} k(t, s) < \infty.$

Lemma 2.2. *Suppose assumptions (A_4), (A_7) and (A_8) hold with*
$\hat{k} = \lim_{t\to\infty} \sum_{s=0}^{t-1} k(t, s).$ *Then for every $x \in K$,*

$$\lim_{t\to\infty} \sum_{s=0}^{t-1} k(t, s)x(s) = \hat{k} \lim_{t\to\infty} x(t).$$

Proof. Let us consider $x \in K$ with $\lim_{t\to\infty} x(t) = \theta$. Since assumption (A_8) and

$$\sum_{s=0}^{t-1} k(t, s)x(s) = \sum_{s=0}^{t-1} k(t, s)[x(s) - \theta] + \sum_{s=0}^{t-1} k(t, s)\theta,$$

it follows that it is enough to prove that

$$\left(\lim_{t\to\infty} x(t) = 0 \right) \Rightarrow \left(\sum_{s=0}^{t-1} k(t, s)x(s) = 0 \right).$$

So, let $x \in K$ be such that $\lim_{t\to\infty} x(t) = 0$. Then for all $\varepsilon > 0$, there is $T = T(\varepsilon) > 0$ such that

$$|x(t)| < \varepsilon \quad \text{for all} \quad t \geq T. \tag{2.4}$$

For all $t \geq T$, we have

$$\left| \sum_{s=0}^{t-1} k(t, s)x(s) \right| \leq \left| \sum_{s=0}^{T-1} k(t, s)x(s) \right| + \left| \sum_{s=T}^{t-1} k(t, s)x(s) \right|.$$

First, by (A_4) and (2.4), we have

$$\left|\sum_{s=T}^{t-1} k(t,s)x(s)\right| \le \sum_{s=T}^{t-1} |k(t,s)||x(s)| \le \varepsilon k^*.$$

It remains to prove that for all $x \in K$,

$$\lim_{t\to\infty} \sum_{s=0}^{T-1} k(t,s)x(s) = 0. \tag{2.5}$$

For $x(t) \equiv 1$, (2.5) holds due to (A_7). Let $n \in \mathbb{N}$ be arbitrary. Summing by parts, we deduce

$$\sum_{s=0}^{T-1} k(t,s)s^n = T^n \sum_{s=0}^{T-1} k(t,s) - \sum_{s=0}^{T-1}\sum_{u=0}^{s-1} k(t,u+1)((s+1)^n - s^n). \tag{2.6}$$

The first term in the right-hand side of (2.6) converges to zero as $t \to \infty$ due to (A_7). Let us consider a sequence $(t_i)_i$ with $\lim_{i\to\infty} t_i = \infty$, and let

$$p_i(s) = \sum_{u=0}^{s-1} k(t_i, u+1)((s+1)^n - s^n), \quad i \in \mathbb{N}, \ s \in [0,T].$$

Then $|p_i(s)| \le ((T+1)^n + T^n) \sum_{u=0}^{T-1} |k(t_i, u+1)|$. By ($A_4$), there is $D > 0$ such that $|p_i(s)| \le D$ for all $i \in \mathbb{N}$ and for all $s \in [0,T]$. In addition, due to (A_7), $\lim_{i\to\infty} p_i(s) = 0$ for all $s \in [0,T]$.

By applying the dominated convergence theorem, we deduce that the second term in the right-hand side of (2.6) converges to zero as $t \to \infty$.

Hence, for each polynomial sequence P, $\lim_{t\to\infty} \sum_{s=0}^{T-1} k(t,s)P(s) = 0$ and from Weierstrass's approximation theorem, $\lim_{t\to\infty} \sum_{s=0}^{T-1} k(t,s)x(s) = 0$ for each $x \in K$.

$\square$

Lemma 2.3. $H(K)$ *is relatively compact where H is defined by (2.2).*

To prove that $H(K)$ is relatively compact, we will use Theorem 2.1 and show that

(i) $H(K)$ is (uniformly) bounded,

(ii) $H(K)$ is equicontinuous on compact intervals of $[0, \infty)$,

(iii) $H(K)$ is equiconvergent.

Proof. The work, we have given earlier to prove that $(H\phi)(t)$ is bounded, can be used to show that $H(K)$ is uniformly bounded. Likewise, the work, for the continuity of $(H\phi)(t)$, shown earlier, can be used to show that $H(K)$ is equicontinuous on compact intervals of $[0, \infty)$.

Now, we show that $H(K)$ is equiconvergent. It follows from the continuity of f that there exists a constant $f_\theta := f(\theta)$ such that $\lim_{t\to\infty} f(\phi(t)) = f_\theta$ uniformly with respect to $\phi \in K$. Using Lemma 2.2, and assumptions (A_5)–(A_6), we get

$$(H\phi)(\infty) = \lim_{t\to\infty}(H\phi)(t) = \theta + \hat{a} - \hat{b}f_\theta + \hat{k}\hat{g} \tag{2.7}$$

for all $\phi \in K$. Therefore, for any $\varepsilon > 0$, one can easily show that there exists a $T(\varepsilon) > 0$ such that $|(H\phi)(t) - (H\phi)(\infty)| < \varepsilon$ for all $t > T$ and for all $\phi \in K$. This proves that $H(K)$ is equiconvergent and hence, the proof of Lemma 2.3 is complete. $\square$

So, we have shown that the mapping $H : \mathbb{B} \to \mathbb{B}$ is completely continuous.

For the parameter λ in Schaefer's theorem, we define the auxiliary equation

$$x_\lambda(t) = \lambda \left[\sum_{s=0}^{t-1} 2^{-(t-s)} x_\lambda(s+1) + \sum_{s=0}^{t-1} 2^{-(t-s)} a(s) \tag{2.8} \right.$$
$$\left. - \sum_{s=0}^{t-1} 2^{-(t-s)} b(s) f(x_\lambda(s+1)) + \sum_{s=0}^{t-1} 2^{-(t-s)} \sum_{u=0}^{s-1} k(s,u) g(u, x_\lambda(u+1)) \right].$$

Notice that (2.8) becomes (2.1) when $\lambda = 1$. Also, notice that (2.1) and (1.1) are equivalent.

Theorem 2.4. *Let assumptions* (A_1)–(A_8) *hold. Suppose there exists a constant* $B > 0$ *such that* $||x_\lambda|| \leq B$ *for all solutions* x_λ *of* (2.8) *for all* $\lambda \in (0, 1]$. *Then* (1.1) *has a bounded solution* x *on* $\mathbb{N}_0$ *with* $||x|| \leq B$.

Proof. In the above work, we have shown that if assumptions (A_1)–(A_8) hold, then the mapping H is completely continuous. Since we assumed that there exists an *a priori* bound B for all solutions x_λ of (2.8) for all $\lambda \in (0, 1]$, the conclusion of Theorem 2.4 follows from Schaefer's theorem. $\qquad\square$

Lemma 2.5. *The function* x *satisfies* (2.8) *if and only if it satisfies the difference equation*

$$\Delta x_\lambda(t) = (\lambda - 1)x_\lambda(t + 1) + \lambda\Big[a(t) - b(t)f(x_\lambda(t + 1))$$
$$+ \sum_{s=0}^{t-1} k(t, s)g(s, x_\lambda(s + 1))\Big], \quad x_\lambda(0) = 0, \tag{2.9}$$

for all $\lambda \in (0, 1]$.

Proof. Suppose x satisfies (2.8). Taking difference on both sides of (2.8) and using (2.8) again in the resulting equation, we obtain

$$\Delta x_\lambda(t) = -\frac{1}{2}x_\lambda(t) + \frac{\lambda}{2}x_\lambda(t + 1) + \frac{\lambda}{2}\Big[a(t) - b(t)f(x_\lambda(t + 1))$$
$$+ \sum_{s=0}^{t-1} k(t, s)g(s, x_\lambda(s + 1))\Big].$$

Now, adding and subtracting $\frac{1}{2}x_\lambda(t+1)$ on the right-hand side of the above equation, we get

$$\Delta x_\lambda(t) = \frac{1}{2}\Delta x_\lambda(t) + \frac{(\lambda - 1)}{2}x_\lambda(t + 1) + \frac{\lambda}{2}\Big[a(t) - b(t)f(x_\lambda(t + 1))$$
$$+ \sum_{s=0}^{t-1} k(t, s)g(s, x_\lambda(s + 1))\Big]$$

and then multiplying by 2 on both sides and subtracting $\Delta x_\lambda(t)$ on both sides of the resulting equation, we obtain

$$\Delta x_\lambda(t) = (\lambda - 1)x_\lambda(t+1) + \lambda \left[a(t) - b(t)f(x_\lambda(t+1)) + \sum_{s=0}^{t-1} k(t,s)g(s,x_\lambda(s+1)) \right]$$

and clearly $x_\lambda(0) = 0$.

Conversely, suppose x satisfies (2.9). Multiplying by 2^t on both sides of the first equation of (2.9), we get

$$\Delta(2^t x_\lambda(t)) = \lambda 2^t x_\lambda(t+1) + \lambda 2^t \left[a(t) - b(t)f(x_\lambda(t+1)) + \sum_{s=0}^{t-1} k(t,s)g(s,x_\lambda(s+1)) \right]$$

for all $\lambda \in (0,1]$. Now, taking sum and then dividing by 2^t on both sides of the above equation yields (2.8). $\qquad\square$

Now, we obtain an *a priori* bound on all bounded solutions of (2.9) for all $\lambda \in (0,1]$. We apply a variant of Lyapunov's method as the mathematical tool.

Theorem 2.6. *In addition to assumptions* (A_1)–(A_8), *suppose the following conditions hold.*

(A_9) *(i) a is summable,*

   (ii) $b(t) > 0$ *for* $t \in \mathbb{N}_0$,

   (iii) $xf(x) > 0$ *for all* $x \neq 0$,

   (iv) $|f(x)| \geq q|x|$ *for some* $q > 0$,

   (v) $g(t,0) = 0$;

(A_{10}) *there exists a constant* $\alpha > 0$ *such that*

$$1 - qb(t) + \bar{g} \sum_{u=1}^{\infty} |k(u+t,t)| \leq -\alpha,$$

   for all $t \in \mathbb{N}_0$, *where* $\bar{g}$ *is the constant in* $(A_2)(\text{ii})$.

Then there exists an a priori bound on all solutions x_λ, of (2.9) for all $\lambda \in (0, 1]$.

Proof. Suppose x satisfies (2.9). Then

$$x_\lambda(t) = (2 - \lambda)x_\lambda(t+1) + \lambda b(t)f(x_\lambda(t+1)) - \lambda \left[a(t) + \sum_{s=0}^{t-1} k(t, s)g(s, x_\lambda(s + 1)) \right].$$

Hence,

$$|x_\lambda(t)| \geq |(2 - \lambda)x_\lambda(t + 1) + \lambda b(t)f(x_\lambda(t + 1))|$$
$$- \left| \lambda \left[a(t) + \sum_{s=0}^{t-1} k(t, s)g(s, x_\lambda(s + 1)) \right] \right|. \tag{2.10}$$

Now, if $x(t + 1) \geq 0$, then using assumption(A_9)(ii)–(iii) and (A_9)(iv), we have

$$|(2 - \lambda)x_\lambda(t + 1) + \lambda b(t)f(x_\lambda(t + 1))|$$
$$= (2 - \lambda)x_\lambda(t + 1) + \lambda b(t)f(x_\lambda(t + 1))$$
$$= (2 - \lambda)|x_\lambda(t + 1)| + \lambda b(t)|f(x_\lambda(t + 1))|$$
$$\geq (2 - \lambda)|x_\lambda(t + 1)| + \lambda q b(t)|x_\lambda(t + 1)|$$
$$= (2 - \lambda + \lambda q b(t))|x_\lambda(t + 1)|. \tag{2.11}$$

Again, if $x(t + 1) \leq 0$, then using assumption(A_9)(ii)–(iii) and (A_9)(iv), we have

$$|(2 - \lambda)x_\lambda(t + 1) + \lambda b(t)f(x_\lambda(t + 1))|$$
$$= - [(2 - \lambda)x_\lambda(t + 1) + \lambda b(t)f(x_\lambda(t + 1))]$$
$$= (2 - \lambda)(-x_\lambda(t + 1)) + \lambda b(t)(-f(x_\lambda(t + 1)))$$
$$= (2 - \lambda)|x_\lambda(t + 1)| + \lambda b(t)|f(x_\lambda(t + 1))|$$
$$\geq (2 - \lambda)|x_\lambda(t + 1)| + \lambda q b(t)|x_\lambda(t + 1)|$$
$$= (2 - \lambda + \lambda q b(t))|x_\lambda(t + 1)|. \tag{2.12}$$

Hence, (2.11) and (2.12) implies

$$|(2 - \lambda)x_\lambda(t+1) + \lambda b(t)f(x_\lambda(t+1))| \geq (2 - \lambda + \lambda q b(t))|x_\lambda(t+1)|. \qquad (2.13)$$

Using (2.13) in (2.10), we obtain

$$|x_\lambda(t)| \geq (2 - \lambda + \lambda q b(t))|x_\lambda(t+1)| - \lambda \left[|a(t)| + \sum_{s=0}^{t-1} |k(t,s)||g(s, x_\lambda(s+1))| \right]. \qquad (2.14)$$

Define a Lyapunov functional by

$$V(t) := V(t, x_\lambda(\cdot)) = |x_\lambda(t)| + \lambda \sum_{s=0}^{t-1} \sum_{u=t-s}^{\infty} |k(u+s, s)||g(s, x_\lambda(s+1))|. \qquad (2.15)$$

Now, taking difference on $V(t)$, using (2.14) along with assumptions $(A_2)(ii)$, and (A_{10}), we obtain

$$
\begin{aligned}
\Delta V(t) &= |x_\lambda(t+1)| + \lambda \sum_{s=0}^{t} \sum_{u=t+1-s}^{\infty} |k(u+s,s)||g(s, x_\lambda(s+1))| \\
&\quad - |x_\lambda(t)| - \lambda \sum_{s=0}^{t-1} \sum_{u=t-s}^{\infty} |k(u+s,s)||g(s, x_\lambda(s+1))| \\
&= |x_\lambda(t+1)| + \lambda \sum_{s=0}^{t-1} \left[\sum_{u=t+1-s}^{\infty} |k(u+s,s)||g(s, x_\lambda(s+1))| \right. \\
&\quad \left. - \lambda \sum_{u=t-s}^{\infty} |k(u+s,s)||g(s, x_\lambda(s+1))| \right] \\
&\quad + \lambda \sum_{u=1}^{\infty} |k(u+t,t)||g(t, x_\lambda(t+1))| - |x_\lambda(t)| \\
&= |x_\lambda(t+1)| - \lambda \sum_{s=0}^{t-1} |k(t,s)||g(s, x_\lambda(s+1))| \\
&\quad + \lambda \sum_{u=1}^{\infty} |k(u+t,t)||g(t, x_\lambda(t+1))| - |x_\lambda(t)| \\
&\leq \left[1 + \lambda \bar{g} \sum_{u=1}^{\infty} |k(u+t,t)| \right] |x_\lambda(t+1)|
\end{aligned}
$$

$$-\lambda \sum_{s=0}^{t-1} |k(t,s)||g(s,x_\lambda(s+1))| - |x_\lambda(t)|$$

$$\leq \left[1 + \lambda\bar{g}\sum_{u=1}^{\infty} |k(u+t,t)|\right] |x_\lambda(t+1)| - \lambda\sum_{s=0}^{t-1} |k(t,s)||g(s,x_\lambda(s+1))|$$

$$- (2 - \lambda + \lambda q b(t))|x_\lambda(t+1)| + \lambda|a(t)| + \lambda\sum_{s=0}^{t-1} |k(t,s)||g(s,x_\lambda(s+1))|$$

$$= \left[-1 + \lambda\left[1 - qb(t) + \bar{g}\sum_{u=1}^{\infty} |k(u+t,t)|\right]\right] |x_\lambda(t+1)| + \lambda|a(t)|$$

$$\leq |a(t)| - (1 + \lambda\alpha)|x_\lambda(t+1)|.$$

Taking sums on the above inequality yields

$$V(t) \leq V(0) - (1 + \lambda\alpha)\sum_{s=0}^{t-1} |x_\lambda(s+1)| + \sum_{s=0}^{t-1} |a(s)|.$$

Since a is summable, there exists a constant $B > 0$ such that

$$V(t) + (1 + \lambda\alpha)\sum_{s=0}^{t-1} |x_\lambda(s+1)| \leq V(0) + \sum_{s=0}^{t-1} |a(s)| \leq B. \tag{2.16}$$

This implies $V(t) \leq B$ for all λ, $0 < \lambda \leq 1$. Therefore, from (2.15), we get $||x_\lambda|| \leq B$ for all λ, $0 < \lambda \leq 1$. This B is the required *a priori* bound on all solutions of (2.9). $\qquad\square$

Theorem 2.7. *It follows from (2.16) that x_λ is summable for all $\lambda \in (0,1]$. Therefore, if the assumptions of Theorems 2.6 hold, then there exists a bounded solution x of (1.1) on $\mathbb{N}_0$, with $\sup\limits_{t\in\mathbb{N}_0} |x(t)| \leq \sum\limits_{s=0}^{t-1} |a(s)|$ and x is summable.*

Example 2.8. Let $a(t) = (-1)^{t+1}\frac{2t+1}{t(t+1)} + \frac{(-1)^{t+1}}{t+1} + \frac{2}{3(t+1)(t+2)}(1 - (\frac{-1}{2})^t)$, $b(t) = 1$, $f(x) = x$, $k(t,s) = \frac{2^{-s}(s^2+2s+2)}{(t+1)(t+2)(s+1)}$, and $g(t,x) = \frac{x}{1+x^2}$. These functions satisfy all the assumptions we used in this article. Note that in this example $\bar{a} = \frac{8}{3}$, $\bar{b} = 1$, $\bar{f} = 1$, $\bar{g} = 2$, $\hat{a} = 0$, $\hat{b} = 1$, $f^* = 0$, $g^* = 0$, $k^* = \frac{1}{2}$, $\hat{g} = \frac{\theta}{1+\theta^2}$, $q = 8$ and $\alpha = 1$. So by

Theorem 2.7

$$\Delta x(t) = (-1)^{t+1} \frac{2t+1}{t(t+1)} - x(t+1)$$

$$+ \sum_{s=0}^{t-1} \frac{2^{-s}(s^2 + 2s + 2)}{(t+1)(t+2)(s+1)} \left(\frac{x(s+1)}{1 + x^2(s+1)} \right), \quad x(0) = 0 \qquad (2.17)$$

has a bounded solution. In fact, $x(t) = \frac{(-1)^t}{t}$ is a bounded oscillatory solution of (2.17).

Example 2.9. Let $a(t) = \frac{1}{(1+t)^2}$, $b(t) = 2^{-t}$, $k(t,s) = \frac{1}{(t-s)(t-s+1)}$, $f(x) = qx + \frac{x^3}{1+x^2}$; $q \geq$ 4, and $g(t,x) = \frac{x^3}{1+x^2} + x2^{-t}$. These functions satisfy all the assumptions we used in this article. Note that in this example $\bar{a} = 1$, $\bar{b} = 1$, $\bar{f} = 5$, $\bar{g} = 2$, $\hat{a} = 0$, $\hat{b} = 0$, $f^* = 0$, $g^* = 0$, $k^* = 1$, $\hat{g} = \frac{\theta^3}{1+\theta^2}$, $q = 4$ and $\alpha = 1$. So, by Theorem 2.7

$$\Delta x(t) = \frac{1}{(1+t)^2} - \left(\frac{4x(t+1)}{2^{-t}} + \frac{x^3(t+1)}{(1 + x^2(t+1))2^{-t}} \right)$$

$$+ \sum_{s=0}^{t-1} \frac{1}{(t-s)(t-s+1)} \left(\frac{x^3(s+1)}{1 + x^2(s+1)} + \frac{x(s+1)}{2^{-s}} \right), \quad x(0) = 0$$

$$(2.18)$$

has a bounded solution.

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V. ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF HIGHER-ORDER INTEGRO-DYNAMIC EQUATIONS

ABSTRACT

In this paper, we establish some new criteria on the asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations on time scales.

1. INTRODUCTION

In this paper, we are concerned with the asymptotic behavior of nonoscillatory solutions of the higher-order integro-dynamic equation on time scales

$$x^{\Delta^n}(t) + \int_0^t a(t,s)F(s,x(s))\Delta s = 0. \tag{1.1}$$

We take $\mathbb{T} \subseteq \mathbb{R}$ to be an arbitrary time scale with $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Whenever we write $t \geq s$, we mean $t \in [s, \infty) \cap \mathbb{T}$. We assume throughout that

(H_1) $a : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is rd-continuous such that $a(t,s) \geq 0$ for $t > s$ and

$$\sup_{t \geq T} \int_0^T a(t,s)\Delta s =: k_T < \infty \quad \text{for all} \quad T \geq 0; \tag{1.2}$$

(H_2) $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous functions $f_1, f_2 :$ $\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that $F(t,x) = f_1(t,x) - f_2(t,x)$ for $t \geq 0$;

(H_3) there exist constants β and γ of ratios of positive odd integers and $p_i \in C_{\mathrm{rd}}(\mathbb{T},(0,\infty))$, $i \in \{1,2\}$ such that

$$f_1(t,x) \geq p_1(t)x^{\beta} \quad \text{and} \quad f_2(t,x) \leq p_2(t)x^{\gamma} \quad \text{for} \quad x > 0 \quad \text{and} \quad t \geq 0,$$

$$f_1(t,x) \leq p_1(t)x^{\beta} \quad \text{and} \quad f_2(t,x) \geq p_2(t)x^{\gamma} \quad \text{for} \quad x < 0 \quad \text{and} \quad t \geq 0.$$

We only consider those solutions of equation (1.1) which are nontrivial and differentiable on $[0,\infty)$. The term solution henceforth applies to such solutions of equation (1.1). A solution x of equation (1.1) is said to be oscillatory if for every $t_0 > 0$, we have $\inf_{t \geq t_0} x(t) < 0 < \sup_{t \geq t_0} x(t)$ and nonoscillatory otherwise. Dynamic equations on time scales are fairly new objects of study and for the general basic ideas and background, we refer to [1, 2].

Oscillation results for integral equations of Volterra type are scant and only a few references exist on this subject. Related studies can be found in [4, 6–8]. To the best of our knowledge, there appear to be no such results on the asymptotic behavior of nonoscillatory solutions of equations (1.1). Our aim here is to initiate such a study by establishing some new criteria for the asymptotic behavior of nonoscillatory solutions of equation (1.1) and some related equations.

2. AUXILIARY RESULTS

We shall employ the following auxiliary results.

Lemma 2.1 (See [3]). *If $X, Y \geq 0$, then*

$$X^\lambda + (\lambda - 1)Y^\lambda - \lambda XY^{\lambda-1} \geq 0 \quad for \quad \lambda > 1 \tag{2.1}$$

and

$$X^\lambda - (1 - \lambda)Y^\lambda - \lambda XY^{\lambda-1} \leq 0 \quad for \quad \lambda < 1, \tag{2.2}$$

and equality holds if and only if $X = Y$.

Lemma 2.2 (See [5, Corollary 1]). *Assume that $n \in \mathbb{N}$, $s, t \in \mathbb{T}$, and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then*

$$\int_s^t \int_{\eta_n}^t \cdots \int_{\eta_2}^t f(\eta_1)\Delta\eta_1\Delta\eta_2 \cdots \Delta\eta_n = (-1)^{n-1} \int_s^t h_{n-1}(s, \sigma(\eta))f(\eta)\Delta\eta.$$

Remark 2.3. Under the conditions of Lemma 2.2, we may reverse all occurring integrals to obtain

$$\int_t^s \int_t^{\eta_n} \cdots \int_t^{\eta_2} f(\eta_1)\Delta\eta_1\Delta\eta_2 \cdots \Delta\eta_n = \int_t^s h_{n-1}(s, \sigma(\eta))f(\eta)\Delta\eta$$

and then replace t by t_0 and s by t to arrive at

$$\int_{t_0}^t \int_{t_0}^{\eta_n} \cdots \int_{t_0}^{\eta_2} f(\eta_1)\Delta\eta_1\Delta\eta_2 \cdots \Delta\eta_n = \int_{t_0}^t h_{n-1}(t, \sigma(\eta))f(\eta)\Delta\eta, \tag{2.3}$$

which is the formula that will be needed in the proofs of our main results in Section 3 below.

In Lemma 2.2 above, the h_n stand for the Taylor monomials (see [1, Section 1.6]) which are defined recursively by

$$h_0(t, s) = 1, \quad h_{n+1}(t, s) = \int_s^t h_n(\tau, s) \Delta \tau \quad \text{for} \quad t, s \in \mathbb{T} \quad \text{and} \quad n \in \mathbb{N}.$$

It follows that $h_1(t, s) = t - s$ for any time scale, but simple formulas, in general, do not hold for $n \geq 2$. We define

$$H_n(t) = h_0(t, 0) + h_1(t, 0) + \ldots + h_n(t, 0). \tag{2.4}$$

Remark 2.4. Note that the properties of the Taylor monomials imply that

$$h_0(t, t_0) + h_1(t, t_0) + \ldots + h_n(t, t_0) \leq H_n(t) \quad \text{for all} \quad t_0 \geq 0. \tag{2.5}$$

3. MAIN RESULTS

In this section, we give the following main results.

Theorem 3.1. *Let conditions (H_1)–(H_3) hold with $\beta > 1$, $\gamma = 1$ and suppose*

$$\lim_{t \to \infty} \frac{1}{H_n(t)} \int_{t_0}^{t} h_{n-1}(t, \sigma(u)) \int_{t_0}^{u} a(u, s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s) \Delta s \Delta u < \infty \tag{3.1}$$

for all $t_0 \geq 0$. If x is a nonoscillatory solution of equation (1.1), then

$$x(t) = \mathrm{O}\left(H_n(t)\right) \quad as \quad t \to \infty. \tag{3.2}$$

Proof. Let x be a nonoscillatory solution of equation (1.1). Hence x is either eventually positive or x is eventually negative.

First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Using conditions (H_2) and (H_3) with $\beta > 1$ and $\gamma = 1$ in equation (1.1), we have

$$\begin{aligned}
x^{\Delta^n}(t) \leq & \int_{t_0}^{t} a(t, s) \left[p_2(s) x(s) - p_1(s) x^{\beta}(s) \right] \Delta s \\
& - \int_{0}^{t_0} a(t, s) F(s, x(s)) \Delta s
\end{aligned} \tag{3.3}$$

for $t \geq t_0$. Let

$$m := \max_{0 \leq t \leq t_0} |F(t, x(t))| < \infty.$$

By assumption (H_1), we have

$$\begin{aligned}
\left| - \int_{0}^{t_0} a(t, s) F(s, x(s)) \Delta s \right| & \leq \int_{0}^{t_0} a(t, s) |F(s, x(s))| \Delta s \\
& \leq m \int_{0}^{t_0} a(t, s) \Delta s \leq m k_T =: b
\end{aligned}$$

for all $t \geq t_0$. Hence from (3.3), we get

$$x^{\Delta^n}(t) \leq \int_{t_0}^t a(t,s)\left[p_2(s)x(s) - p_1(s)x^\beta(s)\right]\Delta s + b \quad \text{for} \quad t \geq t_0. \tag{3.4}$$

By applying (2.1) with

$$\lambda = \beta, \quad X = p_1^{\frac{1}{\beta}}(t)x(t), \quad Y = \left(\frac{1}{\beta}p_2(t)p_1^{-\frac{1}{\beta}}(t)\right)^{\frac{1}{\beta-1}},$$

we obtain

$$p_2(t)x(t) - p_1(t)x^\beta(t) \leq (\beta-1)\beta^{\frac{\beta}{1-\beta}}p_1^{\frac{1}{1-\beta}}(t)p_2^{\frac{\beta}{\beta-1}}(t) \quad \text{for} \quad t \geq t_0. \tag{3.5}$$

Using (3.5) in (3.4), we find

$$x^{\Delta^n}(t) \leq A(t) + b \quad \text{for} \quad t \geq t_0, \tag{3.6}$$

where

$$A(t) = (\beta-1)\beta^{\frac{\beta}{1-\beta}}\int_{t_0}^t a(t,s)p_1^{\frac{1}{1-\beta}}(s)p_2^{\frac{\beta}{\beta-1}}(s)\Delta s.$$

Integrating (3.6) n times from t_0 to t and then using (2.3), we obtain

$$x(t) \leq \int_{t_0}^t \int_{t_0}^{\xi_n} \cdots \int_{t_0}^{\xi_2} A(\xi_1)\Delta\xi_1 \cdots \Delta\xi_n + bh_n(t,t_0)$$

$$+ \sum_{k=0}^{n-1} x^{\Delta^k}(t_0)h_k(t,t_0) \tag{3.7}$$

$$= \int_{t_0}^t h_{n-1}(t,\sigma(u))A(u)\Delta u + bh_n(t,t_0) + \sum_{k=0}^{n-1} x^{\Delta^k}(t_0)h_k(t,t_0).$$

From (3.7), using (2.5), we get

$$|x(t)| \leq \int_{t_0}^t h_{n-1}(t,\sigma(u))A(u)\Delta u + cH_n(t), \tag{3.8}$$

where

$$c := \max\left\{ b, \max_{0 \le k \le n-1} \left| x^{\Delta^k}(t_0) \right| \right\}.$$

Dividing (3.8) by $H_n(t)$ and using (3.1) shows that (3.2) is valid.

Now assume x is eventually negative, say $x(t) < 0$ for $t \ge t_0$ for some $t_0 \ge 0$. Using conditions (H_2) and (H_3) with $\beta > 1$ and $\gamma = 1$ in equation (1.1), we now have

$$
\begin{aligned}
x^{\Delta^n}(t) &\ge \int_{t_0}^{t} a(t,s) \left[p_2(s)x(s) - p_1(s)x^{\beta}(s) \right] \Delta s \\
&\quad - \int_{0}^{t_0} a(t,s)F(s,x(s))\Delta s
\end{aligned}
\tag{3.9}
$$

for $t \ge t_0$. With m defined as before and by assumption (H_1), we have

$$
\begin{aligned}
\left| \int_{0}^{t_0} a(t,s)F(s,x(s))\Delta s \right| &\le \int_{0}^{t_0} a(t,s)|F(s,x(s))|\Delta s \\
&\le m \int_{0}^{t_0} a(t,s)\Delta s \le m k_T =: b
\end{aligned}
$$

for all $t \ge t_0$. Hence from (3.9), we get

$$x^{\Delta^n}(t) \ge \int_{t_0}^{t} a(t,s) \left[p_2(s)x(s) - p_1(s)x^{\beta}(s) \right] \Delta s - b \quad \text{for} \quad t \ge t_0. \tag{3.10}$$

By applying (2.1) with

$$\lambda = \beta, \quad X = -p_1^{\frac{1}{\beta}}(t)x(t), \quad Y = \left(\frac{1}{\beta} p_2(t) p_1^{-\frac{1}{\beta}}(t) \right)^{\frac{1}{\beta-1}},$$

we obtain

$$p_2(t)x(t) - p_1(t)x^{\beta}(t) \ge -(\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(t) p_2^{\frac{\beta}{\beta-1}}(t) \quad \text{for} \quad t \ge t_0. \tag{3.11}$$

Using (3.11) in (3.10), we find

$$x^{\Delta^n}(t) \ge -A(t) - b \quad \text{for} \quad t \ge t_0, \tag{3.12}$$

where A is defined as before. Integrating (3.12) n times from t_0 to t and then using (2.3), we obtain

$$
\begin{aligned}
x(t) \geq & -\int_{t_0}^{t}\int_{t_0}^{\xi_n}\cdots\int_{t_0}^{\xi_2} A(\xi_1)\Delta\xi_1\cdots\Delta\xi_n - bh_n(t,t_0) \\
& -\sum_{k=0}^{n-1} x^{\Delta^k}(t_0)h_k(t,t_0) \\
= & -\left(\int_{t_0}^{t} h_{n-1}(t,\sigma(u))A(u)\Delta u + bh_n(t,t_0) + \sum_{k=0}^{n-1} x^{\Delta^k}(t_0)h_k(t,t_0)\right).
\end{aligned}
\tag{3.13}
$$

From (2.5), we get

$$
x(t) \geq -\left[\int_{t_0}^{t} h_{n-1}(t,\sigma(u))A(u)\Delta u + cH_n(t)\right],
$$

where c is defined as before. This implies (3.8), and thus (3.2) follows as before. $\square$

Theorem 3.2. *Let conditions (H_1)–(H_3) hold with $\beta = 1$, $\gamma < 1$ and suppose*

$$
\lim_{t\to\infty} \frac{1}{H_n(t)}\int_{t_0}^{t} h_{n-1}(t,\sigma(u))\int_{t_0}^{u} a(u,s)p_1^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)\Delta s\Delta u < \infty
\tag{3.14}
$$

for all $t_0 \geq 0$. If x is a nonoscillatory solution of equation (1.1), then (3.2) holds.

Proof. Let x be a nonoscillatory solution of equation (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Using conditions (H_2) and (H_3) with $\beta = 1$ and $\gamma < 1$ in equation (1.1), we have

$$
x^{\Delta^n}(t) \leq \int_{t_0}^{t} a(t,s)\left[p_2(s)x^{\gamma}(s) - p_1(s)x(s)\right]\Delta s - \int_{0}^{t_0} a(t,s)F(s,x(s))\Delta s
$$

for $t \geq t_0$. Hence

$$
x^{\Delta^n}(t) \leq \int_{t_0}^{t} a(t,s)\left[p_2(s)x^{\gamma}(s) - p_1(s)x(s)\right]\Delta s + b \quad \text{for} \quad t \geq t_0,
\tag{3.15}
$$

where b is defined as in the proof of Theorem 3.1. By applying (2.2) with

$$\lambda = \gamma, \quad X = p_2^{\frac{1}{\gamma}}(t)x(t), \quad Y = \left(\frac{1}{\gamma}p_1(t)p_2^{-\frac{1}{\gamma}}(t)\right)^{\frac{1}{\gamma-1}},$$

we obtain

$$p_2(t)x^\gamma(t) - p_1(t)x(t) \leq (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}p_1^{\frac{\gamma}{\gamma-1}}(t)p_2^{\frac{1}{1-\gamma}}(t) \quad \text{for} \quad t \geq t_0. \tag{3.16}$$

Using (3.16) in (3.15), we find

$$x^{\Delta^n}(t) \leq (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}\int_{t_0}^{t} a(t,s)p_1^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)\Delta s + b.$$

The rest of the proof is similar to the proof of Theorem 3.1 and hence is omitted. □

Finally, we present the following result with different nonlinearities, i.e., with $\beta > 1$ and $\gamma < 1$.

Theorem 3.3. *Let conditions (H_1)–(H_3) hold with $\beta > 1$, $\gamma < 1$ and suppose that there exists a positive rd-continuous function $\xi : \mathbb{T} \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{H_n(t)}\int_{t_0}^{t} h_{n-1}(t,\sigma(u))\int_{t_0}^{u} a(u,s)\left[c_1\xi^{\frac{\beta}{\beta-1}}(s)p_1^{\frac{1}{1-\beta}}(s)\right.$$
$$\left. + c_2\xi^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)\right]\Delta s\Delta u < \infty \tag{3.17}$$

for all $t_0 \geq 0$, where $c_1 = (\beta-1)\beta^{\frac{\beta}{1-\beta}}$ and $c_2 = (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}$. If x is a nonoscillatory solution of equation (1.1), then (3.2) holds.

Proof. Let x be a nonoscillatory solution of equation (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Using conditions (H_2) and (H_3)

in equation (1.1), we obtain

$$
\begin{aligned}
x^{\Delta^n}(t) \leq & \int_{t_0}^{t} a(t,s) \left[\xi(s)x(s) - p_1(s)x^{\beta}(s) \right] \Delta s \\
& + \int_{t_0}^{t} a(t,s) \left[p_2(s)x^{\gamma}(s) - \xi(s)x(s) \right] \Delta s \\
& - \int_{0}^{t_0} a(t,s)F(s,x(s))\Delta s \quad \text{for} \quad t \geq t_0.
\end{aligned}
\tag{3.18}
$$

As in the proofs of Theorems 3.1 and 3.2, one can easily find

$$
\begin{aligned}
x^{\Delta^n}(t) \leq & \int_{t_0}^{t} a(t,s) \left[(\beta-1)\beta^{\frac{\beta}{1-\beta}} \xi^{\frac{\beta}{\beta-1}}(s) p_1^{\frac{1}{1-\beta}}(s) \right. \\
& \left. + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \xi^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s) \right] \Delta s + b.
\end{aligned}
\tag{3.19}
$$

The rest of the proof is similar to the proof of Theorem 3.1 and hence is omitted. □

4. REMARKS AND EXTENSIONS

We conclude by presenting several remarks and extensions of the results given in Section 3.

Remark 4.1. The results presented in this paper are new for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Let us therefore rewrite the crucial condition in Theorem 3.1 (this can be done similarly for Theorem 3.2 and Theorem 3.3) for the two special time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = Z$. If $\mathbb{T} = \mathbb{R}$, then (1.1) becomes

$$x^{(n)}(t) + \int_0^t a(t,s)F(s,x(s))\mathrm{d}s = 0$$

and condition (3.1) turns into

$$\lim_{t \to \infty} \frac{1}{\sum_{k=0}^n \frac{t^k}{k!}} \int_{t_0}^t \frac{(t-u)^{n-1}}{(n-1)!} \int_{t_0}^u a(u,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s)\mathrm{d}s\mathrm{d}u < \infty.$$

If $\mathbb{T} = \mathbb{Z}$, then (1.1) becomes

$$\Delta^n x(t) + \sum_{s=0}^{t-1} a(t,s)F(s,x(s)) = 0$$

and condition (3.1) turns into

$$\lim_{t \to \infty} \frac{1}{\sum_{k=0}^n \frac{t^k}{k!}} \sum_{u=t_0}^{t-1} \frac{(t-u-1)^{n-1}}{(n-1)!} \sum_{s=t_0}^{u-1} a(u,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s) < \infty.$$

Remark 4.2. The results of this paper are presented in a form which is essentially new for equation (1.1) with different nonlinearities.

Remark 4.3. The results of this paper will remain the same if we replace (1.2) of assumption (H$_1$) by

$$\sup_{0 \leq s \leq T \leq t} a(t, s) =: K_T < \infty \quad \text{for all} \quad T \geq 0.$$

since then (1.2) is satisfied with $k_T = T K_T$.

Remark 4.4. The results of this paper will remain the same if we replace (1.2) of assumption (H$_1$) by the assumption that there exist rd-continuous functions $\alpha, \beta :$ $\mathbb{T} \to \mathbb{R}^+$ such that $a(t, s) < \alpha(t)\beta(s)$ for all $t \geq s$,

$$\sup_{t \geq 0} \alpha(t) =: K_\alpha < \infty;$$

and

$$\sup_{t \geq 0} \int_0^t \beta(s)\Delta s =: K_\beta < \infty$$

since then (1.2) is satisfied with $k_T = K_\alpha K_\beta$.

Remark 4.5. If we skip (1.2) of assumption (H$_1$) and pick $t_0 = 0$ in Theorem 3.1, Theorem 3.2 and Theorem 3.3, then the results of this paper will remain true for an eventually positive and eventually negative solution.

Remark 4.6. The techniques described in this paper can be employed to Volterra integral equations on time scales of the form

$$x(t) + \int_0^t a(t, s)F(s, x(s))\Delta s = 0. \tag{4.1}$$

As an example illustrating Remark 4.5 and 4.6, we reformulate Theorem 3.1 as follows.

Theorem 4.7. *Let conditions (H$_1$)–(H$_3$) hold with $\beta > 1$, $\gamma = 1$ and assume*

$$\int_0^\infty a(t, s)p_2^{\frac{\beta}{\beta-1}}(s)p_1^{\frac{1}{1-\beta}}(s)\Delta s < \infty.$$

Then any positive solution of equation (4.1) *is bounded.*

Remark 4.8. The results of this paper can be extended easily to delay integro-dynamic equations of the form

$$x^{\Delta^n}(t) + \int_0^t a(t,s)F(s,x(g(s)))\Delta s = 0,$$

where $g : \mathbb{T} \to \mathbb{T}$ is rd-continuous such that $g(t) \le t$ and $g^\Delta(t) \ge 0$ for $t \ge 0$ and $\lim_{t\to\infty} g(t) = \infty$.

Remark 4.9. We note that we can reformulate the obtained results for the time scales $\mathbb{T} = \mathbb{R}$ (the continuous case), $\mathbb{T} = \mathbb{Z}$ (the discrete case), $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ (the quantum calculus case), $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = \mathbb{N}_0^2$ etc.; see [1, 2].

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SECTION

4. CONCLUSION

Some discrete Volterra equations, along with the qualitative and quantitative behavior of their solutions, were examined in this study. Applications of these discrete equations can be found in many areas of study, including biological science. These types of equations occur, most often, during the mathematical modeling of some real-life situations and the numerical approximation of Volterra integral equations. Discrete versions of Volterra equations in this digital era are equally important as continuous versions (Volterra integral equations). Thus, the study of such equations is quite significant.

In the first paper, *Subexponential Solutions of Linear Volterra Difference Equations*, we started by devoting a complete section to subexponential sequences in which we gave a definition, some properties, and also established some fundamental results. The properties of transient renewal equations as well as the rate of convergence of its subexponential solutions were also developed. A Banach space (B_h^l) of sequences, the product of any bounded sequence and a subexponential sequence, equipped with a supremum norm, was also introduced. We have shown that all solutions x are summable, bounded, and asymptotically stable for the scalar linear Volterra sum-difference equation

$$\Delta x(t) = -ax(t) + \sum_{s=0}^{t-1} k(t-1-s)x(s),$$

where $a \in (0,1)$ and the kernel k is assumed to be a summable positive subexponential sequence such that $\sum_{s=0}^{\infty} k(s) < a$. The asymptotic behavior of solutions of transient renewal equations was applied to derive the exact value of the rate of convergence of asymptotically stable solutions. Moreover, we have shown these solutions are in B_h^l and also in the class of subexponential sequences. A more general scalar linear Volterra sum-difference equation was provided. These solutions were obtained

directly from the solutions of the above equation. A general example of subexponential sequences was presented and a proof was given for a particular case to justify all assumptions made. Uniform convergence, shown in Lemma 2.5, was needed in Theorem 3.1. Lemma 2.5 may be generalized by removing the restrictions $\mu < 1$ and $(1 + 4\varepsilon)\mu < 1$. Further study, however, is needed.

The investigation of subexponential solutions of scalar linear Volterra sum-difference equations was continued in both the second and third papers. In the second paper, *Rate of Convergence of Solutions of Linear Volterra Difference Equations*, the same Volterra sum-difference equations were considered as in the first paper. Here, instead of assuming the kernel k is positive subexponential as in the first paper, we assumed the kernel k is a positive, summable sequence and $k(t + 1)/k(t) \to 1$ as $t \to \infty$. Assuming all solutions are asymptotically stable and using elementary analysis, we have found a positive lower bound for the solutions. In contrast, in the first paper we have shown all solutions are asymptotically stable and then observed the exact value for the rate of convergence of asymptotically stable solutions. Following the same pattern as in the first paper, transient renewal equations were studied and a positive lower bound was obtained for the rate of convergence of its subexponential solutions. Finally, a positive lower bound was derived for the rate of convergence of asymptotically stable solutions of scalar linear Volterra sum-difference equations using the results of asymptotic behavior of solutions of transient renewal equations.

The third paper, *Subexponential Solutions of Linear Volterra Delay Difference Equations*, included an examination of the scalar linear Volterra delay sum-difference equation

$$\Delta x(t) = -\sum_{i=1}^{n} a_i x(t - \tau_i) + \sum_{s=0}^{t-1} k(t - 1 - s)x(s), \quad x(t) = \phi(t), \quad -\tau \le t \le 0,$$

where $\tau_i \in \mathbb{N}_0$ and $\tau = \max_{1 \le i \le n} \tau_i$. We supposed that $a_i > 0$ with $\sum_{i=1}^{n} a_i < 1$ and ϕ is a sequence on $[-\tau, 0] \cap \mathbb{Z}$. The exact value for the rate of convergence of

asymptotically stable solutions was established by assuming the kernel k is a non-negative, summable and subexponential sequence (in the sense that $k(t)/h(t) > 0$ as $t \to \infty$, where h is a positive subexponential sequence). Solutions of the more general delay difference equations were expressed in terms of the solutions of the difference equations associated with the purely point delay. We also showed that the solutions of transient renewal delay difference equations are positive, summable, and asymptotically stable. The decay rate of the asymptotically stable solutions of transient renewal delay difference equations was also investigated. This result was used to evaluate the decay rate of solutions of the considered delay difference equations.

In the fourth paper, *Bounded Solutions of a Volterra Difference Equation*, we considered a more general scalar nonlinear Volterra sum-difference equation

$$\Delta x(t) = a(t) - b(t)f(x(t+1)) + \sum_{s=0}^{t-1} k(t,s)g(s, x(s+1)).$$

Schaefer's fixed point theorem was applied to prove the existence of a bounded solution on an unbounded domain. In most cases, the domains are bounded. In our problem, we looked for the solutions on an unbounded domain which is not regular. Lyapunov's direct method was used to determine the a priori bound for the Schaefer fixed point theorem. An upper bound was also identified for all solutions during this process. Some examples, including the closed form of a bounded solution, were provided to illustrate that the assumptions were valid. Some assumptions may be reduced, if either a different fixed point theorem or a different technique is considered. This topic should be examined in future studies.

In the fifth paper, *Asymptotic Behavior of Nonoscillatory Solutions of Higher-order Integro-dynamic Equations*, a higher-order integro-dynamic equation of the form

$$x^{\Delta^n}(t) + \int_0^t a(t,s)F(s, x(s))\Delta s = 0$$

was considered. This work was conducted on time scales, which combine and extend both discrete and continuous calculus. Under various restrictions on constants α and γ, ratios of positive odd integers, we established some new criteria on the asymptotic behavior of nonoscillatory solutions of the considered higher-order integro-dynamic equation. The last section was prepared with many significant and interesting remarks and extensions based on the derived results. Some of those are stated here. The results presented were new for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. They could also be reformulated for $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ (the quantum calculus case), $\mathbb{T} = h\mathbb{Z}$ with $h > 0, \mathbb{T} = \mathbb{N}_0^2$, and so forth. All of the results were presented in a form that was essentially new for the considered equation with different nonlinearities. The techniques used could be applied to Volterra integral equations on time scales. The results we gathered could be extended to delay integro-dynamic equations.

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