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Special subrings of real, continuous functions

Paul Marlin Harms

SPECIAL SUBRINGS OF REAL,

CONTINUOUS FUNCTIONS

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Thesis Corrections

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Page	Place
14	line 4 Replace C by C' under the intersection sign.
15	~~Corrected~~ line 13 Replace in in by is in.
18	Theorem 25, line 1 Replace homeomorphism by homomorphism.
26	line 5 Erase extra equal sign.
31	line 4 Replace the number 0 by the letter O inside the brackets.
31	Definition 13, line 2 Replace P by p as a superscript of script M
37	Proof Theorem 53 line 7 Replace z by Z inside braces.
48	Example 8 line 4 Put in β for $\beta N-N$.
57	Lemma 1 line 4 Replace x by X inside the brackets.
63	Proof Lemma 9 line 3 Put in f as a superscript of I_i.
70	Definition 6 line 2 Replace the second last script S by S.
74	line 5 Replace X by x inside the brackets.
84	line 5 Replace x in $C_k(x)$ by X.
89	line 1 Put in the missing part of the parentheses.
90	last line Replace s by r in the subscript.
94	Proof Theorem 57 lines 6 and 7 Change the number r to $\bar{r}$.
101	line 5 Replace the period by a comma.
102	Theorem 70 line 2 Put in the missing part of the parentheses.
102	second last line Put equal sign under 1.
104	line 1 Put in braces around 0.
105	Proof Theorem 75 line 2 Put equal sign under 0.
106	Proof Theorem 76 line 7 This should read, ...o in A^x if (P') is a maximal ideal in P.
109	line 14 Add the sentence—then $h_i g = Q$ — at the end of the line.
110	line 4 Replace K by K union its interior.
112	line 4 Put in arrow at the end of the line.
124	second last line Replace September 19 by September 10.

ABSTRACT

Some lattice-ordered subrings of $C(X)$ containing $C^*(X)$ are examined where X is a completely regular space. Each realcompact space Y between νX and βX is associated with a lattice-ordered subring of $C(X)$ which is isomorphic to $C(Y)$ and contains $C^*(X)$. The cardinal number of $(\beta X - \nu X)$ is a lower bound for the cardinal number of these subrings. Every prime ideal in each of these subrings is comparable with the intersection of the subring and a maximal ideal in $C(X)$.

The structure space of maximal ideals is studied for special subrings in $C(X)$ containing $C_K(X)$, the continuous functions of compact support, and $C_\infty(X)$, the continuous functions converging to 0 at infinity. Examples of structure spaces are given which are homeomorphic to finite point compactifications of R. A study is made of the free maximal ideals in $C_K(X) + P(X)$ where $P(X)$ is a subring of $C(X)$ such that $C_K(X) \cap P(X) = \{\underline{0}\}$ and X is a locally compact, non-compact, Hausdorff space.

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I. INTRODUCTION

Let X be a completely regular space, let C(X) be the ring of real-valued, continuous functions on X, and let C*(X) be the ring of bounded, real-valued, continuous functions on X. The basic problem is to determine properties for special subrings in C(X) similar to those known for C*(X) and C(X). The properties of prime ideals, convex ideals, and the structure space of maximal ideals are examined.

We first consider properties of some special lattice-ordered subrings of C(X) containing C*(X). An onto iso-morphism is established between the ring of all real-valued, continuous functions on a realcompact subspace of the Stone-Čech compactification of X and a lattice-ordered subring of C(X) containing C*(X). Using this isomorphism we study the prime ideal structure of these lattice-ordered subrings and we also study some conditions on the ideals in these lattice-ordered subrings which make the quotient ring totally ordered. The structure space of these lattice-ordered subrings is easily found with the aid of this isomorphism.

Another problem is to determine the structure space of maximal ideals for different subrings of C(X). A larger problem is the characterizing of different compacti-fications of X using algebraic properties of a subset of

real-valued functions. We consider a lattice-ordered subring of C(R) containing the polynomials. We also study certain subrings of C(X) containing $C_K(X)$, the continuous functions of compact support on X, and $C_\infty(X)$, the continuous functions which converge to 0 at infinity. For a completely regular, non-compact, locally compact space X some of these structure spaces are finite point compactifications of X while other structure spaces are not Hausdorff.

Almost all of the notations and definitions are in [5]. Many of the authors listed in the bibliography follow the notation in [5]. In this dissertation a proper subset is denoted by $\subset$. The basic source for the results involving rings of real-valued, continuous functions and for the background of this dissertation is [5].

II. REVIEW OF THE LITERATURE

In 1937, Stone in [27] and Čech in [2] independently
produced a compactification of a completely regular space
X and proved many of the essential results of this compacti-
fication. We denote this compactification by βX and call
it the Stone-Čech compactification of X. In [27] Stone
also developed much of the theory of the ring C*(X) and
showed that all maximal ideals are fixed if X is a compact
space. In 1939, Gelfand and Kolmogroff in [3] developed
βX as a structure space of C(X) using the Stone topology on
the set of maximal ideals in C(X). They characterized all
maximal ideals in C(X). In 1948, Hewitt in [8] defined and
investigated "Q-spaces", which we call realcompact spaces.
He established the existence of a unique realcompactification
of a completely regular space X and derived many properties
of realcompact spaces. He gave a systematic study of free
ideals and fixed ideals and is responsible for much of the
terminology of free ideals and fixed ideals. Hewitt
studied properties of C(X) using zero sets of the functions
in C(X). Much of the research in rings of continuous
functions has been an outgrowth of the above papers.

In 1958, Kohls in [11] and [12] made the first general
investigation of the prime ideals in C(X). Chapter 14 in
Gillman and Jerison's book [5] has many of Kohls' results.
The 1968 paper by Mandelker [14] investigated the prime
ideal structure of C*(X).

Many different subrings of C(X) have been investigated. $C_K(X)$ or $C_\infty(X)$ were investigated by Kohls in [10] and [11] and by Shanks in [25]. Gillman and Jerison in [5] showed that $C_K(X)$ is the intersection of all free maximal ideals in C(X) whenever X is a realcompact space. In recent papers Pursell in [21] and [22] investigated the structure space and properties of the subrings $C_K(X)$ + (constant functions) and $C_\infty(X)$ + (constant functions). Pursell introduced the concept of a real function ring in [20] and showed that X and the set of fixed maximal ideals in a real function ring are in one-to-one correspondence. Riordan in [23] characterized the ring of functions with pseudo-compact support as $C_K(\upsilon X)$.

III. SPECIAL SUBRINGS CONTAINING C*(X)

A. Realcompact Spaces and Subrings

In this section preliminary theorems are given about maximal ideals, the Stone-Čech compactification, and real-compact spaces. Theorem 15 gives a characterization of realcompact spaces from [5]. Using this characterization, Theorem 16 shows how a realcompact space can be associated with a lattice-ordered subring of $C(X)$, the ring of real-valued continuous functions on X. Theorem 17 gives an isomorphism between the continuous functions on a real-compact space and a lattice-ordered subring. Corollary 19 gives a lower bound for the cardinal number of these lattice-ordered subrings. The isomorphism in Theorem 17 is used throughout Section III C.

In this thesis X denotes a non-empty, completely regular, Hausdorff space and R^X denotes the ring of all real-valued functions on X. Ring operations of addition and multiplication are defined pointwise. An isomorphism denotes a ring isomorphism. There are times when $C(X)$ will be denoted by C and $C^*(X)$ will be denoted by C^*. Consider any function f in R^X and let $Z(f) = \{x \text{ in } X: f(x) = 0\}$. $Z_X(f)$ or $Z(f)$ is called the zero set of f. If A is a subset of R^X, then $Z[A] = \{Z(f): f \text{ is in } A\}$. A z-filter refers to a filter of zero sets. Theorem 1 indicates why

only completely regular spaces are used when dealing with subrings of C(X). Theorem 2 gives some important connections between zero sets and the topology on X.

THEOREM 1. [5, Theorem 3.9] For every topological space X, there exists a completely regular space Y and a continuous mapping θ of X onto Y, such that the mapping $g \mapsto g \circ \theta$ is an isomorphism of C(Y) onto C(X).

THEOREM 2. [5, Theorem 3.2]
(i)   A Hausdorff space X is completely regular if and only if the family Z[C(X)] is a base for the closed sets of X.
(ii)   Every neighborhood of a point in a completely regular space contains a zero set neighborhood.

THEOREM 3. [5, Theorem 2.5] Let Z be a mapping that takes f in C(X) to Z(f) in Z[C(X)].
(i)   If M is a maximal ideal in C(X), then Z[M] is a z-ultrafilter on X.
(ii)   If U is a z-ultrafilter on X, then $Z^{\leftarrow}[U]$ = {f in C(X): Z(f) is in U} is a maximal ideal in C(X).

DEFINITION 1. Let B be an ideal in a subring of R^X. In this thesis an ideal always denotes a proper ideal. B is said to be a fixed ideal if $\bigcap_{f \in B} Z(f) \neq \emptyset$; otherwise, B is said to be a free ideal. In a similar manner we can define a fixed or free z-filter. Since the fixed (free) maximal ideals in C(X) are in one-to-one correspondence

with the fixed (free) z-ultrafilters on X, we will often interchange maximal ideals and z-ultrafilters when working with C(X).

The fixed maximal ideals in C(X) were characterized by Gelfand and Kolmogoroff in [3, Lemma 2] and the fixed maximal ideals in C*(X) were characterized by Stone in [27, Theorems 79, 80].

THEOREM 4.   [5, Theorem 4.6]

(i)   The fixed maximal ideals in C(X) are precisely the sets $M_p = \{f$ in $C(X): f(p) = 0\}$ for p in X. The ideals M_p are distinct for distinct p. For each p, C/M_p is isomorphic with the real field R; in fact, the mapping $M_p(f)$ ↦ $f(p)$ is the unique isomorphism of C/M_p onto R.

(i*)   The fixed maximal ideals in C*(X) are precisely the sets $M_p^* = \{f$ in $C*(X): f(p) = 0\}$ for p in X. The ideals M_p^* are distinct for distinct p. For each p, $C*/M_p^*$ is isomorphic with the real field R; in fact, the mapping $M_p^*(f)$ ↦ $f(p)$ is the unique isomorphism of $C*/M_p^*$ onto R.

From Theorem 4, the fixed maximal ideals (or fixed z-ultrafilters) in C(X) can be associated in a natural way with X, i.e., p in X is associated with M_p. The Stone-Čech compactification of X is denoted by βX. βX is a compact, Hausdorff space in which X is dense. Each point of βX can be associated with exactly one z-ultrafilter on X (or one maximal ideal in C(X)). Each free z-ultrafilter

on X is associated with one point in $\beta X - X$. For p in βX, the associated maximal ideal in $C(X)$ is denoted by M^p where $M^p = M_p$ when p is in X. Theorems 5 and 6 give properties of the Stone-Čech compactification.

DEFINITION 2. Let S be a subspace of X. S is C^*-imbedded in X if every function in $C^*(S)$ can be extended to a function in $C^*(X)$. In a similar manner we define S to be C-embedded in X.

THEOREM 5. [5, Theorems 6.4, 6.7] Let X be dense in T. The following statements are equivalent.

(1) Every continuous mapping τ from X into any compact space Y has an extension to a continuous mapping from T into Y.

(2) The space X is C^*-embedded in T.

(3) Any two disjoint zero sets in X have disjoint closures in T.

(4) For any two zero sets Z_1 and Z_2 in X, $Cl_T (Z_1 \cap Z_2) = Cl_T(Z_1) \cap Cl_T(Z_2)$ where $Cl_T(Z_1)$ is the T-closure of Z_1.

(5) Every point of T is the limit of a unique z-ultrafilter on X.

(6) The spaces satisfy $X \subseteq T \subseteq \beta X$.

(7) The spaces satisfy $\beta T = \beta X$.

THEOREM 6.   [5, Theorem 6.5]   Compactification Theorem.
Every (completely regular) space X has a compactification
βX, with the following equivalent properties.

(I)   (Stone) Every continuous mapping τ from X into any
compact space Y has a continuous extension τ^* from βX into Y.

(II)   (Stone-Čech) Every function f in $C^*(X)$ has an extension
to a function f^β in $C(\beta X)$.

(III)   (Čech) Any two disjoint zero sets in X have disjoint
closures in βX.

(IV)   For any two zero sets Z_1 and Z_2 in X, $Cl_{\beta X}(Z_1 \cap Z_2) = Cl_{\beta X}Z_1 \cap Cl_{\beta X}Z_2$.

(V)   Distinct z-ultrafilters on X have distinct limits on
βX.   Furthermore, βX is unique in the following sense: if
a compactification T of X satisfies any one of the listed
conditions, then there exists a homeomorphism of βX onto
T that leaves X pointwise fixed.

The mapping $(f \mapsto f^\beta)$: $C^*(X) \to C(\beta X)$ is an isomorphism
of $C^*(X)$ onto $C(\beta X)$.

At the present time our main interest will be in real-
compact spaces.   Every residue class field of $C(X)$ or $C^*(X)$
modulo a maximal ideal M contains a copy of the real field
R by identifying $M(\underline{r})$ with r in R where $\underline{r}$ is the "constant"
function on X that takes each x in X to r in R.   Let A
be a commutative ring with unity.   From [5, Section 0.15],
A/M is a field if and only if M is a maximal ideal in A.

DEFINITION 3. An ideal M in a commutative ring A with unity is said to be a real ideal if A/M is isomorphic to the real field. If M is not a real ideal, we call it a hyper-real ideal.

DEFINITION 4. A completely regular space X is said to be realcompact if every free maximal ideal in C(X) is hyper-real; or equivalently, if every real maximal ideal in C(X) is fixed.

From Theorem 4, every fixed maximal ideal in C(X) and also in C*(X) is real. Many of the commonly used spaces are realcompact. The space of real numbers R is a realcompact space [5, Section 5.10]. Every compact space is a realcompact space [5, Theorem 5.8]. An example of a space which is not a realcompact space is the space of all countable ordinals [5, Section 5.12].

DEFINITION 5. Let R* be the one point compactification of R where $R^* = R \cup \{\infty\}$. Let f be in C(X). Using Theorem 6(I), we extend f to a function $f^*: \beta X \to R^*$ such that $f^* | X = f$. f* is called the Stone extension of f.

THEOREM 7. [5, 8.4] The following conditions on p in βX are equivalent.

(1)   The ideal M^p is a real ideal in C(X).

(2)   For all f in C(X), $f^*(p) \neq \infty$

(3)   For all f in C(X), $f^*(p) = M^p(f)$.

(4)   For all f in C(X), $f^*(p) = 0$ implies $M^p(f) = 0$.

The set of points in βX satisfying the conditions of the above theorem is denoted by νX. If p is in X, then $f^*(p) = f(p) \neq \infty$. Thus X is a subset of νX.

COROLLARY 8. [5, Corollary 8.5]

(i) The space νX is the largest subspace of βX in which X is C-embedded.

(ii) The space νX is the smallest realcompact space between X and βX. The space X is realcompact if and only if $\nu X = X$.

The following theorem is similar to the compactification theorem and was first shown by Hewitt. A realcompactification of X is a realcompact space in which X is dense.

THEOREM 9. [5, Theorem 8.7] Every (completely regular) space X has a realcompactification νX, contained in βX, with the following equivalent properties.

(I) Every continuous mapping τ from X into any realcompact space Y has a continuous extension τ^o from νX into Y.

(II) Every function f in C(X) has an extension to a function f^ν in C(νX). (Necessarily, $f^\nu = f^*|\nu X$).

(III) If a countable family of zero sets in X has empty intersection, then their closures in νX have empty intersection.

(IV) For any countable family of zero sets Z_n in X,

$$Cl_{\nu X} \bigcap_n Z_n = \bigcap_n Cl_{\nu X} Z_n.$$

(V) Every point of υX is the limit of a unique z-ultra-filter on X, and it is a real z-ultrafilter.

Furthermore, the space υX is unique in the following sense: if a realcompactification T of X satisfies any one of the listed conditions, then there exists a homeomorphism of υX onto T that leaves X pointwise fixed.

The mapping $(f \mapsto f^\upsilon)$: $C(X) \to C(\upsilon X)$ is an isomorphism of $C(X)$ onto $C(\upsilon X)$. Several useful results involving functions and points in $\beta X - \upsilon X$ are given in Theorem 10.

THEOREM 10.

(i) If p is in $\beta X - \upsilon X$, then there exists a function f in $C^*(X)$ such that $Z(f) = \emptyset$ and $f^\beta(p) = 0$.

(ii) If f is in $C^*(X)$ and f is a unit in $C(X)$ but not a unit in $C^*(X)$, then $f^\beta(p) = 0$ for some p in $\beta X - \upsilon X$.

PROOF. (i) If p is in $\beta X - \upsilon X$, then from Theorem 7, there is a function g in $C(X)$ such that $g^*(p) = \infty$. Let $h(x) = 1 \vee |g(x)|$ for x in X. Then h is in $C(X)$, $h^*(p) = \infty$, and $h \geq \underline{1}$. Let $f(x) = 1/h(x)$ for x in X. Then f is in $C^*(X)$, $Z(f) = \emptyset$, and $f^\beta(p) = 0$.

(ii) If f is not a unit in $C^*(X)$, then using the isomorphism between $C^*(X)$ and $C(\beta X)$, f^β is not a unit in $C(\beta X)$. Since any h in $C(Y)$ is a unit of $C(Y)$ if and only if $Z(h) = \emptyset$, $f^\beta(p) = 0$ for some p in βX. Using the isomorphism between $C(X)$ and $C(\upsilon X)$, f^υ, the image of f, is a unit in $C(\upsilon X)$. Therefore p is not in υX.//

THEOREM 11.   [5, Theorem 8.2]   Every Lindelöf space is realcompact where X is a Lindelöf space if every open cover has a countable subcover.

THEOREM 12.   [5, Theorem 8.9]   An arbitrary inter-section of realcompact subspaces of a given space is real-compact.

THEOREM 13.   [5, Theorem 8.26]   In any space, the union of a compact subspace with a realcompact subspace is realcompact.

Let $\nu_f X = \{p \text{ in } \beta X: f^*(p) \neq \infty\}$.

THEOREM 14.   [5, Problem 8 B.2]   For f in C(X), the space $\nu_f X$ is locally compact and σ-compact.

THEOREM 15.   [5, Problem 8 B.3]   The realcompact subspaces between X and βX are precisely the spaces $\bigcap_{g \in C'} \nu_g X$ for any subset C' of C(X).

PROOF.   From Theorem 14, $\nu_g X$ is σ-compact.   Every σ-compact space is a Lindelöf space.   From Theorem 11, $\nu_g X$ is a realcompact subspace of βX for each g in C(X). From Theorem 12, $\bigcap_{g \in C'} \nu_g X$ is realcompact.   Now let Y be a realcompact   subspace of βX.   From Theorem 5 and Corollary 8, $\nu X \subseteq \nu Y = Y \subseteq \beta X = \beta Y$.   Define C' = {g in C(X): Y is a subset of $\nu_g X$}.   Clearly Y is a subset of $\bigcap_{g \in C'} \nu_g X$.

Let p be in $\beta X - Y = \beta Y - \upsilon Y$. From Theorem 7, there is a function f in $C(Y)$ such that $f^*(p) = \infty$. Let $f' = f|X$. Then p is not in $\upsilon_{f'} X$. Since f is in $C(Y)$, Y is a subset of $\upsilon_{f'} X$ and f' is in C'. Thus p is not in $\bigcap\limits_{g \in C'} \upsilon_g X$. The result follows.//

THEOREM 16. Let Y be a realcompact subspace of βX. Then $C' = \{f \text{ in } C(X): Y \text{ is a subset of } \upsilon_f X\}$ is a lattice-ordered subring of $C(X)$ which contains $C^*(X)$.

PROOF. If f is in C^*, then from Theorem 6, $\upsilon_f X = X$. Therefore C^* is a subset of C' and, in particular, $\underline{1}$ and $\underline{0}$ are in C'. If f is in C', then $(-f)$ is in C' since $\upsilon_f X = \upsilon_{(-f)} X$. Now let f and g be in C' where $X \subseteq \upsilon X \subseteq Y \subseteq \beta X = \beta Y$. Clearly X is dense in the compact, Hausdorff space $\beta X = \beta Y$. R^*, the one point compactification of R, is Hausdorff. A continuous mapping from a space X' into a Hausdorff space is determined by its values on any dense subspace of X', [5, Section 0.12]. On the space X, $(f + g)^* = f^* + g^*$, $(fg)^* = f^* g^*$, $(f \vee g)^* = f^* \vee g^*$, and $(f \wedge g)^* = f^* \wedge g^*$. From above we see that the equalities also are true on the space Y. Since f and g are in C', $f^*[Y]$ and $g^*[Y]$ are contained in R. From the equalities above, $f + g$, fg, $f \vee g$ and $f \wedge g$ are in C'. The other properties of a subring easily follow.//

Example 1 is given to show that not every lattice-ordered subring of C(X) containing C*(X) has the form C' given in Theorem 16.

EXAMPLE 1. Let X = R. Consider the subring A of functions in C(R) which are polynomials in X with coefficients from C*(R). Clearly, C*(R) is a subset of A. We use A to generate a lattice-ordered ring B by taking all functions f ∨ g, f ∧ g, etc., where f and g are in A. For each h in B there are positive integers m, n, and k such that $|h(x)| \leq kx^m$ for all $x \geq n$. Let $exp(x) = e^x$. It is not true that $e^x \leq k\, x^m$ for all large x and positive integers k and m. The function exp is not in B. Since i, the identity function on R, in in B, $\bigwedge\limits_{f \in B} \nu_f\, X = R$. Since $\nu_{exp}\, X$ contains R but exp is not in B, B does not have the form C' in Theorem 16.

THEOREM 17. Let Y be a realcompact subspace of βX and let C' be the subring defined in Theorem 16. Then C' and C(Y) are isomorphic under the mapping that takes f in C' to f*|Y in C(Y).

PROOF. Let $f_Y = f^*|Y$ where f is in C'. The Stone extension f* is unique, thus the mapping described in the theorem is well-defined. Since Y is a subset of $\nu_f\, X$, f_Y is in C(Y). To show that the mapping is one-to-one, consider f and g in C' where $f(x) \neq g(x)$ for some x in X. Then $f_Y \neq g_Y$. To show that the mapping is onto, consider g in C(Y). Since R is a Hausdorff space and

X is dense in Y, $(g|X)_Y = g$ and thus the mapping is onto. For the same reason, $f_Y + g_Y = (f + g)_Y$ and $f_Y g_Y = (fg)_Y$ where f and g are in C'. The desired result follows.//

THEOREM 18. The cardinal number of realcompact subspaces of βX containing νX is at least as large as the cardinal number $|\beta X - \nu X|$.

PROOF. Let p be in $\beta X - \nu X$. νX is realcompact and $\{p\}$ is a compact subspace of βX. From Theorem 13, $\nu X \cup \{p\}$ is realcompact. The above procedure works for any p in $\beta X - \nu X$, and hence the desired result follows.//

COROLLARY 19. The cardinal number of lattice-ordered subrings properly contained in C(X) and containing C*(X) is at least $|\beta X - \nu X|$.

PROOF. From Theorem 17, each realcompact subspace Y of βX can be associated with a lattice-ordered subring $C_Y' = \{f$ in C(X): Y is a subset of $\nu_f X\}$. If Y_1 and Y_2 are realcompact spaces with p in $Y_1 - Y_2$, then there is a function f in $C_{Y_2}' - C_{Y_1}'$ such that $f(p) = \infty$, hence $C_{Y_1}' \neq C_{Y_2}'$. The result follows from Theorem 18.//

From known results about the cardinal number of $|\beta X - \nu X|$ we can use Corollary 19 to determine a lower bound for the cardinal number of lattice-ordered subrings

in C(X) containing C*(X). From [5, Section 9.3],
$|\beta N - N| = |\beta R - R| = 2^c$ where c is the cardinal number of
the continuum and N is the space of positive integers. The
cardinal number of lattice-ordered subrings which contain
C* is at least 2^c for these spaces.

B.   Some Equivalence Relations

First we consider two theorems which will help in
relating several of the equivalence relations with isomorphic
rings.

THEOREM 20.   [5, Theorem 4.9]   Two compact spaces X
and Y are homeomorphic if and only if C(X) and C(Y) are
isomorphic.

Using the Stone-Čech compactification and the isomor-
phism between C(βX) and C*(X), we obtain the following
corollary.

COROLLARY 21.   For any spaces X and Y, βX and βY are
homeomorphic if and only if C*(X) and C*(Y) are isomorphic.

THEOREM 22.   [5, Theorem 8.3]   Two realcompact spaces
X and Y are homeomorphic if and only if C(X) and C(Y)
are isomorphic.

For any space X, C(νX) is isomorphic to C(X). We
obtain the following corollary.

COROLLARY 23. For any spaces X and Y, νX and νY are homeomorphic if and only if C(X) and C(Y) are isomorphic.

We now define two relations between completely regular spaces.

DEFINITION 6. Let R_1 be the following relation: X R_1 Y if βX is homeomorphic to βY; equivalently, from Corollary 21, X R_1 Y if C*(X) and C*(Y) are isomorphic.

DEFINITION 7. Let R_2 be the following relation: X R_2 Y if νX is homemorphic to νY; equivalently, from Corollary 23, X R_2 Y if C(X) and C(Y) are isomorphic.

LEMMA 24. The relations R_1 and R_2 are equivalence relations.

PROOF. This easily follows from definitions.//

THEOREM 25. [5, Theorem 1.9] Let t be a homomorphism from C(Y) to C(X) whose image contains C*(X). Then t takes C*(Y) onto C*(X).

THEOREM 26. If X R_2 Y, then X R_1 Y.

PROOF. From Theorem 25, C*(Y) and C*(X) are isomorphic whenever C(X) and C(Y) are isomorphic. The result follows.//

EXAMPLE 2. This example shows the converse of Theorem 26 is not true. Let X = R and Y = βR where R is the space of real numbers. C*(βR) and C*(R) are isomorphic,

however, from Theorem 25, $C(R)$ and $C^*(\beta R) = C(\beta R)$ are not

isomorphic. This shows $X \not R_1 Y$. Since $\beta X = \beta Y = \beta R$,

$X \ R_2 \ Y$.

The relation R_1 puts all spaces Y such that $X \subseteq Y \subseteq \beta X$

into the same equivalence class while the relation R_2 puts

all spaces Y such that $X \subseteq Y \subseteq \nu X$ into the same equivalence

class.

Realcompactness is a topological invariant since, from

Theorem 20, a homeomorphism induces an isomorphism of

continuous functions, and the homeomorphism together with

the isomorphism take real, fixed maximal ideals into real,

fixed maximal ideals. This implies every realcompact

space between νX and βX can be put into a one-to-one

correspondence with the realcompact spaces between νY and

βY whenever X and Y are homeomorphic. Suppose $C(X)$ and

$C(Y)$ are isomorphic under an isomorphism α. Theorem 25

implies $\alpha[C^*(X)] = C^*(Y)$. In general, α takes a subring

in $C(X)$ into a subring in $C(Y)$. All subrings between

$C^*(X)$ and $C(X)$ can be put into one-to-one correspondence

with subrings between $C^*(Y)$ and $C(Y)$.

Let X and Y be spaces such that the set of subrings

between $C^*(X)$ and $C(X)$ are associated in a one-to-one

correspondence with the set of subrings between $C^*(Y)$

and $C(Y)$ where associated subrings are isomorphic. Since

isomorphisms take proper subrings into proper subrings,

C(X) and C(Y) are isomorphic. From Corollary 23, νX is homeomorphic to νY and from Theorem 26, $\beta Y = \beta \nu Y$ is homeomorphic to $\beta X = \beta \nu X$.

DEFINITION 8. Let R_3 be the following relation: $X R_3 Y$ if the realcompact spaces between νX and βX can be put into a one-to-one correspondence with the realcompact spaces between βY and νY.

LEMMA 27. The relation R_3 is an equivalence relation.

PROOF. This easily follows from definitions.//

THEOREM 28. If $X R_2 Y$, then $X R_3 Y$.

PROOF. This follows from the discussion preceding Definition 8.//.

EXAMPLE 3. This example shows the converse of Theorem 28 does not hold. Let N be the discrete space of positive integers and let $\Sigma = N \cup \{p\}$ where p is in $\beta N - N$. Some of the properties of Σ are in [5, Problem 4M]. Σ is not a discrete space since $\{p\}$ is not open. Σ and N are not homeomorphic, since N is a discrete space while Σ is not a discrete space. N is a realcompact space. From Theorem 13, Σ is also a realcompact space. From Theorem 22, C(N) is not isomorphic to C(Σ). Thus $\Sigma \not R_2 N$. We now show $X R_3 Y$. Let N_1 be a realcompact space between N and βN

with p not in N_1. From Theorem 13, $N_1 \cup \{p\}$ is a real-
compact space and it lies between Σ and $\beta\Sigma$. We conclude
that there are at least as many realcompact spaces between
Σ and $\beta\Sigma$ as there are realcompact spaces between N and βN
which do not contain p. We now consider realcompact
spaces containing p. Let Y be a realcompact space between
Σ and $\beta\Sigma$. Since p is in Σ, p is in Y. Using Theorem 15,
we write $Y = \bigcap\limits_{f \in C'} \nu_f \Sigma$ for some subset C' in $C(\Sigma)$. If f
is in C', let $f_1 = f|N$. f_1 is in $C(N)$ and $f_1^*(p) = f(p)$.
Thus $Y = \bigcap\limits_{f \in C'} \nu_f \Sigma = \bigcap\limits_{f_1} \nu_{f_1} N$. Thus Y is a realcompact
space between N and βN. Now consider a realcompact space Z
between N and βN with p in Z. In a manner similar to
working with Y, we see that Z is a realcompact space between
Σ and $\beta\Sigma$. We have shown a one-to-one correspondence
associating the realcompact spaces between N and βN
containing p with the realcompact spaces between Σ and $\beta\Sigma$.
The cardinal number of realcompact spaces between Σ and $\beta\Sigma$
is at least $|\beta\Sigma - \Sigma| = |\beta N - \Sigma| = 2^C$. Since the cardinal
number is not finite and the cardinal number of realcompact
spaces between N and βN is not greater than "twice" the
cardinal number of realcompact spaces between Σ and $\beta\Sigma$,
$\Sigma R_3 N$.

THEOREM 29. The relations R_3 and R_1 are not comparable, i.e., neither relation implies the other relation.

PROOF. If $X = R$ and $Y = \beta R$, then $X\ R_1\ Y$ but $X\ \not{R}_3\ Y$. To show the other implication does not hold, let X and Y be discrete spaces where $X = \{a,b\}$ and $Y = \{b\}$. Then $X = \nu X = \beta X$ and $Y = \nu Y = \beta Y$. Thus, $X\ R_3\ Y$, however, $\beta X = X$ is not homeomorphic to $\beta Y = Y$. Hence $X\ \not{R}_1\ Y.//$

We now define an equivalence relation on subrings in $C(X)$ for a fixed X.

DEFINITION 9. Let A and B be any subrings of $C(X)$ and let R_4 denote the following relation: $A\ R_4\ B$ if

$$\bigcap_{f \varepsilon A} \nu_f\ X = \bigcap_{g \varepsilon B} \nu_g\ X.$$

LEMMA 30. The relation R_4 is an equivalence relation.

PROOF. This easily follows from definitions.//

THEOREM 31. Each equivalence class of subrings in $C(X)$ induced by R_4 has a maximal element, i.e., there is a subring in the equivalence class which contains all other subrings in the equivalence class. This maximal element is C' defined in Theorem 16.

PROOF. Let A and B be in the same equivalence class. Let $Y = \bigcap_{f \varepsilon A} \nu_f\ X = \bigcap_{g \varepsilon B} \nu_g\ X$. The result follows from Theorems 15 and 16.//

C. Order Structure of Prime Ideals

The order structure of the family of prime ideals in C(X) has been studied by several people. Kohls gives results of this structure in [10], [11], and [12]. Gillman and Jerison devote chapter 14 of [5] to prime ideals in C(X). Mandelker in [13] and [14] proves some results involving prime ideals and prime z-ideals in C*(X) and C(X). The first part of this section is devoted to examples and preliminary results of prime ideals. The latter part of this section states and extends some of Mandelker's results found in [14]. As in [14] a proper subset is denoted by $\subset$.

DEFINITION 10. Let A be a subring of C(X). A is said to be closed under pointwise inversion if $\underline{1}/h$ is in A whenever h is in A and $Z(h) = \emptyset$. A is said to be closed under bounded pointwise inversion if $\underline{1}/h$ is in A whenever h is in A and h is bounded away from $\underline{0}$.

C(X) and the subring in C(X) of all constant functions on X satisfy both parts of Definition 10. C*(X), in general, satisfies only the second part of definition 10.

THEOREM 32. Let A be a lattice-ordered subring in C(X) containing C*(X). If A $\neq$ C(X), then A is not closed under pointwise inversion. A is clearly closed under bounded pointwise inversion.

PROOF. If A ≠ C, then there is a function g in C - A such that |g| is unbounded. Since A contains the constant functions, $\underline{r}$ + g is in C - A for each r in R. We consider two cases to show that if there is a function g in C - A, then there is a function h in A such that Z(h) = ∅ but $\underline{1}$/h is not in A.

Case 1. Suppose g is in C - A and g is bounded below, i.e., g ≥ -$\underline{r}$ for some r in R. A similar proof can be used when g is bounded above. Since $\underline{1}$ + $\underline{r}$ + g ≥ $\underline{1}$, h = $\underline{1}$/($\underline{1}$ + $\underline{r}$ + g) is in C* and **A**. Also $\underline{1}$ + $\underline{r}$ is in C* and A. If $\underline{1}$ + $\underline{r}$ + g is in A, then $\underline{1}$ + $\underline{r}$ + g - ($\underline{1}$ + $\underline{r}$) = g is in A. This contradiction shows that $\underline{1}$/h is not in A. Clearly h is in A and Z(h) = ∅.

Case 2. Suppose g is in C - A where g is not bounded below and is not bounded above. g = (g ∨ $\underline{0}$) + (g ∧ $\underline{0}$). Since g is not in A, the function g ∨ $\underline{0}$ or the function g ∧ $\underline{0}$ is not in A. Now case 1 applies to g ∨ $\underline{0}$ or g ∧ $\underline{0}$ and the desired result follows.//

The conclusion of Theorem 32 applies to rings other than lattice-ordered subrings in C containing C*. For example, consider X = R and let C^{∞}(R) denote the subring of C(R) of all infinitely differentiable functions. Let $C*^{\infty}$(R) denote the subring of all bounded infinitely differentiable functions in C(R).

COROLLARY 33. Let A be a subring containing $C*^\infty(R)$ and properly contained in $C^\infty(R)$. Then A is not closed under pointwise inversion.

PROOF. The subring $C^\infty(R)$ is closed under pointwise inversion.

Case 1. Let g be in $C^\infty(R) - A$ where g is bounded below or above. The proof is the same as case 1 of Theorem 32.

Case 2. Let g be in $C^\infty(R) - A$ where g is not bounded below and is not bounded above. The functions $\ell = e^{-g^2}$ and $g\ell$ are in $C*^\infty(R)$ and hence in A. Also $\underline{1}/\ell$ is in $C^\infty(R)$. If $\underline{1}/\ell$ is in A, then $(g\ell)(\underline{1}/\ell) = g$ is in A. This contradiction shows that $\underline{1}/\ell$ is not in A. Clearly ℓ is in A with $Z(\ell) = \emptyset$.//

Let C' be any subring of C containing C*. Assume the function rings are defined on the same space when the space is not explicitly stated.

THEOREM 34. Let P be an ideal in C. P is a prime ideal in C if and only if $P \cap C'$ is a prime ideal in C'.

PROOF. The procedure of the proof is similar to the case $C' = C*$ in [5, Problem 2 B.1].

Assume P is a prime ideal in C. Let f and g be in C' with fg in $P \cap C'$. Since fg is in P and P is a prime ideal in C, f is in P or g is in P. Suppose f is in P. Then f is in $P \cap C'$. In a similar manner, g in P and g in C' gives g in $P \cap C'$. Clearly $P \cap C'$ is an ideal in C'. The result follows. We note that this part of the proof works for any subring C' of C if $P \cap C' \subset C'$.

Assume $P \cap C'$ is a prime ideal in C'. If P is not a prime ideal in C, then there are functions f and g in C with fg in P but g and f not in P. We know $\underline{1} + f^2$, $\underline{1} + g^2$, $\underline{1}/(\underline{1} + f^2)$, and $\underline{1}/(\underline{1} + g^2)$ are in C. Since P is an ideal, $fg/(\underline{1} + f^2)(\underline{1} + g^2)$ is in P. Also $f/(\underline{1} + f^2)$ and $g/(\underline{1} + g^2)$ are in C* and C'. Thus $fg/(\underline{1} + f^2)(\underline{1} + g^2)$ is in $P \cap C'$ which is a prime ideal in C'. Thus $f/(\underline{1} + f^2)$ or $g/(\underline{1} + g^2)$ is in $P \cap C'$. Suppose $g/(\underline{1} + g^2)$ is in $P \cap C'$. Then $(\underline{1} + g^2)g/(\underline{1} + g^2) = g$ is in P. This type of contradiction can also be found if $f/(\underline{1} + f^2)$ is in $P \cap C'$. The result follows.//

EXAMPLE 4. This example shows that not all prime ideals in C' can be written in the form $P \cap C'$ for some prime ideal P in C. Let X = N and C'(N) = C*(N). The function, j, where j(x) = 1/x for x in N, is in every free ideal in C*(N), [5, Section 4.3]. Since j is a unit of C(N) and C*(N) has free ideals, the free ideals of C*(N) cannot be written in the form $P \cap C*$.

EXAMPLE 5. If P is a prime ideal in C, this example shows $P \cap C'$ is not necessarily an ideal in C'. Let $C' = O^p$ for some p in βX where $O^p = \{f$ in $C(X): Cl_{\beta X} Z_X(f)$ is a neighborhood of p$\}$. Every prime ideal $P \subset C$ is between O^p and M^p for some p in βX, [5, Theorem 7.15]. Hence $P \cap C' = C'$ for all prime ideals P in C containing O^p.

Let α be a mapping from the set of prime ideals in C(X) into the set of prime ideals in C'(X) where C' is a subring of C containing C*. Define α by $P \cap C' = \alpha(P)$.

THEOREM 35. The mapping α is a one-to-one mapping.

PROOF. From Theorem 34, α is a well-defined mapping. Suppose $\alpha(P') = \alpha(P'')$ where P' and P'' are different prime ideals in C. Then $\alpha(P') = \alpha(P'')$ implies $P' \cap C' = P'' \cap C'$. Suppose f is in P' - P''. Then $f/(\underline{1} + f^2)$ is in P' and C'. Since $C' \cap P' = C' \cap P''$, $f/(\underline{1} + f^2)$ is in P''. Thus $(\underline{1} + f^2) f/(\underline{1} + f^2) = f$ is in P''. This contradiction shows that P' is a subset of P''. In a similar manner P'' is a subset of P'. Hence P' = P'' and the result follows.//

Let C' now be a lattice-ordered subring of C containing C*. Let α be the mapping defined preceding Theorem 35.

THEOREM 36. The mapping α is onto if and only if X is pseudocompact.

PROOF. By definition, X is pseudocompact if C(X) = C*(X).

Assume X is pseudocompact. Since C = C*, the result easily follows.

Assume α is onto. Suppose $C \neq C*$ and let $C* \subseteq C' \subset C$. From Theorem 32, there is a function f in C' such that $Z(f) = \emptyset$ and $\underline{1}/f$ is not in C'. The principal ideal (f) is in a maximal ideal in C', since every ideal is contained in

a maximal ideal when working with a commutative ring with unity [5, Section 0.15]. Let P' be one maximal ideal in C' such that (f) is a subset of P'. Then P' is a prime ideal in C'. If α is onto, then there is a prime ideal P in C such that $\alpha(P) = P \cap C' = P'$. This implies f is in P. f is a unit of C and cannot be in any ideal in C. This contradiction establishes the result.//

We now consider a theorem proved by C. W. Kohls. Theorem 38 will extend this result to a certain class of lattice-ordered subrings containing C*.

THEOREM 37. [11, Theorem 2.4] In the ring C(X) and also in C*(X), the prime ideals containing a given prime ideal form a chain.

THEOREM 38. Let X' be a realcompact subspace of βX containing νX. Let C'(X) = {f in C(X): ν_f X contains X'}. Let P' be a prime ideal in C'(X). The prime ideals in C'(X) which contain P' form a chain.

PROOF. From Theorem 16, C'(X) is a lattice-ordered subring. From Theorem 17, C'(X) and C(X') are isomorphic under the mapping t taking g in C'(X) to g*|X' in C(X'). If P' is a subset of P" where P' and P" are prime ideals in C'(X), then t is order-isomorphic, i.e., t(P') is a subset of t(P"). Similarly $t^{\leftarrow} = \alpha$ is an order-isomorphic mapping from C(X') onto C'(X). Let P_1 be a prime ideal in

C(X') containing a prime ideal P. From Theorem 37, the prime ideals in C(X') containing P form a chain. The mapping α takes prime ideals into prime ideals and is order-isomorphic. Each prime ideal P_1 containing P is mapped into a prime ideal $\alpha(P_1)$ in C'(X) containing the prime ideal $P' = \alpha(P)$ in C'(X). Since C'(X) and C(X') are isomorphic, the prime ideals are in a one-to-one correspondence, hence the desired result is obtained.//

DEFINITION 11. [14] An ideal P in a ring A is called a z-ideal if h is in P whenever h is in A and h is contained in the same maximal ideals as some function in P.

All maximal ideals in C(X) are z-ideals and prime ideals. For p in X, $O^p = \{f \text{ in } C(X): Cl_{\beta X} Z(f) \text{ is a}$ neighborhood of p} is a z-ideal in C(X), and in general, is not a prime ideal in C(X). For every z-filter on X there corresponds a unique z-ideal in C(X), namely, all functions in C(X) whose zero set is in the z-filter.

Theorems 39 and 40 will be needed shortly in proving some results on the order structure of prime ideals.

THEOREM 39. [5, Section 14.2(a)] The union and intersection of any chain of prime ideals are prime.

THEOREM 40. [5, Section 14.7] If I is a z-ideal in C(X) and Q is minimal in the class of prime ideals in C(X) containing I, then Q is a z-ideal. Every minimal prime ideal in C(X) is a z-ideal.

THEOREM 41. [14, Theorem I] Let p be in βX. Every prime ideal P in C*(X) contained in $M*^p$ is comparable with $M^p \cap C*$. Specifically, P is a subset of $M^p \cap C*$ if and only if P contains no units of C, while $M^p \cap C*$ is a proper subset of P if and only if P contains a unit of C.

THEOREM 42. [14, Theorem II] Let p be in βX and let X be locally compact and σ-compact, or equivalently, let $\beta X - X$ be a zero set in βX.

(i) The family of prime z-ideals in C*(X) contained in $M^p \cap C*$ is order-isomorphic with the family of prime z-ideals in C(X) contained in M^p.

(ii) The family of prime z-ideals in C*(X) properly containing $M^p \cap C*$ (when p is not in X) is order-isomorphic with the family of prime z-ideals in C($\beta X - X$) contained in $M^p_{\beta X-X}$.

DEFINITION 12. A space Y is an F-space if the prime ideals contained in a given maximal ideal form a chain.

COROLLARY 43. [14, Corollary 1] If X is locally compact and σ-compact, then $\beta X - X$ is a compact F-space.

Let M^p (or M^p_X) denote the z-filter $Z[M^p]$ on X, let Θ^p (or Θ^p_X) denote the z-filter $Z[0^p]$ on X, and let $N^p = \{Z$ in $Z[C(\beta X)]: p$ is in $Cl_{\beta X}(Z \cap X)\}$. Then N^p is a prime z-filter on βX.

DEFINITION 13. [14] For any space T, a point p in βT is called a remote point in βT if every member of M^p_T has a non-empty interior. A point p in T is called a P-point of T if every zero set containing p is a neighborhood of p, or equivalently, if M^p_T is a minimal prime ideal.

COROLLARY 44. [14, Corollary 3] Let X be a locally compact, σ-compact metric space and let p be in βX. Then the following conditions are equivalent.
(i) The prime ideals in C* contained in $M*^p$ form a chain.
(ii) $M^p \cap C*$ is a minimal prime ideal in C*.
(iii) p is a remote point in βX.

COROLLARY 45. [14, Corollary 4] Let X be locally compact and σ-compact, and let p be in $\beta X - X$. Then $M*^p$ is the immediate successor of $M^p \cap C*$ in the family of prime z-ideals in C*(X) if and only if p is a P-point of $\beta X - X$.

COROLLARY 46. [14, Corollary 5] Let X be a locally compact, σ-compact metric space, and let p be in $\beta X - X$. Then the family of prime z-ideals in C* contained in $M*^p$ consists of the ideals $M*^p$ and $M^p \cap$ C* if and only if p is both a remote point in βX and a P-point of $\beta X - X$.

Since any space Y, such that $X \subseteq Y \subseteq \beta X$, has the property $\beta Y = \beta X$, the previous theorems and corollaries give results for the prime ideals in C*(Y). We now consider a realcompact space X_1 where $\upsilon X \subseteq X_1 \subseteq \beta X$. The proofs of the following theorems will be essentially the same as the proofs of the previous theorems and corollaries from [14] when $X_1 = \beta X$. Theorem 38 is basic for the proof of some of the following theorems. Theorem 47 is also used and can be found in [13], [7], and [9].

THEOREM 47. [9, Section 3.1] A prime z-filter Q on a space T is minimal if and only if for every zero set Z in Q there exists a zero set W not in Q such that $Z \cup W = T$.

From Theorem 4, the maximal ideals in C* are $M*^p = \{f$ in C*(X): $f^\beta(p) = 0\}$ and the maximal ideals in C are $M^p = \{f$ in C(X): p is in $Cl_{\beta X} Z_X(f)\}$ for p in βX. The following theorem is often used in proofs without specifically being mentioned.

THEOREM 48. [5, Theorem 7.15] Every prime ideal P in C(X) contains O^p for a unique p in βX, and M^p is the unique maximal ideal containing P.

Let X_1 be a realcompact subspace of βX. From Theorem 17, $C'(X) = \{f \text{ in } C(X) : \nu_f X \text{ containing } X_1\}$ and $C(X_1)$ are isomorphic using the mapping that takes f in $C'(X)$ to $f^\nu = f^*|X_1$ in $C(X_1)$. Let M^p be a maximal ideal in $C(X)$. From Theorem 34, $M^p \cap C'(X)$ is a prime ideal in $C'(X)$. Since $Z_X(f)$ is a subset of $Z_{X_1}(f^\nu)$, f in M^p implies f^ν is in $M^p_{X_1}$ under the isomorphism between $C'(X)$ and $C(X_1)$. Then M'^p is the unique maximal ideal containing the prime ideal $M^p \cap C'(X)$. Under the isomorphism, prime z-ideals in $C'(X)$ are in one-to-one, order-preserving correspondence with the prime z-ideals in $C(X_1)$.

THEOREM 49. Consider M'^p and $C'(X)$ as stated above. Let p be in βX. Every prime ideal P in $C'(X)$ contained in M'^p is comparable with $M^p \cap C'(X)$. P is a subset of $M^p \cap C'(X)$ if and only if P contains no units of $C(X)$, while $M^p \cap C'(X)$ is a proper subset of P if and only if P contains a unit of $C(X)$.

PROOF. Let O'^p correspond to the ideal $O^p_{X_1}$ under the isomorphism between $C'(X)$ and $C(X_1)$. Every prime ideal P in $C'(X)$ is between O'^p and M'^p for some p in βX. Let U be the set of all prime ideals between O'^p and M'^p which are comparable with the prime ideal P. P is in U. Partially order U by set inclusion. From Hausdorff's Maximal Principle, P is in a maximal chain of prime ideals.

From Theorem 39, the intersection of this chain is a prime ideal, Q, which is a minimal prime ideal in $C'(X)$ contained in P. From Theorem 38, the prime ideals in $C'(X)$ containing Q form a chain. If we can show $Q \subsetneq M^P \cap C'(X)$, then Theorem 38 gives the comparability of P and $M^P \cap C'(X)$. The prime ideal in $C(X_1)$ corresponding to $M^P \cap C'(X)$ is $\{g^\upsilon$ in $C(X_1): p$ is in $Cl_{\beta X} Z_X(g^\upsilon)\}$. This ideal is clearly a z-ideal. We denote the corresponding z-filter on X_1 by N_1^p. From Theorem 40, the minimal prime ideal Q^υ in $C(X_1)$ corresponding to Q in $C'(X)$ is a z-ideal. We denote the corresponding minimal prime z-filter on X_1 by Q^υ.

In order to show $Q \subsetneq M^P \cap C'(X)$, we need to show p is in $Cl_{\beta X} Z_X(f)$ for each f in Q. We obtain this result by using z-filters on X_1 and showing that for every element Z in Q^υ, p in in $Cl_{\beta X}(Z \cap X)$. Let Z be in Q^υ and let V be any βX zero-set-neighborhood of p in βX. We have $O'^P \subseteq Q$. The set $V_1 = V \cap X_1$ is not empty, since X_1 is dense in βX and V contains a non-empty interior. Hence V_1 is in $\Theta_{X_1}^p$ and Q^υ is in $Z[M_{X_1}^p]$. Since $O'^P \subseteq Q$, $\Theta_{X_1}^p \subseteq Q^\upsilon$. Using Theorem 47, there is a zero set W in X_1 such that W is not in Q^υ and $(Z \cap V_1) \cup W = X_1$. Suppose $V_1 \cap Z$ has an empty interior in X_1. Then W is dense in X_1. Since W is an X_1 zero set, $Cl_{X_1} W = X_1 = W$. Thus $W = X_1$ is in Q^υ,

but this is a contradiction. Therefore $V_1 \cap Z$ must have a non-empty interior in X_1. Thus $V_1 \cap Z \cap X = V \cap (Z \cap X)$ is not empty. V was an arbitrary βX zero-set-neighborhood of p. From Theorem 2 (ii), every neighborhood of p in βX contains a βX zero-set-neighborhood. Thus every neighborhood of p meets $Z \cap X$ where Z is a fixed member in Q^υ. Thus p is in $Cl_{\beta X}(Z \cap X)$ and the result of the first part of the theorem has been shown.

Assume P is a prime ideal in M'^p which does not contain a unit of C(X). Let f be in P and let V be any βX zero-set-neighborhood of p. Let P^υ be the extensions to X_1 of the functions in P. $Z[P^\upsilon]$ is a prime z-filter on X_1 containing $\Theta^p_{X_1}$. The set $V_1 = V \cap X_1$ is in $\Theta^p_{X_1}$ and thus V_1 is in $Z[P^\upsilon]$. If f is in P, then $V_1 \cap Z(f^\upsilon)$ is in $Z[P^\upsilon]$ or $V \cap Z(f)$ is in $Z[P]$. $V \cap Z(f)$ is not empty since P does not contain a unit of C(X). Hence p is in $Cl_{\beta X}Z(f)$ and f is in M^p. Thus $P \subseteq M^p \cap C'(X)$. Assume $P \subseteq M^p \cap C'(X)$. P clearly has no units of C(X).

If P contains a unit of C(X), then $M^p \cap C'(X) \subset P$. Now assume $M^p \cap C'(X) \subset P$. Then from above, P contains a unit of C(X).//

DEFINITION 14. Let Y be a subset of X. Then Y is said to be Z-embedded in X if for every Z in $Z[C(Y)]$, there exists a W in $Z[C(X)]$ such that $Z = W \cap Y$.

THEOREM 50. [13, Theorem 5.2] If Y is Z-embedded in X and F is a z-filter on X every member of which meets Y, then $F|Y$ is a z-filter on Y; if F is prime, so is $F|Y$.

From [5, Section 8.8], $Z_{\nu X}f^{\nu} = Cl_{\nu X}Z_X(f)$. Thus every non-empty zero set in νX meets X. Mandelker in [14] considers $\beta X - X$ as a zero set in βX for some results. In this case, X can be shown to be realcompact. Let X_1 be a realcompact subspace of βX and let $X_1 - X$ be a non-empty zero set in βX. $X_1 - X$ is closed in βX. βX is a compact, Hausdorff space and hence is normal. From [5, Problem 3D.1], $X_1 - X$ is C*-embedded in βX and hence $X_1 - X$ is C*-embedded in X_1.

LEMMA 51. Let $X_1 - X$ be a non-empty zero set in βX. If p is in $X_1 - X$, then $M^p \cap C'$ is a proper subset of M'^p.

PROOF. Clearly $M^p \cap C' \subseteq M'^p$. Let $Z(g^{\nu}) = X_1 - X$. Then g is in M'^p but g is not in M^p.//

LEMMA 52. If $M^p \cap C'$ is a proper subset of M'^p, then p is not in X.

PROOF. From the hypothesis, there exists a function f in C' such that p is in $Cl_{\beta X}Z_{X_1}(f^{\nu})$ but p is not in $Cl_{\beta X}Z_X(f)$. The function f can be chosen bounded. If p

is in X and p is in $\mathrm{Cl}_{\beta X} Z_{X_1}(f^\cup)$, then p is in $Z_{\beta X}(f^\beta)$.

This means p is in $Z_X(f)$ and $Z_{\beta X}(f^\beta)$, and thus p is in $\mathrm{Cl}_{\beta X} Z_X(f)$. This contradiction establishes the result.//

THEOREM 53. Let $X_1 - X$ be a non-empty zero set in βX where X_1 is a realcompact subspace of βX. Let p be in βX. The family of prime z-ideals in C'(X) contained in $M^p \cap C'(X)$ is order-isomorphic with the family of prime z-ideals in C(X) contained in M^p.

PROOF. Under the isomorphism from C'(X) onto $C(X_1)$, prime z-ideals in C'(X) are in one-to-one correspondence with prime z-ideals in $C(X_1)$. Place the prime z-ideals contained in M'^p into order-preserving correspondence with the z-filters on X_1 contained in $M^p_{X_1} = Z[M^p_{X_1}]$ using the one-to-one mapping $P \mapsto Z[P^\cup]$. Under this mapping, $M^p \cap C'$ goes to $N^p_1 = \{Z \text{ in } Z[C(X_1)] : p \text{ is in } \mathrm{Cl}_{\beta X}(Z \cap X)\}$.

Let P be a prime z-filter on X_1 contained in N^p_1. Each member of P meets X. From Theorem 50, $P|X = \{Z \cap X : Z \text{ is in } P\}$ is a prime z-filter on X. If $P \subseteq N^p_1$, then $P|X \subseteq M^p_X$. If Q is any prime z-filter which is a subset of M^p_X, then the prime z-filter $Q\# = \{Z \text{ in } Z[C(X_1)] : Z \cap X \text{ is in } Q\} \subseteq N^p_1$. The mapping $P \mapsto P|X$ for $P \subseteq N^p_1$ is onto the family of prime z-filters on X which are contained in M^p_X. Clearly $P \subseteq (P|X)\#$. We now show $(P|X)\# \subseteq P$. Let Z be in $(P|X)\#$. Then there exists a W in P such that $Z \cap X = W \cap X$. Since W and Z have the same X points, $W \subseteq Z \cup (X_1 - X)$.

Since P is a z-filter on X_1 containing W, $Z \cup (X_1 - X)$ is in P. Since $P \subseteq N_1^p$ and P is prime, $X_1 - X$ is not in P, and therefore Z is in P. We have shown that if Z is in $(P|X)\#$, then Z is in P. Thus $(P|X)\# \subseteq P$. Thus the mapping $P \mapsto P |X$ is one-to-one and onto for the appropriate domain and range. From this mapping we obtain the order-isomorphism.//

THEOREM 54. Let p be in $X_1 - X$ where X_1 is a real-compact subspace of βX and $X_1 - X$ is a non-empty zero set in βX. The family of prime z-ideals in $C'(X)$ properly containing $M^p \cap C'(X)$ is order-isomorphic with the family of prime z-ideals in $C(X_1 - X)$ contained in $M_{X_1-X}^p$.

PROOF. Let P be a prime z-filter on X_1 which properly contains N_1^p where p is in $X_1 - X$. Then $X_1 - X$ is a closed set in the normal space βX. From [5, Problem 3D.1], $X_1 - X$ is Z-embedded in βX, and also in X_1. From Theorem 50, the trace, $P|(X_1 - X)$ is a prime z-filter on $X_1 - X$. Since $P \subseteq M_{X_1}^p$, $P|(X_1 - X) \subseteq M_{X_1-X}^p$ for p in $X_1 - X$. Let Q be a prime z-filter on $X_1 - X$ which is a subset of $M_{X_1-X}^p$ and let $Q\# = \{Z$ in $Z[C(X_1)]$: $Z \cap (X_1 - X)$ is in $Q\}$. We first show $Q\#$ is prime. Consider $Z_1 \cup Z_2$ in $Q\#$. Then $(Z_1 \cup Z_2) \cap (X_1 - X) = [Z_1 \cap (X_1 - X)] \cup [Z_2 \cap (X_1 - X)]$ is in Q. Since Q is prime, one of the sets, say $Z_1 \cap (X_1 - X)$, is in Q. Thus Z_1 is in $Q\#$. Hence $Q\#$ is a prime z-filter.

From definitions, $Q\# | (X_1 - X) = Q$ and $N_1^p \subseteq Q\#$. If $Q \subseteq M_{X_1-X}^p$, then $Q\# \subseteq M_{X_1}^p$. The zero set $X_1 - X$ is in $Q\#$ but not in N_1^p, hence $Q\# \supset N_1^p$. From Theorem 49, if P is a prime ideal in $C'(X)$ contained in M'^p, then P is comparable with $M^p \cap C'(X)$. Under the mapping $P \mapsto Z[P^\nu]$, $M^p \cap C'(X)$ goes to N_1^p. The mapping $P \mapsto P | (X_1 - X)$ for $P \supset N_1^p$ is onto the set of prime z-filters on $X_1 - X$ which are subsets of $M_{X_1-X}^p$. We will show $P = (P | (X_1 - X))\#$, then $P \mapsto P | (X_1 - X)$ is a one-to-one, onto mapping for the appropriate domain and range. From the definition of "#", $P \subseteq (P | (X_1 - X))\#$. Let Z be in $(P | (X_1 - X))\#$. Then there exists a W in P such that $Z \cap (X_1 - X) = W \cap (X_1 - X)$. Since P properly contains N_1^p, Theorem 49 implies that the prime z-ideal in X corresponding to P contains a unit of $C(X)$. Then $X_1 - X$ is in P. P is a prime z-filter and $W \cap (X_1 - X)$ is in P. Thus $W \cap (X_1 - X) = Z \cap (X_1 - X)$ is in P. Since $Z \supseteq Z \cap (X_1 - X)$, Z is in P. We now have $P \supseteq (P | (X_1 - X))\#$ when $P \supset N_1^p$. From these results, $P = (P | (X_1 - X))\#$. From the mapping $P \mapsto P | (X_1 - X)$ for $P \supset N_1^p$, we obtain the result.//

COROLLARY 55. If $X_1 - X$ is a non-empty zero set in βX where X_1 is a realcompact subspace of βX, then $X_1 - X$ is a compact F-space.

PROOF. Since $X_1 - X$ is a non-empty zero set in the compact space βX, it is a closed subspace of a compact space, and thus is a compact subspace. From Theorems 38 and 54, the prime ideals in $C'(X)$ properly containing $M^p \cap C'(X)$ form a chain, and thus the prime z-ideals of $C(X_1 - X)$ contained in $M^p_{X_1-X}$ form a chain when p is in $X_1 - X$. From Theorem 39, the intersection of this chain of prime z-ideals in $C(X_1 - X)$ is a prime ideal. From [5, Problem 14 B.2], $O^p_{X_1-X}$ is the intersection of the prime z-ideals containing it. Hence $O^p_{X_1-X}$ is prime and from [5, Section 14.12], the ideals between $O^p_{X_1-X}$ and $M^p_{X_1-X}$ form a single chain. Thus $X_1 - X$ is a compact F-space.//

THEOREM 56. [13, Theorem 11.2] Let X be a separable metric space and let p be in βX. The following statements are equivalent.

(i) The prime ideals contained in M^p_X form a chain.

(ii) M^p_X is a minimal prime ideal in $C(X)$.

(iii) p is a remote point in βX.

COROLLARY 57. Let X_1 be a realcompact subspace of βX such that $X_1 - X$ is a non-empty zero set of βX. Let X be a separable metric space and let p be in βX. Then the following statements are equivalent.

(i) The prime ideals in $C'(X)$ contained in M'^p form a chain.

(ii) $M^p_X \cap C'(X)$ is a minimal prime ideal in $C'(X)$.

(iii) p is a remote point in $\beta X_1 = \beta X$.

PROOF. Assume p is a remote point in $\beta X = \beta X_1$. Then M_X^p is a minimal prime ideal in C from Theorem 56. $M_X^p \cap C'$ is a minimal ideal in C' from Theorem 53. All prime ideals contained in M'^p are comparable with the prime ideal $M^p \cap C'$ from Theorem 49. The prime ideals in C' contained in M'^p form a chain from Theorem 38.

Now suppose the prime ideals in C' contained in M'^p form a chain. Under the isomorphism between C'(X) and $C(X_1)$, the prime ideals in $M_{X_1}^p$ form a chain. From Theorem 56, p is a remote point of $\beta X_1 = \beta X.//$

COROLLARY 58. Let X_1 be a realcompact subspace of βX such that $X_1 - X$ is a non-empty zero set in βX and let p be in $X_1 - X$. Then M'^p is the immediate successor of $M^p \cap C'$ in the family of prime z-ideals in C' if and only if p is a P-point of $X_1 - X$.

PROOF. We first show the only if part. From Theorem 54, $M_{X_1-X}^p$ is a minimal prime ideal and the only prime z-ideal in $C(X_1 - X)$. This implies p is a P-point of $X_1 - X$.

Assume p is a P-point of $X_1 - X$. Then from the definition of a P-point, there are no prime z-ideals between $M^p \cap C'$ and $M'^p.//$

COROLLARY 59. Let X be a separable metric space with X_1 and $X_1 - X$ as given in Corollary 58. Let p be in $X_1 - X$. Then the family of prime z-ideals in C' contained in M'^P consists of exactly two ideals--M'^P and $M^P \cap C'$--if and only if p is both a remote point and a P-point of $X_1 - X$.

PROOF. We first show the only if part. From Corollary 58, p is a P-point of $X_1 - X$. Then p is a remote point of $\beta(X_1 - X)$ from Corollary 57 and Theorem 54. Since $X_1 - X$ is a (closed) zero set in the compact space βX, $X_1 - X$ is compact. Therefore, $\beta(X_1 - X) = X_1 - X$.

Assume p is both a remote point and a P-point of $X_1 - X$. Then M'^P is the immediate successor of $M^P \cap C'$ from Corollary 58, and $M^P \cap C'$ is a minimal prime ideal of $C(X_1 - X)$ from Corollary 57. The result now follows from Theorem 54.//

D. Totally Ordered Quotient Rings

In this section we work with convex ideals, absolutely convex ideals, and ideals with a property similar to z-ideals. We consider these ideals with regard to totally ordered quotient rings. In investigating the distribution of ideals in a ring A one advantage in knowing about totally ordered quotient rings is that if A/P is totally ordered, then all ideals containing P form a chain. Let A denote a subring in R^X which has the partial ordering inherited

from R^X, i.e., $f \geq g$ if $f(x) \geq g(x)$ for all x in X. The subring A does not necessarily contain $C(X)$ or $C^*(X)$. Many results in this section are extensions of results in Chapters 5 and 14 of [5] where $A = C(X)$ or $A = C^*(X)$.

DEFINITION 15. An ideal I in A is said to be convex provided x is in I wherever x is in A, $\underline{0} \leq x \leq y$, and y is in I. An ideal I is said to be absolutely convex provided x is in I whenever x is in A, y is in I, and $\underline{0} \leq |x| \leq |y|$. An absolutely convex ideal is also a convex ideal.

EXAMPLE 6. This is an example of a convex ideal which is not absolutely convex. Consider the ideal $I = (i)$ in $C(R)$ where $i(x) = x$ for all x in R. Then I contains all functions in $C(R)$ which vanish at 0 and have a derivative at 0. Hence I is convex. Clearly $\underline{0} \leq ||i|| \leq |i|$ but $|i|$ is not in I. Hence I is not absolutely convex.

DEFINITION 16. Let I be an ideal in a partially ordered ring A. In the quotient ring A/I we say $I(a) \geq I(\underline{0})$ or $I(a) \geq 0$ if there is a function f in A such that $f \geq \underline{0}$ and $a \equiv f \pmod{I}$.

THEOREM 60. [5, Theorem 5.2] Let I be an ideal in a partially ordered ring A. Then A/I is a partially ordered ring according to Definition 16 if and only if I is convex.

DEFINITION 17.   Let A be a subring of a ring B contained in R^X.   A is said to satisfy condition II relative to B if whenever f is in A, then A contains all functions g in B such that $Z(g) = Z(f)$.

EXAMPLE 7.   Let O_0 = {f in C(R): Z(f) is a neighborhood of 0}. O_0 satisfies condition II relative to C(R). An example which does not satisfy condition II relative to a ring B will be given in Example 9 at the end of this section.

THEOREM 61.   Let A satisfy condition II relative to R^X.
(i)   If f is in A, then $|f|$ is in A.
(ii)   Every ideal P in A is absolutely convex.

PROOF.   Part (i) is clearly true.   To show (ii), let $\underline{0} \leq |f| \leq |g|$ with g in P and f in A.   We must show f is in P.   Let $h(x) = f(x)/g(x)$ for x in X - Z(g) and $h(x) = 0$ for x in Z(g).   Clearly h is in R^X.   If $|f| \leq |g|$, then Z(f) contains Z(g).   Thus $Z(h) = Z(f)$.   Since A satisfies condition II relative to R^X, h is in A.   Since P is an ideal in A, $hg = f$ is in P.//

COROLLARY 62.   Let A be a subring in C(X) (or C*(X)) which satisfies condition II relative to C(X) (or C*(X)). Every ideal P in A is absolutely convex.

PROOF. The proof follows the procedure of the proof of Theorem 61 with $h(x) = f^2(x)/g(x)$ for x in X - Z(g). Note that $f(x)/g(x) \leq 1$ for all x. It is clear that h is in C(X) (or C*(X)) except for those x in Z(g) where every neighborhood N of x contains a y' such that $g(y') \neq 0$. For y in N, $|h(y) - h(x)| = |h(y)| \leq |f(y) - f(x)| = |f(y)|$. Since f is continuous, h is in C(X) (or C*(X)).//

DEFINITION 18. Let A be a subring of R^X. Then A is said to satisfy condition III if for every f and g in A such that $\underline{0} \leq |f| \leq |g|$ there exist positive integers m and n and a function h in A such that $h(x) = f^n(x)/g^m(x)$ for x in X - Z(g) and h(x) = 0 for x in Z(g).

THEOREM 63. Let A be a subring of R^X which satisfies condition III. Every prime ideal P in A is absolutely convex.

PROOF. Let $\underline{0} \leq |f| \leq |g|$ where g is in P and f is in A. We must show f is in P. Let m, n, and h be the same as in Definition 18. If m > 1 and g is in P, then g^{m-1} is in A. Since P is an ideal, $(hg^{m-1})g = hg^m = f^n$ is in P. If m = 1, $hg = f^n$ is in P. Since P is a prime ideal in A and f is in A, f is in P.//

The following corollaries give examples which satisfy condition III. In Section IV C we will work with a subring A of R^X which has some absolutely convex prime ideals and some maximal ideals (and thus prime ideals) which are not absolutely convex. From Theorem 63, this A does not satisfy condition III.

COROLLARY 64. Let Y be a realcompact subspace of βX. Let $C'(X)$ and $C(Y)$ be as in Theorems 16 and 17. Every prime ideal P in $C'(X)$ is absolutely convex. In particular, the prime ideals in $C^*(X)$ and $C(X)$ are absolutely convex.

PROOF. An isomorphism takes prime ideals to prime ideals. Let $\underline{0} \leq |f| \leq |g|$ where g is in P and f is in $C'(X)$. Let $h(x) = f^2(x)/g(x)$ for x in $X - Z(g)$ and let $h(x) = 0$ for x in $Z(g)$. Let f^ν be in $C(Y)$ where f is in $C'(X)$ and $f^\nu = f*|Y$. Since X is dense in Y, $\underline{0} \leq |f^\nu| \leq |g^\nu|$. Using the method of proof in Corollary 62, $(f^\nu)^2/g^\nu = h^\nu$ is in $C(Y)$ and hence h is in $C'(X)$. The rest of the proof follows from Theorem 63.//

COROLLARY 65. Let $C^1(R)$ be the subring of all functions with a continuous first derivative on R. Then every prime ideal P in $C^1(R)$ is absolutely convex.

PROOF. Let $\underline{0} \leq |f| \leq |g|$ where g is in P and f is in $C^1(R)$. Let $h(x) = f^4(x)/g(x)$ for x in $R - Z(g)$ and let $h(x) = 0$ for x in $Z(g)$. If h is in $C^1(R)$, then Theorem 63 will give the result. Clearly, $|f(x)/g(x)| \leq 1$ for x in R. The function h is easily seen to be continuous at all points of R except those points a in $Z(g)$ where there is a sequence $\{x_n\}$ with $x_n \to a$ and x_n not in $Z(g)$. First we show $h'(a)$ exists. $\lim\limits_{x_n \to a} f^4(x_n)/(x_n-a)g(x_n) = f'(a) \lim\limits_{x_n \to a} f^2(x_n) \, f(x_n)/g(x_n) = 0$. An arbitrary sequence $\{x_n\}$ such that $x_n \to a$ may have some x_n in $Z(g)$ and some x_n not in $Z(g)$. In any case $h'(a) = \lim\limits_{x_n \to a} (h(x_n) - h(a))/(x_n - a) = 0$.

Now we show h' is continuous at a. Suppose x_n is not in $Z(g)$. $\lim\limits_{x_n \to a} h'(x_n) = \lim\limits_{x_n \to a} [4f'(x_n)g(x_n)f(x_n)(f(x_n)/g(x_n))^2 - f^2(x_n)g'(x_n)(f(x_n)/g(x_n))^2] = 0$. For an arbitrary sequence $\{x_n\}$, $\lim\limits_{x_n \to a} h'(x_n) = 0 = h'(a)$. Thus h' is in $C^1(R)$ and the result follows.//

Theorem 66 gives sufficient conditions for a quotient ring to be totally ordered. The theorem is [5, Problem 5G.1] for the subring C(X).

THEOREM 66. Let A be a subring of R^X such that $|f|$ is in A whenever f is in A. If J is a convex ideal in A containing a prime ideal P, then J is absolutely convex and A/J is totally ordered.

PROOF. To show that J is absolutely convex consider $\underline{0} \leq |f| \leq |g|$ where g is in J and f is in A. Then $(g + |g|)$ $(g - |g|)$ is in the prime ideal P. Suppose $g + |g|$ is in P. Since g is in J, $(g + |g|) - g = |g|$ is in J. Since J is convex, $|f|$ and $- |f|$ are in J. Also $(f - |f|)$ $(f + |f|)$ is in P. In a manner similar to the above procedure, f is always in J. Hence J is absolutely convex. For f in A, $f - |f|$ or $f + |f|$ is in P. This implies $f \equiv - |f|$ (mod J) or $f \equiv |f|$ (mod J). Hence $J(f) \geq 0$ or $J(f) \leq 0$ for each f in A. Since every element of A/J is comparable with 0, A/J is totally ordered.//

EXAMPLE 8. This example shows that an ideal J in A does not need to be prime in order that A/J be totally ordered. The results for this space are in [5, Problems 4M and 5 G.3]. Let $\Sigma = N \cup \{\sigma\}$ where σ is in $\beta N - N$. Let $j = (j^*)$ where j^* is the extension to Σ of $j(x) = 1/x$ for x in N. Then $0_\sigma \subset J_\sigma \subset M_\sigma$. Also J is not a prime ideal in $C(\Sigma)$, 0_σ is a prime ideal in $C(\Sigma)$, and $C(\Sigma)/J$ is totally ordered.

THEOREM 67. [16, Theorem 2.45, Exercise 4.9] Let Θ be a homomorphism of a ring B onto the ring $S = \Theta[B]$, with kernel K. Then each of the following is true.

(i) If I is an ideal in B, then $\Theta[I]$ is an ideal in S.

(ii) If U is an ideal in S, then $\Theta^{\leftarrow}[U]$ is an ideal in B which contains K.

(iii) If I is an ideal in B which contains K, then I = $\Theta^{\leftarrow}[\Theta[I]]$.

(iv) The mapping I $\mapsto$ $\Theta[I]$ defines a one-to-one mapping of the set of all ideals in B which contain K onto the set of all ideals in S.

(v) If I and I_1 are ideals in B which contain K, then I is a subset of I_1 if and only if $\Theta[I]$ is a subset of $\Theta[I_1]$.

(vi) The mapping I $\mapsto$ $\Theta[I]$ defines a one-to-one mapping of the set of all prime ideals in B which contain K onto the set of all prime ideals in S.

Some of the results through Theorem 71 are implied in [5, Section 14.3].

THEOREM 68. [5, Theorem 14.3] If I is convex ideal in a partially ordered ring A and J is any convex ideal containing I, then J/I is a convex ideal in the partially ordered ring A/I.

DEFINITION 19. An interval in a partially ordered set A is a chain Y in A such that if x and y are in Y, t is in A, and x $\leq$ t $\leq$ y, then t is in Y. An interval Y in A is called a symmetric interval if (-y) is in Y whenever y is in Y.

LEMMA 69. An ideal I in a totally ordered ring A is convex if and only if I is a symmetric interval.

PROOF. The totally ordered ring A does not have to be a ring of real-valued functions. Assume I is a symmetric interval and let $0 \leq f \leq g$ where f is in A and g is in I. Since 0 and g are in the interval I, f is in I. Hence I is convex.

Now assume I is convex and let $x \leq t \leq y$ where x and y are in I and t is in A. We must show t and (-t) are in I. Since A is totally ordered, every element is comparable with 0. Consider two cases.

Case 1. $0 \leq t \leq y$. Then t is in I from the definition of a convex ideal and (-t) is in I from the definition of an ideal.

Case 2. Let $x \leq t \leq 0$. Then this inequality yields $0 \leq -t \leq -x$. We know (-x) is in I. From the definition of a convex ideal, t and (-t) are in I.//

LEMMA 70. If Y_1 and Y_2 are symmetric intervals in a totally ordered ring A, then Y_1 is a subset of Y_2 or Y_2 is a subset of Y_1.

PROOF. Suppose s is in $Y_2 - Y_1$ and t is in $Y_1 - Y_2$. Then $s \neq 0$ and $t \neq 0$. Since A (not necessarily a subring of R^X) is totally ordered, s is comparable with 0 and t.

Let s > 0 and s < t. Then -t < -s < 0 < s < t. Since
Y_1 is a symmetric interval, the interval from -s to s is
in Y_1. This contradicts s in $Y_2 - Y_1$. This type of
contradiction can be established for the different order
relations of s, t, and 0.//

THEOREM 71. Let A be a partially ordered ring and let
P be a convex ideal such that A/P is totally ordered. The
convex ideals in A containing P form a chain.

PROOF. Let I be a convex ideal in A containing P.
From Theorem 69, I/P is a convex ideal in A/P. From
Lemma 69, I/P is a symmetric interval. From Lemma 70,
any two symmetric intervals are comparable. This and
Theorem 67 (v) give the desired result.//

We now consider results involving totally ordered
quotient rings of subrings that satisfy condition II with
respect to R^X. Similar results for C(X) are in [5,
Theorem 2.9, Section 5.4].

THEOREM 72. If I is an ideal in a lattice-ordered
subring A of R^X satisfying condition II relative to A,
then the following are equivalent.
(i) The ideal I is prime.

(ii) The ideal I contains a prime ideal.

(iii) For h and g in A, hg = $\underline{0}$ implies g is in I or h
is in I.

(iv) For each f in A, there exists a zero set in Z[I] on which f does not change sign.

PROOF. It is clear that (i) implies (ii) and (ii) implies (iii). To show (iii) implies (iv), we see that $f \vee \underline{0}$ and $f \wedge \underline{0}$ are in A for each f in A, and $(f \vee \underline{0})(f \wedge \underline{0}) = \underline{0}$ in I. From (iii), at least one of the functions is in I. The zero set of this function satisfies (iv). To show (iv) implies (i), let gh be in I where g and h are in A. From (iv), there exists a zero set Z_1 in Z[I] such that $g^2 - h^2$ does not change sign on Z_1. Suppose $g^2 \geq h^2$. Then every zero of g in Z_1 is also a zero in h. Then $Z(h) \supseteq Z_1 \wedge Z(h) = Z_1 \wedge Z(gh)$ in Z[I]. If $Z_1 \wedge Z(gh) = \emptyset$, then I = A. This, however, is a contradiction. Hence $Z(h) \supseteq Z_1 \wedge Z(gh) \neq \emptyset$. From condition II, h is in I. Hence I is a prime ideal in A.//

THEOREM 73. [5, Theorem 5.3] The following conditions on a convex ideal I in a lattice-ordered ring A are equivalent.

(i) The convex ideal I is absolutely convex.

(ii) If x is in I, then $|x|$ is in I.

(iii) If x and y are in I, then $x \vee y$ is in I.

(iv) $I(a \vee b) = I(a) \vee I(b)$.

(v) $I(a) \geq 0$ if and only if $a \equiv |a| \pmod{I}$.

THEOREM 74. Let I be an ideal in a lattice-ordered subring A of R^X which satisfies condition II with respect to A. Then $I(f) \geq 0$ if and only if f is non-negative on some zero set of $Z[I]$.

PROOF. From Theorem 61 (ii), I is absolutely convex. From Theorem 73, $I(f) \geq 0$ if and only if $-f + |f|$ is in I. This is true if and only if f agrees with $|f|$ on the zero set $Z(-f + |f|)$ in $Z[I]$. Thus f is non-negative on a zero set of I.

Assume f is non-negative on some zero set Z_1 in $Z[I]$ where $Z(g) = Z_1$ with g in I. Then $Z(f - |f|)$ contains Z_1, so $Z(g(f - |f|)) = Z(|f| - f)$. Since $g(f - |f|)$ is in I, condition II shows $|f| - f$ in I. Theorem 73 gives the result.//

THEOREM 75. Let I be an ideal in a lattice-ordered subring A of R^X. Let I satisfy condition II with respect to A. Then A/I is totally ordered if and only if I is a prime ideal in A.

PROOF. If A/I is totally ordered, then $I(f) \geq 0$ or $I(-f) \geq 0$ for each f in A. From Theorem 74, f or $(-f)$ is non-negative on some zero set in $Z[I]$. From Theorem 72, I is a prime ideal in A. The previous steps are reversible.//

EXAMPLE 9. The following ideals are examples of absolutely convex ideals in $C(R)$ which neither satisfy condition II relative to $C(R)$ nor are prime ideals. Let $I_n = \{f$ in $C(R): f(x)/x^n$ is bounded on a deleted neighborhood of $0\}$ for $n = 1,2,3,\ldots$. Clearly $|g|$ is in I_n if and only if g is in I_n. Let $\underline{0} \leq |f| \leq |g|$ where f is in $C(R)$ and g is in I_n. $0 \leq ||f(x)|/x^n| \leq ||g(x)|/x^n|$ for $x \neq 0$. This shows $|f|$ is in I_n and thus f is in I_n. Let $i(x) = x$ for x in R. I_n does not satisfy condition II with respect to $C(R)$ since $Z(|i|^{1/2}) = Z(i^n)$ for each n but $(|i|)^{1/2}$ is not in I_n for any n. The function i^n is in $I_n - I_{n+1}$, hence $I_{n+1} \subset I_n$ for each n. In general, $(|i|^{n/2})^2 = |i|^n$ is in I_n but $|i|^{n/2}$ is not in I_n, hence I_n is not a prime ideal. If $I = \bigcap_n I_n$, then $O_0 \subseteq I$. If $k(x) = e^{-1/x^2}$ for $x \neq 0$ and $k(0) = 0$, then k is in $I - O_0$.

EXAMPLE 10. This example involves the real "entire" analytic functions. Properties of special ideals are given in Lemma 76 below. We call A the ring of real "entire" functions on R if whenever f is in A, then for each x_0 in R, f and all of its derivatives exist on R, and the Taylor series of f in powers of $x - x_0$ converges to f on a neighborhood of x_0. Let $M_0 = \{f$ in $A: f(0) = 0\}$. Since A/M_0 is isomorphic to the real field R by using

$M_0(f) = f(0)$, M_0 is a maximal ideal in A and hence is a prime ideal in A. Let M_0^2 be the ideal in A generated by functions of the form fg where f and g are in M_0.

LEMMA 76. Let f be in A where A, M_0, and M_0^2 are defined above.

(i) The function f is in M_0^2 if and only if $f(0) = 0$ and $f'(0) = 0$.

(ii) The ideal M_0^2 is not a prime ideal in A and M_0^2 is a proper subset of M_0.

(iii) The ideals M_0^2 and M_0 are absolutely convex.

(iv) The quotient ring A/M_0^2 is not totally ordered, however, A/M_0 is totally ordered.

PROOF.

(i) The only if part is clearly true. If $f(x) = 0$ and $f'(0) = 0$, then $f(x) = a_2x^2 + a_3x^3 + \ldots$. Let $i(x) = x$ and $g(x) = a_2x + a_3x^2 + \ldots$. Then $g(0) = 0$, and g has the same radius of convergence as f. Hence g is in M_0 and $f = ig$. The function i is clearly in M_0. Thus f is in M_0^2.

(ii) Since i^2 is in M_0^2 but i is in $M_0 - M_0^2$, the results follow.

(iii) Let $\underline{0} \leq |f| \leq |g|$ where f is in A and g is in M_0^2. Then $g(x) = b_2x^2 + b_3x^3 + \ldots$ and $f(x) = a_0 + a_1x + a_2x^2 \ldots$ for a neighborhood of 0. Clearly $f(0) = 0$, since $g(0) = 0$. Hence $a_0 = 0$. Let h be a non-zero number, then $0 \leq |f(h)/h| \leq |g(h)/h|$. This implies $|f'(0)| \leq |g'(0)| = 0$.

Thus $a_1 = 0$ and from above, f is in M_0^2. In a similar

manner M_0 can be shown to be absolutely convex.

(iv)   Suppose $M_0^2(i) \geq M_0^2(\underline{0})$ where i is the identity

function, then there is a g in A such that $g \geq \underline{0}$, g(x) =

$a_0 + a_1x + a_2x^2 + \dots$ in a neighborhood of 0, and g - i

is in M_0^2. Thus (g - i)(0) = (g - i)'(0) = 0. This implies

$a_0 = 0$ and $a_1 = 1$. Then g(0) = 0 and g'(0) = 1, which is

not possible for a non-negative, continuous function.

This type of contradiction can be found for all possible

inequalities.

To show that A/M_0 is totally ordered, it is easy to

show $M_0(f) = M_0(\underline{r})$ where r = f(0).//

IV. STRUCTURE SPACES OF SPECIAL SUBRINGS OF C(X)

A. Real Function Rings

Let A be a ring of real-valued functions on a non-empty set X. Let $M_x^A = \{f \text{ in } A: f(x) = 0\}$. If Y is a subset of X, let $M_{[Y]}^A = \{M_x^A: x \text{ is in } Y\}$. We consider a definition and a lemma from [20].

DEFINITION 1. [20, p. 1] By a real function ring (X,A) we mean a ring A of real-valued functions on a non-empty set X such that,

(i) the ring A separates points of X, i.e., if x and y are in X with $x \neq y$, then there is a function f in A such that $f(x) \neq f(y)$, and

(ii) for each x in X, $\{f(x): f \text{ in } A\}$ is the set of real numbers.

LEMMA 1. [20, Lemma 1.4] Let (X,A) be a real function ring. For each x in X, M_x^A is a fixed maximal ideal; every fixed maximal ideal has the form M_x^A for some x; and the mapping $(x \mapsto M_x^A): X \to M_{[X]}^A$ is one-to-one.

PROOF. From definitions, M_x^A is clearly a fixed ideal. The mapping $(f \mapsto f(x)): A \to R$ is a ring homormorphism with M_x^A as its kernel. From the definition of a real function ring, this mapping is onto R. Hence A/M_x^A is isomorphic to the real field R. Thus M_x^A is a maximal ideal.

Every fixed ideal must be contained in M_x^A for some x.

Let I be a fixed ideal contained in M_x^A. If I is a maximal

ideal, then it must be M_x^A.

To show the mapping that takes x in X to M_x^A in $M_{[X]}^A$

is one-to-one, suppose x and y are in X with x ≠ y. From

the definition of a real function ring there exists a

g in A such that g(x) ≠ g(y). Hence the homomorphisms

f ↦ f(x) and f ↦ f(y) are different homomorphisms onto

R. Their kernels, M_x^A and M_y^A must be different ideals

from [5, Section 0.23].//

DEFINITION 2. Let ν_A X be the set of real homomor-

phisms on A, i.e., homomorphisms from A onto R. Let f^ν

be a function from ν_A X into R defined by $f^\nu(t) = t \circ f$

for each t in ν_A X. Let $A^\nu = \{f^\nu : f \text{ is in } A\}$.

THEOREM 2. Let (X,A) be a real function ring. The

mapping taking f in A to f^ν in A^ν is one-to-one from A

onto A^ν.

PROOF. If f ≠ g, then there exists an x_1 in X such

that $f(x_1) \neq g(x_1)$. Let t_1 be the real homomorphism

taking h in A to $h(x_1)$ in R. From the definition of a

real function ring, t_1 is in ν_A X. Clearly $t_1 \circ f \neq t_1 \circ g$.

If $f^\nu = g^\nu$, then $t \circ f = t \circ g$ for each t in ν_A X. In

particular, $t_1 \circ f = t_1 \circ g$. Thus $f^\nu = g^\nu$ and f ≠ g

cannot occur simultaneously. Therefore the mapping $f \mapsto f^\vee$

is one-to-one. The mapping is onto from the definition of

$A^\vee$.//

THEOREM 3. Let (X,A) be a real function ring. Then

$(\nu_A X, A^\vee)$ is a real function ring where $f^\vee + g^\vee = (f + g)^\vee$

and $f^\vee g^\vee = (fg)^\vee$.

PROOF. Clearly $A^\vee$ is a ring of real-valued functions

on $\nu_A X$. To show $A^\vee$ separates points of $\nu_A X$, let t_1

and t_2 be in $\nu_A X$ where $t_1 \neq t_2$. Then there exists an f

in A with $t_1 \circ f \neq t_2 \circ f$. Thus $t_1 \circ f = f^\vee(t_1) \neq f^\vee(t_2)$

$= t_2 \circ f$. Therefore $A^\vee$ separates points of $\nu_A X$.

For a fixed x, the mapping $(f \mapsto f(x)): A \to R$ is a

real homomorphism, hence $\nu_A X$ is not empty. We now

show that $\{f^\vee(t): f^\vee$ is in $A^\vee\}$ is the set of all real

numbers for each t in $\nu_A X$. $f^\vee(t) = t \circ f$. Since t

maps A onto R, the condition is satisfied.//

COROLLARY 4. The mapping taking f in A to $f^\vee$ in $A^\vee$ is

an isomorphism from A onto $A^\vee$.

PROOF. This follows from Theorem 2 and the equalities

$(f + g)^\vee = f^\vee + g^\vee$ and $(fg)^\vee = f^\vee g^\vee$.//

COROLLARY 5. Let (X,A) be a real function ring. Then

$M_t^{A^\vee}$, for t in $\nu_A X$, is a fixed maximal ideal in $A^\vee$; and

every fixed maximal ideal in $A^\vee$ has the form $M_t^{A^\vee}$ for some

t in $\nu_A X$.

PROOF. The results follow from Theorem 3 and Lemma 1.//

The process in Theorem 3 can be continued, since $(\nu_A X, A^\nu)$ is a real function ring. Denote the real homomorphisms on A^ν by $\nu_{A^\nu} X$. Let $f^{\nu\nu}$ be a function from $\nu_{A^\nu} X$ to R defined by $(f^{\nu\nu})(h) = h \circ f^\nu$ for all h in $\nu_{A^\nu} X$.

THEOREM 6. There exists a one-to-one mapping between the real homomorphisms on A and the real homomorphisms on A^ν, i.e., between $\nu_A X$ and $\nu_{A^\nu} X$.

PROOF. Let t be in $\nu_A X$. Define t^ν to be a real-valued mapping on A^ν defined by $t^\nu(f^\nu) = t \circ f$ for each f^ν in A^ν. The mapping t^ν is onto R since t is a mapping onto R. Since $f^\nu + g^\nu = (f + g)^\nu$ and $f^\nu g^\nu = (fg)^\nu$, t^ν is a real homomorphism on A^ν. If t_1 and t_2 are in $\nu_A X$ with $t_1 \neq t_2$, then $t_1^\nu \neq t_2^\nu$. Thus the mapping taking t to t^ν is one-to-one from $\nu_A X$ into $\nu_{A^\nu} X$.

Now let α be a real homomorphism on A^ν. Let $\alpha(f^\nu) = r_f$ in R. Then $\alpha((f + g)^\nu) = r_{f+g} = \alpha(f^\nu) + \alpha(g^\nu) = r_f + r_g$ and $\alpha((fg)^\nu) = \alpha(f^\nu g^\nu) = r_{fg} = r_f r_g$. Define a mapping t from A into R by $t \circ f = \alpha(f^\nu) = r_f$. Then $t \circ (f + g) = t \circ f + t \circ g$ and $t \circ (fg) = (t \circ f)(t \circ g)$. The mapping t is a real homomorphism on A since α is a real homomorphism on A^ν. Thus $\alpha = t^\nu$. The result follows.//

From the mappings given in Theorem 6 and Corollary 4, $(v_A X, A^v)$ and $(v_{A^v} X, (A^v)^v)$ can be identified with each other. This process is somewhat similar to a completion. We might call it a real completion. We get no new information from taking the real completion of $(v_A X, A^v)$.

THEOREM 7. Every real maximal ideal in A^v is a fixed ideal.

PROOF. Let M^v be a real maximal ideal in A^v. Ring isomorphisms take ideals to ideals and real ideals to real ideals. From the isomorphism given in Corollary 4, $M = \{f$ in A: f^v is in $M^v\}$ is a real ideal in A. Define a mapping α from A onto R by $\alpha(g) = M(g)$ for g in A. Then α is a real homomorphism with M as its kernel. Hence α is in $v_A X$; and if m^v is in M^v, then $m^v(\alpha) = \alpha \circ m = 0$. Therefore M^v is a fixed maximal ideal.//

COROLLARY 8. A maximal ideal in A^v is fixed if and only if it is a real ideal.

PROOF. Using Corollary 5, every fixed maximal ideal in A^v is real. Theorem 7 gives the other result.//

B. Piecewise Rational Functions

DEFINITION 3. The piecewise rational functions on R will be denoted by A. Let A be a subset of C(R) where f is in A if there exists a finite number of closed intervals $I_1, I_2, \ldots, I_n$ of R such that $f(x) = p_i(x) / q_i(x)$ for x in I_i, i = 1,2,3,...,n; $p_i(x)$ and $q_i(x)$ are polynomials with no common factors and with $q_i(x) \neq 0$ when x is in I_i; and $R = \bigcup\limits_{i=1}^{n} I_i$.

Without loss of generality we may assume $I_i \cap I_{i+1}$ is one number of R for i = 1,2,...,n-1; and if x is in I_i and y is in I_{i+1}, then $x \leq y$.

The set A will be shown to be a lattice-ordered subring of C(R) containing the polynomials. Some results on the prime ideals in A will be given and the structure space of A will be shown to be a two point compactification of R.

LEMMA 9.

(i) The set A contains the polynomials and the constant functions on R.

(ii) If f is in A, then df/dx exists at all but a finite number of points of R.

(iii) If f is in A, then $|f|$ is in A.

PROOF. Parts (i) and (ii) are obvious. To show (iii), let f be in A where $f(x) = p_i(x)/q_i(x)$ for x in I_i^f, i = 1,2,3,...,n. In each subinterval I_i^f there exists a finite number of points r in R such that the polynomial p_i satisfies $p_i(r) = 0$ and $p_i(x)$ is positive as well as negative in each neighborhood of r. Since $|f|$ is also continuous, we can construct $|f|$ so that it is in A.//

LEMMA 10. The set A is a lattice-ordered subring of C(R).

PROOF. Clearly A is a subring of C(R). Let f and g be in A where $f(x) = p_i(x)/q_i(x)$ for x in I_i^f, i = 1,2,...,n, and $g(x) = r_j(x)/s_j(x)$ for x in I_j^g, j = 1,2,...,m. In each non-empty subinterval of the form $I_j^g \cap I_i^f$ where f and g are not identical, there exists a finite number of points where f(x) = g(x). Then we can construct $f \vee g$ and $f \wedge g$ so that they are in A.//

The ring A does not contain the arctangent function on R and does not contain the exponential function. Hence A does not contain all of C*(R) and does not contain all of the functions in C(R) - C*(R). //

LEMMA 11. The ring A is a real function ring which has the pointwise inversion property.

PROOF. Since the identity function i is in A, A separates points of R. Clearly A satisfies the other conditions of a real function ring. We must show that $\underline{1}/f$ is in A whenever f is in A and $Z(f) = \emptyset$. If $Z(f) = \emptyset$ and $f(x) = p_i(x)/q_i(x)$ in a subinterval, then $p_i(x) \neq 0$ in this subinterval and $1/f(x) = q_i(x)/p_i(x)$ is well-defined and continuous in this subinterval. Since $\underline{1}/f$ is in C(R), the result follows.//

The following theorem is similar to [8, Theorem 37] originally proved by Hewitt for the ring C(X).

THEOREM 12. Let B be any subring of C(X). An ideal I in B is free if and only if for each compact set S there exists a function f in I having no zeros in S.

PROOF. Assume the ideal I is free. For each x in S there is a function f_x in I such that $f_x(x) \neq 0$. Cover each x in S by an open set of the form $X - Z(f_x)$. From the compactness of S, pick a finite subcover. This finite subcover is generated by a finite number of functions, call them $f_{x_1}, f_{x_2}, \ldots, f_{x_n}$. Then $\sum_{i=1}^{n} f_{x_i}^2$ is in I and is greater than zero on S.

To show the if part we note that $\{x\}$ is a compact subset of X. By hypothesis there exists a function f_x in I such that $f_x(x) \neq 0$ for each x in X. Hence I is a free ideal.//

COROLLARY 13.   If M is a free ideal in any subring
B of C(X) such that B has the pointwise inversion property,
then Z(f) is not compact for each f in M.

PROOF.   If Z(f) is compact, Theorem 12 tells us that
there is a g in M such that g is not zero in Z(f).   Then
$g^2 + f^2$ is in M and $Z(g^2 + f^2) = \emptyset$.   Since B has the
pointwise inversion property, we have a contradiction.//

LEMMA 14.   [5, Corollary 0.17].   Let B be a commutative
ring with a unity element.   Let I be an ideal in B with a
an element in B.   If no power of a belongs to I, then there
exists a prime ideal containing I but not a.

LEMMA 15.   There exist prime ideals in A which are not
maximal ideals.

PROOF.   Let $O_0^A = \{f$ in A: Z(f) is a neighborhood of 0$\}$.
Let i be the identity function on R.   No power of i is in
O_0^A.   From Lemma 14, there is a prime ideal P containing
O_0^A but not i.   One such prime ideal is the following:
P = $\{f$ in A: Z(f) contains [0,r] for some positive number r$\}$.
Clearly P properly contains O_0^A and is properly contained
in M_0^A.//

THEOREM 16.   Every prime ideal P in A is absolutely
convex.

PROOF. Let $|f| \leq |g|$ where g is in P and f is in A. Let $h(x) = f^2(x)/g(x)$ for x not in $Z(g)$ and $h(x) = 0$ for x in $Z(g)$. Then h can be shown continuous using the procedure in the proof of Corollary 62 in Section IIID.

Let I_1^g, I_2^g, ..., I_n^g be the subintervals of g. The zeros of g are either isolated in I_i^g or $g[I_i^g] = \{0\}$. Let I_1, I_2, ..., I_m be the subintervals for f^2. Divide R into subintervals with endpoints at the endpoints of I_i^g and I_j where $i = 1,2,...,n$ and $j = 1,2,...,m$, and also at the isolated zeros of g. There are a finite number of subintervals with the above endpoints since there are only a finite number of isolated zeros of g. Clearly h can be written as a rational function in each sub-interval. Since h is continuous, h is in A. If h is in A and g is in P, then hg is in P. Since $hg = f^2$, f^2 is in P. Since P is a prime ideal in A, f is in P.//

COROLLARY 17. The quotient ring A/P is totally ordered for each prime ideal P in A.

PROOF. This easily follows from Theorem 66 in Section IIID.//

EXAMPLE 1. The ideal O_0^A was defined in the proof of Lemma 15. This ideal satisfies condition II with respect to A, is not prime, is absolutely convex, and A/O_0^A is not totally ordered. The last result can be shown by showing that $O_0^A(i)$ and $O_0^A(-i)$ are not comparable with $O_0^A(\underline{0})$.

EXAMPLE 2. We consider ideals in A similar to those of Example 9 in Section IIID. Let $M_0^A = \{f$ in A: $f(0) = 0\}$ and $I_n^A = \{g$ in A: $g(x)/x^n$ is bounded on a deleted neighborhood of $0\}$. If f is in M_0^A, then $f(x) = xp_i(x)/q_i(x)$ in some interval(s) I_i such that 0 is in I_i. Since $f(x)/x$ is bounded for x in $I_i - \{0\}$, f is in I_1^A. Hence $M_0^A \subseteq I_1^A$. Also $I_1^A \subset M_0^A$. Thus $I_1^A = M_0^A$; therefore I_1^A is a prime ideal and an absolutely convex ideal. Also $I_2^A \subset I_1^A$ and I_n^A is not a prime ideal in A for $n > 1$. Let $I^A = \bigcap_n I_n^A$. In this case $O_0^A = I^A$.

DEFINITION 4. The support of a function f is the Cl $(X - Z(f))$. Let $C_K(X)$ denote the set of all functions in C(X) with compact support. Let A_K denote the set of functions in A with compact support.

LEMMA 18.

(i) The function f is in A_K if and only if there is a number a such that Z(f) contains $(-\infty,-a] \cup [a,\infty)$.

(ii) The set A_k is a free ideal in A.

PROOF. Part (i) easily follows from definitions. To show (ii) we use the fact that a closed subspace of a compact space is compact. Clearly $\underline{0}$ is in A_K. If f is in A_K, then $(-f)$ is in A_K, since the support of $(-f)$ is equal to the support of f. Let f and g be in A_K. Since Z(f) is a subset of Z(fg), the support of (fg) is a subset of the support of f. Thus the support of fg is compact and

fg is in A_K. Since $Z(f + g)$ contains $Z(f) \cap Z(g)$, the support of $(f + g)$ is contained in the compact set which is the union of the support of f and the support of g. Thus $f + g$ is in A_K. The other subring properties are easily shown, hence A_K is a subring of A. From (i), A_K is easily shown to be an ideal or equal to A. Since $\underline{1}$ is in $A - A_K$, A_K is an ideal. For each r in R, f_r is in A_K where $f_r(x) = ((|x - r| - 1) \wedge 0)$. Since $f_r(r) \neq 0$, A_K is a free ideal.//

THEOREM 19. The ideal A_K is in every free maximal ideal in A.

PROOF. Suppose M is a free maximal ideal in A and g is in $A_K - M$. From Lemma 18 (i), there is a real number a such that $Z(g)$ contains $(-\infty, -a] \cup [a, \infty)$. Since M is a maximal ideal with g not in M, $\underline{1}$ is in (M, g). There exist functions f in A and m in M such that $\underline{1} = fg + m$. Then $m(x) = 1$ for x in $(-\infty, -a] \cup [a, \infty)$. Thus $Z(m)$ is compact, but m is in M. From Corollary 13, this is not possible. Hence g is in M for each g in A_K.//

DEFINITION 5. The following notation will be used:
$M^+ = \{h$ in A: $Z(h)$ contains $[a, \infty)$ for some a in R$\}$,
$M^- = \{h$ in A: $Z(h)$ contains $(-\infty, -a]$ for some a in R$\}$.

LEMMA 20. The sets M^+ and M^- are distinct, free, prime ideals in A.

PROOF. The sets M^+ and M^- are clearly ideals containing A_K and are free ideals since A_K is a free ideal. Since $i \vee \underline{0}$ is in $M^- - M^+$ and $(-i) \vee \underline{0}$ is in $M^+ - M^-$, the ideals are distinct. To show M^+ is a prime ideal in A, suppose fg is in M^+. Then $Z(fg)$ contains $[a, \infty)$ for some a in R. If f is not in M^+, then there is an interval $[b, \infty)$ such that $Z(f) \cap [b, \infty) = \emptyset$. Then $Z(g)$ contains $[b, \infty) \cap [a, \infty)$; hence g is in M^+. This type of argument shows M^+ is a prime ideal. In a similar manner we can show that M^- is a prime ideal in A.//

LEMMA 21. The ideals M^+ and M^- are free maximal ideals in A.

PROOF. From Lemma 20, they are free ideals. To show that M^+ is maximal let h be in $A - M^+$. Since M^+ is a prime ideal in A, h^2 is in $A - M^+$ if h is in $A - M^+$. From Lemma 18 (i), $Z(h^2) \cap [r, \infty) = \emptyset$ for some r in R. Define m as follows:

$$m(x) = \begin{cases} 1 \text{ if } x \leq r \\ (-1)(x - (r + 1)) \text{ if } r \leq x \leq r + 1 \\ 0 \text{ if } x \geq r + 1. \end{cases}$$

Then m is in M^+ and $Z(m + h^2) = \emptyset$. Thus $m + h^2$ is a unit in (M^+, h). The result follows.//

THEOREM 22. The ideals M^+ and M^- are the only free maximal ideals in A.

PROOF. From Lemma 21, M^+ and M^- are maximal ideals in A. Let M be a free maximal ideal which is not equal to M^+ or M^-. If f is in $M - M^+$, then, from Corollary 13, f is in M^-. If g is in $M - M^-$, then g is in M^+. Since M is not contained in M^+ or M^-, the functions f and g exist. Then $f^2 + g^2$ is in M, but $Z(f^2 + g^2)$ is compact. This is a contradiction from Corollary 13. The result follows.//

We now give two definitions and one theorem for a general commutative ring B with unity. These results will later be applied to the ring A.

DEFINITION 6. Let S be the set of all maximal ideals in B. Let S be a subset of S. The closure of S in S is defined by $Cl(S) = \{M \text{ in } : M \supseteq \cap S\}$ where $\cap S = \bigcap_{M' \varepsilon S} M'$. This defines a topology on S called the Stone (or hull-kernel) topology. The set S with the Stone topology is called the structure space of B.

The Stone topology was first defined by Stone [27, Theorem 1]. Gelfand and Kolmogoroff in [3] applied the Stone topology to C(X) and C*(X) and used the structure space as a model for βX. Theorem 23 will give some of the properties of this structure space.

THEOREM 23.   [5, Problem 7M] and [3]   Let B be a

commutative ring with unity. The space S will refer to

the set of maximal ideals in B with the Stone topology.

(i)   The space S is a T_1 space.

(ii) The space S is a Hausdorff space if and only if,

for each pair of distinct maximal ideals M and M', there

exist a and a' in B such that a is not in M, a' is not

in M', and aa' is in $\bigcap S$.

(iii)   If S is a Hausdorff space, then it is compact.

(iv)   Any ring which is isomorphic to C(Y) has βY as its

structure space; hence $\beta X_1 = \beta X$ is the structure space of

C' used in theorems following Theorem **49** of Section IIIC.

Some previously shown results will be proved using

the Stone-Čech compactification along with results from

[17].   From [17], M_A denotes the collection of maximal

ideals in A with the Stone topology. For p in βX, $M_A^p =$

{f in A: (fg)*(p) = 0 for all g in A} where (fg)* refers

to the Stone extension of fg.   Let A be the subring of

piecewise rational functions. From Lemma 1, M_A^p is a

fixed maximal ideal for p in X.   For p in βX, M_A^p is a

prime ideal in A.   Let $G_A = \{M_A^p$: p is in $\beta X\}$.   The subring

A is a subalgebra of C(R).

THEOREM 24.   [17, Proposition 2.7] The following are equivalent for a subalgebra A of C(X).

(i)   The subalgebra A is closed under bounded pointwise inversion.

(ii)   If I is an ideal in A, then $\bigcap\limits_{f \in I} Z(f^*) \neq \emptyset$.

(iii)   Every ideal in A is contained in some M_A^p.

(iv)   The set M_A is a subset of G_A.

(v)   Every M in M_A is absolutely convex.

THEOREM 25.   See Theorem 19.   The ideal A_K is contained in every free maximal ideal.

PROOF.   The ring A is a real function ring.   From Lemma 1, $\{M_A^p: p \text{ in } R\}$ contains all of the fixed maximal ideals in A.   From Theorem 24 (iv), every free maximal ideal has the form M_A^p.   Since M_A^p is fixed for p in R, the free maximal ideals have the form M_A^p for p in $\beta R - R$. Since A_K is a free ideal, A contains free maximal ideals. If f is in A_K, then f is in M_A^p for each p in $\beta R - R$.   The result follows.//

THEOREM 26.   See Theorem 22.   The ideals M^+ and M^- are the only free maximal ideals in A.

PROOF.   Suppose there is a function f in some free maximal ideal in A such that f is not in A_K.   Then there is an interval $[a, \infty)$ or $(-\infty, -a]$ for some a in R such that f is non-zero on this interval.   Suppose f is not zero on $[a, \infty)$.   Let R^+ denote the subspace of non-negative

real numbers and R^- denote the subspace of non-positive real numbers. From [5, p. 92], $\beta R - R = (\beta R^+ - R^+) \cup (\beta R^- - R^-)$. Consider p in $\beta R^+ - R^+$. If f is in M_A^p, then $f^*(p) = 0$. If there is no interval $[a,\infty)$ such that $f[[a,\infty)] = \{0\}$, then there is a real number b such that $f(x) \neq 0$ for x in $[b,\infty)$. If $g(x) = 1/f(x)$ for x in $[b,\infty)$ and $g(x) = 1/f(b)$ for x in $[-\infty,b)$, then $(gf)^*(p) = 1$. Thus if f is in M_A^p for p in $\beta R^+ - R^+$, f must be 0 on some interval $[a,\infty)$. Similar results can be obtained for functions in M_A^p with p in $\beta R^- - R^-$. Let p and p' be in $\beta R^+ - R^+$. Then f is in M_A^p if and only if f is in $M_A^{p'}$. A similar result can be stated for p and p' in $\beta R^- - R^-$. The only possibilities for free maximal ideals are M^+ and M^- of Definition 5. Lemma 21 gives the desired result.//

We now check the structure space of A. The fixed maximal ideals have the form $M_A^x = M_x^A$ for x in R.

THEOREM 27. [20] Let X be a completely regular, Hausdorff space and let (X,B) be a real function ring. If $Z[B]$ is a subbase for the closed sets of X, then $M_{[X]}^B$ with the Stone topology is homeomorphic to X.

PROOF. The proof is similar to the proof given by Gelfand and Kolmogoroff [3] for the case where X is a compact Hausdorff space and B = C(X). The proof is also similar to the one given in [21, Theorem 2.3]. From Lemma 1, the fixed maximal ideals are M_x^B for x in X and the mapping

$(x \mapsto M_x^B):$ $X \to M_{[X]}^B$ is one-to-one. We show this mapping is a homeomorphism. From Definition 6, if $M_{[S]}^B$ is a subset of $M_{[X]}^B$, then Cl $(M_{[S]}^B)$ = { M_x^B: M_x^B contains $\cap M_{[S]}^B$ }. Suppose S is a non-empty subset of X and let x be in Cl (S). If f is in $\cap M_{[X]}^B$, then f(s) = 0 for s in S. This means S is a subset of Z(f). Since Z(f) is closed by hypothesis, Cl (S) is a subset of Z(f) and f(x) = 0. Therefore f is in M_x^B. We have shown that $\cap M_{[S]}^B$ is a subset of M_x^B if x is in Cl (S); then from the definition of closure in the Stone topology, M_x^B is in Cl $(M_{[S]}^B)$. Now we start with M_x^B in Cl $(M_{[S]}^B)$ which means M_x^B contains $\cap M_{[S]}^B$. If f is in B such that Z(f) contains S, then f is in $\cap M_{[S]}^B$ and thus f(x) = 0. We know Z[B] is a subbase for the closed sets of X. Since no function in B has a zero set containing S but not containing x under the conditions M_x^B is in Cl $(M_{[S]}^B)$, x must be in Cl (S). Thus the mapping is a homeomorphism.//

COROLLARY 28. The subspace $M_{[R]}^A$ is homeomorphic to ·R where A is the ring of piecewise continuous functions on R.

PROOF. The space R is a completely regular, Hausdorff space and (R,A) is a real function ring. Since Z[A] contains closed sets of the form $(-\infty,-a]$ and $[a,\infty)$ for all a in R, Z[A] is a subbase for the closed sets of R. Using Theorem 27, the set of fixed maximal ideals with the Stone topology is homeomorphic to R.//

THEOREM 29. Let S be the structure space of A. Then S is a two point compactification of R.

PROOF. It is easy to find functions satsifying the condition of Theorem 23 (ii); hence S is Hausdorff. From Theorem 23 (iii), S is compact. From Corollary 28, the subspace of fixed maximal ideals in S is homeomorphic to R. The subspace of fixed maximal ideals in S is dense in S. Since there are only two free maximal ideals in S, S is a two point compactification of R.//

We now consider whether any maximal ideals are principal.

LEMMA 30. Let H be a subring of R^X which is closed under pointwise inversion. If I is a principal ideal in H, then I is fixed.

PROOF. Let $I = (f) = \{gf:$ g is in A$\}$. Since H is closed under pointwise inversion, $Z(f) = \emptyset$ is not possible. Thus $\bigcap_{h \in I} Z(h)$ contains $Z(f)$. Hence I is a fixed ideal in H.//

LEMMA 31. No maximal ideals in A are principal.

PROOF. From Lemma 30, no free maximal ideal in A is principal. Every fixed maximal ideal has the form M_x^A for x in R. Suppose f is a generator for M_0^A. Then Z(f) must be {0} since i is in M_0^A. Thus $f(x) = xp_1(x)/q_1(x)$ for x in $I_1 = [-a,0]$ and $f(x) = xp_2(x)/q_2(x)$ for x in $I_2 = [0,a]$ for some positive number a. Since $|i|$ is in M_0^A, we obtain $-\lim_{x\to 0^-} q_1(x)/p_1(x) = \lim_{x\to 0^+} q_2(x)/p_2(x)$. The functions q_1, p_1, q_2, and p_2 are polynomials, continuous and non-zero at x = 0. We obtain $q_1(0)/p_1(0) = -q_2(0)/p_2(0)$. For i in M_0^A we obtain $q_1(0)/p_1(0) = q_2(0)/p_2(0)$. This contradiction establishes the result for M_0^A. The same procedure shows that M_x^A is not a principal ideal for any x in R.//

C. Special Subrings Containing $C_K(X)$

Let A be a subring of C(X) and let A_K denote the set of functions of A with compact support, i.e., f is in A_K if Cl (X - Z(f)) is compact. We often denote the support of f by Supp (f). The set $C_K(X)$, defined in Definition 4, has been extensively studied. Results involving $C_K(X)$ are given by Kohls in [10] and [11], by Shanks in [25], by Gillman and Jerison in [5], and by Pursell in [21]. Riordan in [23] worked with functions in C(X) with pseudocompact support.

In this section several special subrings containing A_K are considered. Emphasis is given to the structure space of subrings of the form $C_K(X) + P(X)$ where $P(X)$ is a subring of $C(X)$ with $C_K(X) \cap P(X) = \{\underline{0}\}$. The space $X = R$ is used for some of the subrings. In some cases $C_K(X)$ is the only free maximal ideal in $C_K(X) + P(X)$; in other cases, C_K is a proper subset of the free maximal ideals. The results through Theorem 37 are related to the properties of $C_K(X)$ given in [5, Problem 4D].

LEMMA 32. If X is compact, then $A_K = A$.

PROOF. The support of f is a closed subspace of the compact space X, and hence is compact for each f in A.//

LEMMA 33. Let A be a subring of $C(X)$ containing $\underline{1}$. If X is not compact, then A_K is an ideal in A.

PROOF. The proof follows the proof of Lemma 18 (ii).//

THEOREM 34. Let A be a subring of $C(X)$ containing $C^*(X)$. Then A_K is a free ideal in A if and only if X is locally compact but not compact.

PROOF. Assume A_K is a free ideal in A. If X is compact, then $A_K = A$ from Lemma 32. This is not possible; hence X is not compact. We now show that X is locally compact. Since A_K is a free ideal, for each y in X there is a function f_y in A_K such that $f_y(y) \neq 0$. There exists

an open neighborhood N_y containing y such that 0 is not in $f_y[N_y]$. Then Cl (N_y) is a compact neighborhood of y since it is a closed subspace of Supp (f_y). Thus X is locally compact.

Assume X is locally compact but not compact. From Lemma 33, A_K is an ideal in A. Each y in X has an open neighborhood N_y whose closure is compact. Since X is completely regular, there exists a function f_y in C* such that $f_y(y) = 1$ and $f_y[X - N_y] = \{0\}$. Supp (f_y) is compact since it is in Cl (N_y). Thus f_y is in A_K. Hence A_K is a free ideal in A.//

In Section IV B we worked with the subring A of piecewise rational functions on R. This subring did not contain C*(R) and A_K was a free ideal contained in every free maximal ideal in A.

We denote the interior of a set S by int (S) or int S.

LEMMA 35. [5, Problem 1.D.1] Let f and g be in C(X). If Z(f) is a neighborhood of Z(g) and X - int Z(f) is compact, then there exists a function h in C*(X) such that f = hg.

LEMMA 36. Let f be in C(X). Then Cl $(X - Z(f))$ = X - int Z(f).

PROOF. This follows easily from definitions.//

THEOREM 37. Let A be a subring of C(X) containing C*(X). Then A_K is contained in every free ideal in A.

PROOF. Let I be a free ideal in A. Let f be in A_K and let S be the support of f. From Theorem 12, there is a function g in I such that g(x) $\neq$ 0 for each x in S. Then Z(g) $\subseteq$ X - S $\subseteq$ Z(f) and Z(f) is a neighborhood of Z(g). From Lemma 36, X - int Z(f) = S. From Lemma 35, there is a function h in C*(X) such that f = hg. Thus f is in I. This procedure works for any function in A_K, hence A_K is a subset of I.//

1. C_K(X) Plus Constants

In Pursell's paper [21] results are obtained for the ring C_K(X) + R(X) where R(X) is the ring of real constants on X. The main results are stated in the following theorems. We will obtain similar results for subrings containing C_K(R) and C_K(X).

THEOREM 38. [21, Theorem 2.1] Let T be a commutative ring with a unity 1 containing a field K and an ideal M such that 1 is in K and T = K + M. Then:
(i) The ideal M is a maximal ideal in T.
(ii) The ring T is the direct sum of K and M, i.e., every t in T has a representation t = γ(t) + δ(t) where γ(t) is a unique element of K and δ(t) is a unique element of M.

(iii) The mapping γ is a homomorphism from T onto K
having M as its kernel. Hence T/M is isomorphic to K.

PROOF.

(i) Suppose M is not a maximal ideal in T. Let M' be a
maximal ideal containing M. If h is in M' - M, then
h = k' + m where $0 \neq k'$, k' is in K, and m is in M. Hence
h - m = k' is in M'. Then $(k')^{-1} k' = 1$ is in M'. This
implies M' = T. This contradiction establishes the result.

(ii) Since M is an ideal in T, M cannot contain any
element of K except 0. Hence M $\cap$ K = 0. By definition,
every t in T can be written as t = γ(t) + δ(t) where γ(t)
is in K and δ(t) is in M. Suppose t = γ'(t) + δ'(t)
where γ'(t) is in K and δ'(t) is in M. Then γ(t) - γ'(t)
= δ'(t) - δ(t) where γ(t) - γ'(t) is in K and δ'(t) - δ(t)
is in M. Since K $\cap$ M = 0, δ'(t) = δ(t) and γ(t) = γ'(t).
The result follows .

(iii) To show γ is a homomorphism onto K, let x and y
be any elements of T. Then

$$x + y = (\gamma(x) + \delta(x)) + (\gamma(y) + \delta(y)) = (\gamma(x) + \gamma(y))$$

$$+ (\delta(x) + \delta(y)).$$

Since γ(x) + γ(y) is in K and δ(x) + δ(y) is in M, it
follows from (ii) that γ(x + y) = γ(x) + γ(y). Also
x $\cdot$ y = (γ(x) + δ(x)) $\cdot$ (γ(y) + δ(y)) = γ(x) $\cdot$ γ(y) + γ(x)
$\cdot$ δ(y) + δ(x) $\cdot$ γ(y) + δ(x) $\cdot$ δ(y). Since γ(x) $\cdot$ γ(y) is
in K and the sum of the last three terms is in the ideal M,

(ii) implies $\gamma(x \cdot y) = \gamma(x) \cdot \gamma(y)$. Clearly γ is onto K.//

THEOREM 39. [21, Theorem 2.2] Let F be a ring of real functions on a Hausdorff space X such that

(i) F contains R(X),

(ii) F is closed under pointwise inversion, and

(iii) zero sets are closed.

If a proper ideal I in F contains a function f such that Z(f) is compact, then I is fixed.

PROOF. The proof is similar to the proof of Theorem 12.//

THEOREM 40. [21, Theorem 2.3] Let F be a ring of real functions on a regular, Hausdorff space X such that

(i) F contains R(X),

(ii) zero sets are closed, and

(iii) F separates points from closed sets, i.e., if S is any closed subset of X and x is in X - S, there is a function f in F such that $f(x) \neq 0$ and S is a subset of Z(f). Then $M_{[X]}^{F}$, the set of all fixed maximal ideals in F with the relative Stone topology, is homeomorphic to X.

PROOF. The method of proof is similar to the method of proof of Theorem 27.//

Let X be a locally compact, non-compact, Hausdorff space. The ring $C_K(X) + R(X)$ is closed under pointwise inversion and zero sets of $C_K(X) + R(X)$ are closed.

LEMMA 41. [21, Section 3] The set $C_K(X)$ is a real maximal ideal in $C_K(X) + R(X)$.

PROOF. This easily follows from definitions and Theorem 38.//

LEMMA 42. [21, Section 3] Every f in $C_K(X) + R(X)$ has a unique representation $f = \gamma(f) + \delta(f)$ where $\gamma(f)$ is a constant and $\delta(f)$ is a continuous function with compact support.

PROOF. This follows directly from definitions and Theorem 38.//

LEMMA 43. [21, Section 3] For f in $C_K(X) + R(X)$, Supp (f) is compact and Z(f) is not compact if and only if $\gamma(f) = \underline{0}$; Z(f) is compact and Supp (f) is not compact if and only if $\gamma(f) \neq \underline{0}$.

PROOF. Since X is not compact and X = Supp (f) $\cup$ Z(f), Supp (f) and Z(f) cannot both be compact. If $\gamma(f) = \underline{0}$, then Supp (f) is compact. If $\gamma(f) \neq \underline{0}$, then Z(f) is compact since Z(f) is contained in the compact set Supp $(\delta(f))$. The results follow.//

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THEOREM 44. [21, Section 3] The ideal $C_K(X)$ is a free maximal ideal. Every free ideal in $C_K(X) + R(X)$ is contained in $C_K(X)$. Hence $C_K(X)$ is the only free maximal ideal in $C_K(X) + R(X)$.

PROOF. From Lemmas 33 and 41, $C_K(X)$ is a free maximal ideal. From Theorem 39 and Lemma 43, $\gamma(f) = \underline{0}$ for all functions f in a free ideal in $C_K(X) + R(X)$, and hence every free ideal in $C_K(X) + R(X)$ is contained in $C_K(X)$.//

THEOREM 45. [21, Section 3] The space $M_{[X]}$ of all fixed maximal ideals in $C_K(X) + R(X)$ with the relative Stone topology is homeomorphic to X.

PROOF. Since X is locally compact and Hausdorff, it is completely regular. Therefore $C_K(X) + R(X)$ separates points. The result follows from Theorem 40.//

THEOREM 46. [21, Theorem 3.8] If X is a locally compact, non-compact, Hausdorff space, then the space X_K^* of all maximal ideals in $C_K(X) + R(X)$ with the Stone topology is the one point compactification of X.

PROOF. Theorem 23 is used. For two distinct, fixed maximal ideals M_x and M_y in $C_K(X) + R(X)$, we can construct functions f and g in $C_K(X) + R(X)$ such that $f(x) \neq 0$, $g(y) \neq 0$, and $fg = \underline{0}$. From the proof of Theorem 40, the fixed maximal ideals in $C_K(X) + R(X)$ are $M_x = $

{f in $C_K(X)$ + R(X): f(x) = 0}. Thus the fixed maximal ideals satisfy Theorem 23 (ii). Now consider a fixed maximal ideal M_x and the free maximal ideal $C_K(X)$. Let U be an open neighborhood of x with compact closure. There exists a function h in $C_K(X)$ + R(X) such that h(x) = 1 and h[X - U] = {0}. Define V = {y in X: h(y) > 3/4} and V' = {y in X: h(y) < 1/4}. Then V and V' are disjoint open sets in the compact, Hausdorff, hence normal, space Cl (U). From Urysohn's lemma there is a real continuous function ϕ defined on Cl (U) such that ϕ[V] = {0} and ϕ[V'] = {1}. If we define ϕ(y) = g(y) for y in U and g(y) = 1 for y in X - U, then g is in $C_K(X)$ + R(X) but not in $C_K(X)$. There also exists a function f in $C_K(X)$ + R(X) such that f(x) = 1 and f[X - V] = {0}. Then f is not in M_x and fg = $\underline{0}$. Hence X_K^* is compact. From Theorem 45, the mapping that takes x to M_x imbeds X into X_K^*. From Theorem 44, $X_K^* - M_{[X]}$ consists of a single ideal $C_K(X)$. From Theorem 23 (iii), X_K^* is a one point compactification of X. That X_K^* is "the' one point compactification follows from Alexandroff's theorem about the uniqueness of the one point compactification [1, p. 92].//

2. $C_K(R)$ Plus Polynomials

Let B be a commutative ring with B = P + M where P is a subring of a ring H, M is an ideal in H, and P $\cap$ M = {0}.

LEMMA 47. The ring B is the direct sum of P and M.

PROOF. The proof is similar to the proof of Theorem 38 (ii).//

LEMMA 48. Let a be in B where a = p + m with p in P and m in M. Let γ be a mapping from B into P defined by $\gamma(a)$ = p. Then γ is a homomorphism from B onto P.

PROOF. The proof is similar to the proof of Theorem 38 (iii).//

Under the conditions given for B, P, and M, M is not necessarily a maximal ideal in the ring B.

We now restrict ourselves to X = R, the space of real numbers, and consider subrings of the form $C_K(R)$ + A'. Results will be given where A' is the ring of polynomials, and later, where A' is the ring of piecewise rational functions. Let A be a subring of C(R) such that f is in A if f is eventually a polynomial, i.e., there exists a real number r and a polynomial p on R such that f(x)

$= p(x)$ for all $|x| \geq r$. Then A is the subring $C_K(R)$ + P(R) where $C_K(R)$ is the set of continuous functions on R with compact support and P(R) is the set of polynomials on R. We often denote $C_K(R)$ by C_K and P(R) by P. From Lemma 47, A is a direct sum of C_K and P.

LEMMA 49. The set C_K is an absolutely convex, free ideal in A.

PROOF. Clearly C_K is a free ideal in A. If $\underline{0} \leq |f| \leq |g|$ where g is in C_K and f is in A, then Z(f) contains Z(g). Therefore Cl (R - Z(f)) is a subset of Cl (R - Z(g)) and f is in C_K. Since $\underline{1}$ is in $A - C_K$, $C_K \neq A.//$

In [7] the ideal C_K is mentioned as an example of an ideal in $A(R^+)$ which is maximal in the set of absolutely convex ideals but is not a maximal ideal in A. The same situation applies for A as the next theorems show.

THEOREM 50. The ideal C_K is maximal in the class of absolutely convex ideals in A.

PROOF. From Lemma 49, C_K is an absolutely convex ideal. Suppose there is an absolutely convex ideal M such that $C_K \subset M$. Then there is a function f in $M - C_K$ and a real number r such that $f(x) \neq 0$ for $|x| \geq r$. Suppose $f(x) = k$ for $|x| \geq r$ where k is a non-zero constant. Let $g = \underline{k} - f$. Then g is in M and $\underline{k} = g + f$ is in M. Hence $\underline{1}$ is in M. This implies M = A. This contradiction shows that f must eventually be a non-constant polynomial

if C_K is not maximal in the class of absolutely convex

ideals. Suppose f is eventually a non-constant polynomial;

then $\lim\limits_{|x|\to\infty} |f(x)| = \infty$. Then there are positive numbers r'

and k such that $|f(x)| \geq k$ for $|x| \geq r'$. There exists a

function g' in A and a number $r_2 > r'$ such that $g'(x) =$

$|f(x)|$ for $|x| \leq r'$, $g' = |g'| \leq |f|$, and $g'(x) = k$ for

$|x| \geq r_2$. Then $\underline{0} \leq |g'| = g' \leq |f|$. Since f is in M

and M is absolutely convex, g' is in M. Let $h = \underline{k} - g'$.

Then h is in M and $h + g' = \underline{k}$ is in M. Thus $\underline{1}$ is in M and

this is a contradiction. The conclusion of the theorem

follows.//

LEMMA 51. The ideal C_K is not a maximal ideal in A.

PROOF. Let $i(x) = x$ for all x in R. If C_K is a

maximal ideal in A, then $(C_K, i) = A$. This means there are

functions g in A and m in C_K such that $\underline{1} = m + ig$. Since

m is in C_K, there exists a positive number r_m such that

$m(x) = 0$ for $|x| \geq r_m$. This implies $1 = xg(x)$ for

$|x| \geq r_m$. This is not possible for a function in A. Hence

C_K is not a maximal ideal in A.//

THEOREM 52. The ideal C_K is contained in every free

ideal in A.

PROOF. Suppose M is a free ideal in A with f in M - C_K.
Then Z(f) is contained in some compact set of the form
[- r, r] for a positive number r. From Theorem 12, there is
a function ℓ in M which is not 0 in [-r,r]. Let $\ell^2 + f^2 = h$.
Then h is a positive function in M. Let m be any function
in C_K where m(x) = 0 for $|x| \geq r_m > 0$. Let $s \geq r_m$ and
define $\ell_s(x)$ by,

$$\ell_s(x) = \begin{cases} 1/h(x) \text{ for x in } [-s,s] \\ \text{the constant } 1/h(s) \text{ for } x \geq s \text{ and} \\ \quad x \leq -s - 1 \\ \text{a linear function for } -s - 1 \leq x \leq -s. \end{cases}$$

Then ℓ_s is in A, and hence $\ell_s h$ is in M with $\ell_s h(x) = 1$
for x in [-s,s]. Since M is an ideal, $(\ell_s h)m = m$ is in M.
Since m was an arbitrary element of C_K, C_K is a subset of
M.//

From Lemma 1, the fixed maximal ideals in A have the
form $M_x^A = \{f$ in A: $f(x) = 0\}$ for a fixed x in R. The
maximal ideals in the ring of polynomials on R are
principal ideals generated by either f_r or $g_{s,t}$ where
$f_r(x) = x - r$ for r in R and $g_{s,t}(x) = x^2 + sx + t$
where s and t are real numbers with $s^2 - 4t < 0$.

THEOREM 53. Let f_r and $g_{s,t}$ be the functions defined
above. The free maximal ideals in A are the ideals

(C_K, f_r) and $C_K, g_{s,t})$. The mapping that takes the real number r to (C_k, f_r) is one-to-one and the mapping that takes the ordered pair of real numbers (s,t) to $(C_K, g_{s,t})$ is one-to-one if $s^2 - 4t < 0$.

PROOF. From Theorem 52, C_K is in every free maximal ideal in A. From Lemma 51, C_K is not a maximal ideal. The sets in the theorem are clearly free ideals if they are not equal to A. We show that (C_K, f_r) is a maximal ideal for a fixed r. If $(C_K, f_r) = A$, then $\underline{1} = m + f_r g$ for some m in C_K and some g in A. This implies $g(x) = 1/(x - r)$ for large $|x|$. This is not possible, so $(C_K, f_r) \neq A$. Similarly $(C_K, g_{s,t}) \neq A$. We need to show $((C_K, f_r), h) = A$ for all h in $A - (C_K, f_r)$. Let p_h be the "eventual" polynomial of h. Since h is not in (C_K, f_r), $x - r$ is not a factor of $p_h(x)$. Since (f_r) is a maximal ideal in P, there are polynomials p_2 and p_3 such that $\underline{1} = f_r p_2 + p_h p_3$. Let $m = \underline{1} - (p_2 f_r + p_3 h)$. Clearly m is in C_K and $\underline{1} = m + p_2 f_r + p_3 h$. Hence $\underline{1}$ is in $((C_K, f_r), h)$. This procedure works for any h not in (C_K, f_r), hence (C_K, f_r) is a maximal ideal for each r in R. Similarly $(C_K, g_{s,t})$ is a maximal ideal in A for each pair of numbers (s,t) satisfying $s^2 - 4t < 0$.

Now we need to show that there are no other free maximal ideals in A. Suppose M is a free maximal ideal which differs from each maximal ideal listed in the

theorem. For each r in R there is a function f in
$M - (C_K, f_r)$. Let p_f be the "eventual' polynomial of f
for a fixed r. Then p_f is a non-zero polynomial and is
not a multiple of f_r. Suppose f_{r_1}, f_{r_2}, ...,f_{r_n} are
factors of p_f. The following procedure also works with $g_{s,t}$
as factors of f. Since M is not completely contained in
any of the ideals (C_K, f_{r_i}), $i = 1,2,3,...,n$, there are
functions g_i in M such that p_{g_i} does not have f_{r_i} as a
factor. Then $h = \sum_{i=1}^{n} g_i^2$ is in M and p_h does not have
any f_{r_i}, $i = 1,2,3,...,n$, as a factor. Thus p_h and p_f
have no common non-constant factors. Therefore there are
polynomials p_1 and p_4 such that $p_1 p_h + p_4 p_f = \underline{1}$; hence
$\underline{1} - (p_1 h + p_4 f)$ is in $C_K \subseteq M$. We know $p_1 h + p_4 f$ is in M.
Hence $\underline{1}$ is in M. The results follow.//

We now consider the space of maximal ideals in A.
From Theorem 27, the structure space of fixed maximal
ideals in A is homeomorphic to R. Since R is a Hausdorff
space, the subspace of fixed maximal ideals in A is a
Hausdorff subspace. To determine whether the structure
space of A is Hausdorff we use Theorem 23 (ii). We have
$\bigcap S \subseteq \bigcap_{x \in R} M_x^A = \underline{0}$ where S is the structure space of
maximal ideals in A. Let (C_K, f_r) and (C_K, f_s) be two
distinct, free maximal ideals in A. If a is not in (C_K, f_s),
then Z(a) is a compact set, and if a' is not in (C_K, f_r),

then $Z(a')$ is a compact set. Then $Z(aa') = Z(a) \cup Z(a')$ is compact and not R. Hence $aa' \neq \underline{0} = \cap S$ and the structure space of A is not Hausdorff.

We further consider the subspace of free ideals in S. If a set S contains a countable number of free maximal ideals, then $\cap S = C_K$ and Cl (S) is the subspace of free maximal ideals in S. The subspace of free maximal ideals has the finite complement topology, i.e., a subspace is closed if and only if it is a finite set. This subspace is not Hausdorff. The subset of free maximal ideals in S can be identified with a "folded" complex plane by identifying (C_K, f_r) with the complex number $(r, 0)$ and $(C_K, g_{s,t})$ with $(-\frac{s}{2}, \frac{\sqrt{4t - s^2}}{2})$ where the last complex number comes from a solution of the quadratic equation $x^2 + sx + t = 0$.

We consider whether the maximal ideals in S are real. From Lemma 1, the fixed ideals have the form M_x^A. The fixed maximal ideals in A are real since $M_x^A (f) \equiv M_x^A(\underline{r})$ where $f(x) = r$. If (C_K, f_r) is a real ideal, then $(C_K, f_r)(g_{s,t}) \equiv (C_K, f_r)(\underline{r}')$ for some r' in R. Thus $g_{s,t} - \underline{r}'$ is in (C_K, f_r). Then for large $|x|$ there exists a function ℓ in A such that $x^2 + sx + t - r' = (x - r) \ell(x)$. Since ℓ must be a polynomial for large $|x|$, we can use $r' = r^2 + rs + t$. Then $(C_K, f_r)(g_{s,t}) \equiv (C_K, f_r)(\underline{r}')$. Also $(C_K, f_r)(f_s) \equiv (C_K, f_r)(\underline{r} - \underline{s})$. This

shows that (C_K, f_r) is a real ideal for each r in R. If $(C_K, g_{s,t})$ is a real ideal, then $(C_K, g_{s,t})(f_0) \equiv (C_K, g_{s,t})(\underline{r})$ for some r in R. For large $|x|$, $x - r = \ell(x)(x^2 + sx + t)$ for some polynomial $\ell(x)$. This is not possible, hence $(C_K, g_{s,t})$ is a hyper-real ideal.

We now consider whether $A/(C_K, f_r)$ or $A/(C_K, g_{s,t})$ are totally ordered. From Theorem 50, no maximal ideal is absolutely convex. All maximal ideals are prime ideals and no maximal ideal contains $\underline{1}$. A function m in C_K can be found such that $\underline{1} \leq m + f_r^2$ (or $\underline{1} \leq m + g_{s,t}^2$). If the maximal ideal is convex, then $\underline{1}$ would be in it. This contradiction shows that no free maximal ideal is convex and from Theorem 60 in Section III, the quotient rings are not partially ordered.

A subring is now considered that is closely related to the subring just studied. Let A* be the subring of C(R) such that f in A* implies there is a real number r and polynomials p_1 and p_2 on R where $f(x) = p_1(x)$ for $x \geq r$ and $f(x) = p_2(x)$ for $x \leq -r$. The notation M^+ and M^- is analogous to that in Definition 5.

LEMMA 54. The sets C_K, M^+ and M^- are ideals in A* with the following properties.

(i) $M^+ \cap M^- = C_K$. Hence $C_K \subseteq M^+$ and $C_K \subseteq M^-$.

(ii) $C_K \subset M^+$ and $C_K \subset M^-$. Hence C_K is not a maximal ideal in A*.

(iii) C_K, M^+, and M^- are absolutely convex, free ideals in A*.

PROOF. The results easily follow from definitions.//

THEOREM 55. The ideals M^+ and M^- are maximal in the class of absolutely convex ideals in A*.

PROOF. The proof is similar to the proof of Theorem 50.//

THEOREM 56. The ideal C_K is contained in every free ideal in A*.

PROOF. Let M be a free ideal in A* with f in $M - C_K$. Consider two cases.

Case 1. If Z(f) is a compact set, then the proof follows the procedure of the proof of Theorem 52.

Case 2. If Z(f) is not compact, then Z(f) is an unbounded set. Since f is eventually a "pair of polynomials", at least one of the polynomials has an unbounded set of zeros. The only polynomial with this property is the zero polynomial. Hence f is in M^+ or f is in M^-. Let m be in $C_K(X)$. Then m(x) = 0 for $|x| \geq r$ for some r in R. Suppose f is in M and f is in $M^+ - C_K$. Then $\{x$ in R: $x \leq r$ and f(x) = 0 $\}$ is compact. Using the method of proof of Theorem 52, a function h can be found in M such that h > $\underline{0}$ for $x \leq r$. Then there is a function ℓ in A* such that h(x)ℓ(x) = 1 for $|x| \leq r$. Hence (ℓ h)m = m is in M. In this way we can show $C_K \subseteq M$.//

THEOREM 57. The free maximal ideals in A^* are (M^+, f_r), (M^-, f_r), $(M^+, g_{s,t})$, and $(M^-, g_{s,t})$ where $f_r(x) = x - r$ and $g_{s,t}(x) = x^2 + sx + t$ for real numbers r, s, and t such that $s^2 - 4t < 0$. If r_1, s_1, and t_1, are real numbers such that $r \neq r_1$ and $(s,t) \neq (s_1, t_1)$ where $s_1^2 - 4t_1 < 0$, then $(M^+, f_r) \neq (M^+, f_{r_1})$, $(M^-, f_r) \neq (M^-, f_{r_1})$, $(M^+, g_{s,t}) \neq (M^+, g_{s_1, t_1})$, and $(M^-, g_{s,t}) \neq (M^-, g_{s_1, t_1})$.

PROOF. The sets mentioned in the theorem are clearly free ideals. The procedure of the proof of Theorem 53 can be used to show that these ideals are maximal. We must also show that there are no other free maximal ideals in A^*. Suppose M is a free maximal ideal in A^* different from those listed in the theorem. Let f be in $M - (M^+, f_r)$ for some fixed r. Since f^2 is not in M^+, there is a non-zero polynomial p_3 such that $f^2 = m_k + p_3$. Suppose p_3 has factors f_{r_1}, $f_{r_2}, \ldots, f_{r_n}$. Factors of $g_{s,t}$ work just as well. For each f_{r_i} there is a g_i in $M - (M^+, f_{r_i})$ such that p_{g_i} does not have f_{r_i} as a factor, $i = 1, 2, \ldots, n$.

Let $h = \sum_{i=1}^{n} g_i^2$. Then h is in M and $h = h_K + p_h$ is in $C_K + P$. Since p_h and p_3 have no common non-constant factors, there are polynomials p_1 and p_2 such that $p_1 p_3 + p_2 p_h = \underline{1}$. Let $\ell = (p_1 f^2 + p_2 h)^2$. Then $\ell(x) = 1$ for large x. Also ℓ is in M since $p_1 f^2$ and $p_2 h$ are in M. For any r in R there is

a function ℓ_r in $C_K \subseteq M$ such that $\ell(x) + \ell_r(x)$ is greater than $(1/2)$ for all $x \geq r$. A function ℓ_2' can be found in M^- such that $\ell_2'(x) = 1/[\ell(x) + \ell_r(x)]$ for $x \geq r$. Then $\ell_2'(\ell + \ell_r)$ is in M and has the value 1 for $x \geq r$. If m is in M^- with $m(x) = 0$ for $x \leq r$, then $\ell_2'(\ell + \ell_r)m = m$ is in M. We obtain $M^- \subseteq M$. Similarly we can obtain $M^+ \subseteq M$. There are functions ℓ_3' and ℓ_4' where ℓ_3' is in $M^- \subseteq M$, ℓ'_4 is in $M^+ \subseteq M$, and $\ell_3' + \ell_4' = \underline{1}$. This implies $\underline{1}$ is in M or $M = A^*$. This is not possible. The results follow.//

The structure space is similar to the subring just studied. The structure space of free maximal ideals can be identified with two "folded" complex planes. The structure space of free maximal ideals is not Hausdorff, for if a is in $(M^+, f_r) - (M^+, f_s)$ and a' is in $(M^+, f_s) - (M^+, f_r)$, then aa' cannot be zero for arbitrarily large x. Clearly $\underline{0} = \bigcap S^*$.

3. $C_K(R)$ Plus Rational Functions.

We now consider some subrings of rational functions in place of the ring of polynomials. Let A be the subring of $C(R)$ such that f is in A if f is eventually one rational function, i.e., there exists a number r and polynomials p and q such that $f(x) = p(x)/q(x)$ with $q(x) \neq 0$ for $|x| \geq r$. Then A can be considered as the sum of C_K and a subring of piecewise rational functions where the first and last rational functions are the same.

LEMMA 58. The ring A has the property of pointwise inversion.

PROOF. We must show $Z(f) = \emptyset$ if and only if f is a unit of A. The result follows since $C(R)$ and the rational functions have this property.//

LEMMA 59. The set C_K is a hyper-real, free maximal ideal in A.

PROOF. Clearly C_K is a free ideal in A. To show that C_K is a maximal ideal we show that $\underline{1}$ is in (C_K, f) for each f in $A - C_K$. If f is in $A - C_K$, then there is a number r such that $f(x) \neq 0$ for $|x| \geq r$. Let $g(x) = 1/f(x)$ for $|x| \geq r$ and let $g(x)$ be linear for $|x| \leq r$. Then g is in A. Let $m(x) = 1 - f(x)g(x)$. Then $m(x) = 0$ for $|x| \geq r$; hence m is in C_K. Thus $\underline{1} = m + fg$. If $C_K(i) \equiv C_K(\underline{r})$ for some r in R, then $i - \underline{r}$ must be in C_K. This is not possible. The results follow.//

THEOREM 60. The ideal C_K is the only free maximal ideal in A.

PROOF. As in Corollary 13, if M is a free maximal ideal in A containing f, then $Z(f)$ is not compact. Every rational function which is not $\underline{0}$ has its zeros in a compact set. Hence f in M implies f is eventually $\underline{0}$.

Then f is in C_K. Then M is a subset of C_K. This and Lemma 59 give the desired result.//

From Lemma 1, the fixed maximal ideals have the form $M_x = \{f$ in A: $f(x) = 0\}$ for a fixed x in R. We now consider the structure space S of maximal ideals in A.

THEOREM 61. The structure space of A is a Hausdorff space.

PROOF. Theorem 23 (ii) is used. Consider two distinct, fixed maximal ideals M_r and M_s where r < s. There is a function f not in M_s such that R - Z(f) = $(s - \frac{s-r}{2}, s + \frac{s-r}{2})$. There is a function g not in M_r such that R - Z(g) has s - (s - r)/2 as an upper bound. Then fg = $\underline{0}$ is in $\bigcap S$. Now consider the free maximal ideal C_K and a fixed maximal ideal M_r. There exist functions f and g such that f is not in M_r, R - Z(f) = (r - 1, r + 1), g is not in C_K, and Z(g) contains [r - 1, r + 1]. Then fg = $\underline{0}$ is in $\bigcap S$. Thus the structure space is Hausdorff.//

THEOREM 62. The structure space of A is the one point compactification of R.

PROOF. The space S is compact from Theorem 23 (iii). The subspace of fixed maximal ideals in S is homeomorphic to R from Theorem 27. The subspace of fixed maximal ideals is dense in S. The result follows.//

Let A* be the subring of C(R) where f is in A* if
there are numbers r and s such that f is a rational
function for $x \geq r$ and f is a (possibly different) rational
function for $x \leq s$. Then A* can be considered as the
sum of $C_K(R)$ with the subring of piecewise rational functions.
We consider the notation M^+ and M^- analogous to that in
Definition 5.

LEMMA 63. The set C_K equals $M^+ \cap M^-$ and is properly
contained in M^+ and M^-.

PROOF. This lemma follows from definitions and the
observation that $(i \vee \underline{0})$ is in $M^- - C_K$ and $((-i) \vee \underline{0})$
is in $M^+ - C_K$.//

THEOREM 64. The sets M^+ and M^- are the only free
maximal ideals in A*.

PROOF. Clearly M^+ and M^- are free ideals. To show
that M^+ is a maximal ideal, consider f in $A^* - M^+$. Then
there is a number r such that $f(x) \neq 0$ for $x \geq r$. We
must show $\underline{1}$ is in (M^+, f). Define g by $g(x) = 1/f(x)$
for $x \geq r$ and $g(x) = 1/f(r)$ for $x \leq r$. Then g is in A*.
Let $m^+ = \underline{1} - fg$. Since $m^+(x) = 0$ for $x \geq r$, m^+ is in
M^+. Then $\underline{1} = m^+ + fg$, hence M^+ is a free maximal ideal in
A*. In a similar manner M^- can be shown to be a free
maximal ideal in A*.

To show that M^+ and M^- are the only free maximal ideals in $A*$, let M be a free maximal ideal in $A*$. Since $A*$ has the pointwise inversion property, Corollary 13 implies $Z(k)$ is not compact for each k in M. This means k has an unbounded set of zeros. The only rational function with this property is the zero function. Thus k is in M^+ or M^-. Since M cannot properly contain M^+ or M^-, and M^+ and M^- cannot properly contain M, there exist functions ℓ in $M - M^+$ and m in $M - M^-$. The function $h = m^2 + \ell^2$ is in M and $Z(h)$ is compact. This contradicts Corollary 13. Hence M^+ and M^- are the only free maximal ideals in $A*$.//

We now consider the structure space $S*$ of maximal ideals in $A*$. In this case $\bigcap S* = \underline{0}$. The results are similar to the results involving the subring with one eventual rational function.

THEOREM 65. The structure space of $A*$ is a Hausdorff space.

PROOF. Theorem 23 (ii) is used. From Theorem 27, R is homeomorphic to the subspace of fixed maximal ideals in $S*$, and hence this subspace is Hausdorff. Using the method of proof of Theorem 61, functions whose product is $\underline{0}$ can be found satisfying the condition in Theorem 23(ii). Hence $S*$ is a Hausdorff space.//

THEOREM 66. The structure space of A* is a two point compactification of R.

PROOF. The space $S*$ is compact from Theorem 23 (iii). The subspace of fixed maximal ideals is homeomorphic to R from Theorem 27. The subspace of fixed maximal ideals is dense in $S*$. Hence $S*$ is a two point compactification of R.//

LEMMA 67. The free maximal ideals in A* are hyper-real.

PROOF. If M^+ is a real ideal in A*, then $M^+(i) \equiv M^+(\underline{r})$ for some r in R. This means $i - \underline{r}$ is in M^+ for some real number r. This is not possible. Thus M^+ is not a real ideal. Similarly M^- is not a real ideal in A*.//

We now consider the subring A** of C(R) such that f is in A** if there are real numbers r and s where f is a polynomial for $x \leq r$ and f is a rational function for $x \geq s$. The sets C_K, M^+, and M^- will denote sets of functions in A** which are analogous to those defined earlier.

LEMMA 68.

(i) The sets C_K, M^+, and M^- are absolutely convex, free ideals in A**.

(ii) The sets satisfy $C_K \subset M^+$, $C_K \subset M^-$, and $C_K = M^- \cap M^+$.

(iii) The set M^+ is a maximal ideal in A^{**}.

PROOF. Parts (i) and (ii) easily follow from definitions. Part (iii) can be shown using the same procedure as that in the proof of Theorem 64.//

If $(M^-, i) = A^{**}$, then there are functions g in A^{**} and m^- in M^- such that $\underline{1} = m^- + ig$. This implies $g(x) = 1/x$ for x less than some real number. This is not possible, thus M^- is not a maximal ideal in A^{**}.

THEOREM 69. If M is an ideal in A^{**} with f in M such that $Z(f)$ is compact, then M is a fixed ideal in A^{**} or M contains M^-.

PROOF. Consider the following three cases for $Z(f)$. Case 1. Let $Z(f) = \emptyset$ and let the eventual polynomial p_f of f be a nonzero constant function $\underline{k}$ for $x \leq r$. In this case $\underline{1}/f$ is in A^{**}, and hence $\underline{1}$ is in M. This contradiction implies that M contains no functions of this form. Case 2. Let $Z(f) = \emptyset$ and let p_f is a non-constant polynomial for $x \leq r$. Let $g_s(x) = 1/f(x)$ for $x \geq s$ and $g_s(x) = 1/f(s)$ for $x \leq s$. This can be done for each s in R. Then g_s is in A^{**} and $g_s f$ is in M with $(g_s f)(x) = 1$ for $x \geq s$. Let m^- be in M^- such that $m^-(x) = 0$ for $x \leq s$. Then $m^-(g_s f) = m^-$ is in M. This procedure applies for any m^- in M^-. Hence M^- is a subset of M.

Case 3. Let $Z(f) = Z_1$ where Z_1 is a non-empty, compact subspace of R. From Corollary 13, M cannot be a free ideal in A**. Thus M is a fixed ideal.//

THEOREM 70. The free maximal ideals in A** are M^+, (M^-, f_r), and $(M^-, g_{s,t})$ where f_r and $g_{s,t}$ are the functions defined in Theorem 57. If $r_1, s_1,$ and t_1 are numbers such that $(s,t) \neq (s_1, t_1)$, $r \neq r_1$, and $s_1^2 - 4t_1 < 0$, then $(M^-, f_r) \neq (M^-, f_{r_1})$ and $(M^-, g_{s_1, t_1}) \neq (M^-, g_{s,t})$.

PROOF. The above subsets of A** are clearly ideals. Since M^+ and M^- are free ideals, the above ideals are free ideals. Using the procedure in the proofs of Theorems 64 and 53, the above ideals can be shown to be maximal. Suppose M is a free maximal ideal which is not equal to any of the free maximal ideals listed above. Then there are functions f and g in M such that f is not in M^+ and g is not in M^-. Then $f^2 + g^2$ is in M and has a compact zero set. From Theorem 12, there is a function h_1 in M such that $Z(h_1) \cap Z(f^2 + g^2) = \emptyset$. From Theorem 69, M^- is a subset of M. Clearly M cannot contain f_r or $g_{s,t}$. Suppose g is in $M - M^-$ and has an eventual polynomial p_g for $x \leq r_g$ with factors $f_{r_1}, f_{r_2}, \ldots, f_{r_n}$. As in Theorem 53, there is a function ℓ in M such that p_ℓ and p_g have no common non-constant factors. There are polynomials p and q such that $pp_\ell + qp_g = \underline{1}$. Then $\underline{1} = [\underline{1} - (p\ell + qg)] + (p\ell + qg)$ is in M. The results follow.//

The structure space S** of maximal ideals in A** is
one that follows previous results. It can be described
as the one point compactification of R union with a "folded"
complex plane. As before, R is homeomorphic with the
subspace of fixed maximal ideals in S**. The subspace
consisting of $\{M^+\}$ union with the set of fixed maximal
ideals in S** is Hausdorff and compact. The subspace
of free maximal ideals is not Hausdorff and has the finite
complement topology as its subspace topology.

4. $C_K(X)$ plus a Subring of C(X)

Let X be a locally compact, non-compact, Hausdorff
space. Let A* = $C_K(X)$ + P(X) where $C_K(X)$ is the set of
functions in C(X) with compact support, and P(X) is a
subring of C(X) containing $\underline{1}$ with $P(X) \cap C_K(X) = \{\underline{0}\}$. All
of the previous subrings studied in Section IIIC are
special cases of this A*. From Lemma 47, A* is the
direct sum of C_K and P.

LEMMA 71. The set C_K is an ideal in A*.

PROOF. The proof follows that of Lemma 18 (ii).//

LEMMA 72. The ideal C_K is a free ideal in A*.

PROOF. From Lemma 71, C_K is an ideal in A*. For
each y in X there is an open neighborhood N_y such that
Cl (N_y) is compact. Since X is completely regular, there
is a function f_y in C(X) such that $f_y(y) \neq 0$ and

$f[X - N_y] = \{0\}$. Then $Z(f_y)$ contains $X - N_y$. Hence Supp (f_y) is a closed subspace of Cl (N_y) and therefore is compact. Thus f_y is in C_K. This procedure works for any y in X; hence C_K is a free ideal in A*.//

The following lemma is closely related to [26, Theorem 37A].

LEMMA 73. Let S be a compact subspace of X. Any continuous function on S can be extended to a function in $C_K(X)$.

PROOF. The proof is an exercise in general topology and is similar to the proof of [26, Theorem 37A].//

THEOREM 74. The ideal C_K is in every free ideal in A*.

PROOF. Let g be in $C_K(X)$ and let S = Cl $[X - Z(g)]$. Then S is a compact subspace of X. Let M be any free ideal in A*. From Theorem 12, a function h is in M such that 0 is not in h[S]. Let $f_S(x) = 1/h(x)$ for x in S. Then f_S is in C(S). From Lemma 73, f_S can be extended to a continuous function f in $C_K(X)$. Since h is in M, fgh = g is in M. Hence C_K is a subset of M.//

DEFINITION 7. A subring P of C(X) has property P_1 if for each non-zero p in P there is a non-empty compact subspace S of X and a function p' in P such that pp'(x) = 1 for all x in X - S.

THEOREM 75. The ideal C_K is a maximal ideal in A* if and only if P has property P_1.

PROOF. Assume C_K is a maximal ideal in A* and let p be in P with $p \neq \underline{0}$. Then p is not in C_K. Since C_K is a maximal ideal in A*, there are functions $g = m_1 + p_1$ in C_K + P and m in C_K such that $\underline{1} = m + p(m_1 + p_1)$ on X. Let $h = m + m_1 p$ and let S = Cl [X - Z(h)]. Since h is in C_K, S is compact with $pp_1(x) = 1$ for x in X - S. Since X is not compact, $X \neq S$.

Assume P has property P_1. We know C_K is an ideal in A* from Lemma 72. Let f be in A* - C_K. Then there is a non-zero p in P such that $f = f_K + p$ with f_K in C_K. Using property P_1, there is a function p' in P and a compact subspace S such that $pp' = \underline{1}$ on X - S. Define $m_K = \underline{1} - pp'$. Then Supp $(m_K) \subseteq$ Cl (S) = S. Thus m_K is in C_K. The function $\underline{1} = (m_K - p'f_K) + fp'$ is in (C_K, f). Hence C_K is a maximal ideal in A.//

DEFINITION 8. An ideal M in a subring P of C(X) has property P_2 if for each p in M and all non-empty compact subspaces S of X, $p|(X - S) \neq \underline{1}$.

THEOREM 76. Let P' be a non-empty subset of P and let $(P') = \{pp' : p$ is in P and p' is in $P'\}$. Let (C_K, P') $= \{m + gp' : m$ is in C_K, p' is in P', and g is in $A^*\}$.

(i) The set (C_K, P') is a free maximal ideal in A^* if and only if (P') is a maximal ideal in P.

(ii) The free maximal ideals in A^* have the form (C_K, P') where (P') is an ideal in P which is maximal with respect to property P_2 in the sense that there is no function f in $P - P'$ such that (P', f) is an ideal in P satisfying property P_2.

PROOF.

(i) If (C_K, P') is a free maximal ideal in A^*, then for each p in $P - P'$, $\underline{1}$ is in $((C_K, P'), p)$ or $\underline{1} = m + p_1 p'$ $+ p_2 p$ for some functions m in C_K, p' in P', and p_1 and p_2 in P. Then $\underline{1} - p_1 p' - p_2 p = m$ where $\underline{1} - p_1 p' - p_2 p$ is in P and m is in C_K. Since $P \cap C_K = \{\underline{0}\}$, $\underline{1} = p_1 p' + p_2 p$. Hence (P') is a maximal ideal in P. Clearly (C_K, P') is a maximal ideal in A^* if (P') is a maximal ideal in P.

(ii) From Theorem 74, C_K is in every free ideal in A^*. Since $\underline{0}$ is in C_K, (P') is an ideal in P whenever (C_K, P') is an ideal in A^*. If there is a function p' in (P') and a compact subspace S of X such that $p'(x) = 1$ for x in $X - S$, then $\underline{1} = (\underline{1} - p') + p'$ is in (C_K, P'). This is not possible, hence (P') has property P_2. In order that (C_K, P') be a maximal ideal in A^*, part (i) implies that (P') be maximal with respect to P_2. //

We now consider sufficient conditions on a space X such that the structure space is an n point compactification of X. Let X be a locally compact, non-compact, Hausdorff space. Let K be a non-empty, compact subspace of X. Let Y_1, Y_2, Y_3,...,Y_n be distinct, non-compact components of X - K with $X = K \cup (\bigcup_{i=1}^{n} Y_i)$.

Let A be a subring of C(X) with the following properties, where i is any member of the set {1,2,3,...,n}.

(a) If f is in A, then there is a compact subspace K_f containing K such that $f[Y_i - K_f] = \{0\}$ or 0 is not in $f[Y_i - K_f]$.

(b) Let M_i = {f in A: there is a compact subspace L_f with $f[Y_i - L_f] = \{0\}$}. The ring A has the pointwise inversion property; if g is in A - M_i, there is a function h in A and a compact space L such that $(gh)(x) = 1$ for x in $Y_i - L$.

(c) Let S be any compact subspace of X. For each i there is a function g_i in A - M_i such that $Z(g_i)$ contains $S \cup (\bigcup_{\substack{j=1 \\ j \neq i}}^{n} Y_j)$.

(d) The ring A contains $C_K(X)$ and the set of constant functions on X.

THEOREM 77. The sets M_1, M_2,...,M_n are distinct, free maximal ideals in A.

PROOF. Property (c) implies that the M_i are distinct.
We prove M_1 is a free maximal ideal. Since $\underline{1}$ is not in
M_1, $M_1 \neq A$. Clearly M_1 is an ideal which contains C_K.
Since $\{x\}$ is compact for a fixed x in X, M_1 is a free
ideal in A from Lemma 73. To show that M_1
is a maximal ideal in A, let g be in $A - M_1$. From property
(a), $g(x) \neq 0$ for x in $Y_1 - K_g$. From property (b), there
is a function h in A and a compact set L such that $1 =$
$hg(x)$ for x in $Y_1 - L$. Then $\underline{1} = (\underline{1} - hg) + hg$ is in
(M_1, g). Therefore M_1 is a free maximal ideal in $A.//$

THEOREM 78. The ideals M_1, M_2,...,M_n are the only
free maximal ideals in A.

PROOF. Suppose I is a free maximal ideal in A which
differs from each M_i. From Corollary 13 and property (a),
f in I implies f is in some M_i. If $I \neq M_i$ for each i,
then there are functions f_i in $I - M_i$ satisfying $f_i(x) \neq 0$
for all x in $Y_i - K_i$ where K_i is a compact set satisfying
property (a). Let $h = \sum_{i=1}^{n} f_i^2$. Then $Z(h) \subseteq \bigcup_{i=1}^{n} K_i = K$;
hence K is compact. Since $Z(h)$ is closed, $Z(h)$ is
compact. This contradicts Corollary 13. The desired
result follows.//

THEOREM 79. The structure space S of A is an n
point compactification of X.

PROOF. Theorem 23 is used. In [21], $C_K(X) + R(X)$ is stated as having the property of separating points from closed sets. Since $C_K(X) + R(X)$ is a subset of A, Theorem 40 implies that the subspace of fixed maximal ideals in A is homeomorphic to X. Consider two distinct, free maximal ideals in A. From property (c), the condition of Theorem 23 (ii) can be satisfied. Now consider a fixed maximal ideal in S and a free maximal ideal in S. From Lemma 1, the fixed maximal ideals have the form $M_X = \{f \text{ in A}: f(x) = 0\}$. For each x in the locally compact space X there is an open neighborhood N_x about x such that Cl (N_x) is compact. Since X is completely regular, there is a g in $C_K(X)$ with $g(x) = 1$ and $g[X - N_x] = \{0\}$. From property (c), h_i exists in A - M_i with $h_i[Cl\ N_x] = \{0\}$. Since X is homeomorphic to the subspace of fixed maximal ideals, this subspace is Hausdorff in the subspace topology. Since there are only n free ideals and Theorem 23 (ii) is satisfied for the cases described above, S is Hausdorff. From Theorem 23 (iii), S is compact. The subspace of fixed maximal ideals is dense in S and there are n free maximal ideals. Hence S can be considered an n point compactification of X.//

The subrings of $C_K(R)$ plus constant functions and $C_K(R)$ plus rational functions can be set up to satisfy properties (a), (b), (c), and (d).

EXAMPLE 3. Let K be the unit circle in the xy-plane with its center at the origin. Let X be K along with the interior of K, the x-axis, and the y-axis. Each axis intersects the complement of K in 2 components. If A is the set of continuous functions on X which are eventually constant, then the space of maximal ideals is a 4 point compactification of R.

D. Special Subrings Containing $C_\infty(X)$.

Let X be a locally compact, non-compact, Hausdorff space. We say f in C(X) converges to a at infinity and write $\lim_\infty f = a$ if for each $\varepsilon > 0$, $\{x \text{ in } X : |f(x) - a| \geq \varepsilon\}$ is compact. We denote the set of functions converging to 0 at infinity by $C_\infty(X)$ or C_∞. The set C_∞ is a subring of C. Previously we worked with C_K, the functions of compact support. We now obtain some similar results for C_∞ or a subring contained in C_∞.

In [21] results are given for the subring $C_\infty(X) + R(X)$ where R(X) is the set of real constant functions on X. We state some results from [21] and later prove similar results using methods like those employed in the study of subrings containing C_K.

LEMMA 80. [21, pp. 16, 17]

(i) Every function f in $C_\infty(X) + R(X)$ has a unique representation $f = \gamma(f) + \delta(f)$ where $\gamma(f)$ is a constant and $\delta(f)$ is a continuous function vanishing at infinity.

(ii) The ring $C_\infty(X)$ is a real, maximal ideal in $C_\infty(X) + R(X)$.

(iii) The ring $C_\infty(X)$ is the only free maximal ideal in $C_\infty(X) + R(X)$.

PROOF. The proofs of (i) and (ii) follow the pattern of the proof of Theorem 38. To show (iii), we consider the one point compactification of X obtained by adjoining an ideal point ω to X. Let f* be in $C(X \cup \{\omega\})$. There is a function f in $C_\infty(X) + R(X)$ such that $f^*|X = f$ and $f^*(\omega) = \lim_\infty f$. From [26, p. 166], the mapping that takes f in $C_\infty(X) + R(X)$ to f* in $C(X \cup \{\omega\})$ is a one-to-one mapping onto $C(X \cup \{\omega\})$. Since $X \cup \{\omega\}$ is compact, from [3] or [5], all maximal ideals in $C(X \cup \{\omega\})$ have the form $M_X^* = \{f^*$ in $C(X \cup \{\omega\}): f^*(x) = 0\}$ where x is in $X \cup \{\omega\}$. If $x \neq y$, then $M_X^* \neq M_y^*$. The above mapping is an isomorphism onto $C(X \cup \{\omega\})$. From this isomorphism, the only free maximal ideal in $C_\infty(X) + R(X)$ is the maximal ideal corresponding to M_ω^* in $C(X \cup \{\omega\})$. This ideal is $C_\infty(X)$. Hence $C_\infty(X)$ is the only free maximal ideal in $C_\infty(X) + R(X)$.//

THEOREM 81. [21, p. 18] The structure space of $C_\infty(X) + R(X)$ is (homeomorphic to) the one point compactifaction of X.

PROOF. Under the isomorphism $(f^* \mapsto f): C(X \cup \{\omega\}) \to C_\infty(X) + R(X)$, the maximal ideal M_x^* in $C(X \cup \{\omega\})$ corresponds to the fixed maximal ideal $M_x = \{f \text{ in } C(X) + R(X): f(x) = 0\}$ if and only if x is in X. From [3], $X \cup \{\omega\}$ is homeomorphic to the structure space of $C(X \cup \{\omega\})$. Since $C(X \cup \{\omega\})$ is isomorphic to $C(X) + R(X)$, the result follows.//

We now consider X = R and prove some of the previous results using techniques similar to those used in working with C_K. Let R denote the space of real numbers or the subset of real constant functions on the space of real numbers. From Lemma 80 (i), f can be written uniquely as $f_\infty + r$ where f_∞ is in C_∞ and r is in R. For X = R, $C_\infty = \{f$ in $C(R): \lim_{|x| \to \infty} f(x) = 0\}$.

THEOREM 82. The set C_∞ is a free maximal ideal in $C_\infty + R$.

PROOF. Clearly C_∞ is an ideal in $C_\infty + R$. Since C_K is a subset of C_∞ and C_K is a free ideal in $C_\infty + R$, C_∞ must be a free ideal. To show C_∞ is a maximal ideal in $C_\infty + R$, consider $f = f_\infty + \underline{r}$ where $r \neq 0$. Then $\underline{1} = (\underline{1} - (\underline{r}^{-1})f) + (\underline{r}^{-1})f$ is in (C_∞, f). The desired result follows.//

THEOREM 83. If M is an ideal in C_∞ + R with a function f in M - C_∞ , then M is a fixed ideal in C_∞ + R.

PROOF. Let f be in M - C_∞. Then f = f_∞ + r for some non-zero r, and Z(f) is compact. If M is a free ideal in C_∞ + R, then we can find a positive function h in M - C_∞ using Theorem 12. Let h = h_∞ + $\underline{s}$ for some s ≠ 0. Then $\lim\limits_{|x| \to \infty} 1/h(x) = s^{-1}$. The function $\ell = (1/h - (\underline{\underline{s}}^{-1}))$ + $(\underline{\underline{s}}^{-1})$ is in C_∞ + R. Then $h\ell = \underline{\underline{1}}$ is in M. This contradiction shows that M is not a free ideal in C_∞ + R.//

COROLLARY 84. The ideal C_∞ is the only free maximal ideal in C_∞ + R.

PROOF. From Theorem 82, C_∞ is a free maximal ideal in C_∞ + R. From Theorem 83, no free maximal ideal can contain a function which is not in C_∞ ; hence C_∞ is the only free maximal ideal in C_∞ + R.//

THEOREM 85. The structure space of C_∞ + R is the one point compactification of R.

PROOF. The proof is similar to the proofs of Theorems 61 and 62.//

Let M(R) be a subset of C_∞(R) such that f is in M(R) if $\lim\limits_{|x| \to \infty} |x^n f(x)| = 0$ for all non-negative integers n. Let P(R) denote the set of polynomials on R. The sets P(R) and M(R) will often be denoted by P and M respectively.

The subring $C_K(R)$ is in $M(R)$. If $g(x) = e^{-x^2}$, then g is in $M - C_K$. Clearly $M \cap P = \{\underline{0}\}$. Let $A = M + P$.

LEMMA 86.

(i) The set A is a subring of $C(R)$.

(ii) The set A is the direct sum of M and P.

(iii) Let $f = m + p$ in $M + P$ and let $\gamma(f) = p$ in P. Then γ is a homomorphism of A onto P having M as its kernel. Hence A/M is isomorphic to P.

PROOF. The proofs of (i) and (iii) follow the same pattern as the corresponding parts of the proof of Theorem 38. We show A is closed under multiplication. The other properties of a subring easily follow. Let p be in P and m in M. The function mp is a linear combination of terms of the form $x^n m$. From the given condition on M, these terms approach 0 as $|x|$ gets large. Hence mp is in M for all p in P. Clearly mm' is in M if m and m' are in M. The result follows.//

LEMMA 87. The set M is an absolutely convex, free ideal in A.

PROOF. The function $\underline{1}$ is in $A - M$ so $M \neq A$. The set M is clearly an ideal in A and contains the free ideal C_K. Hence M is a free ideal in A. Using the inequality

$\underline{0} \leq |f| \leq |g|$ with g in M and f in A, $\lim\limits_{|x| \to \infty} |x^n f(x)| = 0$

for each fixed, non-negative integer n. The result

follows.//

The ideal M is not a maximal ideal in A. This can

be shown in several ways. One method is to show $\underline{1}$ is in

(M,g) for g in A - M. This conclusion can also be found

with the help of Lemma 86 (iii). From [16, Corollary 2.46],

M is maximal in A if and only if A/M is a simple ring. In

this case A/M is isomorphic to P but P is not a simple

ring. Thus M is not a maximal ideal in A.

THEOREM 88. The free maximal ideals in A are (M, f_r)

and $(M, g_{s,t})$ where f_r and $g_{s,t}$ are the functions defined

in Theorem 57. If r_1, s_1, and t_1 are numbers such that

$r \neq r_1$, $(s_1, t_1) \neq (s,t)$, and $s_1^2 - 4t_1 < 0$, then (M, f_{r_1})

$\neq (M, f_r)$ and $(M, g_{s_1, t_1}) \neq (M, g_{s,t})$.

PROOF. From Lemma 87, M is a free ideal. Since $\underline{1}$

is not in any of the above subsets of A, (M, f_r) and $(M, g_{s,t})$

are free ideals. To show that these ideals are maximal

we can use the same methods as those employed in the proof

of Theorem 53. To show that there are no other free

maximal ideal, suppose M_1 is a free maximal ideal in A

which differs from each of the ideals listed in the theorem.

Let f be in $M_1 - (M, f_1)$ where $f = m_1 + p_1$ in M + P.

Then $p_1 \neq \underline{0}$ and p_1 does not have f_1 as a factor. Working

as we did in the proof of Theorem 53, we can find a non-

negative function h in M_1 such that $h = h_m + p_h$ where p_h in P and p_1 have no common non-constant factors. There are polynomials p_2 and p_3 such that $\underline{1} = p_2 p_h + p_3 p_1$. Then $p_2 h + p_3 f$ is in M_1 and has $\underline{1}$ as its polynomial. Let $m_2 + \underline{1} = (p_2 h + p_3 f)^2$. As in Theorem 52, A_K is a subset of M_1. Then there exists a function $m + \underline{1}$ in M_1 such that m is in M and $m > \underline{0}$. Since $(-m)/(m + \underline{1})$ is in M, $(m + \underline{1})$ $(-m)/(m + \underline{1}) = -m$ is in M_1. This implies $(-m) + (m + \underline{1})$ $= \underline{1}$ is in M_1. The results follow.//

LEMMA 89. The structure space of A is not Hausdorff.

PROOF. Using Theorem 23 (ii) we obtain the result by observing that if a is in A - (M,f_0) and a' is in A - (M,f_1), then $aa' \neq \underline{0} = \bigcap S$.//

To generalize the previous results we consider the following definition. Let X be a locally compact, non-compact, Hausdorff space. Let A* be a subring of C(X) such that A* = M* + P* where M* is an ideal in A* containing $C_K(X)$, P* is a subring of C(X) containing $\underline{1}$, and P* $\bigcap$ M* = $\{\underline{0}\}$.

Using the procedure of the proof of Lemma 80 (i), A* is the direct sum of M* and P*.

THEOREM 90. The set $C_K(X)$ is in every free ideal in A*.

PROOF. The proof is similar to the proof of Theorem 74.//

THEOREM 91. Let P' be a non-empty subset of P*.
Then (M*,P') is a free maximal ideal in A* if and only if
(P') is a maximal ideal in P* where (P') = {pp': p is in
P* and p' is in P'}.

PROOF. The proof is similar to the proof of Theorem
76 (i).//

V. SUMMARY, CONCLUSIONS, AND FURTHER PROBLEMS

The set of continuous functions on any realcompact
space between νX and βX is isomorphic with a lattice-
ordered subring of $C(X)$ containing $C^*(X)$. The cardinal
number of these subrings is at least as large as the
cardinal number of $\beta X - \nu X$. This isomorphism was used
to establish properties of prime ideals and prime z-
ideals of these lattice-ordered subrings. The prime
ideal structure of $C^*(X)$ was examined by Mandelker in
[14] and many of his results generalize to these lattice-
ordered subrings. In [14] the immediate successor of
$M_p \cap C^*(X)$ in the family of prime z-ideals of $C^*(X)$ was
found when $\beta X - X$ was a zero set of βX. Neither a
corresponding result nor a contradiction was found for the
other lattice-ordered subrings that were studied in this
dissertation. Mandelker has recently defined "real-
compact" for non-Hausdorff spaces. One further problem
is to see which results in this dissertation carry over
for these spaces.

Another problem is the characterizing of different
compactifications of a space X by using algebraic pro-
perties of a subset of real-valued functions. The structure
space of the piecewise rational functions was shown to be
a two point compactification of R. The initial interest
in working with the piecewise rational functions was

that this subring was a lattice-ordered subring with the pointwise inversion property which contained the polynomials on R. A further problem is to find proper, lattice-ordered subrings of C(X) which contain the ring of infinitely differentiable functions and to study the structure space of each subring.

The free ideals and the structure space of subrings of C(X) of the form $C_K(X) + P(X)$ were studied where $C_K(X) \cap P(X) = \{\underline{0}\}$ and X is a locally compact, non-compact Hausdorff space. Some results were also given for $C_\infty(X) + P(X)$. The free maximal ideals in these subrings partly depended upon the ideals of P(X). Some of the structure spaces were compact and some were not Hausdorff. A further problem is to find smallest subrings whose structure space is a particular compactification of X.

BIBLIOGRAPHY

1. Bourbaki, N., Elements of mathematics, general topology, part 1, (1966), Addison-Wesley, Reading, Mass.

2. Čech, E., On bicompact spaces, Ann. of Math. 38 (1937), 823-844.

3. Gelfand, I. and Kolmogoroff, A., On rings of continuous functions on topological spaces, Dokl. Akad. Nauk SSSR, 22 (1939), 11-15.

4. Gillman, L. and Henriksen, M., Rings of continuous functions in which every finitely generated ideal is principal, Trans. Amer. Math. Soc. 82 (1956), 366-391.

5. Gillman, L. and Jerison, M., Rings of continuous functions, (1960), D. van Nostrand, Princeton, New Jersey.

6. Henrikson, M. and Johnson, D. G., On the structure of a class of archimedian lattice-ordered algebras, Fund. Math. 50 (1961), 73-94.

7. Henriksen, M. and Jerison, M., The spaces of minimal prime ideals of a commutative ring, Trans. Amer. Math. Soc. 115 (1965), 110-130.

8. Hewitt, E., Rings of real valued continuous functions, I, Trans. Amer. Math. Soc. 64 (1948), 54-99.

9. Kist, J., Minimal prime ideals in commutative semi-groups, Proc. London Math. Soc., (3) 13 (1963), 31-50.

10. Kohls, C. W., Ideals in rings of continuous functions, Fund. Math. 45 (1958), 28-50.

11. Kohls, C. W., Prime ideals in rings of continuous functions, Ill. J. Math. 2 (1958), 505-536.

12. Kohls, C. W., Prime ideals in rings of continuous functions, II, Duke Math. J. 25 (1958), 447-458.

13. Mandelker, M., Prime z-ideal structure of C(R), Fund. Math. 63 (1968), 145-166.

14. Mandelker, M., _Prime ideal structure of rings of bounded continuous functions_, Proc. Amer. Math. Soc. 19 (1968), 1432-1438.

15. McCoy, N. H., _Rings and ideals_, (1948), Carus Monograph No. 8, Buffalo, New York.

16. McCoy, N. H., _The theory of rings_, (1948), Macmillan, New York.

17. Plank, D., _On a class of subalgebras of C(X) with applications to βX - X_, Fund. Math. 64 (1969), 45-45.

18. Pursell, L. E., _An algebraic characterization of fixed ideals in certain function rings_, Pacific J. Math. 5 (1955), 963-969.

19. Pursell, L. E., _The ring C(X,R) considered as a subring of the ring of all real-valued functions_, Proc. Amer. Math. Soc. 8 (1957), 820-821.

20. Pursell, L. E., _Lecture notes from a research seminar_, (1970) unpublished.

21. Pursell, L. E., _The ring of real, continuous functions of the form (constant) + (function of compact support)_, unpublished.

22. Pursell, L. E., _The rings of real, continuous functions which converge at infinity_, unpublished.

23. Riordan, D., _Pseudocompact subsets of X and ideals in C(X)_, (preliminary report), Notices Amer. Math. Soc. 18 (1971), 568.

24. Royden, H. L., _Function algebras_, Bull. Amer. Math. Soc., 69 (1963), 281-288.

25. Shanks, M. E., _Rings of functions on locally compact spaces_, (abstract no. 365), Bull. Amer. Math. Soc. 57 (1951), 295.

26. Simmons, G. F., _Introduction to topology and modern analysis_, (1963), McGraw-Hill, New York.

27. Stone, M. H., _Applications of the theory of Boolean rings to general topology_, Trans. Amer. Math. Soc. 41 (1937), 375-481.

LIST OF SYMBOLS

$\cup$, $\cap$	union, intersection
$\subseteq$, $\subset$	set inclusion, proper set inclusion
$Z(f)$, $Z[A]$	zero sets, Section IIIA
$\leftarrow$	inverse of mapping
$M(f)$	residue class
$Cl(S)$, $Cl\ S$	closure of S
$\underline{r}$	function whose range is $\{r\}$
f^*	Stone extension; Section IIIA, Definition 5
R^*	one point compactification of R
νX	Section IIIA, Corollary 8
$\nu_f X$	Section IIIA, Theorem 14
$\nu_A X$	Section IV A, Definition 2
$|\ |$	cardinal number or absolute value
$f \vee g$, $f \wedge g$	sup $\{f,g\}$, inf $\{f,g\}$
(M,P), (M)	ideal generated by set(s) M and P or the whole ring
(X,A)	real function ring; Section IV A, Definition 1
$Supp(f)$	support of f; Section IV B. Definition 4
$C_K(X)$	functions with compact support; Section IV B, Definition 4

$C_\infty(X)$	functions converging to 0 at infinity; Section IV D
$\circ$	composition
$\mid$	restriction
$\rightarrow$	mapping of one set into another set or approaches
$\mapsto$	mapping of one element to another element

VITA

Paul Marlin Harms was born on October 5, 1934 at Newton, Kansas. He graduated from Whitewater High School of Whitewater, Kansas in May 1952. In May 1956 he received a Bachelor of Arts degree with a major in Mathematics from Bethel College of North Newton, Kansas. In July 1958 he received a Master of Science degree in Mathematics from Iowa State University. He has attended N.S.F. Summer Institutes at Kansas State College of Pittsburg (1960), Georgia Institute of Technology (1961), Iowa State University (1962), and Argonne National Laboratories (1965).

He was employed at Iowa State University as a graduate assistant in Mathematics from 1956 to 1958 and as an instructor in Mathematics from 1958 to 1959. He was employed as an instructor of Mathematics and Physics at Simpson College of Indianola, Iowa from 1959 to 1961. From 1961 to 1967 he taught primarily Mathematics and Physics at Bethel College of North Newton, Kansas. He has been on leave of absence from Bethel College since 1967. Since September 1967, he has been employed as a part-time instructor in Mathematics at the University of Missouri at Rolla where he has also continued graduate study in Mathematics.

He was married to the former Shirley Ellen Funk on September 10, 1956. They have three children, Douglas, Adley, and Gwendolyn.