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The effect and compensation of time-base perturbations in analog recording systems

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THE EFFECT AND COMPENSATION OF TIME-BASE PERTURBATIONS

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ABSTRACT

The mean square error between recorded and reproduced signals is used as an error measure to determine the effects of time-base perturbations on an analog singal. The mean square error caused by time-base perturbations is shown to be proportional to the product of the square of the signal bandwidth and the time-base error variance for the case of low pass signal. When the signal is band pass, there is shown to be an additional error term which is proportional to the square of the signal center frequency and the time-base error variance.

Calculations are also carried out to determine the relative effects of pre-recorder and post-recorder external additive noise. It is found that this external noise adds a term to the mean square error which is approximately equal to the noise power.

An analysis is made to determine the error reduction which is possible by the use of the optimum linear filter. It is shown that a significant improvement is possible for the case where the signal bandwidth is less than the time-base error bandwidth. Practical approximations to the optimum linear filter are also considered and, in some cases, they are found to give a reduction in the mean square error

which is approximately the same as that given by the optimum filter.

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CHAPTER I

INTRODUCTION

In modern communication systems, analog tape recorders are widely employed for signal storage and data processing. Much of the time high data accuracy is required, and it is necessary to employ recorders capable of recording and reproducing a signal without introducing errors which significantly degrade system performance.

In order to determine if this accuracy has been achieved for a particular system, it is necessary to have a method of analysis of the effect of recording and reproducing a signal and of comparing the error introduced by the recorder to other errors. Also, it is desirable to be able to analyze the effects of filters on the signal and the effects of various techniques of compensation which might be employed to correct the errors introduced by the recorder.

Although several techniques have been devised to minimize or compensate for the error introduced by the recorder, no one, to the author's knowledge, has provided an analysis of this error or of its compensation. Neither has there been an analysis of the relative error caused by the recorder as compared to the other errors, or a method of computing the effects of compensating filters.

This research is an attempt to provide such an analysis. A method of computing the effects of the error introduced by the recorder and also of computing the effects of compensating filters connected to the output of the recorder is provided. The theory of mean square optimization is applied to the problem of compensation to ascertain what improvement is possible by the use of optimum (in the mean square sense) linear filters. The relative effects of the error caused by time-base perturbations as compared to the other errors are analyzed, and consideration is also given to the question of what type of signal suffers more distortion from the time-base error. Some calculations are also made on the sensitivity of the error to changes in the signal power spectrum, and on the improvement which is possible by the use of practical approximations to the optimum filter.

CHAPTER II

REVIEW OF THE LITERATURE

Tape recorders have been in extensive use in communications and data processing systems for a number of years and a considerable amount of effort has been devoted to the problems of analysis, reduction, and compensation of error introduced by these recorders. Much of the effort of error analysis has been devoted to the analysis of rather specific systems and to the compilation and study of the data from these systems. However, it has been shown by various authors (1, 5, 7, 9) that a general effect of recording and replaying a signal is the introduction of a random time displacement in the playback signal. This time displacement results from velocity variations (flutter) in the record and in the playback process (1, 2, 3). Its causes are imperfections in the tape transport mechanism and disturbances (vibrations) in the tape itself (3, 7, 14).

Several people have made studies of the nature of flutter and of time-base error (the integral of flutter) (1, 2, 3, 10, 14, 16, 17). It has been found that, although there is a very considerable variation in the flutter and the time-base error spectra from recorder to recorder, the flutter spectrum can often be reasonably approximated by a rectangular model and the time-base error can often be reasonably approximated by a trapezoidal model. Further, measurements on the distribution of the random flutter and the time-base error have shown them to be essentially Gaussian in nature (1, 2, 14).

Analyses of the effects of recorder flutter and of time-base error have been carried out for certain rather specific systems. Nichols and Schmitt (8) conducted an investigation on the cause and effects of time-base errors in coherent demodulation of suppressed carriers in AM multiplex systems. Simpson and Tranter (4, 5) have provided an analysis of the effect of recorder time-base error on an AM baseband telemetry system which also employed suppressed carrier demodulation of the recorded baseband. They also have provided an analysis of the effect of flutter on sinusoidal modulation. Results of a somewhat more general nature are provided by Ratz (3), who gives an analysis of the effects of tape transport flutter on spectrum and correlation analysis and by Chao (2), who attempts to present a unified picture of flutter and time-base errors in a multi-channel longitudinal instrumentation recorder.

Compensation of flutter and time-base error has also been considered by various people. Manufacturers of recorders have attempted to build compensation into the recorder itself, whereas others have considered external compensation. Peshel (15) has considered the application of wow and flutter compensation techniques to FM systems;

a digital system for compensating time-base error in analog tape recorders has been developed by Houts, Burlage, and Simpson (7). Some authors (1, 2, 6, 13, 14) have considered compensation by the use of a pilot signal which has been recorded in synchronism with the data. Chao (6) employed a variable delay line and a phase detector in conjunction with the pilot signal and a reference frequency to achieve a reduction in time-base error by a factor of 20 or more.

However, none of these has been in the nature of a general analysis of the effects of time-base error. Neither have, to the author's knowledge, the extensively developed techniques of mean square optimization been applied to compensation of this error.

CHAPTER III

MODELS

A. The Recorder Model

It has been found by various authors (1, 5, 7, 9) that the principle effect of recording and replaying a signal is to introduce a random time-base error in the output with respect to the input. In mathematical terms, this means that if the combined effect of record and playback flutter is represented by $g(t)$, the signal at the playback heads of the recorder is of the form

$$y_p(t) = K[1 + g(t)]\ x'[t + \int_0^t g(\tau)\ d\tau] \tag{3.1}$$

where $x(t)$ is the input signal to the recorder, K is a constant, and the prime denotes the derivative with respect to the argument. The total time-base error due to both record and playback is the integral of the total flutter (1,5). Thus, denoting the time-base error (TBE) by $h(t)$, one can write

$$h(t) = \int_0^t g(\tau)\ d\tau \tag{3.2}$$

or

$$h'(t) = g(t). \tag{3.3}$$

Then equation (3.1) becomes

$$y_p(t) = K[1 + h'(t)]\ x'[t + h(t)]. \tag{3.4}$$

This is exactly the derivative with respect to t of $x(t + h)$ so that when the signal at the playback heads is integrated with respect to time, it gives the recorder output $y(t)$ which is

$$y(t) = K\ x[t + h(t)]. \tag{3.5}$$

Thus, the output of the recorder can be expressed as the input signal shifted in time by an amount $h(t)$. Since the flutter $g(t)$ does not appear in $y(t)$, it may be concluded that only the integral of the flutter, namely the time-base error (TBE), need be considered when the effects of the recorder are investigated.

B. Signal and Time-Base Error Models

The statistical models for the signal, $x(t)$, and for the TBE, $h(t)$, must also be chosen for the analysis to proceed. It is necessary to choose these models in a way which is realistic enough for the analysis to give useful results while, at the same time, keeping the mathematical complexity of the analysis within reason.

For this analysis, the signal $x(t)$ is taken as a sample function of a random process with zero mean whose power spectrum is ideal band pass or ideal low pass. The TBE, h(t) is taken as a sample function of a Gaussian random process with zero mean whose power spectrum is also ideal band pass (or ideal low pass). Further, the signal and the TBE are taken to be statistically independent, time stationary, and ergodic. It has been found that these models correspond closely to the experimental evidence collected on recording systems and on TBE (1, 2, 3, 14). The use of ideal low pass power spectra for the signal and TBE is probably the least realistic part of the modeling, however, there are good reasons for using these models.

First, the mathematical complexity of the analysis is extremely difficult except for signal and TBE models with very simple forms for their power spectral densities. Also, as shown later, more general spectral densities can be considered by writing the total spectrum (of either signal or TBE) as a sum of band pass spectra. Additionally, the significant components of signals and TBE are usually concentrated in a particular band. Many signals also have relatively flat spectral densities over the band pass region and can be reasonably approximated as being ideal band pass. Those signals which do not fit this model may be represented as a sum of band pass spectra. The TBE spectrum is subject to considerable variability from one recorder to another, but measurements indicate (1, 2, 3, 14) that the

TBE spectrum is usually of the band pass form with a rather sharp cutoff.

Thus, the ideal band pass (or ideal low pass) model should give reasonable results if the cutoff frequency of the model is taken as the equivalent signal or TBE bandwidth. In cases where a more accurate model is needed, the spectra can be written as a sum of band pass spectra.

C. The Error Measure

To complete the modeling, it is necessary to choose an error measure so that the difference between the input and the output can be characterized in terms of the parameters of the signal and TBE models. The primary criteria to be used in the choice of the error measure are the usefulness of the measure and simplicity of the mathematical calculations involved in dealing with the error measure. The mean square error would seem to be a natural choice here.

First, it leads to a problem which is mathematically tractable while most other error measures lead to very complex mathematics. Also, since it is a squared function, it may be interpreted in terms of power. Finally, the concepts of mean square error have been extensively developed in the literature and the solution of the problem of minimum mean square error leads to the concept of an optimum linear filter. For these reasons, the mean square error between input and output is used as the error measure for this analysis.

For the case where the output of the recorder is not filtered, the error $e(t)$ can be written as

$$e(t) = x(t) - y(t), \tag{3.6}$$

and the mean square error $\overline{e^2}$ as

$$\overline{e^2} = E\left\{[x(t) - y(t)]^2\right\}, \tag{3.7}$$

where $E\{\cdot\}$ indicates expected value. In cases where an optimum filter is connected to the output of the recorder, its output will be denoted by $z(t)$. The error $e_f(t)$ between the input to the recorder and the output of the filter will be written as

$$e_f(t) = x(t) - z(t), \tag{3.8}$$

and the mean square error as

$$\overline{e_f^2} = E\left\{[x(t) - z(t)]^2\right\}, \tag{3.9}$$

where the subscript f indicates use of the optimum linear filter.

CHAPTER IV

TBE EFFECTS FOR LOW PASS SPECTRA

A. Spectra

In this chapter both the signal and the TBE will be assumed to have ideal low pass power spectral densities as shown in Figure 1, where $S_{xx}(\omega)$ denotes the signal power spectral density and $S_{hh}(\omega)$ denotes the TBE power spectral density. In equation form, the signal power spectral density can be expressed as

$$S_{xx}(\omega) = \sigma_x^2 \frac{\pi}{\omega_x} P_{\omega_x}(\omega), \tag{4.1}$$

where σ_x^2 is the signal variance, ω_x is the signal bandwidth, and where $P_{\omega_x}(\omega)$ is a function defined by

$$P_A(\omega) = \begin{cases} 1 \text{ for } |\omega|<A \\ \\ 0 \text{ for } |\omega|>A. \end{cases} \tag{4.2}$$

In a similar manner, the TBE power spectral density can be expressed as

$$S_{hh}(\omega) = \sigma_h^2 \frac{\pi}{\omega_h} P_{\omega_h}(\omega), \tag{4.3}$$

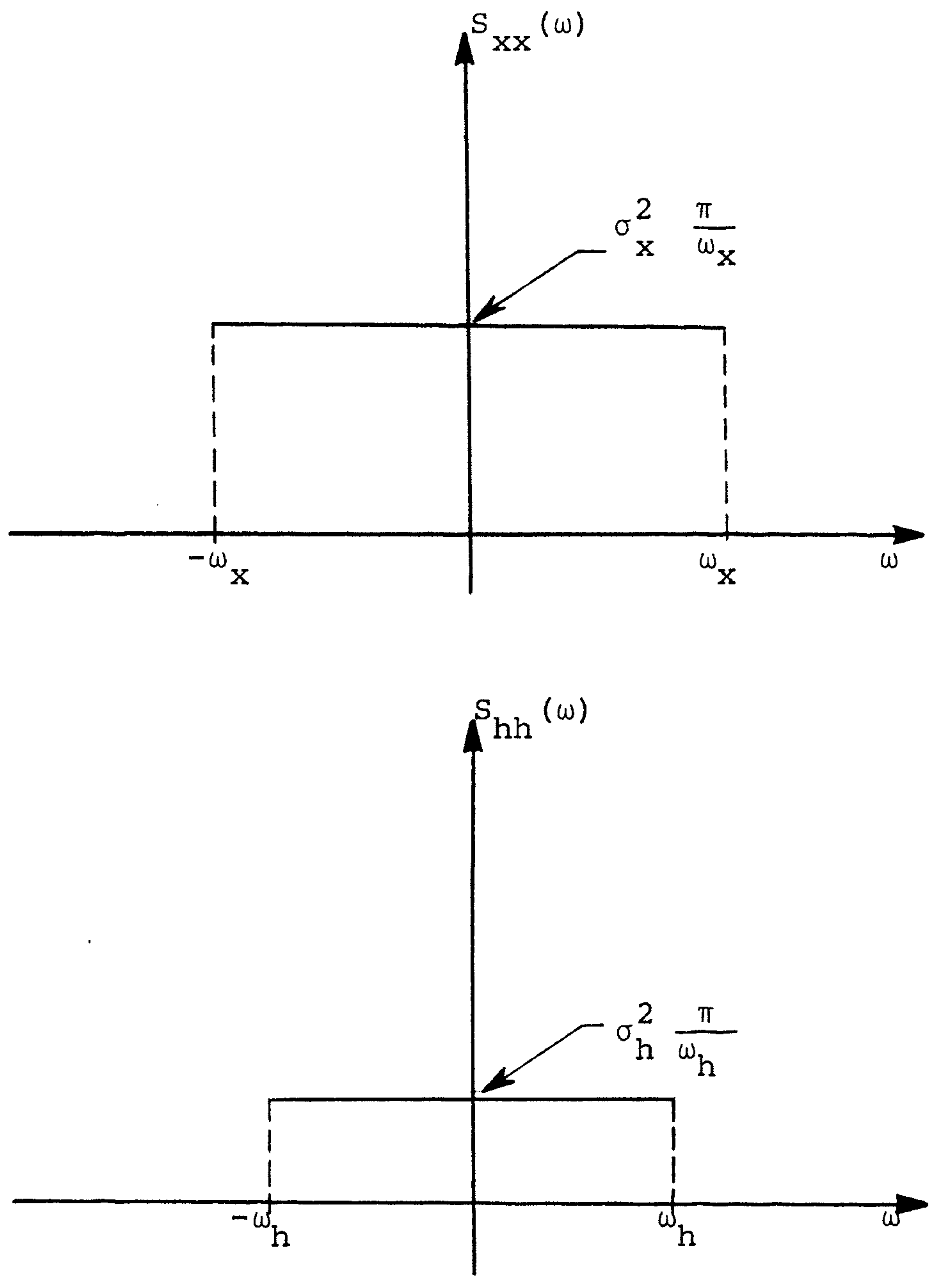


Figure 1. Spectra for low pass signal and time-base error.

where σ_h^2 is the TBE variance, ω_h is the TBE bandwidth, and $P_{\omega_h}(\omega)$ is as defined in (4.2).

B. Mean Square Error For Unfiltered Output

The mean square error for the case where no filter is employed is given by equation (3.7) as

$$\overline{e^2} = E\left\{[x(t) - y(t)]^2\right\},$$

which can be rewritten as

$$\overline{e^2} = E\left\{x^2(t)\right\} + E\left\{y^2(t)\right\} - 2E\left\{x(t)y(t)\right\}. \tag{4.4}$$

Since the signal x(t) is assumed to be a sample function of a stationary random process with zero mean

$$E\left\{x^2(t)\right\} = E\left\{x^2(t+h)\right\} = \sigma_x^2. \tag{4.5}$$

From equation (3.5), the output y(t) is of the form K x(t+h) so that if K is taken to be unity, then equation (4.4) becomes

$$\overline{e^2} = 2\sigma_x^2 - 2E\left\{x(t)y(t)\right\}, \tag{4.6}$$

where the term $E\left\{x(t)y(t)\right\}$ is the cross correlation, denoted by $R_{xy}(\tau)$,

$$R_{xy}(\tau) = E\left\{x(t+\tau)y(t)\right\}. \tag{4.7}$$

Setting $\tau = 0$,

$$E\left\{x(t)y(t)\right\} = R_{xy}(0). \tag{4.8}$$

Now

$$R_{xy}(\tau) = E\left\{x(t+\tau)y(t)\right\} = E\left\{x(t+\tau)x(t+h)\right\} \tag{4.9}$$

and using conditional expectations,

$$E\left\{x(t+\tau)x(t+h)\right\} = E_h\left\{E\left\{(t+\tau)x(t+h)\,|\,h\right\}\right\} \tag{4.10}$$

so that

$$R_{xy}(\tau) = E_h\left\{R_{xx}(\tau-h)\right\}. \tag{4.11}$$

As previously mentioned, $h(t)$ has been assumed to be a sample function of a zero mean normal random process. Therefore,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau-h)\,\frac{e^{-\frac{h^2}{2\sigma_h^2}}}{\sqrt{2\pi\sigma_h^2}}\,dh, \tag{4.12}$$

where σ_h^2 is assumed to be known. For the present development, the power spectral density of x(t) is assumed to be ideal low pass as given by (4.1), so that the autocorrelation function $R_{xx}(\tau)$ is given by

$$R_{xx}(\tau) = \sigma_x^2 \frac{\sin \omega_x \tau}{\omega_x \tau}, \tag{4.13}$$

and equation (4.12) can be written as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} \sigma_x^2 \frac{\sin[\omega_x(\tau-h)]}{\omega_x(\tau-h)} \frac{e^{-\frac{h^2}{2\sigma_h^2}}}{\sqrt{2\pi\sigma_h^2}} \, dh. \tag{4.14}$$

Taking the Fourier transform to obtain the cross spectral density $S_{xy}(\omega)$ gives

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_x^2 \frac{\sin[\omega_x(\tau-h)]}{\omega_x(\tau-h)} \frac{e^{-\frac{h^2}{2\sigma_h^2}}}{\sqrt{2\pi\sigma_h^2}} \, dh \; e^{-j\omega\tau} \, d\tau. \tag{4.15}$$

Rearranging and recognizing the integration with respect to τ as a Fourier transform gives

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} \sigma_x^2 \frac{\pi}{\omega_x} P_{\omega_x}(\omega) \; e^{-j\omega h} \frac{e^{-\frac{h^2}{2\sigma_h^2}}}{\sqrt{2\pi\sigma_h^2}} \, dh, \tag{4.16}$$

which can be integrated to give

$$S_{xy}(\omega) = \sigma_x^2 \frac{\pi}{\omega_x} P_{\omega_x}(\omega)\, e^{-\frac{\sigma_h^2 \omega^2}{2}} . \tag{4.17}$$

Then

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega \tag{4.18}$$

and

$$E\left\{xy\right\} = R_{xy}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega)\, d\omega. \tag{4.19}$$

But $S_{xy}(\omega)$ is an even function which is zero for $|\omega|>\omega_x$ so that (4.19) can be written as

$$E\left\{xy\right\} = (2)\frac{1}{2\pi} \int_{0}^{\omega_x} S_{xy}(\omega)\, d\omega = \frac{\sigma_x^2}{\sigma_h \omega_x} \sqrt{2\pi}\, N(\sigma_h \omega_x), \tag{4.20}$$

where

$$N(u) = \frac{1}{2\pi} \int_{0}^{u} e^{-t^2/2}\, dt. \tag{4.21}$$

Substituting into equation (4.6) gives

$$\overline{e^2} = 2\sigma_x^2 \left\{ 1 - \frac{2\pi}{\sigma_h \omega_x} N(\sigma_h \omega_x) \right\} .$$

This is the general expression for the mean square error which is shown to be a function of the product of $\omega_x \sigma_h$.

It should be noted that in the evaluation of the mean square error, it was necessary to use the expression for the power spectrum for the signal; this required a knowledge of its form and of the parameters ω_x and σ_x but the only knowledge of the TBE which was needed (in addition to its Gaussian density function) was the value of σ_h. Thus, for a given signal spectrum, the mean square error is dependent only on the TBE power and not on its spectral density.

A plot of mean square error as a function of $\omega_x \sigma_h$ is shown in Figure 2. Since $\overline{e^2}$ is a monotonically increasing function of $\omega_x \sigma_h$, it is obvious that wide bandwidth recorders require low TBE. For modern instrumentation recorders where normally $\sigma_h < 10^{-6}$ and thus normally $\omega_x \sigma_h \ll 1$, the error can be calculated from (4.20) using

$$e^{-t^2/2} \approx 1 - t^2/2 \tag{4.22}$$

which is valid for small values of t. Then for $\omega_x \sigma_h \ll 1$, (4.21) becomes

$$\overline{e^2} = 2\sigma_x^2 \left\{ 1/6 \ \sigma_h^2 \omega_x^2 \right\} = 1/3 \ \sigma_x^2 \sigma_h^2 \omega_x^2 , \tag{4.23}$$

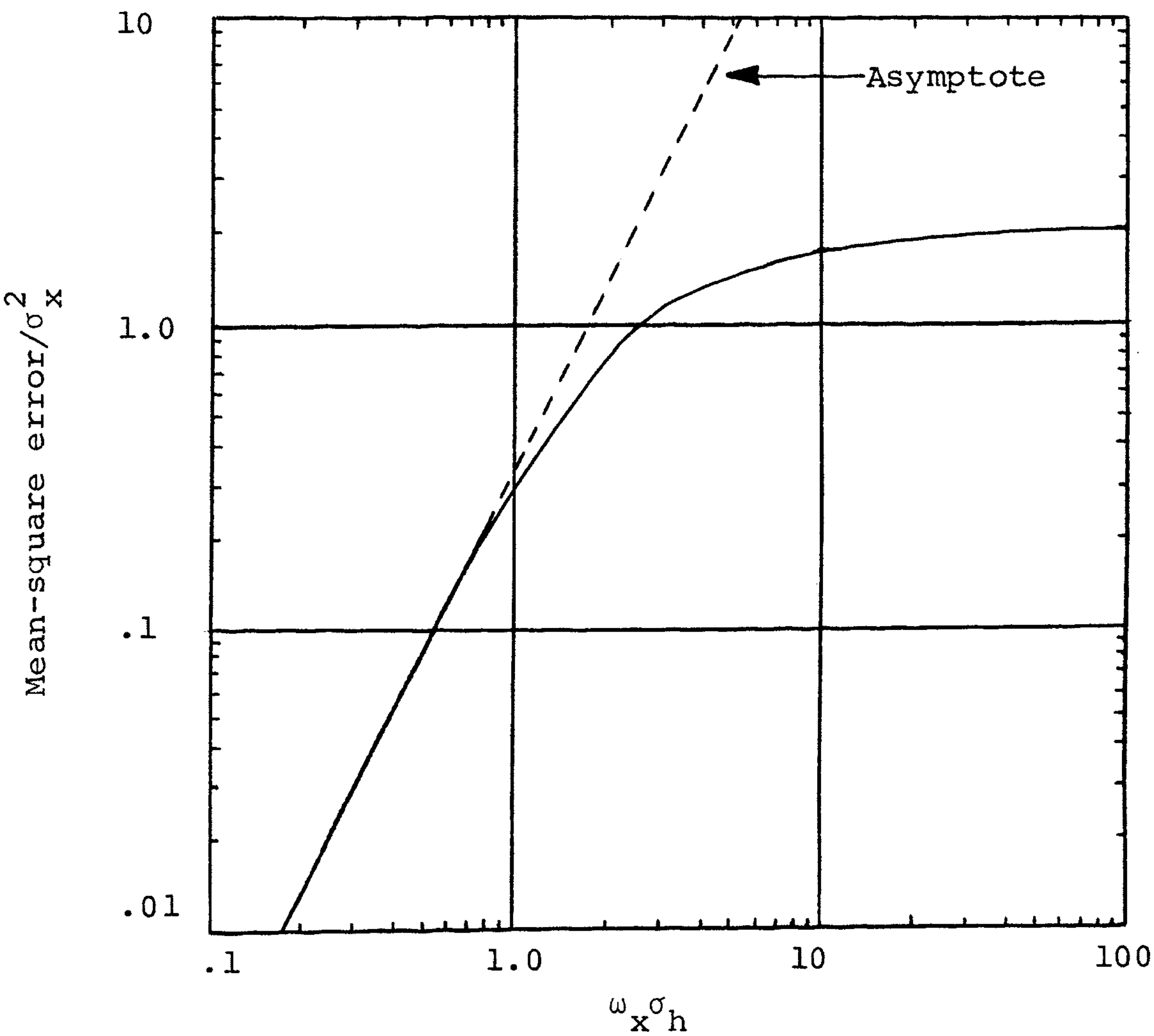


Figure 2. Mean square error due to time-base error.

making the error proportional to the product of the mean square signal (signal power), the mean square TBE (TBE power), and the data bandwidth squared. The use of this approximation corresponds to operation on the lower straight line portion of the plot shown in Figure 2.

This is the error due to TBE, where no consideration has been given to other noise in the system, and no attempt has been made to compensate the output to reduce the error.

C. Mean Square Error With An Optimum Linear Filter

1. The Filter

In order to determine the reduction in mean square error which is possible by use of a filter, consider the case where a filter is connected to the output of the recorder so that the input to the filter is y(t). Using z(t) to denote the output of the filter, the mean square error between the input to the recorder and the output of the filter is given by

$$\overline{e_f^2} = E\left\{[x(t) - z(t)]^2\right\}. \tag{4.24}$$

In order to minimize this error, a filter must be found such that whenever an input y(t) is applied to the filter, the output z(t) of the filter makes (4.24) a minimum. It is known from the orthogonality principle (20) that the error $e_f(t)$ is minimized when the error is orthogonal to the data. That is, the error is minimized when

$$E\left\{[x(t) - z(t)] \quad y(\alpha)\right\} = 0, \tag{4.25}$$

or

$$E[x(t)\ y(\alpha)] - E[z(t)\ y(\alpha)] = 0. \tag{4.26}$$

Now

$$E\left\{x(t)\ y(\alpha)\right\} = R_{xy}(t-\alpha), \tag{4.27}$$

and the response of the filter to y(t) can be written

$$z(t) = \int_{-\infty}^{\infty} \hat{h}(\beta)\ y(t-\beta)\ d\beta \tag{4.28}$$

where $\hat{h}(t)$ is the impulse response of the filter. Thus, equation (4.26) becomes

$$R_{xy}(t-\alpha) = E\left\{y(\alpha) \int_{-\infty}^{\infty} \hat{h}(\beta)\ y(t-\beta)\ d\beta\right\} \tag{4.29}$$

or

$$R_{xy}(t-\alpha) = \int_{-\infty}^{\infty} R_{yy}(t-\alpha-\beta)\, \hat{h}(\beta)\, d\beta\,. \tag{4.30}$$

Using τ as the argument, this can be written as

$$R_{xy}(\tau) = R_{yy}(\tau) * \hat{h}(\tau)\,, \tag{4.31}$$

where $*$ denotes convolution. Upon taking the Fourier transform, this becomes

$$H(j\omega) = \frac{S_{xy}(\omega)}{S_{yy}(\omega)}\,, \tag{4.32}$$

where $H(j\omega)$ is the Fourier transform of $\hat{h}(t)$ and is the transfer function of the optimum linear filter which is being sought. In general, this filter is physically unrealizable since no realizability constraints have been imposed. From (4.32) it can be seen that, since $S_{xy}(\omega)$ has already been calculated, the calculation of $S_{yy}(\omega)$ will complete the determination of $H(j\omega)$. This can be done by determining the autocorrelation, $R_{yy}(\tau)$, of $y(t)$ and taking its Fourier transform.

The autocorrelation of the recorder output is

$$R_{yy}(\tau) = E\left\{y(t+\tau)y(t)\right\} \tag{4.33}$$

which can be written as

$$R_{yy}(\tau) = E_h \left\{ E \left\{ x[t+\tau+h(t+\tau)]x[t+h(t)] \,|\, h(t) \right\} \right\} \tag{4.34}$$

or as

$$R_{yy}(\tau) = E_h \left\{ R_{xx}[\tau+h(t+\tau) - h(t)] \right\} . \tag{4.35}$$

By making the definition

$$u(\tau) = h(t+\tau) - h(t) , \tag{4.36}$$

the autocorrelation can be written as

$$R_{yy}(\tau) = E_h \left\{ R_{xx}(\tau+u) \right\} . \tag{4.37}$$

Since h(t) is stationary with zero mean

$$E[u(\tau)] = E \left\{ h(t+\tau) - h(t) \right\} = 0 \tag{4.38}$$

and the variance of $u(\tau)$ is

$$\sigma_u^2 = E \left\{ u^2(\tau) \right\} = 2[\sigma_h^2 - E \left\{ h(t+\tau)\ h(t) \right\}], \tag{4.39}$$

or

$$\sigma_u^2 = 2[\sigma_h^2 - R_{hh}(\tau)]. \tag{4.40}$$

Since u(t) is a linear transformation of normal random variables, it is normally distributed with zero mean and variance σ_u^2 as given by (4.40).

Using this, along with equations (4.13) and (4.37), $R_{yy}(\tau)$ can be written as

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \sigma_x^2 \frac{\sin[\omega_x(u+\tau)]}{\omega_x(u+\tau)} \frac{e^{-\frac{u^2}{2\sigma_u^2}}}{\sqrt{2\pi\sigma_u^2}} \, du. \tag{4.41}$$

The details of carrying out this integration and of taking the Fourier transform of $R_{yy}(\tau)$ to obtain $S_{yy}(\omega)$ are very tedious and reader is referred to Appendix A. In the appendix, it is shown that (4.41) can be written as

$$R_{yy}(\tau) = \frac{\sigma_x^2}{\omega_x\sigma_u} \int_{0}^{\omega_x\sigma_u} e^{-\beta^2/2} \cos(\beta\frac{\tau}{\sigma_u}) \, d\beta. \tag{4.42}$$

Again, assuming that $\omega_x\sigma_h \ll 1$, this can be calculated to be

$$R_{yy}(\tau) = \sigma_x^2 \frac{\sin \omega_x\tau}{\omega_x\tau} [1 - \frac{(\omega_x\sigma_u)^2}{2} + (\frac{\sigma_u}{\tau})^2]$$

$$- \sigma_x^2 (\frac{\sigma_u}{\tau})^2 \cos \omega_x\tau. \tag{4.43}$$

In order to be able to complete the computation of $S_{yy}(\omega)$, it is necessary to make use of equation (4.3) which gives the power spectral density of h(t). The autocorrelation, $R_{hh}(\tau)$, of the TBE is

$$R_{hh}(\tau) = \sigma_h^2 \frac{\sin \omega_h \tau}{\omega_h \tau} , \tag{4.44}$$

so that σ_u^2 is

$$\sigma_u^2 = 2\sigma_h^2 \left[1 - \frac{\sin \omega_h \tau}{\omega_h \tau}\right] . \tag{4.45}$$

Inserting this in (4.43) above and taking the Fourier transform (see Appendix A for details) gives $S_{yy}(\omega)$ and completes the determination of the optimum filter $H(j\omega)$

2. The Error

To find the mean square error with the above determined optimum filter connected to the recorder, it is necessary to find a means of evaluating equation (4.24). This can be accomplished by expressing the error $\overline{e_f^2}$ in terms of power spectral densities.

Equation (4.24) can be written as

$$\overline{e_f^2} = E\left\{[x(t)-z(t)]\ x(t)\right\} - E\left\{[x(t) - z(t)]\ z(t)\right\}, \tag{4.46}$$

but

$$E\left\{[x(t)-z(t)]\ z(t)\right\} = E[x(t)-z(t)]\int_{-\infty}^{\infty} y(t-\sigma)h(\sigma)d\sigma, \tag{4.47}$$

or

$$E\left\{[x(t)-z(t)]z(t)\right\} = \int_{-\infty}^{\infty} E\left\{[x(t)-z(t)]y(t-\sigma)\right\} h(\sigma)d\sigma. \tag{4.48}$$

From equation (4.25) this is zero so that (4.46) becomes

$$\overline{e_f^2} = E\left\{[x(t)-z(t)]x(t)\right\}, \tag{4.49}$$

or

$$\overline{e_f^2} = R_{xx}(0)-E\left\{x(t)\int_{-\infty}^{\infty} y(t-\alpha)h(\alpha)\ d\alpha\right\}. \tag{4.50}$$

But this can be written as

$$\overline{e_f^2} = R_{xx}(0) - \int_{-\infty}^{\infty} R_{xy}(\alpha)h(\alpha)\ d\alpha. \tag{4.51}$$

Defining $g(\tau)$ as

$$g(\tau) = R_{xx}(\tau) - \int_{-\infty}^{\infty} R_{xy}(\alpha-\tau)\hat{h}(\alpha)\ d\alpha, \tag{4.52}$$

which can be written

$$g(\tau) = R_{xx}(\tau) - R_{xy}(-\tau) * \hat{h}(\tau), \tag{4.53}$$

and taking the Fourier transform of $g(\tau)$ to get

$$G(\omega) = S_{xx}(\omega) - S_{xy}(-\omega)H(j\omega) \tag{4.54}$$

enables the error $\overline{e_f^2}$ to be written as

$$\overline{e_f^2} = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\ d\omega, \tag{4.55}$$

or

$$\overline{e_f^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[S_{xx}(\omega) - S_{xy}(-\omega)H(j\omega)\right]\ d\omega. \tag{4.56}$$

Using (4.32), this becomes

$$\overline{e_f^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{S_{xx}(\omega) - \frac{[S_{xy}(\omega)]^2}{S_{yy}(\omega)}\right\} d\omega \tag{4.57}$$

which, by the use of (4.1) and (4.17), can be written as

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_o^{\omega_x} \frac{e^{-\sigma_h^2 \omega^2}}{S_{yy}(\omega)} \, d\omega. \tag{4.58}$$

In order to complete the evaluation of this error, it is necessary to use this equation along with the equations for $S_{yy}(\omega)$ as given in Appendix A. The results of this are (see Appendix A for details)

$$\overline{e_f^2} = \frac{\omega_x}{\omega_h} \quad \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} \tag{4.59}$$

for $2\omega_x < \omega_h$,

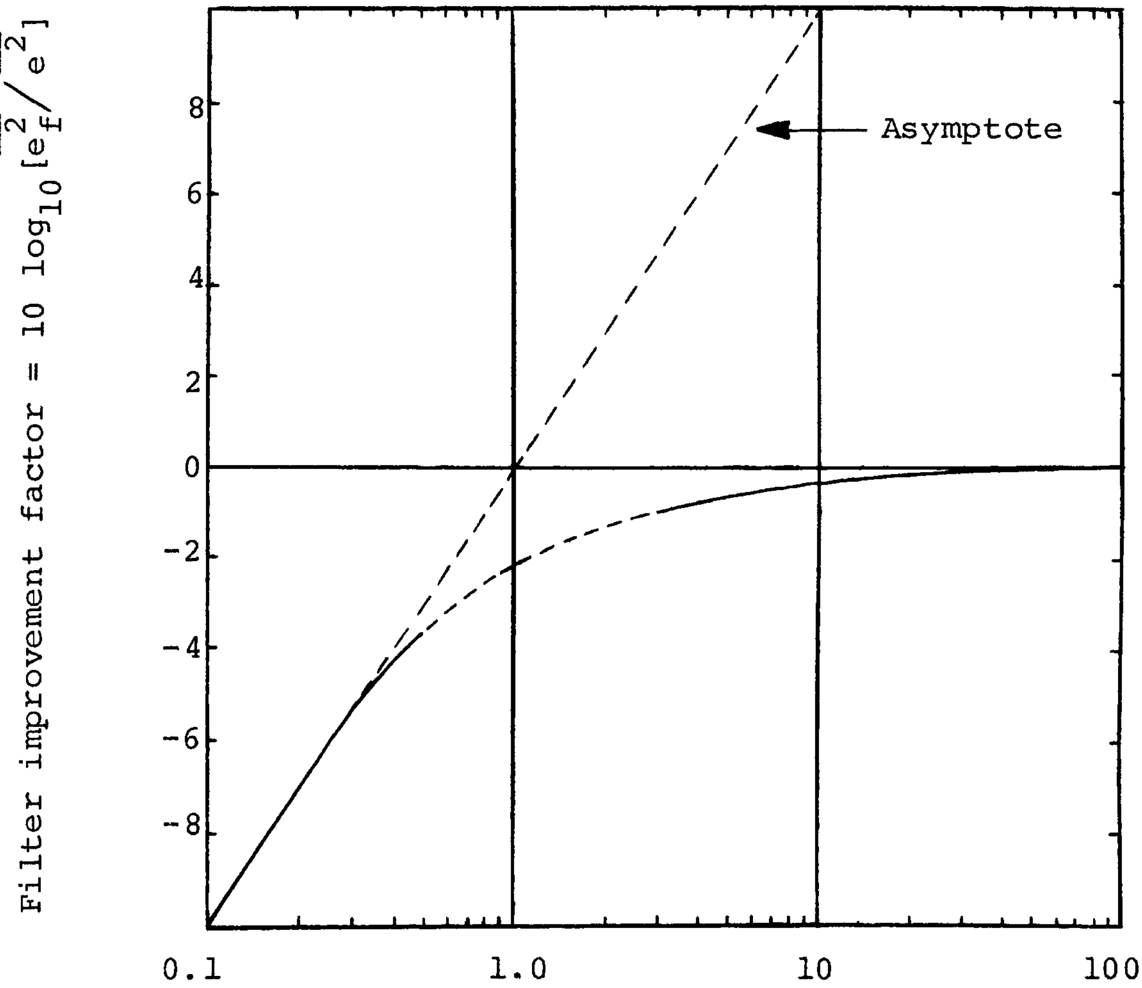
$$\overline{e_f^2} = \frac{5}{24} \quad \sigma_x^2 \sigma_h^2 \omega_x^2 \tag{4.60}$$

for $\omega_x = \omega_h$, and

$$\overline{e_f^2} = \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} - \frac{\sigma_x^2 \sigma_h^2}{24} \frac{\omega_h}{\omega_x} (6\omega_x^2 - 4\omega_x \omega_h + \omega_h^2) \tag{4.61}$$

for the case where $2\omega_h < \omega_x$.

A plot of 10 $\log_{10}$ $(\overline{e_f^2}/\overline{e^2})$ versus $\frac{\omega_x}{\omega_h}$ is shown in Figure 3. This figure shows the improvement in the mean square error which is possible by the use of the optimum linear filter. The figure is shown dashed in the region where $\frac{\omega_h}{2} < \omega_x < \omega_h$ and where $\omega_h < \omega_x < 2\omega_h$ since no calculation was made for $S_{yy}(\omega)$ or



$$\frac{\text{Signal Bandwidth}}{\text{TBE Bandwidth}} = \frac{\omega_x}{\omega_h}$$

Figure 3. System improvement with optimum filter.

for the error $\overline{e_f^2}$ in either of these regions. The calculations for these regions are quite tedious and there would seem to be little information to be gained by them since it seems unreasonable to suppose that there would be a large deviation from the extrapolated (dashed) curves shown in Figure 3.

While it is true that the improvement calculated is the theoretical maximum and no physical realizability conditions have been imposed, the improvement indicated by the calculations could be approached either by processing the data y(t) (as per the equations for the unrealizable optimum filter) rather than using an actual physical filter, or by allowing a time delay in the output z(t). Of course, in the region where $\omega_x > \omega_h$, there would be little point in filtering because the improvement, even with the unrealizable optimum filter, is very small. On the other hand, in the region where ω_x is small compared to ω_h, the filtered error is less than the unfiltered error by the factor ω_x/ω_h so that significant improvement could be expected here. In all cases, the equation for the error contains the product of TBE power and the data bandwidth squared so that the product $\omega_x \sigma_h$ should always be kept as small as possible. Since σ_h is usually fixed, this means that the data bandwidth ω_x should be kept to a minimum.

CHAPTER V

TBE EFFECTS FOR BAND PASS SPECTRA

A. Band Pass Signal--Low Pass TBE

1. Spectra

Since the ideal low pass signal case considered in the previous chapter is not very general, it is of interest to generalize the computations by considering a signal which is band pass rather than low pass. Although there are obviously several different possible specifications on the relative sizes of signal bandwidth, TBE bandwidth, and center frequency of the signal pass band, previous results indicate that the greatest improvement (by filtering) is to be obtained where signal bandwidth is small compared to the TBE bandwidth. Therefore, in this computation, the signal bandwidth will be taken as small compared to the TBE bandwidth. More specifically, the power spectral densities will be taken as shown in Figure 4 and it will be assumed that $\omega_h > 2(\omega_o + \omega_x)$ where ω_o is the center frequency of the signal band. While this does not correspond to the spectra likely to be encountered, it does show the effect of TBE on the various frequency components of the signal.

The equations for the power spectral densities are then

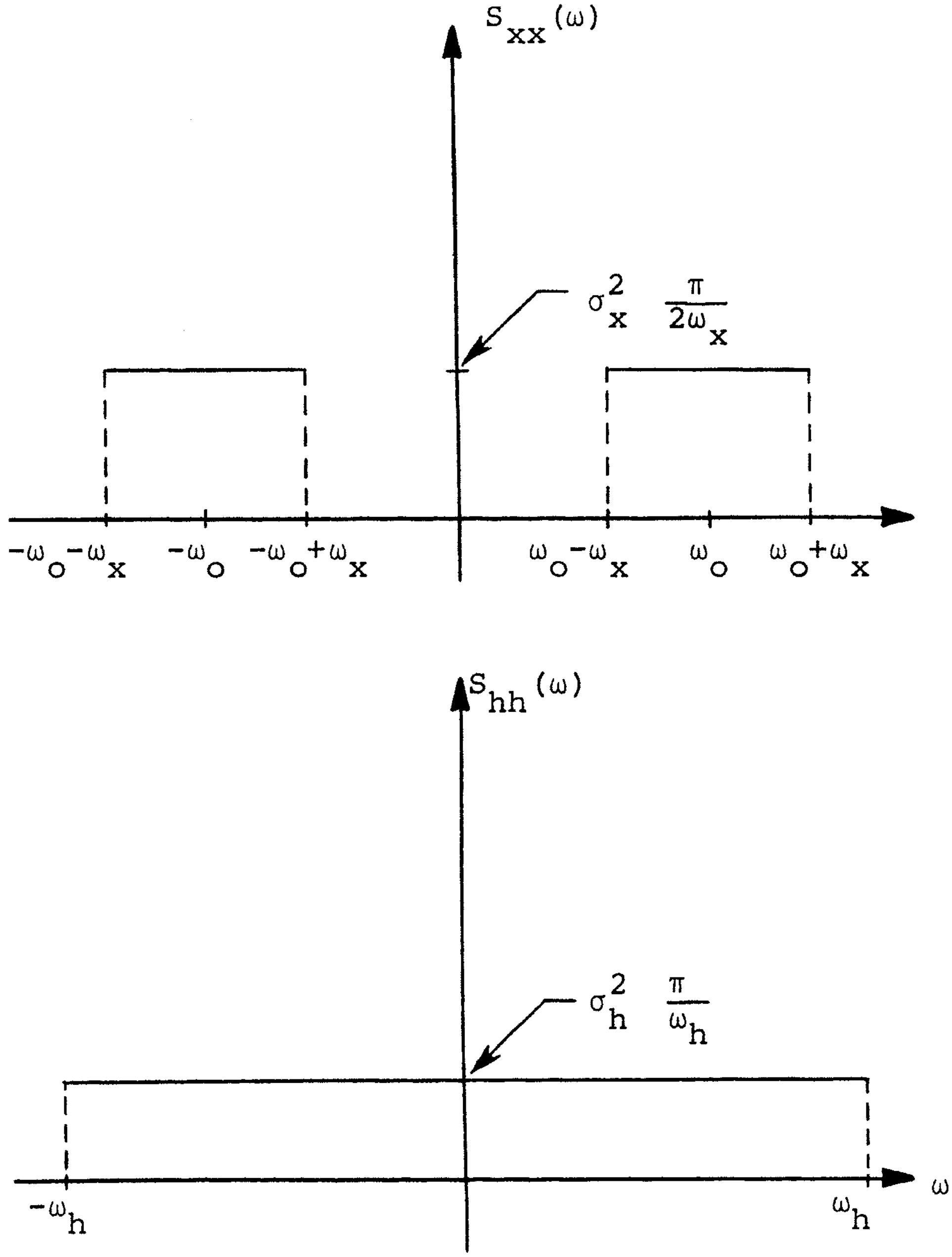


Figure 4. Spectra for band pass signal and low pass time-base error.

$$S_{hh}(\omega) = \sigma_h^2 \frac{\pi}{\omega_h} P_{\omega_h}(\omega) \tag{5.1}$$

for the TBE spectrum, and

$$S_{xx}(\omega) = \sigma_x^2 \frac{\pi}{\omega_x} 1/2 \ [P_{\omega_x}(\omega-\omega_o) + P_{\omega_x}(\omega+\omega_o)] \tag{5.2}$$

for the signal spectrum (the center frequency ω_o is taken to be greater than ω_x).

2. Mean Square Error For Unfiltered Output

As in the previous chapter, the error for the unfiltered recorder is given by

$$\overline{e^2} = 2[\sigma_x^2 - R_{xy}(0)] \tag{5.3}$$

where

$$R_{xy}(\tau) = E_h[R_{xx}(\tau-h)]. \tag{5.4}$$

However, in this case $R_{xx}(\tau)$ is found from (5.2) as

$$R_{xx}(\tau) = \sigma_x^2 \frac{\sin \omega_x\tau}{\omega_x\tau} \cos \omega_o\tau \tag{5.5}$$

so that (5.4) becomes

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} \sigma_x^2 \frac{\sin[\omega_x(\tau-h)]}{\omega_x(\tau-h)} \cos[\omega_o(\tau-h)] \frac{e^{-\frac{h^2}{2\sigma_h^2}}}{\sqrt{2\pi\sigma_h^2}} dh \qquad (5.6)$$

Taking the Fourier transform and integrating in the same manner as in the previous chapter gives

$$S_{xy}(\omega) = \sigma_x^2 \frac{\pi}{\omega_x} \; e^{-\frac{\sigma_h^2\omega^2}{2}} \; 1/2[P_{\omega_x}(\omega-\omega_o)+P_{\omega_x}(\omega+\omega_o)]. \qquad (5.7)$$

Now

$$E(xy) = R_{xy}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\ d\omega, \qquad (5.8)$$

which can be simplified to give

$$R_{xy}(0) = \frac{\sigma_x^2}{2\omega_x}\int_{\omega_o-\omega_x}^{\omega_o+\omega_x} e^{-\frac{\sigma_h^2\omega^2}{2}}\ d\omega. \qquad (5.9)$$

While this can be expressed in terms of the error function, it is convenient to make the approximation that $\sigma_h(\omega_o+\omega_x) << 1$ which would again be valid for most instrumentation recorders. Under this assumption

$$R_{xy}(0) = \frac{\sigma_x^2}{2\omega_x}\int_{\omega_o-\omega_x}^{\omega_o+\omega_x} (1 - \frac{\sigma_h^2}{2}\omega^2)\ d\omega, \qquad (5.10)$$

or

$$R_{xy}(0) = \sigma_x^2 - \frac{\sigma_x^2\sigma_h^2}{6}(3\omega_o^2+\omega_x^2). \tag{5.11}$$

Then using this expression along with (5.2) gives

$$\overline{e^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3}[1+3(\frac{\omega_o}{\omega_x})^2]. \tag{5.12}$$

As in the case of low pass signal, this error is independent of the specific spectral density of $S_{hh}(\omega)$ but dependent on σ_h^2, and on ω_x^2. Additionally, in this case, the center frequency ω_o contributes significantly to the error. By comparing this with (4.23) (which is the error for the unfiltered case with low pass signal), it is evident that the error for this case is considerably larger than for the low pass signal case. In fact, if ω_o is written as $\omega_o = M\omega_x$ where M>>1, the expression for the error becomes

$$\overline{e^2} = \sigma_x^2\sigma_h^2\omega_o^2 = 3M^2(\frac{\sigma_x^2\sigma_h^2\omega_x^2}{3}). \tag{5.13}$$

That is, the error is $3M^2$ times the error for the low pass signal case if all the parameter values are the same. This clearly indicates that it is important to keep the signals to be recorded at as low a frequency as possible in order to keep the error small.

3. Mean Square Error With an Optimum Linear Filter

Since this case (bandpass signal) is one where the signal bandwidth is much smaller than the TBE bandwidth, one might expect a significant improvement using the optimum filter; it will now be shown that this is indeed the case.

Using the same symbols as before, the error with the filter attached is once again

$$\overline{e_f^2} = E[x(t) - z(t)]^2. \tag{5.14}$$

Tracing through the same steps as before for the derivation of the optimum filter and the mean square error $\overline{e_f^2}$ again gives

$$H(j\omega) = \frac{S_{xy}(\omega)}{S_{yy}(\omega)} \tag{5.15}$$

and

$$\overline{e_f^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[S_{xx}(\omega) - \frac{|S_{xy}(\omega)|^2}{S_{yy}(\omega)} \right] d\omega, \tag{5.16}$$

so that once again it is necessary to compute $S_{yy}(\omega)$, where

$$S_{yy}(\omega) = \int_{-\infty}^{\infty} R_{yy}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{5.17}$$

and where, in this case,

$$R_{yy}(\tau) = E_h[R_{xx}(\tau+u)] = E_h \left\{ \frac{\sigma_x^2 \pi}{\omega_x} \frac{\sin[\omega_x(\tau+u)]}{\omega_x(\tau+u)} \cdot \cos\ [\omega_o(\tau+u)] \right\}. \tag{5.18}$$

The details of the computation of $S_{yy}(\omega)$ are shown in Appendix B. The results of using $S_{yy}(\omega)$ as computed in the appendix with equation (5.16) above is

$$\overline{e_f^2} = \frac{\sigma_x^2 \sigma_h^2}{3} [\omega_x^2 + 3\omega_o^2] \frac{2\omega_x}{\omega_h} \tag{5.19}$$

or

$$\overline{e_f^2} = \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} [1+3(\frac{\omega_o}{\omega_x})^2] \frac{2\omega_x}{\omega_h} . \tag{5.20}$$

If once again ω_o is written as $\omega_o = M\omega_x$, where M>>1, the error becomes

$$\overline{e_f^2} = \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} \frac{2\omega_x}{\omega_h} 3M^2 \tag{5.21}$$

where the restrictions on the values for ω_o, ω_x, and ω_h

are

$$\omega_h > 2(\omega_o + \omega_x) \text{ and } \omega_o \geq \omega_x .$$

Comparison of either (5.12) and (5.20) or (5.13) and (5.21) shows that the error for the filtered case is reduced by the factor of $\frac{2\omega_x}{\omega_h}$ (the maximum value for this factor is $\frac{1}{2}$). These results are consistent with the low pass signal case (where the signal bandwidth was less than the TBE bandwidth) because in both cases the filtered error is less than the unfiltered error by the factor of signal bandwidth divided by the TBE bandwidth. Thus, for the wide band TBE case, the effect of adding the optimum linear filter is to significantly reduce the mean square error. It should be noted, however, that whether the filtered or unfiltered case is considered, the situation of a bandpass signal leads to significantly more error than the low pass case (for comparable bandwidths). The high frequency components then, are the ones which suffer the most degradation due to TBE in the process of recording and reproducing a signal. Narrow band signals with large center frequencies are particularly vulnerable. In fact, for this case, the error is approximately dependent only on the center frequency [see (5.21)]. This indicates, for example, that in a situation where several band pass signals are frequency division multiplexed, the high frequency bands suffer much more degradation due to TBE than do the low frequency bands.

B. Low Pass Signal--Band Pass TBE

1. Spectra

To this point, TBE spectra have all been taken to be ideal low pass. In practice, however, the TBE spectra usually more nearly resemble band pass spectra because the low frequency components of TBE can be removed by servos. For this reason, it is desirable to have an analysis of the error (both filtered and unfiltered) for the case of a band pass TBE spectrum. This is true not only because it is important to be able to calculate the effects of narrow band TBE on signals, but also because it can be shown that more general TBE spectra can be represented as a sum of band pass spectra.

In this analysis, the signal and the TBE spectra are taken as ideal low pass and ideal band pass respectively as shown in Figure 5. The complexity of the analysis requires that restrictions be placed on the relative signal and TBE bandwidths and on the TBE center frequency. The restrictions used for this analysis are

$$\omega_x > (\omega_c + \omega_h) \text{ and } \omega_c \geq \omega_h$$

where ω_c is the center frequency of the TBE as shown in

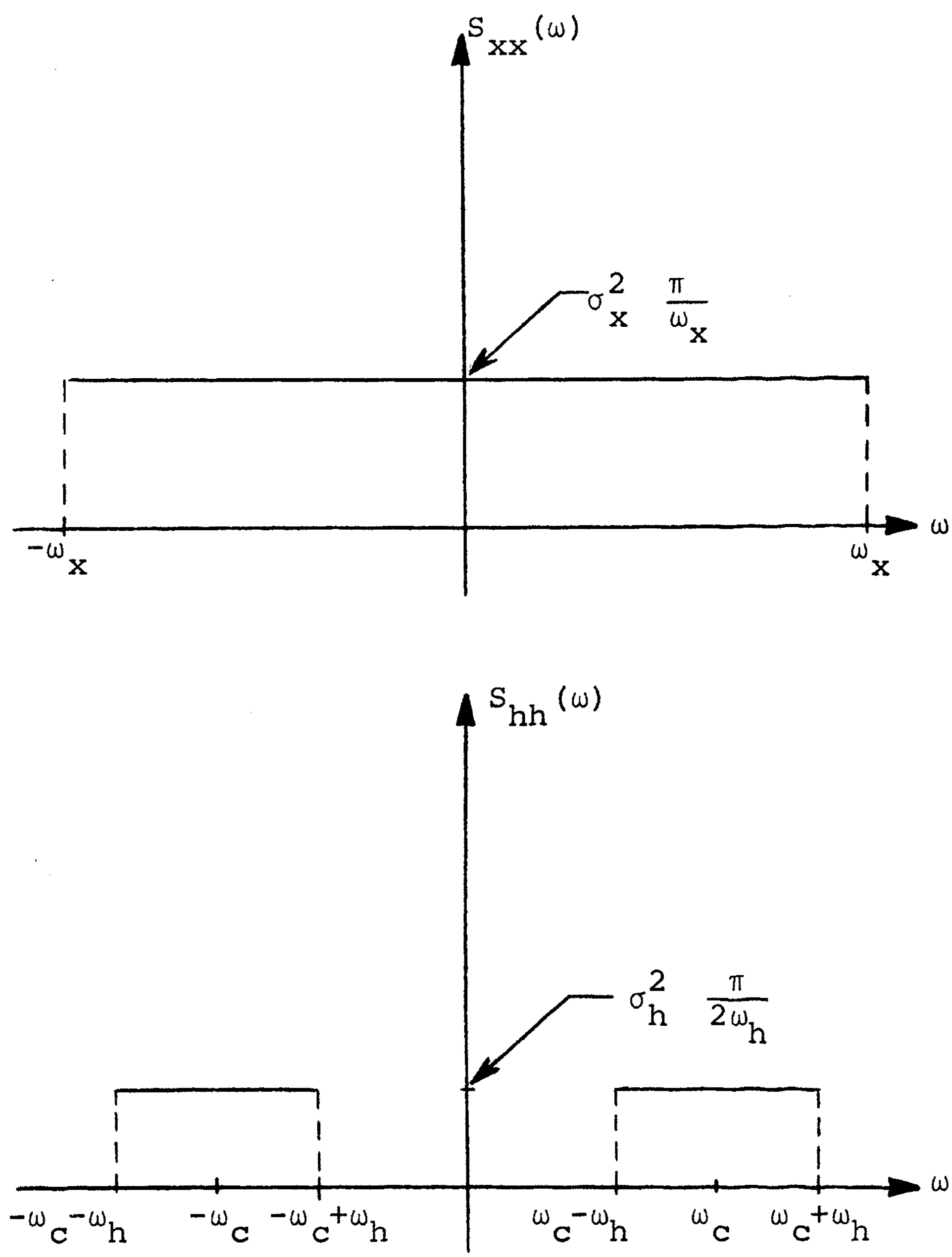


Figure 5. Spectra for low pass signal and band pass time-base error.

Figure 5.

The equations for the spectra for this case are

$$S_{hh}(\omega) = \sigma_h^2 \frac{\sin \omega_h \tau}{\omega_h \tau} \cos \omega_c \tau,$$

for the TBE spectrum, and

$$S_{xx}(\omega) = \sigma_x^2 \frac{\sin \omega_x \tau}{\omega_x \tau}$$

for the signal spectrum.

2. Mean Square Error For Unfiltered Output

It has already been shown that, for both the case of low pass signal and for the case of band pass signal, the mean square error is dependent only on σ_h^2 and not on the spectral density of the TBE. Therefore, the mean square error is given by (4.23) for low pass signal and by (5.12) for the band pass signal regardless of the spectral distribution of $S_{hh}(\omega)$.

3. Mean Square Error With An Optimum Linear Filter

Using the same procedure as in the previous cases, it is found that equations (5.15) for the optimum filter and

(5.16) for the mean square error are still valid. Also, the cross spectral density $S_{xy}(\omega)$ is the same as for the first case considered, namely,

$$S_{xy}(\omega) = \sigma_x^2 \frac{\pi}{\omega_x} P_{\omega_x}(\omega) \, e^{-\frac{\sigma_h^2 \omega^2}{2}} .$$

Once again, due to the tedious nature of the calculations, the details of computing $S_{yy}(\omega)$ and $\overline{e_f^2}$ are shown in the appendix (Appendix C). A result of these calculations is

$$R_{yy}(\omega) = \sigma_x^2 \frac{\sin \omega_x \tau}{\omega_x \tau} [1 - \frac{(\omega_x \sigma_h)^2}{2} + (\frac{\sigma_u}{\tau})^2]$$

$$- \sigma_x^2 (\frac{\sigma_u}{\tau})^2 \cos \omega_x \tau \qquad (5.22)$$

where, in this case,

$$\sigma_u^2 = 2\sigma_h^2 [1 - \frac{\sin \omega_h \tau}{\omega_h \tau} \cos \omega_c \tau]. \qquad (5.23)$$

By computing $S_{yy}(\omega)$ from (5.22) and (5.23) and using the result in (5.16), one obtains the mean square error for this case

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3}\left\{1-\frac{1}{2}\left[\frac{\omega_c}{\omega_x}\left(\frac{\omega_c^2+\omega_h^2+3\omega_x^2}{\omega_x^2}\right)-\left(\frac{\omega_h^2+3\omega_c^2}{\omega_x^2}\right)\right]\right\}. \tag{5.24}$$

Although the complexity of this equation makes conclusions regarding the "best" values of ω_c and ω_h difficult, careful examination reveals that the smallest error occurs when $\omega_x = \omega_c+\omega_h$ which is the minimum value for ω_x. For this case, the error becomes

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3}\left\{1-\frac{1}{2}\left[1-\left(\frac{\omega_h}{\omega_x}\right)^2\left(1-\frac{\omega_h}{\omega_x}\right)\right]\right\} . \tag{5.25}$$

For the small values of ω_h this becomes

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{6} + \frac{\sigma_x^2\sigma_h^2\omega_h^2}{6} \tag{5.26}$$

or approximately

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{6} . \tag{5.27}$$

This shows that for the case where the TBE spectrum is at the upper end of the signal spectrum, the error for the filtered case does depend on the TBE bandwidth. It also shows that use of the optimum filter can reduce the error by a factor of approximately 1/2 for the case where the TBE bandwidth is small (compared to signal bandwidth) and concentrated at the upper end of the signal spectrum [compare (5.26) and (5.27) to equation (4.23)]. Study of equation (5.24) reveals, however, that this improvement is not significant unless the TBE spectrum is concentrated at the upper end of the signal spectrum. In fact, equation (5.24) can be shown to reduce to (4.61) as the TBE spectrum approaches the low pass spectrum considered previously.

C. Composite Spectra

Since the types of signal and TBE spectra encountered in practice are not always ideal low pass or ideal band pass or even a reasonable approximation thereto, questions arise as to the effect of the approximations used and as to the possibility of making computations using more general spectral models. Partial answers can be found for these questions by considering cases where the signal and/or TBE spectra are composed of sums of band pass spectra.

In order to gain some insight about the change in the mean square error with deviations in the form of the power spectrum of the signal from the ideal models previously considered, suppose that a signal has a power spectrum as shown in Figure 6. In this case the signal spectrum can be considered to be composed of a sum of two band pass spectra and written in the form

$$S_{xx}(\omega) = A\, P_{\omega_1}(\omega) + A\, P_{\omega_2}(\omega). \tag{5.28}$$

The autocorrelation function then becomes

$$R_{xx}(\tau) = A\,\frac{\omega_1}{\pi}\,\frac{\sin\,\omega_1\tau}{\omega_1\tau} + A\,\frac{\omega_2}{\pi}\,\frac{\sin\,\omega_2\tau}{\omega_2\tau}\;. \tag{5.29}$$

As in previous cases

$$R_{xy}(\tau) = E_h\,[R_{xx}(\tau-h)], \tag{5.30}$$

which becomes

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau-h)\,\frac{e^{-\frac{h^2}{2\sigma_h^2}}}{\sqrt{2\pi\sigma_h^2}}\;dh. \tag{5.31}$$

Taking the Fourier transform gives

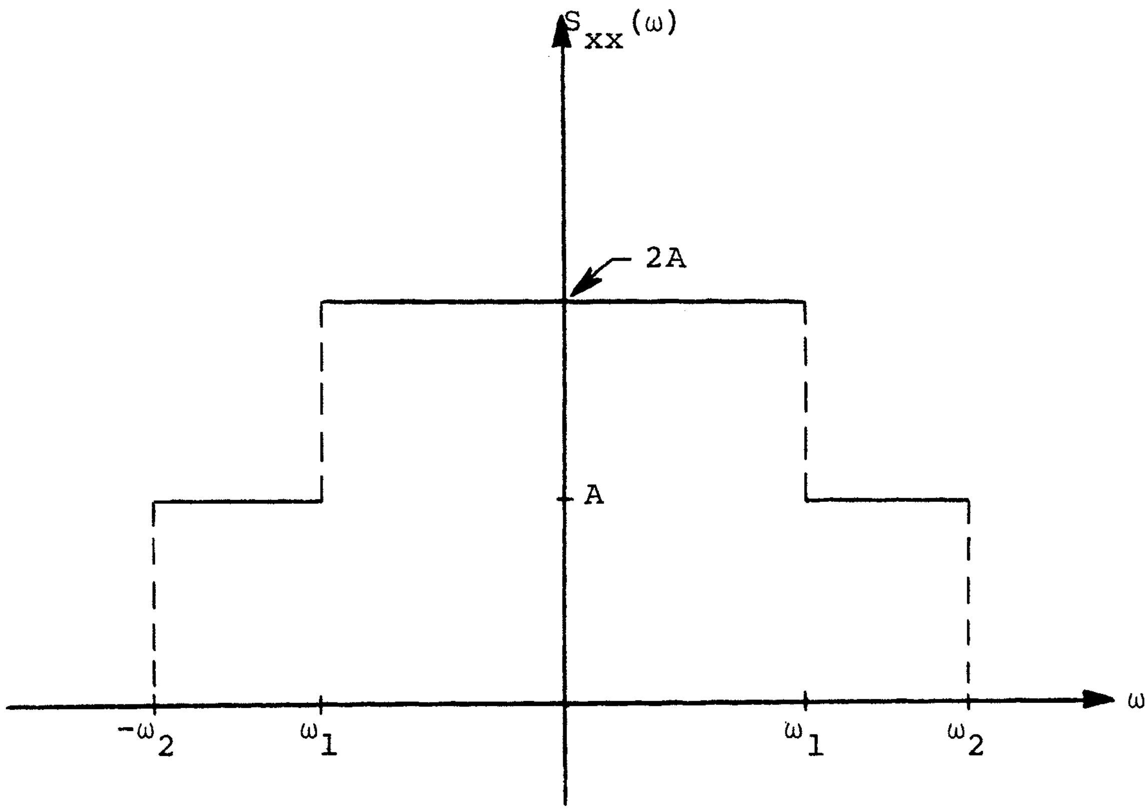


Figure 6. Composite signal spectrum.

$$S_{xy}(\omega) = [A\ P_{\omega_1}(\omega) + A\ P_{\omega_2}(\omega)]\ e^{-\frac{\sigma_h^2\omega^2}{2}} . \tag{5.32}$$

In order to find the error (unfiltered) for this case, it is only necessary to use equations (5.3) and (5.8) along with the above. This gives

$$\overline{e^2} = \frac{\sigma_h^2\sigma_x^2}{3}\ [\omega_1^2+\omega_2^2-\omega_1\omega_2], \tag{5.33}$$

which for $\omega_2 = k\omega_1$ is

$$\overline{e^2} = \frac{\sigma_h^2\sigma_x^2\omega_1^2}{3}\ (1+k^2-k). \tag{5.34}$$

If k>>1, this becomes approximately

$$\overline{e^2} = \frac{\sigma_h^2\sigma_x^2\omega_2^2}{3}, \tag{5.35}$$

which is just the error computed in Chapter IV for an ideal low pass spectrum of bandwidth ω_2. Then if the spectrum of Figure 6 is approximated by an ideal low pass spectrum whose bandwidth is almost ω_2, the error computed will be approximately that given by (5.34). Since it has already been shown that the error (unfiltered) is independent of the shape of the TBE spectrum, it would seem that the use of ω_h and ω_x as equivalent bandwidths of the signal and the TBE will lead to reasonable approximations of the mean square error.

In cases where the signal and/or the TBE spectra are complex or where higher accuracy is desired, it may be necessary to use a more general representation of the power spectra. For these cases, it is still possible to use the techniques employed thus far. This is accomplished in general terms by writing both the signal spectrum and the TBE spectrum as a summation of band pass spectra. It can be shown that this is a procedure which leads to sums of terms of the same form as those already dealt with. This procedure, however, becomes exceedingly cumbersome and tedious if many terms are involved, and the complexity of the resulting equations is such that they are difficult to interpret. It is probable that the best procedure for cases where many terms are necessary to adequately represent the signal and/or the TBE spectrum, is to use a digital computer to calculate the quantities desired.

CHAPTER VI

ADDITIVE EXTERNAL NOISE

A. Sources of Error

To this point the consideration of the effects of time-base perturbations has been strictly in terms of the mean square error caused by time-base error. No consideration has been given to additive noise which is present in the signal before recording takes place or to the noise which is introduced into the signal after playback.

Since it is obviously of no use to attempt to minimize the error caused by time-base perturbations if this error is small compared to other errors, a measure of the relative size of this error is needed. Specifically, criteria are needed to determine when the mean square error due to TBE is significant compared to noise present in the signal before recording and compared to noise introduced after playback.

B. Pre-Recorder Noise

In order to establish the first of these criteria, consideration will now be given to the case where noise is present in the signal before recording; that is, where the input to the recorder is composed of signal, $x(t)$, plus noise, $n(t)$. The noise, $n(t)$, will be assumed to be a sample function of a normal random process which is independent of x(t) and of h(t). The power spectral density of h(t) will be assumed to be ideal low pass and the power spectra for x(t) and n(t) will be taken as ideal low pass of bandwidth ω_x. Since the noise passes through the same channel as the signal prior to recording, the noise bandwidth is assumed to be the same as the signal bandwidth.

For an input to the recorder x(t) + n(t), the output of the recorder will be of the form

$$y(t) = x[t+h(t)] + n[t+h(t)], \tag{6.1}$$

and the mean square error is

$$\overline{e^2} = E\left\{[x(t)-y(t)]^2\right\} = E(x^2) + E(y^2) - 2E(xy). \tag{6.2}$$

Denoting the noise variance by σ_n^2,

$$E\left[x^2\right] = \sigma_x^2 ,$$

$$E\left[n^2\right] = \sigma_n^2 ,$$

and

$$E\left[y^2\right] = E\left\{x(t+h)+n(t+h)\right\}^2 = \sigma_x^2+\sigma_n^2 . \tag{6.3}$$

The spectrum of x(t) is again taken as ideal low pass so that

$$S_{xx}(\omega) = \sigma_x^2 \frac{\pi}{\omega_x} P_{\omega_x}(\omega) , \tag{6.4}$$

$$S_{xy}(\omega) = \sigma_x^2 \frac{\pi}{\omega_x} P_{\omega_x}(\omega) e^{-\frac{\sigma_h^2 \omega^2}{2}} , \tag{6.5}$$

and

$$\overline{e^2} = 2\left[\sigma_x^2 - \frac{1}{2\pi} (2) \int_o^{\omega_x} \sigma_x^2 \frac{\pi}{\omega_x} e^{-\frac{\sigma_h^2 \omega^2}{2}} d\omega\right] . \tag{6.6}$$

Assuming $\sigma_h \omega_x << 1$, (6.6) becomes

$$\overline{e^2} = \sigma_n^2 + \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} . \tag{6.7}$$

This means that the error due to the additive noise which is present in the signal before recording simply adds to the error due to the TBE when no filter is used. Unless σ_n^2 is significant compared to the second term in equation (6.7), the error due to noise already present in the signal could be neglected in comparison to error caused by TBE.

If one considers the addition of the optimum linear filter to the output of this system, it is necessary to minimize the quantity

$$\overline{e_f^2} = E[x(t)-z(t)]^2 \tag{6.8}$$

where z(t) is the output of a filter whose input is

$$y(t) = x[t+h(t)] + n[t+h(t)] . \tag{6.9}$$

The details of the computation of the mean square error $\overline{e_f^2}$ are carried out in Appendix D, but the results are repeated here for convenience.

The optimum filter is again given by

$$H(j\omega) = \frac{S_{xy}(\omega)}{S_{yy}(\omega)},$$

where the cross spectral density $S_{xy}(\omega)$ is given by (6.5). Since n(t) is independent of x(t), the autocorrelation of y(t) can be computed as

$$R_{yy}(\tau) = E_h[R_{xx}(\tau+u) + R_{hh}(\tau+u)]. \tag{6.10}$$

The noise bandwidth is assumed to be the same as the signal bandwidth so the autocorrelation function for the noise can be written

$$R_{nn}(\tau) = \sigma_n^2 \frac{\sin \omega_x\tau}{\omega_x\tau}. \tag{6.11}$$

Taking the Fourier transform gives

$$S_{yy}(\omega) = (\sigma_x^2+\sigma_n^2)\frac{\pi}{\omega_x}[1+ \frac{\sigma_h^2\omega_x^2}{3} - \frac{\sigma_h^2\omega_x}{2}\,\omega - \frac{\sigma_h^2\omega^2}{2} - \frac{\sigma_h^2\omega^3}{6\omega_x}], \tag{6.12}$$

and the mean square error becomes

$$\overline{e_f^2} = \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2} \quad [\sigma_n^2 + \frac{5}{24}\,\sigma_x^2\sigma_h^2\omega_x^2]. \tag{6.13}$$

Assuming that $\sigma_n^2 << \sigma_x^2$, the error can be written as

$$\overline{e_f^2} = \sigma_n^2(1 - \frac{5}{24}\,\sigma_h^2\omega_x^2) + \frac{5}{24}\,\sigma_x^2\sigma_h^2\omega_x^2 \tag{6.14}$$

and, under the assumption that $\sigma_h\omega_x<<1$, this becomes approximately

$$\overline{e_f^2} = \sigma_n^2 + \frac{5}{24}\,\sigma_x^2\sigma_h^2\omega_x^2. \tag{6.15}$$

Therefore, the error due to n(t) is essentially unchanged by either recording or by the optimum linear filter. It simply adds to the error due to the TBE, whether the output is filtered or not. Furthermore, the error due to n(t) is not significant if

$$\sigma_n^2 << \sigma_x^2 \sigma_h^2 \omega_x^2$$

However, if σ_n is large enough so that σ_n^2 is on the same order as $\sigma_x^2 \sigma_h^2 \omega_x^2$, it is of little value to filter the output because (6.13) becomes

$$\overline{e_f^2} \approx \sigma_n^2 + \frac{5}{24} \sigma_x^2 \sigma_h^2 \omega_x^2$$

which is nearly the same as the unfiltered error given by (6.7). If

$$\sigma_n^2 >> \frac{5}{24} \sigma_x^2 \sigma_h^2 \omega_x^2$$

but

$$\sigma_n^2 << \sigma_x^2,$$

the error becomes

$$\overline{e_f^2} \approx \sigma_n^2.$$

In this case, the error due to n(t) is dominant and it is essentially unnecessary to consider the error due to time-base perturbations.

C. Post-Recorder Noise

In many cases, the noise added to the signal after playback is larger than the noise which is present in the signal before recording. In fact, since the signal on the tape is very low level, noise is always introduced in the process of amplifying and integrating the signal at the playback heads. As in the case just considered, it is important to be able to determine when the error due to this noise is significant compared to the error due to the TBE.

In order to establish a criterion for determining when the error added by this noise is significant, consideration will now be given to a system where a noise n(t) is added to the signal after playback. This noise is assumed to be a sample function of a normal random process which is independent of the signal and the TBE. The power spectral density of n(t) will be ideal low pass with bandwidth ω_n and the power spectra for x(t) and h(t) will be taken as ideal low pass of bandwidth ω_x and ω_h, respectively.

Denoting the composite of the recorder output and the additive noise by $y_1(t)$, the composite can be written as

$$y_1(t) = y(t) + n(t) \tag{6.16}$$

or as

$$y_1(t) = x[t + h(t)] + n(t). \tag{6.17}$$

The mean square error to be computed is

$$\overline{e^2} = E[x(t) - y_1(t)]^2 \tag{6.18}$$

which can be written as

$$\overline{e^2} = E(x^2) - 2E\left\{x(t)[x(t+h) + n(t)]\right\} + E\left\{[x(t+h) + n(t)]^2\right\}, \tag{6.19}$$

or

$$\overline{e^2} = 2\sigma_x^2 + \sigma_n^2 - 2R_{xy}(0), \tag{6.20}$$

where again, as in (4.11),

$$R_{xy}(\tau) = E_h\left\{R_{xx}(\tau-h)\right\}. \tag{6.21}$$

Using the same procedure as in (4.12) through (4.23), (6.20) becomes

$$\overline{e^2} = \sigma_n^2 + 1/3\ \sigma_x^2\sigma_h^2\omega_x^2. \tag{6.22}$$

So that once again, for the case where n(t) is independent of x(t) and h(t), the mean square error contributed by the noise added after playback is simply the noise power; the total mean square error is the noise power plus the previously computed value of the mean square error due to the TBE.

In order to find the improvement possible by the use of the optimum linear filter, it is necessary to find the filter which will minimize the quantity

$$\overline{e_f^2} = E[x(t) - z(t)]^2 \tag{6.23}$$

where z(t) is the output of a filter whose input is

$$y_1(t) = x[t+h(t)] + n(t). \tag{6.24}$$

The details are shown in Appendix E and the results are repeated here for convenience.

The cross correlation $R_{xy}(\tau)$ is

$$R_{xy}(\tau) = E_h[R_{xx}(\tau-h)], \tag{6.25}$$

and the autocorrelation $R_{yy}(\tau)$ is

$$R_{yy}(\tau) = E_h[R_{xx}(\tau+u) + R_{nn}(\tau)]. \tag{6.26}$$

Carrying out this computation for $\omega_x = \omega_h$ gives

$$\overline{e_f^2} = \sigma_n^2(1 - \frac{5}{8}\frac{\sigma_h^2\omega_x^2}{3}) + \frac{5}{24}\sigma_x^2\sigma_h^2\omega_x^2 \tag{6.27}$$

for $\omega_n<\omega_x$, and

$$\overline{e_f^2} = \frac{\omega_x}{\omega_n}\sigma_n^2 \quad (1 - \frac{5}{8}\frac{\sigma_h^2\omega_x^2}{3}) + \frac{5}{24}\sigma_x^2\sigma_h^2\omega_x^2 \tag{6.28}$$

for $\omega_n \geq \omega_x$. For $\omega_h > 2\omega_x$

$$\overline{e_f^2} = \sigma_n^2 \quad (1 - \frac{\sigma_h^2\omega_x^2}{3}\frac{\omega_x}{\omega_h}) + \frac{\sigma_h^2\sigma_x^2\omega_x^2}{3}\frac{\omega_x}{\omega_h} \tag{6.29}$$

when $\omega_n<\omega_x$ and

$$\overline{e_f^2} = \frac{\omega_x}{\omega_n}\sigma_n^2 \quad (1 - \frac{\sigma_h^2\omega_x^2}{3}\frac{\omega_x}{\omega_h}) + \frac{\sigma_h^2\sigma_x^2\omega_x^2}{3}\frac{\omega_x}{\omega_h} \tag{6.30}$$

when $\omega_n>\omega_x$. But for $\omega_x = \omega_h$, these are approximately

$$\overline{e_f^2} = \sigma_n^2 + \frac{5}{24}\sigma_x^2\sigma_h^2\omega_x^2 \tag{6.31}$$

for $\omega_n < \omega_x$, and

$$\overline{e_f^2} = \frac{\omega_x}{\omega_n} \sigma_n^2 + \frac{5}{24} \sigma_x^2 \sigma_h^2 \omega_x^2 \tag{6.32}$$

for $\omega_n > \omega_x$. For $\omega_h > 2\omega_x$, the approximations are

$$\overline{e_f^2} = \sigma_n^2 + \frac{\sigma_h^2 \sigma_x^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h} \tag{6.33}$$

for $\omega_n < \omega_x$, and

$$\overline{e_f^2} = \frac{\omega_x}{\omega_n} \sigma_n^2 + \frac{\sigma_h^2 \sigma_x^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h} \tag{6.34}$$

for $\omega_n > \omega_x$.

Then to a first approximation, the additional error caused by the noise which is added after playback is simply the noise power in n(t). To determine the most significant error, comparisons should be made between the relative values of the first and second terms of equations (6.31) through (6.34). [The first term of each of the equations represents the error due to the noise and the second term is the error due to TBE.] In order to insure that no appreciable error is added by noise which is introduced after playback, the noise power σ_n^2 must be much less than $\sigma_x^2 \sigma_h^2 \omega_x^2$. Under this condition, filtering the output signal gives the error shown by equations (4.59), (4.60), and (4.61). If, on the other hand, σ_n^2 is on the same order as $\sigma_x^2\sigma_h^2\omega_x^2$, equations (6.31), (6.32), (6.33), and (6.34) should be used. Finally, if $\sigma_n^2 >> \sigma_x^2\sigma_h^2\omega_x^2$ then

$$\overline{e_f^2} \approx \frac{\omega_n}{\omega_x}\,\sigma_n^2 \quad \text{for } \omega_n < \omega_x, \tag{6.35}$$

and

$$\overline{e_f^2} \approx \sigma_n^2 \qquad \text{for } \omega_n > \omega_x. \tag{6.36}$$

D. Pre-record Plus Post-record Noise

An analysis for a system where both pre-recorder noise $n_1(t)$ and post-recorder noise $n_2(t)$ are present was also carried out. The details are somewhat involved, but, to a first approximation, the total mean square error is the sum of the noise powers and the error due to the time-base error. This is assuming that the noises $n_1(t)$ and $n_2(t)$ are independent of each other and of the signal and TBE. In order to determine the most significant contribution to the total mean square error for this case then, the comparison should be among $\sigma_{n_1}^2$, $\sigma_{n_2}^2$, and $\sigma_x^2\sigma_h^2\omega_x^2$.

CHAPTER VII

PRACTICAL FILTERING

A. General Considerations

It has been demonstrated that some improvement in the mean square error is possible by the use of the optimum linear filter, especially in the case where the signal bandwidth is small compared to the noise bandwidth. However, no consideration has been given to this point as to the improvement which is possible with physically realizable filters. Although it is possible to realize the optimum filter within arbitrary accuracy by allowing a time delay, it would be advantageous to be able to compute the improvement in the mean square error by the use of simple filters that are easily realizable. If, for example, a simple filter gives almost as much improvement as the optimum filter, then there is little use in trying to synthesize a complex filter. It is also desirable to be able to predict the effects on the mean square error which are produced by the channel through which the combined signal and noise output of the recorder pass. Since it has already been shown that it is not possible to achieve much improvement, even using the optimum filter, for the case where $\omega_h<\omega_x$, the cases to be considered here will be confined to the region $\omega_x<\omega_h$ where significant improvement was possible by the use of the optimum filter.

In order to be able to compute the error at the output of a filter which is not the optimum linear filter, it is necessary to use a more general equation than the one used for computing the error when the optimum filter was employed. This general equation will now be found.

Considering the output of the general filter to be $z_1(t)$, the mean square error between the signal and the output of the filter is

$$\overline{e_{f1}^2} = E \left\{ [x(t) - z_1(t)]^2 \right\} , \tag{7.1}$$

where $\overline{e_{f1}^2}$ is used to denote the mean square error when a filter other than the optimum linear filter is used. This can be written as

$$\overline{e_{f1}^2} = E[x^2(t)] - 2E\ [x(t)y(t)] + E[z_1^2(t)] \tag{7.2}$$

or, making the definition

$$e_1(t) = x(t) - z_1(t) , \tag{7.3}$$

one can write the autocorrelation function $R_{ee}(\tau)$ of e_1 as

$$R_{ee}(\tau) = E\Big[[x(t+\tau)x(t)]-x(t+\tau)z_1(t)-z_1(t+\tau)x(t)+z_1(t+\tau)z_1(t)\Big]. \tag{7.4}$$

The mean square error can then be written as

$$\overline{e_{f1}^2} = R_{ee}(0), \tag{7.5}$$

or as

$$\overline{e_{f1}^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ee}(\omega)\, d\omega, \tag{7.6}$$

where $S_{ee}(\omega)$ is the Fourier transform of $R_{ee}(\tau)$. From equation (7.4), $S_{ee}(\omega)$ can be found as

$$S_{ee}(\omega) = S_{xx}(\omega) - S_{xz}(\omega) - S_{zx}(\omega) + S_{zz}(\omega). \tag{7.7}$$

Now

$$S_{xz}(\omega) = S_{xy}(\omega)\ H_1(\omega) \tag{7.8}$$

and

$$S_{zx}(\omega) = S_{xy}(\omega)\ H_1(-\omega) \tag{7.9}$$

where $H_1(j\omega)$ is the transfer function of the filter which is connected to the recorder. Then equation (7.6) can be written as

$$\overline{e_{f1}^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{xx}(\omega) + S_{yy}(\omega)\,|H_1(\omega)|^2 - S_{xy}(\omega)\,[H_1(\omega)+H_1(-\omega)]\right\}d\omega \tag{7.10}$$

which is the general equation for the mean square error when a general (not necessarily optimum) filter is used.

B. Low Pass R-C Filter

If one returns to the case where both signal and TBE spectra are ideal low pass, it is found that the transfer characteristic for the optimum filter is low pass in form. Consideration will now be given to a practical low pass filter in order to determine the usefulness of such a filter. For simplicity, the filter will be considered to be composed of a resistor and a capacitor and to have a transfer characteristic of the form $\frac{\alpha}{\alpha+j\omega}$ where $\alpha = \frac{1}{RC}$. Now for the case where $\omega_x = \omega_h$, the previously calculated equations (see Appendix A) for $S_{yy}(\omega)$ and $S_{xy}(\omega)$ can be used to compute the error $\overline{e_{f1}^2}$ as

$$\begin{aligned}\overline{e_{f1}^2} = {} & \sigma_x^2 - \frac{\sigma_x^2}{\omega_x}\,\alpha\,\mathrm{Tan}^{-1}\left(\frac{\omega_x}{\alpha}\right) + \frac{2}{3}\,\sigma_x^2\sigma_h^2\alpha^2 \\ & + \sigma_x^2\sigma_h^2\alpha\omega_x\,\mathrm{Tan}^{-1}\left(\frac{2\omega_x}{\alpha}\right)\left[\frac{1}{3} - \frac{\alpha^2}{2\omega_x^2}\right] \\ & - \frac{\sigma_x^2\sigma_h^2\alpha^2}{4}\,\ln\left[\frac{\alpha^2+4\omega_x^2}{\alpha^2}\right]\left[1 - \frac{\alpha^2}{3\omega_x^2}\right].\end{aligned} \tag{7.11}$$

The complexity of this equation obscures the exact results but the following cases are of interest. For $\alpha >> \omega_x = \omega_h$ the error reduces to

$$\overline{e^2_{f1}} = \frac{1}{3} \sigma_x^2 \sigma_h^2 \omega_x^2 \tag{7.12}$$

which is, of course, the error for no filter. In this case the bandpass of the filter is much greater than the signal and noise bandwidths and, therefore, has essentially no effect. For $\alpha = \omega_x$ the error becomes

$$\overline{e^2_{f1}} = .215\ \sigma_x^2 + .241\ \sigma_x^2 \sigma_h^2 \omega_x^2, \tag{7.13}$$

and for $\alpha \to 0$ the error is

$$\overline{e^2_{f1}} = \sigma_x^2. \tag{7.14}$$

In fact, the general form of the error is

$$\overline{e^2_{f1}} = K_1 \sigma_x^2 + K_2 \sigma_x^2 \sigma_h^2 \omega_x^2 \tag{7.15}$$

where K_1 and K_2 are dependent on the value of α. As α is increased from zero to infinity, K_1 decreases from one to zero, while K_2 increases from zero to one-third.

This then indicates that for $\omega_x = \omega_h$, the R-C filter is of no value in reducing the error. Indeed, the best it can do is leave the error unchanged from the case of no filter.

Turning consideration to the case where $\omega_x \ll \omega_h$ and performing the same calculations leads to

$$\overline{e^2_{f1}} = \sigma^2_x [1 - \frac{\alpha}{\omega_x} \ \mathrm{Tan}^{-1}(\frac{\omega_x}{\alpha}) + \frac{1}{3} \ \frac{\sigma^2_h \omega^2_x}{\omega_h} \alpha \ \mathrm{Tan}^{-1}(\frac{\omega_h - \omega_x}{\alpha})]. \tag{7.16}$$

Again, the complexity of the equation makes it difficult to assess the behavior of the filter, but it is instructive to consider the case where $\omega_x = \alpha \ll \omega_h$. For this situation, one might expect a reduction in the error since the signal is within the pass region of the filter, but much of the noise is in the stop band. The error for this case is

$$\overline{e^2_{f1}} = \sigma^2_x \ (1 - \frac{\pi}{4}) + \frac{1}{3} \sigma^2_h \omega^2_x \frac{\omega_x}{\omega_h} \ \mathrm{Tan}^{-1}(\frac{\omega_h - \omega_x}{\alpha}), \tag{7.17}$$

or

$$\overline{e^2_{f1}} < \sigma^2_x \ (1 - \frac{\pi}{4}) + \frac{1}{3} \ \sigma^2_h \omega^2_x \ \frac{\omega_x}{\omega_h} \ \frac{\pi}{2}, \tag{7.18}$$

which is quite large if σ^2_x is large. If one considers the case where $\omega_x \ll \alpha \ll \omega_h$ which places the signal well within the pass band but still keeps much of the TBE spectrum in the stop band, it turns out that

$$\overline{e_{f1}^2} < \frac{\sigma_x^2 \omega_x^2}{3\alpha^2} + \frac{1}{3} \sigma_h^2 \sigma_x^2 \omega_x^2 \frac{\alpha}{\omega_h} \frac{\pi}{2} \quad (7.19)$$

This may be less than the unfiltered error but it is not a significant improvement at best because the first term still contains σ_x^2. For example, for the case where $\alpha = 10\omega_x$ and $\omega_h = 10\alpha$, (7.19) becomes

$$\overline{e_{f1}^2} < \frac{\sigma_x^2}{30} + \sigma_x^2 \sigma_h^2 \omega_x^2 \frac{\pi}{30}.$$

Evidently, then, the simple R-C filter is of little benefit in reducing the mean square error due to TBE. One might consider more complex filters whose characteristics more closely approximate the optimum filter, however, there are unlimited possibilities as far as specific filters are concerned and it is probably more instructive to consider a general type of transfer characteristic which is useful in the filtering process.

C. Bandpass Filters

As a first step, suppose that the error is calculated for the case of low pass signal and TBE spectra with an ideal low pass filter. That is, suppose

$$H_1(j\omega)=\begin{cases}1 & |\omega|<\omega_x \\ 0 & |\omega|>\omega_x\end{cases} \quad . \tag{7.20}$$

Carrying out the calculations using (7.10) gives

$$\overline{e_{f1}^2} = \frac{5}{24}\, \sigma_x^2\sigma_h^2\omega_x^2 \tag{7.21}$$

for $\omega_x = \omega_h$, and

$$\overline{e_{f1}^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3}\, \frac{\omega_x}{\omega_h} \tag{7.22}$$

for $\omega_x \ll \omega_h$. Since these values are exactly the same as those calculated for the optimum filter, one might at first suspect an error. However, it should be remembered that in all of the calculations, only the most significant terms have been retained. For an exact relationship, an infinite number of higher order terms of the form $C_3\sigma_x^2\sigma_h^3\omega_x^3$, $C_4\sigma_x^2\sigma_h^4\omega_x^4$, etc. (where C's are constants of decreasing size) would have to be included. Then, to the approximations of the calculations, the ideal low pass characteristic just considered is as good as the optimum filter. Actually, the difference between the error using the optimum filter and the error using the ideal low pass filter should be on the order of some fraction of $\sigma_x^2\sigma_h^3\omega_x^3$ (where $\omega_x\sigma_h \ll 1$).

Since the characteristic of the ideal low pass filter just considered cannot be realized exactly any more than the optimum filter can, one might ask just how sensitive the error is to deviations in the filter characteristics. (Of course some information about this sensitivity is provided by the fact that the optimum filter and the ideal low pass filter just considered have nearly the same error).

In order to better determine this sensitivity, consideration will now be given to an ideal filter with a transfer characteristic as shown in Figure 7. For this case, $H(j\omega)$ is given by

$$H(j\omega) = \begin{cases} \dfrac{\omega_1+\omega}{\omega_1-\omega_x} & -\omega_1<\omega<-\omega_x \\ 1 & |\omega|<\omega_x \\ \dfrac{\omega_1-\omega}{\omega_1-\omega_x} & \omega_x<\omega<\omega_1 \\ 0 & |\omega|>\omega_1 \end{cases} \quad . \qquad (7.23)$$

The use of the equations (for low pass signal and TBE spectra) for $S_{xx}(\omega)$, $S_{xy}(\omega)$, and $S_{yy}(\omega)$ from Appendix A along with equation (7.10) gives

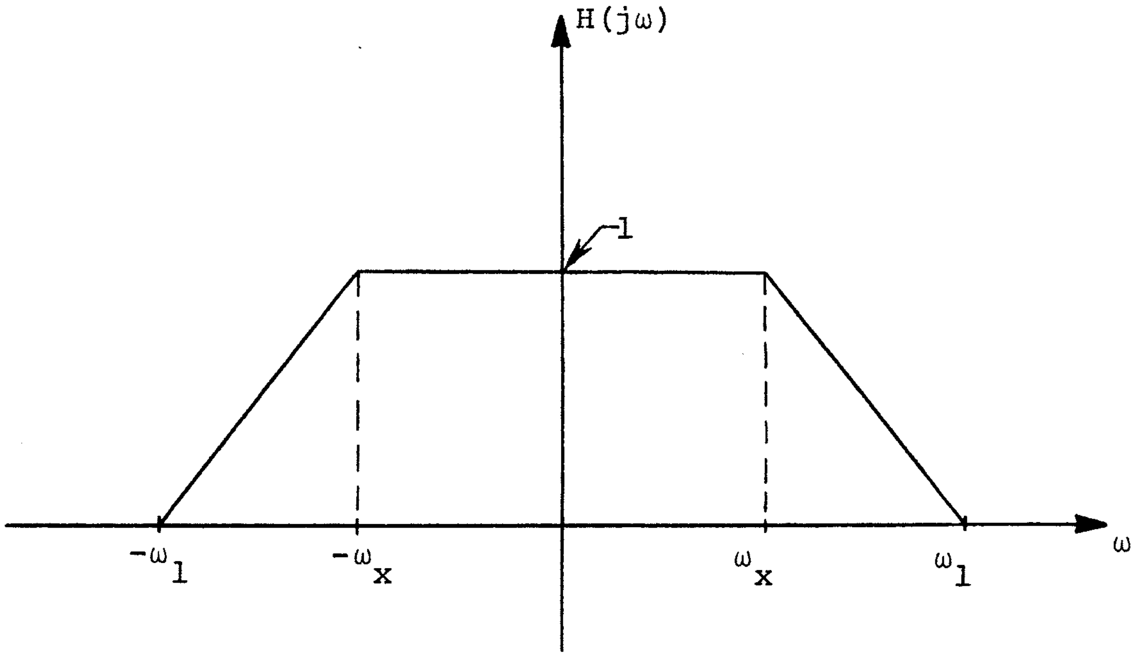


Figure 7. Sub-optimum ideal filter.

$$\overline{e_{f1}^2} = \frac{5}{24} \quad \sigma_x^2\sigma_h^2\omega_x^2$$

$$- \frac{\sigma_x^2\sigma_h^2}{\omega_x} \quad \left(\frac{\omega_1}{\omega_1-\omega_x}\right)\left[\frac{\omega_1^4-\omega_x^4}{24\omega_x} - \frac{\omega_1^3-\omega_x^3}{6} + \frac{\omega_x(\omega_1^2-\omega_x^2)}{4} - \frac{\omega_x^2(\omega_1-\omega_x)}{3}\right]$$

$$+ \frac{\sigma_x^2\sigma_n^2}{\omega_x} \quad \left(\frac{1}{\omega_1-\omega_x}\right)\left[\frac{\omega_1^5-\omega_x^5}{30\omega_x} - \frac{\omega_1^4-\omega_x^4}{8} + \frac{\omega \quad \omega_x(\omega_1^3-\omega_x^3)}{6} - \frac{\omega_x^2(\omega_1^2-\omega_x^2)}{6}\right]$$

for $\omega_x=\omega_h$. Writing $\omega_1=k\omega_x$ gives

$$\overline{e_{f1}^2} = \frac{5}{24} \quad \sigma_x^2\sigma_h^2\omega_x^2$$

$$+ \frac{\sigma_x^2\sigma_h^2\omega_x^2}{120} \quad \left[-k^4+4k^3-6k^2+14k-11\right] \quad . \tag{7.25}$$

Performing the same calculations for $\omega_x \ll \omega_h$ (which is the region where the most improvement is possible by the use of the optimum filter) gives

$$\overline{e_{f1}^2} = \frac{\omega_x}{\omega_x} \quad \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3} \tag{7.26}$$

$$+ \quad \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3\omega_h} \quad \left(\frac{1}{\omega_1-\omega_x}\right)\left[\omega_1(\omega_1-\omega_x) - \frac{1}{2}\ (\omega_1^2-\omega_x^2)\right].$$

Using $\omega_1=k\omega_x$,

$$\overline{e_{f1}^2} = \frac{\omega_x}{\omega_h}\,\frac{\sigma_x^2\sigma_h^2\omega_x^2}{3} + \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3} \quad \frac{\omega_x}{\omega_h} \quad \left(\frac{k-1}{2}\right). \tag{7.27}$$

Thus, the form of the error in either case (i.e., either $\omega_x = \omega_h$ or $\omega_x << \omega_h$) is of the form of the error for the optimum filter plus an additional term which is dependent on ω_1. It should be noted that the increase in error with increase in ω_1 is not too rapid. In fact, even for $\omega_1 = 2\omega_x$, the error for $\omega_x = \omega_h$ is

$$\overline{e_{f1}^2} = \frac{5}{24} \sigma_x^2 \sigma_h^2 \omega_x^2 + \frac{3}{40} \sigma_x^2 \sigma_h^2 \omega_x^2 \tag{7.28}$$

and the error for $\omega_x << \omega_h$ is

$$\overline{e_{f1}^2} = \frac{\omega_x}{\omega_h} \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} + \frac{1}{2} \frac{\omega_x}{\omega_h} \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3}, \tag{7.29}$$

where the second term in each of these equations is the additional error caused by the non-zero filter characteristic between ω_x and ω_1. Therefore, any filter with a transfer characteristic which approximates that shown in Figure 7 might be expected to give mean square error approximating that given by the above equations. Although it is not possible to realize physical filters which exactly duplicate the characteristics in Figure 7, it is not difficult to synthesize filters which approximate this characteristic closely.

Actually, there would be no point in connecting a filter to the output of the recorder if the channel through which the signal passes after playback has a sharp cutoff approximately at ω_x. In this case, the channel itself would act as a filter and, in fact, would be nearly as good as the optimum filter. However, if the channel were broad band, the mean square error could be reduced considerably (for $\omega_x < \omega_h$) by connecting a sharp cutoff filter to the output of the recorder.

CHAPTER VIII

CONCLUSIONS

A. TBE Effects

The mean square error between recorded and reproduced signals is found to be a convenient measure of the error introduced into a signal by time-base perturbations. For low pass signals, this mean square error is proportional to the product of the square of the signal bandwidth and the TBE variance $(\omega_x^2\sigma_h^2)$. For band pass signals, the error has an additional term which increases as the product of the square of the center frequency of the signal and the TBE variance. When the signal is narrow band with a large center frequency, the error is approximately proportional to the product of the square of the signal center frequency and the TBE variance $(\omega_o^2\sigma_h^2)$. The strong dependence of the mean square error on the maximum frequency components present in the signal spectrum shows an urgent need for the reduction of σ_h to a minimum value, especially for wideband recording applications. Since low frequency components of TBE make a rather large contribution to σ_h if they are not removed, it is essential that servos be used to reduce the low frequency components of TBE to as small a value as possible. For a given value of σ_h, it is found that the form of the spectral density of the TBE does not affect the mean square error (assuming that no filter is used). The form of the spectral density of the signal has some effect on the error, however, this effect is found to be rather small as long as there is a well-defined band where the signal power is concentrated.

B. Optimum Filtering

Use of the optimum linear filter can give significant improvement for the case where the signal bandwidth is small compared to the TBE bandwidth. The optimum filter normally gives little improvement when the signal bandwidth is comparable to or larger than the TBE bandwidth. It is true that theoretically the error can be reduced by a factor of 2 by using the optimum filter with narrow band (compared to the signal) TBE if the TBE spectrum lies at the upper end (in frequency) of the signal spectrum. Unfortunately, this improvement decreases rapidly with increases in the TBE bandwidth and/or with downward (in frequency) movement of the center frequency of the TBE spectrum. Therefore, in practice, linear filtering could be expected to give little improvement in the mean square error for any situation where the signal bandwidth is greater than the TBE bandwidth (unless the TBE band lies completely above the frequency band of the signal). The use of servos for removal of the low frequency components of TBE help make this the usual situation encountered in practice.

C. General Analysis

In cases where the signal and the TBE spectra cannot be reasonably represented by ideal band pass spectral models, it is possible to compute the error by representing the signal and the TBE spectra as sums of band pass spectra. This leads to sums of terms of the same form as the ideal band pass models, making the analysis very long and complicated. Also, the form of the resulting equations could be expected to be so complicated as to obscure the results. It is, therefore, probable that, when the spectra are complex enough to require many terms to represent them accurately, it would be better to use a computer to carry out the details of the calculations.

D. External Noise

The calculations of Chapter VI show that the effect of noise present in the signal before recording and/or noise added to the signal after playback is to add the noise powers to the mean square error caused by the time base perturbations. For small external noise, this is approximately true whether the optimum filter is used or not since the optimum filter (which is designed for the reduction of both the error due to TBE and the error due to external noise) has little effect on the external noise for the case where $\sigma_n^2 << \sigma_x^2$ and $\omega_x \sigma_h << 1$. To determine the most significant contribution to the mean square error then, it is only necessary to compare the noise powers (both pre-recorder and post-recorder) to the mean square error caused by the time-base perturbations.

E. Practical Filtering.

Simple physical filters are not generally effective in reducing the mean square error because they do not usually have sharp cutoff frequencies. However, more complex filters which are synthesized to approximate the sharp cutoff of the optimum filter can give a reduction in the mean square error which approaches that of the optimum filter. This is possible because the mean square error is not very sensitive to changes in the filter characteristics as long as the filter has a rather sharp cutoff frequency. The actual synthesis of a filter is not necessarily required since the signal channel itself can serve as a good approximation to the optimum filter if it has a transfer function with a sharp cutoff.

VITA

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APPENDIX A

CALCULATION OF $S_{yy}(\omega)$ AND $\overline{e_f^2}$ FOR LOW PASS SIGNAL AND TIME-BASE ERROR

A. Calculation of $S_{yy}(\omega)$

The autocorrelation function $R_{yy}(\tau)$ is given by equation (4.41) as

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \sigma_x^2 \frac{\sin[\omega_x(u+\tau)]}{\omega_x(u+\tau)} \frac{e^{-\frac{u^2}{2\sigma_u^2}}}{\sqrt{2\pi\sigma_u^2}} du \qquad (A1)$$

where

$$\sigma_u^2 = 2\,\sigma_h^2\,[1 - \frac{\sin\,\omega_h\tau}{\omega_h\tau}]. \qquad (A2)$$

Letting $z = \frac{u}{\sigma_u}$ gives

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \sigma_x^2 \frac{\sin[\omega_x(\sigma_u z+\tau)]}{\omega_x(\sigma_u z+\tau)} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz. \qquad (A3)$$

From Fourier transform theory

$$\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\beta^2}{2}} e^{j\beta z} d\beta \tag{A4}$$

so that

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\beta^2}{2}} \int_{-\infty}^{\infty} \frac{\sin[\omega_x(\sigma_u z+\tau)]}{\omega_x(\sigma_u z+\tau)} e^{j\beta z} dz d\beta. \tag{A5}$$

Defining $\sigma_u z+\tau = \gamma$,

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\frac{\beta^2}{2}}}{\sigma_u} e^{-j\beta\frac{\tau}{\sigma_u}} \int_{-\infty}^{\infty} \frac{\sin \omega_x\gamma}{\omega_x\gamma} e^{j\beta\frac{\gamma}{\sigma_u}} d\gamma d\beta. \tag{A6}$$

Recognizing the integration with respect to γ as a Fourier transform gives

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\frac{\beta^2}{2}}}{\sigma_u} e^{-j\beta\frac{\tau}{\sigma_u}} \frac{\pi}{\omega_x} P_{\omega_x}\left(-\frac{\beta}{\sigma_u}\right) d\beta \tag{A7}$$

where $P_{\omega_x}(\cdot)$ is defined by (4.2).

Then

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\omega_x\sigma_u}\int_{-\omega_x\sigma_u}^{\omega_x\sigma_u} e^{-\frac{\beta^2}{2}} e^{-j\beta\frac{\tau}{\sigma_u}} \, d\beta \tag{A8}$$

or

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\omega_x\sigma_u}(2)\int_{o}^{\omega_x\sigma_u} e^{-\frac{\beta^2}{2}} \cos\left(\beta\frac{\tau}{\sigma_u}\right) d\beta \,. \tag{A9}$$

This can be integrated by parts to give

$$R_{yy}(\tau) = \frac{\sigma_x^2}{\omega_x\sigma_u}\left\{ \sum_{n=0}^{\infty} \left[\frac{d^{2n}}{d\beta^{2n}} e^{-\frac{\beta^2}{2}}\right]\left(\frac{\sigma_u}{\tau}\right)^{2n+1} (-1)^n \right.$$

$$\cdot \sin\left(\beta\frac{\tau}{\sigma_u}\right)\Bigg|_{o}^{\omega_x\sigma_u} + \sum_{n=0}^{\infty}\left[\frac{d^{2n+1}}{d\beta^{2n+1}} e^{-\frac{\beta^2}{2}}\right]\left(\frac{\sigma_u}{\tau}\right)^{2n+2} (-1)^n$$

$$\left. \cdot \cos\left(\beta\frac{\tau}{\sigma_u}\right)\Bigg|_{o}^{\omega_x\sigma_u} \right\} \,. \tag{A10}$$

By taking the definition of Hermite polynomials to be

$$H_k(\beta) = e^{\frac{\beta^2}{2}} (-1)^k \frac{d^k}{d\beta^k} e^{-\frac{\beta^2}{2}} ,$$

$R_{yy}(\tau)$ can be written as

$$R_{yy}(\tau) = \frac{\sigma_x^2}{\omega_x \sigma_u} \left\{ e^{-\frac{\omega_x^2 \sigma_u^2}{2}} \sum_{n=0}^{\infty} (-1)^n H_{2n}(\omega_x \sigma_u) \left(\frac{\sigma_u}{\tau}\right)^{2n+1} \sin \omega_x \tau \right.$$

$$\left. + e^{-\frac{\omega_x^2 \sigma_u^2}{2}} \sum_{n=0}^{\infty} (-1)^n H_{2n+1}(\omega_x \sigma_u) \left(\frac{\sigma_u}{\tau}\right)^{2n+2} \cos \omega_x \tau \right\}. \quad \text{(A11)}$$

For the case $\omega_x \sigma_u << 1$,

$$H_{2n}(\omega_x \sigma_u) \approx (-1)^n \quad (2n+1)!!$$

where $$(2n+1)!! = 1 \cdot 3 \cdot 5 \ . \ . \ . \ (2n+1).$$

Also,

$$H_{2n+1}(\omega_x \sigma_u) \approx (-1)^{n+1} \omega_x \sigma_u \ (2n+1)!!$$

and

$$e^{-\frac{\omega_x^2 \sigma_u^2}{2}} \approx 1 - \frac{\omega_x^2 \sigma_u^2}{2} .$$

Using these approximations

$$R_{yy}(\tau) = \frac{\sigma_x^2}{\omega_x \sigma_u} \left(1 - \frac{\omega_x^2 \sigma_u^2}{2}\right)\left(\sum_{n=0}^{\infty} (2n+1)!!\, \left(\frac{\sigma_u}{\tau}\right)^{2n+1} \sin \omega_x \tau \right.$$

$$\left. + \sum_{n=0}^{\infty} (2n+1)!!\; \omega_x \sigma_u \left(\frac{\sigma_u}{\tau}\right)^{2n+2} \cos \omega_x \tau \right). \qquad \text{(A12)}$$

Since

$$\sigma_u^{2n} = 2^n\, \sigma_h^{2n} \left[1 - \frac{\sin \omega_h \tau}{\omega_h \tau}\right]^n \leq 2^n \sigma_h^{2n}$$

where σ_h is on the order of 10^{-6} or 10^{-7}, the terms containing σ_u^{2n} can be neglected for $n \geq 2$. Using this approximation

$$R_{yy}(\tau) = \sigma_x^2 \left\{ \frac{\sin \omega_x\tau}{\omega_x\tau} \left[1 - \frac{(\omega_x\sigma_u)^2}{2} + \left(\frac{\sigma_u}{\tau}^2\right)\right] - \left(\frac{\sigma_u}{\tau}\right)^2 \cos \omega_x\tau \right\}. \tag{A13}$$

The Fourier transform of a product $g(\tau)\ h(\tau)$ can be written

$$F[g(\tau)h(\tau)] = \frac{1}{2\pi}[G(\omega) * H(\omega)]$$

where F indicates the Fourier transform,

$$G(\omega) = F[g(\tau)],$$

$$H(\omega) = F[h(\tau)],$$

and $*$ denotes convolution.

The convolution process can be used repeatedly to find $S_{yy}(\omega)$. Starting with

$$F[\sigma_u^2] = F[2\sigma_h^2\ (1 - \frac{\sin \omega_h\tau}{\omega_h\tau})] = 2\sigma_h^2[2\pi\delta(\omega) - \frac{\pi}{\omega_h} P_{\omega_h}(\omega)], \tag{A14}$$

the Fourier transform of $\frac{\sigma_u^2}{\tau}$ can be written

$$F[\frac{\sigma_u^2}{\tau}] = \frac{1}{2\pi} [F\ (\sigma_u^2) * F\ (\frac{1}{\tau})]. \tag{A15}$$

Using

$$F[\frac{1}{\tau}] = -j\pi \mathrm{sgn}\ \omega \tag{A16}$$

where $\mathrm{sgn}\ \omega = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$,

gives

$$F(\frac{\sigma_u^2}{\tau}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [-j\pi \mathrm{sgn}(\omega-\beta)] 2\sigma_h^2\ [2\pi\delta(\beta) - \frac{\pi}{\omega_h} P_{\omega_h}(\beta)]\ d\beta. \tag{A17}$$

Integrating

$$F(\frac{\sigma_u^2}{\tau}) = \begin{cases} 0 & \omega < -\omega_h \\ j2\pi\sigma_h^2\ (\frac{\omega}{\omega_h} + 1) & -\omega_h < \omega < 0 \\ j2\pi\sigma_h^2\ (\frac{\omega}{\omega_h} - 1) & 0 < \omega < \omega_h \\ 0 & \omega_h < \omega \end{cases} \tag{A18}$$

In a similar manner, the Fourier transform of $\left(\frac{\sigma_u}{\tau}\right)^2$ can be computed to be

$$F\left[\left(\frac{\sigma_u}{\tau}\right)^2\right] = \begin{cases} 0 & \omega<-\omega_h \\ \frac{\pi\ \sigma_h^2}{\omega_h}\ (\omega+\omega_h)^2 & -\omega_h<\omega<0 \\ \frac{\pi\ \sigma_h^2}{\omega_h}\ (\omega-\omega_h)^2 & 0<\omega<\omega_h \\ 0 & \omega_h<\omega. \end{cases} \tag{A19}$$

Using

$$F\left[\frac{\sin\ \omega_x\tau}{\omega_x\tau}\right] = \frac{\pi}{\omega_x}\ P_{\omega_x}(\omega) \tag{A20}$$

with equation (A14) gives

$$F\left[\sigma_u^2\ \frac{\sin\ \omega_x\tau}{\omega_x\tau}\right] = \frac{1}{2\pi}\ \left[F(\sigma_u^2)\ *\ F\left(\frac{\sin\ \omega_x\tau}{\omega_x\tau}\right)\right].$$

Carrying out this convolution

$$F[\sigma_u^2 \frac{\sin \omega_x \tau}{\omega_x \tau}] = \begin{cases} 0 & \omega<-2\omega_x \\ -\frac{\pi\sigma_h^2}{\omega_x^2}(\omega+2\omega_x) & -2\omega_x<\omega<-\omega_x \\ -\frac{\pi\sigma_h^2}{\omega_x^2}\omega & -\omega_x<\omega<0 \\ \frac{\pi\sigma_h^2}{\omega_x^2}\omega & 0<\omega<\omega_x \\ \frac{\pi\sigma_h^2}{\omega_x^2}(\omega-2\omega_x) & \omega_x<\omega<2\omega_x \\ 0 & 2\omega_x<\omega \end{cases} \tag{A21}$$

for $\omega_x = \omega_h$,

$$
F\left[\sigma_u^2 \frac{\sin \omega_x \tau}{\omega_x \tau}\right] = \begin{cases}
0 & \omega<-(\omega_x+\omega_h) \\[2ex]
-\dfrac{\pi}{\omega_h} \dfrac{\sigma_h^2}{\omega_x} (\omega+\omega_x+\omega_h) & -(\omega_x+\omega_h)<\omega<-\omega_h+\omega_x \\[2ex]
-\dfrac{\pi}{\omega_h} 2\sigma_h^2 & -\omega_h+\omega_x<\omega<-\omega_x \\[2ex]
-\left(\dfrac{\pi}{\omega_h}+\dfrac{\pi}{\omega_x}\right) 2\sigma_h^2 & -\omega_x<\omega<\omega_x \\[2ex]
-\dfrac{\pi}{\omega_h} 2\sigma_h^2 & \omega_x<\omega<\omega_h-\omega_x \\[2ex]
+\dfrac{\pi}{\omega_h} \dfrac{\sigma_h^2}{\omega_x} (\omega-\omega_x-\omega_h) & \omega_h-\omega_x\le\omega<\omega_h+\omega_x \\[2ex]
0 & \omega_h+\omega_x<\omega
\end{cases} \qquad \text{(A22)}
$$

for $\omega_h > 2\omega_x$, and

$$F[\sigma_u^2 \frac{\sin \omega_x\tau}{\omega_x\tau}] = \begin{cases} 0 & \omega<-\omega_h-\omega_x \\ -\frac{\pi\sigma_h^2}{\omega_h\omega_x}(\omega+\omega_x+\omega_h) & -\omega_h-\omega_x<\omega<-\omega_x \\ -\frac{\pi\sigma_h^2}{\omega_h\omega_x}(\omega+\omega_x-\omega_h) & -\omega_x<\omega<-\omega_x+\omega_h \\ 0 & -\omega_x+\omega_h<\omega<\omega_x-\omega_h \\ +\frac{\pi\sigma_h^2}{\omega_h\omega_x}(\omega-\omega_x+\omega_h) & \omega_x-\omega_h<\omega<\omega_x \\ +\frac{\pi\sigma_h^2}{\omega_h\omega_x}(\omega-\omega_x-\omega_h) & \omega_x<\omega<\omega_x+\omega_h \\ 0 & \omega_x+\omega_h<\omega \end{cases} \tag{A23}$$

for $\omega_x>2\omega_h$.

In the same way

$$F[(\frac{\sigma_u}{\tau})^2 (\frac{\sin \omega_x\tau}{\omega_x\tau})] = \frac{1}{2\pi} [F(\frac{\sigma_u}{\tau})^2 * F(\frac{\sin \omega_x\tau}{\omega_x\tau})].$$

This can be computed as

$$F\left[\left(\frac{\sigma_u}{\tau}\right)^2 \frac{\sin \omega_x \tau}{\omega_x \tau}\right] = \begin{cases} 0 & \omega < -2\omega_x \\[2ex] +\dfrac{\pi\sigma_h^2}{6\omega_x^2}(\omega+2\omega_x)^3 & -2\omega_x < \omega < \omega_x \\[2ex] +\dfrac{\pi\sigma_h^2}{6\omega_x^2}\omega^3 + \dfrac{\pi\sigma_h^2\omega_x}{3} & -\omega_x < \omega < 0 \\[2ex] -\dfrac{\pi\sigma_h^2}{6\omega_x^2}\omega^3 + \dfrac{\pi\sigma_h^2\omega_x}{3} & 0 < \omega < \omega_x \\[2ex] -\dfrac{\pi\sigma_h^2}{6\omega_x^2}(\omega-2\omega_x)^3 & \omega_x < \omega < 2\omega_x \\[2ex] 0 & 2\omega_x < \omega \end{cases} \qquad \text{(A24)}$$

for $\omega_x = \omega_h$,

$$F\left[\left(\frac{\sigma_u}{\tau}\right)^2 \frac{\sin \omega_x\tau}{\omega_x\tau}\right] = \begin{cases} 0 & \omega<-\omega_h-\omega_x \\ +\dfrac{\pi\sigma_h^2}{2\omega_x\omega_h}\dfrac{(\omega+\omega_x+\omega_h)^3}{3} & -\omega_h-\omega_x<\omega<-\omega_h+\omega_x \\ +\dfrac{\pi\sigma_h^2}{\omega_h\omega_x}\left[\omega_x\omega^2+2\omega_h\omega_x\omega+\omega_h^2\omega_x+\dfrac{\omega_x^3}{3}\right] & -\omega_h+\omega_x<\omega<-\omega_x \\ +\dfrac{\pi\sigma_h^2}{\omega_h\omega_x}\left[(\omega_x-\omega_h)\omega^2+\omega_h\omega_x(\omega_h-\omega_x)+\dfrac{\omega_x^3}{3}\right] & -\omega_x<\omega<\omega_x \\ +\dfrac{\pi\sigma_h^2}{\omega_h\omega_x}\left[\omega_x\omega^2-2\omega_h\omega_x\omega+\omega_h^2\omega_x+\dfrac{\omega_x^3}{3}\right] & -\omega_x<\omega<\omega_h-\omega_x \\ -\dfrac{\pi\sigma_h^2}{2\omega_x\omega_h}\dfrac{(\omega-\omega_x-\omega_h)^3}{3} & \omega_h-\omega_x<\omega<\omega_h+\omega_x \\ 0 & \omega_h+\omega_x<\omega \end{cases} \quad (A25)$$

for $\omega_h>2\omega_x$, and

$$F\left[\left(\frac{\sigma_u}{\tau}\right)^2 \left(\frac{\sin \omega_x \tau}{\omega_x \tau}\right)\right] = \begin{cases} 0 & \omega < -\omega_h - \omega_x \\ \dfrac{\pi\sigma_h^2}{6\omega_x\omega_h} (\omega+\omega_h+\omega_x)^3 & -\omega_h-\omega_x<\omega<-\omega_x \\ 1/3 \dfrac{\pi}{\omega_x} \sigma_h^2\omega_h^2 + 1/6 \dfrac{\pi\sigma_h^2}{\omega_x\omega_h} (\omega-\omega_h+\omega_x)^3 & -\omega_x<\omega<-\omega_x+\omega_h \\ 1/3 \dfrac{\pi}{\omega_x} \sigma_h^2\omega_h^2 & -\omega_x+\omega_h<\omega<\omega_x-\omega_h \\ 1/3 \dfrac{\pi}{\omega_x} \sigma_h^2\omega_h^2 - 1/6 \dfrac{\pi\sigma_h^2}{\omega_x\omega_h} (\omega+\omega_h-\omega_x)^3 & \omega_x-\omega_h<\omega<\omega_x \\ -\dfrac{\pi\sigma_h^2}{6\omega_x\omega_h} (\omega-\omega_h-\omega_x)^3 & \omega_x<\omega<\omega_h+\omega_x \\ 0 & \omega_h+\omega_x<\omega \end{cases} \tag{A26}$$

for $\omega_x > 2\omega_h$. In the same way

$$
F[(\frac{\sigma_u}{\tau})^2 \cos \omega_x \tau] = \begin{cases} 0 & \omega < -2\omega_x \\ \frac{\pi\sigma_h^2}{2\omega_x} (\omega + 2\omega_x)^2 & -2\omega_x < \omega < -\omega_x \\ \frac{\pi\sigma_h^2}{2\omega_x} \omega^2 & -\omega_x < \omega < \omega_x \\ \frac{\pi\sigma_h^2}{2\omega_x} (\omega - 2\omega_x)^2 & \omega_x < \omega < 2\omega_x \\ 0 & 2\omega_x < \omega \end{cases}
$$

(A27)

for $\omega_x = \omega_h$,

$$
F[(\frac{\sigma_u}{\tau})^2 \cos \omega_x \tau] = \begin{cases} 0 & \omega < -\omega_h - \omega_x \\ \frac{\pi \sigma_h^2}{2\omega_h} (\omega + \omega_h + \omega_x)^2 & -\omega_h - \omega_x < \omega < -\omega_h + \omega_x \\ \frac{\pi \sigma_h^2}{2\omega_h} [(\omega + \omega_h + \omega_x)^2 + (\omega + \omega_h - \omega_x)^2] & -\omega_h + \omega_x < \omega < -\omega_x \\ \frac{\pi \sigma_h^2}{2\omega_h} [(\omega + \omega_h - \omega_x)^2 + (\omega - \omega_h + \omega_x)^2] & -\omega_x < \omega < \omega_x \\ \frac{\pi \sigma_h^2}{2\omega_h} [(\omega - \omega_h + \omega_x)^2 + (\omega - \omega_x - \omega_h)^2] & \omega_x < \omega < \omega_h - \omega_x \\ \frac{\pi \sigma_h^2}{2\omega_h} (\omega - \omega_x - \omega_h)^2 & \omega_h - \omega_x < \omega < \omega_h + \omega_x \\ 0 & \omega_h + \omega_x < \omega \end{cases} \tag{A28}
$$

For $\omega_h > 2\omega_x$, and

$$
F\left[\left(\frac{\sigma_u}{\tau}\right)^2 \cos \omega_x \tau\right] = \begin{cases}
0 & \omega < -\omega_x - \omega_h \\
\dfrac{\pi\sigma_h^2}{2\omega_h} (\omega + \omega_h + \omega_x)^2 & -\omega_h - \omega_x < \omega < -\omega_x \\
\dfrac{\pi\sigma_h^2}{2\omega_h} (\omega + \omega_x - \omega_h)^2 & -\omega_x < \omega < -\omega_x + \omega_h \\
0 & -\omega_x + \omega_h < \omega < \omega_x - \omega_h \\
\dfrac{\pi\sigma_h^2}{2\omega_h} (\omega - \omega_x + \omega_h)^2 & +\omega_x - \omega_h < \omega < \omega_x \\
\dfrac{\pi\sigma_h^2}{2\omega_h} (\omega - \omega_x - \omega_h)^2 & \omega_x < \omega < \omega_x + \omega_h \\
0 & \omega_x + \omega_h < \omega
\end{cases} \qquad \text{(A29)}
$$

for $\omega_x > 2\omega_h$.

Combining these terms to get the Fourier transform of (A13) gives

$$S_{yy}(\omega) = \begin{cases} 0 & \omega < -2\omega_x \\[2ex] \pi\sigma_x^2\sigma_h^2 \left[-\dfrac{(\omega+2\omega_x)^3}{6\omega_x^2} + \dfrac{(\omega+2\omega_x)^2}{2\omega_x} - \dfrac{\omega+2\omega_x}{2} \right] & -2\omega_x < \omega < -\omega_x \\[2ex] \sigma_x^2 \left\{ \pi/\omega_x + \pi\sigma_h^2 \left[\dfrac{\omega^3}{6\omega_x^2} - \dfrac{\omega^2}{2\omega_x} + \dfrac{\omega}{2} + \dfrac{\omega_x}{3} \right] \right\} & -\omega_x < \omega < 0 \\[2ex] \sigma_x^2 \left\{ \pi/\omega_x - \pi\sigma_h^2 \left[\dfrac{\omega^3}{6\omega_x^2} + \dfrac{\omega^2}{2\omega_x} + \dfrac{\omega}{2} - \dfrac{\omega_x}{3} \right] \right\} & 0 < \omega < \omega_x \\[2ex] \pi\sigma_x^2\sigma_h^2 \left[-\dfrac{(\omega-2\omega_x)^3}{6\omega_x^2} - \dfrac{(\omega-2\omega_x)^2}{2\omega_x} - \dfrac{\omega-2\omega_x}{2} \right] & \omega_x < \omega < 2\omega_x \\[2ex] 0 & \omega > 2\omega_x \end{cases} \tag{A30}$$

for $\omega_x = \omega_h$,

$$
S_{yy}(\omega) = \begin{cases}
0 & \omega < -(\omega_x+\omega_h) \\[2ex]
\pi\sigma_x^2\sigma_h^2 \left[\frac{(\omega+\omega_x+\omega_h)^3}{6\omega_h\omega_x} - \frac{(\omega+\omega_h+\omega_x)^2}{2\omega_h} + \frac{\omega_x}{\omega_h}\,\frac{(\omega+\omega_x+\omega_h)}{2}\right] & -(\omega_x+\omega_h) < \omega < -\omega_h+\omega_x \\[2ex]
\pi\sigma_x^2\sigma_h^2 \left[4/3\,\frac{\omega_x^2}{\omega_h} + \omega_h + \frac{\omega^2}{\omega_h} + 2\omega - \frac{(\omega+\omega_h+\omega_x)^2}{2\omega_h} - \frac{(\omega+\omega_h-\omega_x)^2}{2\omega_h}\right] & -\omega_h+\omega_x < \omega < -\omega_x \\[2ex]
\sigma_x^2 \left\{\pi/\omega_x + \pi\sigma_h^2 \left[4/3\,\frac{\omega_x^2}{\omega_h} + \frac{\omega_x-\omega_h}{\omega_x\omega_h}\,\omega^2 + \omega_h - \frac{(\omega+\omega_h-\omega_x)^2}{2\omega_h} - \frac{(\omega-\omega_h+\omega_x)^2}{2\omega_h}\right]\right\} & -\omega_x < \omega < \omega_x \\[2ex]
\pi\sigma_x^2\sigma_h^2 \left[4/3\,\frac{\omega_x^2}{\omega_h} + \omega_h + \frac{\omega^2}{\omega_h} - 2\omega - \frac{(\omega-\omega_h-\omega_x)^2}{2\omega_h} - \frac{(\omega-\omega_h+\omega_x)^2}{2\omega_h}\right] & \omega_x < \omega < \omega_h-\omega_x \\[2ex]
\pi\sigma_x^2\sigma_h^2 \left[-\frac{(\omega-\omega_x-\omega_h)^3}{6\omega_h\omega_x} - \frac{(\omega-\omega_h-\omega_x)^2}{2\omega_h} - \frac{\omega_x}{\omega_h}\,\frac{(\omega-\omega_x-\omega_h)}{2})\right] & \omega_h-\omega_x < \omega < \omega_h+\omega_x \\[2ex]
0 & \omega > \omega_h+\omega_x
\end{cases}
\tag{A31}
$$

for $\omega_h > 2\omega_x$, and

$$
S_{yy}(\omega)=\begin{cases}
0 & \omega<-(\omega_h+\omega_x) \\[2ex]
\pi\sigma_x^2\sigma_h^2\ [\dfrac{(\omega+\omega_h+\omega_x)^3}{6\omega_x\omega_h}-\dfrac{(\omega+\omega_h+\omega_x)^2}{2\omega_h}+\dfrac{(\omega+\omega_h+\omega_x)\,\omega_x}{2\omega_h} & -(\omega_h+\omega_x)<\omega<-\omega_x \\[2ex]
\sigma_x^2\ \dfrac{\pi}{\omega_x}+\pi\sigma_h^2\ [\dfrac{(\omega-\omega_h+\omega_x)^3}{6\omega_x\omega_h}-\dfrac{(\omega-\omega_h+\omega_x)^2}{2\omega_h}+\dfrac{(\omega-\omega_h+\omega_x)\,\omega_x}{2\omega_h}+\dfrac{1}{3}\ \dfrac{\omega_h^2}{\omega_x}] & -\omega_x<\omega<-\omega_x+\omega_h \\[2ex]
\sigma_x^2\ [\dfrac{\pi}{\omega_x}+\dfrac{1}{3}\ \dfrac{\pi}{\omega_x}\ \sigma_h^2\omega_h^2 & -\omega_x+\omega_h<\omega<\omega_x-\omega_h \\[2ex]
\sigma_x^2\ \dfrac{\pi}{\omega_x}+\pi\sigma_h^2[-\dfrac{(\omega+\omega_h-\omega_x)^3}{6\omega_x\omega_h}-\dfrac{(\omega+\omega_h-\omega_x)^2}{2\omega_h}-\dfrac{(\omega+\omega_h-\omega_x)\,\omega_x}{2\omega_h}+\dfrac{1}{3}\ \dfrac{\omega_h^2}{\omega_x}] & \omega_x-\omega_h<\omega<\omega_x \\[2ex]
\pi\sigma_x^2\sigma_h^2\ [-\dfrac{(\omega-\omega_h-\omega_x)^3}{6\omega_x\omega_h}-\dfrac{(\omega-\omega_h-\omega_x)^2}{2\omega_h}-\dfrac{(\omega-\omega_h-\omega_x)\,\omega_x}{2\omega_h} & \omega_x<\omega<\omega_x+\omega_h \\[2ex]
0 & \omega>\omega_h+\omega_x
\end{cases}
\tag{A32}
$$

for $\omega_x>2\omega_h$.

B. Calculation of $\overline{e_f^2}$

From (4.58)

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_o^{\omega_x} \frac{e^{-\sigma_h^2 \omega^2}}{S_{yy}(\omega)} \, d\omega.$$

For $\omega_x \sigma_h <<$, this can be approximated as

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_o^{\omega_x} \frac{1-\sigma_h^2 \omega^2}{S_{yy}(\omega)} \, d\omega. \tag{A33}$$

Equations (A30), (A31), and (A32) can be used to perform the division indicated by the integrand of (A33). Carrying out this division and retaining the most significant terms gives

$$e_f^2 = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_o^{\omega_x} \frac{1}{\sigma_x^2 \frac{\pi}{\omega_x}} \left[1 - \frac{\sigma_h^2 \omega_x^2}{3} + \frac{\sigma_h^2 \omega_x}{2} \omega - \frac{\sigma_h^2 \omega^2}{2} + \frac{\sigma_h^2 \omega^3}{6\omega_x}\right] d\omega \tag{A34}$$

for $\omega_x = \omega_h$,

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_o^{\omega_x} \frac{1}{\sigma_x^2 \frac{\pi}{\omega_x}} [1 - \frac{\sigma_h^2 \omega_x^3}{3\omega_h}] d\omega \tag{A35}$$

for $\omega_h > 2\omega_x$, and

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \left\{ \int_o^{\omega_x - \omega_h} \frac{1}{\sigma_x^2 \frac{\pi}{\omega_x}} [1 - \frac{\sigma_h^2 \omega_h^2}{3} - \sigma_h^2 \omega^2] \; d\omega \right.$$

$$+ \int_{\omega_x - \omega_h}^{\omega_x} \frac{1}{\sigma_x^2 \frac{\pi}{\omega_x}} \left[1 - \frac{\sigma_h^2 \omega_h^2}{3} + \frac{\sigma_h^2}{6\omega_h} (\omega + \omega_h - \omega_x)^3 + \frac{\omega_x^2 \sigma_h^2}{2\omega_h} (\omega + \omega_h - \omega_x)\right.$$

$$\left. \left. + \frac{\sigma_h^2 \omega_x}{2\omega_h} (\omega + \omega_h - \omega_x)^2 - \sigma_h^2 \omega^2 \right] \; d\omega \right\} \tag{A36}$$

for $\omega_x > 2\omega_x$.

Performing these integrations and simplifying leads to

$$\overline{e_f^2} = \frac{5}{24}\,\sigma_x^2\sigma_h^2\omega_x^2 \qquad \text{(A37)}$$

for $\omega_x = \omega_h$,

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3}\;\frac{\omega_x}{\omega_h} \qquad \text{(A38)}$$

for $\omega_h > 2\omega_x$, and

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3} - \frac{\sigma_x^2\sigma_h^2}{24}\;\frac{\omega_h}{\omega_x}\;(6\omega_x^2 - 4\omega_x\omega_h + \omega_h^2) \qquad \text{(A39)}$$

for $\omega_x > 2\omega_h$.

APPENDIX B

CALCULATION OF $S_{yy}(\omega)$ AND $\overline{e_f^2}$ FOR BAND PASS SIGNAL AND LOW PASS TIME-BASE ERROR

A. Calculation of $S_{yy}(\omega)$

From (5.18)

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \frac{\sigma_x^2 \pi}{\omega_x} \frac{\sin[\omega_x(u+\tau)]}{\omega_x(u+\tau)} \cos[\omega_o(u+\tau)] \frac{e^{-\frac{u^2}{2\sigma_u^2}}}{\sqrt{2\pi\sigma_u^2}} du \quad \text{(B1)}$$

where

$$\sigma_u^2 = 2\sigma_h^2 \left[1 - \frac{\sin \omega_h \tau}{\omega_h \tau}\right]. \quad \text{(B2)}$$

Proceeding in the same manner as in Appendix A gives

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{\beta^2}{2}}}{\sigma_u} e^{-j\beta\frac{\tau}{\sigma_u}} \int_{-\infty}^{\infty} \frac{\sin \omega_x \gamma}{\omega_x \gamma} \cos \omega_o \gamma \, e^{j\beta\frac{\gamma}{\sigma_u}} d\gamma d\beta \quad \text{(B3)}$$

or

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\beta^2/2} e^{-j\beta\frac{\tau}{\sigma_u}}}{\sigma_u} \frac{\pi}{\omega_x} 1/2 [P_{\omega_x}(\omega-\omega_o)+P_{\omega_x}(\omega+\omega_o)] \, d\beta. \tag{B4}$$

This can be written as

$$R_{yy}(\tau) = \frac{\sigma_x^2}{4\omega_x\sigma_u}\left\{\int_{-\omega_o-\omega_x}^{-\omega_o+\omega_x} e^{-\beta^2/2} e^{-j\beta\frac{\tau}{\sigma_u}} d\beta + \int_{\omega_o-\omega_x}^{\omega_o+\omega_x} e^{-\beta^2/2} e^{-j\beta\frac{\tau}{\sigma_u}} d\beta\right\} \tag{B5}$$

or as

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\omega_x\sigma_u} \int_{\omega_o-\omega_x}^{\omega_o+\omega_x} e^{-\beta^2/2} \cos\left(\beta\frac{\tau}{\sigma_u}\right) d\beta \tag{B6}$$

which is

$$R_{yy}(\tau) = \frac{\sigma_x^2}{2\omega_x\sigma_u}\left\{\int_{o}^{\omega_o+\omega_x} e^{-\beta^2/2} \cos\left(\beta\frac{\tau}{\sigma_u}\right) d\beta - \int_{o}^{\omega_o-\omega_x} e^{-\beta^2/2} \cos\left(\beta\frac{\tau}{\sigma_u}\right) d\beta\right\}. \tag{B7}$$

By the same procedure as in (A10) through (A13), this can be written as

$$R_{yy}(\tau) = 1/2\sigma_x^2 \left\{ \frac{\sin[(\omega_o+\omega_x)\tau]}{\omega_x\tau} [1-1/2(\omega_o+\omega_x)^2 \; \sigma_u^2+(\frac{\sigma_u}{\tau})^2] \right.$$

$$- \frac{\omega_o+\omega_x}{\omega} (\frac{\sigma_u}{\tau})^2 \cos[(\omega_o+\omega_x)\tau]$$

$$- \frac{\sin[(\omega_o-\omega_x)\tau]}{\omega_x\tau} [1-1/2(\omega_o-\omega_x)^2 \; \sigma_u^2 + (\frac{\sigma_u}{\tau})^2]$$

$$\left. + \frac{\omega_o-\omega_x}{\omega_x} (\frac{\sigma_u}{\tau})^2 \cos[(\omega_o-\omega_x)\tau] \right\} . \tag{B8}$$

The Fourier transform of this function can be computed in the same manner as $S_{yy}(\omega)$ was computed in Appendix A, however, it is more complex for this case. Also in the computation of e_f^2, $S_{yy}(\omega)$ is required only for the range $\omega_o-\omega_x<\omega<\omega_o+\omega_x$. For these reasons, the equation for $S_{yy}(\omega)$ will be given only for $\omega_o-\omega_x<\omega<\omega_o+\omega_x$. This equation is

$$S_{yy}(\omega) = \frac{\sigma_x^2 \pi}{2\omega_x} \Bigg\{ 1 - (\omega_o+\omega_x)^3 \, \sigma_h^2 \left(\frac{1}{\omega_o+\omega_x} - \frac{1}{\omega_h}\right)$$

$$- \frac{(\omega_o-\omega_x)^3 \, \sigma_h^2}{\omega_h} + \frac{\sigma_h^2}{\omega_h} \Bigg[(2\omega_x-\omega_h)\,\omega^2$$

$$+ \frac{(\omega_o+\omega_x)^3}{3} + \frac{(\omega_o-\omega_x)^3}{3} + 2\omega_h(\omega_o-\omega_x)\,\omega$$

$$+ \omega_h(\omega_h\omega_o-\omega_o^2-2\omega_o\omega_x)$$

$$- \frac{\omega_o+\omega_x}{2} (\omega+\omega_h-\omega_o-\omega_x)^2 - \frac{\omega_o+\omega_x}{2} (\omega-\omega_h+\omega_o+\omega_x)^2$$

$$+ \frac{\omega_o-\omega_x}{2} (\omega-\omega_h-\omega_o+\omega_x)^2 + \frac{\omega_o-\omega_x}{2} (\omega-\omega_h+\omega_o-\omega_x)^2 \Bigg] \Bigg\} \qquad \text{(B9)}$$

for $\omega_o-\omega_x \le \omega \le \omega_o+\omega_x$.

B. Calculation of $\overline{e_f^2}$

For this case, equation (5.16) can be simplified to

$$e_f^2 = \sigma_x^2 - \frac{\sigma_x^4 \pi}{2\omega_x^2} \int_{\omega_o-\omega_x}^{\omega_o+\omega_x} \frac{e^{-\sigma_h^2\omega^2}}{S_{yy}(\omega)} \, d\omega. \tag{B10}$$

Again using

$$e^{-\sigma_h^2\omega^2} \approx 1 - \sigma_h^2\omega^2,$$

simplifying (B9), dividing, and dropping higher order terms gives

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^2}{2\omega_x} \int_{\omega_o-\omega_x}^{\omega_o+\omega_x} [1 - \frac{\sigma_h^2}{\omega_h} (2\omega_o^2\omega_x + \frac{2}{3}\, \omega_x^3)] \, d\omega. \tag{B11}$$

This can be calculated to be

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3} [1 + 3 \, (\frac{\omega_o}{\omega_x})^2] \, \frac{2\omega_x}{\omega_h} \, . \tag{B12}$$

APPENDIX C

CALCULATION OF $S_{yy}(\omega)$ AND $\overline{e_f^2}$ FOR LOW PASS SIGNAL AND BAND PASS TIME-BASE ERROR

A. Calculation of $S_{yy}(\omega)$

For this case the calculation of $R_{yy}(\tau)$ is identical to that in Appendix A. Then

$$R_{yy}(\tau) = \sigma_x^2 \left\{ \frac{\sin \omega_x \tau}{\omega_x \tau} \left[1 - \frac{(\omega_x \sigma_u)^2}{2} + \left(\frac{\sigma_u}{\tau}\right)^2\right] - \left(\frac{\sigma_u}{\tau}\right)^2 \cos \omega_x \tau \right\} \tag{C1}$$

where, in this case,

$$\sigma_u^2 = 2\sigma_h^2 \left[1 - \frac{\sin \omega_h \tau}{\omega_h \tau} \cos \omega_c \tau\right], \tag{C2}$$

and where ω_c is the center frequency of the TBE spectrum. Once again, $S_{yy}(\omega)$ can be found by frequency domain convolution. In this case

$$F(\sigma_u^2) = 2\sigma_h^2 \left\{ 2\pi\delta(\omega) - 1/2 \frac{\pi}{\omega_h} [P_{\omega_h}(\omega-\omega_c) + P_{\omega_h}(\omega+\omega_c)] \right\} \quad \text{(C3)}$$

and

$$F(\frac{\sigma_u}{\tau})^2 = \begin{cases} 0 & \omega<\omega_c+\omega_h \\ \frac{\pi\sigma_h^2}{2\omega_h}(\omega+\omega_c+\omega_h)^2 & \omega_c+\omega_h<\omega<-\omega_c+\omega_h \\ 2\pi\sigma_h^2(\omega+\omega_c) & -\omega_c+\omega_h<\omega<0 \\ -2\pi\sigma_h^2(\omega-\omega_c) & 0<\omega<\omega_c-\omega_h \\ \frac{\pi\sigma_h^2}{2\omega_h}(\omega-\omega_c-\omega_h)^2 & \omega_c-\omega_h<\omega<\omega_c+\omega_h \\ 0 & \omega_c+\omega_h<\omega \end{cases} \quad . \text{(C4)}$$

For simplicity, $S_{yy}(\omega)$ is listed only for $|\omega|\leq\omega_x$, since this is all that is necessary for the calculation of $\overline{e_f^2}$.

The Fourier transform of the terms of (Cl) are

$$F\left(\sigma_u^2 \frac{\sin \omega_x \tau}{\omega_x \tau}\right) = \begin{cases} +\dfrac{\pi\sigma_h^2}{\omega_x} & -\omega_x<\omega<-\omega_x+\omega_c-\omega_h \\[2ex] -\dfrac{\pi\sigma_h^2}{2\omega_x\omega_h}(\omega+\omega_x-\omega_c-\omega_h) & -\omega_x+\omega_c-\omega_h<\omega<-\omega_x+\omega_c+\omega_h \\[2ex] 0 & -\omega_x+\omega_c+\omega_h<\omega<\omega_x-\omega_c-\omega_h \\[2ex] +\dfrac{\pi\sigma_h^2}{2\omega_x\omega_h}(\omega-\omega_x+\omega_c+\omega_h) & \omega_x-\omega_c-\omega_h<\omega<\omega_x-\omega_c+\omega_h \\[2ex] \dfrac{\pi\sigma_h^2}{\omega_x} & \omega_x-\omega_c+\omega_h<\omega<\omega_x \quad , \end{cases} \tag{C5}$$

$$
F\left[\left(\frac{\sigma_u}{\tau}\right)^2 \frac{\sin \omega_x\tau}{\omega_x\tau}\right] = \begin{cases}
\dfrac{\pi\sigma_h^2}{2\omega_x}\left[\omega_c^2 + \dfrac{1}{3}\,\omega_h^2 - \omega^2 - 2(\omega_x - \omega_c)\,\omega + 2\omega_x\omega_c - \omega_x^2\right] & -\omega_x < \omega < -\omega_x + \omega_c - \omega_h \\[2ex]
\dfrac{\pi\sigma_h^2}{\omega_x}\left(\omega_c^2 + \dfrac{1}{3}\,\omega_h^2\right) - \dfrac{\pi\sigma_h^2}{12\omega_h\omega_x}\left(\omega_c + \omega_h - \omega_x + \omega\right)^3 & -\omega_x + \omega_c - \omega_h < \omega < -\omega_x + \omega_c + \omega_h \\[2ex]
\dfrac{\pi\sigma_h^2}{\omega_x}\left(\omega_c^2 + \dfrac{1}{3}\,\omega_h^2\right) & -\omega_x + \omega_c + \omega_h < \omega < \omega_x - \omega_c - \omega_h \\[2ex]
\dfrac{\pi\sigma_h^2}{\omega_x}\left(\omega_c^2 + \dfrac{1}{3}\,\omega_h^2\right) - \dfrac{\pi\sigma_h^2}{12\omega_h\omega_x}\left(\omega_c + \omega_h - \omega_x - \omega\right)^3 & \omega_x - \omega_c - \omega_h < \omega < \omega_x - \omega_c + \omega_h \\[2ex]
\dfrac{\pi\sigma_h^2}{2\omega_x}\left[\omega_c^2 + \dfrac{1}{3}\,\omega_h^2 - \omega^2 + 2(\omega_x - \omega_c)\,\omega + 2\omega_x\omega_c - \omega_x^2\right] & \omega_x - \omega_c + \omega_h < \omega < \omega_x,
\end{cases}
\tag{C6}
$$

and

$$F[(\frac{\sigma_u}{\tau})^2 \cos \omega_x \tau] = \begin{cases} -\pi\sigma_h^2(\omega-\omega_c+\omega_x) & -\omega_x<\omega<-\omega_x+\omega_c-\omega_h \\ \frac{\pi\sigma_h^2}{4\omega_h}(\omega+\omega_x-\omega_c-\omega_h)^2 & -\omega_x+\omega_c-\omega_h<\omega<-\omega_x+\omega_c+\omega_h \\ 0 & -\omega_x+\omega_c+\omega_h<\omega<\omega_x-\omega_c-\omega_h \\ \frac{\pi\sigma_h^2}{4\omega_h}(\omega-\omega_x+\omega_c+\omega_h)^2 & \omega_x-\omega_c-\omega_h<\omega<\omega_x-\omega_c+\omega_h \\ \pi\sigma_h^2(\omega+\omega_c-\omega_x) & \omega_x-\omega_c+\omega_h<\omega<\omega_x \,. \end{cases} \tag{C7}$$

Combining (C5), (C6), (C7), and (C1) to get $S_{yy}(\omega)$ gives

$$S_{yy}(\omega) = \begin{cases} \sigma_x^2\left\{\frac{\pi}{\omega_x}+\frac{\pi\sigma_h^2}{\omega_x}(\omega_c^2+1/3\omega_h^2)\right\} & 0<\omega<\omega_x-\omega_c-\omega_h \\ \sigma_x^2\left\{\frac{\pi}{\omega_x}-\frac{\omega_x^2}{4}\frac{\pi\sigma_h^2}{\omega_x\omega_h}[\omega-(\omega_x-\omega_c-\omega_h)] \right. & \\ \quad +\frac{\pi\sigma_h^2}{\omega_x}(\omega_c^2+1/3\omega_h^2)-\frac{\pi\sigma_h^2}{12\omega_h\omega_x}(\omega_c+\omega_h-\omega_x-\omega)^3 & \\ \quad \left. -\frac{\pi\sigma_h^2}{4\omega_h}(\omega-\omega_x+\omega_c+\omega_h)^2\right\} & \omega_x-\omega_c-\omega_h<\omega<\omega_x-\omega_c+\omega_h. \end{cases} \tag{C8}$$

and

$$S_{yy}(\omega) = \sigma_x^2 \left\{ \frac{\pi}{\omega_x} - \frac{\omega_x^2}{2} \frac{\pi\sigma_h^2}{\omega_x} \right.$$

$$+ \frac{\pi\sigma_h^2}{2\omega_x} [\omega_c^2 + 1/3\omega_h^2 - \omega^2 + 2(\omega_x - \omega_c)\omega + 2\omega_x\omega_c - \omega_x^2]$$

$$\left. - \pi\sigma_h^2(\omega + \omega_c - \omega_x) \right\} \tag{C9}$$

for $\omega_x - \omega_c + \omega_h < \omega < \omega_x$.

B. Calculation of $\overline{e_f^2}$

For this case equation (5.16) for $\overline{e_f^2}$ can be written

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \left\{ \int_o^{\omega_x - \omega_c - \omega_h} \frac{e^{-\sigma_h^2 \omega^2}}{S_{yy}(\omega)} d\omega + \int_{\omega_x - \omega_c - \omega_h}^{\omega_x - \omega_c + \omega_h} \frac{e^{-\sigma_h^2 \omega^2}}{S_{yy}(\omega)} d\omega \right.$$

$$\left. + \int_{\omega_x - \omega_c + \omega_h}^{\omega_x} \frac{e^{-\sigma_h^2 \omega^2}}{S_{yy}(\omega)} d\omega \right\} \tag{C10}$$

where the proper $S_{yy}(\omega)$ is used for each interval. Using the same procedure as in Appendix A gives

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^2}{\omega_x}\int_0^{\omega_x-\omega_c-\omega_h} [1 - \sigma_h^2\omega^2 - \sigma_h^2(\omega_c^2 + \tfrac{1}{3}\omega_h^2)]\, d\omega$$

$$- \frac{\sigma_x^2}{\omega_x}\int_{\omega_x-\omega_c-\omega_h}^{\omega_x-\omega_c+\omega_h} \Big\{1-\sigma_h^2\omega^2 + \sigma_h^2\, [\frac{1}{12\omega_h}(\omega-\omega_x+\omega_c+\omega_h)^3$$

$$+ \frac{\omega_x}{4\omega_h}(\omega-\omega_x+\omega_c+\omega_h)^2 + \frac{\omega_x^2}{4\omega_h}(\omega-\omega_x+\omega_c+\omega_h) - (\omega_c^2 + \tfrac{1}{3}\omega_h^2)\Big\}\, d\omega$$

$$- \frac{\sigma_x^2}{\omega_x}\int_{\omega_x-\omega_c+\omega_h}^{\omega_x} \Big\{1-\sigma_h^2\omega^2 + \sigma_h^2\, [\frac{\omega^2}{2} + \omega_c\omega - (\frac{\omega_c^2}{2} + \frac{\omega_h^2}{6})\Big\}\, d\omega. \tag{C11}$$

Performing these integrations and simplifying

$$\overline{e_f^2} = \frac{\sigma_x^2\sigma_h^2\omega_x^2}{3} - \frac{\sigma_x^2\sigma_h^2}{6}\, [\frac{\omega_c}{\omega_x}(\omega_c^2+\omega_h^2+3\omega_x^2) - (\omega_h^2+3\omega_c^2)]$$

APPENDIX D

CALCULATION OF $S_{yy}(\omega)$ AND $\overline{e_f^2}$ FOR LOW PASS SIGNAL AND LOW PASS TIME-BASE ERROR WITH ADDITIVE PRE-RECORD NOISE

A. Calculation of $S_{yy}(\omega)$

From (6.10)

$$R_{yy}(\tau) = E_h[R_{xx}(\tau+u) + R_{hh}(\tau+u)] \tag{D1}$$

For low pass signal and low pass noise of the same band-width

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \sigma_x^2 \frac{\pi}{\omega_x} \frac{\sin[\omega_x(\tau+u)]}{\omega_x(\tau+u)} \frac{e^{-\frac{u^2}{2\sigma_u^2}}}{\sqrt{2\pi\sigma_u^2}}\, du$$

$$+ \int_{-\infty}^{\infty} \sigma_n^2 \frac{\pi}{\omega_x} \frac{\sin[\omega_x(\tau+\omega)]}{\omega_x(\tau+u)} \frac{e^{-\frac{u^2}{2\sigma_u^2}}}{\sqrt{2\pi\sigma_u^2}}\, du \tag{D2}$$

where

$$\sigma_u^2 = 2\sigma_h^2 \; [1 - \frac{\sin \omega_h \tau}{\omega_h \tau}]$$

and where σ_n^2 is the noise variance.

Using the same procedure as in Appendix A, this becomes

$$R_{yy}(\tau) = (\sigma_x^2 + \sigma_n^2) \left\{ \frac{\sin \omega_x \tau}{\omega_x \tau} \; [1 - \frac{(\omega_x \sigma_u)^2}{2} + (\frac{\sigma_u}{\tau})^2 \;] \right.$$

$$\left. - (\frac{\sigma_u}{\tau})^2 \; \cos \omega_x \tau \right\} . \qquad \text{(D3)}$$

By the same procedure as in Appendix A, $S_{yy}(\omega)$ is found to be

$$S_{yy}(\omega) = \begin{cases} (\sigma_x^2+\sigma_n^2)\left\{ \dfrac{\pi}{\omega_x} - \pi\sigma_h^2 \left[\dfrac{\omega^3}{6\omega_x^2} + \dfrac{\omega^2}{2\omega_x} - \dfrac{\omega}{2} - \dfrac{\omega_x}{3}\right] \right\} & -\omega_x<\omega<0 \\ (\sigma_x^2+\sigma_n^2)\left\{ \dfrac{\pi}{\omega_x} - \pi\sigma_h^2 \left[- \dfrac{\omega^3}{6\omega_x^2} + \dfrac{\omega^2}{2\omega_x} + \dfrac{\omega}{2} - \dfrac{\omega_x}{3}\right] \right\} & 0<\omega<\omega_x \end{cases} \tag{D4}$$

for $\omega_x = \omega_h$, and

$$S_{yy}(\omega) = (\sigma_x^2+\sigma_n^2)\left\{ \frac{\pi}{\omega_x} + \pi\sigma_h^2 \left[\omega_x^2 \left(\frac{1}{\omega_h} + \frac{1}{\omega_x}\right) + \omega_h-\omega_x + \frac{\omega_x^2}{3\omega_h} + \frac{\omega_x-\omega_h}{\omega_h\omega_x}\ \omega^2 - \frac{(\omega+\omega_h-\omega_x)^2}{2\omega_h} + \frac{(\omega+\omega_h+\omega_x)^3}{2\omega_h}\right]\right\} \qquad -\omega_x<\omega<\omega_x \tag{D5}$$

for $\omega_h>2\omega_x$.

B. Calculation of $\overline{e_f^2}$

By the same procedure as used in Chapter IV, the optimum filter for this case is found to be

$$H(j\omega) = \frac{S_{xy}(\omega)}{S_{yy}(\omega)} \; .$$

The mean square error using the optimum filter is again

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_0^{\omega_x} \frac{e^{-\sigma_h^2 \omega^2}}{S_{yy}(\omega)} \, d\omega.$$

By the use of (D4), (D5), and the same steps as in Appendix A

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_0^{\omega_x} \frac{1}{(\sigma_x^2+\sigma_n^2)\frac{\pi}{\omega_x}} \left[1 - \frac{\sigma_h^2 \omega_x^2}{3}\,\omega + \frac{\sigma_h^2 \omega_x}{2}\,\omega \right.$$
$$\left. - \frac{\sigma_h^2 \omega^2}{2} + \frac{\sigma_h^2 \omega^3}{6\omega_x}\right] d\omega \qquad \text{(D6)}$$

for $\omega_x = \omega_n = \omega_h$, and

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_o^{\omega_x} \frac{1}{(\sigma_x^2+\sigma_n^2)\frac{\pi}{\omega_x}} [1 - \frac{\sigma_h^2 \omega_x^3}{3\omega_h}] \, d\omega \tag{D7}$$

for $\omega_h > 2\omega_x$ and $\omega_x = \omega_n$.

Integrating

$$\overline{e_f^2} = \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2} [\sigma_n^2 + 5/24 \; \sigma_x^2 \sigma_h^2 \omega_x^2] \tag{D8}$$

for $\omega_x = \omega_h$, and

$$\overline{e_f^2} = \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2} [\sigma_n^2 + \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h}] \tag{D9}$$

for $\omega_h > 2\omega_x$.

For $\sigma_n^2 << \sigma_x^2$, these become

$$\overline{e_f^2} = \sigma_n^2 \; (1 - \frac{5}{24} \sigma_h^2 \omega_x^2) + 5/24 \; \sigma_x^2 \sigma_h^2 \omega_x^2 \tag{D10}$$

for $\omega_x = \omega_h$, and

$$\overline{e_f^2} = \sigma_n^2 \left(1 - \frac{\sigma_h^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h}\right) + \frac{\sigma_x^2 \sigma_h^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h} \tag{D11}$$

for $\omega_h > 2\omega_x$.

APPENDIX E

CALCULATION OF $S_{yy}(\omega)$ AND $\overline{e_f^2}$ FOR LOW PASS SIGNAL AND TIME-BASE ERROR WITH ADDITIVE POST-RECORD NOISE

A. Calculation of $S_{yy}(\omega)$

Using (6.26) and carrying out the same details as in Appendix A gives

$$R_{yy}(\tau) = \sigma_x^2 \left\{ \frac{\sin \omega_x \tau}{\omega_x \tau} \left[1 - \frac{\omega_x^2 \sigma_u^2}{2} + \left(\frac{\sigma_u}{\tau}\right)^2\right] - \left(\frac{\sigma_u}{\tau}\right)^2 \cos \omega_x \tau \right\} + \sigma_n^2 \frac{\sin \omega_n \tau}{\omega_n \tau}. \tag{E1}$$

Taking the Fourier transform

$$S_{yy}(\omega) = \sigma_x^2 \left\{ \pi/\omega_x - \pi\sigma_h^2 \left[\frac{\omega^3}{6\omega_x^2} + \frac{\omega^2}{2\omega_x} + \frac{\omega}{2} - \frac{\omega_x}{3}\right] \right\} + \sigma_n^2 \frac{\pi}{\omega_n} P_{\omega_n}(\omega) \tag{E2}$$

for $0<\omega<\omega_x$ and $\omega_x = \omega_h$, while

$$S_{yy}(\omega) = \sigma_x^2 \left\{ \frac{\pi}{\omega_x} - \pi\sigma_h^2 \left[\omega_x^2 \left(\frac{1}{\omega_h} + \frac{1}{\omega_x}\right) + \omega_h - \omega_x \right.\right.$$

$$+ \frac{\omega_x^2}{3\omega_h} + \frac{\omega_x - \omega_h}{\omega_h \omega_x} \omega^2 - \frac{(\omega+\omega_h-\omega_x)^2}{2\omega_h}$$

$$\left.\left. + \frac{(\omega+\omega_h+\omega_x)^3}{2\omega_h} \right] \right\} + \sigma_n^2 \frac{\pi}{\omega_n} P_{\omega_n}(\omega) \qquad \text{(E3)}$$

for $-\omega_x<\omega<\omega_x$ and $\omega_h>2\omega_x$.

B. Calculation of $\overline{e_f^2}$

The equation for $\overline{e_f^2}$ is

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \int_o^{\omega_x} \frac{e^{-\sigma_h^2 \omega^2}}{S_{yy}(\omega)} \, d\omega \qquad \text{(E4)}$$

For $\omega_h = \omega_x$, this can be written

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4\pi}{\omega_x^2} \int_o^{\omega_x} \frac{1}{\sigma_x^2 \quad \frac{\pi}{\omega_x}} [1- \frac{\omega_x\sigma_n^2}{\omega_n \ \sigma_x^2} - \frac{\sigma_h^2\omega_x^2}{3} + \frac{\sigma_h^2\omega}{2} \omega \tag{E5}$$

$$- \frac{\sigma_h^2\omega^2}{2} + \frac{\sigma_h^2\omega^3}{6\omega_x} \quad d\omega$$

when $\omega_n \geq \omega_x$, and

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4\pi}{\omega_x^2} \left\{ \int_o^{\omega_n} \frac{1}{\sigma_x^2 \quad \frac{\pi}{\omega_x}} [1 - \frac{\omega_x}{\omega_n} \frac{\sigma_n^2}{\sigma_x^2} - \frac{\sigma_h^2\omega_x^2}{3} + \frac{\sigma_h^2\omega_x}{2} \omega - \frac{\sigma_h^2\omega^2}{2} \right.$$

$$+ \frac{\sigma_h^2\omega^3}{6\omega_x}] \ d\omega$$

$$\left. + \int_{\omega_n}^{\omega_x} \frac{1}{\sigma_x^2 \quad \frac{\pi}{\omega_x}} [1 - \frac{\sigma_h^2\omega_x^2}{3} + \frac{\sigma_h^2\omega_x}{2} \omega - \frac{\sigma_h^2\omega^2}{2} + \frac{\sigma_h^2\omega^3}{6\omega_x}] \ d\omega \right\} \tag{E6}$$

when $\omega_n < \omega_x$.

For $\omega_h > 2\omega_x$, (E4) can be written as

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4\pi}{\omega_x^2} \int_o^{\omega_x} \frac{1}{\sigma_x^2 \quad \frac{\pi}{\omega_x}} [1 - \frac{\omega_x}{\omega_n} \frac{\sigma_n^2}{\sigma_x^2} - \frac{\sigma_h^2\omega_x^3}{3\omega_h}] \ d\omega \tag{E7}$$

for $\omega_n \geq \omega_x$, and

$$\overline{e_f^2} = \sigma_x^2 - \frac{\sigma_x^4 \pi}{\omega_x^2} \left\{ \int_o^{\omega_n} \frac{1}{\sigma_x^2 \quad \frac{\pi}{\omega_x}} \left[1 - \frac{\omega_x}{\omega_n} \frac{\sigma_n^2}{\sigma_x^2} - \frac{\sigma_h^2 \omega_x^3}{3\omega_h} \right. d\omega \right. \qquad \text{(E8)}$$

$$\left. + \int_{\omega_n}^{\omega_x} \frac{1}{\sigma_x^2 \quad \frac{\pi}{\omega_x}} \left[1 - \frac{\sigma_h^2 \omega_x^2}{3\omega_h}\right] \, d\omega \right\}$$

for $\omega_n < \omega_x$.

Carrying out these integrations for $\omega_x = \omega_h$

$$\overline{e_f^2} = \frac{\omega_x}{\omega_n} \quad \sigma_n^2 \quad (1 - 5/24 \ \sigma_h^2 \omega_x^2) \quad + \quad 5/24 \quad \sigma_x^2 \sigma_h^2 \omega_x^2 \qquad \text{(E9)}$$

for $\omega_n \geq \omega_x$, and

$$\overline{e_f^2} = \sigma_n^2 \quad (1 - 5/24 \ \sigma_h^2 \omega_x^2) + 5/24 \ \sigma_x^2 \sigma_h^2 \omega_x^2 \qquad \text{(E10)}$$

for $\omega_n < \omega_x$.

When $\omega_h > 2\omega_x$,

$$\overline{e_f^2} = \frac{\omega_x}{\omega_n} \sigma_n^2 \left(1 - \frac{\sigma_h^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h}\right) + \frac{\sigma_h^2 \sigma_x^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h} \tag{E11}$$

for $\omega_n \geq \omega_x$, and

$$\overline{e_f^2} = \sigma_n^2 \left(1 - \frac{\sigma_h^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h}\right) + \frac{\sigma_h^2 \sigma_x^2 \omega_x^2}{3} \frac{\omega_x}{\omega_h} \tag{E12}$$

for $\omega_n \leq \omega_x$.

BIBLIOGRAPHY

1. Davies, G. L., Magnetic Tape Recording. New York: McGraw Hill, 1961.

2. Chao, S. C., "Flutter and Time Errors in Magnetic Data Recorders," International Telemetering Conference Proceedings, vol. 1, pp. 578-594, May, 1965.

3. Ratz, Alfred G., "The Effect of Tape Transport Flutter on Spectrum and Correlation Analysis," IEEE Transactions on Space Electronics and Telemetry, pp. 129-134, December, 1964.

4. Simpson, R. S., and Tranter, W. H., "Carrier Synthesis From Perturbed DSB/SC Signals," Proceedings of the International Telemetering Conference, vol. 3, pp. 371-381, October, 1967.

5. Simpson, R. S., and Tranter, W. H., "Effect of Recorder Time-Base Error on an AM-Baseband Telemetry System," IEEE Transactions on Communication Technology, 16-2, pp. 316-320, April, 1968.

6. Chao, S. C., "Time Base Compensation in Magnetic Tape Recorders," Proceedings of the IEEE, vol. 54, pp. 1572-1573, November, 1966.

7. Houts, R. C., Burlage, D. W., and Simpson, R. S., "A Digital System for Compensating Time-Base Error in Analog Tape Recorders," IEEE Transactions on Communications Technology, vol. COM-18, No. 3, June, 1970.

8. Nichols, M. H., and Schmitt, F. J., "Cause and Effect of a Suppressed Carrier AM Multiplex," Proceedings of the International Telemetering Conference, vol. 4, p. 400, 1968.

9. Nichols, M. H., "Some Analysis of WSMR Test Results on DSB," Proceedings of the International Telemetering Conference, vol. 3, pp. 361-370, October, 1967.

10. Schultze, G. H., "Time Displacement Errors in Instrumentation Tape Recording," Proceedings of the International Telemetering Conference, pp. 263-272, London, 1963.

11. Sweeney, J. R. "A Method for Measuring the Changes Introduced in Recorded Time Intervals by a Recorder-Reproducer," IRE Transactions on Audio, vol. PGA-T,

p. 24, May 1952.

12. Simpson, R. S., and Tranter, W. H., "Effect of Recorder Flutter on Coherent Demodulation in an AM-Baseband System," Proceedings of the National Telemetering Conference, pp. 46-49, May, 1967.

13. Stewart, J. L., "The Power Spectrum of a Carrier Frequency Modulated by Gaussian Noise," Proceedings of the IRE, vol. 42, pp. 1539-1542, October, 1954.

14. Athey, S. W., "Magnetic Tape Recording," National Aeronautics and Space Administration Technology Survey, #SP-5038, 1966.

15. Peshel, L. R., "The Application of Wow and Flutter Compensation Techniques to FM Magnetic Recording Systems," IRE 1957 National Convention Record, vol. 5, pt. 7, pp. 95-110, March, 1957.

16. Pear, C. B., Jr., "Flutter in Magnetic Recording of Data," IRE International Convention Record, pt. 2, pp. 81-88, 1961.

17. Brenner, B. M., and Meyer, G., "Time-Base Errors in Analog Recording," International Conference on Magnetic Recording, pp. 26-29, July, 1964.

18. Chao, S. C., "Flutter and Time Errors in Instrumentation Magnetic Recorders," IEEE Transactions on Aerospace and Electronic Systems, vol. AES-2, pp. 214-223, March, 1966.

19. Panter, Philip F., Modulation, Noise and Spectral Analysis. New York: McGraw Hill, 1965.

20. Papoulis, A., Probability, Random Variables, and Stochastic Processes. New York: McGraw Hill, 1965.

21. Baghdady, Elie J., "Theory of Low-distortion Reproduction of FM Signals in Linear Systems," IRE Transactions on Circuit Theory, vol. CT-5, pp. 202-214, 1958.

22. Carson, J. R., and Fry, T. C., "Variable-Frequency Electric Circuit Theory," Bell System Technical Journal, vol. 16, pp. 513-540, October, 1967.

23. Thomas, John B., An Introduction to Statistical Communication Theory. New York: John Wiley & Sons, 1969.

24. Middleton, D., Introduction to Statistical Communication Theory. New York: McGraw Hill, 1960.

25. Davenport, Wilber B., Jr., and Root, William L., An Introduction to the Theory of Random Signals and Noise. New York: McGraw Hill, 1965.

26. Bendat, J. S., Principles and Applications of Random Noise Theory. New York: John Wiley and Sons, 1958.

27. Gradshteyn, I. S., and Ryzhik, I. M., Tables of Integrals, Series, and Products. New York: Academic Press, 1965.

28. Swartz, M., Bennett, W. R., and Stein, S., Communication Systems and Techniques. New York: McGraw Hill, 1966.