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Characterizing topologies by continuous selfmaps

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CHARACTERIZING TOPOLOGIES BY CONTINUOUS SELFMAPS

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ABSTRACT

Various topological spaces are examined in an effort to describe topological spaces from a knowledge of their class of continuous selfmaps or their class of autohomeomorphisms. Relationships between topologies and their continuous selfmaps are considered. Several examples of topological spaces are given and their corresponding classes of continuous selfmaps are described completely. The problem, given a set X and a topology U when does there exist a topology V either weaker or stronger than U such that the class of continuous selfmaps of (X,V) contains the class of continuous selfmaps of (X,U), is considered.

M* and S** spaces are defined and some of their properties are considered. Two M* (or S**) spaces are shown to be homeomorphic if and only if certain semigroups of continuous selfmaps are isomorphic.

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I. INTRODUCTION

Let C(X) (or C(X,t) if the topology is to be stressed) denote the semigroup under composition of all continuous mappings from a topological space X into X. Likewise, let H(X) denote the group of autohomeomorphisms on X.

Although the problem of determining if a function from a topological space to itself is continuous is basic in the study of topology, it is frequently difficult to determine whether a functon is or is not continuous. Hence several examples of topological spaces are considered from the point of view of obtaining simple rules which characterize their continuous selfmaps. A converse type of a problem, that of finding all topologies which can be assigned to X which have a given semigroup of selfmaps, is considered and some topologies are characterized by their autohomeomorphisms.

It is clear that C(X,D) and $C(X,t_o)$ each consist of all selfmaps, where D and t_o represent the discrete and trivial topologies respectively. Thus for any topology U defined on X, U is stronger than t_o and weaker than D while C(X,U) is a subset of both C(X,D) and $C(X,t_o)$. Problems of examining the effect on C(X,U) by changing the topology U are considered in searching for a topology V either weaker or stronger than U such that C(X,V) contains C(X,U). A similar problem exists when continuous selfmaps are replaced by autohomeomorphisms.

Another important problem in this area is to study the relationship between the topological properties of a space X and the algebraic properties of C(X). To this end we note that if X and Y are homeomorphic then C(X) and C(Y) are isomorphic. In particular if h is a homeomorphism from X to Y then the mapping from C(X) to C(Y) defined by $\Phi(f) = h \circ f \circ h^{-1}$ is an isomorphism. Setting X equal to Y and recalling that $C(X,D) = C(X,t_o)$ it becomes evident that C(X) being isomorphic to C(Y) is not sufficient to imply that X and Y are homeomorphic. Thus it is desirable to find classes of topological spaces and semigroups of continuous selfmaps such that within this class two spaces are homeomorphic if and only if certain semigroups of continuous selfmaps are isomorphic.

II. REVIEW OF THE LITERATURE

In 1948, Everett and Ulam [1] posed the following question. Given the class of all homeomorphisms from a topological space to itself, what other topologies exist on the same set which have the same set of homeomorphisms? No results appeared until 1963 when Whittaker [2] studied a related problem. He proved if X and Y are compact, locally Euclidean manifolds and if Φ is an isomorphism from H(X) to H(Y) then there exists a homeomorphism h from X onto Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for each f in H(X). It follows immediately from this result that if (X,U) and (X,V) are compact, locally Euclidean manifolds such that H(X,U) = H(X,V) then (X,U) and (X,V) are homeomorphic.

Yu-Lee Lee in [3] developed some methods which enabled him to start either with certain locally compact spaces or certain first countable Hausdorff spaces and to construct another topological space with the same class of homeomorphisms. In [4] Yu-Lee Lee showed that given an n-manifold (X,U) there exists no other Hausdorff space (X,V) such that H(X,U) = H(X,V) and one of the following conditions is satisfied: 1) (X,V) is locally compact, 2) (X,V) is first countable, or 3) (X,V) is locally arcwise connected.

In 1964, Magill [5] considered the problem of determining a class of spaces such that if two spaces X and Y belong to this class then X and Y are homeomorphic if and only if C(X) and C(Y) are isomorphic. Magill defined a class of spaces called S-spaces with the required property. The class of S-spaces includes all locally Euclidean spaces and also all 0-dimensional Hausdorff spaces. Later Magill [6,7,8] found other collections of spaces which satisfied his basic problem.

Hicks and Haddock [9] extended Magill's results from S-spaces to M-spaces and obtained several results involving arbitrary semigroups of continuous selfmaps which contain the collection of constant selfmaps.

There were actually forerunners to the problem stated by Magill. In 1939, Gelfand and Kolmogoroff [10] studied the problem from the point of view of considering the ring S(X) of continuous mappings from X to the reals. They proved that two completely regular, compact spaces X and Y are homeomorphic if and only if S(X) is isomorphic to S(Y). Previous to this time Stone [11] studied the same problem but introduced a topology on the ring of continuous functions. Much work has been done in the area of rings of continuous functions and a collection of these results can be found in Gillman and Jerison [12].

Just as Stone introduced a topology on the ring of continuous functions S(X), Weschler [13] also introduced a topology on H(X). He found a family of topological spaces such that if X and Y are spaces belonging to this family and if H(X) and H(Y) are both isomorphic and homeomorphic then X and Y are homeomorphic. Thomas [14] and Wiginton and Shrader [15] obtained similar results over slightly different families of spaces.

III. SEMIGROUPS OF ALL CONTINUOUS SELFMAPS

A. COMPARING TOPOLOGIES WITH THEIR CONTINUOUS SELFMAPS

In what follows C(X,t) will be used to denote the collection of continuous selfmaps from the space (X,t) to (X,t). It is easy to see that for any topology t the constant selfmaps as well as the identity selfmap are continuous. It is also a trivial matter to show that the topologies D and t_o, where D and t_o denote the discrete and trivial topologies respectively, are characterized in the sense that they are the only topologies on X with the property that every selfmap is continuous. To see this assume there exists a topology U such that $t_o < U < D$ and such that $C(X,U) = C(X,D) = C(X,t_o)$. Let O be a proper subset of X such that O is in U, let A be an arbitrary subset of X and let x and y be elements of O and X - O respectively. Then define a function f such that f maps each element of A to x and each element of X - A to y. Since f is assumed to be continuous, $f^{-1}(O) = A$ must be an open set. A is arbitrary and thus U = D, which is a contradiction.

A method for finding topologies U and V which can be put on a set X with the property that C(X,U) = C(X,V) will now be shown.

The following definition is due to Lorrain [16].

DEFINITION 1. A space (X,t) is said to be a saturated space if and only if the intersection of an arbitrary number of open sets is open.

Clearly, any topology on a finite set is saturated, however; there are many other saturated spaces as can be seen by the following theorem.

THEOREM 1. If (X,t) is a topological space such that for each x in X there exists a neighborhood of x, say N_x, which intersects only a finite number of closed sets (the family of closed sets is locally finite), then t is a saturated topology.

PROOF. Let $\{F_a : a \text{ is in } A\}$ be an arbitrary family of closed sets and let x be an element of $X - \bigcup_{a \varepsilon A} F_a$. Then there exists a neighborhood N_x of x which intersects only a finite number of these closed sets, say $F_1, F_2, \ldots, F_n$. Since x is in no F_a, x is in each of the open sets $X - F_i$ for $i = 1, 2, \ldots, n$. Therefore $N_x \cap (\bigcap_{i=1}^{n} (X - F_i))$ is an open neighborhood of x which is a subset of $X - \bigcup_{a \varepsilon A} F_a$. Thus $X - \bigcup_{a \varepsilon A} F_a$ is open and hence $\bigcup_{a \varepsilon A} F_a$ is closed, which is equivalent to t being saturated.//

The converse of this theorem is not true as can be seen by letting (X,t) be the set of positive integers with the saturated topology $t = \{O : 1 \varepsilon O \text{ and } O \subset X\} \cup \{\phi\}$. Since there are infinitely many closed sets containing the element 2, every neighborhood of 2 must intersect infinitely many closed sets.

THEOREM 2. If (X,U) is a saturated space and $V = \{O : X - O \text{ is in } U\}$ then $C(X,U) = C(X,V)$.

PROOF. Let f be an element of C(X,U) and let F be a closed set of (X,V). Hence F is open in U and therefore $f^{-1}(F)$ is in U which implies $f^{-1}(F)$ is closed in V. Thus f is in C(X,V). Similarily C(X,V) is a subset of C(X,U).//

Note that V as defined in theorem 2 is a topology if and only if U is a saturated topology. Also, V is a saturated topology. This implies that if U is a saturated topology different from the discrete topology then U can not be a T_1 space. For if U is assumed to be a saturated T_1 space then V is the discrete topology. This follows since each singleton is closed in U and hence open in V. Because of the way that V is constructed from U, V being discrete implies U is discrete.

Given a set X and a topology U such that $t_o < U < D$, it follows that $C(X,U)$ is a proper subset of both $C(X,t_o)$ and C(X,D). Some results will now be given which are useful when attempting to find a topology V comparable with U such that C(X,V) contains C(X,U).

THEOREM 3. Let X be an arbitrary set and let $\{ t_a :$ a in A$\}$ be a collection of topologies on X, then

i) $\bigcap_{a\varepsilon A} C(X,t_a)$ is a subset of $C(X, \bigcap_{a\varepsilon A} t_a)$ and

ii) $\bigcap_{a\varepsilon A} C(X,t_a)$ is a subset of C(X,T) where T is the least upper bound of the topologies in $\{ t_a :$ a in A$\}$.

PROOF. i) Let $f \varepsilon \bigcap_{a\varepsilon A} C(X,t_a)$ and let $O \varepsilon \bigcap_{a\varepsilon A} t_a$. Then since O is in each t_a and f is in each $C(X,t_a)$ it

follows that $f^{-1}(O)$ is in $\bigcap_{a\varepsilon A} t_a$ and hence $f \varepsilon C(X, \bigcap_{a\varepsilon A} t_a)$.
ii) Again let $f \varepsilon \bigcap_{a\varepsilon A} C(X,t_a)$ and let $O \varepsilon \bigcup_{a\varepsilon A} t_a$. Note $\bigcup_{a\varepsilon A} t_a$ is a subbase for T. Since O is a member of some t_a and since f is also in $C(X,t_a)$ it follows that $f^{-1}(O)$ is in t_a. Therefore $f^{-1}(O)$ is in the subbase $\bigcup_{a\varepsilon A} t_a$ and hence in T. Thus $f \varepsilon C(X,t)$.//

If $C(X,t) = C(X,U)$ then it follows immediately that $C(X,U)$ is a subset of $C(X,t \cap U)$ and $t \cap U$ is a topology comparable to U. Also $C(X,U)$ is a subset of $C(X, \text{lub}\{t,U\})$ and lub $\{t,U\}$ is a topology comparable to U.

Examples illustrating the use of theorems 2 and 3 will now be given.

1). Let X be the set of positive integers. Let t_1 be the topology consisting of supersets of the element 1; that is, $t_1 = \{O : 1 \varepsilon O \text{ and } O \subset X\} \cup \{\phi\}$. Let t_2 be the cofinite topology on X; that is, $t_2 = \{O : O \subset X \text{ and } X - O \text{ is a finite set}\} \cup \{\phi\}$. It will be shown that $C(X,t_1) = \{f : f \text{ is a constant function or } f(1) = 1\}$ and $C(X,t_2) = \{f : f \text{ is a constant function or } f \text{ is a finite to one function}\}$. A function is said to be finite to one if and only if each element of X has at most a finite number of preimages. From theorem 3 it follows that every function f which is constant, or is finite to one and maps 1 onto 1 is in $C(X,t)$ where $t = t_1 \cap t_2 = \{O : 1 \varepsilon O \text{ and } X - O \text{ is finite}\} \cup \{\phi\}$.

2). Let $X = \{1,2,3,4\}$, $t_1 = \{\phi, \{1,2\}, \{3\}, \{1,2,3\},$

$\{3,4\}$, X$\}$ and $t_2 = \{O : X - O \varepsilon t_1\}$. It follows from theorem 2 that $C(X,t_1) = C(X,t_2)$. Note $V = t_1 \cap t_2 = \{\phi, \{1,2\}, \{3,4\}, X\}$ is weaker than t_1 however from theorem 3 it follows that $C(X,V) \supset C(X,t_1)$ while $U = t_1 \cup t_2 = \{\phi, \{1,2\}, \{1,2,3\}, \{1,2,4\}, \{3\}, \{3,4\}, X\}$ is stronger than t_1 such that $C(X,U) \supset C(X,t_1)$.

Other methods for finding a topology V comparable to U such that $C(X,V) \supset C(X,U)$, are given in the next two theorems.

DEFINITION 2. Let (X,U) be a given space then S_s and S_w are defined as follows:

$S_s = \{O : O \varepsilon U\} \cup \{O_i : O_i$ is not in U and i is in a nonempty index set I$\} \cup \{f^{-1}(O_i) : f$ is in $C(X,U)$ and i is in I$\}$.

$S_w = \{O : O \varepsilon U\} - (\{O_i : O_i$ is in U and i is in a nonempty index set I$\} \cup \{O : O \varepsilon U$ and $f^{-1}(O) = O_i$ for some i in I and some f in $C(X,U)\})$.

THEOREM 4. (X,V) is a topological space with a subbase of the form S_s if and only if

i) V is stronger than U and

ii) $C(X,U) \subset C(X,V)$

PROOF. Assuming i) and ii) trivially there exists a subbase for (X,V) of the form S_s, namely $S_s = \{O : O$ is in U$\} \cup \{O_i : O_i$ is in V - U$\} \cup \{f^{-1}(O_i) : f$ is in $C(X,U)$ and O_i is in V - U$\}$.

On the other hand if V has a subbase of the form S_s, it is immediate that V is stronger than U.

To show that $C(X,U) \subset C(X,V)$ let g be a function in $C(X,U)$ and let O be in S_s.

CASE 1. If O is in U than $g^{-1}(O)$ is in U. Therefore $g^{-1}(O)$ is in S_s which is a subset of V.

CASE 2. If $O = O_i$ for some i in I, then $g^{-1}(O)$ is in S_s which is a subset of V.

CASE 3. If $O = f^{-1}(O_i)$ for some f in $C(X,U)$ and i in I, then $g^{-1}(O) = g^{-1}(f^{-1}(O_i)) = h^{-1}(O_i)$ where $h = f \circ g$ is in $C(X,U)$. Therefore $g^{-1}(O) = h^{-1}(O_i)$ is in S_s which is a subset of V.

The three cases taken together indicate that g^{-1} of any subbase element of V is in V. Therefore g is in $C(X,V)$ and hence $C(X,U) \subset C(X,V)$.//

The following example illustrates the use of this theorem. Let X be the set of real numbers and let $U = \{O : X - O \text{ is finite}\} \cup \{\phi\}$. Let $I = \{1\}$ and let $O_1 = X - \{1,2,3,4,...\}$. The topology V obtained from the theorem is $V = \{O : X - O \text{ is countable}\} \cup \{\phi\}$. It will be shown on page 19 that in this example the set $C(X,V)$ actually properly contains the set $C(X,U)$.

A theorem similar to theorem 4 can be obtained for the set S_w.

THEOREM 5. (X,V) is a topological space with a subbase of the form S_w if and only if

i) V is weaker than U and

ii) $C(X,U) \subset C(X,V)$

PROOF. Again assuming i) and ii) trivially there exists a subbase of (X,V) of the form S_w, namely $S_w = \{ O : O \text{ is in } U\} - (\{O_i : O_i \text{ is in } V - U\} \cup \{O : O \in U$ and $f^{-1}(O) = O_i$ for some O_i in $V - U$ and some f in $C(X,U)\})$.

If V has a subbase of the form S_w it is immediate that V is weaker than U.

To show that $C(X,U)$ is a subset of $C(X,V)$ let g be a function from $C(X,U)$ and let O be from S_w. It is necessary to show that $g^{-1}(O)$ belongs to V. From the form of S_w $g^{-1}(O) \neq O_i$ for any i in I otherwise O would not be in S_w. Neither can $g^{-1}(O)$ equal a set O^* where there exists an f in $C(X,U)$ and an i in I such that $f^{-1}(O^*) = O_i$. For this would imply that $f^{-1}(g^{-1}(O)) = O_i$. That is $h^{-1}(O) = O_i$ where $h = g \circ f$ is in $C(X,U)$. This in turn would imply O is not in S_w. Therefore $g^{-1}(O)$ is in S_w and hence in V.//

The following example illustrates the use of this theorem. Let $X = \{a,b,c\}$, $U = \{\phi, \{a\}, \{b\}, \{b,c\}$ $\{a,b\}, X\}$ and let $S_w = \{O : O \text{ is in } U\} - (\{\{b\}\} \cup \{O : f^{-1}(O) = \{b\} \text{ for } f \text{ in } C(X,U)\})$. Since the function f defined by $f(a) = c$, $f(b) = b$ and $f(c) = c$ is in $C(X,U)$ and since $f^{-1}(\{a,b\}) = \{b\}$ the set $\{a,b\}$ will not be in S_w. Then S_w is a subbase for $V = \{\phi, \{a\}, \{b,c\}, X\}$. It is easy to see that $V = U \cap \{O : X - O \text{ is in } U\}$ and hence by theorem 2 and theorem 3 it follows that $C(X,V) \supset C(X,U)$.

Some methods for finding a topology V comparable to U such that $C(X,V) \supset C(X,U)$ have been considered. However, frequently there does not exist a topology V with this property. Several such spaces will be given throughout the remainder of this chapter.

The following definition is given in a paper by Hicks and Haddock [9].

DEFINITION 3. A space (X,U) is called an M space if and only if the set $\{H(f) : f \text{ is in } C(X,U)\}$ forms a basis for the closed sets of (X,U), where $H(f) = \{x : f(x) = x\}$.

This definition will be changed slightly to yield the definition of an M_1 space.

DEFINITION 4. A space (X,U) is called an M_1 space if and only if the set $\{H(f) : f \text{ is in } C(X,U)\}$ forms a subbasis for the closed sets of (X,U).

It follows immediately that if (X,U) and (X,V) are M_1 spaces such that $C(X,U) = C(X,V)$ then $U = V$. This is true since both spaces have the same subbase for their closed sets.

Two theorems concerning M_1 spaces will now be given.

THEOREM 6. If (X,U) is an M_1 space then there does not exist V such that (X,V) is an M_1 space and such that $t_o < V < U$ while $C(X,V) \supset C(X,U)$. (t_o denotes the trivial topology).

PROOF. Assume there exists a topology V satisfying the conclusion. Let $U_F = \{H(f) : f \text{ is in } C(X,U)\}$ and let

V_F = { H(f) : f is in C(X,V)}. Since U_F and V_F are subbases for the closed sets of (X,U) and (X,V) respectively and since $V_F \supset U_F$, it follows that V is at least as strong as U. This is a contradiction.//

THEOREM 7. If (X,U) is an M_1 space and (X,V) is any T_2 space such that $C(X,U) \subset C(X,V)$ then $U \leq V$.

PROOF. Since V is T_2 and $C(X,U) \subset C(X,V)$, H(f) is closed in V for each f in C(X,U). Thus the set of closed sets in V contains a subbase for the closed sets of U and thus $U \leq V$.//

The same proof holds if the condition (X,V) is T_2 is replaced by the weaker condition H(f) is closed in V for each f in C(X,U). An example will be given on page 64 showing that H(f) being closed for each continuous selfmap is a weaker condition then the property of being T_2.

B. THE SPACE OF REAL NUMBERS WITH THE USUAL TOPOLOGY

(R,U) shall denote the space of real numbers with the usual topology. It will be established in this section that there does not exist a topology V weaker than U such that $C(R,V) \supset C(R,U)$. Although it is not known if there exists a topology V stronger than U such that $C(R,V) \supset C(R,U)$, results will be given which will make it unnecessary to examine certain topologies in search for such a V.

THEOREM 8. There does not exist a topology V such that $t_o < V < U$ while C(R,V) contains C(R,U).

PROOF. Assume there exists a topology V such that $t_o < V < U$ and C(R,V) contains C(R,U).

CASE 1. Assume there exists a set in V of the form $\{x : x > a\}$ for some real number "a". (A similar argument can be made if there exists a set in V of the form $\{x : x < b\}$). Note $f_c(x) = x - c + a$ is in C(R,V) for each c in R. Hence $f_c^{-1}(\{x : x > a\}) = \{x : x > c\}$ is in V for each real number c. Likewise $g_c(x) = -x + c + a$ is in C(R,V) for each c in R and hence $g^{-1}(\{x : x > a\}) = \{x : x < c\}$ is in V. Thus a subbase for U is contained in V. This implies V is at least as strong as U which contradicts the hypothesis of this theorem.

CASE 2. Assume there exists an interval (a,b) in V. Again requiring $C(R,V) \supset C(R,U)$ is sufficient to imply that

$$f(x) = \begin{cases} \frac{a+b}{2} & \text{for all } x < 0 \\ x + \frac{a+b}{2} & \text{for all } x > 0 \end{cases} \quad \text{is in } C(R,V).$$

Therefore $f^{-1}((a,b)) = \{x : x < \frac{b-a}{2}\}$ is in V. Using the fact that $\{x : x < \frac{b-a}{2}\}$ is in V and the results of case 1 is sufficient to imply again that V must be at least as strong as U.

CASE 3. Let O be an arbitrary set in V such that $O \neq \phi$ and $O \neq R$. Since V is assumed to be weaker than U, O is also in U. A well-known theorem which states every O in U can be written as a countable union of disjoint open intervals will now be used. This theorem can be found in Royden [17]. The possibilities which have not already been treated in cases 1 and 2 will now be considered.

i) Let $O = (-\infty, a) \cup (b, \infty)$.

The function $f(x) = \begin{cases} x & \text{for all } x \le \frac{a+b}{2} \\ \frac{a+b}{2} & \text{for all } x > \frac{a+b}{2} \end{cases}$ is in $C(X,V)$.

Therefore $f^{-1}(O) = (-\infty, a)$ is in V and hence by case 1, V must be at least as strong as U.

ii) Let $O = \bigcup_{i=1}^{K} (a_i, b_i)$ where K is either finite or ∞, and the intervals are pairwise disjoint. Zero, one or two of these intervals might be unbounded. For some $j, (a_j, b_j)$ is an interval with finite bounds otherwise case 1 or subcase i) of case 3 applies.

The function $f(x) = \begin{cases} a_j & \text{for all } x \le a_j \\ x & \text{for all } x \text{ in } (a_j, b_j) \\ b_j & \text{for all } x \ge b_j \end{cases}$ is in $C(X,V)$.

Since neither a_j nor b_j is in O, $f^{-1}(O)$ is (a_j, b_j). This implies (a_j, b_j) is in V and then by case 2, V is at least as strong as U.

In each case we get a contradiction and thus the theorem is proven.//

THEOREM 9. If $C(R,V) \supset C(R,U)$ and for some O in V, O has a largest element (or smallest element) then V is the discrete topology.

PROOF. Let "a" be the largest element of O. Define

$f(x) = -x + 2a$. Then f is in C(R,V) and $f^{-1}(O)$ has "a" as the smallest element. Therefore $f^{-1}(O) \cap O = \{a\}$ is in V. For each c in R define $h_c(x) = x + a - c$. Then h_c is in C(R,V) and hence $h_c^{-1}(\{a\}) = \{c\}$ is in V for each real number c. Therefore every singleton is in V and hence V is the discrete topology.//

It follows immediately that if $C(R,V) \supset C(R,U)$ and there is a set O in V which is finite or is a closed or half closed interval with respect to U then V is the discrete topology.

In the above theorem it was not necessary to know that V was comparable to U in order to arrive at the conclusion. However this is not true in the following theorem.

THEOREM 10. If $C(R,V) \supset C(R,U)$, $V > U$, and for some O in V there exists real numbers a and b such that $(a,b) \cap O$ is a singleton, then V is the discrete topology.

PROOF. Assume $(a,b) \cap O = \{d\}$. Then $\{d\}$ is in V and by theorem 9,V is the discrete topology.//

In the next two theorems it will again not be necessary to require in advance that V is stronger than U.

THEOREM 11. If $C(R,V) \supset C(R,U)$ and if some O in V is the set of all rational numbers then V is the discrete topology.

PROOF. Clearly $f(x) = e^x$ is in C(R,V). It will now be shown that $f^{-1}(O) \cap O = \{\text{zero}\}$. That is, it will be shown that the only rational number mapped into a rational number by f(x) is zero. Assume $e^x = \frac{r}{s}$ where x is in O and x is different from zero. Note r and s are both positive since e^x is a positive function. Assume $x = p/q$ is a nonzero rational number where p is positive. Now $e^{p/q} = \frac{r}{s}$ implies $e^p = (\frac{r}{s})^q = \frac{m}{n}$ where m is r^q or s^q depending on whether q is positive or negative. n is defined similarly. Now $e^p = \frac{m}{n}$ implies $ne^p - m = 0$ which implies e is a solution to the polynomial equation $nx^p - m = 0$. This implies e is an algebraic number which contradicts the fact that e is a transcendental number. Therefore {zero} is in V and by theorem 9, V must be the discrete topology.//

This theorem can be generalized as follows.

THEOREM 12. If $C(R,V) \supset C(R,U)$ and if some O in V is a subset of rational numbers then V is the discrete topology.

PROOF. If O is a singleton then by theorem 9, V is the discrete topology. On the other hand if there exist distinct rationals "a" and "b" in O then $f(x) = a - 1 + e^{x-b}$ is in C(R,V). Note f(b) = a and it follows from the proof of theorem 11 that b is the only element of O which maps into O. Thus $f^{-1}(O) \cap O = \{b\}$ and again it follows from theorem 9 that V must be the discrete topology.//

C. CHARACTERIZING SPACES (X,t) BY C(X,t)

In this section certain semigroups of continuous selfmaps C(X) will be given and all topologies t will be found so that C(X) = C(X,t).

First of all if $C(X,t)$ consists of all functions from X to X then t could be t_o or D. In fact it was shown on page 6 that these are the only topologies with this property.

On the other hand, if $C(X)$ consists only of the constant selfmaps and the identity function then there may or may not be topologies t such that $C(X,t) = C(X)$. In fact the answer is dependent on the set X. For example there is no topology which can be put on three points such that the only continuous selfmaps are the constant functions and the identity. However de-Groot [18] gives 2^c nonhomeomorphic subspaces of the Euclidean plane with this property.

In the next four theorems spaces will be characterized by their continuous selfmaps, but first some definitions will be given.

DEFINITION 5. A function f is defined to be μ to one if $f^{-1}(\{x\})$ is a set with cardinality μ or less for every x in X.

DEFINITION 6. A space (X,t) is said to have the μ - complement topology if and only if $t = \{O : O = \phi$ or $X - O$ has cardinality μ or less$\}$.

The following is an extension of a result by Hicks and Haddock [9]. Their proof carries through but will be given here for completeness.

THEOREM 13. Let X be an arbitrary set.

i) $C(X,t) = \{f : f$ is a constant or finite to one function$\}$ if and only if t is the finite complement topology.

ii) $C(X,t) = \{ f : f$ is a constant or countable to one function$\}$ if and only if t is the countable complement topology.

iii) $C(X,t) = \{ f : f$ is a constant or μ to one function$\}$ if and only if t is the μ complement topology. (μ is assumed to be a transfinite cardinal number).

PROOF of iii). We will assume the cardinality of X is greater than μ. If not t will become the discrete topology and the result will follow immediately. Assume t is the μ-complement topology. If f is a constant function then f is in $C(X,t)$. Assume f is a nonconstant μ to one function and let O be in t. Note $f^{-1}(X) = f^{-1}(O \cup (X - O)) = f^{-1}(O) \cup f^{-1}(X - O) = X$. Since f is μ to one and since $X - O$ has cardinality less than or equal to μ it follows that $f^{-1}(O) = X - f^{-1}(X - O)$ is open, for it is the complement of $f^{-1}(X - O)$ and $f^{-1}(X - O)$ has cardinality less than or equal to μ. Therefore f is in $C(X,t)$.

Assume f is not a constant function but that there exists a point "a" in X such that $f^{-1}(\{a\})$ has cardinality greater than μ. Let $O = X - \{a\}$ then $f^{-1}(O) = f^{-1}(X - \{a\}) = X - f^{-1}(\{a\})$ fails to be open. Therefore f is not continuous. Thus $C(X,t) = \{ f : f$ is a constant function or f is μ to one$\}$. To show t is the only topology on X with continuous functions $C(X,t)$, assume there exists another topology s such that $C(X,t) = C(X,s)$. Let U be a nonempty set in s and assume $X - U \neq \phi$.

Choose q in U and p in X - U. Define $h(x) = \begin{cases} x & \text{for } x \neq p \\ q & \text{for } x = p \end{cases}$.

Then h is in C(X,t) = C(X,s) and hence $h^{-1}(U) = U \cup \{p\}$ is in s. Since p was an arbitrary element of X - U note the following property. If U is in s then each superset of U is in s. This follows since by arbitrary unions $U \cup \{x : x$ is in A where $A \subset X - U\}$ is in s.

Suppose now that U is in s and X - U has cardinal number greater than μ. Let q be in X - U and define f(x) by

$f(x) = \begin{cases} x & \text{for } x \text{ in } U \\ q & \text{for } x \text{ in } X - U \end{cases}$. Note f is neither constant

nor μ to one. However it will be shown that if such a set U is allowed to be in s then f will be continuous. To see this let V be in s. If q is not in V then $f^{-1}(V) = V \cap U$ which is open. If q is in V then $f^{-1}(V) = V \cup (X - U)$. This set is open since V is in s and it was shown above that supersets of open sets are open. Thus it can be concluded that if C(X,s) = C(X,t) and U is in s then X - U has cardinal number less than or equal to μ. To verify the converse of this statement let $U \subset X$ be such that X - U has cardinality less than or equal to μ. Let V be in s such that $V \neq \phi$ and $V \neq X$. If $V \subset U$ then U is a superset of V and hence U is in s. If $V \not\subset U$ then V - U is nonempty. Let p be in V - U and let q be in X - V. A function g will now be defined which is μ to one and hence g will be in C(X,t) = C(X,s).

$$\text{Let } g(x) = \begin{cases} x \text{ for } x \text{ in } U \cap V \\ x \text{ for } x \text{ in } X - (U \cup V) \\ p \text{ for } x \text{ in } U - V \\ q \text{ for } x \text{ in } V - U . \end{cases}$$

Note $U - V \subset X - V$ and hence $U - V$ has cardinality less than or equal to μ. Since g is continuous $g^{-1}(V) = (U \cap V) \cup (U - V) = U$ is in s. Therefore U is in s if and only if $X - U$ has cardinality less than or equal to μ. Hence $s = t$ and the theorem is established.//

Another topology which can be put on an arbitrary set X will now be considered. This particular topology is always possible since an arbitrary set can be well-ordered.

DEFINITION 7. Let X be a well-ordered set with smallest element s and linear ordering $<$. Then tw and ctw will be defined as follows. $tw = \{\phi, \{s\}, X\} \cup \{\{x : s \leq x < r\} : r \text{ is in } X\}$ and $ctw = \{O : X - O \text{ is in } tw\}$.

LEMMA 1. (X,tw) and (X,ctw) are topological spaces and C(X,tw) = C(X,ctw).

PROOF. In order to prove this lemma, it is sufficient to show that (X,tw) is a saturated space and then apply theorem 2. To this end note that ϕ and X are in tw by definition. Also note that $\bigcup_{a \varepsilon A} \{x : s \leq x < r_a\} = X$ or else, $\bigcup_{a \varepsilon A} \{x : s \leq x < r_a\} = \{x : s \leq x < r\}$ where r is the first element of $X - \bigcup_{a \varepsilon A} \{x : s \leq x < r_a\}$. To see that tw is saturated observe that $\bigcap_{a \varepsilon A} \{x : s \leq x < r_a\} =$

$\{x : s \le x < r\}$ where r is the first element of $X - \bigcap_{a \in A} \{x : s \le x < r_a\}$.//

DEFINITION 8. The topology tw as given in definition 7 and lemma 1 will be called the tower topology.

LEMMA 2. f is in C(X,tw) if and only if f is a nondecreasing function.

PROOF. Assume f is nondecreasing. Let $O_r = \{x : s \le x < r\}$. Since by definition $f^{-1}(O_r) = \{x : f(x) < r\}$ and since f is assumed to be nondecreasing either $f^{-1}(O_r)$ is ϕ or X or else there exists a first element x* not in $f^{-1}(O_r)$. In the latter case $f^{-1}(O_r) = \{x : s \le x < x^*\}$ In any case $f^{-1}(O_r)$ is in tw and thus f is in C(X,tw).

On the other hand assume f is in C(X,tw) and f is not nondecreasing. Then there exist elements y and z such that $y > z$ but $f(z) > f(y)$. Let $O = \{x : s \le x < f(z)\}$. Since f is assumed to be continuous $f^{-1}(O)$ must be in tw. Notice that y is in $f^{-1}(O)$ while z is not in $f^{-1}(O)$. Therefore $f^{-1}(O)$ can not be open since by the definition of tw there is no open set which contains some element but fails to contain a smaller element. (Recall $z < y$). This is a contradiction and hence if f is in C(X,tw) then f must be a nondecreasing function.//

LEMMA 3. If $V > tw$ and $C(X,V) \supsetneq C(X,tw)$ then V is the discrete topology.

PROOF. Let O be an open set from V - tw.

CASE 1. Assume s is in O. Let x be the first element not in O and let y be an element of O such that $y > x$. There is such an element y for otherwise $O = \{ z : s \leq z < x\}$ which would imply O is in tw. Let $f(z) = \begin{cases} x \text{ for all } z < x \\ y \text{ for all } z \geq x \end{cases}$.

From lemma 2, f is continuous and thus $f^{-1}(O) = \{ z : z \geq x\}$ is in V. Let x^* be the first element of $\{ z : z > x\}$. Since $\{ z : s \leq z < x^*\}$ is in tw and since $tw < V$ it follows that $\{ z : s \leq z < x^*\} \cap \{ z : z \geq x\} = \{ x\}$ is in V.

Let $f_r(z) = \begin{cases} s \text{ for } z < r \\ x \text{ for } z = r \\ x^* \text{ for } z > r \end{cases}$. Again by lemma 2, f_r is continuous. Thus $f_r^{-1}(\{ x\}) = \{ r\}$ is in V. Since r was arbitrary this implies each singleton is open in V and thus V is the discrete topology.

CASE 2. Assume s is not in O. Let x be the first element of O. Define $g(z) = \begin{cases} s \text{ for } z < x \\ x \text{ for } z \geq x \end{cases}$. From lemma 2, g is continuous and therefore $g^{-1}(O) = \{ z : z \geq x\}$ is in V. Let x^* be the first element of $\{ z : z > x\}$. As in case 1, $\{ z : s \leq z < x^*\} \cap \{ z : z \geq x\} = \{ x\}$ is in V. Using the continuous function f_r as defined in case 1, yields $f_r^{-1}(\{ x\}) = \{ r\}$ is in V. Again the fact that r is arbitrary is sufficient to imply that V is the discrete topology.//

LEMMA 4. If $V > ctw$ and $C(X,V) \supset C(X,ctw)$ then V is the discrete topology.

PROOF. Let O be a set in V - ctw.

CASE 1. Assume s is not in O. Let x be the first element of O and let y be some element such that y is not in O but such that y > x. This is possible otherwise O would be $\{z : z \geq x\}$ which would imply O is in ctw.

Define $f_r(z) = \begin{cases} s \text{ for } z < r \\ x \text{ for } z = r \\ y \text{ for } z > r \end{cases}$. By lemma 2, f_r is continuous. Therefore $f_r^{-1}(O) = \{r\}$ is in V. The fact that r is arbitrary implies V is the discrete topology.

CASE 2. Assume s is in O. Let x be the first element not in O. Define $g_r(z) = \begin{cases} s \text{ for } z \leq r \\ x \text{ for } z > r \end{cases}$. g_r is continuous and hence $g_r^{-1}(O) = \{z : z \leq r\}$ is in V. Therefore $\{z : z \leq r\} \cap \{z : z \geq r\} = \{r\}$ is in V. Again since r is arbitrary V must be the discrete topology.//

THEOREM 14. Let X be a well-ordered set with linear ordering < and smallest element s. C(X,V) consists precisely of the set of all nondecreasing functions if and only if V is either tw or ctw.

PROOF. It follows immediately from lemmas 1 and 2 that if V is either tw or ctw then C(X,V) consists of the set of all nondecreasing functions.

The proof that if C(X,V) consists of precisely the nondecreasing functions then V is either tw or ctw will be treated in two cases. Although the theorem is true if

X is a singleton, it will be assumed here that X has at least two elements. Then V is clearly neither the discrete nor the trivial topology.

CASE 1. Assume there is a set O in V such that $O \neq \phi$ and $O \neq X$ and s is in O. Let x be an element of X - O.

Define $f(z) = \begin{cases} s & \text{for } z = s \\ x & \text{for } z > s \end{cases}$. Then f is continuous and $f^{-1}(O) = \{s\}$ belongs to V. For $n > s$ define

$f_n(z) = \begin{cases} s & \text{for } z < n \\ x & \text{for } z \geq n \end{cases}$. f_n is continuous and $f_n^{-1}(\{s\}) = \{z : s \leq z < n\}$ belongs to V. Since n is arbitrary it follows that $V \geq tw$. But because of lemmas 2 and 3 it must be concluded that V = tw.

CASE 2. Assume for every O* in V such that $O^* \neq X$ that s does not belong to O*. Let O be one such set and let n be the first element of O. Let m be an arbitrary element of X which follows s. Define $f_m(z) = \begin{cases} s & \text{for } z < m \\ n & \text{for } z \geq m \end{cases}$. Then $f_m^{-1}(O) = \{z : z \geq m\}$ belongs to V. Since m is arbitrary it follows that $V \geq ctw$. However because of lemmas 1, 2 and 4 it must be concluded that V = ctw.//

Another topology which can be put on an arbitrary set X is given in the following definition.

DEFINITION 9. Let X be an arbitrary set with arbitrary subset A. Let $s = \{O : O \subset X \text{ and } O \supset A\} \cup \{\phi\}$. (X,s) is called the "superset of A" topology.

It is clear that (X,s) is a saturated space and hence (X,s_c) is a topological space where $s_c = \{ O : X - O \text{ is in } s\}$.

LEMMA 5. If s is the "superset of A" topology on X then $C(X,s) = C(X,s_c) = \{ f : f \text{ is a constant function or } f(A) \subset A\}$.

PROOF. From theorem 2 it follows that $C(X,s) = C(X,s_c)$.

If f is a constant function then f is clearly continuous. Assume $f(A) \subset A$ and let O be a nonempty open set then $O \supset A$. Therefore $f^{-1}(O) \supset f^{-1}(A) \supset A$. Hence $f^{-1}(O)$ is open and thus f is in C(X,s).

On the other hand if f(A) is not a subset of A for some nonconstant function f then there exists some x in A such that f(x) is not in A and there exists some y in X such that y is an image point of f but y does not equal f(x). Let $O_1 = A \cup \{y\}$ and note that $f^{-1}(O_1)$ is nonempty but does not contain A. Therefore $f^{-1}(O_1)$ is not open and hence f can not be continuous.//

Although some of the lemmas which follow are true for the "superset of A" topology not all of the lemmas hold. Thus in the next three lemmas and in theorem 15 it will be assumed that s is the superset of the singleton set $\{a\}$. s_c is defined similarly.

LEMMA 6. Let s be the "superset of $\{a\}$" topology on X. There does not exist a topology V such that $t_o < V < s$ and such that $C(X,V) \supset C(X,s)$. (This lemma could be obtained in the more general setting of lemma 5).

PROOF. Assume there does exist a topology V which satisfies the conditions of this lemma. Let O be a nonempty open set of $V \cap s$ such that $O \neq X$, let c be an element of $X - O$ and let B be an arbitrary subset of X such that a is in B. Define $f_B(x) = \begin{cases} a \text{ for } x \text{ in } B \\ c \text{ for } x \text{ not in } B \end{cases}$. From lemma 5 it follows that f_B is continuous. Therefore $f_B^{-1}(O) = B$ belongs to V. Since B was an arbitrary subset of X which contains $\{a\}$ it follows that $V \geq s$ which contradicts the assumption that $V < s$.//

LEMMA 7. Let s be the "superset of $\{a\}$" topology on X and let s_c be defined as usual. There does not exist a topology V such that $t_o < V < s_c$ with $C(X,V) \supset C(X,s_c)$.

PROOF. Assume there does exist a topology V satisfying the conditions of this lemma. Then there exists a nonempty O in V such that a is in $X - O$. Let x be some element of O and let t be an arbitrary element of X such that $t \neq a$. Define $f_t(z) = \begin{cases} x \text{ for } z = t \\ a \text{ for } z \neq t \end{cases}$. It follows from lemma 5 that f_t is continuous. Therefore $f_t^{-1}(O) = \{t\}$. Since t is an arbitrary element different from a, every singleton different from $\{a\}$ is open and hence $V \geq s_c$. This contradicts the assumption that $V < s_c$.//

LEMMA 8. Let s be the "superset of $\{a\}$" topology on X. There does not exist a topology V such that $s < V < D$ and such that $C(X,V) \supset C(X,s)$.

PROOF. Assume there exists a topology V satisfying the conditions of the lemma. Let O be a set in $V - s$. Then notice that a is not in O. Let x be an element of O and define $f(z) = \begin{cases} x \text{ for } z = x \\ a \text{ for } z \neq x \end{cases}$. Then f is continuous and $f^{-1}(O) = \{x\}$ is in V. Let t be an arbitrary element of X such that $t \neq a$ and define $f_t(z) = \begin{cases} x \text{ for } z = t \\ a \text{ for } z \neq t \end{cases}$. Since $f_t(a) = a$, f_t is continuous. Therefore $f_t^{-1}(\{x\}) = \{t\}$ is in V. Since {a} is known to be in V, V contains each singleton set and thus V = D. This contradicts the assumption that V < D.//

This lemma is not true if s is the "superset of A" topology. To see this let X = {1,2,3} and A = {1,2}. Then $s = \{\phi, \{1,2\}, \{1,2,3\}\}$ and $s_c = \{\phi, \{3\}, \{1,2,3\}\}$. Note from the results of theorem 3 that $V = s \cup s_c = \{\phi, \{1,2\}, \{3\}, \{1,2,3\}\}$ satisfies the conditions that V > s and $C(X,V) \supset C(X,s)$.

THEOREM 15. Let X be an arbitrary set with arbitrary element "a". Then C(X,t) = {f : f is a constant function or f(a) = a} if and only if t is either s or s_c where s is the "superset of {a}" topology.

PROOF. It follows immediately from lemma 5 that if t is s or s_c then C(X,t) = {f : f is a constant function or f(a) = a}.

To show that there are exactly two topologies whose class of continuous selfmaps is C(X,t) assume V is a topology such that C(X,V) = C(X,t). Clearly V is neither t_o nor D.

CASE 1. Assume $V \cap s \neq t_o$. By theorem 3 it follows that

$C(X, V \cap s) \supset C(X,s)$. Now $(V \cap s) \geq s$ otherwise lemma 6 is contradicted. Therefore $V \geq s$ and hence by lemma 8, $V = s$.

CASE 2. Assume $V \cap s = t_o$. Then there exists a nonempty set O in V such that a is in X - O. Therefore $V \cap s_c \neq t_o$. By theorem 3, $C(X, V \cap s_c) \supset C(X, s_c)$. Thus $(V \cap s_c) \geq s_c$ otherwise lemma 7 will be contradicted. Hence $V \geq s_c$. Actually $V = s_c$. For, if V properly contained s_c there would be a set O in V such that $O \neq X$ and such that O contains the element "a". This would contradict $V \cap s = t_o$.//

Note that if the element "a" of the superset topology is the same as the element "s" of the tower space then the tower space is strictly weaker than the "superset of {a}" topology while the sets C(X,tw) and C(X,s) are noncomparable in the sense that neither is a subset of the other.

The last problem to be considered in this section is that of finding a topology t on the real numbers such that $C(R,t) = \{ f : f \text{ is a constant function or } |f(x)| \to \infty \text{ as } |x| \to \infty \}$.

DEFINITION 10. Let R be the set of real numbers and let $Z = \{\phi\} \cup \{ O : \text{there exist real numbers a and b satisfying } O \supset ((-\infty, a) \cup (b, \infty)) \}$. The space (R,Z) is called the fuzzy space.

Note a set F is closed in the fuzzy space if and only if F is R or F is bounded. This space is nonsaturated for let $O_n = (-\infty, -1) \cup (n, \infty)$ and note that $\bigcap_{n=1}^{\infty} O_n$ is not in Z.

LEMMA 9. If (R,Z) is the fuzzy space then $C(R,Z) = \{ f : f$ is a constant function or $|f(x)| \to \infty$ as $|x| \to \infty\}$.

PROOF. Assume f is a function such that $|f(x)| \to \infty$ as $|x| \to \infty$. Let F be a closed set. Then F is bounded and hence there exists a positive real number "a" such that $F \subset (-a,a)$. Since $|f(x)| \to \infty$ as $|x| \to \infty$ there exists a positive real number "b" such that $|f(x)| > a$ for all x with $|x| > b$. Therefore $f^{-1}(F) \subset f^{-1}(\{(-a,a)\}) \subset (-b,b)$, hence $f^{-1}(F)$ is bounded and thus closed. Thus f is continuous.

On the other hand, assume f is a nonconstant continuous selfmap and assume $|f(x)|$ does not approach ∞ as $|x| \to \infty$. Then for some $a > 0$ there does not exist b such that $|f(x)| \geq a$ for all x satisfying $|x| > b$. That is, for every b there exists some x such that $|x| > b$ and $|f(x)| < a$. Let F be the closed set $(-a,a)$. Then $f^{-1}(F)$ is not bounded and hence $f^{-1}(F)$ is not closed. Therefore f is not continuous.//

LEMMA 10. Let (R,Z) be the fuzzy space. There does not exist a topology V such that $t_o < V < Z$ and such that $C(R,V) \supset C(R,Z)$.

PROOF. Assume there is a topology V satisfying the conditions of this lemma. Let O be a set of $V - t_o$ and let O_z be an arbitrary set in Z. Then there exist real numbers a_1, a_2, b_1 and b_2 such that $O \supset ((-\infty,a_1) \cup (b_1,\infty))$ and $O_z \supset ((-\infty,a_2) \cup (b_2,\infty))$. Choose p and q such that q is in O and p is in R - O and define f as follows.

$$\text{Let } f(x) = \begin{cases} x \text{ for all } x < a = \min \{a_1, a_2\} \\ x \text{ for all } x > b = \max \{b_1, b_2\} \\ q \text{ for all } x \text{ in } O_z - ((-\infty, a) \cup (b, \infty)) \\ p \text{ for all } x \text{ in } R - O_z \end{cases}$$

Now f is continuous and $f^{-1}(O) = O_z$. Since O_z is arbitrary it follows that V = Z. This is a contradiction since it was assumed that $V < Z$.//

LEMMA 11. Let (R,Z) be the fuzzy space. There does not exist a topology V such that $Z < V < D$ while $C(R,V) \supset C(R,Z)$.

PROOF. Assume there is a topology V which satisfies the conditions of the lemma. Let O be in V - Z. Since O is not in Z, for each positive integer n there exists a real number b_n such that $|b_n| > n$ and b_n is in X - O. Let "a" be an arbitrary element in O and define f as follows.

$$f(x) = \begin{cases} a \text{ for } x = a \\ b_n \text{ for } x \text{ in } \{[-n, -n+1) \cup [n-1, n)\} - \{a\} \end{cases}.$$

Then f is continuous and $f^{-1}(O) = \{a\}$ belongs to V. Let "c" be an arbitrary real number different from "a" and

$$\text{define } g_c(x) = \begin{cases} a \text{ for } x = c \\ a + 1 \text{ for } x = a \\ |x| \text{ for } x \text{ in } R - \{a, c\} \end{cases}.$$

Then g is continuous and $g_c^{-1}(\{a\}) = \{c\}$ is in V. Since "c" is arbitrary this implies V = D which is a contradiction.//

THEOREM 16. Let R be the set of real numbers. Then $C(R,t) = \{ f : f$ is a constant function or $|f(x)| \to \infty$ as

$|x| \to \infty\}$ if and only if (R,t) is the fuzzy space (R,Z).

PROOF. It follows from lemma 9 that if $t = Z$, then $C(R,t) = \{f : f$ is a constant function or $|f(x)| \to \infty$ as $|x| \to \infty\}$.

On the other hand assume there exists a topology V different from Z such that $C(R,V) = C(R,Z)$. Clearly V is neither t_o nor D.

CASE 1. Assume $V \cap Z \neq t_o$. Then from theorem 3 it follows that $C(R,V \cap Z) \supset C(R,Z)$. Therefore $(V \cap Z) = Z$ or else lemma 10 is contradicted. However if $(V \cap Z) = Z$ then $V > Z$ and lemma 11 is contradicted.

CASE 2. Assume $V \cap Z = t_o$. Let O be a set of $V - t_o$. Note O is in $V - Z$, thus a repeat of the argument used in the proof of lemma 11 will force $V = D$. This will contradict the fact that $V \cap Z = t_o$. In either case a contradiction is reached and hence the desired conclusion follows.//

IV. HOMEOMORPHISMS FROM A SPACE TO ITSELF

A. PARTIALLY ORDERED ARRAY OF SPACES

The class of all homeomorphisms from a space (X,t) to itself shall be denoted by H(X,t) or H(X) if the topology is not essential to the argument. By assigning different topologies t to an arbitrary set X an array of spaces is obtained. The relation (X,t) follows (X,s) if and only if $H(X,t) \supset H(X,s)$ is both reflexive and transitive. Hence this relation defines a partial order on the array of spaces. It will be shown in this chapter that this array has both largest elements, that is there exist topologies on X such that H(X,t) consists of all 1-1 onto functions, and smallest elements, that is there exist topologies on X such that H(X,t) consists of only the identity function. Since $C(X,t_o)$ and C(X,D) both consist of all functions it is obvious that (X,t_o) and (X,D) are largest elements in this array. Other largest elements and some smallest elements will be found in the work which follows.

THEOREM 1. If $C(X,t) \subset C(X,s)$ then $H(X,t) \subset H(X,s)$.

PROOF. Let f be an element of H(X,t). Then f is 1-1, onto and bicontinuous. Therefore f and f^{-1} are elements of C(X,t) and hence elements of C(X,s). Thus f is in H(X,s).//

The following is an obvious corollary.

COROLLARY. If C(X,t) = C(X,s) then H(X,t) = H(X,s). Neither the converse of the theorem nor the converse of the corollary are true. Examples will be given on page 37 pointing this out.

THEOREM 2. Let $\{t_a : a \varepsilon A\}$ be a collection of topologies on X, then

i) $\bigcap_{a\varepsilon A} H(X,t_a)$ is a subset of $H(X, \bigcap_{a\varepsilon A} t_a)$ and

ii) $\bigcap_{a\varepsilon A} H(X,t_a)$ is a subset of H(X,t) where t is the least upper bound of the collection of topologies $\{t_a, a\varepsilon A\}$.

PROOF. i) Let f be an element of $\bigcap_{a\varepsilon A} H(X,t_a)$. Then f and f^{-1} both belong to $\bigcap_{a\varepsilon A} C(X,t_a)$. Hence by theorem 3 of chapter III, f and f^{-1} both belong to $C(X, \bigcap_{a\varepsilon A} t_a)$. Thus f belongs to $H(X, \bigcap_{a\varepsilon A} t_a)$.

ii) The proof is similar to the proof of part i).//

It follows immediately from this theorem that if t and s are topologies on X such that H(X,t) = H(X,s), then i) H(X,t) is a subset of $H(X,t \cap s)$ and ii) H(X,t) is a subset of H(X, lub. {t,s}).

Assume (X,U) is a given space. It was seen in chapter III that if V has a subbase of the form S_w or S_s where S_w and S_s are defined on page 10 then $C(X,V) \supset C(X,U)$. Therefore by theorem 1, (X,V) follows (X,U) in the partially ordered array of spaces.

Another class of topologies such that $H(X,t) \subset H(X,t^*)$ where $t^* \leq t$ will be given after the following definition due to Levine [19].

DEFINITION 1. Let (X,t) be a given space. The complementary topology t^* is defined to be the topology whose base is $\{ \mathrm{Int}_t\, F : F \text{ is closed in } (X,t)\}$.

THEOREM 3. Let (X,t) be an arbitrary space. Then $H(X,t) \subset H(X,t^*)$ and $t^* \leq t$ where t^* denotes the complementary topology for t.

PROOF. It follows immediately from definition 1 that $t^* \leq t$. To show that $H(X,t)$ is a subset of $H(X,t^*)$, let h be an element of $H(X,t)$ and let O be an open set of t^*. Then $O = \bigcup_{a\varepsilon A} \mathrm{Int}_t\, F_a$ and hence $h(O) = h(\bigcup_{a\varepsilon A} \mathrm{Int}_t\, F_a)$

$= \bigcup_{a\varepsilon A} h(\mathrm{Int}_t\, F_a) = \bigcup_{a\varepsilon A} \mathrm{Int}_t\, h(F_a)$ which is in t^*.

The last equality holds since it will now be established that $h(\mathrm{Int}_t\, F_a) = \mathrm{Int}_t\, h(F_a)$.

i) Since $h(\mathrm{Int}_t\, F_a) \subset h(F_a)$ and since $h \,\varepsilon\, H(X,t)$ it follows that $h(\mathrm{Int}_t\, F_a)$ is open. Therefore since $\mathrm{Int}_t\, h(F_a)$ is the largest open subset of $h(F_a)$, the result $h(\mathrm{Int}_t\, F_a) \subset \mathrm{Int}_t\, h(F_a)$ is obtained.

ii) On the other hand if $x \,\varepsilon\, \mathrm{Int}_t\, h(F_a)$ then there is a set O in t such that $x \,\varepsilon\, O \subset h(F_a)$. Thus $h^{-1}(x) \,\varepsilon\, h^{-1}(O) \subset F_a$. Since $h^{-1}(O)$ is in t it follows that $h^{-1}(x) \,\varepsilon\, \mathrm{Int}_t\, F_a$ and thus $x \,\varepsilon\, h(\mathrm{Int}_t\, F_a)$. Therefore $\mathrm{Int}_t\, h(F_a) \subset h(\mathrm{Int}_t\, F_a)$.

From the above we can conclude that h^{-1} is continuous. It follows by a similar argument that h is continuous. Since h is clearly 1-1 and onto, h must be an element of

H(X,t*).//

The following two examples illustrate theorem 3.

1) Let X be the set of integers and let t be the finite complement topology. Then t* is the trivial topology. It will be shown in theorem 5 that H(X,t) = H(X,t*).

2) Let X be the set of integers and let t be the "superset of { 1}" topology. Then t* is again the trivial topology and H(X,t) is a proper subset of H(X,t*).

The next example points out that if t* is the complementary topology of t then it is not necessary for C(X,t) to be a subset of C(X,t*) even though it frequently happens.

3) Let X = {1,2,3,4}, t = {ϕ,{1}, {2}, {1,2}, {1,2,3}, {1,2,3,4}} then t* = {ϕ, {1}, {2}, {1,2}, {1,2,3,4}}. Define the function f by f(x) = 1 if x belongs to {1,2,3} and f(4) = 4. Then f is in C(X,t) - C(X,t*).

This example also points out that the converse of theorem 1 does not hold. By examining the topologies of theorem 5 it will be immediate that the converse to the corollary of theorem 1 also can not be established.

Since every set can be well-ordered the following theorem yields smallest elements in the partially ordered array of spaces.

THEOREM 4. Let X be an arbitrary set. Then the tower topology tw and the topology t_c = {O : X - O is in tw} yield spaces (X,tw) and (X,t_c) such that H(X,tw) = H(X,t_c) = {the identity function}.

PROOF. From theorem 14 of chapter III, C(X,tw) consists of all nondecreasing function. The identity selfmap is the only continuous selfmap which can be 1-1 and onto.//

It is to be understood that these may not be the only smallest topologies on X, in fact 12 of the 29 possible topologies defined on {a,b,c} have only the identity function as an autohomeomorphism.

In the following section all largest elements will be found.

B. COMPLETELY HOMOGENEOUS SPACES

DEFINITION 2. A space (X,t) is said to be completely homogeneous if and only if every 1-1 onto selfmap is a homeomorphism.

The following theorem characterizes all completely homogeneous topologies which can be put on a set X if the cardinality of X is less than or equal to c.

THEOREM 5. Let X be a set of cardinality less than or equal to c. Assuming the continuum hypothesis then (X,t) is completely homogeneous if and only if t is the trivial topology, the finite complement topology, the countable complement topology or the discrete topology. (F_c and C_c will denote the finite complement and countable complement topologies respectively).

PROOF. From theorem 13 of chapter III each of the four topologies listed in the theorem are completely homogeneous.

To show that these are the only completely homogeneous topologies assume that V is completely homogeneous and $V \neq t_o$. The proof is divided into three parts depending on the cardinality of X.

PART 1. Assume X is finite. Let O be from $V - t_o$. Then $O = \{a_1, a_2, \ldots, a_n\}$ where n is less than the cardinality of X. Since each 1-1 onto function is a homeomorphism it follows that each subset of X containing n elements is open. Then by taking finite intersections it follows that each singleton subset of X is open. Therefore V = D.

PART 2. Assume X is an infinite countable set and let O be from $V - t_o$.

If O is a finite set then using the argument in part 1 it follows that V = D.

On the other hand if O is countable say $O = \{a_1, a_2, \ldots\}$ then either $X - O = \{b_2, b_3, \ldots\}$ is infinite or $X - O = \{c_1, c_2, \ldots c_n\}$ is finite. If X - O is infinite define f such that $f(a_1) = a_1$, $f(a_i) = b_i$ for $i \geq 2$, $f(b_i) = a_i$ for $i \geq 2$ and $f(x) = x$ for all other x in X. Then f is in H(X,V) and hence $f(O) = \{a_1, b_2, b_3 \ldots\}$ is open. Thus $f(O) \cap O = \{a_1\}$ is open. Since (X,V) is homogeneous every singleton is open. Therefore V = D. If X - O is finite with n elements then each set with n elements is closed. By finite intersections of these closed sets it follows that each singleton is closed. Thus $V \geq F_c$ since every T_1 space is stronger than F_c. If $V = F_c$ this part is completed. However if $V > F_c$ then there exists an open set O_V such that $X - O_V$ is infinite and countable. By making an argument similar to the argument at the beginning of this paragraph it will follow that $V = D$.

PART 3. Assume X has cardinality c. Let O be from $V - t_o$.

CASE 1. If O is finite then by the argument in part 1, $V = D$.

CASE 2. If O is countably infinite, say $O = \{a_1, a_2, \ldots\}$, then let $D^* = \{d_2, d_3, \ldots\}$ be a countable set disjoint from O. Define f such that $f(a_1) = a_1$, $f(d_i) = a_i$ for $i \geq 2$, $f(a_i) = d_i$ for $i \geq 2$ and $f(x) = x$ for all other x in X. Since f is assumed to be a homeomorphism, $f(O) = \{a_1\} \cup D^*$ is open. Therefore $f(O) \cap O = \{a_1\}$ is open. By homogeneity each singleton is open and thus $V = D$.

CASE 3. Assume O has cardinality c.

SUBCASE 1. Assume $X - O = \{d_1, d_2, \ldots, d_n\}$ is finite. As was shown in part 2, $V \geq F_c$. If $V = F_c$ this subcase is completed. If $V > F_c$ then there exists O^* in $V - F_c$ such that $X - O^*$ is infinite.

i) If $X - O^*$ is countable then $X - O^* = \{a_1, a_2, \ldots\} = A$ is closed. It will now be shown that if $B = \{b_1, b_2, \ldots\}$ then B is closed. In order to show that B is closed let $C = \{c_1, c_2, c_3, \ldots\}$ where $C \subset X$ and $C \cap (A \cup B) = \phi$. There exists a set C of this type since the cardinality of X is c.

Define f by $f(a_i) = c_i$ for $i \geq 1$, $f(c_i) = a_i$ for $i \geq 1$ and $f(x) = x$ for all other x in X. Then f is a homeomorphism and $f(A) = C$. Therefore C is closed. Thus define a homeomorphism h by $h(c_i) = b_i$ for $i \geq 1$, $h(b_i) = c_i$ for $i \geq 1$ and $h(x) = x$ for all other x in X. Note $h(C) = B$ and hence B is closed.

Since an arbitrary countably infinite set is closed and since $V > F_c$ it is necessary that $V \geq C_c$. If $V = C_c$ this subcase is completed. If $V > C_c$ then there exists O' in V such that X - O' is uncountable. This leads to three possibilities.

a) If O' is finite then by case 1, V = D.

b) If O' is countably infinite then by case 2, V = D.

c) If O' is uncountable then $O' = \{a\} \cup (O' - \{a\})$ where "a" is an element of O. Note X - O' and O' - {a} each have cardinality c.

Since each of these sets has cardinality c there exist 1-1 onto functions f_1 and f_2 which map respectively, X - O' and O' - {a} onto the set R of real numbers. Define a homeomorphism h from X to X as follows. Let h(a) = a, let $h(x) = f_2^{-1}(f_1(x))$ for x in X - O' and let $h(x) = f_1^{-1}(f_2(x))$ for x in O' - {a}. Then $h(O') \cap O' = \{a\}$ must be in V. Therefore, by homogeneity each singleton is open and thus V = D.

ii) If X - O* has cardinality c then by repeating the argument in c) above it follows that V = D.

SUBCASE 2. Assume X - O is countably infinite. Let X - O = $\{a_1, a_2, \ldots\}$ = A and let D = $\{d_2, d_3, \ldots\}$ be a subset of of X disjoint from A. Define a homeomorphism g by $g(a_1) = a_1$, $g(a_i) = d_i$ for $i \geq 2$, $g(d_i) = a_i$ for $i \geq 2$ and $g(x) = x$ for all other x in X. Then g(A) is closed and hence $g(A) \cap A = \{a\}$ is closed. By homogeneity it follows that $V \geq F_c$. Actually $V > F_c$ since X - O is a countably infinite closed set. This subcase is now completed by repeating the proof of case 3 starting at i) of subcase 1.

SUBCASE 3. Assume X - O has cardinality c. By an argument similar to that given in part c) of subcase 1 it follows that V = D.

All possibilities have now been considered and thus the theorem is established.//

Although the previous theorem was proposed and proven by this author the following more general theorem has been recently announced. The following theorem by Larson [20] was announced without proof in the abstract section of the Notices of the American Mathematical Society.

THEOREM 6. The only completely homogeneous topologies on X are the following: (1) the trivial topology, (2) the discrete topology and (3) topologies of the form $t = \{G : G \subset X \text{ and card } (X - G) \leq m\}$ where $\aleph_0 \leq m <$ card X.

C. 1-1, ONTO, CONTINUOUS SELFMAPS

In most of the examples considered so far every 1-1, onto, continuous selfmap actually turns out to be a homeomorphism. The following question might be asked. When are all 1-1, onto, continuous selfmaps homeomorphisms?

The following example shows that 1-1, onto, continuous selfmaps need not be homeomorphisms. Let X be the set of nonnegative real numbers and let $t = \{\phi, X, [0,1], [0,2], [0,3], \ldots\}$ then $f(x) = \frac{x}{2}$ is a 1-1, onto function in $C(X,t)$ which is not a homeomorphism. Let $t_1 = \{\phi, X, [0,2], [0,4], [0,6], \ldots\}$ and note that $t_1 < t$ while the spaces (X,t) and (X,t_1) are homeomorphic. It also turns out that $C(X,t)$ and $C(X,t_1)$ are noncomparable in the sense that neither is a subset of the other.

In partial answer to the above question it is known that, if X is a compact, T_2 space then each continuous, 1-1, onto map is a homeomorphism. This theorem can be found in Kelley [21]. This theorem will now be extended and two other results will be given.

The following definition is due to Levine [22].

DEFINITION 3. A space (X,t) is called a CC space if and only if closed and compact sets coincide.

It is well known that every compact, T_2 space is a CC space and also that many theorems defined on compact, T_2 spaces can be extended to CC-spaces. The following is one such theorem.

THEOREM 7. If (X,t) is a CC space then every 1-1, onto, continuous selfmap is a homeomorphism.

PROOF. One way to obtain the desired conclusion is to show that if f is a 1-1, onto, continuous selfmap then f is a closed function. To this end let F be a closed set in (X,t). Then F is compact and thus since f is continuous, f(F) is compact. Since f(F) is compact, f(F) is closed. Therefore f is a homeomorphism.//

Let (X,t) be the space where X = {rational numbers} $\cup$ {∞} and t is the one point compactification of the rational numbers where the topology on the rational numbers is the relative topology from the real numbers. This space is shown in Levine [22] to be a CC space which is not a compact, T_2 space.

The following definition is due to Wilansky [23].

DEFINITION 4. A space (X,t) is called a US space if every convergent sequence has exactly one limit to which it converges.

THEOREM 8. If (X,t) is a first countable, US, sequentially compact space then every 1-1, onto, continuous selfmap is a homeomorphism.

PROOF. Let F be a closed subset of X and let f be a 1-1, onto, continuous selfmap. f will be a homeomorphism if it can be shown that f(F) is closed. To this end let $x \in Cl(f(F))$. Then there exists a sequence (x_n) in f(F) such that (x_n) converges to x. Since f is 1-1 and onto, the sequence (y_n), defined by $y_n = f^{-1}(x_n)$, is a subset of F. Since (X,t) is sequentially compact there exists a subsequence (y_{n_k}) of (y_n) which converges. Assume (y_{n_k}) converges to y, then y must be an element of F since F is closed. Since f is continuous, $(x_{n_k}) = (f(y_{n_k}))$ converges to f(y). Recall that f(y) is in f(F). Since (x_{n_k}) converges to f(y) and since (x_n) converges, it follows that (x_n) converges to f(y). Since (x_n) converges to both x and f(y) it follows by the US property that x = f(y). Therefore x is in f(F) and hence f(F) is closed. Thus f is a homeomorphism.//

The space whose elements consist of all ordinal numbers less than the first uncountable ordinal number and whose topology is the order topology is considered on page 163 of Kelley [21]. It is claimed that this space is first countable, T_2 and sequentially compact, but not compact. Thus this example satisfies the hypothesis of theorem 8, but fails to satisfy the hypothesis of theorem 7.

The following theorem concludes this chapter.

THEOREM 9. If f is a 1-1, onto, continuous function from (X,t) to (Y,V) where (X,t) is homeomorphic to (Y,V) and if V has a finite number of open sets V_1, V_2, ..., V_n then f is a homeomorphism.

PROOF. Clearly t has n open sets, say O_1, O_2, ..., O_n, labeled such that $f^{-1}(V_i) = O_i$ for i = 1, 2, ..., n. Since f is 1-1, $f^{-1}(V_i) \neq f^{-1}(V_j)$ for $i \neq j$. Note $f(O_i) = f(f^{-1}(V_i)) = V_i$. Therefore f is an open mapping and hence a homeomorphism.//

COROLLARY. If f is a 1-1, onto, continuous selfmap of (X,t) where t has a finite number of open sets then f is a homeomorphism.

V. α-SEMIGROUPS OF CONTINUOUS SELFMAPS

A. SOME KNOWN RESULTS

It is the authors intent in this chapter to extend some of the results by Magill [5,7,8] and Hicks and Haddock [9].

Results from the paper by Hicks and Haddock will be used in obtaining these extensions. Several of these results will now be listed without proof.

DEFINITION 1. Let (X,t) be an arbitrary space. A semigroup of continuous selfmaps which contains the set of constant selfmaps is called an α-semigroup and is denoted by $\alpha(X)$.

A constant selfmap whose only image point is "a" will be denoted by $\bar{a}$, that is $\bar{a}(x) = a$ for all x in X.

LEMMA 1. Let f be in $\alpha(X)$. Then $f \circ g = f$ for all g in $\alpha(X)$ if and only if $f = \bar{x}$ for some x in X.

In the discussion to follow $Z(X)$ is used to denote the set of all constant selfmaps.

Suppose Φ is an isomorphism from $\alpha(X)$ onto $\alpha(Y)$ and assume Φ^* is Φ restricted to $Z(X)$. Consider the following diagram.

DIAGRAM 1.

$$\begin{array}{ccc}
\alpha(X) & \xrightarrow{\quad\Phi\quad} & \alpha(Y) \\
\cup & & \cup \\
Z(X) & \xrightarrow{\quad\Phi^*\quad} & Z(Y) \\
\uparrow x^* & & \uparrow y^* \\
X & \xrightarrow{\quad h\quad} & Y
\end{array}$$

x^* and y^* are defined in the following manner.

$x^*(a) = \bar{a}$ for each a in X and $y^*(b) = \bar{b}$ for each b in Y. The mapping h is defined by $h = y^{*-1} \circ \Phi \circ x^*$ and it follows that $h(x) = y$ if and only if $\phi(\bar{x}) = \bar{y}$.

LEMMA 2. Φ^* maps Z(X) onto Z(Y). Consequently, h maps X 1-1 and onto Y.

THEOREM 1. Suppose $\beta(X)$ and $\beta(Y)$ are semigroups of selfmaps such that $\alpha(X) \subset \beta(X)$ and $\alpha(Y) \subset \beta(Y)$. Φ can be extended to an isomorphism Ψ from $\beta(X)$ onto $\beta(Y)$ if and only if

(1) $h \circ f \circ h^{-1} \in \beta(Y)$ for every $f \in \beta(X)$, and

(2) $h^{-1} \circ g \circ h \in \beta(X)$ for every $g \in \beta(Y)$.

Furthermore if Φ can be extended then Ψ is unique and $\Psi(f) = h \circ f \circ h^{-1}$ for every $f \in \beta(X)$.

B. S**-SPACES

In [7], Magill gives the following definition.

DEFINITION 2. X is said to be an S*-space if it is T_1 and for each closed subset F of X and each point p in X - F there exists f in C(X) and y in X such that f(x) = y for each x in F and $f(p) \neq y$.

Magill then shows that every 0-dimensional Hausdorff space is an S*-space. He also shows that every completely regular Hausdorff space containing at least two distinct points which are connected by an arc is an S*-space.

Magill's main theorem is the following.

THEOREM 2. Assume X and Y are S*-spaces. Φ is an isomorphism from C(X) onto C(Y) if and only if there exists a homeomorphism h from X onto Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in C(X).

The following theorem gives a characterization of S*-spaces.

THEOREM 3. A space X is an S*-space if and only if the family of sets $\{f^{-1}(x) : f \in C(X), x \in X\}$ is a basis for the closed sets of X.

PROOF. If X is an S*-space, it is T_1 and therefore any set of the form $f^{-1}(x)$ where $f \in C(X)$ is closed. Let F be a proper closed subset of X and $p \in X - F$. Since X is an S*-space there exists f_p in C(X) and y_p in X such that $f_p(x) = y_p$ for all x in F and such that $f_p(p) \neq y_p$. Since $F = \bigcap \{f_p^{-1}(y_p) : p \in X - F\}$ the family of sets is a basis for the closed sets of X.

If the family of sets given in this theorem is assumed to be a basis for the closed sets of X then since the identity map i is in C(X), $i^{-1}(x) = \{x\}$ is closed and hence X is T_1. Let F be a proper closed subset of X and let p be in X - F. Then $F = \bigcap \{f_a^{-1}(x_a)$: "a" is in an index set A, x_a is in X and f_a is in C(X)$\}$. Since $p \notin F$ there exists $a \in A$ such that $f_a^{-1}(x_a) \supset F$ but $f_a^{-1}(x_a)$ does not contain p. That is $f_a(x) = x_a$ for all x in F while $f_a(p) \neq x_a$. Therefore X is an S*-space.//

The above theorem motivates the following definition.

DEFINITION 3. X is called an S**-space if and only if there exists an α-semigroup $\alpha_I(X)$ of continuous selfmaps such that the collection of sets $\{f^{-1}(x) : x \in X, f \in \alpha_I(X)$ forms a subbasis for the closed sets of X. The semigroup $\alpha_I(X)$ is said to generate a subbasis for the closed sets of X.

Note i need not belong to $\alpha_I(X)$ and thus X need not be a T_1-space. The following is an example of an S**-space which is not T_1. Let $X = \{a,b,c\}$ and let $t = \{\phi, \{a\}, \{b,c\}, X\}$ and choose $\alpha_I(X) = \{\bar{a}, \bar{b}, \bar{c}, f\}$ where f is defined by $f(a) = a$, $f(b) = c$ and $f(c) = c$.

LEMMA 3. Let X and Y be arbitrary topological spaces. If Φ is an isomorphism between arbitrary α-semigroups $\alpha(X)$ and $\alpha(Y)$, then there exists a bijection h from X onto Y such that

i) $\Phi(f) = h \circ f \circ h^{-1}$ for each f in $\alpha(X)$,

ii) $h(f^{-1}(x)) = (\Phi(f))^{-1}(h(x))$ for all f in $\alpha(X)$ and x in X, and

iii) $h^{-1}(g^{-1}(y)) = (\Phi^{-1}(g))^{-1}(h^{-1}(y))$ for all g in $\alpha(Y)$ and y in Y.

PROOF. Define h by $h(x) = y$ if and only if $\Phi(\bar{x}) = \bar{y}$. Since Φ is an isomorphism, h is a bijection from X onto Y by lemma 2. Note $\Phi(\bar{x}) = \overline{h(x)}$. To establish i), let f be an element of $\alpha(X)$ and let y be an element of Y.

Then $(h \circ f \circ h^{-1})(y) = \overline{h(f(h^{-1}(y)))}(y) = (\Phi(\overline{f(h^{-1}(y))}))(y)$

$= (\Phi(f \circ \overline{h^{-1}(y)}))(y) = (\Phi(f) \circ \Phi(\overline{h^{-1}(y)}))(y) =$

$(\Phi(f) o \overline{y})(y) = (\Phi(f))(y)$.

Therefore $\Phi(f) = h \circ f \circ h^{-1}$ for each f in $\alpha(X)$.

To establish ii) note that each of the following statements are equivalent.

$$y \in h(f^{-1}(x))$$

$$y = h(z) \text{ and } f(z) = x$$

$$y = h(z) \text{ and } f \circ \overline{z} = \overline{x}$$

$$y = h(z) \text{ and } \Phi(f) \circ \Phi(\overline{z}) = \Phi(\overline{x})$$

$$y = h(z) \text{ and } \Phi(f) \circ \overline{h(z)} = \overline{h(x)}$$

$$y = h(z) \text{ and } (\Phi(f))(h(z)) = h(x)$$

$$y \in (\Phi(f))^{-1}(h(x)).$$

iii) is established by a similar argument.//

THEOREM 4. Let X and Y be S**-spaces generated by $\alpha_I(X)$ and $\alpha_I(Y)$ respectively. Then a bijection Φ from $\alpha_I(X)$ onto $\alpha_I(Y)$ is an isomorphism if and only if there exists a homeomorphism h from X to Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in $\alpha_I(X)$.

PROOF. Assume Φ is an isomorphism from $\alpha_I(x)$ onto $\alpha_I(Y)$. By i) of lemma 3 there exists a bijection h from X onto Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in $\alpha_I(X)$.

To show that h is bicontinuous let $g^{-1}(y)$ be a subbase element for the closed sets of Y. From iii) of lemma 3 $h^{-1}(g^{-1}(y)) = (\Phi^{-1}(g))^{-1}(h^{-1}(y))$. Note there exists $x \in X$ and $f \in \alpha_I(X)$ such that $h^{-1}(y) = x$ and $\Phi^{-1}(g) = f$. Hence $h^{-1}(g^{-1}(y)) = f^{-1}(x)$. Since $f^{-1}(x)$ is a subbase element for the closed sets of X, it follows that h is continuous.

Likewise h^{-1} is seen to be continuous by using ii) of lemma 3.

On the other hand if Φ is a bijection from $\alpha_I(X)$ onto $\alpha_I(Y)$ and if there exists a homeomorphism h from X to Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in $\alpha_I(X)$ then $\Phi(f \circ g) = h \circ f \circ g \circ h^{-1} = (h \circ f \circ h^{-1}) \circ (h \circ g \circ h^{-1})$ $= \Phi(f) \circ \Phi(g)$ and hence Φ is an isomorphism.//

COROLLARY 1. Let X and Y be S**-spaces. X is homeomorphic to Y if and only if there exist isomorphic $\alpha_I(X)$ and $\alpha_I(Y)$ which generate subbases for the closed sets of X and Y respectively.

PROOF. If there exist isomorphic $\alpha_I(X)$ and $\alpha_I(Y)$ which generate subbases for the closed sets of X and Y respectively then from theorem 4, X and Y are homeomorphic.

On the other hand if h is a homeomorphism from X to Y then Φ defined by $\Phi(f) = h \circ f \circ h^{-1}$ is an isomorphism from C(X) onto C(Y). Since X is an S**-space there exists an α-semigroup $\alpha_I(X)$ which generates the closed sets of X. Define $\alpha_I(Y) = \Phi(\alpha_I(X))$. It is claimed that $\alpha_I(Y)$ is an α-semigroup of continuous selfmaps which generates a subbasis for the closed sets of Y. From the way Φ was defined, it is immediate that $\alpha_I(Y)$ is an α-semigroup isomorphic to $\alpha_I(X)$. To see that $\alpha_I(Y)$ generates the closed sets of Y, let F be a closed set in Y. Then $h^{-1}(F)$ is closed in X and hence can be written in the form

$h^{-1}(F) = \bigcap_{a \varepsilon A} \left(\bigcup_{n=1}^{k_a} f_{a_n}^{-1}(x_{a_n}) \right)$ where each f_{a_n} is in $\alpha_I(X)$ and each x_{a_n} is in X. Taking h of each side and using the fact that h is 1-1 yields

$$F = \bigcap_{a\varepsilon A} \left(\bigcup_{n=1}^{k_a} h(f_{a_n}^{-1}(x_{a_n})) \right) = \bigcap_{a\varepsilon A} \left(\bigcup_{n=1}^{k_a} (\Phi(f_{a_n}))^{-1} h(x_{a_n}) \right).$$

The last equality follows by ii) of lemma 3, since all that was required in lemma 3 was the fact that $\Phi(\bar{x}) = \bar{y}$ if and only if $h(x) = y$. Since Φ is an isomorphism and h is a homeomorphism there exist g_{a_n} in $\alpha_I(Y)$ and y_{a_n} in Y such that $\Phi(f_{a_n}) = g_{a_n}$ and $h(x_{a_n}) = y_{a_n}$.

Therefore $F = \bigcap_{a\varepsilon A} \left(\bigcup_{n=1}^{k_a} g_{a_n}^{-1}(y_{a_n}) \right)$. It should be noted that $g_{a_n}^{-1}(y_{a_n})$ is closed since it is the homeomorphic image of the closed set $f_{a_n}^{-1}(x_{a_n})$. Thus $\alpha_I(Y)$ generates the closed sets of Y.//

COROLLARY 2. Let X and Y be S**-spaces which are also T_1. Then X is homeomorphic to Y if and only if C(X) is isomorphic to C(Y).

PROOF. Since the spaces are T_1, let $\alpha_I(X) = C(X)$ and let $\alpha_I(Y) = C(Y)$.//

This corollary does not hold without the condition T_1. For let X = Y and assign the discrete and trivial topologies respectively.

COROLLARY 3. If (X,t_1) and (X,t_2) are S**-spaces which are also T_1 and if $C(X,t_1) = C(X,t_2)$ then $t_1 = t_2$.

PROOF. Note $\Phi = i$, therefore $i(f) = h \circ f \circ h^{-1} = f$ for all f in $C(X,t_1)$ which implies h is the identity homeomorphism.//

COROLLARY 4. Let X and Y be S**-spaces. Then any isomorphism from $\alpha_I(X)$ onto $\alpha_I(Y)$ has a unique extension to an isomorphism from C(X) onto C(Y).

PROOF. Let Φ be an isomorphism from $\alpha_I(X)$ onto $\alpha_I(Y)$. From theorem 4 there exists a homeomorphism h from X to Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in $\alpha_I(X)$. Note if f is in C(X) then $h \circ f \circ h^{-1}$ is in C(Y) and if g is in C(Y) then $h^{-1} \circ g \circ h$ is in C(X). Therefore the corollary follows from theorem 1.//

COROLLARY 5. If X is an S**-space then the automorphism group of $\alpha_I(X)$ is isomorphic to the group, under composition, of all autohomeomorphisms of X.

PROOF. Let A denote the automorphism group of $\alpha_I(X)$ and let G denote the group of all autohomeomorphisms of X. From theorem 4 for each Φ in A there exists an autohomeomorphism h of X such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in $\alpha_I(X)$. This autohomeomorphism is unique, for assume $h_1 \circ f \circ h_1^{-1} = h \circ f \circ h^{-1}$ for all f in $\alpha_I(X)$. This implies $f = h_1^{-1} \circ h \circ f \circ h^{-1} \circ h_1$ for each f in $\alpha_I(X)$. Since $h(x) = y$ if and only if $\Phi(\overline{x}) = \overline{y}$ by letting $f = \overline{x}$ it will follow that $h_1(x) = y$. To see this let "a" be an arbitrary element of X. Then $x = \overline{x}(a) = (h_1^{-1} \circ h \circ \overline{x} \circ h^{-1} \circ h_1)(a) = h_1^{-1}(h(x)) = h_1^{-1}(y)$ and thus $h_1(x) = y$. Therefore $h_1 = h$.

Define the function B from A to G by $B(\Phi) = h$ where $\Phi(\overline{x}) = \overline{y}$ if and only if $h(x) = y$. By direct calculation it follows that B is a homomorphism. B is seen to be onto by theorem 4. To see that B is 1-1, assume B(Φ) = i the identity in G. Then $\Phi(f) = i \circ f \circ i^{-1} = f$ for each f in $\alpha_I(X)$. That is Φ must be the identity automorphism. Hence since the kernel of B consists of exactly the identity element, B must be 1-1.//

COROLLARY 6. Let X be an S**-space. If $\alpha_I(X) = C(X)$ then every automorphism on C(X) is an inner automorphism.

PROOF. Let Φ be an automorphism on C(X). From theorem 4 there exists a homeomorphism h from X to X such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in C(X). Note h and h^{-1} are elements of C(X) and therefore Φ is an inner automorphism.//

In the following $\Gamma(X)$ and T(X) denote the closed selfmaps and the connected selfmaps respectively.

COROLLARY 7. Let X and Y be S**-spaces such that $\alpha_I(X) \subset \Gamma(X)$. Then any isomorphism from $\alpha_I(X)$ onto $\alpha_I(Y)$ has a unique extension to an isomorphism from $\Gamma(X)$ onto $\Gamma(Y)$. (Here $\alpha_I(X)$ and $\alpha_I(Y)$ generate the closed sets of X and Y respectively).

PROOF. From theorem 4, X and Y are homeomorphic with homeomorphism h satisfying the equation $\Phi(f) = h \circ f \circ h^{-1}$ for all f in $\alpha_I(X)$. Since $\Phi(\alpha_I(X)) = \alpha_I(Y)$ it follows that $\alpha_I(Y) \subset \Gamma(Y)$ and then the result follows from theorem 1.//

$\alpha_I(X)$ will always be a subset of $\Gamma(X)$ in a compact, T_2 space. Although these conditions are not necessary since the countable complement topology is neither compact nor T_2 while every continuous function is a closed function.

A corollary similar to corollary 6 can be obtained by replacing $\Gamma(X)$ and $\Gamma(Y)$ by $T(X)$ and $T(Y)$. Here it is always true that $\alpha_I(X) \subset T(X)$.

In the following $I_s(X)$ will denote the semigroup under composition which is generated by the idempotent continuous selfmaps.

Magill in [8] considers a class D of spaces consisting of topological spaces which are Hausdorff and 0-dimensional together with all spaces which are Hausdorff, normal and contain an arc.

Magill then shows that if X is in D then $\{f^{-1}(x) : x$ is in X and f is in $I_s(X)\}$ is a subbasis for the closed sets of X. His main theorem is then the following.

THEOREM 5. Assume X and Y belong to the class D. Then a bijection Φ from $I_s(X)$ onto $I_s(Y)$ is an isomorphism if and only if there exists a homeomorphism h from X onto Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for each f in $I_s(X)$.

It should be noted here that each space in D is an S**-space and hence theorem 5 is actually an immediate corollary of theorem 4.

DEFINITION 4. Let C be the collection of all spaces X which have the property that there exists an α-semigroup of continuous, idempotent selfmaps $\alpha_s(X)$ such that the collection of sets $\{f^{-1}(x) : x$ is in X and f is in $\alpha_s(X)\}$

forms a subbasis for the closed sets of X. A set of the form $\alpha_s(X)$ is said to generate the closed sets of X.

Again note that each space in C is also an S**-space. Because of theorem 4, the sets $I_s(X)$ and $I_s(Y)$ of theorem 5 can be replaced by $\alpha_s(X)$ and $\alpha_s(Y)$ respectively where X and Y are topological spaces in C.

Clearly each space in D is also in C. The following theorem gives additional spaces in C and hence additional S**-spaces.

THEOREM 6. Let (X,t) be a space satisfying the condition, "if O is an open set and F is a closed set such that $O \cap F \neq \phi$ then $O \cup F$ is open", then each closed set is of the form $\{f^{-1}(x) : x \text{ is in } X \text{ and } f \text{ is in } I_s(X)\}$.

PROOF. Let F be a closed set such that $\phi \neq F \neq X$ and let q be an element of F. Define $f(x) = \begin{cases} q & \text{for } x \text{ in } F \\ x & \text{for } x \text{ in } X - F \end{cases}$.

f is idempotent since it is well known that a selfmap is idempotent if and only if it is the identity function when restricted to its range. Note $f^{-1}(q) = F$. To see that f is continuous let O be an open set.

CASE 1. Assume q is in O. Then $f^{-1}(O) = F \cup O$ is in t by hypothesis.

CASE 2. Assume q is not in O. Then $f^{-1}(O) = O - F$ which is open.

If F = X then $\bar{x}^{-1}(x) = X$, while if $F = \phi$ then $\bar{x}^{-1}(y) = \phi$ where $x \neq y$.//

Note the superset topology, the tower space topology, the finite complement topology and the countable complement topology all satisfy the conditions of theorem 6.

COROLLARY 1. Let (X,t) be a space satisfying the hypothesis of theorem 6. The collection of set $\{f^{-1}(x)$: x is in X and f is in $I_s(X)\}$ is a base for the closed sets of (X,t) if and only if (X,t) is a T_1-space.

PROOF. If the collection of sets $\{f^{-1}(x)$: x is in X and f is in $I_s(X)\}$ is a basis for the closed sets then $i^{-1}(x)=\{x\}$ is closed and hence the space is T_1. On the other hand if the space is T_1 then each $f^{-1}(x)$ is closed and hence from theorem 6 the collection of sets $\{f^{-1}(x)$: x is in X and f is in $I_s(X)\}$ forms a base for the closed sets of (X,t).//

COROLLARY 2. Let (X,t) be a T_1-space such that supersets of all nonempty open sets are again open, then the collection of sets $\{f^{-1}(x)$: x is in X and f is in $I_s(X)\}$ forms a base for the closed sets of X.

PROOF. Follows immediately from theorem 6 and corollary 1.//

The finite complement topology and the countable complement topology obviously satisfy this corollary.

Clearly the finite complement space belongs to C where $\alpha_s(X) = I_s(X)$. Also $\alpha_s(X)$ could be defined as $\alpha_s(X) = \{f$: f is the identity function or f is a constant function$\}$. Hence it is possible to find non-isomorphic semigroups of idempotents which generate the closed sets of X.

Let $ID_s(X)$ denote the class of continuous idempotent selfmaps of X.

THEOREM 7. Let X and Y be arbitrary spaces. Assume Φ is an isomorphism from $C(X)$ onto $C(Y)$. Then Φ restricted to $I_s(X)$ yields an isomorphism from $I_s(X)$ to $I_s(Y)$. Furthermore Φ maps $ID_s(X)$ onto $ID_s(Y)$ in a 1-1 manner.

PROOF. Note $I_s(X)$ is clearly isomorphic to $\Phi(I_s(X))$. Thus it must be shown that $\Phi(I_s(X)) = I_s(Y)$.

To do this let $\Phi(f) \varepsilon \Phi(I_s(X))$ and note that $f \varepsilon I_s(X)$.

CASE 1. Assume $f \varepsilon ID_s(X)$. Then $\Phi(f) = \Phi(f \circ f) = \Phi(f) \circ \Phi(f)$ and hence $\Phi(f) \varepsilon ID_s(Y) \subset I_s(Y)$.

CASE 2. Assume $f \varepsilon I_s(X) - ID_s(X)$. Then $f = h_1 \circ h_2 \circ \ldots \circ h_m$. Note f is written as products of powers of elements of $ID_s(X)$. Only first powers are needed since $h_i^n = h_i$. Hence $\Phi(f) = \Phi(h_1) \circ \Phi(h_2) \ldots \circ \Phi(h_m) = g_1 \circ g_2 \circ \ldots g_m$ where from case 1 it is clear that $g_i \varepsilon ID_s(Y)$. Therefore $\Phi(f) \varepsilon I_s(Y)$. From cases 1 and 2 we can conclude that $\Phi(I_s(X)) \subset I_s(Y)$. To establish equality let $g \varepsilon I_s(Y)$. Since Φ is an isomorphism from $C(X)$ to $C(Y)$ there exists $h \varepsilon C(X)$ such that $\Phi(h) = g$.

CASE 3. Assume $g \varepsilon ID_s(Y)$. Then $g \circ g = g$ which implies $\Phi(h) \circ \Phi(h) = \Phi(h)$ and hence $\Phi(h) = \Phi(h \circ h)$. Since Φ is 1-1 we get $h = h \circ h$. Thus $h \varepsilon ID_s(X)$ and we can conclude that $g = \Phi(h) \varepsilon \Phi(I_s(X))$.

CASE 4. Assume $g \in I_s(Y) - ID_s(Y)$. Then $g = f_1 o\ f_2 o \ldots o\ f_n$ where each $f_i \in ID_s(X)$. Again since Φ is 1-1 there exists corresponding $q_1, q_2, \ldots, q_n$ such that $\Phi(q_i) = f_i$. From case 3 we see that each $q_i \in ID_s(X)$. Therefore, $g = \Phi(q_1)\ o\ \Phi(q_2)\ o \ldots o\ \Phi(q_n) = \Phi(q_1 o \ldots o\ q_n)$ where $q_1 o\ q_2 o \ldots o\ q_n \in I_s(X)$ and hence $g \in \Phi(I_s(X))$. From cases 3 and 4 we get $I_s(Y) \subset \Phi(I_s(X))$. Thus $\Phi(I_s(X)) = I_s(Y)$ and the isomorphism is established. From cases 1 and 3 we see that Φ maps $ID_s(X)$ onto $ID_s(Y)$ in a 1-1 manner.//

COROLLARY 1. Let X and Y be elements of C. Assume there exists isomorphic $\alpha_s(X)$ and $\alpha_s(Y)$ which generate the closed sets of X and Y respectively, then $I_s(X)$ and $I_s(Y)$ are isomorphic.

PROOF. Follows immediately from corollary 4 of theorem 4 along with theorem 7.//

C. M*-SPACES

Another class of topological spaces which admit theorems similar to those for S**-spaces will be considered in this section.

The following definitions are due to Magill in [5].

DEFINITION 5. Let X be a topological space and let x be an element of X. An open set G containing x is an S-neighborhood of x if it consists of x alone or if there exists a continuous function f mapping the closure of G into X such that $f(x) \neq x$, but $f(y) = y$ for each y in (closure of G) - G.

DEFINITION 6. A topological space is an S-space if it is Hausdorff and every point has a basis of S-neighborhoods.

Examples of S-spaces as given by Magill are all 0-dimensional Hausdorff spaces and all locally Euclidean spaces.

Magill's main theorem is the following.

THEOREM 8. Two S-spaces X and Y are homeomorphic if and only if C(X) and C(Y) are isomorphic.

Hicks and Haddock in [9] defined M-spaces as follows.

DEFINITION 7. X is called an M-space if the collection of sets of the form {H(f) : f is in C(X)} is a basis for the closed sets of X. Recall H(f) = {x : f(x) = x}.

Hicks and Haddock pointed out that all S-spaces are M-spaces and that theorem 8 can be extended to M-spaces.

They also give the following lemma.

LEMMA 4. Let X and Y be arbitrary topological spaces. If Φ is an isomorphism between arbitrary α-semigroups $\alpha(X)$ and $\alpha(Y)$ then there exists a bijection h from X onto Y such that

i) $\Phi(f) = h \circ f \circ h^{-1}$ for each f in $\alpha(X)$,

ii) $h(H(f)) = H(\Phi(f))$ for every f in $\alpha(X)$, and

iii) $h^{-1}(H(g)) = H(\Phi^{-1}(g))$ for every g in $\alpha(Y)$.

DEFINITION 8. A space (X,t) is called an M*-space if there exists an α-semigroup of continuous selfmaps $\alpha_F(X)$ such that the collection of sets of the form {H(f) : f is in $\alpha_F(X)\}$ is a subbase for the closed sets of X.

The proofs of theorem 9 and its corollaries parallel the proofs of theorem 4 and its corollaries and hence will not be given. Lemma 4 is used to prove theorem 9 in the same manner that lemma 3 was used in proving theorem 4.

THEOREM 9. Let X and Y be M*-spaces generated by $\alpha_F(X)$ and $\alpha_F(Y)$ respectively. Then a bijection Φ from $\alpha_F(X)$ onto $\alpha_F(Y)$ is an isomorphism if and only if there exists a homeomorphism h from X to Y such that $\Phi(f) = h \circ f \circ h^{-1}$ for all f in $\alpha_F(X)$.

COROLLARY 1. Let X and Y be M*-spaces. X is homeomorphic to Y if and only if there exist isomorphic $\alpha_F(X)$ and $\alpha_F(Y)$ which generate subbases for the closed sets of X and Y respectively.

COROLLARY 2. Let X and Y be M*-spaces generated by $\alpha_F(X)$ and $\alpha_F(Y)$ where $\alpha_F(X) = C(X)$ while $\alpha_F(Y) = C(Y)$. Then X is homeomorphic to Y if and only if C(X) is isomorphic to C(Y).

COROLLARY 3. Let X and Y be M* -spaces. Then any isomorphism from $\alpha_F(X)$ onto $\alpha_F(Y)$ has a unique extension to an isomorphism from C(X) onto C(Y).

COROLLARY 4. If X is an M*-space then the automorphism group of $\alpha_F(X)$ is isomorphic to the group, under composition, of all autohomeomorphisms of X.

COROLLARY 5. Let X be an M*-space. If $\alpha_F(X) = C(X)$ then every automorphism on C(X) is an inner automorphism.

COROLLARY 6. Let X and Y be M*-spaces such that $\alpha_F(X) \subset \Gamma(X)$. Then any isomorphism from $\alpha_F(X)$ onto $\alpha_F(Y)$ has a unique extension to an isomorphism from $\Gamma(X)$ onto $\Gamma(Y)$.

Let X be the set of integers with the finite complement topology and let $\alpha_F(X) = \{f\colon f$ is either the identity selfmap or a constant selfmap$\}$. This space is an M*-space. Note $H(i) = X$ and $H(\bar{a}) = \{a\}$ for each a in X. Since a set F is closed in this space if and only if $F = \phi$, $F = X$ or F is a finite set, it follows that $\alpha_F(X)$ generates a subbase for the closed sets of X. Furthermore this space fails to be an M-space or an S-space. To see that it is not an M-space define f by

$$f(x) = \begin{cases} x & \text{for } x \text{ an even integer} \\ x+1 & \text{for } x \text{ an odd integer} \end{cases}.$$

Then f is a continuous function but $H(f) = \{\text{even integers}\}$ is not a closed set.

D. EXAMPLES

EXAMPLE 1. Let (X,t) be the one-point compactification of the rational numbers with the relative topology. Let ∞ be the point of compactification. This example is known to be a CC space and hence has the US property.

CLAIM 1. If f is in C(X,t) then $f(\infty) = \infty$ or f is a constant function.

PROOF. Assume f is not a constant function and $f(\infty) = a \neq \infty$.

CASE 1. Assume there exists $b \neq \infty$ such that $b \neq a$ and b is an image point of f. Let O_b and O_a be disjoint open intervals of rationals around "b" and "a" respectively. Then $f^{-1}(O_b)$ and $f^{-1}(O_a)$ are disjoint. Note $f^{-1}(O_a) = O_a^* \cup \{\infty\}$ is a neighborhood of ∞ and $f^{-1}(O_b) = O_b^*$ contains an open interval of rationals. Since $O_a^* \cup \{\infty\}$ is a neighborhood of ∞, the complement of O_a^* denoted by $O_a^{*\prime}$ is closed and compact in the rationals. Note $O_b^* \subset O_a^{*\prime}$. This is a contradiction since a compact set can not contain an interval of rationals.

CASE 2. Assume ∞ and "a" are the only two image points of f. Since the space is T_1, $f^{-1}(\infty) = F_\infty$ and $f^{-1}(a) = F_a$ are disjoint closed sets such that $F_\infty \cup F_a = X$. Note F_∞ and F_a must also be open and hence they contain an interval. Thus these sets can not be compact. This is a contradiction since this is a CC space.//

CLAIM 2. $H(f) = \{x : f(x) = x$ where f is in $C(X,t)\}$ is a closed set.

PROOF. Let y be an accumulation point of H(f) such that y is different from ∞. Also assume f is not a constant function. Then there exists a net $\{x_n\}$ in H(f) which converges to y. For each positive integer k select x_{n_k} from $(y - \frac{1}{k}, y + \frac{1}{k})$. Then the sequence $\{x_{n_k}\}$ converges to y. Therefore $\{x_{n_k}\} = \{f(x_{n_k})\}$ converges to f(y). Since this space has the US property each convergent sequence converges to a unique limit. Thus since $\{x_{n_k}\}$ converges to both y and f(y), it must be concluded that y = f(y) and hence y is in H(f).//

It is well known that each H(f) is closed in a T_2-space. This example exhibits a non T_2-space with this property. However it will follow from claim 3 that this space fails to be an S-space, an M-space or an M*-space.

CLAIM 3. The collection of sets of the form H(f) does not form a subbasis for the closed sets.

PROOF. Let $O = \{x : x < 0\} \cup (\bigcup_{n=2}^{\infty} (\frac{1}{n}, \frac{1}{n-1})) \cup \{x : x > 1\}$ be an open subset of the rational numbers. Then the set of rationals - $O = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ is closed and compact in the rationals. This is true since any neighborhood of zero with respect to the relative topology on the rationals contains all but a finite number of elements of the sequence. Therefore $O \cup \{\infty\}$ is in t and hence $F = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ is thus closed in (X,t). The set F can not be expressed in terms of an arbitrary intersection of finite unions of sets of the form H(f). The reason for this is that the only continuous functions which do not fix ∞ are the constant functions and they each have one fixed point.//

EXAMPLE 2. Let X = {rational numbers} ∪ {∞} and let t = {O : O is in the relative topology for the rationals} ∪ {A : ∞ is in A and X - A is finite}.

CLAIM 1. If f is in $C(X,t)$ then $f(\infty) = \infty$ or f is a constant function.

PROOF. Assume f is not a constant function and $f(\infty) = a \neq \infty$.

CASE 1. Assume there exists "b" which is an image point of f where $b \neq a$ and $b \neq \infty$. Let O_b and O_a be disjoint open intervals of rational numbers around "b" and "a" respectively.

Then $f^{-1}(O_b)$ and $f^{-1}(O_a)$ are disjoint open sets. Furthermore $f^{-1}(O_a)$ is a neighborhood of ∞ and $f^{-1}(O_b)$ contains an open interval of rationals. This is impossible since neighborhoods of ∞ contain all but a finite number of points.

CASE 2. Assume "a" and ∞ are the only two image points. Since this space is T_1, $f^{-1}(\infty) = F_\infty$ and $f^{-1}(a) = F_a$ are closed and disjoint sets. Furthermore, since $F_a \cup F_\infty = X$ the sets F_a and F_∞ are both open and thus both contain open intervals of rational numbers. Again this is impossible since one of these sets is the complement of a neighborhood of ∞.//

CLAIM 2. Not all sets of the form $H(f)$ are closed. Hence (X,t) fails to be an M-space or an S-space.

PROOF. Let $f(x) = \begin{cases} x & \text{for } x < 0 \\ \infty & \text{for all other } x \in X \end{cases}$. Let O_1 be an open set about ∞ then $f^{-1}(O_1)$ contains all but a finite number of points and hence is open. Let O_2 be an open set such that ∞ is not in O_2. Note $f^{-1}(O_2) = O_2 \cap \{x : x < 0\}$ which is an open set and hence f is continuous. Note $H(f) = \{x : x < 0\} \cup \{\infty\}$ and this is not a closed set for otherwise $\{x : x \geq 0\}$ would be an open set. $\{x : x \geq 0\}$ is not open since zero is not an interior point.//

CLAIM 3. (X,t) is an M*-space.

PROOF. Let $\alpha_F(X)$ be the semigroup generated by {f : f is a constant function or f is the identity function} $\cup$ $\{f_b : f_b(x) = x$ for all $x \leq b$, $f_b(\infty) = \infty$ and $f(x) = \frac{x+b}{2}$ for $x > b$, where b is an arbitrary rational number$\}$ $\cup$ $\{f_a : f_a(x) = x$ for all $x \geq a$, $f_a(\infty) = \infty$ and $f_a(x) = \frac{x+a}{2}$ for $x < a$ where a is an arbitrary rational number$\}$. Note $H(f_b) = \{x : x \leq b\} \cup \{\infty\}$ and $H(f_a) = \{x : x \geq a\} \cup \{\infty\}$. The set $\{H(f_b) : b$ is a rational number$\} \cup \{H(f_a) : a$ is a rational number$\} \cup \{H(\bar{a}) : a$ is in X$\} \cup \{H(i) : i$ is the identity function$\}$ forms a subbasis for the closed sets of (X,t). $\alpha_F(X)$ actually contains the above collection. Each additional f in $\alpha_F(X)$ is a polygonal function and hence H(f) is closed. Therefore (X,t) is an M*-space.//

EXAMPLE 3. Let X = {real numbers} and let $t = \{O :$ there exist real numbers a and b with $O \supset ((-\infty,a) \cup (b,\infty))\}$. This space is called the fuzzy space and it was shown in chapter III that f is in C(X,t) if and only if f is a constant function or $|f(x)| \to \infty$ as $|x| \to \infty$.

Some properties of this space will now be noted.

1) The continuous functions in (X,t) can become unbounded

in the usual sense even when x is in a bounded set.

2) (X,t) is a T_1 space which fails to be T_2.

3) The supersets of open sets are open therefore by corollary 2 of theorem 6 this space belongs to the class C of spaces and hence is an S**-space.

4) H(f) need not be a closed set. Therefore (X,t) is not an M-space nor an S-space. To see this let f(x) = x for $x \neq 1$ and let f(1) = 2 then f is continuous but H(f) is not bounded and hence not closed.

5) (X,t) is an M*-space. To see this let F be a closed subset of X and define $f_F(x) = \begin{cases} x \text{ for } x \text{ in } F \\ x+1 \text{ for } x \text{ not in } F \end{cases}$.

Then f_F is continuous and $H(f_F) = F$. Let $\alpha_{F_1}(X)$ be the semigroup generated by the set $\{f$: f is a constant function$\} \cup \{f_F$: F is a closed subset of X and f_F is as defined above$\}$. Note each element of $\alpha_{F_1}(X)$ is of the form $f_{F_1} \circ f_{F_2} \circ f_{F_3} \circ \ldots \circ f_{F_n}$ and $H(f_{F_1} \circ f_{F_2} \circ \ldots \circ f_{F_n}) = \bigcap_{i=1}^{n} H(f_{F_i}) = \bigcap_{i=1}^{n} F_i$ is therefore closed.

6) Define $g_F(x) = \begin{cases} x \text{ for } x \in F \\ x-1 \text{ for } x \notin F \end{cases}$ where F is a closed subset of (X,t). Then in a manner analogous to that used above it follows that if $\alpha_{F_2}(X)$ is the semigroup generated by $\{g$: g is a constant$\} \cup \{g_F$: F is a closed subset of X$\}$ then $\alpha_{F_2}(X)$ generates the closed sets of (X,t). However note that $\alpha_{F_1}(X) \cap \alpha_{F_2}(X) = \{f$: f is a constant function

or f is the identity function does not generate a subbasis for the closed sets of (X,t).

7) No closed and bounded subset of X is compact unless it is finite.

8) The set of integers is a compact subset of X.

VI. SUMMARY, CONCLUSIONS AND FURTHER PROBLEMS

It is to be noted that for a given set X and a given semigroup $\alpha(X)$ of selfmaps there sometimes exists a topology t such that $C(X,t) = \alpha(X)$. The topology t might be uniquely determined as was the case with the fuzzy space or t might be determined only to within a saturation as was the case with the tower space. Also because of de Groot's theorem in [18], which implies that there exist 2^c nonhomeomorphic subspaces of the real line which have as their continuous selfmaps only the identity function and the constant functions, it follows that there could be several topologies defined on a set X such that $C(X,t) = \alpha(X)$. On the other hand the set $X = \{1,2,3\}$ admits no topology t such that $C(X,t)$ consists of only the identity function and the constant functions. Thus it must be concluded that the existence of t such that $C(X,t) = \alpha(X)$ is dependent both on X and $\alpha(X)$.

The problem of finding methods to construct topologies t and s such that $C(X,t) = C(X,s)$ is related to the problem of finding a topology t such that $C(X,t) = \alpha(X)$ and is a problem which merits further consideration.

Although methods are given in this dissertation which are useful in finding a topology V comparable to a given topology U such that $C(X,V)$ contains $C(X,U)$ it should be noted that the success of these methods is dependent on the topology U. It has still not been determined whether there exists a topology V such that V is strictly stronger than the usual topology U for the real numbers and such that C(R,V) contains C(R,U).

Many of the examples given in chapters III, IV and V have the property that every 1-1, onto, continuous selfmap is a homeomorphism. Some theorems are given in chapter IV which point out when every 1-1, onto, continuous selfmap must be a homeomorphism. However it appears that other theorems might be found since many simple examples, which possess the property that every 1-1, onto, continuous selfmap is a homeomorphism, are not covered by any known theorems.

The classes of S** and M* spaces are fairly extensive classes of spaces which satisfy the property that two spaces are homeomorphic if and only if there exist certain isomorphic α-semigroups. Since there exist S**-spaces which fail to be T_1 spaces, it follows that the class of S**-spaces is not a subset of the class of M*-spaces. However it is not known if every M*-space is an S**-space. This question was originally raised by Magill in [7], when he asked if there exist S-spaces which are not S*-spaces.

REFERENCES

1. Everett, C. J. and Ulam, S. M. (1948) On the Problem of Determination of Mathematical Structures by their Endomorphisms, Bulletin of the American Mathematical Society, 54, Abstract 285t.

2. Whittaker, J. V. (1963) On Isomorphic Groups and Homeomorphic Spaces, Annals of Mathematics, 78, p. 74-91.

3. Lee, Yu-Lee (1967) Topologies with the Same Class of Homeomorphisms, Pacific Journal of Mathematics, 20, p. 77-88.

4. Lee, Yu-Lee (1968) Characterizing the Topology by the Class of Homeomorphisms, Duke Mathematics Journal, 35, p. 625-629.

5. Magill, K. D. (1964) Semigroups of Continuous Functions, The American Mathematical Monthly, 71, p. 984-988.

6. Magill, K. D. (1966) Semigroups of Functions on Topological Spaces, Proceedings of the London Mathematical Society, 3, p. 507-518.

7. Magill, K. D. (1967) Another S - Admissible Class of Spaces, Proceedings of the American Mathematical Society, 18, p. 295-298.

8. Magill, K. D. (1969) Semigroups of Functions Generated by Idempotents, Journal of the London Mathematical Society, 44, p. 236-242.

9. Hicks, T. L. and Haddock, A. G. (1969) On Semigroups of Functions on Topological Spaces, Journal of Mathematical Analysis and Applications, 28, to appear.

10. Gelfand, I. and Kolmogoroff, A. (1939) On Rings of Continuous Functions on Topological Spaces, Dokl. Akad. Nauk SSSR, 22, p. 11-15.

11. Stone, M. H. (1937) Applications of the Theory of Boolean Rings to General Topology, Transactions of the American Mathematical Society, 41, p. 375-481.

12. Gillman, L. and Jerison, M. (1960) Rings of Continuous Functions, D. Van Nostrand, Princeton, New Jersey.

13. Wechsler, M. T. (1955) Homeomorphism Groups of Certain Topological Spaces, Annals of Mathematics, 62, p. 360-377.

14. Thomas, E. S. (1967) Spaces Determined by their Homeomorphism Groups, Transactions of the American Mathematical Society, 126, p. 244-250.

15. Wiginton, C. L. and Shrader, S. (1968) Spaces Determined by a Group of Functions, Bulletin of the American Mathematical Society, 74, p. 1110-1112.

16. Lorrain, F. (1969) Notes on Topological Spaces with Minimum Neighborhoods, The American Mathematical Monthly, 76, p. 616-627.

17. Royden, H. L. (1963) Real Analysis, Macmillan, New York.

18. de-Groot, J. (1959) Groups Represented by Homeomorphism Groups I, Mathematische Annalen, 138, p. 80-102.

19. Levine, N. (1967) The Complementary Topology, Rendiconti Del Circolo Matematico Di Palermo, Series 2, Volume 16, p. 247-251.

20. Larson, R. E. (1969) Minimum and Maximum Topological Spaces, Notices of the American Mathematical Society, Vol. 16, Abstract 69T-G99, p. 847.

21. Kelley, J. L. (1955) General Topology, D. Van Nostrand, Princeton, New Jersey.

22. Levine, N. (1965) When are Compact and Closed Equivalent? The American Mathematical Monthly, 72, p. 41-44.

23. Wilansky, A. (1967) Between T_1 and T_2, The American Mathematical Monthly, 74, p. 261-266.

VITA

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