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COMPLETENESS AND RELATED TOPICS
IN A QUASI-UNIFORM SPACE

by

JOHN WARNOCK CARLSON, 1940 -

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI - ROLLA

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

1970

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ABSTRACT

Completions and a strong completion of a quasi-uniform space are constructed and examined. It is shown that the trivial completion of a T_0 space is T_0. Examples are given to show that a T_1 space need not have a T_1 strong completion and a T_2 space need not have a T_2 completion. The nontrivial completion constructed is shown to be T_1 if the space is T_1 and the quasi-uniform structure is the Pervin structure. It is shown that a space can be uniformizable and admit a strongly complete quasi-uniform structure and not admit a complete uniform structure.

Several counter-examples are provided concerning properties which hold in a uniform space but do not hold in a quasi-uniform space. It is shown that if each member of a quasi-uniform structure is a neighborhood of the diagonal then the topology is uniformizable.

ACKNOWLEDGEMENTS

The author wishes to express his deep appreciation to Dr. Troy L. Hicks for his inspiration and guidance during these years of graduate study. The value of his many penetrating suggestions and consistent patience in the preparation of this thesis can not easily be measured.

The author also wishes to express his appreciation to his wife and family for their understanding and encouragement.

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I. INTRODUCTION

Let $(X,\mathcal{U})$ be a quasi-uniform space. The primary problem in Chapter III is to construct a completion for $(X,\mathcal{U})$. It is shown that it is impossible in general to construct a completion which preserves the Hausdorff separation property. A trivial completion for any $(X,\mathcal{U})$ is given which preserves the T_0 separation property, but the trivial completion is not T_1. Several nontrivial completions are given for special classes of quasi-uniform spaces. One of these constructions preserves the T_1 separation property when $\mathcal{U}$ is the Pervin quasi-uniform structure. A parallel discussion of strong completions is also considered, and an example is given to show that not every T_1 space need have a T_1 strong completion. A sufficient condition for the concepts of complete and strongly complete to coincide is given. An example of a uniformizable space which does not admit a complete uniform structure is shown to admit a strongly complete quasi-uniform structure.

In Chapter IV we consider some well-known properties of uniform space which fail to carry over for quasi-uniform spaces. It is of interest to have conditions on $\mathcal{U}$ that will guarantee that $\mathcal{U}$ is compatible with a uniform structure. It is shown that if each $U \in \mathcal{U}$ is a neighborhood of the diagonal in $X \times X$ then $\mathcal{U}$ is compatible with a uniform structure. A necessary and sufficient condition for $\mathcal{U} \wedge \mathcal{U}^{-1}$ to be a quasi-uniform structure is presented. A class of separation properties which is dependent on the particular quasi-uniform structure under consideration is defined and some of their properties are studied.

Finally, an example is given to show that a sequence of quasi-uniformly continuous, (continuous) functions may converge quasi-uniformly to a function g and g not be quasi-uniformly continuous, (continuous).

The topological terminology found in this thesis is consistent with the definitions found in Gaal [11]. The definitions of concepts related to quasi-uniform spaces can be found in Murdeshwar and Naimpally [16], with the following exception. By $U \circ V$ we mean $\{ (x,y) :$ there exists $z \in X$ such that $(x,z) \in U$ and $(z,y) \in V\}$. This is the definition of $U \circ V$ found in Gaal [11].

II. REVIEW OF THE LITERATURE

In 1955, Krishnan [14] essentially showed that every topological space admitted a compatible quasi-uniform structure. In 1960, Császár [3] showed this more explicitly and in 1962, Pervin [19] gave a useful and simple construction of a compatible quasi-uniform structure for a given topology. Fletcher [7] gave a construction for a family of compatible quasi-uniform structures.

In 1960, Császár [3] extended the notions of a Cauchy filter and completeness to a quasi-uniform space. Isbell [13] noted that the convergent filters were not necessarily Cauchy, so Sieber and Pervin [20] in 1965 proposed the definition of Cauchy filter which is now in use. They defined a space to be complete if every Cauchy filter converged. We will call a space strongly complete if it has this property. In this paper, they showed that every ultrafilter is Cauchy in a pre-compact space. They obtained the following generalization of the Niemytzki-Tychonoff Theorem [18]. A topological space is compact if and only if it is strongly complete with respect to every compatible quasi-uniform structure.

In 1966, Murdeshwar and Naimpally [16] continued to use the definition of Cauchy filter proposed by Sieber and Pervin [20] but defined a quasi-uniform space to be complete if every Cauchy filter had a nonvoid adherence.

By making use of the fact that every ultrafilter in a pre-compact space is Cauchy, the corresponding generalization of the Niemytzki-Tychonoff Theorem carried over for this new definition of completeness.

Sieber and Pervin [20] proposed the question, does every quasi-uniform space have a completion? Stoltenberg [22] in 1967 showed that every quasi-uniform space had a strong completion. However his construction left open the question of whether every Hausdorff quasi-uniform space had a Hausdorff completion. The techniques used by Liu [15] motivated the completions developed in this thesis.

In 1965, Naimpally [17] showed that if $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ were quasi-uniform spaces and Y is T_3 and $\mathcal{V}$ the Pervin structure then U and C are closed in $(F,\mathcal{W})$, where $F = Y^X$ and $\mathcal{W}$ is the quasi-uniform structure of quasi-uniform convergence and U (C) is the set of all quasi-uniformly continuous (continuous) mappings from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$.

III. COMPLETIONS OF A QUASI-UNIFORM SPACE

A. TRIVIAL COMPLETION AND SOME EXAMPLES

DEFINITION 1. Let X be a nonempty set. A quasi-uniform structure for X is a family $\mathcal{U}$ of subsets of $X \times X$ such that:

(1) $\Delta = \{ (x,x) : x \in X\} \subset U$ for each $U \in \mathcal{U}$;

(2) if $U \in \mathcal{U}$ and $U \subset V$, then $V \in \mathcal{U}$;

(3) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;

(4) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

DEFINITION 2. If $\mathcal{U}$ is a quasi-uniform structure for a set X, let $t_{\mathcal{U}} = \{ A \subset X :$ if $a \in A$ there exists $U \in \mathcal{U}$ such that $U[a] \subset A\}$. Then $t_{\mathcal{U}}$ is the quasi-uniform topology on X generated by $\mathcal{U}$.

DEFINITION 3. Let (X,t) be a topological space and let $\mathcal{U}$ be a quasi-uniform structure for X. Then $\mathcal{U}$ is compatible if $t = t_{\mathcal{U}}$.

If (X,t) is a topological space and $O \in t$, let

$$S(O) = O \times O \cup (X-O) \times X.$$

In [19], Pervin showed that $\{ S(O) : O \in t\}$ is a subbase for a compatible quasi-uniform structure for X. We will refer to this as the Pervin structure. The following definition is due to Sieber-Pervin [20] and is equivalent to the usual definition of a Cauchy filter in a uniform space.

DEFINITION 4. Let $(X,\mathcal{U})$ be a quasi-uniform space and $\mathcal{F}$ a filter on X. $\mathcal{F}$ is $\mathcal{U}$-Cauchy if for every $U \in \mathcal{U}$ there exists $x = x(U)$ such that $U[x] \in \mathcal{F}$.

DEFINITION 5. A quasi-uniform space $(X,\mathcal{U})$ is strongly complete if every $\mathcal{U}$-Cauchy filter converges. $(X,\mathcal{U})$ is said to be complete if every $\mathcal{U}$-Cauchy filter has an adherent point.

DEFINITION 6. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and f a mapping from X to Y. The mapping f is said to be quasi-uniformly continuous if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $(a,b) \in U$ implies that $(f(a),f(b)) \in V$.

DEFINITION 7. Two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are said to be quasi-uniformly isomorphic relative to $\mathcal{U}$ and $\mathcal{V}$ if there exists a one-to-one mapping f of X onto Y such that f and f^{-1} are quasi-uniformly continuous.

DEFINITION 8. A completion of a quasi-uniform space $(X,\mathcal{U})$ is a complete quasi-uniform space $(Y,\mathcal{V})$ such that X is quasi-uniformly isomorphic (relative to $\mathcal{U}$ and $\mathcal{V}$) to a dense subset of Y.

In [22], Stoltenberg proved that every quasi-uniform space has a strong completion. The proof is long and involved and it is not clear if any separation properties the space may possess carry over to the completion. The following construction shows that every quasi-uniform space has a rather simple completion.

CONSTRUCTION 1. Let $(X,\mathcal{U})$ be a quasi-uniform space and put $X^* = X \cup \{\beta\}$ where $\beta \notin X$. For $U \in \mathcal{U}$, let

$$S(U) = U \cup \{ (\beta,x) : x \in X^*\}.$$

Then $\mathcal{B} = \{ S(U) : U \in \mathcal{U}\}$ forms a base for a quasi-uniform structure $\mathcal{U}^*$ for X^*. Note that $S(U)[\beta] = X^*$ for each $U \in \mathcal{U}$, and $S(U)[x] = U[x]$ if $x \in X$. Clearly, every filter $\mathcal{F}$ on X^* converges to β. Hence $(X^*,\mathcal{U}^*)$ is strongly complete. Also, $\mathcal{U} = \{ U^* \cap X \times X : U^* \in \mathcal{U}^*\}$ and

$$i : (X,\mathcal{U}) \to (X^*,\mathcal{U}^*)$$

is a quasi-uniform isomorphism. Since X is dense in X^*, $(X^*,\mathcal{U}^*)$ is a completion of $(X,\mathcal{U})$. In fact, X^* is compact and $t_{\mathcal{U}^*} = t_{\mathcal{U}} \cup \{X^*\}$.

Fletcher [7], working independently, also discovered the above construction. If X is T_0, then X^* is T_0. X^* is never T_1 since the only open set containing β is X^*. Thus the following question naturally arises. Does a $T_2(T_1)$ quasi-uniform space have a $T_2(T_1)$ completion or strong completion?

EXAMPLE 1. We give an example of a Hausdorff quasi-uniform space that does not have a Hausdorff completion and consequently does not have a Hausdorff strong completion. Let X denote the real numbers and $A = \{ 1/n : n = 1,2,\ldots\}$. Set

$$t = \{ E-B : B \subset A \text{ and } E \text{ is open in the usual topology}\}.$$

(X,t) is Hausdorff but not T_3. Let $\mathcal{U}$ denote the Pervin quasi-uniform structure generated by t. Suppose $(X^*,\mathcal{U}^*)$

is a completion for $(X,\mathcal{U})$. We may assume that $\overline{X} = X^*$ and $\mathcal{U} = \{ U^* \cap X \times X : U^* \in \mathcal{U}^*\}$. Let $\mathcal{F}$ be the filter on X generated by $\{ A, \{1/2, 1/3, \ldots\}, \{1/3, 1/4, \ldots\}, \ldots\}$. Then there exists an ultrafilter $\mathcal{M}$ on X such that $\mathcal{M} \supset \mathcal{F}$. Since $\mathcal{U}$ is pre-compact, $\mathcal{M}$ is a Cauchy filter. Let $\tilde{\mathcal{M}}$ be the ultrafilter on X^* generated by $\mathcal{M}$. If $U^* \in \mathcal{U}^*$, $U^* \cap X \times X = U \in \mathcal{U}$ so that there exists $x \in X$ such that $U[x] \in \mathcal{M}$. Clearly $U^*[x] \in \tilde{\mathcal{M}}$. Hence $\tilde{\mathcal{M}}$ is Cauchy. Since $(X^*,\mathcal{U}^*)$ is complete, there exists $x^* \in X^*$ such that $\tilde{\mathcal{M}}$ converges to x^*. Thus every open neighborhood of x^* must meet every set of the form $\{ 1/n, 1/(n+1), \ldots\}$, $(n \in N)$. We show that 0 and x^* can not be separated by disjoint open sets. Let

$$U = (X-A \times X-A) \cup (A \times X).$$

Then there exists $U^* \in \mathcal{U}^*$ such that $U = U^* \cap X \times X$. Now $U^*[0] \cap A = \emptyset$. Therefore, $0 \neq x^*$. Suppose $0_1, 0_2 \in t^*$ such that $0 \in 0_1$ and $x^* \in 0_2$. Then $0_1 \cap X \in t$ so there exists $\varepsilon > 0$ such that $((-\varepsilon,\varepsilon) - A) \subset 0_1 \cap X$. Now there exists positive integers N and k such that

$$1/N < \varepsilon \text{ and } 1/k \in 0_2 \cap X \cap \{ 1/N, 1/(N+1), \ldots\}.$$

Since $1/k \in 0_2 \cap X \in t$ there exists $\delta > 0$ such that $(1/k - \delta, 1/k + \delta) - A \subset 0_2 \cap X$. Clearly

$$[(-\varepsilon,\varepsilon) - A] \cap [(1/k - \delta, 1/k + \delta) - A] \neq \emptyset.$$

Therefore, $0_1 \cap 0_2 \neq \emptyset$ and $(X^*,\mathcal{U}^*)$ is not Hausdorff.

EXAMPLE 2. We give an example of a quasi-uniform structure $\mathcal{U}$ for the set N of natural numbers such that:

(1) $t_{\mathcal{U}}$ is the discrete topology, and

(2) $(N,\mathcal{U})$ does not have a T_1 strong completion. Let $U_n = \{ (x,y) : x = y \text{ or } x \geq n\}$, $\mathcal{B} = \{ U_n : n \varepsilon N\}$, and let $\mathcal{U}$ denote the quasi-uniformity generated by the base $\mathcal{B}$. Suppose $(N^*,\mathcal{U}^*)$ is a strong completion for $(N,\mathcal{U})$. We may assume that $\overline{N} = N^*$ and $\mathcal{U} = \mathcal{U}^* \cap N \times N$. If $\mathcal{F} = \{N\}$, $\mathcal{F}$ is a Cauchy filter on $(N,\mathcal{U})$. Let $\tilde{\mathcal{F}}$ denote the Cauchy filter on $(N^*,\mathcal{U}^*)$ generated by $\mathcal{F}$. Since $(N^*,\mathcal{U}^*)$ is strongly complete, there exists $n^* \varepsilon N^*$ such that $\tilde{\mathcal{F}}$ converges to n^*. Clearly, $n^* \varepsilon N^* - N$. Let 0^* be any open set in N^* containing n^*. Then $0^* \varepsilon \tilde{\mathcal{F}}$ and $0^* \cap N \varepsilon \mathcal{F}$. Thus $0^* \cap N = N$. For each $n \varepsilon N$, we have $n \varepsilon 0^*$ for every open set in N^* containing n^*. Thus N^* is not T_1.

B. CONSTRUCTIONS OF A COMPLETION

LEMMA 1. Let $(X,\mathcal{U})$ be a quasi-uniform space and $\mathcal{S}$ a subbase for $\mathcal{U}$. Then a filter $\mathcal{F}$ converges to x if and only if for each $S \varepsilon \mathcal{S}$ we have $S[x] \varepsilon \mathcal{F}$.

Throughout this section we will let $(X,\mathcal{U})$ denote a quasi-uniform space with a base $\mathcal{B}$ such that for any $V \varepsilon \mathcal{B}$ we have that $V \circ V = V$. If $\mathcal{U}$ has a subbase with this property then the base generated by the subbase also has this property. It is clear that the Pervin quasi-uniform structure has such a base as well as the class of quasi-uniform structures introduced by Fletcher in [7]. We will say that two Cauchy ultrafilters $\mathcal{M}_1$ and $\mathcal{M}_2$ on X are equivalent provided $U[x] \varepsilon \mathcal{M}_1$ if and only if $U[x] \varepsilon \mathcal{M}_2$.

Certainly this is an equivalence relation on the set of all Cauchy ultrafilters on X. We will denote the equivalence class containing $\mathcal{M}$ by $\hat{\mathcal{M}}$.

Set $\Lambda = \{ \hat{\mathcal{M}} : \mathcal{M}$ is a nonconvergent Cauchy ultrafilter on $X\}$ and $X^* = X \cup \Lambda$. $D(V)$ will denote the set of all mappings δ from Λ to X such that $V[\delta(\hat{\mathcal{M}})] \in \mathcal{M}$ where $V \in \mathcal{B}$. Since $\mathcal{M}$ is Cauchy, $D(V) \neq \emptyset$ for each $V \in B$. For $V \in \mathcal{B}$ and $\delta \in D(V)$, we set

$$S(V,\delta) = V \cup \Delta \cup \{ (\hat{\mathcal{M}},y) : \hat{\mathcal{M}} \in \Lambda \text{ and } y \in V[\delta(\hat{\mathcal{M}})]\}.$$

LEMMA 2. $\mathcal{S}^* = \{ S(V,\delta) : V \in B, \delta \in D(V)\}$ forms a subbase for a quasi-uniform structure $\mathcal{U}^*$ on X^*.

PROOF. It is clear that $\mathcal{S}^* \neq \emptyset$. Let $S(V,\delta) \in \mathcal{S}^*$. Then $S(V,\delta) \supset \Delta$ and we show that $S(V,\delta) \circ S(V,\delta) \subset S(V,\delta)$. Suppose $(x^*,y^*) \in S(V,\delta)$ and $(y^*,z^*) \in S(V,\delta)$. Case (a). If $x^* = x \in X$, then $y^* = y \in X$ and $z^* = z \in X$. Thus $(x,y) \in V$, $(y,z) \in V$ and since $V \circ V = V$, we have $(x^*,z^*) = (x,z) \in S(V,\delta)$. Case (b). $x^* = \hat{\mathcal{M}} \in \Lambda$. If $y^* = \hat{\mathcal{M}}$ we have $(x^*,z^*) = (y^*,z^*) \in S(V,\delta)$. If $y^* = y \in X$, then $z^* = z \in X$, $y \in V[\delta(\hat{\mathcal{M}})]$, and $(y,z) \in V$. Hence $(\delta(\hat{\mathcal{M}}),y) \in V$ and consequently $(\delta(\hat{\mathcal{M}}),z) \in V$ since $V \circ V = V$. Then $z^* = z \in V[\delta(\hat{\mathcal{M}})]$ and $(x^*,z^*) = (\hat{\mathcal{M}},z) \in S(V,\delta)$. Therefore, $S(V,\delta) \circ S(V,\delta) \subset S(V,\delta)$.//

THEOREM 1. $(X^*,\mathcal{U}^*)$ is complete.

PROOF. Let $\mathcal{F}$ be a Cauchy filter on X^*. Then $\mathcal{F} \subset \mathcal{M}$ where $\mathcal{M}$ is an ultrafilter. If $\mathcal{M}$ converges, $\mathcal{F}$ has an adherent point and $\mathcal{U}^*$ is complete. Suppose $\mathcal{M}$ is not convergent. We show that $X \varepsilon \mathcal{M}$. Let $S(U,\delta) \varepsilon \mathcal{U}^*$. Since $\mathcal{M}$ is Cauchy, there exists $x^* \varepsilon X^*$ such that $S(U,\delta)[x^*] \varepsilon \mathcal{M}$. If $x^* = \hat{\mathcal{M}}_1 \varepsilon \Lambda$, then

$$S(U,\delta)[x^*] = \{\hat{\mathcal{M}}_1\} \cup U[\delta(\hat{\mathcal{M}}_1)].$$

$\mathcal{M}$ does not converge to $\hat{\mathcal{M}}_1$ implies there exists $S(V,\delta) \varepsilon \mathcal{S}^*$ such that $S(V,\delta)[\hat{\mathcal{M}}_1] \notin \mathcal{M}$. Since $\mathcal{M}$ is Cauchy, there exists $z^* \neq \hat{\mathcal{M}}_1$ such that $S(V,\gamma)[z^*] \varepsilon \mathcal{M}$. Now if $z^* \varepsilon X$ there is nothing to show; so we suppose that $z^* = \hat{\mathcal{M}}_2 \varepsilon \Lambda$. Then $\hat{\mathcal{M}}_1 \neq \hat{\mathcal{M}}_2$ and we have

$$(V[\gamma(\hat{\mathcal{M}}_2)] \cup \{\hat{\mathcal{M}}_2\}) \cap (U[\delta(\hat{\mathcal{M}}_1)] \cup \{\hat{\mathcal{M}}_1\}) \varepsilon \mathcal{M}.$$

Thus $X \varepsilon \mathcal{M}$ since

$$X \supset V[\gamma(\hat{\mathcal{M}}_2)] \cap U[\delta(\hat{\mathcal{M}}_1)] \varepsilon \mathcal{M}.$$

$\mathcal{M}_0 = \{ M \varepsilon \mathcal{M} : M \subset X\}$ is an ultrafilter on X. We show that $\mathcal{M}_0$ is $\mathcal{U}$-Cauchy. If $U \varepsilon \mathcal{U}$, there exist $V \varepsilon \mathcal{B}$ with $V \subset U$. Let $\delta \varepsilon D(V)$. Then $S(V,\delta) \varepsilon \mathcal{U}^*$ so there exists $x^* \varepsilon X^*$ such that $S(V,\delta)[x^*] \varepsilon \mathcal{M}$. If $x^* \varepsilon X$ we have $V[x^*] \varepsilon \mathcal{M}$ and thus $U[x^*] \varepsilon \mathcal{M}_0$. If $x^* = \underline{\hat{\mathcal{M}}} \varepsilon \Lambda$, then $V[\delta(\underline{\hat{\mathcal{M}}})] \cup \{\underline{\hat{\mathcal{M}}}\} \varepsilon \mathcal{M}$ and since $X \varepsilon \mathcal{M}$, we have $V[\delta(\underline{\hat{\mathcal{M}}})] \varepsilon \mathcal{M}$. Consequently, $U[\delta(\underline{\hat{\mathcal{M}}})] \varepsilon \mathcal{M}_0$. Thus $\mathcal{M}_0$ is Cauchy on X. Either $\mathcal{M}_0$ is convergent on X or it is not. Case (1) $\mathcal{M}_0$ converges to x. Let $S(V,\delta) \varepsilon \mathcal{S}^*$. Then $S(V,\delta)[x] = V[x] \varepsilon \mathcal{M}_0$. Thus $S(V,\delta)[x] \varepsilon \mathcal{M}$ and we have that $\mathcal{M}$ converges to x which is a contradiction. Case (2). $\mathcal{M}_0$ is nonconvergent on X. Then $\hat{\mathcal{M}}_0 \varepsilon \Lambda$ and we show that $\mathcal{M}$ converges to $\hat{\mathcal{M}}_0$. If $S(V,\delta) \varepsilon \mathcal{S}^*$,

$$S(V,\delta)[\hat{\mathcal{M}}_0] = V[\delta(\hat{\mathcal{M}}_0)] \cup \{\hat{\mathcal{M}}_0\}.$$

Now $V[\delta(\hat{M}_0)] \in M_0$ and consequently $S(V,\delta)[\hat{M}_0] \in M$. Therefore, M converges to $\hat{M}_0$ and this is a contradiction.//

THEOREM 2. (X^*,U^*) is a completion for (X,U).

PROOF. By theorem 1, (X^*,U^*) is complete. Let $\hat{M} \in \Lambda$ and $U^* \in U^*$. Now there exists $S(V_1,\delta_1),\ldots,$ $S(V_n,\delta_n) \in S^*$ such that $\bigcap_1^n S(V_i,\delta_i) \subset U^*$. We have

$$S(V_i,\delta_i)[\hat{M}] = \{\hat{M}\} \cup V_i[\delta_i(\hat{M})] \text{ for } i = 1,2,\ldots n.$$

Since $V_i[\delta_i(\hat{M})] \in M$ for each $i = 1,2,\ldots,n$, we have $\bigcap_1^n V_i[\delta_i(\hat{M})] \neq \emptyset$. Now

$$\begin{aligned} U^*[\hat{M}] \cap X &\supset (\bigcap_1^n S(V_i,\delta_i))[\hat{M}] \cap X \\ &= \bigcap_1^n S(V_i,\delta_i)[\hat{M}] \cap X \\ &= \bigcap_1^n V_i[\delta_i(\hat{M})] \neq \emptyset. \end{aligned}$$

Thus $\hat{M} \in \overline{X}$ and $\overline{X} = X^*$. Let U' denote the induced quasi-uniform structure of U^* on X. If $U \in U$ there exists $V \in B$ with $V \subset U$. Let $\delta \in D(V)$. Then $S(V,\delta) \in U^*$ and $V = S(V,\delta) \cap X \times X$. Thus $V \in U'$ and hence $U \in U'$. Suppose $U \in U'$. Then there exists $U^* \in U^*$ such that $U = U^* \cap X \times X$. Since $U^* \in U^*$ there exists $S(V_1,\delta_1),\ldots,$ $S(V_n,\delta_n) \in S^*$ such that $U^* \supset \bigcap_1^n S(V_i,\delta_i)$. Consequently,

$$(\bigcap_1^n S(V_i,\delta_i)) \cap X \times X \subset U^* \cap X \times X = U.$$

Therefore, $\bigcap_1^n V_i \subset U$ and hence $U \in U$. Thus we have shown that $U = U'$. From this it follows that the identity mapping $i\colon (X,U) \to (X,U^* \cap X \times X)$ is a quasi-uniform isomorphism.//

One can let $\Lambda = \{\, \hat{\mathcal{F}} : \mathcal{F}$ is a nonconvergent Cauchy filter on $(X,\mathcal{U})\}$. Then using the same construction we get a strong completion of $(X,\mathcal{U})$. Denote it by $(\mathring{X},\mathring{\mathcal{U}})$. It is clear that $(\mathring{X},\mathring{\mathcal{U}})$ is not in general the trivial strong completion given in construction 1.

EXAMPLE 3. A space $(X,\mathcal{U})$ may be T_1 and $(X^*,\mathcal{U}^*)$ not T_1. Consider the space $(N,\mathcal{U})$ described in example 2. It is clear that $U_n \circ U_n = U_n$ for each U_n in the base $\mathcal{B}$. Thus we may construct the completion $(N^*,\mathcal{U}^*)$. There exists an ultrafilter $\mathcal{M}$ containing the filter base $\{\, \{n,n+1,\ldots\} : n \in N\}$. Since $\mathcal{U}$ is pre-compact, we have that $\mathcal{M}$ is a Cauchy filter. Since $\mathcal{M}$ is nonconvergent, $\hat{\mathcal{M}} \in N^*$. Let $S(U_n,\delta) \in \mathcal{S}^*$. Clearly, $\delta(\hat{\mathcal{M}}) \geq n$. Therefore,

$$\begin{aligned} S(U_n,\delta)[\hat{\mathcal{M}}] &= U_n[\delta(\hat{\mathcal{M}})] \cup \{\hat{\mathcal{M}}\} \\ &= N \cup \{\hat{\mathcal{M}}\}. \end{aligned}$$

Hence $(N^*,\mathcal{U}^*)$ is not T_1.

THEOREM 3. Let (X,t) be a T_1 topological space and $\mathcal{U}$ the Pervin quasi-uniform structure. Then $(X^*,\mathcal{U}^*)$ is a T_1 completion for $(X,\mathcal{U})$.

PROOF. If $x,y \in X$, $x \neq y$, there exists $O,W \in t$ such that $x \in O$, $y \in W$, $x \notin W$, and $y \notin O$. Since $X \in t^*$, we have $O,W \in t^*$. Let $\hat{\mathcal{M}}_1 \neq \hat{\mathcal{M}}_2$ be points in Λ and $S(U,\delta) \in \mathcal{S}^*$. Then

$$S(U,\delta)[\hat{\mathcal{M}}_1] = \{\hat{\mathcal{M}}_1\} \cup U[\delta(\hat{\mathcal{M}}_1)] \text{ and,}$$

$$S(U,\delta)[\hat{\mathcal{M}}_2] = \{\hat{\mathcal{M}}_2\} \cup U[\delta(\hat{\mathcal{M}}_2)].$$

Suppose $x \in X$ and $\hat{M} \in \Lambda$. Since M does not converge to x, we have $\{x\} \notin M$. Now $X - \{x\} \in M$ since M is an ultrafilter. Set

$$U = (X - \{x\}) \times (X - \{x\}) \cup \{x\} \times X.$$

Let $x_1 \in X$ and $x_1 \neq x$; if $X = \{x\}$ we would have $X^* = X$. Define δ by $\delta(\hat{M}) = x_1$ for each $\hat{M} \in \Lambda$. Clearly, $U[x_1] = X - \{x\} \in N$ for each nonconvergent ultrafilter N. Thus $\delta \in D(U)$ and $S(U,\delta) \in S^*$. Now $S(U,\delta)[\hat{M}] = \{\hat{M}\} \cup (X - \{x\})$ and X is a neighborhood of x that does not contain $\hat{M}$. Therefore, $(X^*,\mathcal{U}^*)$ is T_1.//

DEFINITION 9. $(X,\mathcal{U})$ is R_3 if given $x \in X$ and $U \in \mathcal{U}$, there exists a symmetric $V \in \mathcal{U}$ such that $(V \circ V)[x] \subset U[x]$.

The above condition was introduced in [16]. It might be described as a local symmetric triangle inequality. In [16, p.41], it was shown that if (X,t) is regular, then the Pervin structure is R_3.

THEOREM 4. Let $(X,\mathcal{U})$ be a T_1 and R_3 quasi-uniform space. Then $(X^*,\mathcal{U}^*)$ is a T_1 space.

PROOF. It suffices to show that for each $x \in X$ and $\hat{M} \in \Lambda$ there exists O_1, $O_2 \in t^*$ such that $x \in O_1$, $\hat{M} \in O_2$, $x \notin O_2$, and $\hat{M} \notin O_1$. Let x and $\hat{M}$ be given. Since M does not converge to x, there exists $U \in \mathcal{U}$ such that $U[x] \notin M$. There exists a symmetric $V \in \mathcal{U}$ such that $(V \circ V)[x] \subset U[x]$. M is Cauchy implies that there exists $x_0 \in X$ with $V[x_0] \in M$. We show that $x \notin V[x_0]$. Suppose $x \in V[x_0]$ and let $a \in V[x_0]$. Then $(x,x_0) \in V$ and $(x_0,a) \in V$.

Hence $a \varepsilon (V \circ V)[x] \subset U[x]$. But this implies that $V[x_0] \subset U[x]$ which is impossible since $U[x] \notin M$. There exist $\delta \varepsilon D(V)$ with $\delta(\hat{M}) = x_0$. Then

$$x \notin V[x_0] \cup \{\hat{M}\} = S(V,\delta)[\hat{M}]$$

while $\hat{M} \notin X$, an open neighborhood of x. Thus $(X^*,\mathcal{U}^*)$ is T_1.//

THEOREM 5. Listed here are some easily verified properties of $(X^*,\mathcal{U}^*)$.

(1) X is an open dense subset of X^*.

(2) $(X,\mathcal{U})$ is T_0 if and only if $(X^*,\mathcal{U}^*)$ is T_0.

(3) If $(X^*,\mathcal{U}^*)$ has property P and P is an open hereditary property, then $(X,\mathcal{U})$ has property P.

(4) Λ is closed in X^* and the subspace topology on Λ is the discrete topology.

(5) If $(X^*,\mathcal{U}^*)$ is pre-compact and Hausdorff, then $(X,\mathcal{U})$ is completely regular.

These properties also hold for $(\mathring{X},\mathring{\mathcal{U}})$.

PROOF. (1). By theorem 2, $\overline{X} = X^*$. Let $x \varepsilon X$ and $S(U,\delta) \varepsilon \mathcal{U}^*$. Then $S(U,\delta)[x] = U[x] \subset X$. Hence X is open in X^*. (2). Since T_0 is a hereditary property the sufficiency is clear. Suppose $(X,\mathcal{U})$ is T_0, then since X is open in X^* it suffices to consider the case where $x^* \neq \hat{M} \varepsilon \Lambda$, $x^*,\hat{M} \varepsilon X^*$. Let $S(U,\delta) \varepsilon \mathcal{U}^*$ and we have that $\hat{M} \notin S(U,\delta)[x^*]$. Hence X^* is T_0. Statement (3) follows from (1). (4). Λ is closed in X^* by (1), and for any $S(U,\delta) \varepsilon \mathcal{U}^*$ we have that $\{\hat{M}\} = \Lambda \cap S(U,\delta)[\hat{M}]$ for any

$\hat{M} \varepsilon \Lambda$. Therefore the subspace topology on Λ is discrete. (5). If $(X^*,\mathcal{U}^*)$ is pre-compact and Hausdorff then since it is complete it must be a compact Hausdorff space. Therefore the subspace $(X,\mathcal{U})$ must be completely regular.//

C. A COMPLETION FOR A PRE-COMPACT STRUCTURE AND FURTHER EXAMPLES

Let $(X,\mathcal{U})$ be a pre-compact quasi-uniform space. Define X^* as before and set $S(U) = U \cup \Delta \cup \{ (\hat{M},y) : \hat{M} \varepsilon \Lambda$ and $y \varepsilon U[x] \varepsilon \mathcal{M}$ for some $x \varepsilon X\}$.

LEMMA 3. $\mathcal{B}^* = \{ S(U) : U \varepsilon \mathcal{U}\}$ forms a base for a quasi-uniform structure for X^*. Denote it by $\tilde{\mathcal{U}}$.

PROOF. $\mathcal{B}^* \neq \emptyset$ and $\Delta \subset S(U)$ for each $S(U) \varepsilon \mathcal{B}^*$. Suppose $S(U)$ and $S(V)$ belong to $\mathcal{B}^*$. Then $S(U \cap V) \varepsilon \mathcal{B}$ and we show that

$$S(U \cap V) \subset S(U) \cap S(V).$$

Let $(x,y) \varepsilon S(U \cap V)$. If $x \varepsilon X$, then $(x,y) \varepsilon U \cap V$ and thus $(x,y) \varepsilon S(U) \cap S(V)$. If $x = \hat{M} \varepsilon \Lambda$, then $y = \hat{M}$, in which case $(x,y) \varepsilon S(U) \cap S(V)$, or there exists $z \varepsilon X$ such that $y \varepsilon (U \cap V)[z] \varepsilon \mathcal{M}$. Thus $y \varepsilon U[z] \varepsilon \mathcal{M}$ and $y \varepsilon V[z] \varepsilon \mathcal{M}$. Hence $(x,y) \varepsilon S(U) \cap S(V)$. Let $S(U) \varepsilon \mathcal{B}^*$. Then there exists $V \varepsilon \mathcal{U}$ such that $V \circ V \subset U$. We show that $S(V) \circ S(V) \subset S(U)$, let $(x,y) \varepsilon S(V)$ and $(y,z) \varepsilon S(V)$. If $x \varepsilon X$ then $(x,y) \varepsilon V$ and $(y,z) \varepsilon V$ and therefore $(x,z) \varepsilon V \circ V \subset U \subset S(U)$. Now if $x = \hat{M} \varepsilon \Lambda$ and $y = \hat{M}$ then $(x,z) = (y,z) \varepsilon S(V) \subset S(U)$. If $x = \hat{M}$ and

$y \in X$, then there exist $s \in X$ such that $y \in V[s] \in \mathcal{M}$ and $(y,z) \in V$. Hence $(s,z) \in V \circ V \subset U$, that is, $z \in U[s] \supset V[s]$. Thus $U[s] \in \mathcal{M}$ and we have that $(x,z) = (\hat{\mathcal{M}},z) \in S(U)$. Therefore $S(V) \circ S(V) \subset S(U)$ and we have that $\mathcal{B}^*$ is a base for a quasi-uniform structure on X^*.//

THEOREM 6. $(X^*,\tilde{\mathcal{U}})$ is complete.

PROOF. Suppose $(X^*,\tilde{\mathcal{U}})$ is not complete. Then there exists a nonconvergent Cauchy ultrafilter $\mathcal{M}^*$ on X^*. Using the same argument as in the proof of theorem 1 we have that $\mathcal{M} = \{ M \in \mathcal{M}^* : M \subset X\}$ is an ultrafilter on X. Since $(X,\mathcal{U})$ is pre-compact it follows that $\mathcal{M}$ is Cauchy [16, p. 51]. Case (1). $\mathcal{M}$ converges to $x \in X$. Let $U \in \tilde{\mathcal{U}}$. Then there exists $S(V)$ such that $S(V) \subset U$. Since $x \in \lim \mathcal{M}$, $V[x] \in \mathcal{M}$. Hence $V[x] \in \mathcal{M}^*$, and $U[x] \supset S(V)[x] = V[x]$ implies that $U[x] \in \mathcal{M}^*$. Thus $x \in \lim \mathcal{M}^*$ but this is impossible. Case (2). $\lim \mathcal{M} = \emptyset$. Then $\hat{\mathcal{M}} \in X$. Let $U \in \tilde{\mathcal{U}}$, then there exists $S(V) \subset U$. Now $S(V)[\hat{\mathcal{M}}] = \{\hat{\mathcal{M}}\} \cup (\bigcup \{ V[y] : V[y] \in \mathcal{M}\})$. Hence $S(V)[\hat{\mathcal{M}}] \in \mathcal{M}$ and therefore $U[\hat{\mathcal{M}}] \in \mathcal{M}$. Consequently $\hat{\mathcal{M}} \in \lim \mathcal{M}$ which is a contradiction. Thus there are no nonconvergent Cauchy ultrafilters on $(X^*,\tilde{\mathcal{U}})$; that is, $(X^*,\tilde{\mathcal{U}})$ is complete.//

THEOREM 7. $(X^*,\tilde{\mathcal{U}})$ is a completion for $(X,\mathcal{U})$.

PROOF. By the previous theorem $(X^*,\tilde{\mathcal{U}})$ is complete. It is clear that $\overline{X} = X^*$. We show that $\mathcal{U} = \tilde{\mathcal{U}}_X$. Let $U \in \tilde{\mathcal{U}}_X$, then there exists $V \in \tilde{\mathcal{U}}$ such that $U = V \cap X \times X$. Now there exists $W \in \mathcal{U}$ such that $S(W) \subset V$, hence $W \subset U$ and we have that $\tilde{\mathcal{U}}_X \leq \mathcal{U}$. Let $U \in \mathcal{U}$, then $S(U) \in \tilde{\mathcal{U}}$ and $U = S(U) \cap X \times X \in \mathcal{U}_X$. Thus $\mathcal{U} \leq \mathcal{U}_X$. Thus the identity mapping $i : (X,\mathcal{U}) \rightarrow (X,\tilde{\mathcal{U}}_X)$ is a quasi-uniform isomorphism.//

We note that the Pervin quasi-uniform structure is pre-compact. A proof similar to the proof of theorem 4 shows that if $(X,\mathcal{U})$ is T_1 and R_3, then $(X^*,\tilde{\mathcal{U}})$ is T_1.

THEOREM 8. Let $(X,\mathcal{U})$ be a pre-compact quasi-uniform space.

(1) $(X^*,\tilde{\mathcal{U}})$ is pre-compact if and only if Λ is finite if and only if $(X^*,\tilde{\mathcal{U}})$ is compact.

(2) X^* is T_3 implies that X^* is compact.

PROOF. (1) follows from the definition of $\tilde{\mathcal{U}}$ and the fact that completeness and pre-compactness are equivalent to compactness. Since every ultrafilter on X is Cauchy, it must converge to a point in X^* and hence (2) follows from theorem 4.17 in [16].//

EXAMPLE 4. Let $N = \{1,2,3,\ldots\}$ and let $\mathcal{U}$ denote the quasi-uniform structure generated by the base $\mathcal{B} = \{ U_n : n \in N\}$, where $U_n = \{ (x,y) : x = y \text{ or } y \geq x \geq n\}$. Let $\mathcal{M}_0$ be an ultrafilter containing $\{N,\{2,3,4,\ldots\}, \{3,4,5,\ldots\}, \ldots\}$. Then $\hat{\mathcal{M}}_0 \in \Lambda$. We show that $\Lambda = \{\hat{\mathcal{M}}_0\}$ and $(N^*,t_{\mathcal{U}^*})$ is homeomorphic to (N_∞,t_∞), the one-point compactification of (N,t).

Let $\mathcal{M}$ be a nonconvergent Cauchy ultrafilter on N. Note that $U_n[k] = \{k\}$ if $k \leq n$, and $U_n[k] = \{k, k+1,\ldots\}$ if $k \geq n$. It follows that $U_n[k] \in \mathcal{M}$ if and only if

$U_n[k] \in \mathcal{M}_0$. Therefore, $\mathcal{M} \sim \mathcal{M}_0$ and $\Lambda = \{\hat{\mathcal{M}}_0\}$. Let $i : N^* \to N_\infty$ be the identity on N and $i(\hat{\mathcal{M}}_0) = \infty$. Now i and i^{-1} are continuous at each $n \in N$. We show that i is continuous at $\hat{\mathcal{M}}_0$. Let $O_\infty = (X - O) \cup \{\infty\} \in t_\infty$. Since O is compact, there exists k such that $n \in O$ implies $n < k$. Now there exists $\delta \in D(U_k)$ such that $S(U_k,\delta)[\hat{\mathcal{M}}_0] = \{k,k+1,\ldots\} \cup \{\hat{\mathcal{M}}_0\}$. δ is defined by $\delta(\hat{\mathcal{M}}) = k$ for each $\hat{\mathcal{M}} \in \Lambda$. Now $i(S(U_k,\delta)[\hat{\mathcal{M}}_0]) = \{k,k+1,\ldots\} \cup \{\infty\} \subset O_\infty$. We show that i^{-1} is continuous at ∞. Let $S(U_k,\delta)$ be given. Now $\delta(\hat{\mathcal{M}}_0) \geq k$, so let $O = \{1,2,\ldots,\delta(\hat{\mathcal{M}}_0)-1)\}$. (If $\delta(\hat{\mathcal{M}}_0) = 1$, then $k = 1$ and $S(U_1,\delta)[\hat{\mathcal{M}}_0] = N^*$ and there is nothing to show.) Let $O_\infty = \{\infty\} \cup (N - O)$. Then $O_\infty \in t_\infty$ and $i^{-1}(O_\infty) = \{\hat{\mathcal{M}}_0\} \cup (N - O) = S(U_k,\delta)[\hat{\mathcal{M}}_0]$.

EXAMPLE 5. Let I denote the set of integers and $U_n = \Delta \cup \{ (x,y) : x = y,\ y \geq x \geq n, \text{ or } y \leq x \leq -n\}$. Let $\mathcal{U}$ denote the quasi-uniform structure generated by the base $\{ U_n : n = 0,1,2,\ldots\}$. Then I has the discrete topology and $\mathcal{U}$ is pre-compact. Let $\mathcal{M}_\infty$ be an ultrafilter containing the filter base $\{\{2,3,4,\ldots\},\{3,4,5,\ldots\},\ldots\}$ and $\mathcal{M}_{-\infty}$ be an ultrafilter containing the filter base $\{\{-2,-3,\ldots\},\{-3,-4,\ldots\},\ldots\}$. Then $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ are nonconvergent Cauchy ultrafilters. Now $U_2[2] \in \mathcal{M}_\infty$ while $U_2[2] \notin \mathcal{M}_{-\infty}$. Hence $\hat{\mathcal{M}}_\infty \neq \hat{\mathcal{M}}_{-\infty}$. Now by considering the various cases, as in example 3, we have that $\Lambda = \{\hat{\mathcal{M}}_\infty,\hat{\mathcal{M}}_{-\infty}\}$. Thus $(I^*,\mathcal{U}^*)$ is compact. Now we show that I^* is Hausdorff. Since I is discrete and open in I^*, it is clear that distinct points in I are separated by disjoint open sets in I*. Now let $k \in I$. Choose $n > k$ and $-m < k$. There exists $\delta_1 \in D(U_n)$ and $\delta_2 \in D(U_m)$ such that $\delta_1(\hat{M}_\infty) = n$, $\delta_1(\hat{M}_{-\infty}) = -n$, $\delta_2(\hat{M}_\infty) = m$, and $\delta_2(\hat{M}_{-\infty}) - m$. Now

$$\{k\} \cap S(U_n,\delta_1)[\hat{M}_\infty] = \emptyset \text{ and}$$
$$\{k\} \cap S(U_m,\delta_2)[\hat{M}_{-\infty}] = \emptyset.$$

There exists $\delta \in D(U_2)$ where $\delta(\hat{M}_\infty) = 2$ and $\delta(\hat{M}_{-\infty}) = -2$. Then

$$S(U_2,\delta)[\hat{M}_\infty] \cap S(U_2,\delta)[\hat{M}_{-\infty}]$$
$$= (\{\hat{M}_0\} \cup \{2,3,4,\ldots\}) \cap (\{\hat{M}_{-\infty}\} \cup \{-2,-3,-4,\ldots\}) = \emptyset.$$

Hence I* is Hausdorff.

D. CAUCHY FILTERS

In a uniform space the adherence of a Cauchy filter equals its limit. The following example shows that this is not necessarily the case in a quasi-uniform space.

EXAMPLE 6. Let $X = \{1,2,3,4,5\}$ and $W = \{(x,y) : x \leq y\}$. Since $W \circ W = W$ we have that $\{W\}$ forms a base for a quasi-uniform structure. Denote the structure by $\mathcal{U}$. Set $\mathcal{F} = \{X,\{2,3,4,5\}\}$. Then $\mathcal{F}$ is a Cauchy filter since $W[2] \in \mathcal{F}$. It is clear that $\lim \mathcal{F} = \{1,2\}$ while $\operatorname{adh} \mathcal{F} = X$. Hence we have a Cauchy filter whose limit is not equal to its adherence.

It is natural to wonder if there are quasi-uniform spaces, that are not uniform spaces, in which the limit of every Cauchy filter equals its adherence. The following theorem shows that such spaces do exist.

THEOREM 9. Let $(X,\mathcal{U})$ be a R_3 quasi-uniform space. If $\mathcal{F}$ is a Cauchy filter then $\lim \mathcal{F} = \operatorname{adh} \mathcal{F}$.

PROOF. Let $\mathcal{F}$ be a Cauchy filter. It suffices to prove that $\operatorname{adh} \mathcal{F} \subset \lim \mathcal{F}$. Let $x \in \operatorname{adh} \mathcal{F}$ and $U \in \mathcal{U}$. Since $(X,\mathcal{U})$ is R_3 there exists a symmetric $V \in \mathcal{U}$ such that $(V \circ V \circ V)[x] \subset U[x]$. Since $\mathcal{F}$ is Cauchy there exists $a \in X$ such that $V[a] \in \mathcal{F}$. We will show that $V[a] \subset U[x]$. Since $x \in \operatorname{adh} \mathcal{F}$ there exists $b \in V[x] \cap V[a]$. Hence $(x,b) \in V$ and $(a,b) \in V$. Let $c \in V[a]$. Then $(a,c) \in V$. Since V is symmetric, $(b,a) \in V$. Thus $(x,b) \in V$, $(b,a) \in V$, and $(a,c) \in V$. Therefore $c \in (v \circ V \circ V)[x] \subset U[x]$. Hence $V[a] \subset U[x]$, and we have $U[x] \in \mathcal{F}$. Thus $x \in \lim \mathcal{F}$.//

COROLLARY. Let $(X,\mathcal{U})$ be a R_3 quasi-uniform space. $(X,\mathcal{U})$ is complete if and only if it is strongly complete.

PROOF. The result is obvious by theorem 9. and the definitions.//

In a complete uniform space if $\lim \mathcal{F} \neq \emptyset$ then it follows that $\mathcal{F}$ is Cauchy. It is natural then to ask if in a complete quasi-uniform space we have a filter $\mathcal{F}$ such that $\operatorname{adh} \mathcal{F} \neq \emptyset$, does it follow that $\mathcal{F}$ is Cauchy. The following example shows that such a filter need not be Cauchy.

EXAMPLE 7. Let X denote the set of positive integers. Set $U_n = \{ (x,y) : x = y$, or $x = 1$ and $y = 2n + 2k$, or $x = 2$ and $y = 2n + 2k + 1$, where $k = 1,2,\ldots\}$. For example, $U_3 = \Delta \cup \{(1,8), (1,10),\ldots\} \cup \{(2,9), (2,11),\ldots\}$.

Let $\mathcal{B} = \{ U_n : n = 1,2,\ldots\}$. To show $\mathcal{B}$ is a base for a quasi-uniform structure it suffices to prove that $U_n \circ U_n \subset U_n$ for each $n = 1,2,\ldots$. Let U_n be given, and $(x,y) \in U_n$ and $(y,z) \in U_n$. If $x = y$, then $(x,z) = (y,z) \in U_n$. If $x \neq y$ then either $x = 1$ or $x = 2$. Suppose $x = 1$, then $y \geq 2n + 2 > 2$. Hence $z = y$ and we have that $(x,z) = (x,y) \in U_n$. On the other hand, if $x = 2$ then $y \geq 2n + 2 + 1 > 3$ and therefore $y = z$. Thus $(x,z) = (x,y) \in U_n$. Therefore $\mathcal{B}$ generates a quasi-uniform structure which we will denote by $\mathcal{U}$. The topology generated by $\mathcal{U}$ is compact since given any open cover there exists O_1 containing 1 and a O_2 containing 2. Clearly $O_1 \cup O_2$ contain all but at most a finite number of members of X. Hence $(X,\mathcal{U})$ is compact and therefore strongly complete.

The sets of the form $\{ n,n+1,\ldots\}$ $(n \in X)$ generate a filter $\mathcal{F}$ and $1 \in \text{adh } \mathcal{F}$, but $\mathcal{F}$ is not Cauchy.

In a uniform space it is well known that the neighborhood filters are minimal among the Cauchy filters. This does not hold in general for quasi-uniform spaces as the following example indicates.

EXAMPLE 8. Consider the space in example 6. Let $\mathcal{N}(4)$ denote the neighborhood filter of 4. Clearly $\mathcal{N}(4)$ is the collection of super sets of $\{4,5\}$. $\mathcal{N}(5)$ is the collection of all super sets of $\{5\}$. Now $\mathcal{N}(4) \subsetneq \mathcal{N}(5)$ and thus $\mathcal{N}(5)$ is not minimal among the Cauchy filters.

E. A COUNTER-EXAMPLE

The following theorem shows that:

(1) A uniformizable space can admit a strongly complete quasi-uniform structure and not admit a complete uniform structure.

(2) A space can be uniformizable and admit a strongly complete quasi-uniform structure and not be real compact.

(3) A countably compact space may admit a quasi-uniform structure that is not pre-compact. (This can not happen with uniform structures.)

Let W denote the ordinals less than Ω, the first uncountable ordinal. t will denote the order topology for W. It is well known [12, p. 74] that (W,t) is normal, countably compact, not metrizable, and not real compact. Dieudonne [5] showed that (W,t) admits a unique compatible uniform structure $\mathcal{U}$ and $(W,\mathcal{U})$ is not complete.

THEOREM 10. (W,t) admits a strongly complete quasi-uniform structure.

PROOF. Let $\mathcal{P}$ denote the Pervin quasi-uniform structure for t. Set $L = \{(x,y) : x \geq y\}$. Let $\mathcal{S} = \{U \cap L : U \in \mathcal{P}\}$. If $A \in \mathcal{S}$, then $A \supset \Delta$. If $U \cap L \in \mathcal{S}$ and $V \cap L \in \mathcal{S}$, then $U \cap V \in \mathcal{P}$ and $(U \cap L) \cap (V \cap L) = (U \cap V) \cap L \in \mathcal{S}$. Suppose $U \cap L \in \mathcal{S}$, then there exists $V \in \mathcal{P}$ such that $V \circ V \subset U$. Thus $V \cap L \in \mathcal{S}$ and $(V \cap L) \circ (V \cap L) \subset U \cap L$. For suppose $(a,b) \varepsilon V \cap L$ and $(b,c) \varepsilon V \cap L$. Then $(a,c) \varepsilon U$. Also $(a,b) \varepsilon L$ implies $a \geq b$ and $(b,c) \varepsilon L$ implies $b \geq c$. Hence $a \geq c$; that is, $(a,c) \varepsilon L$. Therefore $(a,c) \varepsilon U \cap L$. Thus $\mathcal{S}$ is a base for a quasi-uniform structure. Denote the generated quasi-uniform structure by $\mathcal{U}$. We will show that $t = t_{\mathcal{U}}$. Since $\mathcal{P} \leq \mathcal{U}$ we have that $t \leq t_{\mathcal{U}}$. Let $O \varepsilon t_{\mathcal{U}}$ and $x \varepsilon O$. Then there exists $L \cap U \varepsilon \mathcal{U}$ such that $x \varepsilon (L \cap U)[x] \subset O$. Now $U[x]$ is a neighborhood of x with respect to t. $L[x] = [1,x] = [1,x+1)$ is also a neighborhood of x with respect to t. Thus

$$L[x] \cap U[x] = (L \cap U)[x]$$

is a neighborhood of x with respect to t. Hence $O \varepsilon t$ and we have that $t = t_{\mathcal{U}}$.

We now show that $(W,\mathcal{U})$ is complete. Let $W^* = [1,\Omega]$ and let $\mathcal{M}$ be a Cauchy ultrafilter on W. Let $\mathcal{M}^*$ denote the generated ultrafilter on W^*. Since W^* is compact there exists $w \varepsilon W^*$ such that $w \varepsilon \lim \mathcal{M}^*$. Case 1. If $w \varepsilon W$, then $w \varepsilon \lim \mathcal{M}$. Case 2. Suppose $w = \Omega$. Since $\mathcal{M}$ is Cauchy and $L \varepsilon \mathcal{U}$, there exists $x \varepsilon W$ such that $L[x] = [1,x] \varepsilon \mathcal{M}$. Hence $[1,x] \varepsilon \mathcal{M}^*$. Now $(x,\Omega]$ is a neighborhood of Ω and since $\mathcal{M}^*$ converges to Ω we have that $(x,\Omega] \varepsilon \mathcal{M}^*$, but this is impossible. Hence $w \varepsilon W$ and by case 1, we have that $w \varepsilon \lim \mathcal{M}$. Therefore $(W,\mathcal{U})$ is complete.

An alternate proof that $(W,\mathcal{U})$ is complete can be given. Let $\mathcal{M}$ be a Cauchy ultrafilter on W. Since $L \varepsilon \mathcal{U}$ there exists $x \varepsilon W$ such that $[1,x] = L[x] \varepsilon \mathcal{M}$. Let $\mathcal{M}_1$

be the trace of $\mathcal{M}$ on $[1,x]$, then $\mathcal{M}_1$ is an ultrafilter on $[1,x]$. Since $[1,x]$ is compact there exists $z \in [1,x]$ such that $z \in \lim \mathcal{M}_1$. Hence $z \in \lim \mathcal{M}$.

We now show that $(X,\mathcal{U})$ is R_3. Let $L \cap U \in \mathcal{U}$ and $w \in W$. Then there exists $x \in W$ such that $(x,w] \subset (L \cap U)[w]$. Let $O = [1,x]$, $S = (x,w]$, and $T = (w,\Omega)$. Then O, S, and T are open. Set

$$V_1 = O \times O \cup (W-O) \times W$$

$$V_2 = S \times S \cup (W-S) \times W$$

$$V_3 = T \times T \cup (W-T) \times W$$

Then V_1, V_2, V_3 belong to $\mathcal{P}$ and hence to $\mathcal{U}$. It is easy to see that

$$\begin{aligned} Z &= V_1 \cap V_2 \cap V_3 \\ &= (O \times O) \cup (S \times S) \cup (T \times T). \end{aligned}$$

Thus $Z \in \mathcal{U}$, Z is symmetric, and $Z \circ Z = Z$. Now $Z[w] = (x,w] \subset (L \cap U)[w]$. Therefore $(W,\mathcal{U})$ is R_3 and by the corollary to theorem 9 we have that $(W,\mathcal{U})$ is strongly complete.//

F. GENERAL PROPERTIES FOR A COMPLETION

It is apparent that not all of the properties of a completion for a uniform space carry over for a quasi-uniform space. In this section we note that irregardless of the definitions of "Cauchy" filter and "completeness" not all of the pleasant properties of a "completion" are going to be preserved in a quasi-uniform space.

We would like a definition of "Cauchy" filter and "completeness" that would satisfy the following conditions.

(a) The definitions would reduce to the ordinary definitions on a uniform space.

(b) There exists at least one completion for every quasi-uniform space.

(c) If $(X,\mathcal{U})$ is T_2, then there exists a T_2 completion.

(d) If $(X,\mathcal{U})$ is compact, then $(X,\mathcal{U})$ is complete.

(e) If $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is quasi-uniformly continuous where $(Y,\mathcal{V})$ is complete and T_2, then there exists a quasi-uniformly continuous extension $\tilde{f} : (X^*,\mathcal{U}^*) \to (Y,\mathcal{V})$ where $(X^*,\mathcal{U}^*)$ is any completion of $(X,\mathcal{U})$.

(f) If $(X,\mathcal{U})$ is pre-compact then $(X^*,\mathcal{U}^*)$ is compact where $(X^*,\mathcal{U}^*)$ is any completion of $(X,\mathcal{U})$.

(g) Every convergent filter is Cauchy.

It is clear that each of these conditions hold in a uniform space.

THEOREM 11. (c) and (f) can not both hold (for all spaces $(X,\mathcal{U})$).

PROOF. Let (X,t) be a T_2 space that is not T_3 and $\mathcal{U}$ the Pervin quasi-uniform structure for t. By (c) there exists a T_2 completion $(X^*,\mathcal{U}^*)$ for $(X,\mathcal{U})$ and by (f) $(X^*,\mathcal{U}^*)$ is compact. Hence $(X,\mathcal{U})$ is T_3 which is impossible.//

THEOREM 12. (d) and (e) can not both hold (for all space $(X,\mathcal{U})$).

PROOF. Let $X = (0,1]$ and $\mathcal{U}$ the Pervin quasi-uniform structure associated with the usual topology. Let Y = [-1,1] and let $\mathcal{V}$ be the Pervin structure associated with the usual topology on Y. (Y,$\mathcal{V}$) is T_2 and compact and by (d) it is complete. Define f : X → Y by f(x) = sin 1/x. Now f is continuous and since $\mathcal{U}$ and $\mathcal{V}$ are the Pervin structures it follows that f is quasi-uniformly continuous.

Now X* = [0,1] is compact with the usual topology. Let $\mathcal{U}$* be the Pervin structure for X*. By (d) we have that (X*,$\mathcal{U}$*) is complete, hence it is a completion of (X,$\mathcal{U}$). Now by (e) there exists a quasi-uniformly continuous extension $\tilde{f}$: X* → Y, but this implies that f has a continuous extension to [0,1] which is impossible. Therefore (d) and (e) can not both hold.//

THEOREM 13. If (d) holds then (b) holds.

PROOF. The trivial completion is compact and by (d) it is complete.//

Considering our usual definition of Cauchy filter we obtain the following summary.

Complete (or Strongly Complete)

(a) holds

(b) holds

(c) does not hold

(d) holds

(e) does not hold (See theorem 12.)

(f) does not hold

(g) holds

To see that (f) does not hold consider the space in example 2. Since every filter is Cauchy in this space it is clear that there are infinitely many nonconvergent Cauchy filters. Let Λ denote this collection. Now the construction of the strong completion $(\overset{\circ}{X},\overset{\circ}{\mathcal{U}})$ carries over with Λ redefined as above. If $(\overset{\circ}{X},\overset{\circ}{\mathcal{U}})$ is compact then it must be pre-compact and hence Λ would be finite, which it is not.

IV. SOME THEOREMS AND EXAMPLES REGARDING QUASI-UNIFORM SPACES

A. NEIGHBORHOODS OF Δ

One of the points of interest is the following. Given a quasi-uniform structure $\mathcal{U}$ for a set X, when is $\mathcal{U}$ compatible with a uniform structure? By definition $\mathcal{U}$ is compatible with a uniform structure if and only if the topology generated is uniformizable. Other sufficient conditions will be obtained.

DEFINITION 1. A quasi-uniform space $(X,\mathcal{U})$ is said to have property P if each $U \in \mathcal{U}$ is a neighborhood of Δ in $X \times X$ with respect to the product topology.

It is well known that a compact uniform space has property P. The following example shows that this need not be the case in a quasi-uniform space. It also demonstrates that a quasi-uniform structure may be compatible with a uniform structure and not have property P.

EXAMPLE 1. Let $X = [0,2]$ with the usual topology and let $\mathcal{U}$ denote the Pervin quasi-uniform structure. Now $O = (\frac{1}{2},\frac{3}{2}) \in t$ so $U = O \times O \cup (X-O) \times X \in \mathcal{U}$. Suppose that U is a neighborhood of Δ, then there exists $\varepsilon > 0$ such that $(\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon) \times (\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon) \subset U$. Now $p = (\frac{1}{2}+\frac{\varepsilon}{2},\frac{1}{2}-\frac{\varepsilon}{2}) \in (\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon) \times (\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon)$, but $p \notin U$. Hence U is not a neighborhood of Δ.

The following example shows that a quasi-uniform structure can have property P and not be a uniform structure.

EXAMPLE 2. Let N denote the positive integers and set $U_n = \{ (x,y) : x = y \text{ or } n \leq x \leq y\}$. Then $\{ U_n : n = 1,2,\ldots\}$ forms a quasi-uniform base. Let $\mathcal{U}$ denote the quasi-uniform structure generated by this base. It is clear that $\mathcal{U}$ is not a uniform structure and the topology generated is the discrete topology. Hence Δ is open in $X \times X$, and therefore each $U \in \mathcal{U}$ is a neighborhood of Δ.

In the above example we note that $\mathcal{U}$ is compatible with a uniform structure. The following theorem shows that if $\mathcal{U}$ satisfies property P then this is always the case.

DEFINITION 2. Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}^{-1} = \{ U^{-1} : U \in \mathcal{U}\}$ is a quasi-uniform structure on X and it is called the conjugate quasi-uniform structure of $\mathcal{U}$. $\mathcal{U} \vee \mathcal{U}^{-1}$ denotes the smallest quasi-uniform structure which contains both $\mathcal{U}$ and $\mathcal{U}^{-1}$.

THEOREM 1. Let $(X,\mathcal{U})$ be a quasi-uniform space satisfying property P. Then $\mathcal{U}$ is compatible with a uniform structure.

PROOF. Let t and s denote the topologies generated by the quasi-uniform structures $\mathcal{U}$ and $\mathcal{U} \vee \mathcal{U}^{-1}$, respectively. It is clear that $\mathcal{U} \vee \mathcal{U}^{-1}$ is a uniform structure and that $t \leq s$. Thus it suffices to show that $s \leq t$. It is clear that sets of the form $U \cap U^{-1}$, where $U \in \mathcal{U}$, form a uniform base for $\mathcal{U} \vee \mathcal{U}^{-1}$. Let $O \in s$ and $x \in O$. Then there exists $U \cap U^{-1}$ such that $x \in (U \cap U^{-1})[x] \subset O$. Since U is a neighborhood of Δ with respect to $t \times t$, there exists $Q \in t$ such that $x \in Q$ and $Q \times Q \subset U$. Hence $Q \times Q \subset U^{-1}$. Therefore,

$$\begin{aligned} x \in Q \subset U[x] &\cap U^{-1}[x] \\ &= (U \cap U^{-1})[x] \\ &\subset O \end{aligned}$$

Hence $O \in t$ and thus $t = s$.//

COROLLARY. Let $(X,\mathcal{U})$ be a complete quasi-uniform space with property P. Then there exists a compatible complete uniform structure.

PROOF. In the above proof we showed that $\mathcal{U} \vee \mathcal{U}^{-1}$ is a compatible uniform structure. Since $\mathcal{U}$ is complete and $\mathcal{U} \leq \mathcal{U} \vee \mathcal{U}^{-1}$ it follows that $\mathcal{U} \vee \mathcal{U}^{-1}$ is complete.//

Every closed subspace of a complete quasi-uniform space is complete. A complete subspace of a Hausdorff uniform space is closed. The following theorem gives an analogous result for a complete Hausdorff quasi-uniform space that satisfies property P.

THEOREM 2. Let $(X,\mathcal{U})$ be an arbitrary Hausdorff quasi-uniform space with property P. If $Y \subset X$, and $(Y,\mathcal{U}_Y)$ is strongly complete then Y is closed in X.

PROOF. Let $a \in \bar{Y}$. Now $U[a] \cap Y \neq \emptyset$, and $\mathcal{B} = \{ U[a] \cap Y : U \in \mathcal{U}\}$ forms a filter base on Y. Let $\mathcal{F}$ denote the filter on Y generated by $\mathcal{B}$. If $U \in \mathcal{U}$, then $V = U \cap Y \times Y \in \mathcal{U}_Y$. Since U is a neighborhood of Δ, there exists $O \in t$ such that $a \in O$ and $O \times O \subset U$. Now $O \cap Y \neq \emptyset$ and $O \cap Y \times O \cap Y \subset V$.

Let $b \in O \cap Y$. Then $V[b] \supset O \cap Y$ and hence $\mathcal{F}$ is Cauchy on Y. Since Y is strongly complete there exists $c \in Y$ such that $c \in \lim \mathcal{F}$. Let $\mathcal{F}'$ be the filter on X generated by $\mathcal{F}$. Then $c \in \lim \mathcal{F}'$ and $a \in \lim \mathcal{F}'$ and since X is Hausdorff we have that $a = c \in Y$. Hence Y is closed in X.//

DEFINITION 3. Let $(X,\mathcal{U})$ be a quasi-uniform space. We will say that $(X,\mathcal{U})$ has property S if for each x, the collection $\{ V[x] : V \in \mathcal{U}, V \text{ is symmetric}\}$ forms a fundamental system of neighborhoods for x with respect to the topology generated by $\mathcal{U}$.

The following question arises naturally. If $\mathcal{U}$ has property P, then does $\mathcal{U}^{-1}$ have property P? Since this type of question will be of interest later, we make the following definition.

DEFINITION 4. A property Q will be called a quasi-conjugate invariant if a quasi-uniform structure $\mathcal{U}$ has property Q implies that $\mathcal{U}^{-1}$ also has property Q.

The space considered in example 2, Chapter III, clearly has property P since its topology is discrete. Now $U_5^{-1} \in \mathcal{U}^{-1}$. Suppose U_5^{-1} is a neighborhood of Δ with respect to the product conjugate topology. Then there exists $O \in t_{\mathcal{U}^{-1}}$ with $2 \in O$ and $O \times O \subset U_5^{-1}$. Thus there exists positive integer $k > 2$ such that $\{2,k,k+1,\ldots\} \subset O$. Hence $(2,k) \in U_5$ which is impossible. Therefore $\mathcal{U}^{-1}$ does not have property P, that is property P is not a quasi-conjugate invariant.

THEOREM 3. Let $(X,\mathcal{U})$ be a quasi-uniform space satisfying properties P and S. Then $\mathcal{U}^{-1}$ satisfies property P.

PROOF. Let $U^{-1} \in \mathcal{U}^{-1}$. Since $\mathcal{U}$ has property P, we have that for each $x \in X$ there exist $V(x) \in \mathcal{U}$ such that $V(x)[x] \times V(x)[x] \subset U$. Hence $V(x)[x] \times V(x)[x] \subset U^{-1}$ for each $x \in X$. By property S there exists a symmetric $T(x) \in \mathcal{U}$ such that $T(x)[x] \subset V(x)[x]$ for each $x \in X$. Since $T(x)$ is symmetric it follows that $T(x) \in \mathcal{U}^{-1}$ for each $x \in X$. Thus $\bigcup\{ T(x)[x] \times T(x)[x] : x \in X\} \subset \bigcup\{ V(x)[x] \times V(x)[x] : x \in X\} \subset U^{-1}$. Hence U^{-1} is a neighborhood of Δ with respect to the product conjugate topology and therefore $\mathcal{U}^{-1}$ has property P.//

COROLLARY. Let $(X,\mathcal{U})$ be a quasi-uniform space. If $\mathcal{U}$ satisfies properties P and S then $\mathcal{U}^{-1}$ is compatible with a uniform structure.

PROOF. The result is an immediate consequence of theorems 1 and 3.//

Theorem 1.47 in [16] shows that if $t_{\mathcal{U}}$ is weaker than the conjugate topology then the conjugate topology is uniformizable . Since $\mathcal{U} = (\mathcal{U}^{-1})^{-1}$ it follows that if $t_{\mathcal{U}}$ is stronger than the conjugate topology, then $t_{\mathcal{U}}$ is uniformizable.

THEOREM 4. Let $(X,\mathcal{U})$ be a quasi-uniform space. If $\mathcal{U}$ $(\mathcal{U}^{-1})$ has property S then $t_{\mathcal{U}^{-1}}$ $(t_{\mathcal{U}})$ is uniformizable.

PROOF. Suppose $\mathcal{U}$ has property S. Then $t_{\mathcal{U}}$ is weaker than the conjugate topology, for let $O \in t_{\mathcal{U}}$ and $x \in O$. Since $\mathcal{U}$ satisfies property S we have that there exists a symmetric $V \varepsilon \mathcal{U}$ such that $x \varepsilon V[x] \subset O$. Hence $V \varepsilon \mathcal{U}^{-1}$ and $O \varepsilon t_{\mathcal{U}^{-1}}$. Then by theorem 1.47 in [16] we have that $t_{\mathcal{U}^{-1}}$ is uniformizable. Now suppose that $\mathcal{U}^{-1}$ has property S and let $O \varepsilon t_{\mathcal{U}^{-1}}$ with $x \varepsilon O$. Then there exists a symmetric $V \varepsilon \mathcal{U}^{-1}$ with $x \varepsilon V[x] \subset O$. Since V is symmetric we have that $V \varepsilon \mathcal{U}$ and thus $O \varepsilon t_{\mathcal{U}}$. Therefore $t_{\mathcal{U}^{-1}} \leq t_{\mathcal{U}}$ and consequently we have that $t_{\mathcal{U}}$ is uniformizable.//

COROLLARY 1. If (X,t) is T_3 then it has a uniformizable conjugate topology.

PROOF. By theorem 3.17 in [16] $\mathcal{U}$, the Pervin quasi-uniform structure, has property S. $t_{\mathcal{U}^{-1}}$ is uniformizable by theorem 4.//

COROLLARY 2. Let $(X,\mathcal{U})$ be a quasi-uniform space such that $\mathcal{U}$ and $\mathcal{U}^{-1}$ have property S. Then $t_{\mathcal{U}} = t_{\mathcal{U}^{-1}}$ and $t_{\mathcal{U}}$ is uniformizable.

PROOF. Since $\mathcal{U}$ has property S we have $t_{\mathcal{U}} \leq t_{\mathcal{U}^{-1}}$ and since $\mathcal{U}^{-1}$ has property S we obtain $t_{\mathcal{U}^{-1}} \leq t_{\mathcal{U}}$. Hence $t_{\mathcal{U}} = t_{\mathcal{U}^{-1}}$ and thus $t_{\mathcal{U}} = t_{\mathcal{U}} \vee t_{\mathcal{U}^{-1}} = t_{\mathcal{U} \vee \mathcal{U}^{-1}}$. Therefore $t_{\mathcal{U}}$ is uniformizable.//

B. $\mathcal{U} \wedge \mathcal{U}^{-1}$

DEFINITION 5. Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniform structures on X. Then $\mathcal{U} \wedge \mathcal{V} = \{ U : U \varepsilon \mathcal{U} \cap \mathcal{V}\}$.

THEOREM 5. Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniform structures on X. If $\mathcal{U} \wedge \mathcal{V}$ is a quasi-uniform structure on X then $\mathcal{U} \wedge \mathcal{V}$ is the greatest lower bound of the quasi-uniform structures $\mathcal{U}$ and $\mathcal{V}$.

PROOF. It is clear that $\mathcal{U} \wedge \mathcal{V} \leq \mathcal{U}$ and $\mathcal{U} \wedge \mathcal{V} \leq \mathcal{V}$. Now suppose that $\mathcal{S}$ is a quasi-uniform structure such that $\mathcal{S} \leq \mathcal{U}$ and $\mathcal{S} \leq \mathcal{V}$. If $S \in \mathcal{S}$, then $S \in \mathcal{U} \cap \mathcal{V}$. Hence $\mathcal{S} \leq \mathcal{U} \wedge \mathcal{V}$.// It is well known that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are quasi-uniform structures for X then $\mathcal{U}_1 \wedge \mathcal{U}_2$ need not be a quasi-uniform structure. The following example shows that even $\mathcal{U} \wedge \mathcal{U}^{-1}$ need not be a quasi-uniform structure.

EXAMPLE 3. Let X denote the natural numbers and set $U_n = \{ (x,y) : x = y \text{ or } x = 1 \text{ and } y \geq n\}$. Let $\mathcal{U}$ be the quasi-uniform structure generated by $\{ U_n : n = 1,2,\ldots\}$. Now

$$U = \{ (x,y) : x = y \text{ or } x = 1 \text{ and } y \geq 4$$
$$\text{or } y = 1 \text{ and } x \geq 4\} \in \mathcal{U} \wedge \mathcal{U}^{-1}.$$

Suppose there exists $V \in \mathcal{U} \wedge \mathcal{U}^{-1}$ such that $V \circ V \subset U$. Since $V \in \mathcal{U}$ there exists $n \in X$ such that $(1,k) \in V$ for each $k \geq n$. Similarly, since $V \in \mathcal{U}^{-1}$ there exists $m \in X$ such that $(k,1) \in V$ for each $k \geq m$. Let $t = \max\{m,n\}$. Then $t \geq 4$ since $V \subset U$. Now $(t,1) \in V$ and $(1,t+1) \in V$ and therefore $(t,t+1) \in U$ which is impossible. Hence $\mathcal{U} \wedge \mathcal{U}^{-1}$ does not form a quasi-uniform structure.

LEMMA 1. If $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a quasi-uniform structure then $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a uniform structure.

PROOF. Let $U \in \mathcal{U} \wedge \mathcal{U}^{-1}$. Then $U \in \mathcal{U}$ and hence $U^{-1} \in \mathcal{U}^{-1}$. Also, $U \in \mathcal{U}^{-1}$ and thus $U^{-1} \in \mathcal{U}$. Therefore $U^{-1} \in \mathcal{U} \wedge \mathcal{U}^{-1}$.//

DEFINITION 6. A quasi-uniform space $(X,\mathcal{U})$ is said to have property * if for each symmetric $U \in \mathcal{U}$ there exists a symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$.

It is apparent that each uniform structure possesses property *. Not every quasi-uniform structure has property * as the space in example 1 demonstrates. It is clear that property * is a quasi-conjugate invariant property. It is an easy exercise to show that property * does not imply and is not implied by any of the usual separation properties. However, as the next theorem shows it does characterize those structures $\mathcal{U}$ for which $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a quasi-uniform structure.

THEOREM 6. Let $(X,\mathcal{U})$ be a quasi-uniform structure. $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a quasi-uniform structure if and only if $(X,\mathcal{U})$ satisfies property *.

PROOF. Suppose $(X,\mathcal{U})$ satisfies property *. Clearly $\mathcal{U} \wedge \mathcal{U}^{-1} \neq \emptyset$ and if $U \in \mathcal{U} \wedge \mathcal{U}^{-1}$ then $U \supset \Delta$. Now $U \in \mathcal{U} \wedge \mathcal{U}^{-1}$ and $V \supset U$ implies $V \in \mathcal{U}$ and $V \in \mathcal{U}^{-1}$. Hence $V \in \mathcal{U} \wedge \mathcal{U}^{-1}$. Now $U, V \in \mathcal{U} \wedge \mathcal{U}^{-1}$ implies $U \cap V \in \mathcal{U}$ and $U \cap V \in \mathcal{U}^{-1}$. Thus $U \cap V \in \mathcal{U} \wedge \mathcal{U}^{-1}$. Let $U \in \mathcal{U} \wedge \mathcal{U}^{-1}$. Then $U^{-1} \in \mathcal{U}$, and $T = U \cap U^{-1} \in \mathcal{U}$ and T is symmetric. By hypothesis there exists a symmetric $V \in \mathcal{U}$ such that $V \circ V \subset T$. Since V is symmetric, we have $V \in \mathcal{U}^{-1}$ and thus $V \in \mathcal{U} \wedge \mathcal{U}^{-1}$. Hence $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a quasi-uniform structure.

Now suppose that $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a quasi-uniform structure. Let $U \in \mathcal{U}$ and U be symmetric. Then $U \in \mathcal{U} \wedge \mathcal{U}^{-1}$ and there exists $V \in \mathcal{U} \wedge \mathcal{U}^{-1}$ such that $V \circ V \subset U$. Now $V \in \mathcal{U}$ and $V \in \mathcal{U}^{-1}$. Hence $V^{-1} \in \mathcal{U}$, $V^{-1} \in \mathcal{U}^{-1}$, and therefore $V^{-1} \in \mathcal{U} \wedge \mathcal{U}^{-1}$. Let $S = V \cap V^{-1}$. Then $S \in \mathcal{U}$ and S is symmetric and more over $S \circ S = (V \cap V^{-1}) \circ (V \cap V^{-1}) \subset V \circ V \subset U$. Hence $(X,\mathcal{U})$ satisfies property *.//

THEOREM 7. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be quasi-uniform structures on X. Suppose $\mathcal{U}_1 \wedge \mathcal{U}_2$ is a quasi-uniform structure, and denote the topology it generates by t. Then $t = t_1 \wedge t_2$, where t_1 and t_2 are the topologies generated by $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively.

PROOF. Let $x \in O \in t_1 \wedge t_2$. Then there exists $U \in \mathcal{U}_1$ and $V \in \mathcal{U}_2$ such that $x \in U[x] \subset O$ and $x \in V[x] \subset O$. Then $U \cup V \in \mathcal{U}_1 \wedge \mathcal{U}_2$ and $x \in (U \cup V)[x] \subset O$. Therefore $O \in t$. Now suppose $x \in O \in t$. Then there exists $U \in \mathcal{U}_1 \wedge \mathcal{U}_2$ such that $x \in U[x] \subset O$. Since $U \in \mathcal{U}_1$ and $U \in \mathcal{U}_2$, we have that $O \in t_1$ and $O \in t_2$ and hence $O \in t_1 \wedge t_2$. Therefore, $t = t_1 \wedge t_2$.//

COROLLARY. Let $\mathcal{G}$ denote the family of all quasi-uniform structures which generate the topology t on the set X. Then if $\mathcal{U}_1$ and $\mathcal{U}_2$ are in $\mathcal{G}$ and $\mathcal{U}_1 \wedge \mathcal{U}_2$ is a quasi-uniform structure, then $\mathcal{U}_1 \wedge \mathcal{U}_2 \in \mathcal{G}$.

THEOREM 8. Let $(X,\mathcal{U})$ be a quasi-uniform space satisfying property *.

(i) If $\mathcal{U} \wedge \mathcal{U}^{-1}$ generates $t_{\mathcal{U}}$ then $t_{\mathcal{U}^{-1}}$ is uniformizable.

(ii) If $\mathcal{U} \wedge \mathcal{U}^{-1}$ generates $t_{\mathcal{U}^{-1}}$ then $t_{\mathcal{U}}$ is uniformizable.

PROOF. (i) By hypothesis $t_{\mathcal{U}} \leq t_{\mathcal{U}^{-1}}$ and from theorem 1.47 in [16] we have that $t_{\mathcal{U}^{-1}}$ is uniformizable. (ii). By hypothesis $t_{\mathcal{U}^{-1}} \leq t_{\mathcal{U}}$ and by the remark following theorem 3 we have that $t_{\mathcal{U}}$ is uniformizable.//

THEOREM 9. Let $(X,\mathcal{U})$ be a quasi-uniform structure. $\mathcal{U} \vee \mathcal{U}^{-1}$ generates $t_{\mathcal{U}}$ if and only if there exists a compatible uniformity stronger than $\mathcal{U}$.

PROOF. The necessity is obvious. Suppose there exists a compatible uniform structure $\mathcal{V}$ such that $\mathcal{U} \leq \mathcal{V}$. Let $U^{-1} \in \mathcal{U}^{-1}$, then $U \in \mathcal{U}$ and thus $U^{-1} \in \mathcal{V}$. Hence $\mathcal{U} \leq \mathcal{U} \vee \mathcal{U}^{-1} \leq \mathcal{V}$. Now let t denote the topology generated by $\mathcal{U} \vee \mathcal{U}^{-1}$. Then $t_{\mathcal{U}} \leq t \leq t_{\mathcal{U}}$ since $\mathcal{V}$ generates $t_{\mathcal{U}}$. Hence $\mathcal{U} \vee \mathcal{U}^{-1}$ generate $t_{\mathcal{U}}$.//

THEOREM 10. Let $(X,\mathcal{U})$ be a quasi-uniform space satisfying property *. Then there exists a weaker compatible uniform structure if and only if $\mathcal{U} \wedge \mathcal{U}^{-1}$ generates $t_{\mathcal{U}}$.

PROOF. The sufficiency is clear. Suppose $\mathcal{V}$ is a compatible uniform structure such that $\mathcal{V} \leq \mathcal{U}$. If $V \in \mathcal{V}$, we have that $V^{-1} \in \mathcal{V}$ and hence $V \in \mathcal{U}$ and $V^{-1} \in \mathcal{U}$. That is, $V \in \mathcal{U}$ and $V \in \mathcal{U}^{-1}$. Thus $V \in \mathcal{U} \wedge \mathcal{U}^{-1}$ and we have that $\mathcal{V} \leq \mathcal{U} \wedge \mathcal{U}^{-1}$. Denote the topology generated by $\mathcal{U} \wedge \mathcal{U}^{-1}$ by t. Then $t_{\mathcal{U}} \leq t \leq t_{\mathcal{U}}$ since $\mathcal{V} \leq \mathcal{U} \wedge \mathcal{U}^{-1} \leq \mathcal{U}$ and the hypothesis that $\mathcal{V}$ generates $t_{\mathcal{U}}$. Thus $\mathcal{U} \wedge \mathcal{U}^{-1}$ generates $t_{\mathcal{U}}$.//

C. SATURATED QUASI-UNIFORM SPACES

DEFINITION 7. A topological space (X,t) is called saturated if for each $x \in X$ there exists a minimal open set containing x. That is, for each $x \in X$ there exists an open set O_x containing x such that if $O \in t$ and $x \in O$, then $O_x \subset O$.

It is clear that a topological space (X,t) is saturated if and only if every intersection of open sets is open.

DEFINITION 8. A quasi-uniform space $(X,\mathcal{U})$ is called quasi-saturated if $\bigcap\{ U : U \in \mathcal{U}\} \in \mathcal{U}$.

LEMMA 2. A quasi-uniform space $(X,\mathcal{U})$ is quasi-saturated if and only if there exists a unique base for $\mathcal{U}$ consisting of a single set.

PROOF. The sufficiency is clear. Let $V = \bigcap\{ U : U \in \mathcal{U}\}$. By hypothesis $V \in \mathcal{U}$. Set $\mathcal{B} = \{V\}$. It suffices to show that $V \circ V \subset V$. Since $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that $U \circ U \subset V$. But $V \subset U$ and thus $V \circ V \subset U \circ U \subset V$. Hence $\mathcal{B}$ is a base for $\mathcal{U}$. The uniqueness is obvious.//

LEMMA 3. If $(X,\mathcal{U})$ is quasi-saturated then $t_{\mathcal{U}}$ is a saturated topology.

PROOF. Let $\mathcal{B} = \{W\}$ be the base for $\mathcal{U}$ developed in lemma 2. Let $x \in O$, where $O \in t_{\mathcal{U}}$. Then $x \in W[x] \subset O$, and $W[x]$ is the minimal open set containing the point x.//

THEOREM 11. If $(X,\mathcal{U})$ is quasi-saturated then $(X,\mathcal{U})$ is strongly complete.

PROOF. Let $\mathcal{F}$ be a Cauchy filter. Let $\mathcal{B} = \{W\}$ be the base for $\mathcal{U}$ consisting of a single set. Then, since $\mathcal{F}$ is

Cauchy, there exists $x \in X$ such that $W[x] \in F$. Hence for each $U \in \mathcal{U}$ we have $U[x] \supset W[x]$, and therefore $U[x] \in F$ and $x \in \lim F$.//

THEOREM 12. Let (X,t) be a saturated topological space. Then there exists a strongly complete compatible quasi-saturated quasi-uniform structure.

PROOF. Let O_x denote the minimal open set containing x. Set $W = \{ (x,y) : y \in O_x, x \in X\}$. Then $\{W\}$ forms a base for a quasi-uniform structure which we will denote by $\mathcal{U}$. It suffices to show that $W \circ W \subset W$. Let $(a,b) \in W$ and $(b,c) \in W$. Then $b \in O_a$ and $c \in O_b$. Since O_b is the minimal open set containing b we have that $c \in O_b \subset O_a$. Hence $(a,c) \in W$. If $O \in t$ and $x \in O$ then $x \in O_x \subset O$, but $O_x = W[x]$. Thus $t \leq t_{\mathcal{U}}$. If $O \in t_{\mathcal{U}}$ and $x \in O$ then $x \in O_x = W[x] \subset O$ and hence $t_{\mathcal{U}} \leq t$. By theorem 11 $(X,\mathcal{U})$ is strongly complete .//

Fletcher has shown in [7] that if a topological space has property Q then it admits a compatible strongly complete quasi-uniform structure. Fletcher's theorem is a clear generalization of theorem 12.

The following example shows that metacompact and paracompact are not necessary conditions for a topological space to admit a strongly complete structure.

EXAMPLE 4. Let N denote the natural numbers and $t = \{N,\emptyset,\{1\},\{1,2\},\{1,2,3\},\ldots\}$. (N,t) is a saturated topological space. Set $O_i = \{1,2,\ldots,i\}$, then $\{ O_i : i = 1,2,\ldots\}$

is an open cover for (N,t). Let $\{ Q_\alpha : \alpha \in \Omega\}$ be an open refinement of $\{ O_i : i = 1,2 \ldots\}$. Suppose that 1 is contained in only a finite number of the Q_α, say $Q_{\alpha_1}, \ldots, Q_{\alpha_n}$. Clearly each $Q_\alpha \nsupseteq N$ and further the set $\bigcup_1^n Q_{\alpha_i}$ has a maximum element, say r. Now r is contained in some $Q_\beta \nsupseteq Q_{\alpha_i}$ for $i = 1, \ldots, n$. Then $\{1,2,\ldots,r\} \subset Q_\beta$, that is $1 \in Q_\beta$ which is a contradiction. Hence (N,t) is a saturated space that is not metacompact.

THEOREM 13. Quasi-saturated is a quasi-conjugate invariant.

PROOF. Suppose $(X,\mathcal{U})$ is a quasi-saturated quasi-uniform structure. Then there exists $W \in \mathcal{U}$ such that if $U \in \mathcal{U}$ then $W \subset U$. It is clear that $\{W^{-1}\}$ forms a base for $\mathcal{U}^{-1}$ and hence by lemma 2, $\mathcal{U}^{-1}$ is quasi-saturated.//

THEOREM 14. Let $(X,\mathcal{U})$ be a quasi-saturated quasi-uniform space with a base $\{W\}$. Let O_x denote the minimal open set in $t_\mathcal{U}$ containing x. Then W is a neighborhood of Δ; that is, $\mathcal{U}$ has property P if and only if $W = \bigcup\{ O_x \times O_x : x \in X\}$.

PROOF. The sufficiency is evident. If W is a neighborhood of Δ then $W \supset \bigcup\{ O_x \times O_x : x \in X\}$. If $(a,b) \in W$, then $b \in W[a] \subset O_a$. Hence $(a,b) \in O_a \times O_a \subset \bigcup\{ O_x \times O_x : x \in X\}$.//

THEOREM 15. Let $\mathcal{U}$ and $\mathcal{V}$ be compatible quasi-uniform structures on X. If $\mathcal{V}$ is quasi-saturated then $\mathcal{U} \leq \mathcal{V}$.

PROOF. Let $U \in \mathcal{U}$ and $\{V\}$ the base for $\mathcal{V}$. If $(x,y) \in$

V, then $y \in V[x] \subset U[x]$. This follows since $V[x]$ is the smallest open set containing x. Hence $(x,y) \in U$ and we have $V \subset U$ and thus $U \in \mathcal{V}$.//

COROLLARY. If (X,t) is saturated then there exists one and only one compatible quasi-saturated quasi-uniform structure.

PROOF. The result is an immediate consequence of theorems 11 and 15.//

Fletcher [6] working independently has obtained similar results for finite topological spaces.

D. O-COMPLETE

DEFINITION 9. A filter $\mathcal{F}$ will be called an open filter if it has a base consisting of open sets.

DEFINITION 10. Let $(X,\mathcal{U})$ be a quasi-uniform space. $(X,\mathcal{U})$ will be called o-complete if every open Cauchy filter has a nonempty adherence. $(X,\mathcal{U})$ will be called strongly o-complete if every open Cauchy filter has a nonempty limit.

DEFINITION 11. A topological space (X,t) is called generalized absolutely closed if every open cover $\{O_\alpha\}$ has a finite subcollection $O_{\alpha_1},\dots,O_{\alpha_n}$ such that $X = \bigcup_1^n \bar{O}_{\alpha_i}$.

DEFINITION 12. A filter $\mathcal{F}$ in a quasi-uniform space $(X,\mathcal{U})$ is said to contain arbitrarily small open sets if for each $U \in \mathcal{U}$ there exists $x \in X$ and $O_x \in t_{\mathcal{U}}$ such that $x \in O_x \subset U[x]$ and $O_x \in \mathcal{F}$.

It is clear that if $(X,\mathcal{U})$ is complete, (strongly complete), then it is o-complete, (strongly o-complete).

LEMMA 4. Let $(X,\mathcal{U})$ be a quasi-uniform space such that every Cauchy filter contains arbitrarily small open sets. Then $(X,\mathcal{U})$ is strongly o-complete implies that $(X,\mathcal{U})$ is strongly complete.

PROOF. Let $\mathcal{F}$ be a Cauchy filter and let $\mathcal{B} = \{ O : O \in \mathcal{F}$ and O is open$\}$. Let $\mathcal{G}$ be the filter generated by $\mathcal{B}$. Since $\mathcal{F}$ contains arbitrarily small open sets we have that $\mathcal{G}$ is an open Cauchy filter. The result follows from the fact that $\lim \mathcal{G} = \lim \mathcal{F}$.//

It is an easy observation that if (X,t) is a generalized absolutely closed topological space and $\mathcal{U}$ is any compatible quasi-uniform structure then $(X,\mathcal{U})$ is o-complete. This is true since in a generalized absolutely closed space every open filter has a nonempty adherence.

LEMMA 5. If $(X,\mathcal{U})$ is pre-compact then every open ultrafilter is Cauchy.

PROOF. Let $\mathcal{M}$ be an open ultrafilter on X; that is, $\mathcal{M}$ is a maximal element in the class of all open filters on X. Let $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$. Since $\mathcal{U}$ is pre-compact there exists $x_1, \ldots, x_n \in X$ such that $X = \bigcup_1^n V[x_i]$. Let $O_i = \operatorname{int} U[x_i]$. Then $V[x_i] \subset O_i$ for $i = 1,2,\ldots,n$. Hence $\bigcup_1^n O_i = X$ and so there exists $O_k \in \mathcal{U}$ since $\mathcal{M}$ is an open ultrafilter. Now $U[x_k] \supset O_k$ and therefore $U[x_k] \in \mathcal{M}$. Consequently $\mathcal{M}$ is Cauchy.//

LEMMA 6. Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(X,\mathcal{U})$ is pre-compact and o-complete then X is generalized absolutely closed.

PROOF. It suffices to show that every open ultrafilter has a nonempty adherence. Let $\mathcal{M}$ be an open ultrafilter, then since $\mathcal{U}$ is pre-compact $\mathcal{M}$ is Cauchy by lemma 5. Since $(X,\mathcal{U})$ is o-complete we have that adh $\mathcal{M} \neq \emptyset$.//

THEOREM 16. Let $(X,\mathcal{U})$ be a uniform space. $(X,\mathcal{U})$ is generalized absolutely closed if and only if $(X,\mathcal{U})$ is pre-compact and o-complete.

PROOF. Lemma 3 is a generalization of the sufficiency. Suppose $(X,\mathcal{U})$ is generalized absolutely closed, then $(X,\mathcal{U})$ is o-complete by a previous comment. We now show that $(X,\mathcal{U})$ is pre-compact. Let $U \in \mathcal{U}$, then there exists a symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$. Now $\{ V[x] : x \in X\}$ is a neighborhood cover of X. Since X is generalized absolutely closed we have that there exists $x_1,\ldots,x_n \in X$ such that $X = \bigcup_1^n \overline{V[x_i]}$. If $y \in \overline{V[x_i]}$, then $V[y] \cap V[x_i] \neq \emptyset$. Let $z \in V[y] \cap V[x_i]$. Then $(y,z) \in V$, $(x_i,z) \in V$ and since V is symmetric we have that $(x_i,z) \in V$ and $(z,y) \in V$. Thus $(x_i,y) \in V \circ V \subset U$ or $y \in U[x_i]$. Hence $X = \bigcup_1^n U[x_i]$ and therefore $(X,\mathcal{U})$ is pre-compact.//

Fletcher and Naimpally [10] working independently obtained analogous results to those found in this section. They call o-complete, almost complete and generalized absolutely closed, almost compact. They defined a quasi-uniform space $(X,\mathcal{U})$ to be almost pre-compact if for each $U \in \mathcal{U}$ there exists $x_1,\ldots,x_n \in X$ such that $X = \bigcup_1^n \overline{U[x_i]}$. Using these definitions they obtained the following generalization to lemma 6. A topological space is almost-compact if and only if every compatible quasi-uniform structure is almost complete and almost pre-compact.

E. q-T_i SEPARATION AXIOMS

Let $(X,\mathcal{U})$ be a quasi-uniform space. X is t_0 if and only if given $x \neq y$ there exists $U \in \mathcal{U}$ such that either $x \notin U[y]$ or $y \notin U[x]$. X is T_1 if and only if given $x \neq y$ there exists $U \in \mathcal{U}$ such that $x \notin U[y]$ and $y \notin U[x]$. Similarly, X is T_2 if and only if given $x \neq y$ there exists $U \in \mathcal{U}$ such that $U[x] \cap U[y] = \emptyset$. The following example shows that this characterization does not carry over for an arbitrary T_3 quasi-uniform space.

EXAMPLE 5. Let X denote the natural numbers and let $\mathcal{U}$ be the quasi-uniform structure generated by the base consisting of all sets of the form $U_n = \{(x,y) : x = y \text{ or } x \geq n\}$, for $n \in X$. Now $t_{\mathcal{U}}$ is the discrete topology, so $F = \{2,4,6,\ldots\}$ is closed in X. If $U \in \mathcal{U}$, there exists $U_n \subset U$. $U_n[2n] = X$, so $U[F] = X$. Thus $(X,\mathcal{U})$ is T_3, but there does not exist $U \in \mathcal{U}$ such that $U[1] \cap U[F] = \emptyset$.

DEFINITION 13. A quasi-uniform space $(X,\mathcal{U})$ is called q-T_3 if given $x \notin F$, F closed, there exists $U \in \mathcal{U}$ such that $U[x] \cap U[F] = \emptyset$.

It is clear that if a space is q-T_3 then it is T_3. Example 1 showed that a space can be T_3 but not q-T_3. We will show that every uniform space is q-T_3.

THEOREM 17. Let $(X,\mathcal{U})$ be a R_3 quasi-uniform space. Then $(X,\mathcal{U})$ is q-T_3.

PROOF. Let F be closed in X and $x \notin F$. Since $X - F$ is open there exists $U \in \mathcal{U}$ such that $U[x] \cap F = \emptyset$. Since $(X,\mathcal{U})$ is R_3 there exists a symmetric $V \in \mathcal{U}$ such that $(V \circ V)[x] \subset U[x]$. Suppose $a \in V[x] \cap V[F]$. Then there exists $f \in F$ such that $(f,a) \in V$ and $(x,a) \in V$. Since V is symmetric we have $(x,a) \in V$ and $(a,f) \in V$. Thus $f \in (V \circ V)[x] \subset U[x]$. But this is impossible since $U[x] \cap F = \emptyset$. Hence $V[x] \cap V[F] = \emptyset$.//

COROLLARY. Every uniform space is q-T_3.

DEFINITION 14. A quasi-uniform space is called q-T_4 if for every pair of disjoint closed sets F and G there exists $U \in \mathcal{U}$ such that $U[F] \cap U[G] = \emptyset$.

It is clear that every q-T_4 space is T_4 and moreover every q-T_4 + T_0 space is a q-T_3 space; that is, quasi-normal implies quasi-regular.

LEMMA 7. Let $(X,\mathcal{U})$ be a q-T_4 quasi-uniform space and F closed in X. Then $(F,\mathcal{U}_F)$ is a q-T_4 space.

PROOF. Let G and H be closed disjoint subsets of F, then G and H are closed and disjoint in X. Since $(X,\mathcal{U})$ is q-T_4 there exists $U \in \mathcal{U}$ such that $U[G] \cap U[H] = \emptyset$. Let $V = U \cap F \times F$. Then $V \in \mathcal{U}_F$ and $V[G] \cap V[H] = \emptyset$. Hence

$(F,\mathcal{U}_F)$ is q-T_4.//

A space can be T_4 and not q-T_4; consider the space in example 5 with $F = \{2,4,\ldots\}$ and $G = \{1,3,\ldots\}$.

DEFINITION 15. A quasi-uniform space $(X,\mathcal{U})$ is called q-T_5 if for every pair of separated sets F and G there exists $U \in \mathcal{U}$ such that $U[F] \cap U[G] = \emptyset$.

Clearly every q-T_5 space is T_5. The space in example 1 is T_5 but not q-T_5. It is also evident that each space is a q-T_4 space.

LEMMA 8. If $(X,\mathcal{U})$ is q-T_5 then every subspace is q-T_5.

PROOF. Let $Y \subset X$ and F and G separated in $(Y,\mathcal{U}_Y)$. Now $(\mathrm{cl}_X F) \cap Y = \mathrm{cl}_Y F$. Since F and G are separated in Y we have that $(\mathrm{cl}_Y F) \cap G = \emptyset$. Now $(\mathrm{cl}_X F) \cap G \subset (\mathrm{cl}_X F) \cap G \cap Y = (\mathrm{cl}_Y F) \cap G = \emptyset$. Similarly $(\mathrm{cl}_X G) \cap F = \emptyset$. Thus F and G are separated in X and hence there exists $U \in \mathcal{U}$ such that $U[G] \cap U[F] = \emptyset$. Let $V = U \cap Y \times Y$. Then $V \in \mathcal{U}_Y$ and $V[G] \cap V[F] = \emptyset$. Hence $(Y,\mathcal{U}_Y)$ is q-T_5.//

THEOREM 18. Let (X,t) be a topological space and $\mathcal{U}$ the Pervin quasi-uniform structure for t. If (X,t) is T_i, then $(X,\mathcal{U})$ is q-T_i for $i = 3,4,5$.

PROOF. For $i = 3$ the result follows by theorem 17, since the Pervin structure of a T_3 space is R_3. Let $i = 4$. Let F, G be disjoint closed sets in X. Then there exists $O, Q \in t$ such that $F \subset O$, $G \subset Q$ and $O \cap Q = \emptyset$. Let $S = O \times O \cup (X-O) \times X$ and $T = Q \times Q \cup (X-Q) \times X$. Then set

$$U = S \cap T$$
$$= [O \times O \cup Q \times Q] \cup [X - (O \cup Q) \times X].$$

Now $U[F] = O$ and $U[G] = Q$, hence $U[F] \cap U[G] = \emptyset$. Hence $(X,\mathcal{U})$ is q-T_4. The proof for $i = 5$ follows in the same manner.//

LEMMA 9. Let $(X,\mathcal{U})$ be a saturated quasi-uniform space. If X is T_i then it is q-T_i for $i = 3,4,5$.

PROOF. If $\mathcal{U}$ is quasi-saturated then there exists a base consisting of a single set W. Suppose X is T_3 and $x \notin F$ where F is closed. Then there exists open sets O and Q such that $x \in O$, $F \subset Q$, and $O \cap Q = \emptyset$. Now $W[x] \subset O$ and $W[F] \subset Q$. Hence $(X,\mathcal{U})$ is q-T_3. The proofs for q-T_4 and q-T_5 follow in a similar manner.//

EXAMPLE 6. Let X denote the natural numbers. Let $\mathcal{U}$ be the uniform structure generated by the base consisting of the sets $U_n = \{ (x,y) : x = y \text{ or } x \geq n \text{ and } y \geq n\}$. The topology generated is discrete and hence T_5 and T_4. $(X,\mathcal{U})$ is clearly not q-T_4 and hence not q-T_5. Thus we see that a uniform space may be T_4 (T_5) and not q-T_4 (q-T_5).

THEOREM 19. Let $(X,\mathcal{U})$ be a compact q-T_3 quasi-uniform space. Then $(X,\mathcal{U})$ is q-T_4.

PROOF. Let F and G be disjoint closed sets. Suppose that for each $U \in \mathcal{U}$ we have $U[F] \cap U[G] \neq \emptyset$. $\mathcal{B} = \{ U[F] \cap U[G] : U \in \mathcal{U}\}$ is a filter base. Since X is compact there exists $x \in \operatorname{adh} \mathcal{B}$. We may suppose that $x \notin F$, since clearly $x \notin F \cap G$. By hypothesis X is q-T_3 so there exists

$U \in \mathcal{U}$ such that $U[x] \cap U[F] = \emptyset$. Then $U[x] \cap (U[F] \cap U[G]) = \emptyset$, but this is impossible since $x \in \operatorname{adh} \mathcal{B}$. Therefore $(X,\mathcal{U})$ is q-T_4.//

COROLLARY. Let $(X,\mathcal{U})$ be a compact uniform space. Then $(X,\mathcal{U})$ is q-T_4.

The following characterization of q-T_3 seems complicated but it proves a helpful tool in the following theorem.

LEMMA 10. $(X,\mathcal{U})$ is q-T_3 if and only if for each open set O and $x \in O$ there exists $U \in \mathcal{U}$ such that $U[x] \subset X - U[X - O]$.

PROOF. The proof follows immediately from the definition of q-T_3.//

THEOREM 20. $(\Pi X_i, \Pi \mathcal{U}_i)$ is q-T_3 if and only if $(X_i,\mathcal{U}_i)$ is q-T_3 for each i.

PROOF. Let $X = \Pi X_i$ and $\mathcal{U} = \Pi \mathcal{U}_i$. Sufficiency. Let $x \in O$ where O is open in X. Then there exists $\Pi_i O_i$ where O_i is open in t_i and O_i is X_i except for $i = i_1,\ldots,i_k$ and $x \in \Pi_i O_i \subset O$. For each $i = i_1,\ldots,i_k$ there exists $U_{i_k} \in \mathcal{U}_{i_k}$ such that $U_{i_k}[x_{i_k}] \subset X_{i_k} - U_{i_k}[X_{i_k} - O_{i_k}]$. Let $U = \Pi U_i$ where $U_i = X_i \times X_i$ for $i \neq i_1,\ldots,i_k$. Then $U \in \mathcal{U}$. Suppose that $y \in U[x] \cap U[X - O]$. Then there exists $a \in X - O$ such that $y \in U[a]$. If $a \notin O$ then there exists i_k such that $a_{i_k} \notin O_{i_k}$. Thus $y_{i_k} \in U_{i_k}[x_{i_k}]$ and $y_{i_k} \in U[a_{i_k}]$. Thus $y_{i_k} \in U_{i_k}[x_{i_k}] \subset X_{i_k} - U_{i_k}[X_{i_k} - O_{i_k}]$. But $y_{i_k} \in U[a_{i_k}] \subset U_{i_k}[X_{i_k} - O_{i_k}]$ which is impossible. Therefore $U[x] \cap U[X - O] = \emptyset$ and hence $U[x] \subset X - U[X - O]$ and by lemma 10 we have that $(X,\mathcal{U})$ is q-T_3.

Necessity. Let $x_i \in O_i$, where O_i is open in X_i. Let x be any point in X with the i^{th} coordinate equal to x_i. Then there exists $U \in \mathcal{U}$ such that $U[x] \subset X - U[X - \pi_i^{-1}(O_i)]$. There exists $\Pi_k U_k \subset U$ where $U_k = X_k \times X_k$ except for a finite number of indices and $U_k \in \mathcal{U}_k$ for each k.

Then

$$\begin{aligned}(\Pi_k U_k)[x] &\subset U[x] \\ &\subset X - U[X - \pi_i^{-1}(O_i)] \\ &\subset X - (\Pi_k U_k)[X - \pi_i^{-1}(O_i)]\end{aligned}$$

We will show that $U_i[x_i] \subset X_i - U_i[X_i - O_i]$. Suppose $y_i \in U_i[x_i] \cap U_i[X_i - O_i]$. Then there exists $a_i \in X_i - O_i$ such that $(a_i,y_i) \in U_i$. Choose $y_j = x_j$ for all $j \neq i$. Then $y = (y_k) \in (\Pi_k U_k)[x]$ and therefore $y \in X - (\Pi_k U_k)[X - \pi_i^{-1}(O_i)]$. Define $a = (a_j)$ by $a_j = y_j$ for each $j \neq i$. Then $a \in X - \pi_i^{-1}(O_i)$ since $a_i \in X_i - O_i$. Now $(a,y) \in \Pi_k U_k$ since for each $k \neq i$ $(a_k,y_k) = (y_k,y_k) \in U_k$ and for $k = i$ we have $(a_i,y_i) \in U_i$. Therefore $(a,y) \in \Pi_k U_k$ where $a \in X - \pi_i^{-1}(O_i)$. Hence $y \in \Pi_k U_k[X - \pi_i^{-1}(O_i)]$ which is a contradiction. Therefore

$$U_i[x_i] \cap U_i[X_i - O_i] = \emptyset.$$

Thus $U_i[x_i] \subset X_i - U_i[X_i - O_i]$ and by lemma 10 we have that each factor space is q-T_3.//

It is easy to see that the product of a family of a family of q-T_4 spaces need not be q-T_4. Let (X_α,t_α) be a family of T_4 topological spaces such that $(\Pi_\alpha X_\alpha,\Pi_\alpha t_\alpha)$ is not

T_4. Let $\mathcal{U}_\alpha$ be the Pervin structure associated with t_α. Then each factor space $(X_\alpha, \mathcal{U}_\alpha)$ is q-T_4 by theorem 18. Hence if $\Pi_\alpha \mathcal{U}_\alpha$ were q-T_4 then $\Pi_\alpha t_\alpha$ would be T_4 which is impossible.

F. A COUNTER-EXAMPLE IN $(F, \mathcal{W})$

DEFINITION 16. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces. Let F denote the set of all functions from X to Y. For each $V \in \mathcal{V}$ we let

$W(V) = \{ (f,g) \in F \times F : (f(x), g(x)) \in V \text{ for each } x \in X\}$.

The collection of all such $W(V)$ forms a base for a quasi-uniform structure which we denote by $\mathcal{W}$. $(F, \mathcal{W})$ is then a quasi-uniform space and $\mathcal{W}$ is called the quasi-uniform structure of quasi-uniform convergence.

The following example shows that neither the set of all continuous mappings from X to Y nor the set of all quasi-uniformly continuous mappings from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$ need be closed in $(F, \mathcal{W})$.

EXAMPLE 7. Let X and Y denote the integers. Let

$U_n = \{ (x,y) \in X \times X : x = y \text{ or } x = 1 \text{ and } y \geq n\}$.

Set $\mathcal{B} = \{ U_n : n = 1, 2, \ldots\}$. Then $\mathcal{B}$ is a base for a quasi-uniform structure on X, which we will denote by $\mathcal{U}$.

The set of all

$$V_n = \{ (x,y) \in Y \times Y : x = y \text{ or } x \geq n\}$$

where $n = 1, 2, \ldots$, forms a base for a quasi-uniform structure for Y, which we will denote by $\mathcal{V}$. The topology on Y is discrete. Define $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ by $f(x) = x$ for each

$x \in X$. The function f is not continuous since $f^{-1}(1) = \{1\} \notin t_X$, and hence not quasi-uniformly continuous.

For each natural number n, define

$$f_n : (X,\mathcal{U}) \to (Y,\mathcal{V}) \text{ by,}$$
$$f_n(x) = x \text{ for } x < n$$
$$= 1 \text{ for } x \geq n.$$

We will show that each f_n is quasi-uniformly continuous. Let f_n and $V_m \in \mathcal{V}$ be given. It suffices to show that if $(a,b) \in U_n$ then $(f_n(a),f_n(b)) \in V_m$. If $a = b$, then $f_n(a) = f_n(b)$ and thus $(f_n(a),f_n(b)) \in V_m$. If $a \neq b$, then $a = 1$ and $b \geq n$. Therefore $(f_n(a),f_n(b)) = (1,1) \in V_m$. Thus each f_n is quasi-uniformly continuous and by theorem 1.24, in [16] it is continuous.

Let F denote the set of all functions from X to Y and $\mathcal{W}$ the quasi-uniform structure of quasi-uniform convergence. Let $U \subset F$ denote the set of all quasi-uniformly continuous functions from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$, and let $C \subset F$ denote the collection of all continuous functions from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$. We have shown that each $f_n \in U \subset C$, and $f \notin C$. We now show that $f \in \overline{U}$. Let V_n be given. We claim that $f_n \in W(V_n)[f]$. Consider $(f(x),f_n(x))$; if $x < n$ then $(f(x),f_n(x)) = (x,x) \in V_n$, and if $x \geq n$ then $(f(x),f_n(x)) = (x,1) \in V_n$. Hence $f_n \in W(V_n)[f]$ and since V_n was arbitrary, we have that $f \in \overline{U} \subset \overline{C}$. But $f \notin C$. Hence neither U nor C is closed in $(F,\mathcal{W})$.

V. SUMMARY, CONCLUSIONS AND FURTHER PROBLEMS

We noted that a quasi-uniform space need not have a T_1 or T_2 strong completion nor a T_2 completion. It is of interest to know if an arbitrary space has a T_1 completion. Our construction showed that for certain classes a T_1 completion exists. No one as yet has exhibited a topological space that does not admit a complete or strongly complete quasi-uniform structure. Our example demonstrates that a space can be uniformizable, admit a strongly complete structure and not admit a complete uniform structure.

It seems that the present definition of Cauchy filter may admit too many filters. In Chapter III, example 2, we saw that $F = \{X\}$ was a Cauchy filter. Since the topology on X is discrete this seems a bit unreasonable. Thus the study of other classes of "Cauchy" filters would appear to be worthwhile. Since Section F, Chapter III showed that regardless of the definition of "Cauchy" filter and "completeness" not all of the pleasant properties of the completion of a Hausdorff uniform space could be preserved for a quasi-uniform space, it seems logical for one to decide which of these properties should be preserved, if possible, before a new definition of "Cauchy" filter is proposed.

On the other hand it can be noted that for special classes the completion constructed in Chapter III, Section

B, has many of the desired properties.

In Chapter IV several topics were considered. Many other areas in quasi-uniform spaces also remain open to investigation. No one has as yet characterized the universal quasi-uniform structure associated with a given topology, and no necessary and sufficient conditions are known for when a topology admits a minimal quasi-uniform structure. Necessary and sufficient conditions were given for $\mathcal{U} \wedge \mathcal{U}^{-1}$ to be a quasi-uniform structure. We showed that U and C need not be closed in $(F,\mathcal{W})$ and Naimpally [17] showed that if $(Y,\mathcal{V})$ is T_3 and $\mathcal{V}$ is the Pervin structure then U and C are closed in $(F,\mathcal{W})$. More general conditions to insure that U and C be closed or complete seem desirable.

The q-T_i separation properties seem to this author to at least be an interesting concept when one is more concerned about the particular quasi-uniform structure than the generated topology.

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VITA

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