1972

The scattering of elastic waves by void cavities

Jerry Lee Davis

Department: Geosciences and Geological and Petroleum Engineering

THE SCATTERING OF ELASTIC WAVES BY VOID CAVITIES

by

JERRY LEE DAVIS, 1942-

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI-ROLLA

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

GEOPHYSICS

1972

Advisor

ABSTRACT

Elastodynamic multipole theory, the theory of least squares, and the theory of integral representation of solutions are employed in solving certain problems involving an elastic solid containing a source and a scatterer. Both the source and scatterer are of finite geometrical extent; they occupy non-intersecting regions. The source is separable, i.e., its mathematical specification consists of an arbitrary vector function of position multiplied by a time function, which is further assumed to be a sinusoid. The scatterer emphasized is a finite void cavity of arbitrary shape; however, scatterers composed of rigid material may also be treated. The calculation of a Green's Function is emphasized; in this case the fields incident upon the scatterer are dipole fields. However, the method presented is amenable to arbitrary specification of the incident field; plane wave scattering is discussed as an example. While scattering from a single object is emphasized, the case where two or more scatterers exist is discussed briefly. The so-called cavity-source problem is also discussed briefly.

In all cases, a first approximation to the solution in the form of a linear combination of multipole fields is derived using least squares. An improvement in this approximation is derived using an integral representation of the exact solution. The second and final approximation is in the form of a multipole series in which the terms are the fields of fundamental force systems, i.e., dipoles, quadrupoles, etc.

ACKNOWLEDGEMENTS

Sincere and profound gratitude is extended to Dr. Richard D. Rechtien, whose wisdom and advice were prerequisite for completion of the work leading to this dissertation. In the dual role as advisor and friend, his academic, spiritual, and material assistance was immeasurable.

Sincere appreciation must also be extended to several other professors for their dedication and professional competence in the classroom. Particularly outstanding in this respect are Professors S.J. Pagano, A.J. Penico, J.L. Rivers, G.B. Rupert, and R.E. Ziemer. Without the academic grounding made possible through the efforts of these men and others, this dissertation would not exist.

In addition, the contributions of Dr. David M. Stewart of the University of North Carolina at Chapel Hill must be warmly acknowledged. While at the University of Missouri-Rolla, Dr. Stewart was a continuous source of academic help, inspiration, and the self-confidence required of Ph.D. candidates.

Finally, profound appreciation is extended to the author's wife, Jeannie, whose love, companionship, and tolerance invariably overwhelmed the usual frustrations connected with four years of graduate study.

TABLE OF CONTENTS

Page

Page

Page

LIST OF FIGURES

Page

LIST OF TABLES

LIST OF SYMBOLS

$\#$	Denotes complex conjugate (see Section IVC-3)
$\sim$	Denotes spatial factor of a quantity with sinusoidal time dependence (see Section IVB)
$\int_{a.s.}$	Denotes volume integration over all space (see Section ID)
$\int_V$	Denotes volume integration over a region named V (see Section ID)
$\int_S$	Denotes surface integration over a surface named S (see Section ID)
A^j_{sk}, A^j_{skp}	Numerical coefficients in linear combinations of multipole fields (see Equation (4.16))
A_{sk}, A_{skp}	The same coefficients as those directly above but with the superscript omitted; or, coefficients analogous to these but requiring no superscript whatsoever (see Equation (6.13))
$A_\alpha, A_\beta, B_\alpha, B_\beta, C_\alpha, C_\beta$	Special functions defined in Section 2, Appendix C
a^j_s	The result of renaming A^j_{sk} and A^j_{skp} (see Equations (4.19) and Table (D.1))
a_s	The result of omitting the superscript in a^j_s; or, analogous coefficients requiring no superscript whatsoever (see Table (D.1))
$a_m(\bar{r}), b_m(\bar{r})$	Real and imaginary parts of the m displacement component (see Equation (7.9))
α	Compressional wave speed (see Section IIIA-1)
b_s, c_s	Real and imaginary parts of the numbers a_s (see Equation (4.27))
b, c	Two special constants defined by Equations (D.5)
B^j_{sk}, B^j_{skq}	Numerical coefficients in the multipole series for the scattered part of a Green's Function (see Equation (5.46))

$B^{jp}_{sk}, B^{jp}_{skq}, B^{jpq}_{sk}, B^{jpq}_{skm}$ Numerical coefficients in the multipole series for scattered quadrupole and octupole fields (see Equations (5.47) and (5.48))

β Shear wave speed (see Section IIIA-1)

$c_{mn}(\bar{r}), d_{mn}(\bar{r})$ Real and imaginary parts of a stress tensor (see Equation (7.10))

$D_i, D^o_i, D^1_i, D'_i, D_{ij}$, etc. Denote differentiation with respect to various sets of coordinates (see Section ID)

$\delta(\bar{r} - \bar{r}_o)$ The Dirac delta function (see Section 1, Appendix A)

δ_{ij} Kronecker's delta (see Section ID)

ΔS_n A finite surface element used in numerical integration (see Section IVC-4)

$E(\bar{r})$ Time averaged energy per unit area per unit time carried by elastic waves (see Equation (7.6))

$\tilde{E}^j_m(\bar{r}')$ Components of the traction-error vector arising in least squares theory (see Equation (4.38))

$\tilde{E}_m(\bar{r}')$ The result of omitting the superscript in $\tilde{E}^j_m(\bar{r}')$; or, the components of a traction-error vector requiring no superscript whatsoever (see Section IVC-3)

E_{mn} The value of $\tilde{E}_m(\bar{r}')$ at a point $\bar{r}_n'$ (see Equation (4.24))

$\hat{e}_i$ Base vectors in a rectangular coordinate system (see Section ID)

$\tilde{\varepsilon}_{ij}(\bar{r})$ An indexed set of displacement-error fields corresponding to traction-error vectors (see Section 3, Appendix E)

η_o A special function defined in Section IIIA-2

η_s, η_{sk}, etc. Special functions related to dipoles, quadrupoles, etc., defined in Section IIIA-2

$\bar{F}(\bar{r})$ A static body force per unit volume (see Section IIA)

$F_s(\bar{r})$ The components of $\bar{F}(\bar{r})$

$\tilde{F}_m(\bar{r})$ The spatial part of a dynamic body force having sinsusoidal time dependence (see Section VA)

$\bar{F}_i(\bar{r}|\bar{r}_o), \bar{F}_{ij}(\bar{r}|\bar{r}_o)$, etc. Dipole, quadrupole, etc., body forces (see Section IIB-1)

$f(t)$ An arbitrary time function (see Equation III-3)

G_s, G_{si}, G_{sij}, etc. Numerical coefficients in multipole series corresponding to given body forces (see Section IIC-1)

$G^j_s, G^j_{sk}, G^j_{skp}$, etc. Numerical coefficients analogous to those directly above but used for a special purpose (see Appendix F)

$g_{sm}(\bar{r}')$ The m component of the s traction vector in an approximating set (see Equation (4.18) and Table (D.1))

g_{smn} The value of $g_{sm}(\bar{r}')$ at a point $\bar{r}'_n$ (see Section IVC-3)

H_{sk} An array defined in Section VD

h A small positive number (see Section IIA)

$\tilde{I}_{ij}(\bar{r}|\bar{r}_1)$ A special set of integrals defined by Equation (5.8)

i The imaginary unit, i.e., $\sqrt{-1}$, when not used as a subscript or superscript

$\bar{K}(\bar{r},t)$ A dynamic body force (see Section IIIA-1)

L_s A multipole series force operator (see Equation (2.51))

L_s^j Force operator analogous to L_s (see Appendix F)

λ,μ Lame's elastic constants (see Section IIIA-1)

$M_s^j, M_s'^j, M_s^{1j}$ Force operators defined in Section VB

M_s, N_s Force operators defined in Section VIC

$\hat{n}$ An arbitrary constant unit vector (see Section IIB-1)

N_s^j Force operator involved in Green's Functions, and obtained by correcting M_s^j (see Section VB)

N_s^{jp}, N_s^{jpq} Force operators involved in scattering of higher order multipole fields (see Section VD)

$^1N_s^j, {}^2N_s^j$ Force operators involved in Green's Functions for the two-cavity case (see Section VIB)

ω Angular frequency

P Magnitude of a constant body force in Section IIB-1; temporal period of waves in Section VIIB

P_n The remainder after n terms in a Taylor series (see Section 3, Appendix A).

P_s A force operator, analogous to L_s, defined in Section VD

$\phi(\bar{r})$ Scalar source potential for an arbitrary static body force (see Section IIA)

$\phi_i(\bar{r}|\bar{r}_o), \phi_{ij}(\bar{r}|\bar{r}_o)$, etc. Scalar source potentials corresponding to static dipoles, quadrupoles, etc. (see Section IIB-1)

$\Phi(\bar{r},t)$ Scalar displacement potential for an arbitrary displacement field (see Section IIIA-1)

Φ_o A special function defined in Section IIIA-2

$\Phi_i(\bar{r},t|\bar{r}_o), \Phi_{ij}(\bar{r},t|\bar{r}_o)$, etc. Scalar displacement potentials corresponding to static dipoles, quadrupoles, etc. (see Section IIIA-2)

$\bar{\psi}(\bar{r})$ Vector source potential for an arbitrary static body force (see Section IIA)

$\bar{\psi}_i(\bar{r}|\bar{r}_o), \bar{\psi}_{ij}(\bar{r}|\bar{r}_o)$, etc. Vector source potentials corresponding to static dipoles, quadrupoles, etc. (see Section IIB-1)

$\bar{\Psi}(\bar{r},t)$ Vector displacement potential for an arbitrary displacement field (see Section IIIA-1)

$\bar{\Psi}_i(\bar{r},t|\bar{r}_o), \bar{\Psi}_{ij}(\bar{r},t|\bar{r}_o)$, etc. Vector displacement potentials corresponding to dipoles, quadrupoles, etc. (see Section IIIA-2)

$Q_{ij}(\bar{r},t|\bar{r}_o)$ The fundamental solution for arbitrary time dependence (see Section IIIB)

$\tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$ The steady state fundamental solution (see Section IVB)

$\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$ A Green's Function (see Sections IVA and IVB)

$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ The scattered part of $\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$ (see Section IVB)

q_i, q'_i The increments $x_i - x_{oi}$ and $x'_i - x_{oi}$, respectively (see Section 3, Appendix A)

$\bar{r}, \bar{r}_o, \bar{r}_1, \bar{r}', \bar{r}''$ Position vectors of points where rectangular coordinates are $x_m, x_{om}, x_{1m}, x'_m$, and x''_m, respectively, for m = 1,2,3

$\bar{r}_n'$ Position vectors of certain points on surfaces (see Section IVC-2)

$\bar{r}_n'$ Position vectors of certain points within body forces (see Section VC-1)

$\bar{r}_m$ Position vectors of certain field points (see Section VC-2)

R Distance between $\bar{r}$ and $\bar{r}'$, i.e., $|\bar{r} - \bar{r}'|$ (see Section IIB-1)

R Radius of a sphere (see Section VIIC)

R_o Distance between $\bar{r}$ and $\bar{r}_o$, i.e., $|\bar{r} - \bar{r}_o|$ (see Section IIB-1)

R_o' Distance between $\bar{r}'$ and $\bar{r}_o$, i.e., $|\bar{r}' - \bar{r}_o|$ (see Section I, Appendix C)

ρ Mass density of an elastic solid medium (see Section IIIA-1)

S The name of a surface

$\bar{S}(\bar{r},t)$ Instantaneous energy flow density vector (see Section VIIB)

$S_n(\bar{r},t)$ The components of $\bar{S}(\bar{r},t)$

$\Sigma^j_{mn}(\bar{r},t|\bar{r}_o)$ Stress tensors corresponding to $Q_{ij}(\bar{r},t|\bar{r}_o)$ (see Section IIIB)

$\tilde{\Sigma}^j_{mn}(\bar{r}|\bar{r}_o)$ Stress tensors corresponding to $\tilde{Q}_{ij}(\bar{r}|r_o)$ (see Section IVB)

$\tilde{\Sigma}^{*j}_{mn}(\bar{r}|\bar{r}_o)$ Stress tensors corresponding to $\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$ (see Section IVB)

$\tilde{\Sigma}^{**j}_{mn}(\bar{r}|\bar{r}_o)$ Stress tensors corresponding to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ (see Section IVB)

$\bar{T}(\bar{r}')$ Traction vector acting on a surface (see Section IIB-2)

$\tilde{T}_m(\bar{r}')$ Components of a sinusoidal traction vector (see Section IVC-1 and VIC)

$\tilde{T}^j_m(\bar{r}')$ Components of special traction vectors (see Section IVC-1)

$T_{mn}(\bar{r})$ Stress tensor corresponding to an arbitrary displacement field (see Section IIIA-1)

$\tilde{T}_{mn}(\bar{r})$ Steady state stress tensor (see Section IVC-1)

T_{mn} Value of $\tilde{T}_m(\bar{r}')$ at a point $\bar{r}_n'$ (see Section IVC-3)

$\hat{\tau}(\bar{r}')$ Unit normal vector to a surface

$\tau_n(\bar{r}')$ Components of $\hat{\tau}(\bar{r}')$

θ A number in the range $0 < \theta < 1$ (see Section 3, Appendix A)

$\bar{U}(\bar{r},t)$ Arbitrary displacement field (see Section IIIA-1)

$U_m(\bar{r},t)$ The components of $\bar{U}(\bar{r},t)$

$\tilde{U}_m(\bar{r},t)$ Steady state displacement components (see Section VA)

V Name of a region in space (see Section IIB-1)

$\bar{V}(\bar{r},t)$ Velocity vector in a medium (see Section VIIB)

$V_m(\bar{r},t)$ Components of $\bar{V}(\bar{r},t)$

V_n Finite volume elements (see Section VC-1)

$\bar{V}_\alpha(\bar{r},t), \bar{V}_\beta(\bar{r},t)$ Parent displacement potentials for an arbitrary displacement field (see Section IIIA-1)

$\bar{V}_{\alpha s}(\bar{r},t|\bar{r}_o), \bar{V}_{\beta s}(\bar{r},t|\bar{r}_o)$ Parent displacement potentials corresponding to dipoles (see Section IIIA-2)

$\overline{W}(\overline{r})$	Parent source potential for an arbitrary static body force (see Section IIA)
$\overline{W}_s(\overline{r})$	Components of $\overline{W}(\overline{r})$
$\overline{W}_i(\overline{r}\|\overline{r}_o), \overline{W}_{ij}(\overline{r}\|\overline{r}_o)$, etc.	Parent source potentials corresponding to dipoles, quadrupoles, etc. (see Section IIA)
$x_i, x_{oi}, x_{1i}, x'_i, x''_i$	Components of $\overline{r}, \overline{r}_o, \overline{r}_1, \overline{r}'$, and $\overline{r}''$, respectively

INTRODUCTION

The primary goal of this dissertation is the presentation of a method for the calculation of displacement fields arising in a perfectly elastic solid medium which, except for the presence of a finite void cavity of arbitrary shape, is infinite, isotropic, and homogeneous. The fields arise due to the application of body forces acting throughout some limited region within the medium. This region does not intersect the cavity. The body force consists of an arbitrary vector force per unit volume multiplied by a sinusoidal time factor $\exp(-i\omega t)$. The primary problem to be solved then, is one of scattering. The problem is of a general nature in the sense that one is allowed a high degree of flexibility in specifying both the cavity shape and the spatial part of the body force. The problem is of a restricted nature in the sense that both the cavity and the body force are of finite geometrical extent, while the latter is further restricted with regard to its time dependence.

The method developed herein for the solution of the above problem also may be used to solve the scattering problem arising when a rigid body replaces the cavity. Being less valuable from a pragmatic standpoint, this latter problem will be discussed only briefly.

Another problem related to cavities, the so-called cavity-source problem, may also be solved by the method developed herein, providing the time dependence is sinusoidal. This problem, which arises when specified tractions are applied to the boundary of a cavity in order to create a displacement field in the medium, will be discussed in more detail.

The method is based upon the unification of two general theories existing in virtually all branches of classical physics. These are multipole theory and the theory of integral representations of solutions. All final solutions will be approximate. The first step entails the calculation of a first approximation to a Green's Function; this is done through the use of multipole theory. Then, a second and final approximation is calculated through the use of integral representations. The Green's Function becomes available in multipole series form.

Calculation of the Green's Function in scattering problems is roughly analogous to completely solving the cavity-source problem. The latter is completely solved by the exercise outlined in the preceding paragraph, though the result is not a Green's Function.

In scattering problems, one writes down the ultimate solution as an integral over the applied body force. As an alternative to evaluating the integral at each field point, it may be replaced by a multipole series. This multipole series consists of the Green's Function and all its derivatives with respect to the spatial coordinates of an expansion point within the body force. Owing to certain properties of the Green's Function, these derivatives cannot be evaluated directly; each must be calculated using the same method employed in finding the Green's Function itself.

When the body force is not specified explicitly, but some given incident field is assumed to exist at the scattering object, the method may be used to calculate the scattered field without the use of Green's Functions. In fact, this provides an additional alternative for calculating solutions even when the body force is given. The situation is

illustrated using plane wave scattering as an example.

The research topic described above was chosen in the hope that it would constitute a non-trivial contribution to a larger program of research underway in the Geophysics Department at the University of Missouri-Rolla. This work (Stewart (1971), Rechtien and Stewart (1971), and Stewart and Rechtien (1971)) is aimed at establishing a seismic method for the detection and delineation of subterranean cavities in the earth. The method, based on the presumption of cavity resonance, excludes the study of seismic sources, at least at the outset. In contrast, the present dissertation is source oriented from the beginning. Hopefully, the two approaches will ultimately be found to supplement each other. A successful method for cavity detection and delineation would indeed be a useful commodity in many areas, as amply pointed out by Stewart (1971).

A review of existing literature on various aspects of the problems considered in the current dissertation forms part of Chapter I. The review is incorporated into a brief discussion of how the current dissertation is organized.

I. ORGANIZATION; LITERATURE REVIEW; NOTATION

A. Multipole Theory

As mentioned in the Introduction, elastodynamic multipole theory is used in this dissertation. Multipole theory exists also in Electromagnetics (Morse and Feshbach (1953), p. 1276) and in acoustics (Oestreicher (1957)). Also, it has been developed for both the static and dynamic cases in elasticity (Archambeau (1968)). However, for reasons to be explained shortly, the form of multipole theory used in the current dissertation is developed in Chapters II and III.

In texts and reference books concerning electromagnetics, it is common to find two independent developments of the theory. The first, based on a 3-dimensional Taylor series expansion of 1/R, where R is the distance between two points in space, is carried out mainly for the purpose of providing a clear physical understanding of multipoles. Solutions in terms of infinite series of multipole fields are easy to understand on a term by term basis. The intuitive approach is possible throughout. However, as pointed out by Morse and Feshbach (1953), p. 1279, the series are redundant (also see Appendix D here). That is, other series expansions of the function 1/R exist, and multipole theory based upon them can result in multipole series solutions requiring fewer terms to express information requiring more terms of the former series. The most common example found in the literature is the expansion of 1/R in spherical harmonics. In view of the fact that there exist at least seven ways in which 1/R may be expanded in infinite series (Van Nostrand and Cook (1966), p. 75), it is not quite clear why spherical harmonics have received the most attention.

The same situation carries over into elastodynamics, where again spherical harmonics are the favored functions. Archambeau (1968) gives multipole series solutions for various field quantities arising in infinite media due to distributed body forces occupying a non-zero volume. The series involve spherical harmonics.

The current dissertation relies on multipole theory arising from a Taylor's expansion of 1/R. In the resulting multipole series, each term is the field of a fundamental multipole force. It is not difficult to gain a physical understanding of individual terms in the series; a feat not easily attainable in other types of expansions.

As stated earlier, the development of the form of the theory used later in this dissertation is developed in Chapters II and III. These two chapters concern fully homogeneous infinite media. Multipole series solutions for various field quantities arising due to body forces, i.e., sources, of finite geometrical extent are derived. Chapter II concerns the static case, which is also applicable in the dynamic case covered in Chapter III. This applicability stems from the fact that the dynamic sources throughout the entire dissertation are formed by the multiplication of a static body force term by a time function. Such sources are termed separable by Archambeau (1968). No displacement fields for the static case are actually calculated in Chapter II since these may be specialized from dynamic solutions given in Chapter III.

It seems that usage of multipole theory in elastodynamics is fairly recent, although the fields for any individual multipole were available in the time of Stokes (Love (1944), p. 305). Its application

in earthquake seismology as illustrated by papers such as Randall (1971a and 1971b) has been the main source of employment.

The primary application of multipole theory in the current dissertation is encountered in Chapter IV. There, linear combinations of multipole fields are used to describe other fields known to arise at scattering surfaces. These linear combinations are not multipole series in the sense of Chapters II and III. In Chapter IV, the traction vectors arising on scattering surfaces are approximately nullified by linear combinations of multipole traction vectors. Coefficients corresponding to each multipole traction vector are calculated using a least mean squared error criterion. Then, a highly important feature of multipole series which carries over to the calculated linear combinations gives the approximate scattered displacement fields with no additional work. A standard text such as Hildebrand (1956) is adequate background for the least squares calculations.

B. Integral Representations

Integral representations of solutions arise often in applied mathematics. A well known example is Kirchoff's Formula (Love (1944), p. 301), which represents solutions of the scalar wave equation. When it is the more general vector elastic equation of motion whose solutions are to be represented, the more complicated representation introduced in Chapter V is required.

Often integral representations serve as intermediate steps in the derivation of integral equations describing a system from differential equations and boundary conditions also describing the system. This is not their value to the current dissertation; they are used to improve

upon the first approximations derived via least squares in Chapter IV. The calculation of the second and final approximations to solutions occupies Chapter V. That an improvement is indeed attained is argued in Appendix E, where a brief derivation of the integral representation is also presented following Case and Colewell (1967).

Integral representations have long been associated with Green's Theorem and Green's Functions in Potential Theory (Love (1944), p. 231). In discussing this, Love points out that it was Betti, the author of Betti's Reciprocal Theorem (Love (1944), p. 173), who first applied similar notions to elasticity. In fact, the integral representation used here in Chapter V can be obtained from Betti's Theorem. In more recent times, other authors have also written on integral representations in elasticity in order to enlarge the scope of their application, e.g., to include anisotropic media, and to apply them in solving problems. These authors include DeHoop (1958), whose interest was in diffraction of plane waves by screens; Korringa (1965), who was interested in variational principles related to diffraction; Case and Colewell (1967), who derived integral equations from the representations of solutions to cavity-source problems and solved them; and Gangi (1970), whose primary interest was in showing that when certain reciprocity principles are assumed, then the integral representations follow as consequences. Some authors consider the more simple steady state representations such as those of the current dissertation while others, e.g., Gangi (1970), consider the time dependence to be arbitrary.

Banaugh (1964) has used integral representations also to treat scattering problems involving surfaces of arbitrary shape. However, his representations are of displacement potentials rather than of

displacement itself as in the current dissertation and the above works.

C. Other Topics

Among additional applications of the current work, Chapter VI includes a suggested method for the treatment of multiple cavities, a discussion of how the cavity-source problem may be solved, and a few remarks on scattering from rigid bodies.

Chapter VII presents some numerical results computed using the theory developed in the current dissertation. A summary of some sources of error and some background material required for understanding the diagrams presented there are also included.

D. Notation

Throughout this dissertation, only rectangular coordinate systems are used. Coordinate variables and base vectors are denoted by x_i and $\hat{e}_i$, respectively, for i = 1,2,3. The position vector of a point (x_1,x_2,x_3) is denoted by $\bar{r}$. Almost always, such a point will be verbally referred to as "the point $\bar{r}$," following customary usage.

The vector $\bar{r}$ is $x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$, which can be written as $\bar{r} = x_i\hat{e}_i = x_m\hat{e}_m$ using the so-called summation convention. This convention is that a repeated index in a term implies a summation over the values of that index. Thus $A_{mn}B_{mn}$ stands for an expression consisting of the sum of nine terms. In this dissertation, no distinction will be made between superscripts and subscripts insofar as the summation convention is concerned. That is, $A_{mnj}B^{j}_{mn}$ will stand for the sum of 27 terms for example. The summation convention is in effect throughout the entire dissertation.

Often it is necessary to discuss several spatial points at the same time. These are distinguished by such symbols as $\bar{r}_o = x_{oi}\hat{e}_i$, $\bar{r}_1 = x_{1i}\hat{e}_i$, $\bar{r}' = x'_i\hat{e}_i$, or perhaps $\bar{r}'' = x''_i\hat{e}_i$. In this connection, it must also be pointed out that partial differentiation is done at various stages with respect to all the spatial coordinates forming the components of the various position vectors just given. For this, operator notation will be used, i.e., $D_i = \partial/\partial x_i$, $D^o{}_i = \partial/\partial x_{oi}$, $D^1{}_i = \partial/\partial x_{1i}$, etc. Additional indices on the D imply additional differentiation, e.g., $D_{ij} = \partial^2/\partial x_i \partial x_j$, etc.

Functions of position are denoted by symbols such as $A(\bar{r})$ or $\bar{B}(\bar{r}) = B_m(\bar{r})\hat{e}_m$, depending on whether the function is a scalar or vector. Components of tensors or other functions requiring more than one index will be encountered often. In cases where a function depends on the components of $\bar{r}'$, say, rather than $\bar{r}$, then of course symbols such as $A(\bar{r}')$ will be used.

When a function depends on two sets of coordinates, say the components of $\bar{r}$ and $\bar{r}_o$, then symbols such as $A(\bar{r}|\bar{r}_o)$ may be used. This rule is not rigidly followed in cases where the dependence on the second set of coordinates does not need to be emphasized. A consistent exception to the rule occurs with the Dirac delta function, which is always denoted by $\delta(\bar{r} - \bar{r}_o)$ or other symbols involving other position vectors. The Dirac delta function should not be confused with Kronecker's delta, δ_{ij}, which also will be encountered often. The latter quantity is the number 1 when its two subscripts happen to be numerically equal and zero otherwise.

There is a mixture of subscript notation and so-called dyadic notation used in this dissertation. For example, if $\overline{A} = A_m \hat{e}_m$, then the divergence of $\overline{A}$ is both $\nabla \cdot \overline{A}$ and $D_m A_m$. No apology is made for this mixture; some ideas are best presented and understood in one notation while other ideas favor the other notation.

Surface integrals appear often in this dissertation. The subscript S on a single integral sign will imply integration over all of a surface named S. On volume integrals, the subscript V on a single integral sign implies integration over a volume named V. The subscript a.s. implies integration over all space.

II. STATIC BODY FORCES AND SOURCE POTENTIALS

This dissertation is a study of the displacement field generated in an elastic solid by body forces. The present chapter defines the various types of forces to be considered. All body forces will be of the type Archambeau (1968) has termed separable, i.e., each body force will have a mathematical specification or formula which consists of two factors. The first factor, to be called the "space part", depends only on spatial coordinates while the second factor is a function of time alone. This chapter is concerned only with the space part; the time factor will be introduced in Chapter III.

A. Forces Distributed Through Finite Volumes

A static, i.e., time independent, body force $\overline{F}(\overline{r})$ is a vector function of position having the dimensions of force per unit volume. This section will consider only those body forces which are non-zero throughout a finite volume V. In this section then, V is neither infinite nor infinitesimal. The body force may be called a volume source density.

Using Helmholtz's Theorem (Appendix A), the vector function $\overline{F}$ may be written in terms of a scalar potential and a vector potential. These potentials may be called source potentials to distinguish them from other potentials to be considered later.

$$\overline{F}(\overline{r}) = -\nabla\phi(r) + \nabla\times\overline{\psi}(\overline{r}), \tag{2.1}$$

where $\phi(\overline{r})$ and $\overline{\psi}(\overline{r})$ are the scalar and vector source potentials, respectively. It is shown in Appendix A that

$$\phi(\overline{r}) = \nabla\cdot\overline{W}(\overline{r}) \tag{2.2}$$

and

$$\bar{\psi}(\bar{r}) = \nabla\times\bar{W}(\bar{r}), \tag{2.3}$$

where

$$\bar{W}(\bar{r}) = \frac{1}{4\pi}\int_{a.s.} \frac{\bar{F}(\bar{r}')}{R}\, dv'. \tag{2.4}$$

In Equation (2.4), $\bar{r}'$ is the position of the volume element dv'. The factor 1/R is the reciprocal of the distance between $\bar{r}'$ and $\bar{r}$, i.e., $R = |\bar{r} - \bar{r}'|$.

It is demonstrated in Appendix A that if the expressions for ϕ and $\bar{\psi}$ in terms of $\bar{W}$ are inserted into Equation (2.1), then indeed

$$\bar{F} \equiv -\nabla(\nabla\cdot\bar{W}) + \nabla\times\nabla\times\bar{W}. \tag{2.5}$$

Just as Equation (2.1) gives $\bar{F}$ in terms of source potentials, Equation (2.5) serves the same purpose. One may refer to $\bar{W}$ as the "parent" source potential.

In a later section it will be shown that body forces distributed through finite volumes may be written as infinite series (multipole series) having particularly useful properties.

B. Forces Distributed Through Vanishing Volumes

1. Point Distributions.--Consider a body force of the type just discussed defined throughout a volume V. In particular, let

$$\bar{F}(\bar{r}) = P\hat{n},\ \bar{r} \text{ in } V$$

$$= 0,\ \bar{r} \text{ not in } V, \tag{2.6}$$

where P is a constant and $\hat{n}$ is a constant unit vector in some direction. Now, remove the restrictions that V be finite and P be constant. Let P become infinite and V infinitesimal in such a way that

$$\lim_{\substack{P \to \infty \\ V \to 0}} P \int_V dv' = 1. \tag{2.7}$$

The body force defined by such a limiting procedure is non-zero only at a single point in space, namely the point to which V was "shrunk". Call this point $\bar{r}_o$. The resultant force is a point force, of magnitude unity and in the direction of $\hat{n}$, located at $\bar{r}_o$. The meaning of such point forces is best understood in terms of volume integrals. For example, consider an integral of the type

$$\bar{I} = \int_{a.s.} \bar{F}(\bar{r}')A(\bar{r}')dv', \tag{2.8}$$

where $A(\bar{r})$ is some function of position. Let $\bar{F}(\bar{r})$ be given by Equation (2.6), where V contains $\bar{r}_o$. Then,

$$\bar{I} = P\hat{n} \int_V A(\bar{r}')dv'. \tag{2.9}$$

If it is further assumed that $A(\bar{r})$ is sufficiently well behaved that it has a finite average value, say $\{A(\bar{r})\}$, over every infinitesimal neighborhood of $\bar{r}_o$, then as $V \to 0$, this average will become $A(\bar{r}_o)$. Since

$$\bar{I} \simeq \{A(\bar{r})\}\hat{n}P \int_V dv', \tag{2.10}$$

then as $P \to \infty$ and $V \to 0$,

$$\bar{I} \to \hat{n}A(\bar{r}_o). \tag{2.11}$$

If it had been specified, prior to writing Equation (2.8), that $\bar{F}(\bar{r})$ was ultimately going to be a point force of unit magnitude at $\bar{r}_o$ in the direction of $\hat{n}$, one would now write

$$\bar{I} = \hat{n}A(\bar{r}_o). \tag{2.12}$$

In order to avoid the limiting process each time it is necessary to consider a volume integral of a point force, one may use the Dirac delta "function" in writing the mathematical specification of the point force. Some comments concerning the delta function are made in Appendix A. The particular point force just discussed would be

$$\bar{F}(\bar{r}) = \hat{n}\delta(\bar{r} - \bar{r}_o). \tag{2.13}$$

$\bar{F}(\bar{r})$ in Equation (2.13) is called the "equivalent volume source density" corresponding to the force described by the limiting process. This terminology has been used by Stakgold (1967); the concept will be encountered several times in this dissertation. Using Equation (2.13) in Equation (2.8), agreement with Equation (2.12) is shown:

$$\bar{I} = \hat{n} \int_{a.s.} \delta(\bar{r}' - \bar{r}_o)A(\bar{r}')dv' = \hat{n}A(\bar{r}_o). \tag{2.14}$$

Notice from Equation (2.8) that $\bar{I}$ should have the physical dimensions of force multiplied by those of A. Moreover, the fact that $\delta(\bar{r}' - \bar{r}_o)dv'$ has no dimensions means that the dimensions of reciprocal volume should be associated with $\delta(\bar{r} - \bar{r}_o)$ itself. In order to make Equation (2.13) define a force per unit volume, it is necessary only to attach a multiplicative factor of unity having the dimensions of force. However, the quantity defined by Equation (2.13) will be called

a dipole force (or just "dipole") regardless whether or not the units of force have been attached. Sometimes it will be convenient to neglect doing so; in such cases, the fields arising in a medium, e.g., displacement, would be viewed on a "per unit force" basis.

Of particular interest are the three dipoles formed by letting $\hat{n} = \hat{e}_i$, $i = 1,2,3$, in Equation (2.13). These three dipoles will be denoted by

$$\bar{F}_i(\bar{r}|\bar{r}_o) = \hat{e}_i\delta(\bar{r} - \bar{r}_o). \tag{2.15}$$

The symbol $\bar{F}_i$ will be used to denote such dipoles throughout this dissertation although the position vectors in the arguments may at times be different from those just used. For example, the two dipoles sketched in Figure 1 below are given by $-(1/h)\bar{F}_2(\bar{r}|\bar{r}_o)$ and $(1/h)\bar{F}_2(\bar{r}|\bar{r}_o - h\hat{e}_3)$, where h is a small positive length. These two dipoles may be used to illustrate the construction of higher order multipoles. Letting h remain finite and fixed, one can carry out the limiting process employed in Equations (2.8) through (2.12) for each dipole. Adding the results leads to an expression of the form

$$\bar{I} = -(1/h)\hat{e}_2[A(\bar{r}_o) - A(\bar{r}_o - h\hat{e}_3)]. \tag{2.16}$$

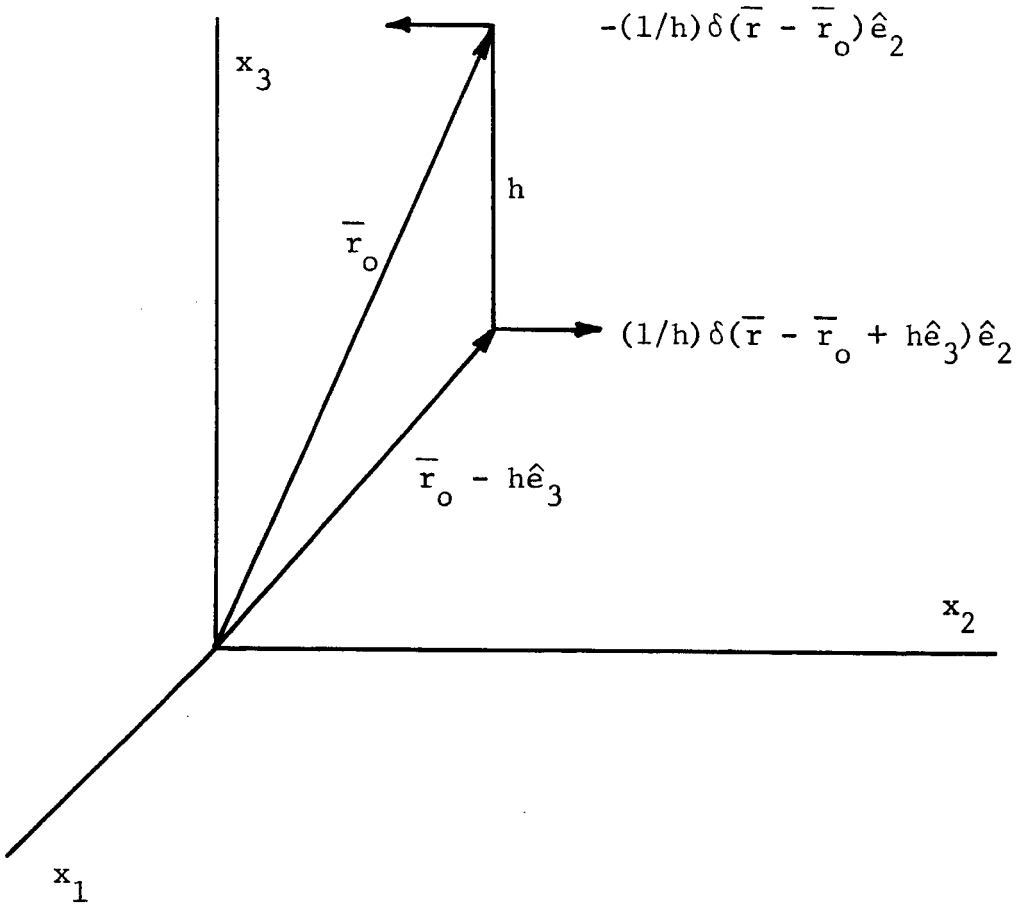

Figure 2.1 Two Dipoles Which Form a Quadrupole as $h \to 0$

The force system shown in Figure (2.1) becomes a quadrupole in the limit $h \to 0$. Quadrupoles are point forces; after the limit $h \to 0$ is taken both dipoles in Figure (2.1) (i.e., the quadrupole) are associated with the point $\bar{r}_o$ only. One may learn of the quadrupole's effect in volume integrals by taking $h \to 0$ Equation (2.16). The result, by definition of partial differentiation, is

$$\lim_{h \to 0} \bar{I} = -\hat{e}_2(\partial/\partial x_{o3})A(\bar{r}_o) = -\hat{e}_2 D^o{}_3 A(\bar{r}_o). \tag{2.17}$$

One may again avoid the limiting processes leading to Equation (2.16), and also the additional limit giving Equation (2.17), by making further use of the properties of the delta function. Referring to Figure (2.1), let the entire force system be denoted by $\bar{F}(\bar{r})$, i.e.,

$$\bar{F}(\bar{r}) = (1/h)[\bar{F}_2(\bar{r}|\bar{r}_o - h\hat{e}_3) - \bar{F}_2(\bar{r}|\bar{r}_o)] \tag{2.18}$$

Now, since

$$\bar{F}_2(\bar{r}|\bar{r}_o - h\hat{e}_3) = \hat{e}_2\delta(\bar{r} - \bar{r}_o + h\hat{e}_3) = \bar{F}_2(\bar{r} + h\hat{e}_3|\bar{r}_o), \tag{2.19}$$

Equation (2.18) can be written

$$\bar{F}(\bar{r}) = (1/h)[\bar{F}_2(\bar{r} + h\hat{e}_3|\bar{r}_o) - \bar{F}_2(\bar{r}|\bar{r}_o)]. \tag{2.20}$$

Letting $h \to 0$,

$$\bar{F}(\bar{r}) \to D_3\bar{F}_2(\bar{r}|\bar{r}_o) = \hat{e}_2 D_3\delta(\bar{r} - \bar{r}_o). \tag{2.21}$$

It is a simple matter to verify that the quadrupole expressed by Equation (2.21) produces the correct results when used in volume integrals. When the expression is substituted into Equation (2.8), Equation (2.17) results immediately.

Notice that the quadrupole just discussed is only one of a set on nine quadrupoles, all of which can be symbolized by

$$\bar{F}_{ij}(\bar{r}|\bar{r}_o) = D_j\bar{F}_i(\bar{r}|\bar{r}_o),\ i = 1,2,3;\ j = 1,2,3. \tag{2.22}$$

Now dipoles are forces per unit volume. Since the quadrupoles are constructed by taking a spatial derivative, they possess an additional dimension of reciprocal length.

Octupoles and all higher order multipoles are constructed by successive differentiation. For example, the set of 27 octupoles is given by

$$\bar{F}_{ijk}(\bar{r}|\bar{r}_o) = D_k\bar{F}_{ij}(\bar{r}|\bar{r}_o). \tag{2.23}$$

Before leaving this section, it will be shown how one may quickly calculate the source potentials for all multipoles. The parent source potential for an arbitrary force is given by Equation (2.4), viz.,

$$\bar{W}(\bar{r}) = \frac{1}{4\pi}\int_{a.s.}\frac{\bar{F}(\bar{r}')}{R}\,dv', \tag{2.24}$$

in which $R = |\bar{r} - \bar{r}'| = [(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2]^{1/2}$. When $\bar{F}(\bar{r})$ in Equation (2.24) is a dipole $\bar{F}_i(\bar{r}|\bar{r}_o)$, the corresponding parent potential will be denoted by $\bar{W}_i(\bar{r}|\bar{r}_o)$. Substitution of Equation (2.15) into the integral in Equation (2.24) yields

$$\bar{W}_i(\bar{r}|\bar{r}_o) = \hat{e}_i/4\pi R_o,\ R_o = |\bar{r} - \bar{r}_o|. \tag{2.25}$$

Substituting $\bar{F}_{ij}(\bar{r}|\bar{r}_o) = \hat{e}_iD_j\delta(\bar{r} - \bar{r}_o)$ into Equation (2.24) and denoting the result by $\bar{W}_{ij}(\bar{r}|\bar{r}_o)$ results in

$$\overline{W}_{ij}(\overline{r}|\overline{r}_o) = -(1/4\pi)\hat{e}_i D^o{}_j(1/R_o) = (1/4\pi)\hat{e}_i D_j(1/R_o). \qquad (2.26)$$

Comparison of Equations (2.25) and (2.26) shows that

$$\overline{W}_{ij}(\overline{r}|\overline{r}_o) = D_j\overline{W}_i(\overline{r}|\overline{r}_o). \qquad (2.27)$$

It is equally easy to show that

$$\overline{W}_{ijk}(\overline{r}|\overline{r}_o) = D_k\overline{W}_{ij}(\overline{r}|\overline{r}_o) = D_{kj}\overline{W}_i(\overline{r}|\overline{r}_o) \qquad (2.28)$$

is the parent source potential for the octupole $\overline{F}_{ijk}(\overline{r}|\overline{r}_o)$. Similarly, one can define corresponding $\overline{W}$ fields for all higher order multipoles; their calculation proceeds by successive differentiation. These facts are easily understood by writing Equation (2.5) for the dipole case, namely,

$$\overline{F}_i(\overline{r}|\overline{r}_o) = -\nabla[\nabla\cdot\overline{W}_i(\overline{r}|\overline{r}_o)] + \nabla\times\nabla\times\overline{W}_i(\overline{r}|\overline{r}_o). \qquad (2.29)$$

Repeated differentiation of both members of this equation reveals the validity of Equations (2.27), (2.28), and the analogous equations involving higher order multipoles.

The calculation of all scalar and vector source potentials $\phi(\overline{r})$ and $\overline{\psi}(\overline{r})$ for all multipoles is accomplished by applying Equations (2.2) and (2.3) to the appropriate source potentials. In doing so, let $\phi_i(\overline{r}|\overline{r}_o)$ correspond to $\overline{F}_i(\overline{r}|\overline{r}_o)$, etc., with similar notation regarding the vector potentials.

There follows

$$\phi_i(\overline{r}|\overline{r}_o) = \nabla\cdot\overline{W}_i(\overline{r}|\overline{r}_o) = (1/4\pi)D_i(1/R_o), \qquad (2.30)$$

$$\phi_{ij}(\overline{r}|\overline{r}_o) = \nabla\cdot\overline{W}_{ij}(\overline{r}|\overline{r}_o) = \nabla\cdot[D_j\overline{W}_i(\overline{r}|\overline{r}_o)]$$

$$= D_j\nabla\cdot\overline{W}_i(\overline{r}|\overline{r}_o) = D_j\phi_i(\overline{r}|\overline{r}_o). \qquad (2.31)$$

Also,

$$\bar{\psi}_i(\bar{r}|\bar{r}_o) = \nabla\times\bar{W}_i(\bar{r}|\bar{r}_o) = (1/4\pi)\nabla\times(\hat{e}_i/R_o)$$

$$= (1/4\pi)\nabla(1/R_o)\times\hat{e}_i. \qquad (2.32)$$

Using $\nabla(1/R_o) = \hat{e}_s D_s(1/R_o) = 4\pi\hat{e}_s\phi_s(\bar{r}|\bar{r}_o)$ in Equation (2.32),

$$\bar{\psi}_i(\bar{r}|\bar{r}_o) = \phi_s(r|r_o)(\hat{e}_s\times\hat{e}_i). \qquad (2.33)$$

Also,

$$\bar{\psi}_{ij} = \nabla\times\bar{W}_{ij}(\bar{r}|\bar{r}_o) = \nabla\times[D_j\bar{W}_i(\bar{r}|\bar{r}_o)]$$

$$= D_j[\nabla\times\bar{W}_i(\bar{r}|\bar{r}_o)] = D_j\bar{\psi}_i(\bar{r}|\bar{r}_o). \qquad (2.34)$$

It should be clear that all higher order source potentials are given by successive differentiation of the lower order source potentials.

At this point a brief demonstration that will be of use later, as well as serving to verify some of the results so far obtained, will be presented. Reconsider Equation (2.1), namely,

$$-\nabla\phi(\bar{r}) + \nabla\times\bar{\psi}(\bar{r}) = \bar{F}(\bar{r}). \qquad (2.35)$$

If the potentials $\phi_i(\bar{r}|\bar{r}_o)$ and $\bar{\psi}_i(\bar{r}|\bar{r}_o)$ are substituted into Equation (2.35), the result should be $\bar{F}_i(\bar{r}|\bar{r}_o)$. This may be verified by making the necessary substitutions from Equations (2.30) and (2.33):

$$-\nabla\phi_i(\bar{r}|\bar{r}_o) + \nabla\times\bar{\psi}_i(\bar{r}|\bar{r}_o) = -\nabla\phi_i + \nabla\times[\phi_s(\hat{e}_s\times\hat{e}_i)]$$

$$= -\nabla\phi_i + (\nabla\phi_s)\times(\hat{e}_s\times\hat{e}_i) = -\hat{e}_m\phi_{im} + \phi_{sm}\hat{e}_m\times(\hat{e}_s\times\hat{e}_i)$$

$$= -\hat{e}_m\phi_{im} + \phi_{sm}(\hat{e}_s\delta_{im} - \hat{e}_i\delta_{sm})$$

$$= -\hat{e}_m\phi_{im} + \hat{e}_s\phi_{si} - \hat{e}_i\phi_{ss}. \qquad (2.36)$$

Comparing Equations (2.30) and (2.31), it is clear that

$$\phi_{is}(\bar{r}|\bar{r}_o) = \phi_{si}(\bar{r}|\bar{r}_o), \tag{2.37}$$

and,

$$\phi_{ss}(\bar{r}|\bar{r}_o) = (1/4\pi)\nabla^2(1/R_o). \tag{2.38}$$

Now, as indicated in Appendix A,

$$\nabla^2(1/R_o) = -4\pi\delta(\bar{r} - \bar{r}_o). \tag{2.39}$$

Thus, Equation (2.38) is

$$\phi_{ss}(\bar{r}|\bar{r}_o) = -\delta(\bar{r} - \bar{r}_o). \tag{2.40}$$

Using Equations (2.37) and (2.40) in Equation (2.36) results in

$$-\nabla\phi_i(\bar{r}|\bar{r}_o) + \nabla\times\bar{\psi}_i(\bar{r}|\bar{r}_o) = \hat{e}_i\delta(\bar{r} - \bar{r}_o) = \bar{F}_i(\bar{r}|\bar{r}_o), \tag{2.41}$$

completing the verification.

2. Surface Distributions.--Let $\bar{r}'$ be the location of an area element dS' on an arbitrary finite surface S. Suppose that a traction (force per unit area), denoted by $\bar{T}(\bar{r}')$ is applied on S. Then, the product $\bar{T}(\bar{r}')dS'$ is a vector force located at $\bar{r}'$. This elemental force is equivalent in its effect to a dipole force density given by

$$\bar{T}(\bar{r}')\delta(\bar{r} - \bar{r}')dS'. \tag{2.42}$$

One says that the traction upon dS' gives rise to this "equivalent volume source density" when $\bar{T}(\bar{r}')$ is applied on S. The equivalent volume source density corresponding to all of S is

$$\bar{F}(\bar{r}) = \int_S \bar{T}(\bar{r}')\delta(\bar{r} - \bar{r}')dS'. \tag{2.43}$$

Equation (2.43) is the formula for a layer of dipoles applied on S. This may be called a single layer of traction. Similarly, one may define double layers on S; these are layers of quadrupoles (or double forces) spread over the surface. An example of such an equivalent volume source density is

$$\bar{F}(\bar{r}) = \int_S \bar{T}(\bar{r}')D_1\delta(\bar{r} - \bar{r}')dS'. \tag{2.44}$$

Other double layers and layers of higher order multipoles can also be defined.

C. Multipole Series for Forces and Source Potentials

1. Finite Volume Distributions.-In this section it will be shown that a general body force $\bar{F}(\bar{r})$ defined throughout a finite volume V may be decomposed into an infinite series of multipole forces. By "decomposed" it is meant that the infinite series produces the same result as the original body force when the former replaces the latter in volume integrals.

Returning to Equation (2.4), the parent potential for $\bar{F}(\bar{r})$ is

$$\bar{W}(\bar{r}) = \frac{1}{4\pi}\int_{a.s.} \frac{\bar{F}(\bar{r}')}{R}\,dv', \quad R = |\bar{r} - \bar{r}'|. \tag{2.45}$$

As shown in Appendix A, the function 1/R may be expanded in a 3-dimensional Taylor series about a point $\bar{r}_o$ within the region V. The series is in powers of the increments $q_i' = x_i' - x_{oi}$, because in making the expansion $\bar{r}'$ is the independent variable. That is, the field point $\bar{r}$ is considered fixed, appearing in the series only parametrically.

After the expansion is made, certain manipulations transform it into the following form:

$$1/R = 1/R_o - q_i' D_i(1/R_o) + (1/2!) q_i' q_j' D_{ij}(1/R_o)$$

$$-(1/3!) q_i' q_j' q_k' D_{ijk}(1/R_o) + \ldots - \ldots \,. \qquad (2.46)$$

Here, $R_o = |\bar{r} - \bar{r}_o|$. The summation convention is in effect.

The series (2.46) will converge at points $\bar{r}'$ located such that

$$|\bar{r}' - \bar{r}_o| < |\bar{r} - \bar{r}_o|, \qquad (2.47)$$

as indicated in Appendix A. A verbal statement of the inequality (2.47) is that the field point $\bar{r}$ is outside the smallest sphere centered at $\bar{r}_o$ which completely contains the source (and hence all source points $\bar{r}'$).

The series (2.46) is unique in the sense that no other power series taken about $\bar{r}_o$ will represent 1/R. It is, of course, not unique in the sense that other points $\bar{r}_o$ could be selected as expansion points.

Now the series (2.46) is inserted into the integral in Equation (2.45), being careful to use only those field points $\bar{r}$ which satisfy inequality (2.47) for all $\bar{r}'$ in V. The integration is done term by term. Factors such as $1/R_o$, $D_i(1/R_o)$, $D_{ij}(1/R_o)$, etc., are fixed during integration and may be removed from within their respective integrals. Thus, Equation (2.45) becomes

$$\bar{W}(\bar{r}) = [\int_{a.s.} \bar{F}(\bar{r}')\,dv'](1/4\pi R_o)$$

$$-[\int_{a.s.} q_i' \bar{F}(\bar{r}')\,dv'] D_i(1/4\pi R_o) + \ldots - \ldots \,. \qquad (2.48)$$

Let $\bar{F}(\bar{r}') = \hat{e}_s F_s(\bar{r}')$ and $\bar{W}(\bar{r}) = \hat{e}_s W_s(\bar{r})$. Also make the definitions

$$G_s = \int_{a.s.} F_s(\bar{r}')dv',$$

$$G_{si} = \int_{a.s.} q_i' F_s(\bar{r}')dv',$$

$$G_{sij} = \int_{a.s.} q_i' q_j' F_s(\bar{r}')dv', \text{ etc.} \tag{2.49}$$

In these definitions, the terms F_s and W_s are merely the rectangular components of $\bar{F}$ and $\bar{W}$, respectively, and <u>not</u> the dipoles $\bar{F}_s(\bar{r}|\bar{r}_o)$ and their parent potentials. Upon using the definitions (2.49) in Equation (2.48), it follows that the rectangular components of $\bar{W}$ are given by

$$W_s(\bar{r}) = [(G_s - G_{si}D_i + (1/2!)G_{sij}D_{ij} - \ldots + \ldots](1/4\pi R_o) \tag{2.50}$$

Now define the operators in Equation (2.50) to be L_s, for s = 1,2,3. That is, let

$$L_s = G_s - G_{si}D_i + (1/2!)G_{sij}D_{ij} - (1/3!)G_{sijk}D_{ijk} + \ldots \tag{2.51}$$

Now, Equation (2.50) is

$$W_s(\bar{r}) = L_s[1/4\pi R_o]. \tag{2.52}$$

Equation (2.52) is the multipole series representation for the components of $\bar{W}$ due to a general body force acting in a finite region. $\bar{W}$ itself is given by

$$\bar{W}(\bar{r}) = L_s[\hat{e}_s/4\pi R_o], \tag{2.53}$$

which clearly shows the relevance of the name "multipole series." The term in brackets is merely $\bar{W}_s(\bar{r}|\bar{r}_o)$, the parent potential due to

the dipole $\overline{F}_s(\overline{r}|\overline{r}_o)$ as shown by Equation (2.25). Thus, each individual term in the series (2.53) is the parent potential of some multipole. Equation (2.53) may be written

$$\overline{W}(\overline{r}) = L_s[\overline{W}_s(\overline{r}|\overline{r}_o)]. \tag{2.54}$$

To calculate multipole series forms for the scalar source potential $\phi(\overline{r})$, Equation (2.2) is used:

$$\phi(\overline{r}) = \nabla\cdot\overline{W}(\overline{r}) = L_s[\nabla\cdot\overline{W}_s(\overline{r}|\overline{r}_o)] = L_s[\phi_s(\overline{r}|\overline{r}_o)], \tag{2.55}$$

where $\phi_s(\overline{r}|\overline{r}_o)$ is given by Equation (2.30). Similarly, the vector source potential $\overline{\psi}(\overline{r})$ is given by

$$\overline{\psi}(\overline{r}) = \nabla\times\overline{W}(\overline{r}) = L_s[\nabla\times\overline{W}_s(\overline{r}|\overline{r}_o)] = L_s[\overline{\psi}_s(\overline{r}|\overline{r}_o)], \tag{2.56}$$

where $\overline{\psi}_s(\overline{r}|\overline{r}_o)$ is given by Equation (2.32).

The body force $\overline{F}(\overline{r})$ itself is given by

$$\begin{aligned}\overline{F}(\overline{r}) &= -\nabla\phi(\overline{r}) + \nabla\times\overline{\psi}(r) \\ &= L_s[-\nabla\phi_s(\overline{r}|\overline{r}_o) + \nabla\times\overline{\psi}_s(\overline{r}|\overline{r}_o)] = L_s[\overline{F}_s(\overline{r}|\overline{r}_o)],\end{aligned} \tag{2.57}$$

where Equation (2.41) was used in the last step. Also by Equation (2.41), the components of $\overline{F}$ are given by

$$F_s(\overline{r}) = L_s[\delta(\overline{r} - \overline{r}_o)]. \tag{2.58}$$

In writing Equations (2.57) and (2.58) it must remembered that both are to be interpreted in a "generalized" sense, i.e., the equalities hold because the series on the right sides produce the same results as the corresponding left sides when used in volume integrals over the

region V.

Another point to be made concerns the physical dimensions associated with quantities expressed in multipole series form. By observing the dimensions of the G's defined by Equations (2.49), it may be seen from Equation (2.51) that L_s effectively carries the dimensions of force. This is because the dimension of D_i is reciprocal length in its effect. Now, looking at the definition of $\bar{F}_s(\bar{r}|\bar{r}_o)$, Equation (2.15), recall that the dipoles have the dimension of force only if a multiplicative factor of unity carrying that dimension is attached. In interpreting Equation (2.57), this factor is unnecessary because L_s supplied the dimension of force. The operators L_s will sometimes be referred to as "force operators".

A final point concerning multipole series in general relates to the coefficients, the G's, defined by Equations (2.49). These coefficients did not stem from the original Taylor coefficients in expansion (2.46). Instead, they evolved from the "powers" of the increments q_i'. The Taylor coefficients in the power series went on to become multipole field quantities, a fact made possible by their parametric dependence upon the field point $\bar{r}$.

2. Point Distributions.--It should be obvious that this subsection is redundant: A multipole force and all its fields are their own multipole expansions. A pause will be made here, however, to verify this for quadrupoles--any other verification would proceed along the same lines.

The s component of a quadrupole $\bar{F}_{mn}(\bar{r}|\bar{r}_o)$ is given by

$$\delta_{sm} D_n \delta(\bar{r} - \bar{r}_o), \tag{2.59}$$

as may be seen by comparing Equations (2.22) and (2.15). Using Equation (2.59), the multipole coefficients are easily calculated from the definitions (2.49):

$$G_s = \int_{a.s.} \delta_{sm} D'_n \delta(\bar{r}' - \bar{r}_o) dv'$$

$$= -\delta_{sm} \int_{a.s.} \delta(\bar{r}' - \bar{r}_o) D'_n(1) dv' = 0, \text{ for all } s. \qquad (2.60)$$

$$G_{si} = \delta_{sm} \int_{a.s.} q_i' D'_n(\bar{r}' - \bar{r}_o) dv'$$

$$= -\delta_{sm} \int_{a.s.} \delta(\bar{r}' - \bar{r}_o) D'_n(q_i') dv'. \qquad (2.61)$$

Now, $q_i' = x_i' - x_{oi}$, so that $D'_n(q_i') = \delta_{in}$. Thus Equation (2.61) becomes

$$G_{si} = -\delta_{sm}\delta_{in}. \qquad (2.62)$$

Continuing,

$$G_{sij} = \delta_{sm} \int_{a.s.} q_i' q_j' D'_n \delta(\bar{r}' - \bar{r}_o) dv'$$

$$= -\delta_{sm} \int_{a.s.} \delta(\bar{r}' - \bar{r}_o)(q_i'\delta_{jn} + q_j'\delta_{in}) dv' = 0 \qquad (2.63)$$

because q_i' evaluated at $\bar{r}' = \bar{r}_o$ is zero for all i. All higher order coefficients, G_{sijk}, etc., are zero for the same reason. Thus, only the G_{si} are nonzero. From Equations (2.51) and (2.62), the operators L_s are

$$L_s = -G_{si}D_i = -(-\delta_{sm}\delta_{in})D_i = \delta_{sm}D_n. \qquad (2.64)$$

Using Equation (2.64) in Equations (2.55) through (2.58) completes the verification.

3. Surface Distributions.--Multipole series for surface distributions are obtained simply by substituting the corresponding equivalent volume source density into the definition (2.49). For example, if the layer is the single layer given by Equation (2.43), then substitution into the first of Equations (2.49) leads to

$$G_s = \int_{a.s.} \{\int_S T_s(r')\delta(r'' - r')dS'\}dv''$$

$$= \int_S T_s(r')\{\int_{a.s.} \delta(r'' - r')dv''\}dS'$$

$$= \int_S T_s(r')dS'. \qquad (2.65)$$

The remaining G's may be expressed as surface integrals in a similar manner. The treatment of double or higher order layers follows the same lines.

III. DISPLACEMENT IN AN INFINITE ELASTIC SOLID

This chapter introduces time dependence into the body forces, i.e., sources, described in the preceding chapter. Such sources will be assumed operating in an infinite, isotropic, homogeneous, perfectly elastic solid medium. Both integral and multipole series solutions for the displacement field will be given. Compatibility between the two types will be shown. The dynamic solutions of the present chapter include static solutions as special cases.

A. Equation of Motion and Displacement Potentials

1. General.--The displacement field due to body forces acting in media such as that described above is governed by the following equation of motion (Love (1944), p. 293):

$$(\partial^2/\partial t^2)\bar{U} = \alpha^2\nabla(\nabla\cdot\bar{U}) - \beta^2\nabla\times\nabla\times\bar{U} + (1/\rho)\bar{K}, \qquad (3.1)$$

in which

$\bar{U} = \bar{U}(\bar{r},t)$ is the displacement field vector;

$\bar{K} = \bar{K}(\bar{r},t)$ is a dynamic source term;

ρ = mass density of the medium;

$\alpha^2 = (\lambda + 2\mu)/\rho$;

$\beta^2 = \mu/\rho$; where λ and μ are Lame's constants.

In most situations, α and β are the velocities of propagation of waves.

It is shown in Appendix B how Equation (3.1) may be written in another well known form in terms of U_m, the components of $\bar{U}$, and T_{mn}, the stress tensor components calculated from $\bar{U}$. That form is (Love (1944), p. 85),

$$(\partial^2/\partial t^2)U_m = (1/\rho)D_n T_{mn} + (1/\rho)K_m. \tag{3.2}$$

Recall that this dissertation deals only with separable sources, i.e., those which may be factored as follows:

$$\bar{K}(\bar{r},t) = \bar{F}(\bar{r})f(t). \tag{3.3}$$

In Equation (3.3), $\bar{F}(\bar{r})$ is any vector force per unit volume, i.e., volume source density, which vanishes outside a finite region in space. The dimensionless function f of time t is arbitrary, with the stipulation that certain integrals and derivatives (to be encountered later) have meaning.

In order to solve Equation (3.1), one may express the displacement $\bar{U}$ in terms of potentials $\Phi(\bar{r},t)$ and $\bar{\Psi}(\bar{r},t)$ using Helmholtz's Theorem. That is,

$$\bar{U}(\bar{r},t) = -\nabla\Phi(\bar{r},t) + \nabla\times\bar{\Psi}(r,t). \tag{3.4}$$

As shown in Appendix B, one may also define a vector field $\bar{V}_\alpha(\bar{r},t)$ such that

$$\Phi(\bar{r},t) = \nabla\cdot\bar{V}_\alpha(\bar{r},t). \tag{3.5}$$

The parameter α appears in the expression for $\bar{V}_\alpha(\bar{r},t)$. By replacing α with β in $\bar{V}_\alpha$, one can construct a new field vector denoted say, by $\bar{V}_\beta(\bar{r},t)$. It then happens that

$$\bar{\Psi}(r,t) = \nabla\times\bar{V}_\beta(\bar{r},t). \tag{3.6}$$

Equations (3.5) and (3.6) are analogs of Equations (2.2) and (2.3); $\bar{V}_\alpha$ is a parent displacement potential.

Each of the quantities Φ, $\overline{\Psi}$, $\overline{V}_\alpha$, and $\overline{V}_\beta$ satisfy certain wave equations given in Appendix B. As shown there, $\overline{V}_\alpha(\overline{r},t)$ satisfies

$$(\partial^2/\partial t^2)\overline{V}_\alpha - \alpha^2\nabla^2\overline{V}_\alpha = (1/\rho)f(t)\overline{W}, \tag{3.7}$$

where $\overline{W}(\overline{r})$ is given by Equation (2.4). Also,

$$(\partial^2/\partial t^2)\overline{V}_\beta - \beta^2\nabla^2\overline{V}_\beta = (1/\rho)f(t)\overline{W}. \tag{3.8}$$

Notice that the only difference in Equations (3.7) and (3.8) is in the occurence of α and β.

2. Point Forces.--Since rectangular coordinates are being used, vector Equation (3.7) is satisfied on a component by component basis. Moreover, since only particular solutions are sought here, if a component of $\overline{W}$ were to vanish everywhere then the corresponding components of $\overline{V}_\alpha$ and $\overline{V}_\beta$ would also be taken to be zero everywhere. This situation arises when one considers the special case in which $\overline{W}$ corresponds to a dipole $\overline{F}_s(\overline{r}|\overline{r}_o)$. Denoting the resulting $\overline{V}$ fields as $\overline{V}_{\alpha s}(\overline{r},t|\overline{r}_o)$ and $\overline{V}_{\beta s}(\overline{r},t|\overline{r}_o)$, Equations (3.7) and (3.8) become

$$(\partial^2/\partial t^2 - \alpha^2\nabla^2)\overline{V}_{\alpha s}(\overline{r},t|\overline{r}_o) = (1/\rho)f(t)\overline{W}_s(\overline{r}|\overline{r}_o) \tag{3.9}$$

and

$$(\partial^2/\partial t^2 - \beta^2\nabla^2)\overline{V}_{\beta s}(\overline{r},t|\overline{r}_o) = (1/\rho)f(t)\overline{W}_s(\overline{r}|\overline{r}_o). \tag{3.10}$$

From Equation (2.25), it may be seen that $\overline{W}_s(\overline{r}|\overline{r}_o)$ has only the component $(1/4\pi R_o)\hat{e}_s$ for a given value of s. Thus, $\overline{V}_{\alpha s}$ and $\overline{V}_{\beta s}$ will have only the $\hat{e}_s$ component; it must be the same in each case for the three values of s = 1,2,3. Therefore $\overline{V}_{\alpha s}$ is of the form $\hat{e}_s\Phi_o$ while

$\bar{V}_{\beta s}$ is of the form $\hat{e}_s \eta_o$, where Φ_o and η_o are properly chosen functions. Clearly, whatever the form of Φ_o, one may obtain η_o from it by simply replacing the parameter α with β. Upon using Equations (3.5) and (3.6) the displacement potentials are

$$\Phi_s(\bar{r},t|\bar{r}_o) = \nabla\cdot\bar{V}_{\alpha s} = \nabla\cdot(\hat{e}_s\Phi_o) = D_s\Phi_o, \tag{3.11}$$

and

$$\bar{\Psi}_s(\bar{r},t|\bar{r}_o) = \nabla\times\bar{V}_{\beta s} = \nabla\times(\hat{e}_s\eta_o) = D_m\eta_o(\hat{e}_m\times\hat{e}_s), \tag{3.12}$$

in which Φ_s and $\bar{\Psi}_s$ are the scalar and vector displacement potentials for a dipole along $\hat{e}_s$. Since η_o may be obtained from Φ_o using β for α, the same is true for $D_m\eta_o$ and all other derivatives. By Equation (3.11), Φ_m is $D_m\Phi_o$; therefore it is appropriate to denote $D_m\eta_o$ as η_m.

It is not necessary here to actually construct the fields $\bar{V}_{\alpha s}$ and $\bar{V}_{\beta s}$, for they are not specifically required. Using them in Equations (3.9) through (3.12) showed that only the three scalar potentials Φ_m, m = 1,2,3 are needed; the three vector potential $\bar{\Psi}_m$ may be obtained via Equation (3.12).

The wave equation determining the three dipole scalar potentials $\Phi_s(r,t|r_o)$ is found by taking the divergence of both members of Equation (3.9). Using Equation (2.30) afterward leads to

$$(\partial^2/\partial t^2 - \alpha^2\nabla^2)\Phi_s(\bar{r},t|\bar{r}_o) = (1/\rho)f(t)\phi_s(\bar{r}|\bar{r}_o). \tag{3.13}$$

Similarly, by taking the curl of both members of Equation (3.10) and using Equation (2.32),

$$(\partial^2/\partial t^2 - \beta^2\nabla^2)\bar{\Psi}_s(\bar{r},t|\bar{r}_o) = (1/\rho)f(t)\bar{\psi}_s(\bar{r}|\bar{r}_o). \tag{3.14}$$

By operating through both Equations (3.13) and (3.14) with D_k, and then comparing the resulting right hand sides with Equations (2.31) and (2.34), it follows that the quadrupole displacement potentials are given by

$$\Phi_{sk}(\bar{r},t|\bar{r}_o) = D_k\Phi_s(\bar{r},t|\bar{r}_o), \tag{3.15}$$

and

$$\bar{\Psi}_{sk}(\bar{r},t|\bar{r}_o) = D_k\bar{\Psi}_s(\bar{r},t|\bar{r}_o). \tag{3.16}$$

The higher order potentials, which are obtained by continued differentiation, will be denoted by symbols such as $\Phi_{skn}(\bar{r},t|\bar{r}_o)$, $\bar{\Psi}_{skn}(\bar{r},t|\bar{r}_o)$, etc. One will always be able to use Equation (3.12), e.g.,

$$\bar{\Psi}_{skn}(\bar{r},t|\bar{r}_o) = \eta_{mkn}(\hat{e}_m \times \hat{e}_s), \tag{3.17}$$

in which η_{mkn} is obtained from $\Phi_{mkn} = D_m\Phi_{kn}$ by replacing α with β.

It should be quite clear now that all multipole displacement potentials may be calculated through the solution of a single wave equation, Equation (3.13). This is done in Appendix C.

B. The Fundamental Solution

The displacement fields due to the dipoles $f(t)\bar{F}_j(\bar{r}|\bar{r}_o)$ will be seen to be of fundamental importance throughout this dissertation. When the three displacement components for the j dipole are placed in column j of a 3 x 3 array, the latter is easily seen to be a symmetric second rank tensor. This tensor is sometimes called the "Green's Function for infinite space," especially when f(t) is a delta function. It will be called the fundamental solution, regardless of the nature of f(t),

where it must be realized that in fact there will be a family of fundamental solutions; one for each f(t). Note carefully that the fundamental solution is an array of nine functions. The fact that this array happens to qualify as a tensor is not particularly useful within the context of this dissertation; therefore, very little future reference will be made to it. It is more important for the reader to focus upon the fact that the fundamental solution consists of three sets of displacement components, each set corresponding to one of the three fundamental dipoles.

In order to emphasize the special nature of the fundamental solution, it will be given a special symbol. The i component of displacement due to a dipole $f(t)\overline{F}_j(\overline{r}|\overline{r}_o)$ will be denoted by $Q_{ij}(\overline{r},t|\overline{r}_o)$. Clearly, the explicit functional form of Q_{ij} will depend upon the nature of f(t); the detailed form of Q_{ij} is shown in Appendix C.

The displacement vector due to the dipole $f(t)\overline{F}_j(\overline{r}|\overline{r}_o)$ is given by

$$\hat{e}_i Q_{ij}(\overline{r},t|\overline{r}_o) = -\nabla\Phi_j(\overline{r},t|\overline{r}_o) + \nabla\times\overline{\Psi}_j(\overline{r},t|\overline{r}_o), \tag{3.18}$$

where Equation (3.4) was used. Now, from Equation (3.12),

$$\nabla\times\overline{\Psi}_j(r,t|r_o) = \hat{e}_i(\eta_{ij} - \delta_{ij}\eta_{mm}). \tag{3.19}$$

Then, from Equation (3.18),

$$Q_{ij}(\overline{r},t|\overline{r}_o) = -\Phi_{ij}(\overline{r},t|\overline{r}_o) + \eta_{ij}(\overline{r},t|\overline{r}_o) - \delta_{ij}\eta_{mm}(\overline{r},t|\overline{r}_o). \tag{3.20}$$

As may be seen by observing the specific form of Φ_{ij} given in Appendix C, the following properties of Q_{ij} are obvious:

$$Q_{ij}(\bar{r},t|\bar{r}_o) = Q_{ij}(\bar{r}_o,t|\bar{r}) = Q_{ij}(\bar{r}_o,t|\bar{r}) = Q_{ji}(\bar{r},t|\bar{r}_o). \quad (3.21)$$

The fundamental solution was published by Stokes in 1849 (Love (1944), p. 305). Showing that the result in Appendix C here is the same as Stokes's involves only a change in notation.

The displacement due to a quadrupole $f(t)\bar{F}_{jk}(\bar{r}|\bar{r}_o)$ has the i component

$$D_k Q_{ij} = -\Phi_{ijk} + \eta_{ijk} - \delta_{ij}\eta_{kmm}. \quad (3.22)$$

In expressions such as (3.22) one may occasionally need to use the facts that Φ_{ij}, Φ_{ijkm}, etc., are all completely symmetric with respect to any permutation of their indices.

Before leaving this section, the stress tensor components calculated from the fundamental solution will be written down. Since the latter is a set of three displacement fields, there will be three corresponding stress tensors. The symbol $\Sigma^j_{mn}(\bar{r},t|\bar{r}_o)$ will be used to denote these three tensors for j = 1,2,3.

For any displacement field having components U_i, the stress tensor is given by Hooke's Law, namely,

$$T_{mn} = \lambda\delta_{mn}D_iU_i + \mu D_mU_n + \mu D_nU_m, \quad (3.23)$$

in which T_{mn} are the stress tensor components. Therefore,

$$\Sigma^j_{mn} = \lambda\delta_{mn}D_iQ_{ij} + \mu D_mQ_{nj} + \mu D_nQ_{mj}. \quad (3.24)$$

It may be verified that for a given value of j, Σ^j_{mn} is a second rank tensor, as are all stress tensors. Moreover, with j a free index, Σ^j_{mn} is easily found to be a third rank tensor. This latter property is irrelevant to our work here; it is best to adopt the point of view that Σ^j_{mn} represents three distinct stress tensors, for j = 1,2,3, corresponding to the three fundamental dipoles.

Explicit expressions for Σ^j_{mn} for a given time dependence are given in Appendix C. It may be observed there that

$$\Sigma^j_{mn}(\bar{r},t|\bar{r}_o) = -\Sigma^j_{mn}(\bar{r}_o,t|\bar{r}). \tag{3.25}$$

The equation of motion, Equation (3.2), satisfied by the fundamental solution is

$$(\partial^2/\partial t^2)Q_{ij}(\bar{r},t|\bar{r}_o) = (1/\rho)D_n\Sigma^j_{in}(\bar{r},t|\bar{r}_o) + (1/\rho)f(t)\delta_{ij}\delta(\bar{r} - \bar{r}_o), \tag{3.26}$$

from which it follows, incidently, that

$$D_n\Sigma^j_{mn} = D_n\Sigma^m_{jn}. \tag{3.27}$$

C. Multipole Series for Displacement

The multipole series for displacement potentials, displacement, and stress due to the body force $\bar{K}$ of Equation (3.3) will be derived in this section.

The components $F_i(\bar{r})$ of $\bar{F}(\bar{r})$ can be written as

$$F_i(\bar{r}) = L_i[\delta(\bar{r} - \bar{r}_o)] \tag{3.28}$$

by Equation (2.58). With this in mind, operation upon both members of Equation (3.26) with L_j, results in

$$(\partial^2/\partial t^2)L_j[Q_{ij}(\bar{r},t|\bar{r}_o)] = (1/\rho)D_n L_j[\Sigma^j_{in}(\bar{r},t|\bar{r}_o)]$$

$$+ (1/\rho)f(t)L_i[\delta(\bar{r} - \bar{r}_o)]. \qquad (3.29)$$

Now if,

$$U_i(\bar{r},t) = L_j[Q_{ij}(\bar{r},t|\bar{r}_o)], \qquad (3.30)$$

so that the stress tensor components are

$$T_{in}(\bar{r},t) = L_j[\Sigma^j_{in}(\bar{r},t|\bar{r}_o)], \qquad (3.31)$$

then Equation (3.29) is just Equation (3.2) in multipole series form, where $\bar{r}_o$, the expansion point, is taken as somewhere within or near the region in which $\bar{F}(\bar{r})$ is nonzero.

It does happen that Equations (3.30) and (3.31) are in fact the multipole series forms for the displacement and stress components due to the body force $\bar{K}$; the preceding paragraph may be considered as a formal derivation. Equation (3.30) is obtained from a different viewpoint from the series forms for the displacement potentials. The latter series are

$$\Phi(\bar{r},t) = L_s[\Phi_s(\bar{r},t|\bar{r}_o)], \qquad (3.32)$$

and

$$\bar{\Psi}(r,t) = L_s[\bar{\Psi}_s(r,t|r_o)], \qquad (3.33)$$

which are obtained from Equations (3.13) and (3.14), respectively. In doing so, one must take into account the wave equations satisfied by Φ and $\bar{\Psi}$, which appear in Appendix B. Using Equation (3.4) with help

from Equation (3.19) leads back to Equation (3.30). Then Equation (3.31) follows from Hooke's Law. Still additional routes available for arriving at Equation (3.30) are mentioned in Section D following this section.

Note that the results of this section do not distinguish between "actual" and "equivalent" volume source densities $\overline{F}(\overline{r})$. That is, the results presented here apply to the equivalent volume source densities of Equations (2.43) and (2.44) for example, as well as other separable body forces distributed through finite volumes or on finite surfaces. In any case, the field point $\overline{r}$ must be outside the smallest sphere completely enclosing the source in order to assure convergence of the series solutions.

D. Integral Solutions

Through the use of elementary physical arguments, it is possible to write down well known integral solutions for every field quantity so far discussed in multipole series form. The situation will be illustrated using displacement as an example.

Let a separable source density (actual or equivalent) denoted by $\overline{K}(\overline{r}',t)$ act in a finite region. Let the volume element dv' occupy the point $\overline{r}'$. Then, at $\overline{r}'$, there exists a point force $f(t)\overline{F}(\overline{r}')dv'$. This force has three components, each being a dipole multiplied by a "weighting factor". The displacement produced by each such dipole then will be given by the fundamental solution multiplied by the same weighting factor. Thus, the i component of displacement due to $f(t)\overline{F}(\overline{r}')dv'$ is

$$F_s(\overline{r}')dv'Q_{is}(\overline{r},t|\overline{r}'). \tag{3.34}$$

The displacement due to the entire force distribution will be the volume integral of expression (3.34), i.e.,

$$U_i(\bar{r},t) = \int_{a.s.} F_s(\bar{r}')Q_{is}(\bar{r},t|\bar{r}')dv'. \qquad (3.35)$$

This is an integral solution for the displacement components.

If a point $\bar{r}_o$ is chosen as an expansion point while $\bar{r}$ is taken outside the smallest sphere centered at $\bar{r}_o$ completely enclosing the source, then a Taylor series for $Q_{is}(\bar{r},t|\bar{r}')$ in powers of $x_k' - x_{ok}$ will converge at all $\bar{r}'$ within the source. When this Taylor series is substituted into Equation (3.35), the multipole series, Equation (3.30) results.

Finally, one may substitute the multipole series for $F_s(\bar{r})$, Equation (3.28), into Equation (3.35). This procedure also results in Equation (3.30), the multipole series for displacement.

IV. VOID CAVITIES--FIRST APPROXIMATIONS TO GREEN'S FUNCTIONS

This chapter begins the treatment of the complications which arise when void finite cavities of arbitrary shape are introduced into otherwise infinite, isotropic, homogeneous, perfectly elastic solid media. Only separable body forces will be treated. Furthermore, the time factor f(t) in these will be $\exp(-i\omega t)$. This specialization permits, along with other simplifications, the scattered fields to be interpreted as due to certain separable force layers acting on the cavity boundary, i.e., the layers will consist of a spatial part multiplied by the time factor $\exp(-i\omega t)$.

A. Green's Functions

The counterpart of the fundamental solution under these new circumstances is a Green's Function. This latter term refers to an indexed set of three displacement fields, i.e., to an array of nine functions, just as did the term fundamental solution. Three dipole forces are set in action, one by one, outside the cavity; the resulting displacement fields form the components of the Green's Function.

Obviously the Green's Function must satisfy the same equation of motion as the fundamental solution. In addition, the Green's Function must satisfy whatever boundary conditions one imposes at the cavity boundary. The use of the word "void" is meant here to imply the condition that traction vanishes on the boundary. This boundary condition, along with the equation of motion, determines the Green's Function.

It should be clear that each Green's Function will depend quite heavily on the boundary shape (and extent). Also, the relative spatial location of the source point $\bar{r}_o$ will greatly influence it. The fundamental

solution of course also depends parametrically upon $\bar{r}_o$; however, the fundamental solution corresponding to some specific source point is merely a spatial translate of the fundamental solution correspoinding to any other source point. Since cavities of arbitrary shape are now involved, Green's Functions cannot have this property.

As mentioned earlier, the separable body forces will have a complex exponential time dependence. Therefore, so will the Green's Functions. In most branches of physics, the term Green's Function is reserved for the special case in which the time factor is a delta function. Under the point of view being used here, there is a family of Green's Functions; one for each f(t) one might desire to use. Each Green's Function will be of the "steady state" variety, and will correspond to a given angular frequency, a given boundary (shape, size, and boundary conditions), and finally to a given source point $\bar{r}_o$.

In this chapter, a method for calculating a first approximation to the Green's Function will be formulated. The method, based upon least squares, expresses this first approximation as a finite linear combination of multipole fields whose singular points lie outside the medium,i.e., within the cavity. The first approximation therefore will resemble, but definitely will not be, a truncated multipole series. In Chapter V an improved second approximation will be derived; it will be a true multipole series. Another matter to be discussed in Chapter V is the value of being able to construct Green's Functions.

B. The Scattered Part of a Green's Function

As stated above, sinusoidal time dependence is assumed; that is,

$$f(t) = \exp(-i\omega t). \tag{4.1}$$

When no cavity is present in the infinite medium, the displacement fields due to the dipoles $\hat{e}_i \delta(\bar{r} - \bar{r}_o)\exp(-i\omega t)$ are given by the steady state fundamental solution, which satisfies the equation

$$\rho(\partial^2/\partial t^2)Q_{ij}(\bar{r},t|\bar{r}_o) = D_n \Sigma^j_{in}(\bar{r},t|\bar{r}_o) + \delta_{ij}\delta(\bar{r} - \bar{r}_o)\exp(-i\omega t). \tag{4.2}$$

The solutions of Equation (4.2) may be written as $\tilde{Q}_{ij}(\bar{r}|\bar{r}_o)\exp(-i\omega t)$, in which $\tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$ is independent of time but depends parametrically upon the frequency ω. The same statement holds also for the three stress tensors Σ^j_{in}; the "space parts" will be denoted by $\tilde{\Sigma}^j_{in}(\bar{r}|\bar{r}_o)$. Explicit functional forms for $\tilde{Q}_{ij}$ and $\tilde{\Sigma}^j_{in}$ are given in Appendix C. Upon the cancellation of $\exp(-i\omega t)$, Equation (4.2) becomes

$$-\rho\omega^2 \tilde{Q}_{ij}(\bar{r}|\bar{r}_o) = D_n \tilde{\Sigma}^j_{in}(\bar{r}|\bar{r}_o) + \delta_{ij}\delta(\bar{r} - \bar{r}_o). \tag{4.3}$$

When a void cavity is introduced into the medium and $\bar{r}_o$ is outside the cavity, the dipoles represented in Equation (4.3) by the term $\delta_{ij}\delta(\bar{r} - \bar{r}_o)$ produce displacement fields no longer given by the fundamental solution $\tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$. Instead, the fields are given by a new array, denoted say by $\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$, which is the steady state Green's Function for the given boundary, boundary conditions, source point, and frequency. $\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$ is the i component of displacement at $\bar{r}$ due to a dipole of unit magnitude oriented along $\hat{e}_j$ at $\bar{r}_o$. The three stress tensors calculated from $\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$ will be denoted by $\tilde{\Sigma}^{*j}_{mn}(\bar{r}|\bar{r}_o)$. The relevant differential equation is

$$-\rho\omega^2 \tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o) = D_n \tilde{\Sigma}^{*j}_{in}(\bar{r}|\bar{r}_o) + \delta_{ij}\delta(\bar{r} - \bar{r}_o). \tag{4.4}$$

The boundary conditions at the boundary S of the cavity are that the traction vectors calculated from each of the three stress tensors $\tilde{\Sigma}^{*j}_{mn}(\bar{r}|\bar{r}_o)$ vanish on S. In symbols, this is

$$\tilde{\Sigma}^{*j}_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}') = 0, \ \bar{r}' \text{ on S}, \tag{4.5}$$

where $\tau_n(\bar{r}')$ are the components of the unit normal vector on S. The expression on the left side of Equation (4.5) is the m component of the traction vector due to the dipole along $\hat{e}_j$. The boundary condition, Equation (4.5), must hold for all m and j, and all points $\bar{r}'$ on S.

Instead of attempting a direct solution for $\tilde{Q}^{*}_{ij}$, it may first by split into an "incident" field and a "scattered" field. The former is just the field that would exist if the cavity were not present, i.e., the fundamental solution. The scattered field is due to the presence of the cavity. The scattered part will be denoted by $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$, so that

$$\tilde{Q}^{*}_{ij}(\bar{r}|\bar{r}_o) = \tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) + \tilde{Q}_{ij}(\bar{r}|\bar{r}_o). \tag{4.6}$$

By using Hooke's Law in Equation (4.6), there follows

$$\tilde{\Sigma}^{*j}_{mn}(\bar{r}|\bar{r}_o) = \tilde{\Sigma}^{**j}_{mn}(\bar{r}|\bar{r}_o) + \tilde{\Sigma}^{j}_{mn}(\bar{r}|\bar{r}_o), \tag{4.7}$$

where $\tilde{\Sigma}^{**j}_{mn}(\bar{r}|\bar{r}_o)$ are the mn components of the three scattered stress tensors. Using Equations (4.6) and (4.7) in Equation (4.4) and then invoking Equation (4.3), it follows that $\tilde{Q}^{**}_{ij}$ must satisfy the homogeneous equation of motion. That is,

$$-\rho\omega^2\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = D_n\tilde{\Sigma}^{**j}_{in}(\bar{r}|\bar{r}_o). \tag{4.8}$$

By forming $\tilde{\Sigma}^{*j}_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')$, $\bar{r}'$ on S, from Equation (4.7) and then invoking Equation (4.5), the boundary conditions for the scattered field are found to be

$$\tilde{\Sigma}^{**j}_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}') = -\tilde{\Sigma}^{j}_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}'), \ \bar{r}' \text{ on S.} \tag{4.9}$$

C. First Approximation to the Scattered Part of the Green's Function

In this section the scattered part of the Green's Function will be approximated. This, of course, constitutes the approximation of the Green's Function itself because of Equation (4.6). The first subsection below presents a motivation for the method to be used, followed by a general statement of the method. Details regarding the computation of certain numbers appear in other subsections.

1. General.--In order to provide a motivation for what will follow, one must return to the no cavity case, writing down the multipole series form for the displacement field due to some finite steady state source. Let the expansion point be $\bar{r}_1$ in Equation (3.30). Then

$$\tilde{U}_i(\bar{r}) = L_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]. \tag{4.10}$$

More explicitly, using the definition of L_s from Equation (2.51),

$$\tilde{U}_i(\bar{r}) = G_s\tilde{Q}_{is}(\bar{r}|\bar{r}_1) - G_{sk}D_k\tilde{Q}_{is}(\bar{r}|\bar{r}_1) + (1/2!)G_{skp}D_{kp}\tilde{Q}_{is}(\bar{r}|\bar{r}_1) - \ldots + \ldots . \tag{4.11}$$

Also, the corresponding series for the stress tensor calculated from $\tilde{U}_i(\bar{r})$ is

$$\tilde{T}_{mn}(\bar{r}) = G_s\tilde{\Sigma}^{s}_{mn}(\bar{r}|\bar{r}_1) - G_{sk}D_k\tilde{\Sigma}^{s}_{mn}(\bar{r}|\bar{r}_1) + \ldots - \ldots . \tag{4.12}$$

The traction on any surface whose unit normal has components $\tau_n(r')$ is given by

$$\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}') = G_s\tau_n(\bar{r}')\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1) - G_{sk}\tau_n(\bar{r}')D'_k\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1) + \ldots . \tag{4.13}$$

A highly important feature of Equations (4.11) through (4.13) is that the numbers G_s, G_{sk}, G_{skp}, etc., are the same in each equation. Thus, for example, if the traction given by Equation (4.13) were known to arise on some surface due to the existence of a displacement field, then that displacement field would at once be known via Equation (4.11).

Now, assume that by some means it is learned that traction components, say $\tilde{T}_m(\bar{r}')$, on a surface S, are given by linear combinations of terms such as

$$\tilde{T}_m(\bar{r}') = A_s\tau_n(\bar{r}')\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1) + A_{sk}\tau_n(\bar{r}')D'_k\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1). \tag{4.14}$$

These linear combinations have 12 terms; the A_s and A_{sk} are merely numbers. Then, by inspection, a displacement field giving rise to this traction has i component

$$\tilde{U}_i(\bar{r}) = A_s\tilde{Q}_{is}(\bar{r}|\bar{r}_1) + A_{sk}D_k\tilde{Q}_{is}(\bar{r}|\bar{r}_1), \tag{4.15}$$

as a direct calculation of traction will verify. With this in mind, consider the equation of motion. The displacement field given by Equation (4.15) clearly satisfies the equation of motion because each of the 12 terms does; the equation of motion is linear. In regions not containing the point $\bar{r}_1$, it is the homogeneous equation of motion that is satisfied; none of the 12 terms has its singular point in

such regions.

Now the discussion can be more specific. Returning to the scattering problem involving a void cavity, recall that what are sought are displacement fields which satisfy the homogeneous equation of motion outside the cavity while giving rise to tractions $-\tilde{\Sigma}^{j}_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')$ on S, the cavity boundary. To find these, it is possible to choose a point $\bar{r}_1$ within the cavity, i.e., not in the medium, and write down an expression analogous to Equation (4.14) for the traction just mentioned. Then, it is possible to devise a way of estimating the coefficients, i.e., the A's, in such a way that this linear combination of multipole tractions will approximate the desired traction over all of S. Then, the analog of Equation (4.15) will be the first approximation to the scattered part of the Green's Function. The governing differential equation will be satisfied exactly; the boundary conditions will be satisfied approximately.

In the numerical examples given later in this dissertation, the linear combination of multipole tractions used is

$$-\tilde{\Sigma}^{j}_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}') \simeq A^{j}_{sk}\tau_n(\bar{r}')D'_k\tilde{\Sigma}^{s}_{mn}(\bar{r}'|\bar{r}_1) + A^{j}_{skp}\tau_n(\bar{r}')D'_{kp}\tilde{\Sigma}^{s}_{mn}(\bar{r}'|\bar{r}_1), \qquad (4.16)$$

in which all A^{j}_{skp} for all j and s are taken as zero at the outset when $k > p$. Thus a total of 27 multipole tractions appear in the linear combination for each value of j. There are nine quadrupole terms, i.e., nine A^{j}_{sk}, and 18 octupole terms, i.e., 18 A^{j}_{skp}. All the dipole terms and some of the octupole terms were omitted in order to make the set of terms

remaining an independent set. The reasons why some multipole fields are linear combinations of others is discussed in Appendix D. More important at the moment is the fact that the method to be used to calculate the A's in approximation (4.16) fails if the terms in the series form a dependent set. Note that the independent set that remains has the same amount of "flexibility" as would the set formed by all dipole, quadrupole, and octupole tractions. That is, any traction that could be described exactly using one set could also be described exactly using the other.

As one may notice from Equation (4.16), the notational burden has become rather heavy. A pause will be made here to revise this notation, for the work to come is entirely too messy otherwise. First of all, the left side of representation (4.16) will be denoted by

$$\tilde{T}^j_m(\bar{r}') = -\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}'),\ \bar{r}' \text{ on } S. \tag{4.17}$$

Secondly, let

$$g_{1m}(\bar{r}') = \tau_n(\bar{r}')D'_1\tilde{\Sigma}^1_{mn}(\bar{r}'|\bar{r}_o),$$

$$g_{2m}(\bar{r}') = \tau_n(\bar{r}')D'_1\tilde{\Sigma}^2_{mn}(\bar{r}'|\bar{r}_o), \text{ etc.,} \tag{4.18}$$

according to a bookkeeping scheme given in its entirety in Appendix D. As explained also in that appendix, the A's are replaced by

$$a^j_1 = A^j_{11},$$

$$a^j_2 = A^j_{21}, \text{ etc.,} \tag{4.19}$$

This results in representation (4.16) becoming

$$\tilde{T}^j_m(\bar{r}') \simeq a^j_s \, g_{sm}(\bar{r}'), \ \bar{r}' \text{ on } S. \tag{4.20}$$

In this, the implied summation on the index s of course runs from s = 1 to the number of traction vectors in the approximating set.

Finally, the superscript j will temporarily be dropped. The process to be described shortly must be carried out three times anyway, once for each value of j. The details in each case are the same; the superscript is not explicitly required. For what follows then, assume j fixed at some value and write

$$\tilde{T}_m(\bar{r}') \simeq a_s \, g_{sm}(\bar{r}'), \ \bar{r}' \text{ on } S, \tag{4.21}$$

in place of representation (4.20).

At this point, a clear statement of intentions can be made. Algebraic equations to be solved for the a_s will be derived. For definiteness, it will be assumed that there are 27 vectors in the approximating set, i.e., s = 1,...,27. Therefore a set of 27 algebraic equations is required. Three methods for deriving this set of equations will be mentioned, each requiring a different amount of computational labor. The one requiring the most labor, least squares over the entire surface, is thought to be the best of the three; however, in special cases it is reducible to each of the others. This makes the latter important as computational checkpoints.

2. Collocation.--The simplest method one can devise for deriving the 27 equations, or conditions, determining the a_s is to choose nine points $\bar{r}'_n$, n = 1,...,9, somewhere on S, requiring representation (4.21) to hold exactly at these points. Then,

$$\tilde{T}_m(\bar{r}'_n) = a_s\, g_{sm}(\bar{r}'_n),\ m = 1,2,3;\ n = 1,\ldots,9, \tag{4.22}$$

is a set of 27 equations in the 27 unknowns a_s. This procedure, which is very inaccurate at most points $\bar{r}'$ other than the nine $\bar{r}'_n$, will provide results in situations where the number of vectors in the approximating set is a multiple of three.

3. Least Squares Over a Finite Set of Surface Points.--Instead of choosing only nine points $\bar{r}'_n$ on S, one may choose N points, where N is much larger than nine. Then, denoting $\tilde{T}_m(\bar{r}'_n)$ and $g_{sm}(\bar{r}'_n)$ as T_{mn} and g_{smn} respectively, there follows a set of 3N equations:

$$T_{mn} = a_s\, g_{smn},\ m = 1,2,3;\ n = 1,\ldots,N. \tag{4.23}$$

The set of Equations (4.23) of course has no solution in general. The set is overdetermined; however, one may seek an approximate solution which is the best available in some sense. Here, the "least squares" criterion will be used. A traction error vector having the m component $\tilde{E}_m(\bar{r}'_n) = E_{mn}$ at each of the N points $\bar{r}'_n$ is defined by

$$T_{mn} - a_s\, g_{smn} = E_{mn}. \tag{4.24}$$

Now, the magnitude of this traction error vector can be minimized in the least squares sense over the points $\bar{r}'_n$, n = 1,...,N.

Before attempting the detailed calculations, one must first consider the facts that $T_m(\bar{r}')$ and all $g_{sm}(\bar{r}')$ are complex-valued functions. Therefore, the a_s will be complex numbers; $\tilde{E}_m(\bar{r}')$ are complex-valued. The "ordinary" squared magnitude of the traction error vector at a point $\bar{r}'$ is of course $\tilde{E}_m(\bar{r}')\tilde{E}_m(\bar{r}')$, which is complex. Since the word

"minimize" cannot be applied to complex-valued quantities, one must work with the real valued quantity $\tilde{E}_m(\bar{r}')\tilde{E}_m^{\#}(\bar{r}')$, where the symbol "#" implies complex conjugate. This is the square of the so-called Hermitian magnitude (Hildebrand (1965), p. 24) of the traction-error vector.

The first step involves writing down the quantity to be minimized, i.e., the sum of the numerical values of the squared Hermitian magnitudes calculated at all the N points $\bar{r}'_n$. The average, or mean, value of the sum could then be found by dividing by N, but this step is unnecessary. A convenient way to write the sum involves using the notation of Equation (4.24) while employing the summation convention on both the subscripts m and n. That is, the quantity to be minimized is

$$E_{mn}E_{mn}^{\#} = \text{sum of all squared Hermitian magnitudes.} \tag{4.25}$$

From Equation (4.24),

$$E_{mn}E_{mn}^{\#} = (T_{mn} - a_s\, g_{smn})(T_{mn}^{\#} - a_p^{\#}\, g_{pmn}^{\#}). \tag{4.26}$$

As stated earlier, the a_s are complex, so that

$$a_s = b_s + ic_s,$$

and

$$a_s^{\#} = b_s - ic_s, \tag{4.27}$$

where b_s and c_s are real numbers. Instead of seeking 27 equations determining the a_s directly, one can at first seek 54 equations determining the b_s and c_s. This is done by writing down 54 conditions necessary for $E_{mn}E_{mn}^{\#}$ to be a minimum with respect to the b_s and c_s. These

conditions are

$$(\partial/\partial b_k)(E_{mn}E^{\#}_{mn}) = (\partial/\partial c_k)(E_{mn}E^{\#}_{mn}) = 0,\ k = 1,\ldots,27. \qquad (4.28)$$

In carrying out the differentiation implied in Equations (4.28), it is convenient to note that

$$(\partial/\partial b_k)a_s = \delta_{sk};\ (\partial/\partial c_k)a_s = i\delta_{sk};$$

and

$$(\partial/\partial b_k)a^{\#}_s = \delta_{sk};\ (\partial/\partial c_k)a^{\#}_s = -i\delta_{sk}. \qquad (4.29)$$

Now,

$$\begin{aligned}(\partial/\partial b_k)(E_{mn}E^{\#}_{mn}) &= E_{mn}(\partial/\partial b_k)E^{\#}_{mn} + E^{\#}_{mn}(\partial/\partial b_k)E_{mn}\\ &= (T_{mn} - a_s\, g_{smn})(-g^{\#}_{kmn}) + (T^{\#}_{mn} - a^{\#}_s\, g^{\#}_{smn})(-g_{kmn})\\ &= 0,\ k = 1,\ldots,27. \qquad (4.30)\end{aligned}$$

Also,

$$\begin{aligned}(\partial/\partial c_k)(E_{mn}E^{\#}_{mn}) &= (T_{mn} - a_s\, g_{smn})(-ig^{\#}_{kmn})\\ &\quad + (T^{\#}_{mn} - a^{\#}_s\, g^{\#}_{smn})(ig_{kmn}) = 0,\ k = 1,\ldots,27.\end{aligned}$$

(4.31)

Equations (4.30) and (4.31) are 54 algebraic equations in the 54 unknowns b_s and c_s. Notice, however, that Equations (4.30) simply state that the real parts of the complex quantities $(T_{mn} - a_s\, g_{smn})g^{\#}_{kmn}$, $k = 1,\ldots,27$, must be zero. Moreover, Equations (4.31) state that the imaginary parts of these same quantities must also

be zero. Taken together, the two sets imply that

$$(T_{mn} - a_s\, g_{smn})g^{\#}_{kmn} = 0,\ k = 1,\ldots,27, \tag{4.32}$$

which is a set of 27 equations to be solved for the a_s directly. Written in another form, these equations are

$$a_s g_{smn} g^{\#}_{kmn} = T_{mn} g^{\#}_{kmn},\ k = 1,\ldots,27. \tag{4.33}$$

It should be pointed out that when N = 9, Equations (4.33) are equivalent to Equations (4.22) derived in the collocation procedure. Equations (4.22) are

$$a_s g_{smn} = T_{mn},\ m = 1,2,3;\ n = 1,\ldots,9. \tag{4.34}$$

Multiplication of these by $g^{\#}_{kmn}$ followed by summation over the indices m and n would not change the values of the a_s determined by them. Equations (4.33) for the special case N = 9 would be reproduced in the process, showing the equivalence mentioned.

4. Least Squares Over the Entire Surface.--In the preceding section, the traction-error vector's squared Hermitian magnitude $\tilde{E}_m(\bar{r}')\tilde{E}^{\#}_m(\bar{r}')$ was evaluated at N points $\bar{r}'_n$ on S. The results were summed; the sum was minimized. In this section, the squared Hermitian magnitude will be evaluated at all points $\bar{r}'$ on S; the sum is replaced by integration. Thus, the quantity to be minimized is

$$\int_S \tilde{E}_m(\bar{r}')\tilde{E}^{\#}_m(\bar{r}')\,dS', \tag{4.35}$$

which is the analog of expression (4.25). The results of the last section permit the work here to proceed by inspection. Also, see Stakgold (1967), pp. 124-125.

The a_s are determined by

$$a_s \int_S g_{sm}(\bar{r}')g^{\#}_{km}(\bar{r}')dS' = \int_S \tilde{T}_m(\bar{r}')g^{\#}_{km}(\bar{r}')dS', \qquad (4.36)$$

for k = 1,...,27. Equations (4.36) are analogous to Equations (4.33).

If the integrals in Equations (4.36) were to be performed numerically by approximations with finite sums, and if the elements of area ΔS_n associated with each of the N points $\bar{r}'_n$ involved in the finite sums were all equal, then Equations (4.36) would become equivalent to Equations (4.33). This may be seen by using the notation of Equation (4.23) in writing the finite sums which would replace equations (4.36):

$$a_s g_{smn} g^{\#}_{kmn} \Delta S_n = T_{mn} g^{\#}_{kmn} \Delta S_n, \; k = 1,...,27. \qquad (4.37)$$

In Equation (4.37) it is necessary to excuse the unusual notation: The summation over the index n is to be done only once, with each term in the sum having the same value of n in all of the three positions. If ΔS_n is the same number for all n, it may be divided out of the equations, whereupon they become identical to Equations (4.33), as stated.

If, on the other hand, ΔS_n is different for each n, then it plays the role of a "weighting function" that weights each term in the sum according to the amount of area on S represented by the term. The numerical examples given later in this dissertation are all based upon Equations (4.36). Since the surface integration had to be done numerically, it was in fact Equations (4.37) that were employed. In doing so, it did not happen that ΔS_n was the same at each integration point; making it so is impractical with most surfaces. While the advantages

of using Equations (4.37) instead of Equations (4.33) are thought to be appreciable, one special case (not presented in the numerical examples given later) was worked out in which ΔS_n was in fact the same at each integration point. This was done as one means of testing the programming. Equations (4.37) reduced to Equations (4.33). Then, N was taken as nine, reducing the equations to Equations (4.22). Subsequent calculation of $\tilde{E}_m(\bar{r}'_n)$ at the nine points $\bar{r}'_n$ after solving for the a_s gave the correct results, i.e., $\tilde{E}_m(\bar{r}'_n) = 0$, for all m and n. This provided a measure of confidence in the computer programs.

Now it is time to recall that Equation (4.21) temporarily dropped a superscript j, i.e., the above discussions dealt only with a single dipole. There are three fundamental dipoles, hence three traction error vectors $\tilde{E}^j_m(\bar{r}')$. By Equations (4.17) and (4.20),

$$-\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}') - a^j_s g_{sm}(\bar{r}') = \tilde{E}^j_m(\bar{r}'). \qquad (4.38)$$

V. VOID CAVITIES--FINAL APPROXIMATION OF GREEN'S FUNCTIONS AND THEIR USE

In this chapter, a method for deriving a second, final, and substantially better approximation to the Green's Function for a finite void cavity will be developed. The development is somewhat more complicated than that leading to the first approximation; in practice, however, there is no great increase in the computational workload. In computer realizations of the theory, proper storage of intermediate results leading to the first approximation makes them available in calculating the second approximation. The second approximation, of course, evolves directly from the first approximation derived in Chapter IV using least squares.

A. Some General Results Required Later

The development in this chapter is based upon what will be called Green's Integral Representation. Suppose a body force having components $\tilde{F}_m(\bar{r})\exp(-i\omega t)$ acts in an elastic solid having a boundary S. Let $\tau_n(\bar{r}')$, $\bar{r}'$ on S, be the components of the unit normal vector on S which points out of the medium. Let $\tilde{U}_i(\bar{r})$ and $\tilde{T}_{mn}(\bar{r})$ be the resulting space parts of the displacement and stress components respectively. Then,

$$\tilde{U}_i(\bar{r}) = \int_{a.s.} \tilde{F}_m(\bar{r}')\tilde{Q}_{mi}(\bar{r}'|\bar{r})\,dv'$$

$$+ \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}')\,dS'$$

$$- \int_S \tilde{U}_m(\bar{r}')\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')\,dS', \qquad (5.1)$$

where $\tilde{Q}_{mi}(\bar{r}'|\bar{r})$ is the fundamental solution and $\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})$ are the corresponding stress tensors. Equation (5.1) is the well known Green's

Integral Representation for the displacement components; it derivation and some references to the literature are presented in Appendix E. The representation (5.1) clearly incorporates boundary conditions, for one may observe $\tilde{U}_m(\bar{r}')$ and $\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}')$ in the surface integrals. The latter quantities are the traction components exerted on S by external means (see the discussion related to Equation (6.11)).

Corresponding to three fundamental problems in elasticity, one may: 1) Specify $\tilde{U}_m(\bar{r}')$ only over all of S; 2) specify $\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}')$ only over all of S; or 3) specify displacement over part of S and traction over the remainder. Arbitrary specification of both over the same portion of S is not permissible.

The work here is concerned with the case where S is made up of two parts, the boundary of a finite void cavity and a "surface at infinity." The surface integrals in Equation (5.1) vanish automatically on the part at infinity. In the remaining integrals, the unit normal points into the cavity.

Since the cavity is void, the relevant boundary conditions are that no traction exist on S. This causes the vanishing of the first of the two surface integrals in Equation (5.1).

Because of the fact that a Green's Function is a set of three displacement fields, three representations such as Equation (5.1) are required for its representation. These representations may be found by letting the body force be three dipoles in succession, each being parallel to a coordinate axis. These are, of course, the three fundamental dipoles of Chapter II; the j dipole has m component

$$\delta_{mj}\delta(\bar{r} - \bar{r}_o), \tag{5.2}$$

where $\bar{r}_o$ is the point of application located somewhere in the medium and not on S. By using the expression (5.2) for $\tilde{F}_m(r)$ in Equation (5.1), while at the same time requiring traction on S to vanish, it follows that the Green's Function $\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$ is represented by

$$\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o) = \tilde{Q}_{ij}(\bar{r}|\bar{r}_o) - \int_S \tilde{Q}^*_{mj}(\bar{r}'|\bar{r}_o)\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS'. \tag{5.3}$$

Comparing this with Equation (4.6), namely,

$$\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o) = \tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) + \tilde{Q}_{ij}(\bar{r}|\bar{r}_o), \tag{5.4}$$

it becomes clear that the surface integral in Equation (5.3) is merely the scattered part of the Green's Function. That is,

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = - \int_S \tilde{Q}^*_{mj}(\bar{r}'|\bar{r}_o)\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS'. \tag{5.5}$$

Since Equation (5.4) holds on S as well as elsewhere, it may be used in Equation (5.5), leading to

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = - \int_S \tilde{Q}^{**}_{mj}(\bar{r}'|\bar{r}_o)\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS' - \int_S \tilde{Q}_{mj}(\bar{r}'|\bar{r}_o)\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS'. \tag{5.6}$$

This is an integral representation for $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$; however, it is not of the form of a Green's Representation obtainable directly from Equation (5.1). To obtain the latter representation, note that there are no sources for $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ in the medium, i.e., it satisfies the homogeneous equation of motion there. Thus, there is no volume integral

in Green's Representation. Also, as noted explicitly in Equation (4.9), the traction on S due to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ must be $-\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')$, $\bar{r}'$ on S. Using these facts in Equation (5.1), the Green's Representation is

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = -\int_S \tilde{Q}^{**}_{mj}(\bar{r}'|\bar{r}_o)\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS'$$

$$-\int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')dS'. \qquad (5.7)$$

There is an apparent discrepancy between Equations (5.6) and (5.7). However, Part 1 of the following theorem, which is proven in Appendix E, shows the two representations to be identical.

THEOREM: Let i) S be a closed surface dividing all of space into two regions;

ii) $\hat{\tau}$ be the unit normal vector on S;

iii) $\bar{r}$ and $\bar{r}_1$ be two points in space; and

iv) $$\tilde{I}_{ij}(\bar{r}|\bar{r}_1) = \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_1)\tau_n(\bar{r}')dS'. \qquad (5.8)$$

Then: <u>Part 1</u>: If neither $\bar{r}$ nor $\bar{r}_1$ is on S, but both are on the same side of S,

$$\tilde{I}_{ij}(\bar{r}|\bar{r}_1) - \tilde{I}_{ji}(\bar{r}_1|\bar{r}) \equiv 0. \qquad (5.9)$$

<u>Part 2</u>: If neither $\bar{r}$ nor $\bar{r}_1$ is on S, but if they are on opposite sides of S while $\hat{\tau}$ points into the region containing $\bar{r}_1$,

$$\tilde{I}_{ij}(\bar{r}|\bar{r}_1) - \tilde{I}_{ji}(\bar{r}_1|\bar{r}) \equiv \tilde{Q}_{ij}(\bar{r}|\bar{r}_1). \qquad (5.10)$$

The above theorem will find additional application shortly. Note in passing that if S is a surface enclosing a source point $\bar{r}_1$, then Equation (5.10) is merely Green's Representation for the fundamental

solution throughout the region not containing $\bar{r}_1$.

B. The Second Approximation

It is necessary now to briefly return to Chapter IV, recalling that by Equation (4.16), the traction on S due to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ is given approximately by

$$\tau_n(\bar{r}')M'^j_s[\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)], \tag{5.11}$$

where $\bar{r}'$ is on S, $\bar{r}_1$ is inside the cavity, and M'^j_s is the operator given by

$$M'^j_s = A^j_{sk}D'_k + A^j_{skp}D'_{kp}. \tag{5.12}$$

In M'^j_s, all A^j_{skp} for $k > p$ are zero, while the remaining coefficients are determined through the least squares procedure summarized by Equation (4.36).

The first approximation to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ itself is

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) \simeq M^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \tag{5.13}$$

regardless whether $\bar{r}$ is on or off S. The operator M^j_s is, of course, merely M'^j_s with all primes removed, i.e.,

$$M^j_s = A^j_{sk}D_k + A^j_{skp}D_{kp}. \tag{5.14}$$

Now, since $\tilde{Q}_{is}(\bar{r}|\bar{r}_1)$ is a function only of the increments which are components of the vector $\bar{r} - \bar{r}_1 = (x_m - x_{1m})\hat{e}_m$, it is clear that

$$D_k\tilde{Q}_{is}(\bar{r}|\bar{r}_1) = -D^1_k\tilde{Q}_{is}(\bar{r}|\bar{r}_1), \tag{5.15}$$

where D^1_k means $\partial/\partial x_{1k}$. For this reason, it is a fact that

$$M_s^j[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)] = M_s^{1j}[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \tag{5.16}$$

in which

$$M_s^{1j} = -A^j_{sk}D^1{}_k + A^j_{skp}D^1{}_{kp}. \tag{5.17}$$

With Equation (5.17) in mind, reconsider now the integral representation, Equation (5.7). Substituting $M_s^{1j}[\tilde{Q}_{ms}(\bar{r}'|\bar{r}_1)]$ for $\tilde{Q}^{**}_{mj}(\bar{r}'|\bar{r}_o)$ in the first integral in Equation (5.7) results in

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = -M_s^{1j}[\int_S \tilde{Q}_{ms}(\bar{r}'|\bar{r}_1)\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS']$$

$$- \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')dS', \tag{5.18}$$

in which the operator was removed from within the integral. Equation (5.18) is the second (and final) approximation to the scattered part of the Green's Function. Since it is still an approximation, the equal sign employed therein is technically incorrect; this will be disregarded.

The form of Equation (5.18) is not particularly convenient for purposes of interpretation or computation. In the equation, $\bar{r}$ and $\bar{r}_1$ are on opposite sides of S; $\bar{r}$ is the field point in medium outside the cavity while $\bar{r}_1$ is inside the cavity. The vector $\hat{\tau}$ points into the cavity. Part 2 of the theorem given in the preceding section is applicable to the operand of M_s^{1j} in Equation (5.18). The vectors $\bar{r}$ and $\bar{r}_1$ here correspond to the vectors $\bar{r}$ and $\bar{r}_1$, respectively, of Equation (5.10). The index s here is the index j there. The operand of M_s^{1j} is therefore

$$\tilde{I}_{si}(\bar{r}_1|\bar{r}) = \tilde{I}_{is}(\bar{r}|\bar{r}_1) - \tilde{Q}_{is}(\bar{r}|\bar{r}_1). \tag{5.19}$$

Hence, Equation (5.18) may be written in the form

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = M^{1j}_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)] - M^{1j}_s[\int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)\tau_n(\bar{r}')dS']$$

$$- \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')dS'. \tag{5.20}$$

Equation (5.20) may be rewritten in still another form by using Equation (5.16) and by moving M^{1j}_s back under the integral sign in the expression where it now appears. The result is

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = M^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]$$

$$+ \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\{-\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}') - \tau_n(\bar{r}')M^{1j}_s$$

$$[\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)]\}dS'. \tag{5.21}$$

Now, the property expressed by Equation (5.15) is also valid for $\tilde{\Sigma}^s_{mn}(\bar{r}|\bar{r}_1)$. It is therefore a fact that

$$M^{1j}_s[\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)] = M'^j_s[\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)]. \tag{5.22}$$

Thus, the expression $\tau_n(\bar{r}')M^{1j}_s[\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)]$ in Equation (5.21) is merely the right side of the approximation (4.16) of Chapter IV. A brief review of Chapter IV, especially Equation (4.38), shows that the expression in curly brackets in Equation (5.21) above is just $\tilde{E}^j_m(\bar{r}')$, the m component of the j traction-error vector defined in the least squares procedure leading to the first approximation. Equation (5.21) may be rewritten in the form

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = M^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)] + \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{E}^j_m(\bar{r}')dS'. \tag{5.23}$$

Notice that if the first approximation to $\tilde{Q}^{**}_{ij}$ were to be considered as exact, so that $\tilde{E}^{j}_{m}$ would be taken as zero everywhere on S, then Equation (5.23) would reduce at once to an identity.

Equation (5.23) expresses the second, and final, approximation to $\tilde{Q}^{**}_{ij}$ as the sum of the first approximation and the correction term expressed by the integral. At each field point $\bar{r}$ where a value of the scattered field is desired, the integral must be evaluated. This leads to much labor in practice even if only a few field points are considered. For this reason, the correction term will be replaced by a multipole series.

First of all, note the physical significance, or interpretation, of the correction term. It is the displacement that would arise in an infinite, homogeneous medium (i.e., no cavity) if a traction $\tilde{E}^{j}_{m}(\bar{r}')$ were to be applied on S. This fact is discussed in Appendix E, where the physical interpretation of other integrals encountered in the current chapter is also discussed. An important feature of the correction integral is that it gives the displacement due to a <u>separable</u> layer of traction on S, making the multipole series form for it easy to calculate.

The details concerning the calculation of the multipole series form appear in Appendix F. The series may be truncated at any point one chooses; for the purposes of this dissertation, all terms of higher order than the octupole terms are deleted. Furthermore, owing to the fact, explained in Appendix D, that some multipole fields are linear combinations of others, it is possible to combine some of the terms in the remaining finite series. In this way, the correction to the first approximation reduces to correcting the coefficients A^{j}_{sk} and

A^j_{skp} in that first approximation. Also, as discussed in Appendix F, there is not much additional labor connected with computing these corrections.

Upon correcting the coefficients in the operator M^j_s as discussed above, the resulting operator will be denoted by N^j_s. Therefore, since the Green's Function is the sum of its direct and scattered parts, the final approximation to the Green's Function is

$$\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o) = \tilde{Q}_{ij}(\bar{r}|\bar{r}_o) + N^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]. \qquad (5.24)$$

The following section concerns the use of Equation (5.24).

C. Using the Green's Function

One of the most important points to be borne in mind when using Equation (5.24) is that the operator N^j_s is made up partly of a truncated multipole series. This means that, even when the entire series is present, Equation (5.24) is not valid everywhere in space, even approximately. The field point $\bar{r}$ must be more distant from $\bar{r}_1$ than are all points $\bar{r}'$ on the cavity boundary S. This condition is required in order to assure convergence of the series to the integral it replaced. The truncated series implied in Equation (5.24) will be the more accurate the farther $\bar{r}$ is from $\bar{r}_1$.

In Appendix E, along with the derivation of Equation (5.1), it is shown that if a body force having components $\tilde{F}_s(\bar{r})\exp(-i\omega t)$ acts in the medium enclosing a void cavity, then the space part of the displacement components are given by

$$\tilde{U}_i(\bar{r}) = \int_{a.s.} \tilde{F}_s(\bar{r}')\tilde{Q}^*_{si}(\bar{r}'|\bar{r})dv'. \qquad (5.25)$$

When the Green's Function $\tilde{Q}^*_{si}(\bar{r}'|\bar{r})$ is known, Equation (5.25) is an integral solution.

An important property possessed by all Green's Functions can be derived from Equation (5.25) by substituting the body force components $\tilde{F}_s(\bar{r}') = \delta_{js}\delta(\bar{r} - \bar{r}_o)$ into the integral. Since these are the components of the j fundamental dipole, the result of the substitution must be $\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o)$ by definition. Upon making the substitution, there follows

$$\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o) = \tilde{Q}^*_{ji}(\bar{r}_o|\bar{r}). \tag{5.26}$$

This is an example of a so-called "reciprocity principle" of the type discussed by White (1965), p. 229. Equation (5.26) is analogous to a part of Equations (3.21); however, the latter equations show that the fundamental solution possesses some properties not mentioned in Equation (5.26) for the Green's Function.

Upon using Equation (5.26) in Equation (5.25), the latter becomes

$$\tilde{U}_i(\bar{r}) = \int_{a.s.} \tilde{F}_s(\bar{r}')\tilde{Q}^*_{is}(\bar{r}|\bar{r}')dv' \tag{5.27}$$

Equation (5.27) is the analog of the steady state counterpart of Equation (3.34). All the optional methods available for using the Green's Function are based on either Equation (5.25) or Equation (5.27).

1. Using Equation (5.27) Directly.--Assume that the integrals in Equation (5.27) are to be evaluated by approximation with finite sums. Let the finite region in which $\tilde{F}_s(\bar{r}')$ are nonzero be divided into N elements of volume V_n, n = 1,...,N. Let $\bar{r}'_n$ be the locations of these elements. Then, the displacement is (approximated by)

$$\tilde{U}_i(\bar{r}) = \tilde{F}_s(\bar{r}'_n)\tilde{Q}^*_{is}(\bar{r}|\bar{r}'_n)V_n, \tag{5.28}$$

in which the sum over the index n is performed only once; the value of n in each term is the same in all three positions.

A tremendous disadvantage of Equation (5.28) is that the Green's Function $\tilde{Q}^*_{is}(\bar{r}|\bar{r}'_n)$ must be known for each of the N source points $\bar{r}'_n$. If N is large, then the method presented in this dissertation for calculating the Green's Function will lead to an enormous quantity of labor. Equation (5.28) will not, in general, provide a satisfactory means for calculating $\tilde{U}_i(\bar{r})$.

2. Using Equation (5.25) Directly.--The main disadvantage of Equation (5.28) is that the Green's Function for a large number of source points must be known. In case one desires values of $\tilde{U}_i(\bar{r})$ at only a few field points, say $\bar{r}_m$, m = 1,...,M, then Equations (5.25) are appropriate. The displacement at a point $\bar{r}_m$ would be, using the analog of Equation (5.28),

$$\tilde{U}_i(\bar{r}_m) = \tilde{F}_s(\bar{r}'_n)\tilde{Q}^*_{si}(\bar{r}'_n|\bar{r}_m)V_n. \tag{5.29}$$

In using Equation (5.29), one would calculate the Green's Function corresponding to the source points $\bar{r}_m$, the actual field points. Once done for a point $\bar{r}_m$, then values of $\tilde{Q}^*_{si}(\bar{r}'_n|\bar{r}_m)$ are known at each of the N integration points. Thus, only a total of M Green's Functions are required.

3. Using the Body Force Multipole Series.--By Equation (2.58) the body force components are

$$\tilde{F}_s(\bar{r}) = L_s[\delta(\bar{r} - \bar{r}_o)], \tag{5.30}$$

in which $\bar{r}_o$ is some "typical" point in the region where the $\tilde{F}_s(\bar{r})$ are nonzero. L_s is given by Equation (2.51) as

$$L_s = G_s - G_{sk}D_k + (1/2!)G_{skp}D_{kp} - \ldots + \ldots , \tag{5.31}$$

where the G_s, etc., are defined by Equations (2.49). Upon using Equations (5.30) and (5.4) in Equation (5.27), there follows

$$\tilde{U}_i(\bar{r}) = (G_s + G_{sk}D^o{}_k + (1/2!)G_{skp}D^o{}_{kp} + \ldots)\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o) + (G_s + G_{sk}D^o{}_k + (1/2!)G_{skp}D^o{}_{kp} + \ldots)\tilde{Q}_{is}(\bar{r}|\bar{r}_o). \tag{5.32}$$

Just as Equations (5.25) and (5.27) possess inherent computational drawbacks, so does Equation (5.32). Since the fundamental solution $\tilde{Q}_{is}(\bar{r}|\bar{r}_o)$ has the property

$$D^o{}_k\tilde{Q}_{is}(\bar{r}|\bar{r}_o) = -D_k\tilde{Q}_{is}(\bar{r}|\bar{r}_o), \tag{5.33}$$

the second series in Equation (5.32) is simply $L_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_o)]$, which presents no computational difficulties whatsoever. This part of Equation (5.32) is the direct field part; it can be easily evaluated at any field point $\bar{r}$ outside the smallest sphere centered at $\bar{r}_o$ completely containing the source. However, in the scattered part of Equation (5.32), i.e., the first series, one is forced to take literally the implied differentiation of $\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$ with respect to the components x_{ok} of $\bar{r}_o$. There is no reason to expect that $\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$ can have the property expressed by Equation (5.33) for the fundamental solution. On the other hand, it is quite clear that the method presented in this dissertation for the calculation of $\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$ requires $\bar{r}_o$ to be a given constant vector; numerical values of its components are required as

"input" information. While the symbol $\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$ continues to indicate the dependence on $\bar{r}_o$, there is in fact no easy way to differentiate the end result with respect to the components of $\bar{r}_o$.

Since the differentiation implied by D^o_k, etc., can neither be performed nor transferred to the field coordinates, Equation (5.32) is, at the moment, somewhat worthless from a computational viewpoint. It is valid nevertheless; in Section D below its meaning will be explored in detail. In its present form, Equation (5.32) may in some cases serve as the basis for approximations. For example, if the G_s are found to be much larger than all other coefficients, then dropping all other terms but the first of both series in Equation (5.32) may be justified. If $\bar{r}_o$ is very far from the cavity while $|\bar{r} - \bar{r}_o|$ is small, it may prove feasible to retain only the dipole terms in the first series while retaining more terms in the second. Such approximations are best made on an ad hoc basis; the matter will not be discussed further.

D. Scattering of Higher Order Multipole Fields

It may happen that for one reason or another, none of the methods suggested in Section C above for the calculation of the displacement due to an arbitrary body force are acceptable. There are at least two other routes one might take. One of these methods involves taking the complete field from the source (instead of only dipole fields) as the input to Chapter IV. This is best discussed in the next chapter; the technique may be useful in cases in which only one arbitrary source is to be studied. If, on the other hand, one intends to study more than one source in a series of problems related to the same cavity, then the contents of this section should be of more interest. What follows

here amounts to an outline of how one would go about analysing the scattered field of an arbitrary source on a multipole component by multipole component basis.

Reconsider Equation (5.32), recalling that only the scattered field, i.e., the second series, offers any computational difficulty. Only this scattered field need be considered here; it is

$$(G_s + G_{sk}D^o_k + (1/2!)G_{skp}D^o_{kp} + \ldots)\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o), \tag{5.34}$$

in which $\bar{r}_o$ is an expansion point somewhere in the finite region where the body force is nonzero. It is important to realize that if one were clever enough to calculate $\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$ in a functional form containing $\bar{r}_o$ parametrically, then Equation (5.34) would become immediately useful. What follows here would become redundant, for it is a study of some of the consequences of the numerical procedures used to calculate $\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$. The meaning of the symbol $D^o_p\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$ must first be explained from a computational viewpoint. Clearly, from a "theoretical" viewpoint it is merely the scattered field due to an incident field $D^o_p\tilde{Q}_{is}(\bar{r}|\bar{r}_o)$, i.e., $-D_p\tilde{Q}_{is}(\bar{r}|\bar{r}_o)$.

Returning to Chapter IV, recall that the first approximation to $\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o)$ is given by

$$A^j_{sk}D_k\tilde{Q}_{is}(\bar{r}|\bar{r}_1) + A^j_{skp}D_{kp}\tilde{Q}_{is}(\bar{r}|\bar{r}_1), \tag{5.35}$$

where $A^j_{skp} = 0$ for $k > p$. $\bar{r}_1$ is a point within the cavity. As indicated in Equation (4.19) and explained thoroughly in Appendix D, the A^j_{sk} and A^j_{skp} are temporarily replaced with a^j_s, $s = 1,\ldots,27$, as a matter of notational convenience. The a^j_s are then determined from Equation (4.36),

namely,

$$\{\int_S g_{sm}(\bar{r}')g^{\#}_{km}(\bar{r}')dS'\}a^j_s = \int_S \tilde{T}^j_m(\bar{r}')g^{\#}_{km}(\bar{r}')dS'. \qquad (5.36)$$

Notice that for each value of j, Equation (5.36) has a simple interpretation in the language of matrices. A formal solution for the a^j_s may be written in the form

$$a^j_s = H_{sk}\int_S \tilde{T}^j_m(\bar{r}')g^{\#}_{km}(\bar{r}')dS', \qquad (5.37)$$

in which H_{sk} is the sk element of a 27 x 27 array of complex numbers. The matrix formed from this array is the inverse of the matrix whose ks element is

$$\int_S g_{sm}(\bar{r}')g^{\#}_{km}(\bar{r}')dS'. \qquad (5.38)$$

It is important to realize that since the expression (5.38) is independent of $\bar{r}_o$, so is H_{sk}. Once a cavity and a frequency are given, then H_{sk} may be calculated; this part of the work does not depend upon the nature of the incident field. (Note: As pointed out in Appendix F, some of the work involved in computing the second approximation also has this property.) Information concerning the incident field enters Equation (5.37) through the factors $\tilde{T}^j_m(\bar{r}')$, the traction on S due to the incident field. Using Equation (4.17), the a^j_s are given more explicitly by

$$a^j_s = H_{sk}\int_S -\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')g^{\#}_{km}(\bar{r}')dS'. \qquad (5.39)$$

In Equation (5.39) it is clear that the first approximation depends on $\bar{r}_o$ only through the factor $-\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)$ which helps determine the a^j_s.

Meaning may be assigned to the operation $D^{o}_{p} a^{j}_{s}$ in the following way:

$$D^{o}_{p} a^{j}_{s} = H_{sk} \int_{S} D'_{p} \tilde{\Sigma}^{j}_{mn}(\bar{r}'|\bar{r}_{o}) \tau_{n}(\bar{r}') g^{\#}_{km}(\bar{r}') dS'. \qquad (5.40)$$

in which D^{o}_{p} was first applied to the right side of Equation (5.39) under the integral sign. Then the property

$$D^{o}_{p} \tilde{\Sigma}^{j}_{mn}(\bar{r}'|\bar{r}_{o}) = -D'_{p} \tilde{\Sigma}^{j}_{mn}(\bar{r}'|\bar{r}_{o}) \qquad (5.41)$$

was used. It must be borne in mind that the symbolic procedure above does not imply that the numbers $D^{o}_{p} a^{j}_{s}$ are derivable from the numbers a^{j}_{s} after the latter are calculated. The integrals shown explicitly in Equation (5.40) are independent of those in Equation (5.39). They must be evaluated separately.

By means of Equation (5.40), it is possible to assign a definite meaning to the result of operating upon expression (5.35) with D^{o}_{p}. That is, the first approximation to the Green's Function can be differentiated with respect to the source coordinates in this way. It is clear that doing so results in a first approximation to the scattered field when the field incident upon the cavity is $-D_{p} \tilde{Q}_{ij}(\bar{r}|\bar{r}_{o})$. This may be understood by noting that this incident field gives rise to tractions $-D'_{p} \tilde{\Sigma}^{j}_{mn}(\bar{r}'|\bar{r}_{o}) \tau_{n}(\bar{r}')$ at $\bar{r}'$ on S. In order to calculate linear combinations of multipole tractions to nullify these tractions, their negatives can be substituted for $\tilde{T}^{j}_{m}(\bar{r}')$ in Equation (5.37), whereupon the right sides of Equations (5.37) and (5.40) become identical.

Now, the second approximation to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_{o})$ is given by Equation (5.23) as

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_{o}) = M^{j}_{s}[\tilde{Q}_{is}(\bar{r}|\bar{r}_{1})] + \int_{S} \tilde{Q}_{mi}(\bar{r}'|\bar{r}) \tilde{E}^{j}_{m}(\bar{r}') dS', \qquad (5.42)$$

in which $\tilde{E}^j_m(\bar{r}')$ is given by Equation (4.38). The part $M^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]$ is the first approximation. From Equation (4.38), it follows that

$$D^o_p \tilde{E}^j_m(\bar{r}') = D'_p \tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}') - (D^o_p a^j_s)g_{sm}(\bar{r}'), \qquad (5.43)$$

which clearly are the traction error vectors arising when the tractions $-[D'_p \tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')]$ are approximated by linear combinations of the multipole tractions $g_{sm}(\bar{r}')$.

At this stage it is quite clear that when the field $-D_p\tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$ scatters from the cavity, an appropriate unambiguous symbol for the scattered field calculated using the method of this dissertation is $D^o_p \tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$. While that is true, this scattered field cannot be calculated by differentiating $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ directly owing to the unavoidable unfortunate feature of the computational technique itself. In order to calculate $D^o_p \tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$, Equation (5.40) is employed first to find the numbers $D^o_p a^j_s$. Then the analog of Equation (5.42) gives the final approximation. The correction term can be replaced with its multipole series, which would be truncated in practice. The remaining terms, assumed here to be the octupole terms and lower, could be combined in such a way that they could then be further combined with the $D^o_p a^j_s$. Ultimately, in direct analogy with the operator N^j_s of Equation (5.24), an operator N^{jp}_s could be formed. That is, from Equation (5.24),

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = N^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \qquad (5.44)$$

and now also,

$$D^o_p \tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = N^{jp}_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]. \qquad (5.45)$$

Now, N_s^j of course has the form

$$N_s^j = B_{sk}^j D_k + B_{skq}^j D_{kq}, \tag{5.46}$$

in which the B_{sk}^j and B_{skp}^j result from correcting the A_{sk}^j and A_{skq}^j of the first approximation to $\tilde{Q}_{ij}^{**}(\bar{r}|\bar{r}_o)$. It will happen then that N_s^{jp} will have the form

$$N_s^{jp} = B_{sk}^{jp} D_k + B_{skq}^{jp} D_{kq}. \tag{5.47}$$

Thus far in the current section, it has been shown that when the incident fields at the cavity are $-D_p \tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$, i.e., the negative quadrupole displacements from a source point $\bar{r}_o$, then the resultant scattered fields are given by Equation (5.45). This required retracing the work of Chapter IV and most of the current chapter, which together treated the case in which the incoming fields were $\tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$. Clearly the same process can be done for the case in which the incident fields are the octupole fields $+D_{pq} \tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$, $p \leq q$. Without much effort it can be seen that the resulting scattered fields may be expressed by

$$D^o_{pq} \tilde{Q}_{ij}^{**}(\bar{r}|\bar{r}_o) = N_s^{jpq}[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \tag{5.48}$$

where

$$N_s^{jpq} = B_{sk}^{jpq} D_k + B_{skm}^{jpq} D_{km}, \; p \leq q. \tag{5.49}$$

The process can be continued for higher order incident multipole fields, the workload increasing each time. There are 81 coefficients in the N_s^j, 243 in the N_s^{jp}, and 486 in the N_s^{jpq}.

At this point, a more definite computational meaning may be associated with Equation (5.32), the multipole series form for the displacement due to an arbitrary body force. As a matter of simplicity, assume the body force of Equation (5.30) is given sufficiently well by

$$\tilde{F}_s(\bar{r}) = (G_s - G_{sp}D_p)\delta(\bar{r} - \bar{r}_o). \tag{5.50}$$

Equation (5.32) becomes

$$\tilde{U}_i(\bar{r}) = G_s\tilde{Q}_{is}(\bar{r}|\bar{r}_o) - G_{sp}D_p\tilde{Q}_{is}(\bar{r}|\bar{r}_o)$$
$$+ G_s\tilde{Q}^{**}_{is}(\bar{r}\,\bar{r}_o) + G_{sp}D^o{}_p\tilde{Q}^{**}_{is}(\bar{r}|\bar{r}_o). \tag{5.51}$$

By Equations (5.44) and (5.45) this is

$$\tilde{U}_i(\bar{r}) = G_s\tilde{Q}_{is}(\bar{r}|\bar{r}_o) - G_{sp}D_p\tilde{Q}_{is}(\bar{r}|\bar{r}_o)$$
$$+ G_jN^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)] + G_{jp}N^{jp}_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]. \tag{5.52}$$

In Equation (5.52), all differentiation is with respect to the field coordinates. No computational difficulties arise, though for purposes of convenience it may prove useful to notice that

$$G_jN^j_s + G_{jp}N^{jp}_s = G_jB^j_{sk}D_k + G_jB^j_{skm}D_{km} + G_{jp}B^{jp}_{sk}D_k + G_{jp}B^{jp}_{skm}D_{km}$$
$$= (G_jB^j_{sk} + G_{jp}B^{jp}_{sk})D_k + (G_jB^j_{skm} + G_{jp}B^{jp}_{skm})D_{km} = P_s \tag{5.53}$$

is just an operator analogous to L_s. Regardless of the number of terms retained in L_s, one will always be able to express the displacement

in the form

$$\tilde{U}_i(\bar{r}) = L_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_o)] + P_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \tag{5.54}$$

in which the multipole series about $\bar{r}_o$ in the source is the direct field while the second multipole series about $\bar{r}_1$ in the cavity is the final approximation to the scattered field.

Equation (5.54) has several convenient computational features. Once L_s and P_s are known, the displacement, stress, or other quantities caused by the body force are known at all field points $\bar{r}$ outside the smallest sphere completely enclosing the cavity while also outside the smallest sphere completely containing the source. Numerical values for any of the field quantities are easily obtained.

As stated earlier, if one desires to study the effect of many different sources, one at a time, all operating at the same frequency near the same point $\bar{r}_o$ in the vicinity of a given cavity, then Equation (5.49) is particularly useful. Only the numbers G_s, G_{sk}, etc., need be recalculated upon introducing a different source.

VI. ADDITIONAL APPLICATIONS OF THE THEORY

In the preceding chapters, attention has been focused primarily upon scattering of waves from arbitrary finite sources by void finite cavities of arbitrary shape. Only cases in which the mathematical nature of the source was given were considered. In particular, the last two sections of Chapter V concerned the final steps involved in writing down solutions. The current chapter discusses some other problems which may be handled by the analytical apparatus of the earlier chapters.

A. Additional Scattering Problems--Plane Waves

It is assumed here that there exists a void finite cavity in a medium of the type considered earlier. However, instead of taking the source to be known, only the incident displacement field is considered as given. The procedure will be discussed using plane wave scattering as the example incident field; however, the technique provides an additional alternative to Section (V-D). That is, when a source is given, one may study its scattered field as a whole using the current section rather than studying the scattering of the individual multipole components as in Section (V-D).

No essential modifications of the method are necessary in order to treat problems from the new viewpoint. One merely calculates the traction on the cavity boundary due to the given incident field, constructs a scattered field via Chapter IV to approximately nullify this traction, and derives the correction to the scattered field via Chapter V.

Suppose the incident field is given by

$$U_m(\bar{r},t) = \delta_{m3}\exp(-i\omega(t - x_3/\alpha)). \tag{6.1}$$

This is the displacement due to a plane compressional wave traveling in the direction of $\hat{e}_3$. The space part is

$$\tilde{U}_m(\bar{r}) = \delta_{m3}\exp(i\omega x_3/\alpha). \tag{6.2}$$

By applying Hooke's Law, the space part of the stress tensor is

$$\tilde{T}_{mn}(\bar{r}) = (i\omega/\alpha)(\lambda\delta_{mn} + 2\mu\delta_{m3}\delta_{n3})\exp(i\omega x_3/\alpha). \tag{6.3}$$

Then, the traction components on a surface whose unit normal vector is $\hat{\tau}$ are

$$\begin{aligned}\tilde{T}_m(\bar{r}') &= \tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}') \\ &= (i\omega/\alpha)[\lambda\tau_m(\bar{r}') + 2\mu\delta_{m3}\tau_3(\bar{r}')]\exp(i\omega x_3'/\alpha),\end{aligned} \tag{6.4}$$

where $\bar{r}'$ is on the surface.

Now, taking the surface to be the boundary of a void cavity, a return to Chapter IV is made in order to derive the first approximation to the scattered field. In particular, the beginning point is the analog of representation (4.20). Since it is required that the constructed linear combination of multipole tractions represent $-\tilde{T}_m(\bar{r}')$, the representation is

$$-\tilde{T}_m(\bar{r}') \approx a_s g_{sm}(\bar{r}'), \tag{6.5}$$

where $\tilde{T}_m(\bar{r}')$ is now given by Equation (6.4). There is no superscript j here because only one boundary value problem is being solved; Chapter IV concerned a set of three boundary value problems. The rest of the work routinely follows the remainder of Chapter IV and all of Chapter V except the last two sections, which do not apply.

Consider now an incident field given by

$$U_m(\bar{r},t) = \delta_{m1}\exp(-i\omega(t - x_3/\beta)), \tag{6.6}$$

which is the displacement due to a plane shear wave traveling parallel to $\hat{e}_3$ while polarized parallel to $\hat{e}_1$. In this case the space parts of the traction components on the cavity boundary are

$$\tilde{T}_m(\bar{r}') = (i\omega/\beta)[\mu\delta_{m1}\tau_3(\bar{r}') + \mu\delta_{m3}\tau_1(\bar{r}')]\exp(i\omega x_3'/\beta). \tag{6.7}$$

Again a return to Chapter IV is made; the work is routine.

It may be noted that the method outlined here for the treatment of one plane wave scattering problem is in principle quite analogous to calculating one third of a Green's Function. However, the plane wave problems are in fact easier computationally because the structure of the incident field is simpler in each case.

Some numerical results regarding the plane wave problems just discussed are presented in the next chapter.

B. Scattering by Multiple Cavities

When two or more void cavities exist in the medium, it is likely that fields scattered from some of them would be non-negligible at the boundaries of others. Thus, secondary scattered fields may have to be calculated. In turn, these secondary scattered fields may be sufficiently strong to create still other scattered fields, and so on. In this manner, the cavities may be said to interact with each other.

To solve source problems in such media, one may desire a Green's Function corresponding to the total free boundary. This total boundary is the "sum" of all the individual cavity boundaries. A method for

estimating such Green's Functions will now be presented. With a modification in viewpoint analogous to that described in the preceding section, it can also be used to solve problems in which prescribed incident fields, such as plane waves, traverse the region with cavities. Also, as discussed in the next section, the method may be used to treat cavity-source problems in which the field generated by such a source scatters back upon it from some other object, the resulting secondary scattered field being non-negligible.

In all problems related to a specified source operating in the vicinity of two or more cavities, one will always have the choice of treating incident fields on a multipole component by multipole component basis or of treating the incident fields as whole entities.

What follows now is the outline of a method for calculating the approximate Green's Function for the two cavity case. The geometrical aspects of the problem are shown in Figure (6.1).

In the figure, S_1 and S_2 are the boundaries of the two void finite cavities. The points $\bar{r}_1$ and $\bar{r}_2$ are "expansion points" chosen within S_1 and S_2 respectively; they each correspond to the point $\bar{r}_1$ used previously in connection with one-cavity problems. $\bar{r}_o$ is the source point, i.e., the point where the three fundamental dipoles are located. $\bar{r}$ is the field point.

In the absence of cavity 2, one can calculate the following Green's Function, corresponding to cavity 1, by means of Chapters IV and V:

$$ {}^1\tilde{Q}^*_{ij}(\bar{r}|\bar{r}_o) = \tilde{Q}_{ij}(\bar{r}|\bar{r}_o) + {}^1N^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]. \qquad (6.8) $$

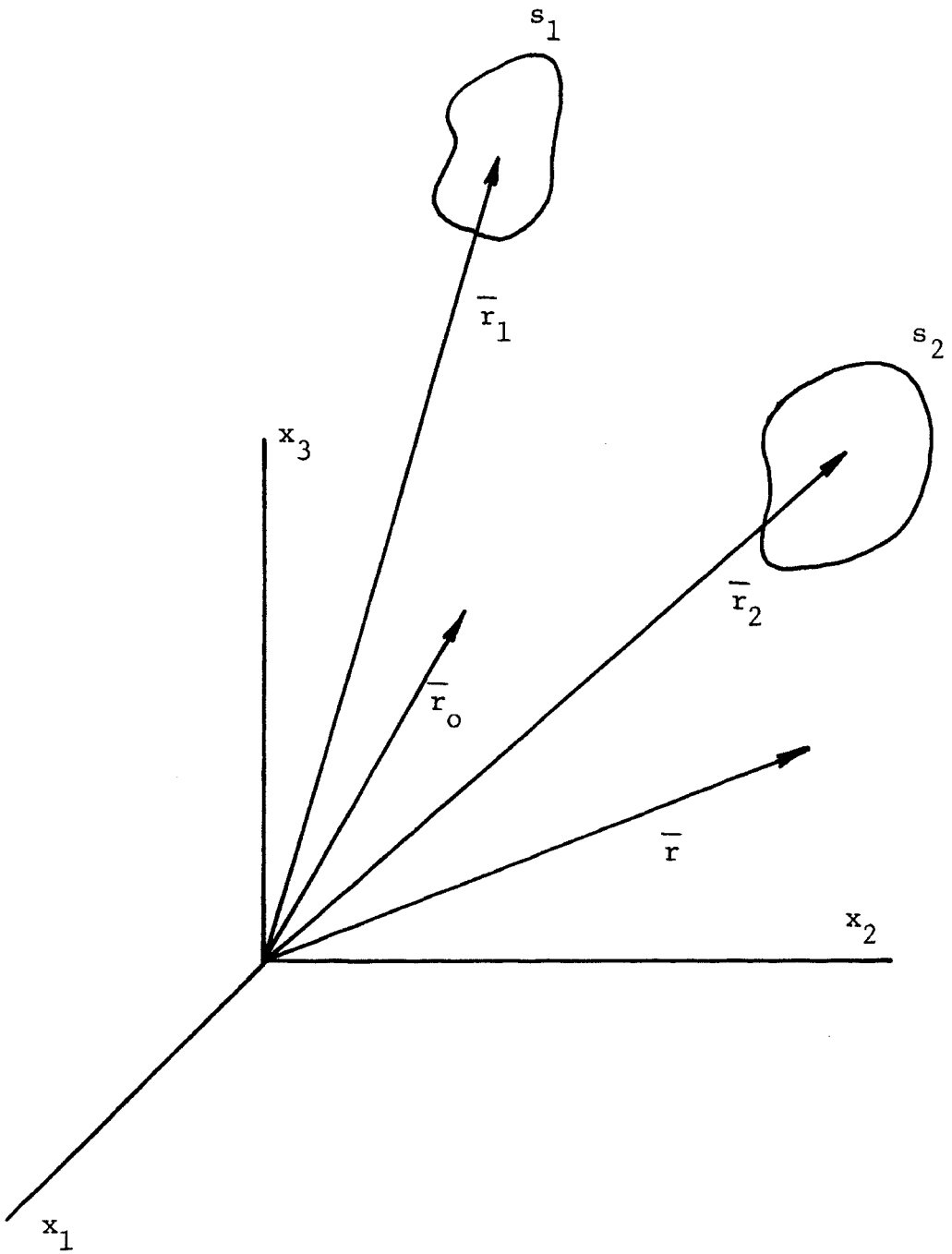

Figure 6.1 The Two-Cavity Scattering Problem

Similarly, in the absence of cavity 1, one can calculate

$$ {}^{2}\tilde{Q}^{*}_{ij}(\bar{r}|\bar{r}_o) = \tilde{Q}_{ij}(\bar{r}|\bar{r}_o) + {}^{2}N^{j}_{s}[\tilde{Q}_{is}(\bar{r}|\bar{r}_2)]. \tag{6.9} $$

the Green's Function corresponding to cavity 2 alone. Assuming these two Green's Functions are known, the method for calculating $\tilde{Q}^{*}_{ij}(\bar{r}|\bar{r}_o)$, the "total" Green's Function, is one of successive approximations. First, $\tilde{Q}^{*}_{ij}(\bar{r}|\bar{r}_o)$ is approximated in the following way:

$$ \tilde{Q}^{*}_{ij}(\bar{r}|\bar{r}_o) \approx \tilde{Q}_{ij}(\bar{r}|\bar{r}_o) + {}^{1}N^{j}_{s}[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)] + {}^{2}N^{j}_{s}[\tilde{Q}_{is}(\bar{r}|\bar{r}_2)]. \tag{6.10} $$

On the right side of the approximation, the second part is the scattered field from S_1 when S_2 is absent; it comes from Equation (6.8). The third part is the scattered field from S_2 in the absence of S_1; it is taken from Equation (6.9). Now, the first two parts on the right side of approximation (6.10) taken together give rise to negligible traction on cavity 1 by construction of the Green's Function (6.8). However, the third part gives rise to non-negligible traction at cavity 1. Similarly, it is the second part alone that gives rise to non-negligible traction at the boundary of cavity 2. Thus, there exists a situation in which multipole fields are incident upon each cavity. Suitable scattered fields must be constructed to cancel the consequent traction in each case.

Such construction was the subject of Section (V-D). In that section it was the multipole fields contained in $L_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_o)]$ that arrived at the cavity. Here it is the fields contained in ${}^{2}N^{j}_{s}[\tilde{Q}_{is}(\bar{r}|\bar{r}_2)]$ that arrive at cavity 1 requiring nullification. In the meantime, the fields contained in ${}^{1}N^{j}_{s}[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]$ are incident upon cavity 2; their

tractions must be nullified by suitably constructed scattered fields. When the two new sets of scattered multipole fields are added to the right side of approximation (6.10), a second approximation to the Green's Function results. However, the new fields just constructed which represent scattering from cavity 1 now create a (perhaps) non-negligible traction at cavity 2, and vice versa. Therefore, the next approximation is obtained by the calculation of still two more sets of scattered multipole fields. The process is continued until the most recently constructed fields representing scattering from cavity 1 are negligible at cavity 2, and vice versa.

C. The Cavity-Source Problem

The cavity-source problem involves the calculation of the displacement field arising in a medium due to the application of known tractions upon the boundary of a cavity situated in the medium. In what follows, the cavity will be taken as being of finite extent; the surrounding medium will be infinite, isotropic, homogeneous, and perfectly elastic.

The applied traction components are taken as $\tilde{T}_m(\bar{r}')\exp(-i\omega t)$, $\bar{r}'$ on S, the cavity boundary. With $\tau_n(\bar{r}')$ the components of the inward unit normal vector on S, the boundary conditions on the stress tensor field $\tilde{T}_{mn}(\bar{r}')$ are given by

$$\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}') = \tilde{T}_m(\bar{r}'). \tag{6.11}$$

Equation (6.11) may be understood as follows: The traction components $\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}')$ are, by definition, exerted upon matter on the side of S out of which $\hat{\tau}$ is pointing by whatever is on the other side of S. In the present case, some kind of machinery is on this latter side of S;

it is the traction with components $\tilde{T}_m(\bar{r}')$ that this machinery is applying. Hence Equation (6.11) follows.

Introducing the source through the boundary conditions, Equation (6.11), means that no volume integral is present in Green's Representation, Equation (5.1), for the displacement components $\tilde{U}_i(\bar{r})$. That representation becomes

$$\tilde{U}_i(\bar{r}) = \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{T}_m(\bar{r}')dS' - \int_S \tilde{U}_m(\bar{r}')\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS', \tag{6.12}$$

which is the form used by Case and Colewell (1967). Notice that if $\tilde{T}_m(\bar{r}')$ were given as $-\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')$, where $\bar{r}_o$ is outside the cavity, then Equation (6.12) would become identical to Equation (5.7). In making the comparison of the two equations, one should note that $\tilde{U}_i(\bar{r})$ here would be replaced with $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$. This comparison clearly shows that calculation of the scattered part of a Green's Function involves the solution of three particular cavity-source problems. It is no surprise then, that the methods used in Chapters IV and V are immediately applicable here.

The first step in solving the cavity-source problem is the calculation of the first approximation via Chapter IV. Representation (4.16) becomes

$$\tilde{T}_m(\bar{r}') \simeq A_{sk}\tau_n(\bar{r}')D'_k\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1) + A_{skp}\tau_n(\bar{r}')D'_{kp}\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1). \tag{6.13}$$

Thus, a linear combination of multipole tractions that best approximates $\tilde{T}_m(\bar{r}')$ in the least squares sense is being sought. The absence of the

superscript j on the A_{sk} and A_{skp} here is indicative of the fact that only one boundary value problem is being solved here as opposed to the three problems involved with Green's Functions.

The first approximation to $\tilde{U}_i(\bar{r})$ may be denoted as

$$M_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \tag{6.14}$$

where

$$M_s = A_{sk}D_k + A_{skp}D_{kp}. \tag{6.15}$$

These expressions should be compared with Equation (5.13) and expression (5.14). Upon using the method of Chapter V in connection with Equation (6.12) here, the first approximation may be corrected to yield a second, and final, approximation in multipole series form:

$$\tilde{U}_i(r) = N_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \tag{6.16}$$

where N_s arises from M_s upon applying the corrections.

An important asset accrued by being able to solve cavity-source problems concerns the development of a test which can be used to check certain portions of the computations made in solving other problems. That is, one can solve a cavity-source problem that has a known solution. Such a case will now be discussed.

Consider the three problems arising when $\tilde{T}_m(\bar{r}')$ are specified as $\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_1)\tau_n(\bar{r}')$, successively as j = 1,2,3. Here, $\bar{r}_1$ is inside the cavity so that this is not the scattering problem. $\hat{\tau}$ points into the cavity. The three displacement fields for j = 1,2,3 will be denoted by $\tilde{U}_{ij}(\bar{r}|\bar{r}_1)$. Green's Representations for these fields are, by Equation (6.12)

$$\tilde{U}_{ij}(\bar{r}|\bar{r}_1) = \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_1)\tau_n(\bar{r}')\,dS'$$

$$- \int_S \tilde{U}_{mj}(\bar{r}'|\bar{r}_1)\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')\,dS'. \qquad (6.17)$$

Comparison of this with Equation (5.10) shows that

$$\tilde{U}_{ij}(\bar{r}|\bar{r}_1) = \tilde{Q}_{ij}(\bar{r}|\bar{r}_1), \qquad (6.18)$$

in accordance with the well known (Love (1944), p. 185) and intuitively clear fact that $\tilde{Q}_{ij}(\bar{r}|\bar{r}_1)$ can be generated by application of the traction components $\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_1)\tau_n(\bar{r}')$ to the boundary of a cavity containing $\bar{r}_1$. Therefore, the known solution with which the computed results will be compared is merely the fundamental solution in this case.

As shown in Appendix D, the fundamental solution can be written as a certain linear combination of octupole terms. Thus, the dipole tractions, i.e., those calculated from the fundamental solution, are linear combinations of octupole tractions. All the independent octupole tractions are in the approximating set used in the least squares process giving the first approximation. This means that the solutions to the present cavity-source problems can be represented exactly by certain linear combinations of displacement fields whose corresponding tractions are in the approximating set. If the theory is valid and the computational procedures correct, then the first approximation, calculated by least squares, should be exact in each of the three present examples. The coefficients in the linear combinations calculated by least squares should be identical to those exact coefficients calculated by using Appendix D. Finally, the corrections applied via Chapter V should be zero.

The numerical calculations described above were in fact carried out for a given cavity. The least squares process generated the correct coefficients within four significant figures of accuracy. The corrections were so small that they did not change these four figures in any case.

These results point out that when the given applied traction $\tilde{T}_m(\bar{r}')$ in a cavity-source problem closely resembles some linear combination of multipole tractions contained in the approximating set, then one may expect good results from the computations. Another point to be made concerns numerical work; the computations just described required about seven minutes on an IBM 360-50 computer. The results help generate confidence in a rather large amount of programming used in all other problems. Also, they indicate that roundoff error is not intolerably severe.

Before closing this section, it should be mentioned that in some problems it may happen that the field generated by the cavity-source strikes some other object, thereby causing non-negligible energy to return to the source. One may be required to calculate the field resulting when this returning energy is scattered from the cavity-source itself. Here, it will be assumed that the returning field has been calculated, and postulated that the mechanism within the cavity-source continues to apply the same traction to the boundary that it does in the absence of any returning energy. Under this postulate, it is necessary only to construct a scattered field whose traction cancels that due to the returning energy. This approach leads back to the preceding section, which may be applied directly.

D. Scattering from Rigid Bodies

If any of the finite void cavities so far discussed in this dissertation were to be replaced with rigid bodies, a slight modification of the method would again provide solutions. This is true because of the similarity in the boundary conditions in the two instances. The boundary conditions are vastly different from a physical viewpoint, but quite similar mathematically. In the case of void cavities, the traction must vanish at the boundary; with rigid bodies, displacement must vanish. Green's Functions and other fields must be constructed in such a way that this latter condition is fulfilled.

In Chapter IV, one would use the least squares process to find a linear combination of multipole displacements that would approximately cancel the incident displacement field at the boundary. Then, in Chapter V, the traction on the boundary due to this first approximation would be calculated using Hooke's Law. This traction would be used to replace the unknown traction appearing in a Green's Representation of the exact scattered field, leading to the second and final approximation.

VII. NUMERICAL RESULTS

A. Sources of Error

Computer programs have been written (by the author) in order to compute numerical solutions for some selected problems. Before presenting these results however, it might be well to summarize the approximations made during the entire process.

The first approximation comes in the least squares process of Chapter IV. Only in rare instances would one expect to be able to represent the negative incident traction by any linear combination of multipole tractions. The least squares procedure, while almost never producing an exact result, could be improved upon by adding more independent multipole traction vectors to the approximating set. The computational burden would increase, along with roundoff error. This latter type of error is present in all phases of the numerical work.

In Chapter V, the output from the least squares process is used in integral representations, whereupon they become approximate integral solutions. This improves upon the original approximations as shown in Appendix E; however, even if it were possible to evaluate the integrals exactly, the final solutions would still be approximate.

The new solutions were seen to have the form of the least squares approximations to the displacements plus correction integrals. The burden of having to evaluate the latter at each field point encouraged their replacement with multipole series. Multipole series, however, must be truncated in practice; doing so resulted in still other errors. The antidote for these errors is the retention of more terms in the multipole series.

Finally, there are errors arising from the method used to evaluate the integrals arising during the calculation of the first approximation and also during the calculation of the multipole series replacing the correction integrals. In general, such surface integrals cannot be evaluated in closed form. For the numerical examples presented here, the integrals were approximated by replacing them with finite sums. The surface was broken into N surface elements whose areas were computed. N was taken as 392 here. The integrands were evaluated near the centroid of each element, the result being multiplied by the element's area in each case. The results were summed to complete the approximate integration.

In summary, there are at least four ways in which accuracy may be improved:

1) By using more multipole tractions in the approximating set;
2) By using more terms in the multipole series replacing the correction integrals, or by using the latter directly;
3) By using more surface elements in the numerical integration;
4) By using more significant figures in all the several million arithemetic operations involved throughout.

B. Energy Carried by Seismic Waves

In presenting numerical results related to 3-dimensional elastic wave problems, there are several physical quantities whose values can be computed and sketched. Examples include the magnitude of the displacement vector, components of displacement, dilatation, components of rotation, potentials, etc. One could discuss quantities that propagate with velocity α and quantities propagating with velocity β. Source

fields, scattered fields, or their sum could be depicted.

Of all the available choices, elastic wave energy has been chosen as the vehicle used to pictorially summarize the overall nature of the scattering. The calculations leading to the diagrams will now be explained.

The energy carried by elastic waves is perhaps best described by a vector function $\bar{S}(\bar{r},t)$, which gives the time rate and direction of energy flow per unit area at any point $\bar{r}$ in the medium. The vector has dimensions of energy per unit time per unit area, i.e., power per unit area. It may be called the density of energy flow; it is the direct analog of Poynting's Vector of Electrodynamics. The form of $\bar{S}(\bar{r},t)$ is given by Morse and Feshbach (1953), p. 151. An intuitive derivation of the formula will be given here in the hope that it will help clarify the diagrams given later.

Whatever the form of $\bar{S}$, the scalar quantity $\bar{S}\cdot\hat{\tau}$, where $\hat{\tau}(\bar{r})$ is a unit vector at $\bar{r}$, must be the energy per unit time per unit area flowing in the direction of $\hat{\tau}$. If $\hat{\tau}$ is the unit normal to an element of area dS at $\bar{r}$, then $\bar{S}\cdot\hat{\tau}dS$ must be the energy per unit time crossing dS in the direction of $\hat{\tau}$. Now, energy is transmitted in an elastic medium by tractile forces, i.e., traction. When forces are exerted upon matter on the side of dS toward which $\hat{\tau}$ is drawn, by matter on the other side of dS, then there is a positive flow of energy crossing dS in the direction of $\hat{\tau}$. The traction exerted upon matter on the side of dS toward which $\hat{\tau}$ is drawn, by matter on the other side of dS, is $-\hat{e}_m T_{mn} \tau_n$, by definition of the stress tensor T_{mn}. The force on dS is $-\hat{e}_m T_{mn} \tau_n dS$.

The power delivered by this force is $\bar{V}\cdot(-\hat{e}_m T_{mn}\tau_n dS)$, where $\bar{V}$ is the velocity of the medium at dS. This power is the energy per unit time crossing dS in the direction of $\hat{\tau}$. Hence

$$-V_m T_{mn}\tau_n dS = \bar{S}\cdot\hat{\tau}dS = S_n\tau_n dS, \tag{7.1}$$

where

$$S_n(\bar{r},t) = -V_m(\bar{r},t)T_{mn}(\bar{r},t) \tag{7.2}$$

is the definition of $\bar{S}$. Since Equation (7.2) does not necessarily follow from Equation (7.1), the above derivation is non-rigorous; nevertheless, Equation (7.2) does correctly give $S_n(\bar{r},t)$. The discussion above explains physically why it does so.

Now, the velocity components are given by

$$V_m(\bar{r},t) = (\partial/\partial t)U_m(\bar{r},t), \tag{7.3}$$

so that

$$S_n = -(\partial U_m/\partial t)T_{mn}. \tag{7.4}$$

Reconsider now the scalar quantity

$$\bar{S}\cdot\hat{\tau} = -(\partial U_m/\partial t)T_{mn}\tau_n, \tag{7.5}$$

which is the instantaneous energy flow density in the direction of $\hat{\tau}$. If the waves were plane waves traveling in the direction of $\hat{\tau}$, then some authors would call the right side of Equation (7.5) the intensity of the waves. Others might prefer to call the time-averaged version by the name intensity. The time-averaged version of $|\bar{S}|$ is called intensity by some authors when $\bar{S}$ is Poynting's Vector in Electrodynamics.

The quantity to be dealt with here is the time average of $\bar{S}\cdot\hat{\tau}$ where $\hat{\tau}$ is the outward unit normal vector to a sphere completely enclosing the scatterer. Thus, the time-averaged values of the energy per unit time per unit area crossing the surface in a direction away from the scatterer will be graphed.

If P is the temporal period of the waves, then the quantity $E(\bar{r})$ to be plotted is

$$E(\bar{r}) = (1/P) \int_0^P \bar{S}(\bar{r},t)\cdot\hat{\tau}(\bar{r})\,dt. \tag{7.6}$$

The angular dependence of E in planes through the source, scatterer, etc., will be sketched. The intersection of these planes with the sphere whose unit normal is $\hat{\tau}$ are circles. Several values of E will be computed for a given circle. Then all values on the circle will be divided by the largest value in the group. When these normalized values are plotted on a circle using polar graph paper, a "radiation pattern" will emerge. This will show the angular dependence of E in the chosen plane. Comparison of two or more diagrams for other planes will provide an idea of the total angular dependence in three dimensions. By adjusting the radius of the spheres upon which E is calculated, one can study the "near", "far", or "intermediate" fields.

Returning to the computational aspects now, recall that

$$U_m(\bar{r},t) = \tilde{U}_m(\bar{r})\exp(-i\omega t) \tag{7.7}$$

and

$$T_{mn}(\bar{r},t) = \tilde{T}_{mn}(\bar{r})\exp(-i\omega t), \tag{7.8}$$

in which $\tilde{U}_m(\bar{r})$ and $\tilde{T}_{mn}(\bar{r})$ are complex valued functions, say

$$\tilde{U}_m(\bar{r}) = a_m(\bar{r}) + ib_m(\bar{r}) \tag{7.9}$$

and

$$\tilde{T}_{mn}(\bar{r}) = c_{mn}(\bar{r}) + id_{mn}(\bar{r}), \tag{7.10}$$

in which a_m, b_m, c_{mn}, and d_{mn} are real valued.

One must use either the real or the imaginary parts of $\tilde{U}_m$ and $\tilde{T}_{mn}$ in calculations of E. The real parts will be used here, meaning that the sources actually have a time factor $\cos(\omega t)$ rather than the $\exp(-i\omega t)$ used in the theory.

Now,

$$\mathrm{Re}[U_m(\bar{r},t)] = a_m(\bar{r})\cos(\omega t) + b_m(\bar{r})\sin(\omega t), \tag{7.11}$$

and

$$\mathrm{Re}[T_{mn}(\bar{r},t)] = c_{mn}(\bar{r})\cos(\omega t) + d_{mn}(\bar{r})\sin(\omega t). \tag{7.12}$$

Thus, using these real parts in Equation (7.5),

$$\begin{aligned}\bar{S}\cdot\hat{\tau} &= \omega[a_m\sin(\omega t) - b_m\cos(\omega t)][c_{mn}\cos(\omega t) + d_{mn}\sin(\omega t)]\tau_n \\ &= \omega[a_m d_{mn}\sin^2(\omega t) + (a_m c_{mn} - b_m d_{mn})\sin(\omega t)\cos(\omega t) \\ &\quad - b_m c_{mn}\cos^2(\omega t)]\tau_n. \end{aligned} \tag{7.13}$$

Using Equation (7.6) to take a time average over a period $P = 2\pi/\omega$,

$$E = (\omega/2)(a_m d_{mn} - b_m c_{mn})\tau_n. \tag{7.14}$$

Equation (7.14) is the equation used in the computer programs. In applying it, one must realize that the calculation of E involves nonlinear operations upon the displacement field. Thus, values of E due to the existence of two superimposed displacement fields (such as an incident field and its scattered field) is not the sum of the values of E calculated for each individual field. The total displacement must be calculated first, with E following from this total displacement. The same precautionary note applies to individual fields given in multipole series form. If E is calculated for each multipolar component, then the sum of these constituents will not represent the correct value of the total E.

C. Radiation Patterns

Since the development of an analytic technique is the goal of this dissertation, not many specific numerical results will be presented. All the figures to follow, except Figure (7.1), concern scattering from a void spherical cavity having a radius of 100 feet. The radiation patterns depict the scattered fields only. In all cases, the frequency is 90 cps. Wave speeds α and β were taken as 20,000 and 10,000 feet/sec, respectively. The mass density ρ was taken as unity. Any other values could have been used for any of these parameters.

In the diagrams, coordinate axes are labeled to indicate the plane in which the figure is drawn. The value of R given in each case denotes the radius of the spherical surface, concentric with the cavity except in Figure (7.1), upon which E was calculated. Where appropriate, the maximum value of E encountered in a given diagram is given as E_{max}. This number is useful for comparing two or more diagrams for the same

incident field; it is otherwise meaningless.

The first figure below is the field of a dipole force alone, i.e., it pertains to the fundamental solution. In computations leading to the remaining figures, the correction derived in Chapter V was applied.

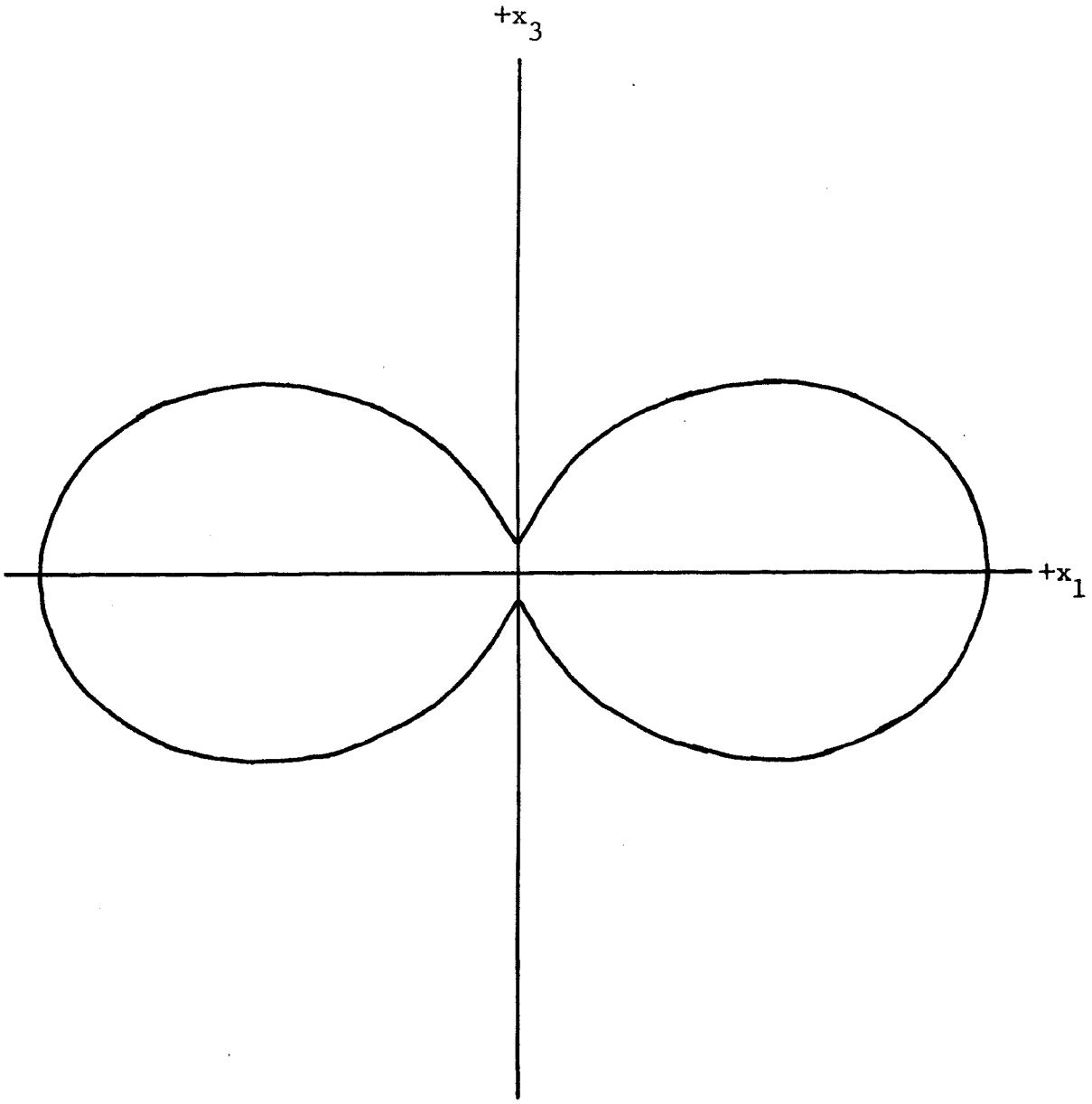

Figure 7.1 Direct E Field for a Dipole Source Along $\hat{e}_3$ in Homogeneous Media

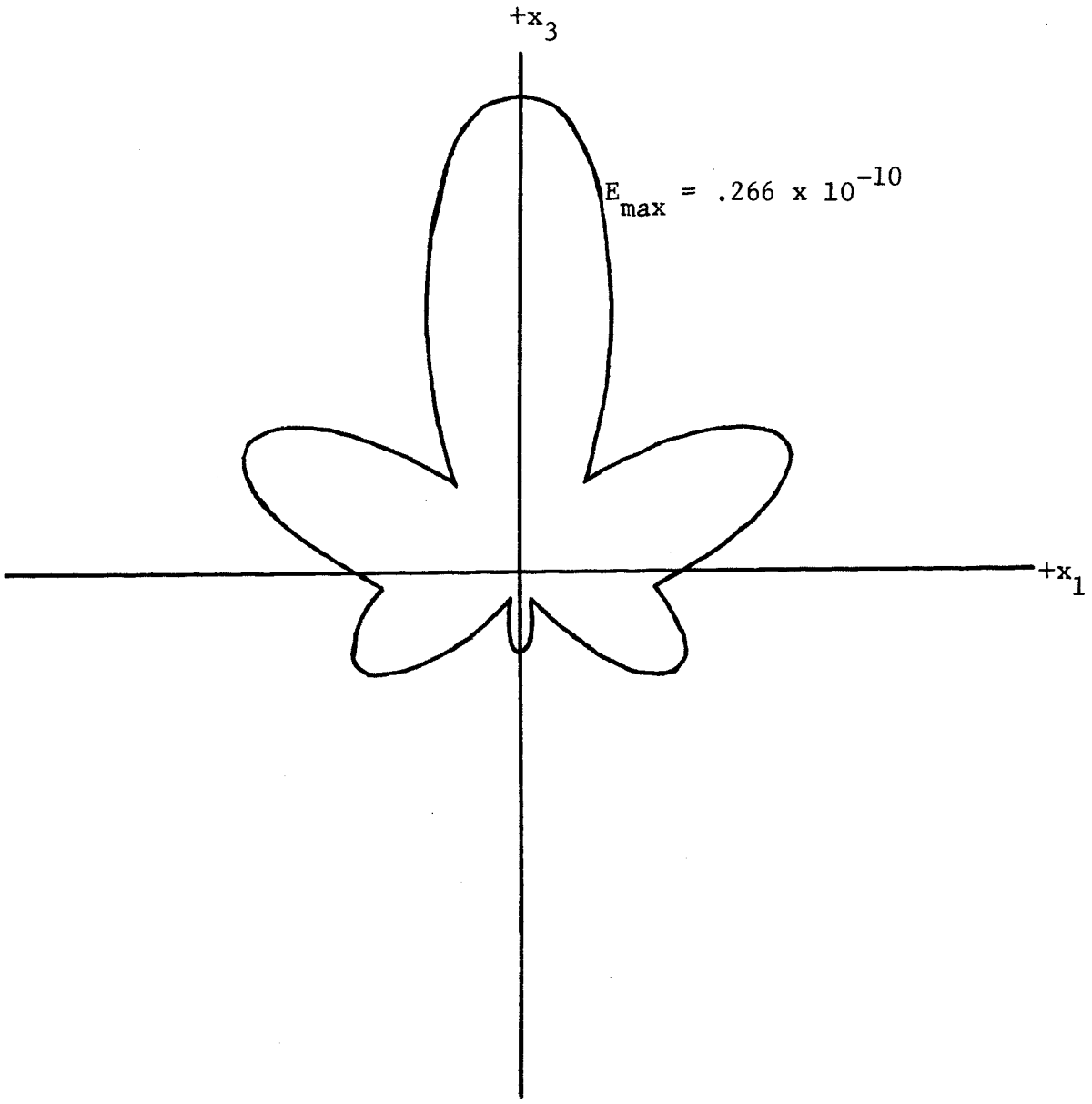

Figure 7.2 Scattered E Field in 1-3 Plane at R = 200 Feet Due to Incoming Displacement $\hat{e}_i Q_{il}$ From Dipole at (0,0,-125)

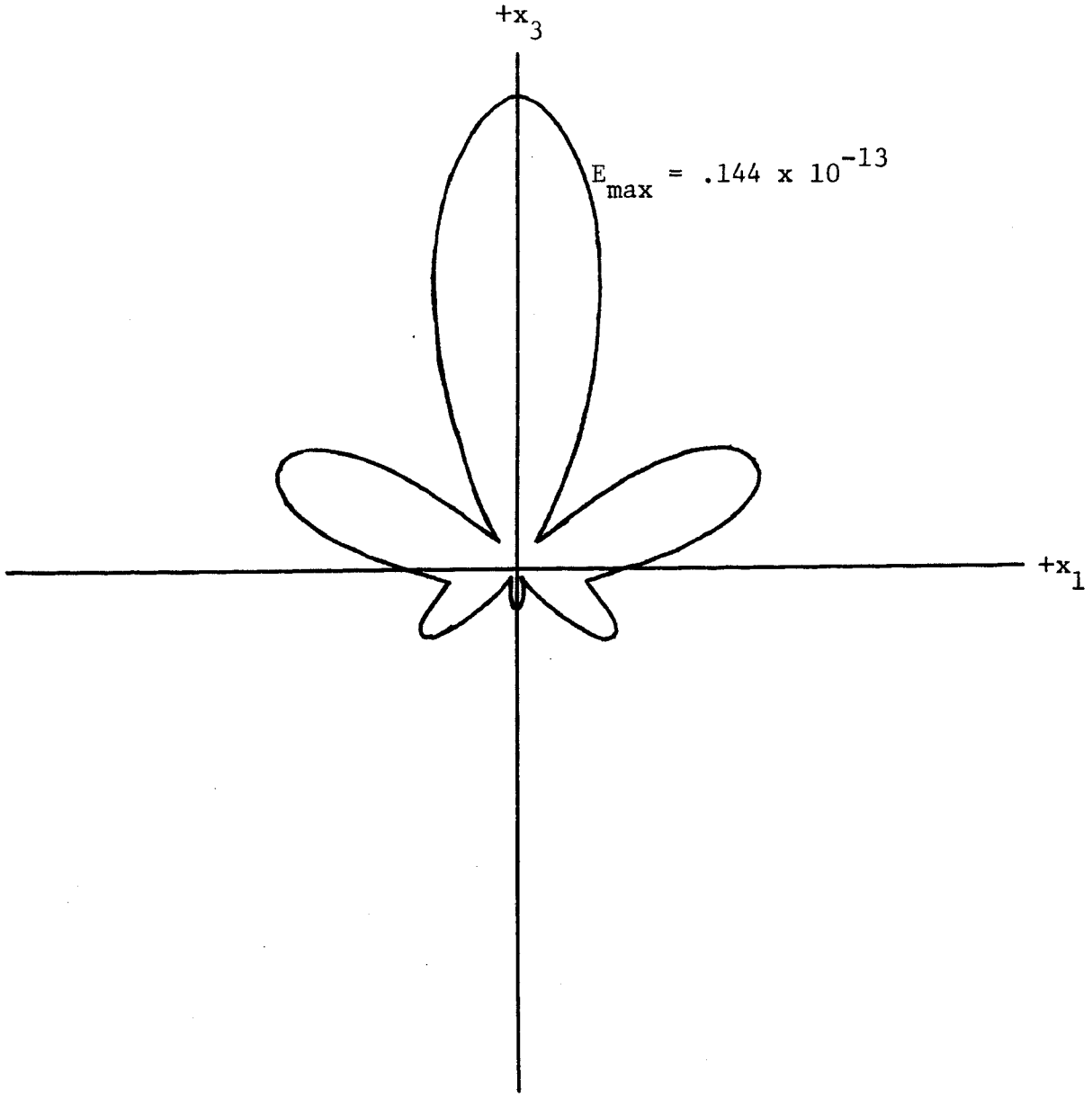

Figure 7.3 Scattered E Field in 1-3 Plane at R = 10,000 Feet Due to Incoming Displacement $\hat{e}_i \tilde{Q}_{i1}$ From Dipole at (0,0,-125)

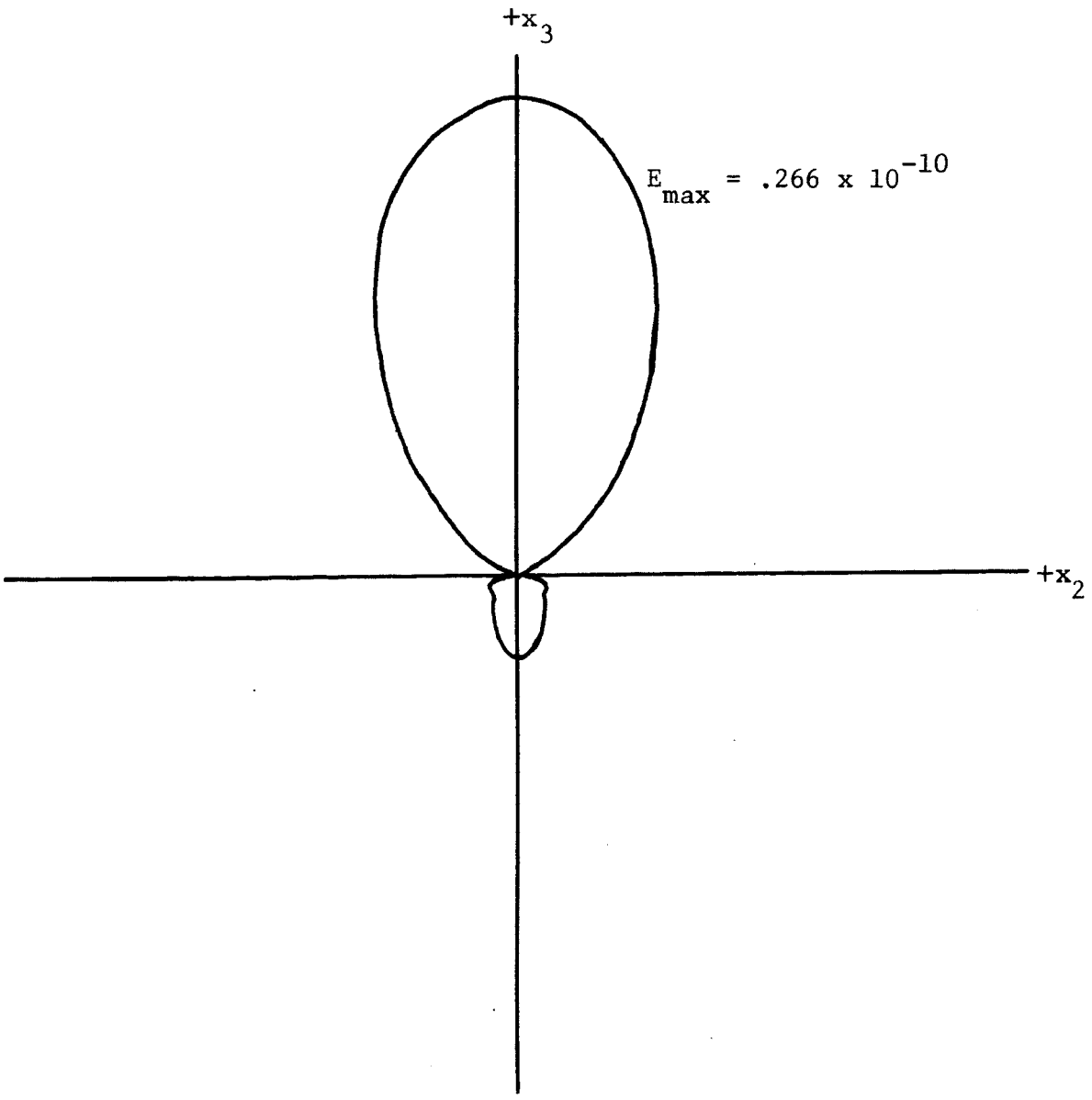

Figure 7.4 Scattered E Field in 2-3 Plane at R = 200 Feet Due to Incoming Displacement $\hat{e}_i \tilde{Q}_{il}$ From Dipole at (0,0,-125)

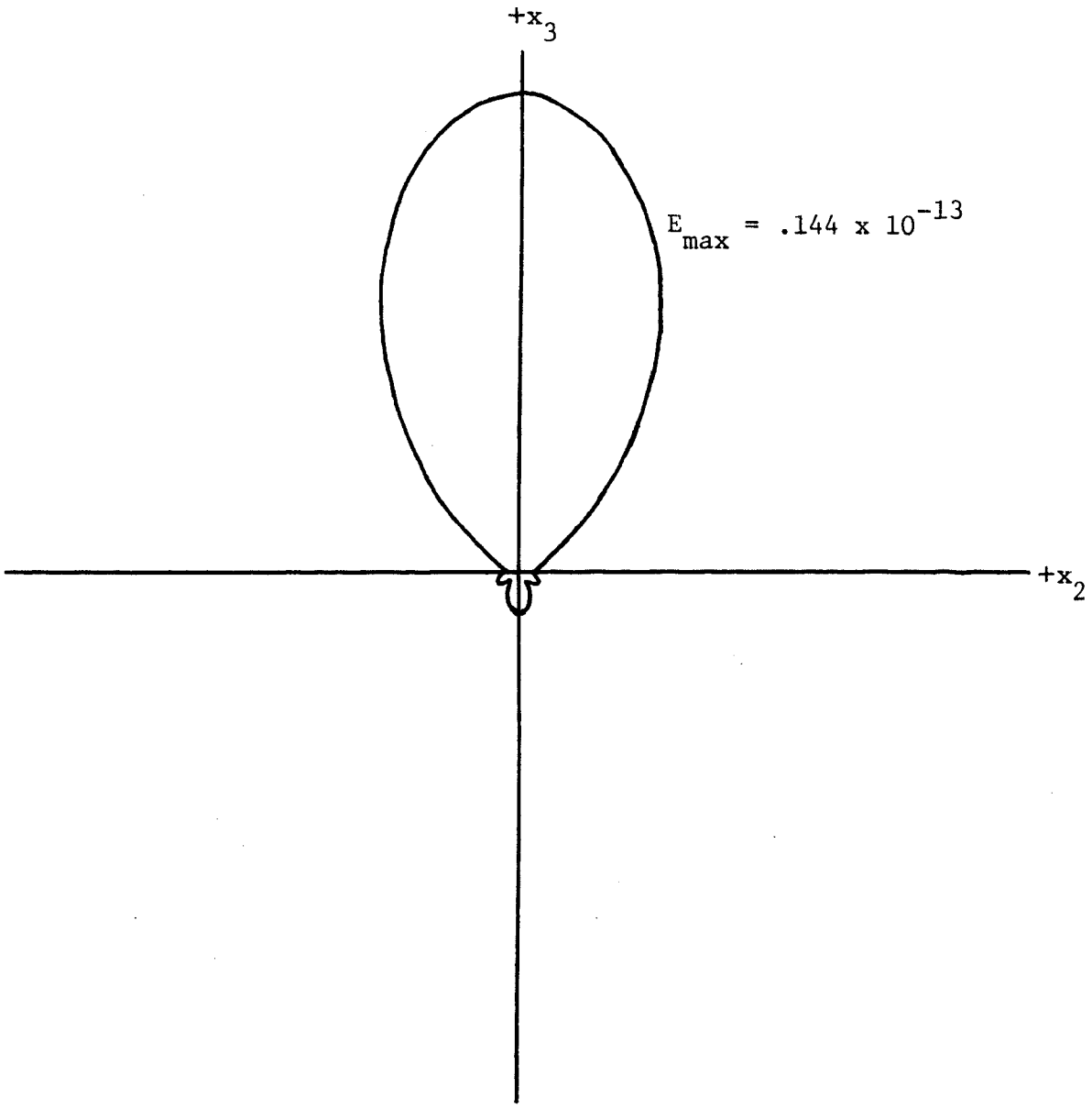

Figure 7.5 Scattered E Field in 2-3 Plane at R = 10,000 Feet Due to Incoming Displacement $\hat{e}_i \tilde{Q}_{i1}$ From Dipole at (0,0,-125)

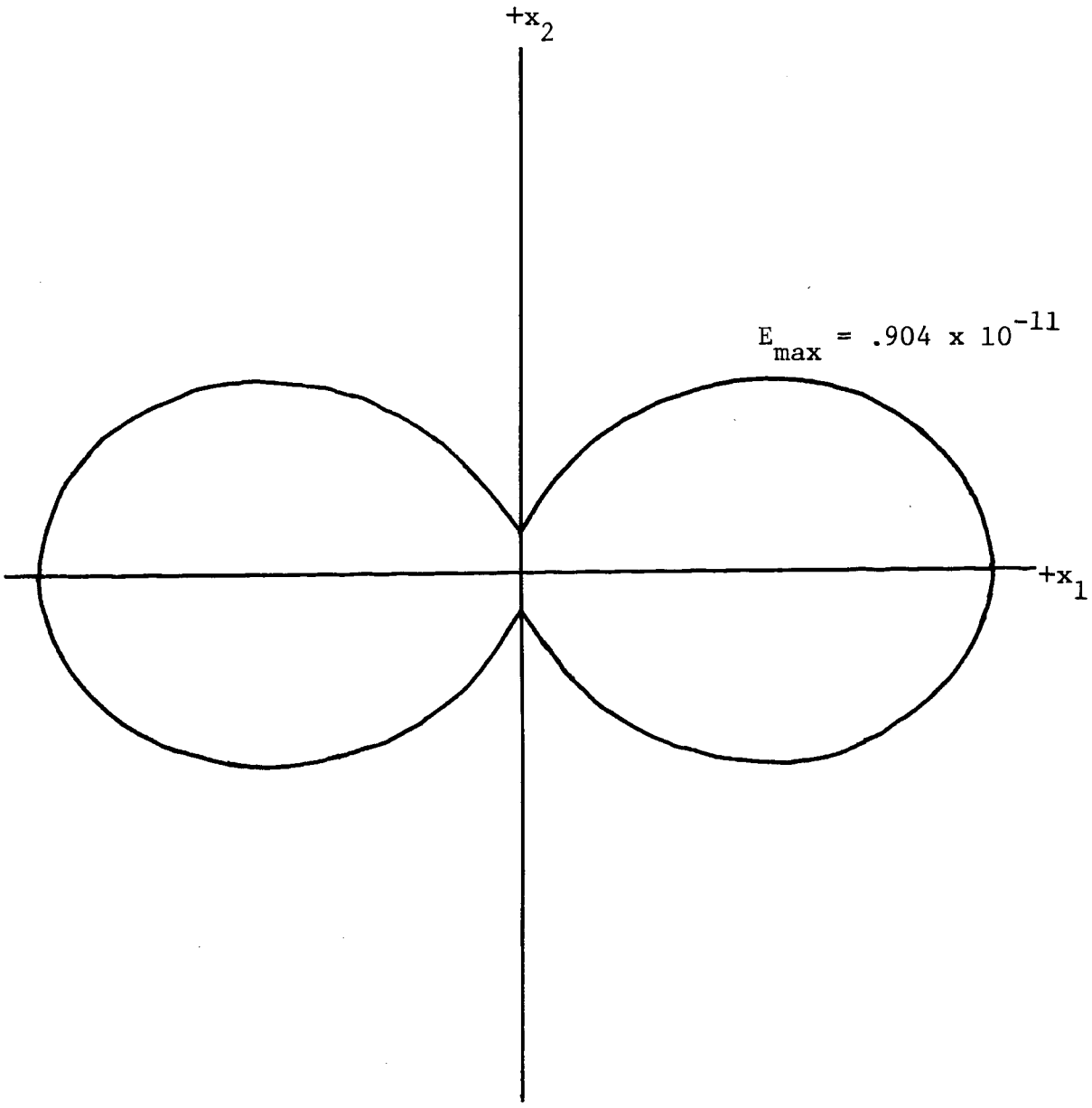

Figure 7.6 Scattered E Field in 1-2 Plane at R = 200 Feet Due to Incoming Displacement $\hat{e}_i \tilde{Q}_{i1}$ From Dipole at (0,0,-125)

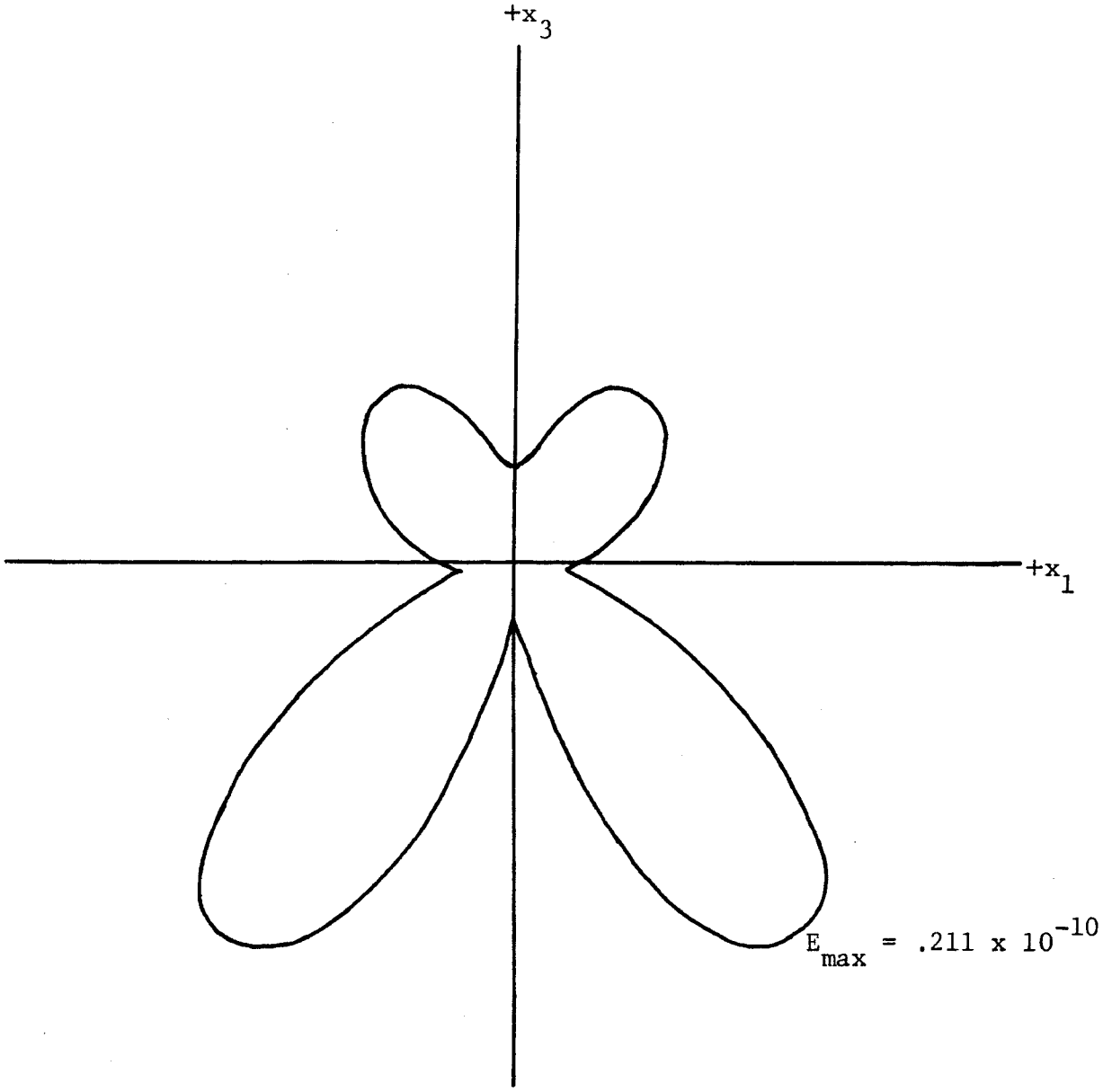

Figure 7.7 Scattered E Field in 1-3 Plane at R = 200 Feet Due to Incoming Displacement $\hat{e}_i \tilde{Q}_{i3}$ From Dipole at (0,0,-125)

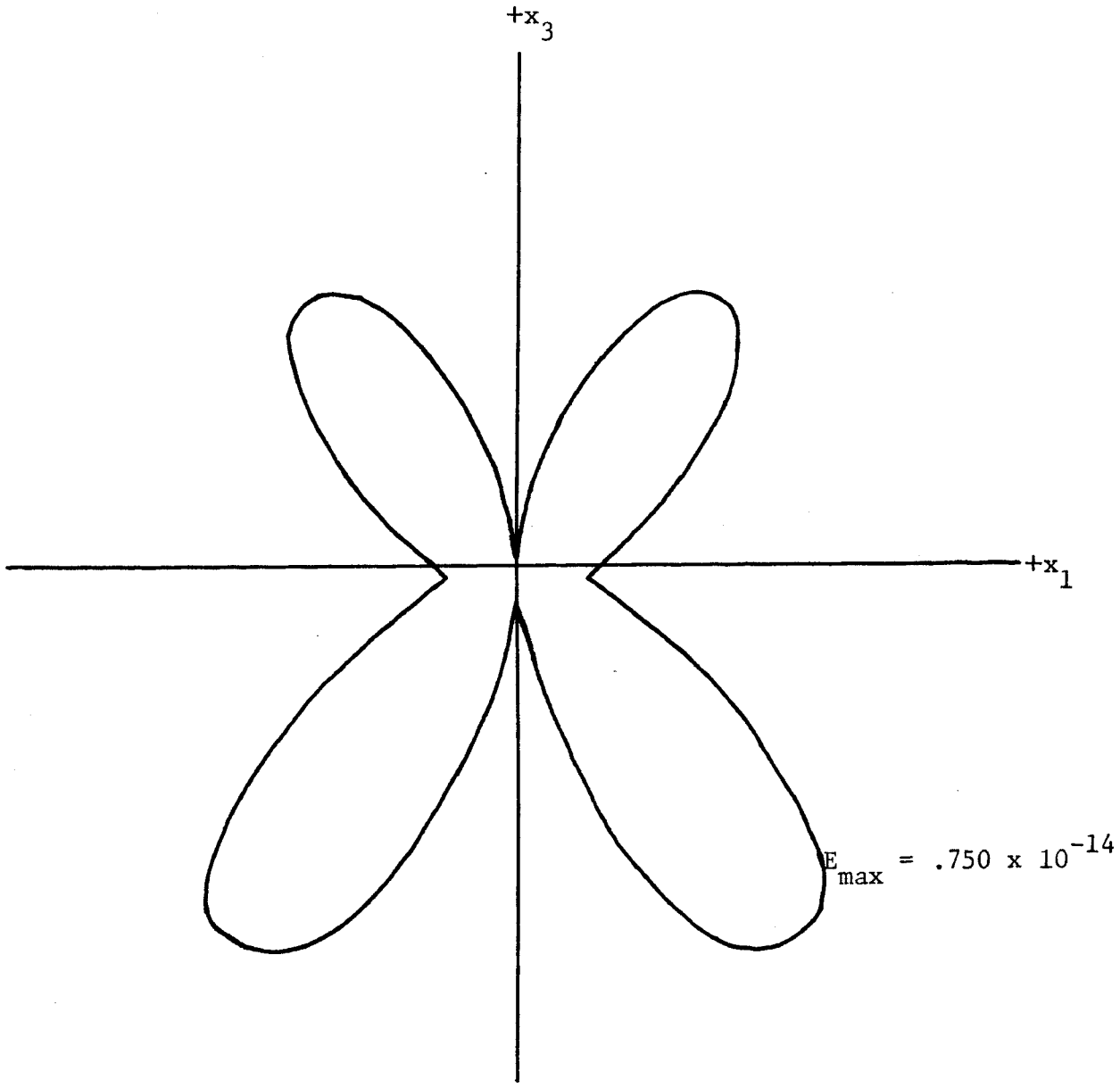

Figure 7.8 Scattered E Field in 1-3 Plane at R = 10,000 Feet Due to Incoming Displacement $\hat{e}_i \tilde{Q}_{i3}$ From Dipole at (0,0,-125)

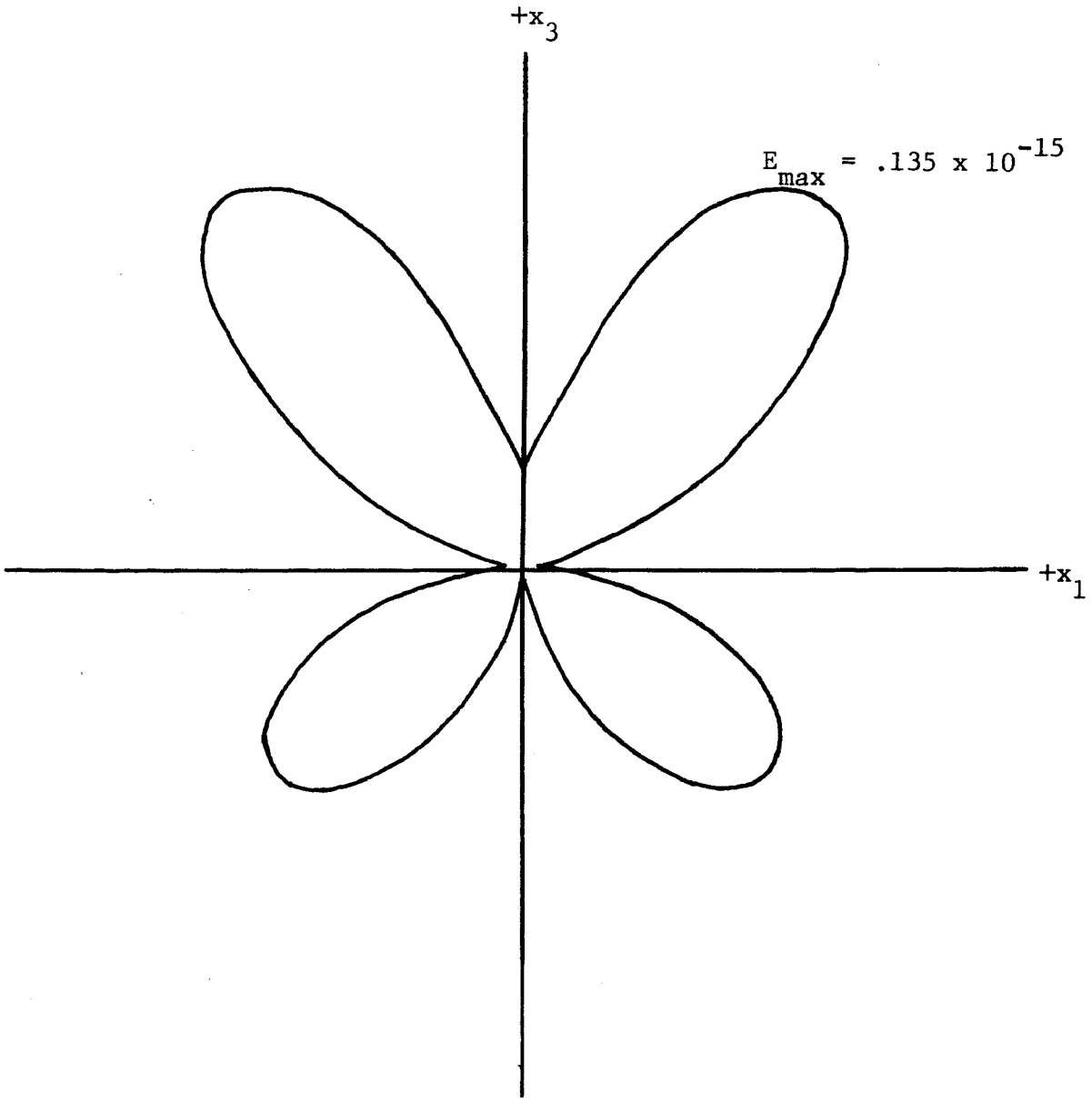

Figure 7.9 Scattered E Field in 1-3 Plane at R = 200 Feet Due to Incoming Plane Compressional Wave Traveling in the $\hat{e}_3$ Direction

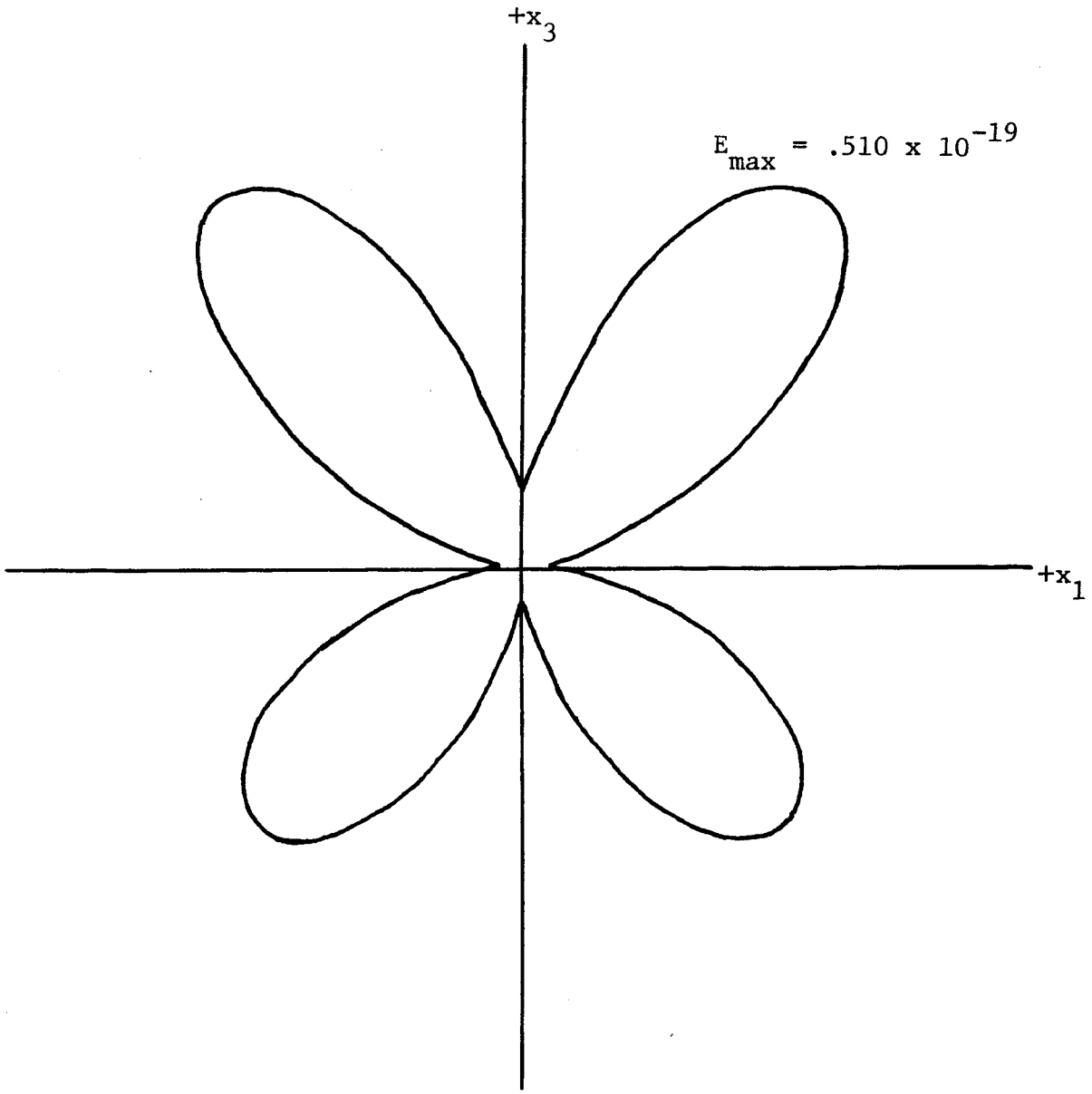

Figure 7.10 Scattered E Field in 1-3 Plane at R = 10,000 Feet Due to Incoming Plane Compressional Wave Traveling in the $\hat{e}_3$ Direction

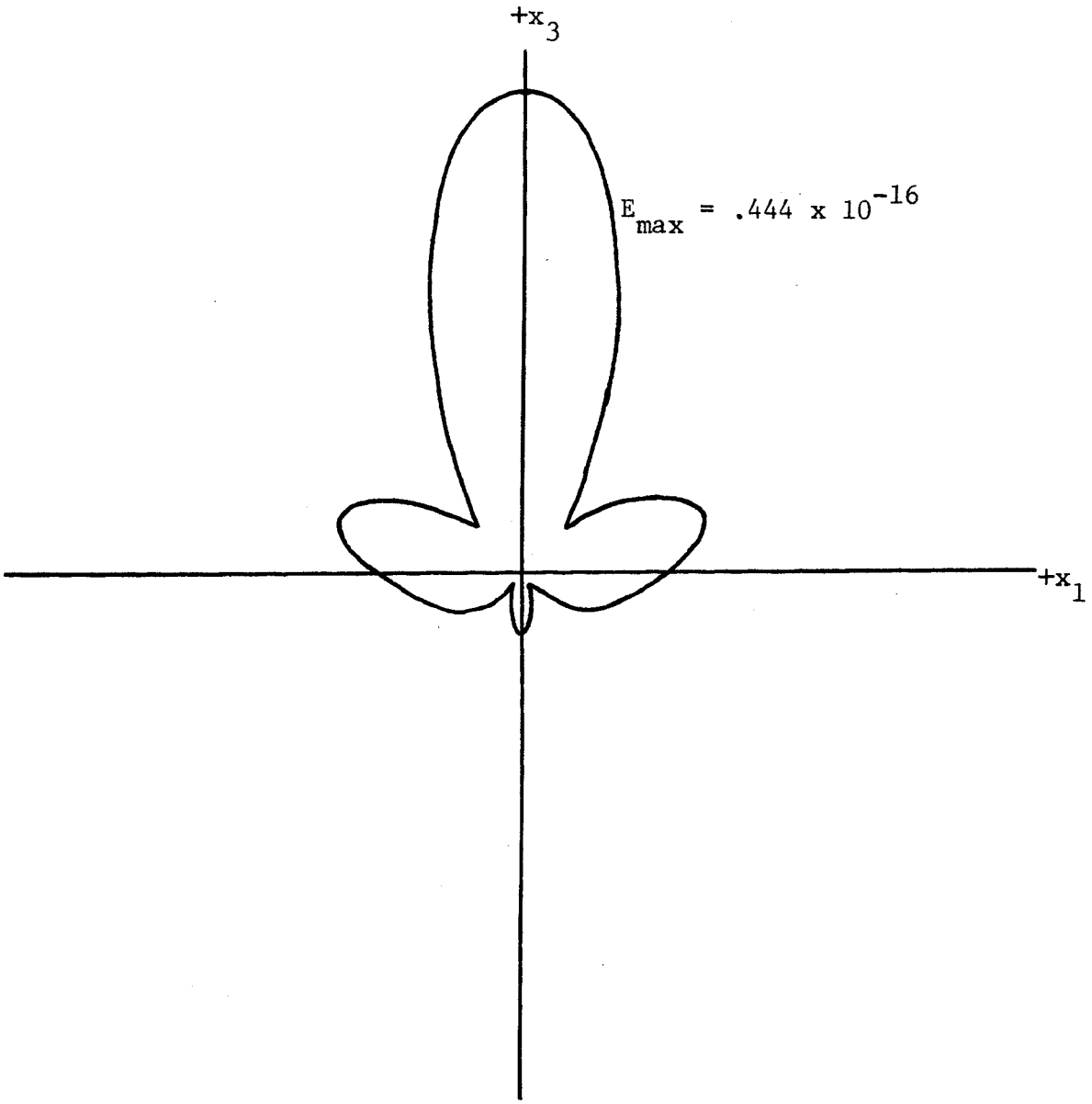

Figure 7.11 Scattered E Field in 1-3 Plane at R = 200 Feet Due to Incoming Plane Shear Wave Traveling in the Direction of $\hat{e}_3$ While Polarized Parallel to $\hat{e}_1$

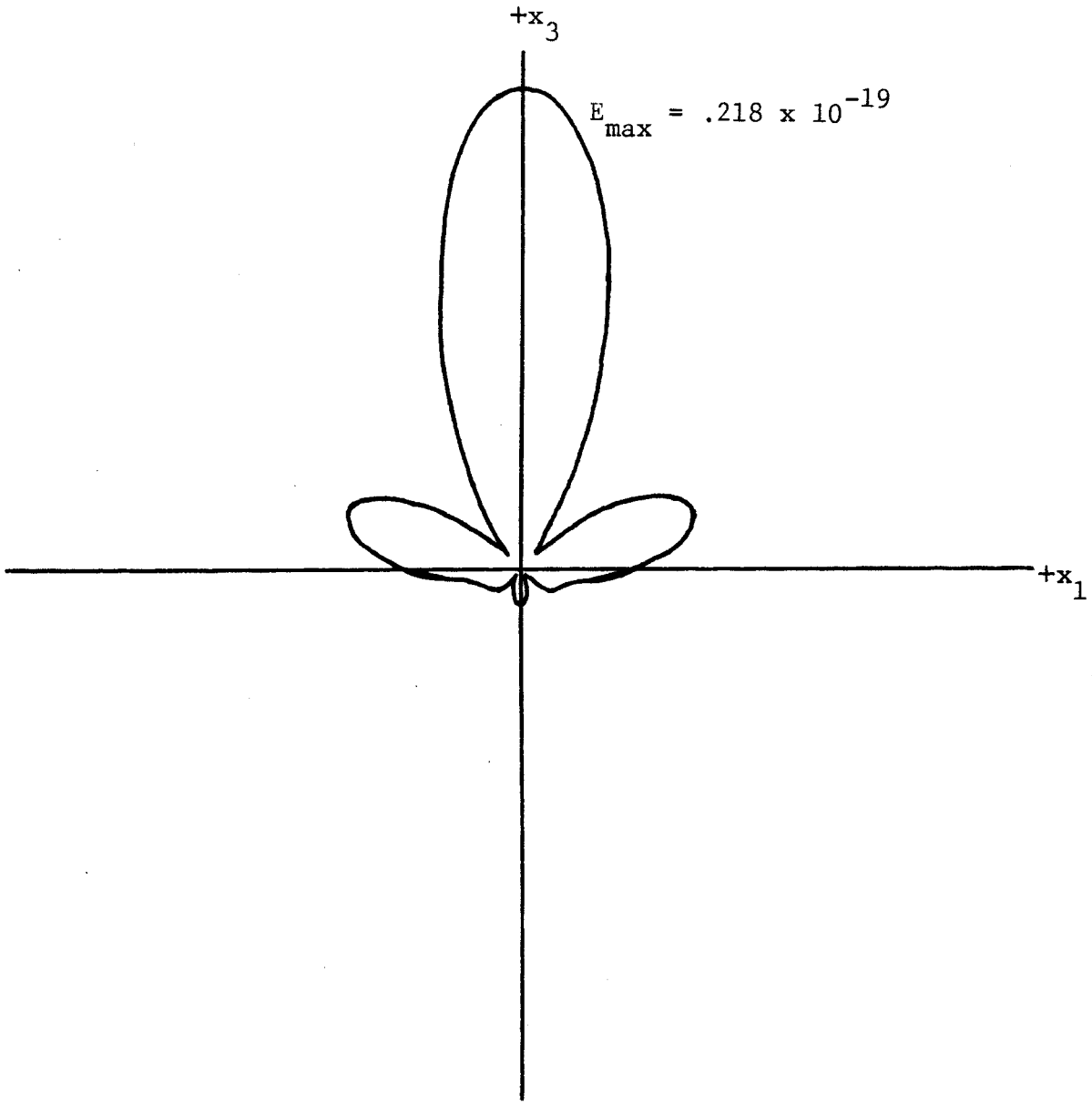

Figure 7.12 Scattered E Field in 1-3 Plane at R = 10,000 Feet Due to Incoming Plane Shear Wave Traveling in the Direction of $\hat{e}_3$ While Polarized Parallel to $\hat{e}_1$

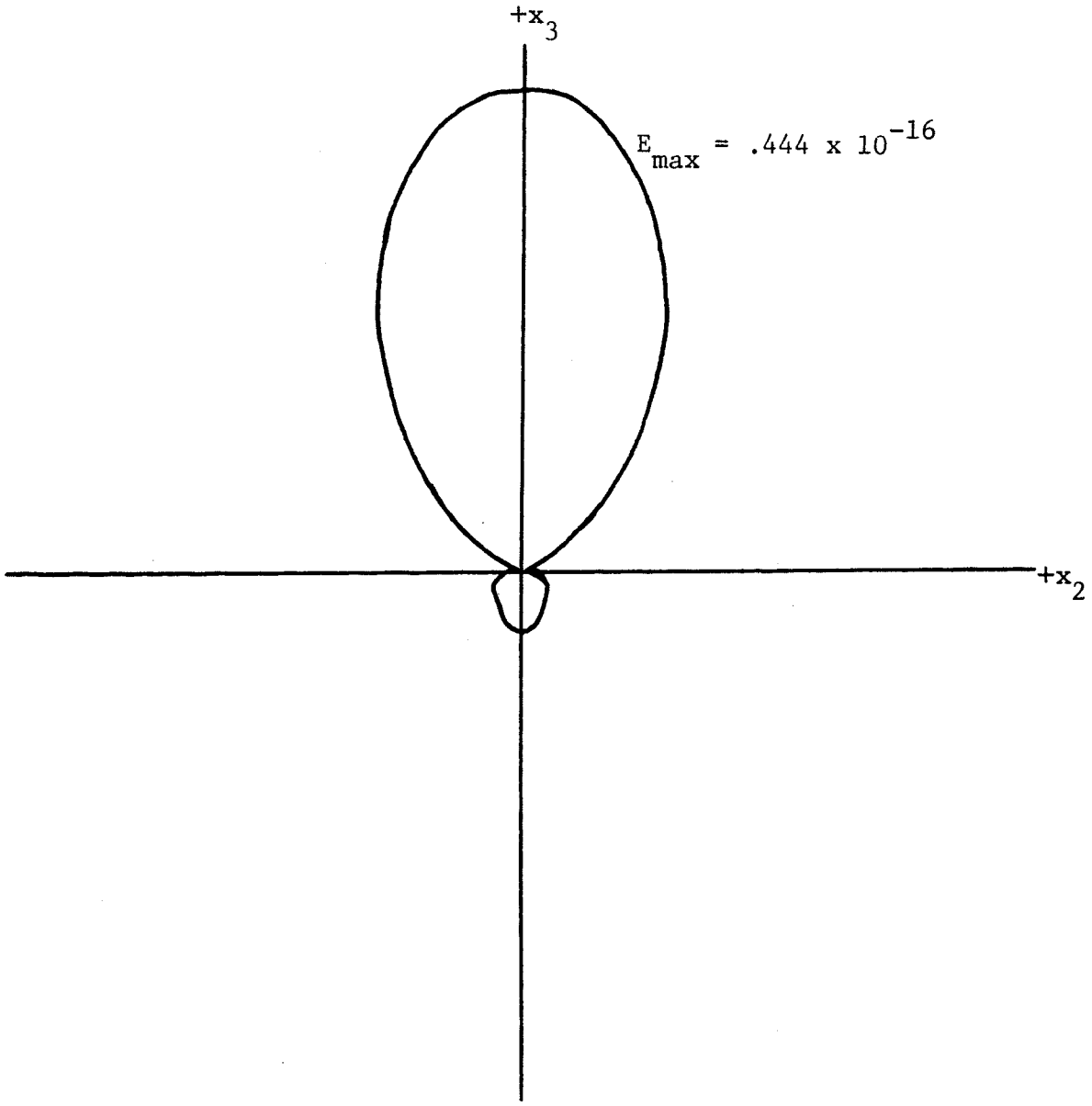

Figure 7.13 Scattered E Field in 2-3 Plane at R = 200 Feet Due to Incoming Plane Shear Wave Traveling in the Direction of $\hat{e}_3$ While Polarized Parallel to $\hat{e}_1$

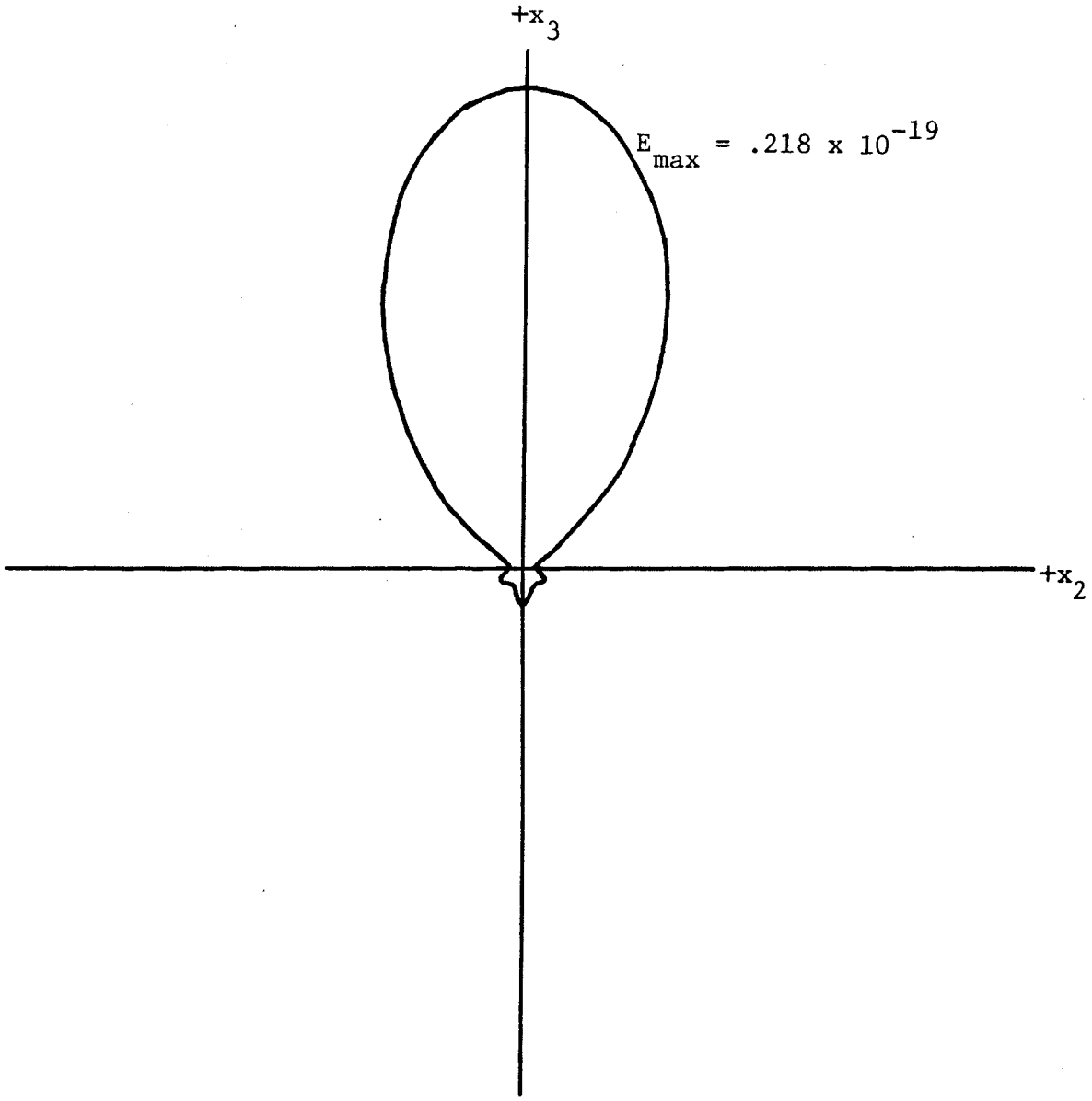

Figure 7.14 Scattered E Field in 2-3 Plane at R = 10,000 Feet Due to Incoming Plane Shear Wave Traveling in the Direction of $\hat{e}_3$ While Polarized Parallel to $\hat{e}_1$

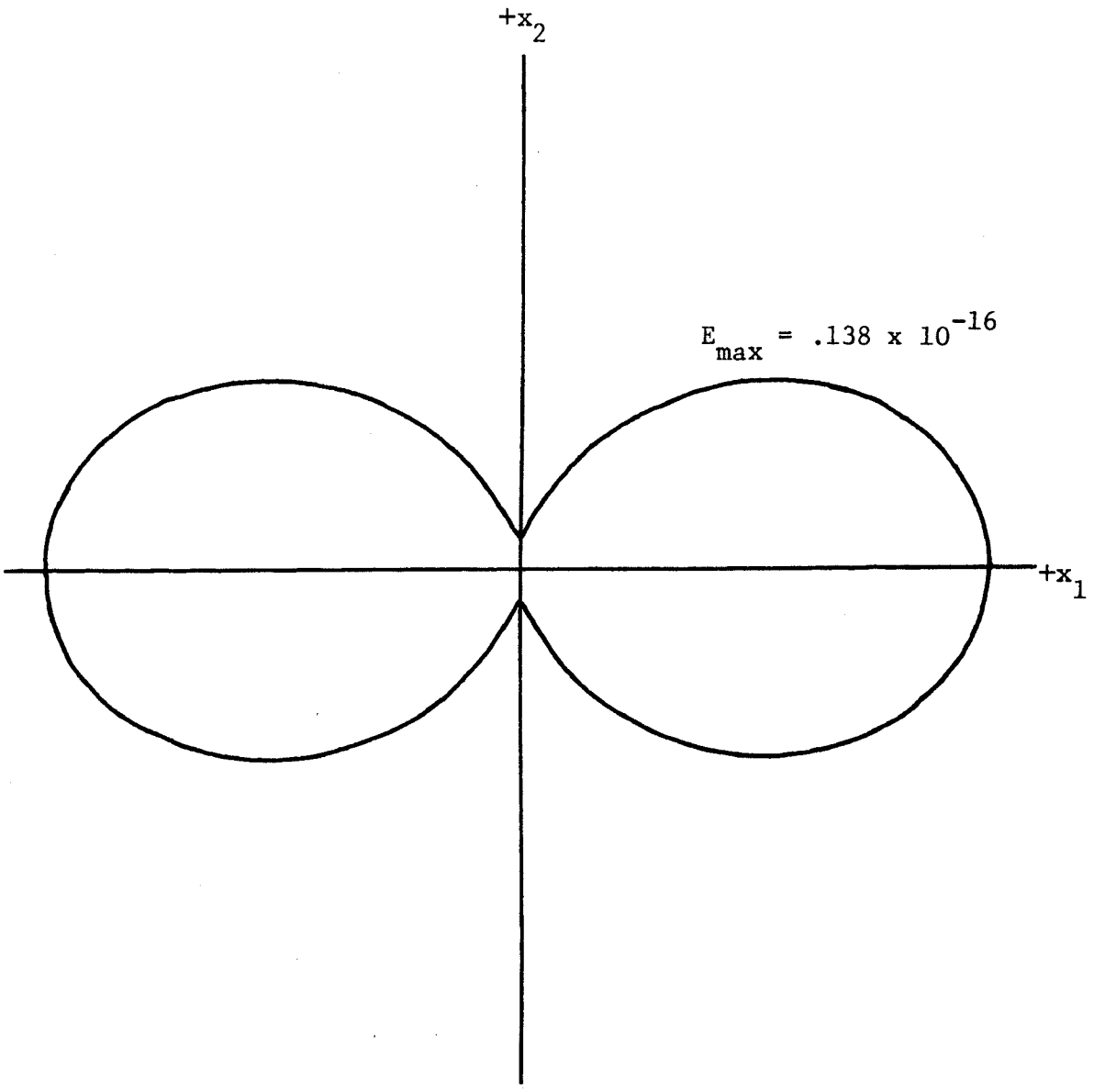

Figure 7.15 Scattered E Field in 1-2 Plane at R = 200 Feet Due to Incoming Plane Shear Wave Traveling in the Direction of $\hat{e}_3$ While Polarized Parallel to $\hat{e}_1$

VIII. SUMMARY; CONCLUSIONS; SUGGESTIONS FOR FURTHER RESEARCH

A. Summary

The preceding chapters have reported upon a technique devised for the solution of a certain class of elastic wave scattering problems. The scattering object was assumed to be situated in an otherwise infinite, isotropic, perfectly elastic, solid medium. While a void finite cavity of arbitrary shape was the scatterer of primary interest, it was pointed out that scattering from arbitrarily shaped finite rigid bodies may also be treated by using a slightly modified approach.

Displacement fields incident upon the scatterer were arbitrarily specified except for their sinusoidal time dependence. The incident fields emphasized throughout the dissertation were those arising directly from specified sources distributed through finite or infinitesimal volumes. Where nonzero, these sources were separable, i.e., their mathematical specification was the product of a spatial factor and a time function. A sinusoidal time function was employed in order to make the scattering problem tractable. It was pointed out that such a source may have an arbitrary vector function, which is zero outside a finite region, as its spatial factor. The sources may be distributed through finite volumes, act only at points, consist of surface layers of traction in homogeneous media, or arise through the application of traction on the boundary of a second arbitrarily shaped finite cavity. Plane wave scattering was discussed as an example in which the incident field was given directly; no source problem had to be solved first.

The chief mathematical concepts employed were multipole theory, least squares theory, and the theory of integral representations of

solutions. All three have found prior application in elastodynamics.

The development of elastodynamic multipole theory based upon a Taylor's expansion of a certain reciprocal distance factor occupied Chapters II and III. This work concentrated solely on the study of separable sources in unbounded media. The three displacement fields due to dipole forces oriented along three mutually perpendicular axes were seen to be of major importance. The rectangular components of these fields were combined into a 3 x 3 array called the fundamental solution. It was mentioned that while a fundamental solution corresponding to any time dependence may be easily calculated, the steady state fundamental solution was found to be the most useful. It was shown that any field (displacement or other field) determined through linear equations from a separable body force of finite extent may be calculated in the form of a multipole series. In such an infinite series for displacement, the first three terms are properly weighted components of the fundamental solution. All other terms are derivatives of these same components, each also weighted by a multiplicative coefficient. Every term in such a series is the field of a certain point force system, e.g., a dipole, quadrupole, octupole, etc.

A major useful feature of multipole series solutions is the fact that the coefficients weighting each elemental term are the same regardless of which type of field is under consideration. For example, a series for displacement has the same coefficients as those in the series for the stress tensor calculated from that displacement. This feature was utilized in Chapter IV, where a void finite cavity was introduced into the medium. In this new situation, the total displacement field

consists of the incident field plus a properly constructed scattered field. The scattered field satisfies the homogeneous equation of motion while also giving rise to traction at the cavitiy boundary that nullifies the traction due to the incident field. Because all multipole fields whose singular points lie inside the cavity satisfy the homogeneous equation of motion in the medium, the scattered field was assumed expressible as a linear combination of such multipole fields. Coefficients within the linear combinations were calculated by minimizing the total traction on the boundary using a least mean squared error criterion. This resulted in what was called the first approximation to the solution. The calculation of Green's Functions, i.e., the bounded media counterparts of the fundamental solution, was consistently emphasized owing to their intrinsic utilitarian value. There were three incident fields in this case, namely those embodied in the fundamental solution corresponding to a source point somewhere outside the cavity. However, the formulation was seen to be applicable to more general incident fields, as was more clearly pointed out later in Chapter VI.

It was shown in Chapter V how part of the inevitable error incurred during the least squares process of Chapter IV can be eliminated by use of Green's Integral Representation of the exact scattered field. The result so obtained was called the second and final approximation. An explanation of the improvement formed part of Appendix E.

Some miscellaneous topics such as plane wave scattering, scattering from rigid bodies, scattering by multiple cavities, and the so-called cavity-source problem were discussed in Chapter VI. Chapter VII contained a brief summary of error sources along with several diagrams sketched from numerical data computed using the theory developed in earlier parts of this dissertation.

B. Conclusions; Suggestions for Further Research

One of the principal attributes of the technique described in this dissertation is the fact that it unequivocally produces solutions. Moreover, once the computer programs required for its implementation have been written, the technique assumed a routine nature. Very little additional programming is required in each new problem. As pointed out in Chapter VII. There are several steps that could be taken in an effort to improve upon numerical results such as those presented. Further research should include a careful assessment of error sources, their effect, and measures to combat them. Increasing the accuracy of results will almost certainly lead to increased computational loads.

Throughout the foregoing work, the sinusoidal sources were assumed to be operating at a given fixed frequency. No restrictions whatsoever were made concerning the magnitude of this frequency, regardless of the cavity size or shape. It would not be surprising if some range of frequencies gave better results than others for a given cavity. Further research should look into this matter.

Although not specifically mentioned elsewhere in this dissertation, it is quite clear that non-sinusoidal scattering problems could be handled through the use of the Fourier Integral in tandem with results obtained herein. For example, a given transient time function forming the temporal factor of a separable body force could be sampled at discrete times and then analyzed into a certain minimum number of frequency components. The technique of this dissertation could be applied for each frequency utilized. Then, at a given field point, an appropriate Fourier synthesis computed numerically would result in a time history of motion at that field point. The amount of computational labor

involved in such a process may well render it impractical; however, the subject does warrant further investigation.

Another subject not referred to previously concerns the use of the results of this dissertation in a half space rather than in the full space considered here. This extension will certainly be challenging; it may be impossible in any practical sense. In this connection, it will be stated without proof that a careful study of Chapter II and Appendix C reveals that if any separable source of finite extent operates in a homogeneous half space, then an image source located across the boundary can be found which nullifies the normal component of traction due to the original source. The tangential components of traction are doubled in the process. By changing the sign of the image source, the tangential boundary traction components become zero while the normal component is doubled. It appears that some additional technique utilizing these facts may find successful application.

BIBLIOGRAPHY

1. Archambeau, C.B., 1968. General Theory of Elastodynamic Source Fields, Rev. of Geoph., Vol. 6, pp. 241-288.

2. Arfken, G., 1970. Mathematical Methods for Physicists, Academic Press, New York.

3. Banaugh, R.P., 1964. Application of Integral Representations of Displacement Potentials in Elastodynamics, Bull. of the Seis. Soc. of Am., Vol. 54, No. 4, pp. 1073-1086.

4. Case, K.M., and Colewell, J.F., 1967. Elastic Waves Generated by a Small Source, Geophysics, Vol. 32, No. 1, pp. 52-59.

5. DeHoop, A.T., 1958. Representation Theorems for the Displacement in an Elastic Solid and Their Application to Elastodynamic Diffraction Theory, Thesis, Delft Technical Institute, The Hague, Excelsior.

6. Gangi, A.F., 1970. A Derivation of the Seismic Representation Theorem Using Seismic Reciprocity, Jour. of Geoph. Res., Vol. 75, No. 11, pp. 2088-2095.

7. Hildebrand, F.B., 1956. Introduction to Numerical Analysis, McGraw-Hill, New York.

8. Hildebrand, F.B., 1965. Methods of Applied Mathematics, Prentice Hall, Englewood Cliffs, NJ.

9. Korringa, J., 1965. Variational Principles for the Diffraction of Elastic Waves, Jour. of Math. Phys., Vol. 6, No. 7, pp. 1107-1114.

10. Love, A.E.H., 1944. A Treatise on the Mathematical Theory of Elasticity, Dover, New York.

11. Morse, P.M., and Feshbach, H., 1953. Methods of Theoretical Physics, McGraw Hill, New York.

12. Oestreicher, H.L., 1957. Representation of the Fields of an Acoustic Source as a Series of Multipole Fields, Jour. of the Acous. Soc. of Am., Vol. 29, No. 11, pp. 1219-1222.

13. Olmsted, J.M.H., 1956. Advanced Calculus, Appleton-Century-Crofts.

14. Randall, M.J., 1971a. Shear Invariant and Seismic Moment for Deep-Focus Earthquakes, Jour. of Geoph. Res., Vol. 76, No. 20, pp. 4991-4992.

15. Randall, M.J., 1971b. Elastic Multipole Theory and Seismic Moment, Bull. of the Seis. Soc. of Am., Vol. 61, No. 5, pp. 1321-1326.

16. Rechtien, R.D., and Stewart, D.M., 1971. Detection and Delineation of Subterranean Cavities, Oral Presentation, 41st Annual Meeting, SEG, Houston, Texas, Nov. 7-11, 1971.

17. Stakgold, I., 1967. Boundary Value Problems of Mathematical Physics, Vol. I, MacMillan, New York.

18. Stakgold, I., 1968. Boundary Value Problems of Mathematical Physics, Vol. II, MacMillan, New York.

19. Stewart, D.M., 1971. The Vibrational Modes of Spheroidal Cavities in Elastic Solids, Thesis, University of Missouri-Rolla, Rolla, Missouri.

20. Stewart, D.M., and Rechtien, R.D., 1971. Theory of Cavity Resonance, Oral Presentation, 41st Annual Meeting, SEG, Houston, Texas, Nov. 7-11, 1971.

21. Van Nostrand, R.G., and Cook, K.L., 1966. Interpretation of Resistivity Data, Geol. Survey Prof. Paper 499, U.S. Gov. Printing Office, Washington, D.C.

22. White, J.E., 1965. Seismic Waves; Radiation, Transmission, and Attenuation, McGraw Hill, New York.

VITA

Jerry L. Davis was born on December 16, 1942. He graduated from Crab Orchard High School, R.R. #2, Marion, Illinois in 1960. He received a B.S. degree in Mining Engineering, Geology Option, at the University of Missouri at Rolla, Rolla, Missouri, in 1964. He worked for Geophysical Service, Incorporated, Dallas, Texas, as a Junior Geophysical Engineer for about 8 months during 1964-65. He served three years with the U.S. Army during 1965-68, being discharged as a Captain in the Corps of Engineers. He received an M.S. degree in Geology, Geophysics Option, in 1970 at the University of Missouri-Rolla.

APPENDIX A

Basic Mathematical Tools

1. The Dirac Delta Function

This quantity, which is not a function in any usual sense, is a continuous linear functional (Stakgold (1968), p. 5). Other names for continuous linear functionals are generalized functions and distributions; the mnemonic nature of the latter terminology is well illustrated in this dissertation. For those interested in a rigorous development of the properties of the delta function, the book by Stakgold is recommended. The theory is not specifically required in this dissertation; however, the following facts will be used:

a) $\int_{a.s.} A(\bar{r})\delta(\bar{r} - \bar{r}_o)dv = A(\bar{r}_o)$, (Stakgold (1968), p. 6.)

(A.1)

b) $$\int_{a.s.} A(\bar{r})D_i\delta(\bar{r} - \bar{r}_o) = -\int_{a.s.}[D_iA(\bar{r})]\delta(\bar{r} - \bar{r}_o)dv$$

$$= -D^o{}_iA(\bar{r}_o). \qquad (A.2)$$

$$\int_{a.s.} A(\bar{r})D_{ij}\delta(\bar{r} - \bar{r}_o)dv = +\int_{a.s.}[D_{ij}A(\bar{r})]\delta(\bar{r} - \bar{r}_o)dv$$

$$= +D^o{}_{ij}A(\bar{r}_o). \qquad (A.3)$$

$$\int_{a.s.} A(\bar{r})D_{ijk}\delta(\bar{r} - \bar{r}_o)dv = -\int_{a.s.}[D_{ijk}A(\bar{r})]\delta(\bar{r} - \bar{r}_o)dv$$

$$= -D^o{}_{ijk}A(\bar{r}_o). \qquad (A.4)$$

etc., (Stakgold (1968), p. 8).

c) The use of the word "dipole" in this dissertation does not correspond to the use by Stakgold. The difference stems from the fact that his sources are scalar (e.g. Stakgold (1968), p. 6), while those

here are vector (e.g. Equation (2.13)).

d) The treatment of surface layers here, e.g. Equations (2.43) and (2.65), follows Stakgold (1968), pp. 6 and 7.

e) The presentation of the point force defined here by Equations (2.6) and (2.7) follows Love (1944), p. 184. Love also shows how to construct higher order multipoles without the use of the delta function. On p. 186, he uses the phrase "synthesis of singularities" to describe the process. The resulting multipoles are called "nuclei of strain".

Within the context of the present dissertation the term "monopole" does not arise. The term, however, seems appropriate to the force system $D_{ii}\delta(\bar{r} - \bar{r}_o)$, which is a linear combination of quadrupoles. Love discusses this system on pp. 187 and 306 under the name "center of compression".

f) $$\nabla^2(1/|\bar{r} - \bar{r}_o|) = \nabla^2(1/R_o) = D_{ii}(1/R_o) = -4\pi\delta(\bar{r} - \bar{r}_o)$$

$$= -4\pi\delta(\bar{r}_o - \bar{r}). \quad \text{(Arfken (1970), p. 69)} \tag{A.5}$$

2. Helmholtz's Theorem

One way of stating this theorem is as follows (see Arfken (1970, p. 67): If $\bar{F}(\bar{r})$ is a vector field whose curl and divergence vanish at infinity, then

$$\bar{F}(\bar{r}) = -\nabla\phi(\bar{r}) + \nabla\times\bar{\psi}(\bar{r}), \tag{A.6}$$

in which $\phi(\bar{r})$ and $\bar{\psi}(\bar{r})$ are given as follows:

$$\phi(\bar{r}) = (1/4\pi) \int_{a.s.} \frac{\nabla'\cdot\bar{F}(\bar{r}')}{R}\, dv' \tag{A.7}$$

$$\bar{\psi}(\bar{r}) = (1/4\pi) \int_{a.s.} \frac{\nabla' \times \bar{F}(\bar{r}')}{R} \, dv' \tag{A.8}$$

In these formulas, $\nabla' = \hat{e}_s D'_s = \hat{e}_s (\partial/\partial x'_s)$, $R = |\bar{r} - \bar{r}'|$, while ϕ and $\bar{\psi}$ are called potentials.

The parent potential $\bar{W}(\bar{r})$ will now be introduced (see also Arfken (1970), p. 71). It is easily verified by direct calculation that the following identities are valid:

$$\frac{\nabla' \cdot \bar{F}(\bar{r}')}{R} = \nabla \cdot [\frac{\bar{F}(\bar{r}')}{R}] + \nabla' \cdot [\frac{\bar{F}(r')}{R}], \tag{A.9}$$

and

$$\frac{\nabla' \times \bar{F}(\bar{r}')}{R} = \nabla \times [\frac{\bar{F}(\bar{r}')}{R}] + \nabla' \times [\frac{\bar{F}(\bar{r}')}{R}]. \tag{A.10}$$

Using these identities and Gauss' Theorem (in the general form given for example by Arfken (1970), pp. 48 and 49) results in

$$\phi(\bar{r}) = (1/4\pi)\nabla \cdot \int_{a.s.} \frac{\bar{F}(\bar{r}')}{R} \, dv' + (1/4\pi) \int_S \frac{\bar{F}(\bar{r}')}{R} \cdot d\bar{S}', \tag{A.11}$$

$$\bar{\psi}(\bar{r}) = (1/4\pi)\nabla \times \int_{a.s.} \frac{\bar{F}(\bar{r}')}{R} \, dv' - (1/4\pi) \int_S \frac{\bar{F}(\bar{r}')}{R} \times d\bar{S}'. \tag{A.12}$$

In these equations, the surface integrals are over the infinite sphere. In this dissertation $\bar{F}(\bar{r}) = \bar{0}$ there so that the surface integrals vanish. Then, with the definition

$$\bar{W}(\bar{r}) = (1/4\pi) \int_{a.s} \frac{\bar{F}(\bar{r}')}{R} \, dv', \tag{A.13}$$

there follows

$$\phi(\bar{r}) = \nabla\cdot\bar{W}(\bar{r}) \text{ and } \bar{\psi}(\bar{r}) = \nabla\times\bar{W}(\bar{r}). \tag{A.14}$$

Now also,

$$\bar{F} = -\nabla(\nabla\cdot\bar{W}) + \nabla\times\nabla\times\bar{W}. \tag{A.15}$$

In rectangular coordinates, this is $\bar{F} = -\nabla^2\bar{W}$ by an identity. Note that this is easily verified as follows:

$$-\nabla^2\bar{W}(\bar{r}) = -(1/4\pi)\nabla^2 \int_{a.s.} \frac{\bar{F}(\bar{r}')}{R}\, dv' = -(1/4\pi) \int_{a.s.} \bar{F}(\bar{r}')\nabla^2(1/R)\, dv' \tag{A.16}$$

which is $\bar{F}(\bar{r})$ since

$$\nabla^2(1/R) = -4\pi\delta(\bar{r} - \bar{r}') = -4\pi\delta(\bar{r}' - \bar{r}). \tag{A.17}$$

3. Taylor Series in Three Dimensions

Under suitable continuity and differentiability conditions, the Extended Law of the Mean, i.e., Taylor's Formula with a Remainder, may be applied to a function $A(\bar{r})$ in the form (Olmsted (1961), p. 280):

$$A(\bar{r}) = A(\bar{r}_o) + \sum_{j=1}^{n-1} (1/j!)[(q_1D_1 + q_2D_2 + q_3D_3)^j A(\bar{r})]_{x_k = x_{ok}} + P_n, \tag{A.18}$$

where $\bar{r}_o = x_{os}\hat{e}_s$ is an expansion point, $q_k = x_k - x_{ok}$, and

$$P_n = (1/n!)[(q_1D_1 + q_2D_2 + q_3D_3)^n A(\bar{r})]_{x_k = x_{ok} + \theta q_k}. \tag{A.19}$$

It must be pointed out that in the finite series and in P_n the evaluation of the bracketed portions at $x_k = x_{ok}$ and $x_k = x_{ok} + \theta q_k$, respectively, must not include evaluation of the q_i. That is, only the

derivatives of $A(\bar{r})$ are to be evaluated at these points. In the expression for P_n, which is the remainder after n terms, θ is an appropriate number in the range $0 < \theta < 1$. The series plus remainder, Equation (A.18), is an exact expression involving a finite number of terms. If P_n as given by Equation (A.19) approaches zero as the number of terms n in the finite series increases indefinitely at some point $\bar{r}$, then the resulting infinite series, i.e., the Taylor series, converges and represents $A(\bar{r})$.

The infinite series may be expressed in various notation, one being

$$A(\bar{r}) = A(\bar{r}_o) + \sum_{j=1}^{\infty} (1/j!)(q_1 D^o{}_1 + q_2 D^o{}_2 + q_3 D^o{}_3)^j A(\bar{r}_o). \quad (A.20)$$

This may be further modified by using the summation convention in expressions such as

$$q_1 D^o{}_1 + q_2 D^o{}_2 + q_3 D^o{}_3 = q_i D^o{}_i, \quad (A.21)$$

$$(q_1 D^o{}_1 + q_2 D^o{}_2 + q_3 D^o{}_3)^2 = q_i q_j D^o{}_{ij}, \quad (A.22)$$

$$(q_1 D^o{}_1 + q_2 D^o{}_2 + q_3 D^o{}_3)^3 = q_i q_j q_k D^o{}_{ijk}, \text{ etc.} \quad (A.23)$$

Using identities such as these results in

$$A(\bar{r}) = [1 + (1/1!)q_i D^o{}_i + (1/2!)q_i q_j D^o{}_{ij} + \ldots]A(\bar{r}_o). \quad (A.24)$$

When the function A depends on $\bar{r}'$ rather than $\bar{r}$, then of course it is necessary only to rename the q_i, using $q_i' = x_i' - x_{oi}$. This bit of renaming is convenient when expanding the function $1/R = 1/|\bar{r} - \bar{r}'|$ in which $\bar{r}'$ is the variable point while $\bar{r}$ is held fixed. There follows

$$1/R = [1 + q_i' D^o{}_i + (1/2!) q_i' q_j' D^o{}_{ij} + (1/3!) q_i' q_j' q_k' D^o{}_{ijk} + \ldots]$$

$$(1/R_o), \qquad (A.25)$$

in which $R_o = |\bar{r} - \bar{r}_o|$. Moreover, since $D^o{}_i(1/R_o) = -D_i(1/R_o)$, there also follows

$$1/R = [1 - q_i' D_i + (1/2) q_i' q_j' D_{ij} - (1/3) q_i' q_j' q_k' D_{ijk} + \ldots - \ldots]$$

$$(1/R_o). \qquad (A.26)$$

A rigorous discussion of the above series for 1/R would have to include proof that the remainder after n terms indeed vanishes as n increases indefinitely. A proof will not be attempted here; the Taylor series for 1/R is a well known series. It converges at all points $\bar{r}'$ located such that $|\bar{r}' - \bar{r}_o| < |\bar{r} - \bar{r}_o|$ (Morse and Feshbach (1953), p. 1277). In regions of convergence the series may be integrated and differentiated term by term.

APPENDIX B

Equation of Motion and Displacement Potentials

1. The Equation of Motion

As stated in Equations (3.1) and (3.2), the equation of motion governing infinite, isotropic, homogeneous, perfectly elastic solid media can be written in the following forms:

$$(\partial^2/\partial t^2)\overline{U} = \alpha^2\nabla(\nabla\cdot\overline{U}) - \beta^2\nabla\times\nabla\times\overline{U} + (1/\rho)\overline{K} \tag{B.1}$$

and

$$(\partial^2/\partial t^2)U_m = (1/\rho)D_n T_{mn} + (1/\rho)K_m. \tag{B.2}$$

Using the following two identities, it will be shown how equation (B.2) follows from Equation (B.1):

$$\nabla\times\phi\overline{A} = \nabla\phi\times\overline{A} + \phi\nabla\times\overline{A} \tag{B.3}$$

and

$$\overline{A}\times(\overline{B}\times\overline{C}) = \overline{B}(\overline{A}\cdot\overline{C}) - \overline{C}(\overline{A}\cdot\overline{B}). \tag{B.4}$$

Identities (B.3) and (B.4) hold for any vectors $\overline{A},\overline{B},\overline{C}$ and functions ϕ.

Since $\overline{U} = U_m\hat{e}_m$ and $\nabla = \hat{e}_s D_s$,

$$\nabla\times\overline{U} = \nabla\times U_m\hat{e}_m = \nabla U_m\times\hat{e}_m = D_s U_m \hat{e}_s\times\hat{e}_m, \tag{B.5}$$

and

$$\nabla\times\nabla\times\overline{U} = \nabla(D_s U_m)\times(\hat{e}_s\times\hat{e}_m) = D_{ks}U_m\hat{e}_k\times(\hat{e}_s\times\hat{e}_m)$$

$$= D_{ks}U_m(\hat{e}_s\delta_{km} - \hat{e}_m\delta_{ks}) = D_{ms}U_m\hat{e}_s - D_{ss}U_m\hat{e}_m$$

$$= D_{sm}U_s\hat{e}_m - D_{ss}U_m\hat{e}_m. \tag{B.6}$$

Also,

$$\nabla(\nabla\cdot\bar{U}) = \nabla(D_sU_s) = D_{sm}U_s\hat{e}_m. \tag{B.7}$$

Using Equations (B.6) and (B.7), the m component of Equation (B.1) may be written as

$$(\partial^2/\partial t^2)U_m = \alpha^2 D_{sm}U_s - \beta^2 D_{sm}U_s + \beta^2 D_{ss}U_m + (1/\rho)K_m. \tag{B.8}$$

Since $\alpha^2 = (\lambda + 2\mu)/\rho$ and $\beta^2 = \mu/\rho$, this is

$$\rho(\partial^2/\partial t^2)U_m = (\lambda + \mu)D_{sm}U_s + \mu D_{ss}U_m + K_m. \tag{B.9}$$

Now the stress tensor T_{mn} is given by Hooke's Law as

$$T_{mn} = \lambda\delta_{mn}D_sU_s + \mu D_mU_n + \mu D_nU_m, \tag{B.10}$$

so that

$$D_nT_{mn} = \lambda D_{ms}U_s + \mu D_{mn}U_n + \mu D_{nn}U_m = (\lambda + \mu)D_{ms}U_s + \mu D_{ss}U_m. \tag{B.11}$$

Upon using Equation (B.11) in Equation (B.9), the latter becomes Equation (B.2).

2. Displacement Potentials

Using Helmholtz's Theorem (Appendix A), the displacement vector may be expressed in terms of potentials Φ and $\bar{\Psi}$ in the form

$$\bar{U} = -\nabla\Phi + \nabla\times\bar{\Psi}. \tag{B.12}$$

In Equation (B.1) $\overline{K}$ is separable, i.e., $\overline{K}(\overline{r}) = \overline{F}(\overline{r})f(t)$. Using the notation of Equation (2.1),

$$\overline{K} = (-\nabla\phi + \nabla\times\overline{\psi})f(t). \tag{B.13}$$

Upon substituting from Equations (B.12) and (B.13) into Equation (B.1), wave equations may be derived for the displacement potentials. In making the substitutions it is necessary to note that for any vector $\overline{A}$ and scalar B, the following two identities hold:

$$\nabla\cdot(\nabla\times\overline{A}) \equiv 0, \tag{B.14}$$

and

$$\nabla\times(\nabla B) \equiv \overline{0}. \tag{B.15}$$

Equation (B.1) becomes

$$(\partial^2/\partial t^2)(-\nabla\Phi + \nabla\times\overline{\Psi}) = \alpha^2\nabla(-\nabla^2\Phi) - \beta^2\nabla\times(\nabla\times\nabla\times\overline{\psi}) +$$

$$(-\nabla\phi + \nabla\times\overline{\psi})f/\rho, \tag{B.15}$$

which may be rewritten as

$$-\nabla[(\partial^2/\partial t^2)\Phi - \alpha^2\nabla^2\Phi - \phi f/\rho] + \nabla\times[(\partial^2/\partial t^2)\overline{\Psi} + \beta^2\nabla\times\nabla\times\overline{\Psi} - \overline{\psi}f/\rho]$$

$$= \overline{0}. \tag{B.17}$$

Equation (B.17) implies that Equation (B.1) will be satisfied if

$$(\partial^2/\partial t^2)\Phi - \alpha^2\nabla^2\Phi = \phi f/\rho, \tag{B.18}$$

and

$$(\partial^2/\partial t^2)\overline{\Psi} + \beta^2\nabla\times\nabla\times\overline{\Psi} = \overline{\psi}f/\rho. \tag{B.19}$$

Equations (B.18) and (B.19) are wave equations for the displacement potentials. Equation (B.19) may be written in a different form by requiring $\overline{\Psi}$ to have zero divergence. This may be done arbitrarily; however, if one calculated $\overline{\Psi}$ by an analog of Equation (A.8) then its divergence is necessarily zero. In either event, $\nabla\cdot\overline{\Psi} = 0$ in this dissertation. Then, since for any vector $\overline{A}$,

$$\nabla^2\overline{A} = \nabla(\nabla\cdot\overline{A}) - \nabla\times\nabla\times\overline{A}, \tag{B.20}$$

it follows that Equation (B.19) is

$$(\partial^2/\partial t^2)\overline{\Psi} - \beta^2\nabla^2\overline{\Psi} = \overline{\psi}f/\rho. \tag{B.21}$$

3. The Parent Displacement Potential

Let $\overline{W}(\overline{r})$ be given by Equation (2.4). Let $\overline{V}_\alpha(\overline{r})$ satisfy

$$(\partial^2/\partial t^2)\overline{V}_\alpha - \alpha^2\nabla^2\overline{V}_\alpha = \overline{W}f/\rho. \tag{B.22}$$

Upon taking the divergence of both members of this equation and using Equation (2.2), i.e., $\phi = \nabla\cdot\overline{W}$, there follows

$$(\partial^2/\partial t^2)(\nabla\cdot\overline{V}_\alpha) - \alpha^2\nabla^2(\nabla\cdot\overline{V}_\alpha) = \phi f/\rho. \tag{B.23}$$

Since Equation (B.18) satisfied by Φ is the same as Equation (B.23) satisfied by $\nabla\cdot\overline{V}_\alpha$, it follows that the two functions differ only by a solution to the homogeneous wave equation. This will be taken as zero so that

$$\Phi = \nabla\cdot\overline{V}_\alpha. \tag{B.24}$$

$\overline{V}_\alpha$ is a parent displacement potential, the analog of $\overline{W}$, a parent source potential.

Now, let a vector $\overline{V}_\beta$ satisfy

$$(\partial^2/\partial t^2)\overline{V}_\beta - \beta^2\nabla^2\overline{V}_\beta = \overline{W}f/\rho. \tag{B.25}$$

Comparison of Equations (B.22) and (B.25) shows that $\overline{V}_\alpha$ and $\overline{V}_\beta$ are the same except that the number α upon which $\overline{V}_\alpha$ must depend parametrically is different from the number β upon which $\overline{V}_\beta$ must depend parametrically. Therefore, $\overline{V}_\beta$ is obtainable from $\overline{V}_\alpha$ by replacement of α with β.

Upon taking the curl of both members of Equation (B.25) and using Equation (2.3), i.e., $\overline{\psi} = \nabla\times\overline{W}$, there follows

$$(\partial^2/\partial t^2)(\nabla\times\overline{V}_\beta) - \beta^2\nabla^2(\nabla\times\overline{V}_\beta) = \overline{\psi}f/\rho. \tag{B.26}$$

Comparison of Equations (B.21) and (B.26) shows that $\overline{\Psi}$ and $\nabla\times\overline{V}_\beta$ differ only by a solution to the homogeneous wave equation. Taking this as zero, there follows

$$\overline{\Psi} = \nabla\times\overline{V}_\beta \tag{B.27}$$

This shows that one need not solve Equation (B.21) directly in order to find $\overline{\Psi}$. But first solving Equation (B.22), $\overline{V}_\beta$ becomes known. Then $\overline{\Psi}$ follows by Equation (B.27).

APPENDIX C

Explicit Expressions for the Fundamental Solution and Its Stress Tensors

1. The Fundamental Solution for Arbitrary Time Dependence

In order to calculate the fundamental solution, it is necessary only to solve the wave Equations (3.13), which are of the form

$$(\partial^2/\partial t^2 - \alpha^2\nabla^2)A = B. \tag{C.1}$$

This equation, in which A is the solution while B is the inhomogeneous term, has a well known retarded integral solution (Love (1944), p. 304 or Morse and Feshbach (1953), p. 206). A particular solution to Equation (C.1) is

$$A(\bar{r},t) = (1/4\pi\alpha^2) \int_{a.s.} \frac{B(\bar{r}',t-R/\alpha)}{R}\, dv', \tag{C.2}$$

in which $R = |\bar{r} - \bar{r}'|$.

The solutions to Equations (3.13), i.e., to

$$(\partial^2/\partial t^2 - \alpha^2\nabla^2)\Phi_s(\bar{r},t|\bar{r}_o) = \phi_s(\bar{r}|\bar{r}_o)f(t)/\rho, \tag{C.3}$$

are, using the definition $\phi_s(\bar{r}|\bar{r}_o) = (1/4\pi)D_s(1/R_o)$, $R_o = |\bar{r} - \bar{r}_o|$,

$$\Phi_s(\bar{r},t|\bar{r}_o) = (1/16\pi^2\alpha^2\rho) \int_{a.s.} (1/R)[D'_s(1/R'_o)]f(t - R/\alpha)dv', \tag{C.5}$$

where $R'_o = |\bar{r}' - \bar{r}_o|$. Also since

$$D'_s(1/R'_o) = -D^o_s(1/R'_o), \tag{C.6}$$

$$\Phi_s(\bar{r},t|\bar{r}_o) = -(1/16\pi^2\alpha^2\rho)D^o_s \int_{a.s.} (1/RR'_o)f(t - R/\alpha)dv'. \tag{C.7}$$

The volume integral in Equation (C.7) may be evaluated directly following Love (1944), pp. 304-305. There all space is divided into concentric shells centered at the field point $\bar{r}$. Only the factor $1/R_o'$ is non-constant on each such shell. The surface integral over a typical shell of radius R is calculated first. Then integration over R from zero to infinity completes the evaluation. The solution becomes

$$\Phi_s(\bar{r},t|\bar{r}_o) = (1/4\pi\rho)[D_s(1/R_o)] \int_o^{R_o/\alpha} t'f(t - t')dt'. \quad (C.8)$$

Upon replacing α with β, the $\eta_s(\bar{r},t|\bar{r}_o)$ defined in the text following Equation (3.12) are found to be

$$\eta_s(\bar{r},t|\bar{r}_o) = (1/4\pi\rho)[D_s(1/R_o)] \int_o^{R_o/\beta} t'f(t - t')dt'. \quad (C.9)$$

Now, according to Equation (3.15),

$$\begin{aligned}\Phi_{sk}(\bar{r},t|\bar{r}_o) &= D_k\Phi_s(\bar{r},t|\bar{r}_o) \\ &= (1/4\pi\rho)[D_{sk}(1/R_o)] \int_o^{R_o/\alpha} t'f(t - t')dt' \\ &\quad + (1/4\pi\alpha^2\rho)[D_s(1/R_o)]q_k f(t - R_o/\alpha), \end{aligned} \quad (C.10)$$

in which $q_k = x_k - x_{ok}$. The $\eta_{sk}(\bar{r},t|\bar{r}_o)$ result from this upon the replacement of α with β. In particular,

$$\eta_{mm}(\bar{r},t|\bar{r}_o) = -(1/4\pi\beta^2\rho R_o)f(t - R_o/\beta). \quad (C.11)$$

From Equation (3.20), the fundamental solution is found to be

$$\begin{aligned}Q_{sk}(\bar{r},t|\bar{r}_o) &= (1/4\pi\rho)[D_{sk}(1/R_o)] \int_{R_o/\alpha}^{R_o/\beta} t'f(t - t')dt' \\ &\quad +(1/4\pi\rho)[(1/\beta^2)f(t - R_o/\beta) - (1/\alpha^2)f(t - R_o/\alpha)\end{aligned}$$

$$q_k D_s(1/R_o)] + (1/4\pi\beta^2\rho R_o)\delta_{sk} f(t - R_o/\beta). \qquad (C.12)$$

No explicit computational use of the fundamental solution for arbitrary time dependence will be made in this dissertation. The tedious (but simple) calculations leading to the stress tensors will be omitted.

2. The Steady State Fundamental Solution

Much use of $\tilde{Q}_{sk}(\bar{r}|\bar{r}_o)$, the space part of $Q_{sk}(\bar{r},t|\bar{r}_o)$ arising when $f(t) = \exp(-i\omega t)$, is encountered in this dissertation. Therefore it will be discussed in more detail.

Upon substituting $f(t) = \exp(-i\omega t)$ into Equation (C.8), it is easily found that the space part of $\Phi_s(\bar{r},t|\bar{r}_o)$ is

$$\tilde{\Phi}_s(\bar{r}|\bar{r}_o) = -A_\alpha q_s, \qquad (C.13)$$

where

$$A_\alpha = (1/4\pi\rho\omega^2 R_o^3)[(1 - i\omega R_o/\alpha)\exp(i\omega R_o/\alpha) - 1], \qquad (C.14)$$

and

$$q_s = x_s - x_{os}. \qquad (C.15)$$

If a function B_α is defined from

$$D_k A_\alpha = B_\alpha q_k, \qquad (C.16)$$

then it happens that

$$B_\alpha = (1/4\pi\rho\alpha^2 R_o^3)\exp(i\omega R_o/\alpha) - 3A_\alpha/R_o^2. \qquad (C.17)$$

Another similar definition will be useful shortly. Defining C_α from $D_k B_\alpha = C_\alpha q_k$, then

$$C_\alpha = (i\omega/\alpha - 6/R_o)B_\alpha/R_o + (i\omega/\alpha - 1/R_o)3A_\alpha/R_o^3. \qquad (C.18)$$

Corresponding functions A_β, B_β, and C_β are formed by replacing α with β in A_α, B_α, and C_α. All six quantities are functions of R_o alone. All depend parametrically upon the frequency ω.

From Equation (C.13),

$$\tilde{\Phi}_{sk}(\bar{r}|\bar{r}_o) = -B_\alpha q_s q_k - A_\alpha \delta_{sk}. \qquad (C.19)$$

Then, from Equation (3.20), it follows that

$$\tilde{Q}_{sk}(\bar{r}|\bar{r}_o) = (B_\alpha - B_\beta)q_s q_k + (A_\alpha + 2A_\beta + R_o^2 B_\beta)\delta_{sk}. \qquad (C.20)$$

Operation by D_p in Equation (C.20) yields

$$D_p\tilde{Q}_{sk}(\bar{r}|\bar{r}_o) = (C_\alpha - C_\beta)q_p q_s q_k + (B_\alpha - B_\beta)(q_s\delta_{kp} + q_k\delta_{sp}) + (B_\alpha + 4B_\beta + R_o^2 C_\beta)q_p\delta_{sk}, \qquad (C.21)$$

from which

$$D_p\tilde{Q}_{pk}(\bar{r}|\bar{r}_o) = (5B_\alpha + R_o^2 C_\alpha)q_k. \qquad (C.22)$$

Since Hooke's Law given $\tilde{\Sigma}^j_{mn}(\bar{r}|\bar{r}_o)$ as

$$\tilde{\Sigma}^j_{mn}(\bar{r}|\bar{r}_o) = \lambda\delta_{mn}D_s\tilde{Q}_{sj}(\bar{r}|\bar{r}_o) + \mu D_m\tilde{Q}_{nj}(\bar{r}|\bar{r}_o) + \mu D_n\tilde{Q}_{mj}(\bar{r}|\bar{r}_o), \qquad (C.23)$$

it follows from Equations (C.21) and (C.22) that

$$\tilde{\Sigma}^j_{mn}(\bar{r}|\bar{r}_o) = 2\mu(C_\alpha - C_\beta)q_m q_n q_j$$

$$+ [\lambda(5B_\alpha + R_o^2 C_\alpha) + 2\mu(B_\alpha - B_\beta)]q_j \delta_{mn}$$

$$+ \mu(2B_\alpha + 3B_\beta + R_o^2 C_\beta)(q_m \delta_{nj} + q_n \delta_{mj}). \qquad (C.24)$$

The properties of $\tilde{Q}_{ij}(\bar{r}|\bar{r}_o)$ expressed by Equation (3.21) are apparent from Equation (C.20). Also, the same properties of the fundamental solution for arbitrary time dependence are evident from Equation (C.12).

The symmetry of the stress tensors with respect to the indices m and n, as well as the property expressed by Equation (3.25), is apparent in the steady state case from Equation (C.24). Both properties exist also for arbitrary time dependence as direct construction of the stress tensors corresponding to Equation (C.12) would quickly show.

APPENDIX D

The Approximating Set of Traction Vectors

1. Linear Dependence of Multipole Fields

For field points $\bar{r} \neq \bar{r}_o$, the fundamental solution satisfies the equation of motion (4.3) in the form

$$-\rho\omega^2 \tilde{Q}_{im}(\bar{r}|\bar{r}_o) = D_n \tilde{\Sigma}^m_{in}(\bar{r}|\bar{r}_o). \tag{D.1}$$

Since, by Hooke's Law,

$$\tilde{\Sigma}^m_{in}(\bar{r}|\bar{r}_o) = \lambda\delta_{in} D_s \tilde{Q}_{sm}(\bar{r}|\bar{r}_o) + \mu D_i \tilde{Q}_{nm}(\bar{r}|\bar{r}_o) + \mu D_n \tilde{Q}_{im}(\bar{r}|\bar{r}_o), \tag{D.2}$$

it follows that Equation (D.1) is

$$-\rho\omega^2 \tilde{Q}_{im}(\bar{r}|\bar{r}_o) = \lambda D_{is} \tilde{Q}_{sm}(\bar{r}|\bar{r}_o) + \mu D_{in} \tilde{Q}_{nm}(\bar{r}|\bar{r}_o) + \mu D_{nn} \tilde{Q}_{im}(\bar{r}|\bar{r}_o). \tag{D.3}$$

Equation (D.3) clearly exhibits the fundamental solution as a linear combination of octupole fields. By using the property $\tilde{Q}_{im}(\bar{r}|\bar{r}_o) = \tilde{Q}_{mi}(\bar{r}|\bar{r}_o)$, this equation may be expressed in the form

$$\tilde{Q}_{im}(\bar{r}|\bar{r}_o) = (b\delta_{mk}\delta_{sp} + c\delta_{ms}\delta_{kp})\, D_{kp} \tilde{Q}_{is}(\bar{r}|\bar{r}_o), \tag{D.4}$$

where

$$b = -(\lambda + \mu)/\rho\omega^2 \quad \text{and} \quad c = -\mu/\rho\omega^2. \tag{D.5}$$

Also, from Equation (D.4),

$$\tilde{\Sigma}^m_{in}(\bar{r}|\bar{r}_o) = (b\delta_{mk}\delta_{sp} + c\delta_{ms}\delta_{kp}) D_{kp} \tilde{\Sigma}^s_{in}(\bar{r}|\bar{r}_o), \tag{D.6}$$

from which

$$\tau_n(\bar{r})\tilde{\Sigma}^j_{mn}(\bar{r}|\bar{r}_o) = (b\delta_{jk}\delta_{sp} + c\delta_{js}\delta_{kp})\tau_n(\bar{r})D_{kp}\tilde{\Sigma}^s_{mn}(\bar{r}|\bar{r}_o). \quad (D.7)$$

In this dissertation, the approximating set of traction vectors is composed of the nine quadrupole traction vectors

$$\tau_n(\bar{r})D_k\tilde{\Sigma}^j_{mn}(\bar{r}|\bar{r}_1), \quad (D.8)$$

and the 18 octupole traction vectors

$$\tau_n(\bar{r})D_{kp}\tilde{\Sigma}^j_{mn}(\bar{r}|\bar{r}_1), \; k < p, \quad (D.9)$$

where $\bar{r}_1$ is in the cavity, $\bar{r}$ (usually denoted by $\bar{r}'$) is on the boundary, and m indexes the traction components. Equation (D.7) shows why the dipole traction vectors are deleted; each one is a linear combination of some of the tractions (D.9). The octupole tractions in which $k > p$ are also deleted because they are each identical to a retained traction vector for which $k < p$; $D_{kp} = D_{pk}$ here.

In future work, if one intended to increase the flexibility of the approximating set by adding the independent 16-pole traction vectors, then all the quadrupole tractions would have to be deleted. Operation by D_j in Equation (D.6) followed by multiplication by τ_n shows that they are linear combinations of the 16-pole vectors.

2. The Compressed Notation and Bookkeeping Scheme

Equations (4.18) and (4.19) indicate a change in notation made to facilitate part of the work in Chapter IV. Table (D.1) below gives the complete scheme. In reading the table, it should be borne in mind that the subscript on the a_s runs from 1 to 27, so that a_{23} for example

Table D.1 The Bookkeeping Scheme

Serial No.	Old coeff.	Old symbol	New coeff.	New symbol
1	A_{11}	$\tau_n D_1 \tilde{\Sigma}^1_{mn}$	a_1	g_{1m}
2	A_{12}	$\tau_n D_2 \tilde{\Sigma}^1_{mn}$	a_2	g_{2m}
3	A_{13}	$\tau_n D_3 \tilde{\Sigma}^1_{mn}$	a_3	g_{3m}
4	A_{21}	$\tau_n D_1 \tilde{\Sigma}^2_{mn}$	a_4	g_{4m}
5	A_{22}	$\tau_n D_2 \tilde{\Sigma}^2_{mn}$	a_5	g_{5m}
6	A_{23}	$\tau_n D_3 \tilde{\Sigma}^2_{mn}$	a_6	g_{6m}
7	A_{31}	$\tau_n D_1 \tilde{\Sigma}^3_{mn}$	a_7	g_{7m}
8	A_{32}	$\tau_n D_2 \tilde{\Sigma}^3_{mn}$	a_8	g_{8m}
9	A_{33}	$\tau_n D_3 \tilde{\Sigma}^3_{mn}$	a_9	g_{9m}
10	A_{111}	$\tau_n D_{11} \tilde{\Sigma}^1_{mn}$	a_{10}	g_{10m}
11	A_{112}	$\tau_n D_{12} \tilde{\Sigma}^1_{mn}$	a_{11}	g_{11m}
12	A_{113}	$\tau_n D_{13} \tilde{\Sigma}^1_{mn}$	a_{12}	g_{12m}
13	A_{122}	$\tau_n D_{22} \tilde{\Sigma}^1_{mn}$	a_{13}	g_{13m}
14	A_{123}	$\tau_n D_{23} \tilde{\Sigma}^1_{mn}$	a_{14}	g_{14m}
15	A_{133}	$\tau_n D_{33} \tilde{\Sigma}^1_{mn}$	a_{15}	g_{15m}
16	A_{211}	$\tau_n D_{11} \tilde{\Sigma}^2_{mn}$	a_{16}	g_{16m}
17	A_{212}	$\tau_n D_{12} \tilde{\Sigma}^2_{mn}$	a_{17}	g_{17m}
18	A_{213}	$\tau_n D_{13} \tilde{\Sigma}^2_{mn}$	a_{18}	g_{18m}
19	A_{222}	$\tau_n D_{22} \tilde{\Sigma}^2_{mn}$	a_{19}	g_{19m}
20	A_{223}	$\tau_n D_{23} \tilde{\Sigma}^2_{mn}$	a_{20}	g_{20m}
21	A_{233}	$\tau_n D_{33} \tilde{\Sigma}^2_{mn}$	a_{21}	g_{21m}
22	A_{311}	$\tau_n D_{11} \tilde{\Sigma}^3_{mn}$	a_{22}	g_{22m}
23	A_{312}	$\tau_n D_{12} \tilde{\Sigma}^3_{mn}$	a_{23}	g_{23m}
24	A_{313}	$\tau_n D_{13} \tilde{\Sigma}^3_{mn}$	a_{24}	g_{24m}
25	A_{322}	$\tau_n D_{22} \tilde{\Sigma}^3_{mn}$	a_{25}	g_{25m}
26	A_{323}	$\tau_n D_{23} \tilde{\Sigma}^3_{mn}$	a_{26}	g_{26m}
27	A_{333}	$\tau_n D_{33} \tilde{\Sigma}^3_{mn}$	a_{27}	g_{27m}

refers to the case s = 23 and does not imply that two subscripts are present. The same precautionary note applies to the first subscript on the g_{sm}.

Finally, the superscript j appearing on the A's and a's in Equations (4.19) was purposely omitted in Table (D.1) in order to indicate that the same approximating set is used in this dissertation for purposes additional to those requiring the superscript.

3. The Omission (Redistribution) of Terms in a Multipole Series

The linear dependence discussed in Section 1 above permits the redistribution of some terms in a multipole series arising directly from a given body force. Some terms may be omitted while at the same time their contribution to the total field is transferred to other terms and not lost.

Suppose a given finite steady state source gives rise to a displacement field

$$\tilde{U}_i(\bar{r}) = L_s[\tilde{Q}_{is}(r|r_o)], \tag{D.10}$$

in which L_s has been truncated to

$$L_s = G_s - G_{sk}D_k + (1/2)G_{skp}D_{kp}. \tag{D.11}$$

Such series as this will be encountered in Appendix F, where it will prove useful to omit the dipole terms and some of the octupole terms. The remaining 27 displacement fields will have to correspond to the 27 traction vectors listed in Table (D.1).

From the definitions (2.49), it is clear that $G_{skp} = G_{spk}$. This means that the G_{skp} for k > p may be omitted providing that the G_{skp}

for $k < p$ are each multiplied by 2. This is permissible of course because $D_{kp}\tilde{Q}_{is} = D_{pk}\tilde{Q}_{is}$.

In eliminating the dipole displacements, Equation (D.4) is used:

$$\begin{aligned} G_m\tilde{Q}_{im} &= (bG_k\delta_{sp} + cG_s\delta_{kp})D_{kp}\tilde{Q}_{is} \\ &= (bG_1\delta_{s1} + cG_s)D_{11}\tilde{Q}_{is} \\ &\quad + (bG_1\delta_{s2} + 0)D_{12}\tilde{Q}_{is} \\ &\quad + (bG_1\delta_{s3} + 0)D_{13}\tilde{Q}_{is} \\ &\quad + (bG_2\delta_{s1} + 0)D_{21}\tilde{Q}_{is} \\ &\quad + (bG_2\delta_{s2} + cG_s)D_{22}\tilde{Q}_{is} \\ &\quad + (bG_2\delta_{s3} + 0)D_{23}\tilde{Q}_{is} \\ &\quad + (bG_3\delta_{s1} + 0)D_{31}\tilde{Q}_{is} \\ &\quad + (bG_3\delta_{s2} + 0)D_{32}\tilde{Q}_{is} \\ &\quad + (bG_3\delta_{s3} + cG_s)D_{33}\tilde{Q}_{is}. \end{aligned} \tag{D.12}$$

Since the octupole displacements $D_{21}\tilde{Q}_{is}$, $D_{31}\tilde{Q}_{is}$, and $D_{32}\tilde{Q}_{is}$ have been deleted, Equation (D.12) needs to be rewritten in the form

$$\begin{aligned} G_m\tilde{Q}_{im} &= (bG_1\delta_{s1} + cG_s)D_{11}\tilde{Q}_{is} \\ &\quad + (bG_1\delta_{s2} + bG_2\delta_{s1})D_{12}\tilde{Q}_{is} \\ &\quad + (bG_1\delta_{s3} + bG_3\delta_{s1})D_{13}\tilde{Q}_{is} \\ &\quad + (bG_2\delta_{s2} + cG_s)D_{22}\tilde{Q}_{is} \end{aligned}$$

$$+ (bG_2\delta_{s3} + bG_3\delta_{s2})D_{23}\tilde{Q}_{is}$$

$$+ (bG_3\delta_{s3} + cG_s)D_{33}\tilde{Q}_{is}. \qquad (D.13)$$

From Equation (D.13), it is apparent that the addition of $bG_1\delta_{s1} + cG_s$ to G_{s11}, $bG_1\delta_{s2} + bG_2\delta_{s1}$ to G_{s12}, etc., along with proper redistribution of the octupole terms as mentioned earlier, permits Equation (D.10) to continue to describe exactly the same displacement while having only 27 terms on the right side. These 27 multipole fields correspond to the 27 tractions in the approximating set of Table (D.1).

APPENDIX E

Green's Integral Representation

1. The Derivation

Green's Representation as given in Equation (5.1) may be obtained quite simply by proper specialization of Betti's Reciprocal Theorem (Love (1944), p. 173). However, a derivation following Case and Colewell (1967) will be given here in order to provide a base for the proof of a later theorem. It may be noticed that the derivation here is quite similar to the application of Green's Theorem in the general form given by Stakgold (1968), p. 40.

When $\exp(-i\omega t)$ is canceled from the equation resulting when $K_m(\bar{r})$ of Equation (3.2) is replaced with $\tilde{F}_m(\bar{r})\exp(-i\omega t)$, there follows

$$-\rho\omega^2\tilde{U}_m(\bar{r}) = D_n\tilde{T}_{mn}(\bar{r}) + \tilde{F}_m(\bar{r}). \tag{E.1}$$

This is the steady state equation of motion due to a body force having components $\tilde{F}_m(\bar{r})\exp(-i\omega t)$. When the body force components are $\delta_{mi}\delta(\bar{r} - \bar{r}_o)\exp(-i\omega t)$, then Equation (4.3) results:

$$-\rho\omega^2\tilde{Q}_{mi}(\bar{r}|\bar{r}_o) = D_n\tilde{\Sigma}^i_{mn}(\bar{r}|\bar{r}_o) + \delta_{mi}\delta(\bar{r} - \bar{r}_o). \tag{E.2}$$

If Equation (E.1) were to be multiplied by $\tilde{Q}_{mi}(\bar{r}|\bar{r}_o)$ and Equation (E.2) by $\tilde{U}_m(\bar{r})$, then the left sides would be identical. Equation (E.1) would have, on the right side, an expression

$$\tilde{Q}_{mi}(\bar{r}|\bar{r}_o)D_n\tilde{T}_{mn}(\bar{r}) = D_n[\tilde{Q}_{mi}(\bar{r}|\bar{r}_o)\tilde{T}_{mn}(\bar{r})] - \tilde{T}_{mn}(\bar{r})D_n\tilde{Q}_{mi}(\bar{r}|\bar{r}_o). \tag{E.3}$$

Also, Equation (E.2) would contain an expression

$$\tilde{U}_m(\bar{r})D_n\tilde{\Sigma}^i_{mn}(\bar{r}|\bar{r}_o) = D_n[\tilde{U}_m(\bar{r})\tilde{\Sigma}^i_{mn}(\bar{r}|\bar{r}_o)] - \tilde{\Sigma}^i_{mn}(\bar{r}|\bar{r}_o)D_n\tilde{U}_m(\bar{r}). \tag{E.4}$$

Upon using Hooke's Law to reduce the rightmost expressions in Equations (E.3) and (E.4) to forms in which combinations of derivatives of displacement replace the stress tensors, a direct comparison shows these two rightmost expressions to be identical. Therefore, multiplication of Equation (E.1) by $\tilde{Q}_{mi}(\bar{r}|\bar{r}_o)$ and Equation (E.2) by $\tilde{U}_m(\bar{r})$ followed by subtraction of the two results yields.

$$0 = D_n[\tilde{Q}_{mi}(\bar{r}|\bar{r}_o)\tilde{T}_{mn}(\bar{r}) - \tilde{U}_m(\bar{r})\tilde{\Sigma}^i_{mn}(\bar{r}|\bar{r}_o)] + \tilde{F}_m(\bar{r})\tilde{Q}_{mi}(\bar{r}|\bar{r}_o) - \tilde{U}_i(\bar{r})\delta(\bar{r} - \bar{r}_o). \tag{E.5}$$

Replacing $\bar{r}$ with $\bar{r}'$, integrating over all of space within the bounded medium, using Gauss' Theorem on the integral containing the above divergence of a vector, and rearranging transforms Equation (E.5) into the form

$$\tilde{U}_i(\bar{r}_o) = \int_S [\tilde{Q}_{mi}(\bar{r}'|\bar{r}_o)\tilde{T}_{mn}(\bar{r}') - \tilde{U}_m(\bar{r}')\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r}_o)]\tau_n(\bar{r}')dS' + \int_{a.s.} \tilde{F}_m(\bar{r}')\tilde{Q}_{mi}(\bar{r}'|\bar{r}_o)dv'. \tag{E.6}$$

Equation (5.1) results from this when $\bar{r}_o$ is relabeled as $\bar{r}$. Here $\hat{\tau}$ of course is the unit normal which points out of the medium on the boundary S.

Two important points concerning Equation (E.6) can be noticed immediately. If the boundary S exists only at infinity, then the surface integral vanishes because displacement and traction due to finite sources are zero at infinity. What remains is the steady state

counterpart of Equation (3.35).

Secondly, if Equation (E.2) had been the equation of motion satisfied by the Green's Function for a finite void cavity, i.e.,

$$-\rho\omega^2\tilde{Q}^*_{mi}(\bar{r}|\bar{r}_o) = D_n\tilde{\Sigma}^{*i}_{mn}(\bar{r}|\bar{r}_o) + \delta_{mi}\delta(\bar{r} - \bar{r}_o), \tag{E.7}$$

then no essential features of the derivation of Equation (E.6) would be changed. However, in place of Equation (5.1) one would have

$$\tilde{U}_i(\bar{r}) = \int_S [\tilde{Q}^*_{mi}(\bar{r}'|\bar{r})\tilde{T}_{mn}(\bar{r}') - \tilde{U}_m(\bar{r}')\tilde{\Sigma}^{*i}_{mn}(\bar{r}'|\bar{r})]\tau_n(\bar{r}')dS'$$

$$+ \int_{a.s.} \tilde{F}_m(\bar{r}')\tilde{Q}_{mi}(\bar{r}'|\bar{r})dv'. \tag{E.8}$$

In Equation (E.8), the surface integral is identically zero when S is a free surface; $\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}') = 0$ as a boundary condition while $\tilde{\Sigma}^{*i}_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}') = 0$ be definition (and subsequent construction) of the Green's Function. What remains is Equation (5.25).

2. Proof of the Theorem of Section (V-A)

To prove Parts 1 and 2 of the Theorem, one may note that if Equation (E.1) had been

$$-\rho\omega^2\tilde{Q}_{mj}(\bar{r}|\bar{r}_1) = D_n\tilde{\Sigma}^{j}_{mn}(\bar{r}|\bar{r}_1) + \delta_{mj}\delta(\bar{r} - \bar{r}_1), \tag{E.9}$$

then Equation (E.5) would have been

$$0 = D_n[\tilde{Q}_{mi}(\bar{r}|\bar{r}_o)\tilde{\Sigma}^{j}_{mn}(\bar{r}|\bar{r}_1) - \tilde{Q}_{mj}(\bar{r}|\bar{r}_1)\tilde{\Sigma}^{i}_{mn}(\bar{r}|\bar{r}_o)]$$

$$+ \tilde{Q}_{ji}(\bar{r}|\bar{r}_o)\delta(\bar{r} - \bar{r}_1) - \tilde{Q}_{ij}(\bar{r}|\bar{r}_1)\delta(\bar{r} - \bar{r}_o). \tag{E.10}$$

Now, as assumed in the Theorem, S is a closed surface dividing all space into two regions. Whether media is contained in both regions

is irrelevant. One of the regions has its boundary formed by S and the infinite sphere; however, since $\bar{r}_o$ and $\bar{r}_1$ are not at infinity the surface integral emerging from Equation (E.10) is zero over the infinite sphere.

Upon forming the volume integral of Equation (E.10) over one of the regions and ultimately relabeling $\bar{r}_o$ as $\bar{r}$ (the old $\bar{r}$ becomes $\bar{r}'$), there follows

$$\tilde{I}_{ij}(\bar{r}|\bar{r}_1) - \tilde{I}_{ji}(\bar{r}_1|\bar{r}) = \int_V \tilde{Q}_{ij}(\bar{r}'|\bar{r}_1)\delta(\bar{r}' - \bar{r})dv' + \int_V \tilde{Q}_{ji}(\bar{r}'|\bar{r})\delta(\bar{r}' - \bar{r}_1)dv'. \quad \text{(E.11)}$$

The integral $\tilde{I}_{ij}(\bar{r}|\bar{r}_1)$ is defined in Equation (5.8), in which $\hat{\tau}$ points out of the region V over which the integration was performed. If $\bar{r}$ and $\bar{r}_1$ are on the same side of S, then regardless which region V happens to be, Part 1 of the Theorem follows immediately from Equation (E.11). That is, if both points are in V, the volume integrals are

$$\tilde{Q}_{ij}(\bar{r}|\bar{r}_1) - \tilde{Q}_{ji}(\bar{r}_1|\bar{r}) = 0 \quad \text{(E.12)}$$

by Equations (3.21). If both points are outside V, the volume integrals are zero be definition of the delta function.

The hypothesis of Part 2 of the Theorem states that $\hat{\tau}$ points into the region containing $\bar{r}_1$, so that $\bar{r}$ is in V while $\bar{r}_1$ is not. Then, the non-vanishing volume integral in Equation (E.11) is $\tilde{Q}_{ij}(\bar{r}|\bar{r}_1)$, proving Part 2.

3. Physical Interpretation of Green's Representation

This section is included for the purposes of interpreting Green's Representation in terms of surface layers while at the same time offering evidence to support the claim that the use of the results of Chapter V provides a better approximation to the exact scattered fields than do those of Chapter IV alone.

Suppose that a traction $\tilde{T}_m(\bar{r}')\exp(-i\omega t)$ acts at a point $\bar{r}'$ on a surface S in a fully infinite homogeneous medium. An element of area dS' at $\bar{r}'$ gives rise to three dipole forces of magnitudes $\tilde{T}_m(\bar{r}')dS'$, for m = 1,2,3, each along a coordinate axis. The first of these, i.e., $\tilde{T}_1(\bar{r}')dS'$, causes a displacement field having an i component given by $\tilde{T}_1(\bar{r}')dS'\tilde{Q}_{i1}(\bar{r}|\bar{r}')$ by definition of the fundamental solution. The other two give rise to analogous displacements, the contributions from all three adding up to $\tilde{T}_m(\bar{r}')dS'\tilde{Q}_{im}(\bar{r}|\bar{r}')$. All other elements dS' on S also contribute to the total field, which is

$$\tilde{U}_i(\bar{r}) = \int_S \tilde{T}_m(\bar{r}')\tilde{Q}_{im}(\bar{r}|\bar{r}')dS'. \qquad (E.13)$$

Now, Equation (E.13) can also be derived in a slightly different fashion. The equivalent volume source density generated by dS' at $\bar{r}'$ on S is $\tilde{T}_m(\bar{r}')dS'\delta(\bar{r} - \bar{r}')$. The entire surface gives rise to a total equivalent volume source density

$$\tilde{F}_m(\bar{r}) = \int_S \tilde{T}_m(\bar{r}')\delta(\bar{r} - \bar{r}')dS', \qquad (E.14)$$

which represents a force per unit volume corresponding to the layer of dipoles spread on S. According to Equation (3.35), the displacement due to a force per unit volume is

$$\tilde{U}_i(\bar{r}) = \int_{a.s.} \tilde{F}_m(\bar{r}')\tilde{Q}_{im}(\bar{r}|\bar{r}')dv'. \tag{E.15}$$

Upon substituting Equation (E.14) into Equation (E.15) and subsequently interchanging the order of integration, there follows

$$\begin{aligned}\tilde{U}_i(\bar{r}) &= \int_{a.s.}\{\int_S \tilde{T}_m(\bar{r}'')\delta(\bar{r}' - \bar{r}'')dS''\}\tilde{Q}_{im}(\bar{r}|\bar{r}')dv' \\ &= \int_S \tilde{T}_m(\bar{r}'')\{\int_{a.s.}\delta(\bar{r}' - \bar{r}'')\tilde{Q}_{im}(\bar{r}|\bar{r}')dv'\}dS'' \\ &= \int_S \tilde{T}_m(\bar{r}'')\tilde{Q}_{im}(\bar{r}|\bar{r}'')dS'', \end{aligned} \tag{E.16}$$

which of course is the same as Equation (E.13).

Both the above viewpoints are available also for the treatment of double layers, i.e., layers of quadrupoles, acting on S in unbounded media. In particular, suppose that a double layer having an equivalent volume source density with s component

$$\int_S \tilde{P}_{mn}(\bar{r}')(\lambda\delta_{mn}D_s + \mu\delta_{ms}D_n + \mu\delta_{ns}D_m)\delta(\bar{r} - \bar{r}')dS' \tag{E.17}$$

acts on S. Substitution of this into Equation (E.15) gives the i component of displacement in the forms

$$\begin{aligned}&\int_{a.s.}\{\int_S \tilde{P}_{mn}(\bar{r}'')(\lambda\delta_{mn}D'_s + \mu\delta_{ms}D'_n + \mu\delta_{ns}D'_m)\delta(\bar{r}' - \bar{r}'')dS''\} \\ &\quad \tilde{Q}_{is}(\bar{r}|\bar{r}')dv' = -\int_S \tilde{P}_{mn}(\bar{r}'')(\lambda\delta_{mn}D''_s + \mu\delta_{ms}D''_n + \mu\delta_{ns}D''_m) \\ &\quad \tilde{Q}_{is}(\bar{r}|\bar{r}'')dS'' = -\int_S \tilde{P}_{mn}(\bar{r}'')\tilde{\Sigma}^i_{mn}(\bar{r}''|\bar{r})dS''. \end{aligned} \tag{E.18}$$

The second of expressions (E.18) follows from the first by interchanging the order of integration while the third follows from the second by the use of Hooke's Law in the integrand. In connection with this last

step, it is easy to verify by direct calculation that when the operator $(\lambda\delta_{mn}D''_s + \mu\delta_{ms}D''_n + \mu\delta_{ns}D''_m)$ is applied to either $\tilde{Q}_{is}(\bar{r}|\bar{r}'')$, $\tilde{Q}_{si}(\bar{r}|\bar{r}'')$, $\tilde{Q}_{is}(\bar{r}''|\bar{r})$, or $\tilde{Q}_{si}(\bar{r}''|\bar{r})$ the result is always $\tilde{\Sigma}^i_{mn}(\bar{r}''|\bar{r})$, and never $\tilde{\Sigma}^i_{mn}(\bar{r}|\bar{r}'')$ which is the negative of the correct result.

Now, it may be seen that when $\tilde{T}_m(\bar{r}')$ in Equation (E.16) is $\tilde{T}_{mn}(\bar{r}')\tau_n(\bar{r}')$ while $\tilde{P}_{mn}(\bar{r}'')$ in Equation (E.18) is $\tilde{U}_m(\bar{r}'')\tau_n(\bar{r}'')$, the two displacements (E.16) and (E.18) become the two surface integrals in Green's Representation (5.1). Therefore, it may be said that Green's Representation gives the total displacement in <u>bounded</u> media as the sum of the fields due to a double layer, a single layer, and an actual volume source density as if all were acting in <u>unbounded</u> media.

This interpretation is particularly illuminating in relation to the cavity-source problem considered in Section (6-C). Equation (6.12) gives the resultant displacement as the unbounded media displacement due to the applied traction plus a second integral which arises due to the cavity's presence. This latter integral, which is the sole difference between the cavity and no-cavity cases, is the no-cavity displacement due to a certain double layer on S.

With the above background, it is possible to reexamine the results of Chapters IV and V in more physical terms. Such a reexamination will occupy the rest of this Appendix, which will terminate with arguments to support the validity and accuracy of those results.

From either argument leading to Equation (E.16), it is clear that when the three traction-error vectors $\tilde{E}^j_m(\bar{r}')$, j = 1,2,3, act on a surface S in unbounded media, the displacement fields have the i components given by

$$\int_S \tilde{E}^j_m(\bar{r}')\tilde{Q}_{im}(\bar{r}|\bar{r}')dS', \tag{E.19}$$

which are the same as the correction terms of Equation (5.23). Obviously these traction-error vectors correspond to three displacement error vectors, say with i components $\tilde{\varepsilon}_{ij}(\bar{r})$. It should be anticipated that the addition of

$$-\int_S \tilde{\varepsilon}_{mj}(\bar{r}')\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS' \tag{E.20}$$

to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ as given by Equation (5.23) would yield the <u>exact</u> scattered part of the Green's Function. This will now be shown to be true; in doing so, the exact result will be denoted by $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$.

The first approximation to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ is given by Equation (5.13) as $M^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]$. In order that the displacement-errors $\tilde{\varepsilon}_{ij}$ will correspond to the traction errors $\tilde{E}^j_m$, Equation (4.38) dictates that

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = M^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)] + \tilde{\varepsilon}_{ij}(\bar{r}) \tag{E.21}$$

be the defining relation for $\tilde{\varepsilon}_{ij}$. In order to write an integral representation for the exact $\tilde{Q}^{**}_{ij}$ in terms of $\tilde{\varepsilon}_{ij}$, the expression (E.21) may be used in Equation (5.7), resulting in

$$\begin{aligned}\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = &-\int_S M'^j_s[\tilde{Q}_{ms}(\bar{r}'|\bar{r}_1)]\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS' \\ &-\int_S \tilde{\varepsilon}_{mj}(\bar{r}')\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS' \\ &-\int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')dS'.\end{aligned} \tag{E.22}$$

Comparison of Equation (E.22) with Equation (5.18) shows that indeed the second approximation to $\tilde{Q}^{**}_{ij}$ given by Equation (5.23) needs only the

addition of expression (E.20) in order to become exact.

From perhaps a more concise point of view, it may be noted that the Green's Representation for $\tilde{\varepsilon}_{ij}$ itself, written directly from Equation (5.1), is

$$\tilde{\varepsilon}_{ij}(\bar{r}) = \int_S \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{E}^j_m(\bar{r}')dS'$$

$$- \int_S \tilde{\varepsilon}_{mj}(\bar{r}')\tilde{\Sigma}^i_{mn}(\bar{r}'|\bar{r})\tau_n(\bar{r}')dS'. \qquad (E.23)$$

When expression (E.20) is omitted from this while the remainder is used in Equation (E.21), the latter becomes the second approximation given by Equation (5.23).

The point is this: By using only the first approximation derived in Chapter IV, the entire displacement-error $\tilde{\varepsilon}_{ij}$ for each j is unaccounted for. But, by using the second approximation from Chapter V, a part of $\tilde{\varepsilon}_{ij}$ is calculated--namely the displacement that would arise in unbounded media if the traction error vector were to be applied on S. Only a part of $\tilde{\varepsilon}_{ij}$ is unattainable--namely the part given by expression (E.20). This lost part is the part interpretable as being due to a double layer acting on S.

As a final optimistic note, it may be pointed out that expression (E.20) can be viewed as a weighted average taken over S of the displacement errors incurred in the least squares process of Chapter IV. It may happen that at a field point $\bar{r}$ the errors in the second approximation are very small owing to the fact that in expression (E.20), which gives these final errors, the original errors $\tilde{\varepsilon}_{mj}$ have the opportunity to cancel themselves during the integration.

APPENDIX F

Multipole Series for the Correction Term

According to Equation (5.23), the final approximation to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ is

$$\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o) = M^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)] + \int_s \tilde{Q}_{mi}(\bar{r}'|\bar{r})\tilde{E}^j_m(\bar{r}')dS', \qquad (F.1)$$

where $\tilde{E}^j_m(\bar{r}')$ are the components of the three traction-error vectors.

To use Equation (F.1) as it is, the surface integral must be evaluated at each field point $\bar{r}$ where values of $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ are desired. It will prove convenient computationally to approximate the integral, i.e., the correction term, by a truncated multipole series containing all octupole and lower order terms. It will be wise, of course, to calculate this multipole series using as an expansion point the same point $\bar{r}_1$ used in calculating the first approximation.

The series will be

$$L^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)], \qquad (F.2)$$

in which

$$L^j_s = G^j_s - G^j_{sk}D_k + (1/2)G^j_{skp}D_{kp}. \qquad (F.3)$$

Notice that in fact expression (F.2) stands for three multipole series, one for each j = 1,2,3.

Now, as discussed in Section 3, Appendix E, the surface integral in Equation (F.1) gives the i components of displacement due to the three equivalent volume source densities

$$\tilde{F}^j_m(\bar{r}) = \int_S \tilde{E}^j_m(\bar{r}')\delta(\bar{r} - \bar{r}')dS'. \qquad (F.4)$$

The coefficients G_s^j are defined by Equations (2.49), i.e.,

$$G_s^j = \int_{a.s.} \tilde{F}_s^j(\bar{r}')\,dv' = \int_{a.s.} \{\int_S \tilde{E}_s^j(\bar{r}'')\,\delta(\bar{r}' - \bar{r}'')\,dS''\}dv'$$

$$= \int_S \tilde{E}_s^j(\bar{r}'')\{\int_{a.s.} \delta(\bar{r}' - \bar{r}'')\,dv'\}dS'' = \int_S \tilde{E}_s^j(\bar{r}'')\,dS''. \qquad (F.5)$$

Also,

$$G_{sk}^j = \int_{a.s.} q_k' \tilde{F}_s^j(\bar{r}')\,dv', \quad q_k' = x_k' - x_{1k}, \qquad (F.6)$$

That is,

$$G_{sk}^j = \int_{a.s.} q_k' \{\int_S \tilde{E}_s^j(\bar{r}'')\,\delta(\bar{r}' - \bar{r}'')\,dS''\}dv'$$

$$= \int_S \tilde{E}_s^j(\bar{r}'')\{\int_{a.s.} q_k' \delta(\bar{r}' - \bar{r}'')\,dv'\}dS''$$

$$= \int_S q_k'' \tilde{E}_s^j(\bar{r}'')\,dS'', \quad q_k'' = x_k'' - x_{1k}. \qquad (F.7)$$

Finally,

$$G_{skp}^j = \int_S q_k'' q_p'' E_s^j(\bar{r}'')\,dS''. \qquad (F.8)$$

In order to apply formulas (F.5), (F.7), and (F.8) in their present form, the complete first approximation must be known. This is true because the traction-error vectors must be evaluated in order to construct the integrands. However, some saving of computational labor can be made by properly using Equation (4.38), namely

$$\tilde{E}_m^j(\bar{r}) = -\tilde{\Sigma}_{mn}^j(\bar{r}|\bar{r}_o)\tau_n(\bar{r}) - \tau_n(\bar{r})M_s^j[\tilde{\Sigma}_{mn}^s(\bar{r}|\bar{r}_1)]. \qquad (F.9)$$

Since M_s^j is merely $A_{sk}^j D_k + A_{skp}^j D_{kp}$ where $A_{skp}^j = 0$ for $k > p$, the G_s^j are given more explicitly by

$$G_m^j = -\int_S \tilde{\Sigma}^j_{mn}(\bar{r}'|\bar{r}_o)\tau_n(\bar{r}')dS'$$

$$-A^j_{sk}\int_S \tau_n(\bar{r}')D'_k\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)dS'$$

$$-A^j_{skp}\int_S \tau_n(\bar{r}')D'_{kp}\tilde{\Sigma}^s_{mn}(\bar{r}'|\bar{r}_1)dS'. \qquad (F.10)$$

While the A's are not known until the completion of the least squares process of Chapter IV, the integrands of each integral in Equation (F.10) must be constructed at each integration point $\bar{r}'$ during the formulation of the least squares process itself. Thus, the integrals in Equation (F.10) may be calculated with only a small amount of labor additional to that already required. Their values must be remembered so that the G^j_s can be calculated later after the A's do become known.

Roughly the same remarks apply to the G^j_{sk} and G^j_{skp}; however, the presence of q''_k and $q''_kq''_p$ in Equations (F.7) and (F.8) does require some extra arithmetic operations.

Once all the G's are known, Section 3, Appendix D, may be applied in order to omit all the dipole terms and the octupole terms for $k > p$. This leaves exactly 27 terms in L^j_s for each j, giving it the form of M^j_s. Then the two series may be added term by term, which amounts to adding the corresponding coefficients. The result is N^j_s of Equation (5.46), namely

$$N^j_s = B^j_{sk}D_k + B^j_{skp}D_{kp}, \qquad (F.11)$$

in which $B^j_{skp} = 0$ for $k > p$.

Clearly,

$$B^j_{sk} = A^j_{sk} - G^j_{sk}, \text{ for all s,k, and j.} \tag{F.12}$$

The redistribution of the G^j_s and G^j_{skp} according to Section 3, Appendix D, followed by the addition of the results to A^j_{skp} leads to

$$\begin{aligned}
B^j_{s11} &= A^j_{s11} + (bG^j_1\delta_{s1} + cG^j_s) + (1/2)G^j_{s11},\\
B^j_{s12} &= A^j_{s12} + (bG^j_1\delta_{s2} + bG^j_2\delta_{s1}) + G^j_{s12},\\
B^j_{s13} &= A^j_{s13} + (bG^j_1\delta_{s3} + bG^j_3\delta_{s1}) + G^j_{s13},\\
B^j_{s22} &= A^j_{s22} + (bG^j_2\delta_{s2} + cG^j_s) + (1/2)G^j_{s22},\\
B^j_{s23} &= A^j_{s23} + (bG^j_2\delta_{s3} + bG^j_3\delta_{s2}) + G^j_{s23},\\
B^j_{s33} &= A^j_{s33} + (bG^j_3\delta_{s3} + cG^j_s) + (1/2)G^j_{s33},
\end{aligned} \tag{F.13}$$

for all s and j. The b and c are given by Equations (D.5).

In this way, the second approximation to $\tilde{Q}^{**}_{ij}(\bar{r}|\bar{r}_o)$ becomes

$$\tilde{Q}^{**}_{ij}(r|r_o) = N^j_s[\tilde{Q}_{is}(\bar{r}|\bar{r}_1)]. \tag{F.14}$$