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Parameter identification of nonuniform distributed RC networks

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PARAMETER IDENTIFICATION
OF NONUNIFORM DISTRIBUTED RC NETWORKS

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ABSTRACT

The problem of the unique parameter identification of nonuniform distributed RC networks is addressed in this dissertation. Previous literature provides for the equivalence of distributed networks with respect to the terminal characteristics. Therefore, the question is whether unique parameter identification is even possible. This work settles that question by presenting the sufficient conditions to insure a unique solution to the parameter identification problem. One theorem requires the knowledge of a driving-point impedance, the knowledge of the physical length, and the constraint that the r(x)c(x) product remains constant to insure uniqueness of the parameters r(x) and c(x). Relaxation of the r(x)c(x) product constraint allows for other combinations of r(x) and c(x) which produce identical terminal characteristics, thus destroying the uniqueness property. However, knowledge of a driving-point impedance, of the physical length, and of r(x) is sufficient to uniquely determine c(x). Sufficiency theorems which involve one of the [A,B,C,D] parameters rather than a driving-point impedance are also included.

A practical parameter identification routine is then presented in order to find the unknown parameters. A Fletcher-Powell/Davidon unconstrained minimization technique is used although any routine with appropriate convergence properties may be utilized. Examples of parameter identification of known networks are given.

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CHAPTER I: INTRODUCTION

A. Introduction to Distributed Networks

The advent of microelectronic technology has made several devices physically practical which were only of theoretical interest before this technology became available; for example, the traveling-wave transistor, the metal-oxide-semiconductor field-effect transistor, the distributed RC network, etc. Much literature has been devoted to the distributed RC network. Such work includes that of analysis. synthesis, approximation, stability, sensitivity, and fabrication. For a treatment of the history, theory, and applications of distributed RC networks, [1] and [2] should be consulted. It is the purpose of this dissertation to present some results in parameter identification of distributed RC networks, an area which has received little attention in the literature.

B. Governing Equations

One form of the distributed RC network is the three-layered device depicted in Figure 1.1. Many variations of this structure and its characteristics are made possible by attaching terminals at chosen points on the device and also by adding additional layers to the basic structure.

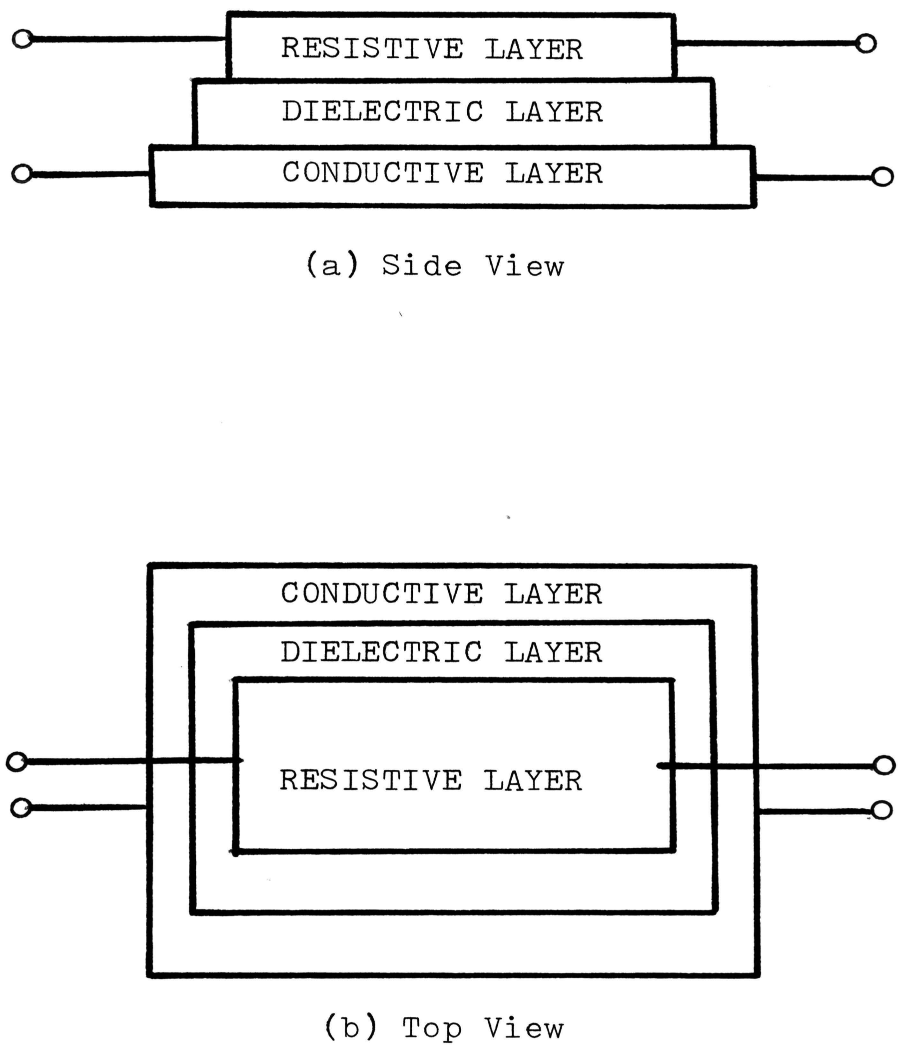


Figure 1.1. Structure of a Distributed RC Network

Figure 1.1 provides a description of the device considered in this paper.

The defining mathematical relationships between voltages and currents of a distributed RC network are not derived here, as ample space has already been devoted to such derivations [1,pp.27,28], [2,p.19], and [3,pp.130,131].

The incremental model of the distributed RC circuit is indicated in Figure 1.2, from which the following defining equations can be derived:

$$\frac{\partial v(x,t)}{\partial x} = -r(x)i(x,t) \tag{1.1a}$$

$$\frac{\partial i(x,t)}{\partial x} = -c(x)\frac{\partial v(x,t)}{\partial t} \tag{1.1b}$$

Taking the Laplace transform of (1.1) with respect to t (with zero initial conditions) gives:

$$\frac{d}{dx}\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = -\begin{bmatrix} 0 & r(x) \\ sc(x) & 0 \end{bmatrix}\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} \tag{1.2}$$

where s is the Laplace transform variable and V(x,s) and I(x,s) are the transformed voltages and currents, respectively. (1.1) may also be combined to form the following sets of equations:

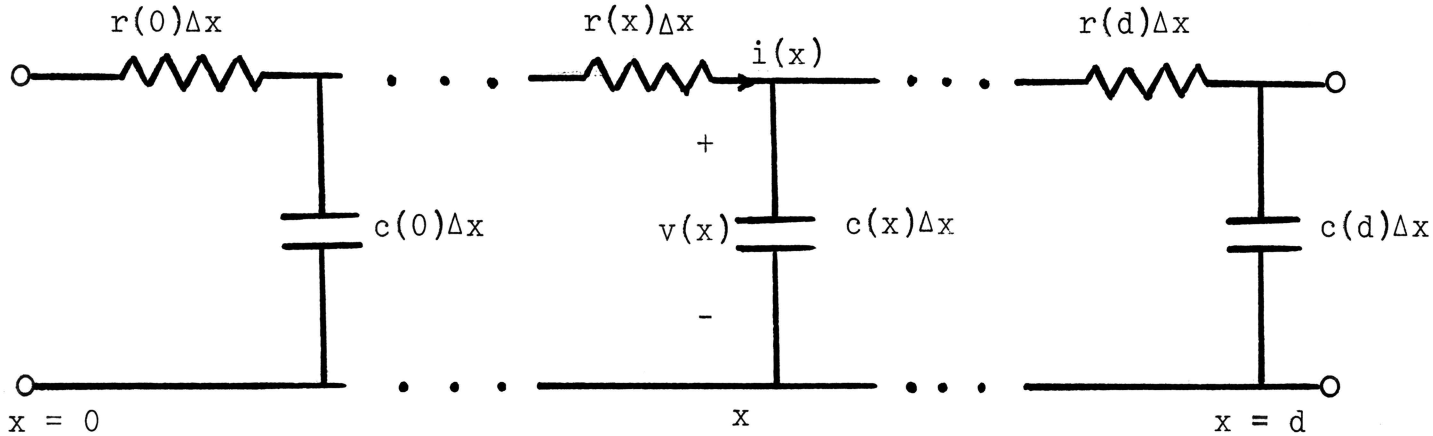


Figure 1.2. Incremental RC Transmission Line

$$\frac{\partial^2 v(x,t)}{\partial x^2} - \frac{1}{r(x)} \frac{dr(x)}{dx} \frac{\partial v(x,t)}{\partial x} = r(x)c(x) \frac{\partial v(x,t)}{\partial t} \tag{1.3a}$$

$$\frac{\partial^2 i(xt)}{\partial x^2} - \frac{1}{c(x)} \frac{dc(x)}{dx} \frac{\partial i(x,t)}{\partial x} = r(x)c(x) \frac{\partial i(x,t)}{\partial t} \tag{1.3b}$$

Application of the Laplace transform with respect to time to (1.3) gives

$$\frac{d^2 V(x,s)}{dx^2} - \frac{1}{r(x)} \frac{dr(x)}{dx} \frac{dV(x,s)}{dx} - sr(x)c(x)V(x,s) = 0 \tag{1.4a}$$

$$\frac{d^2 I(x,s)}{dx^2} - \frac{1}{c(x)} \frac{dc(x)}{dx} \frac{dI(x,s)}{dx} - sr(x)c(x)I(x,s) = 0 \tag{1.4b}$$

Thus, four sets of equations are available which describe the voltage-current characteristics of a general distributed RC network: a) two sets of second-order differential equations, one in the time domain and the other in a transformed domain, and b) two sets of first-order coupled differential equations, one in the time domain and the other in a transformed domain.

C. Physical Assumptions

All of the preceding governing equations are based on certain assumptions which must be considered in the modeling process. The incremental resistance, r(x), and the incremental capacitance, c(x), are assumed to be invariant with respect to frequency, time, temperature, and the magnitudes of the voltage and current in this dissertation.

Another assumption made in the governing equations is that of one-dimensional current flow in the device. Ghausi and Kelley [1,p.239] consider this assumption and derive governing equations where the one-dimensional current assumption is usually considered sufficient for r(x) and c(x) of small variations or where the length-to-width ratio is large; multidimensional current flow is a topic for further research as mentioned in Chapter V.

D. Review of the Literature

Many articles have been published related to distributed networks as shown by the bibliography of [1] alone. Only literature relating to the identification of the incremental resistance and capacitance is considered here.

R. A. Rohrer [4] casts the synthesis problem as a parameter-optimization problem and presents a solution by use of variational calculus. The parameters r(x), c(x), g(x), and l(x) are constrained to be greater than zero. Hellstrom [5] considers a method for finding equivalent functions for given r(x) and c(x), which when viewed from the terminals produce identical electrical characteristics. Implications of Hellstrom's work suggest that the identification of r(x) and c(x) for a specific network may not be possible. Part of the work of this dissertation proves that such identification is possible under certain constraints. Karnik and Cohen [6] approach the problem by using classical optimal-control techniques and use a gradient technique for the final solution. Jain [7] converts the identification problem into a network-matching problem, which results in solving an initial-value Riccati equation. Protonotarios and Wing [8] consider a synthesis problem in which poles and zeros of driving-point functions of RC distributed networks are given. Wohlers [9] utilizes a procedure given the input reflection coefficient. Sondhi and Gopinath [10] determine the shape of the vocal tract by use of an LC line model and driving-point impulse response data.

E. Purpose and Scope of the Dissertation

Very little space has been devoted in the literature to conditions sufficient to guarantee unique parameter identification of systems governed by partial differential equations. It is the purpose of this dissertation to present some work done in this area by considering sufficiency theorems for unique parameter identification of systems governed by equations of the form (1.1) - (1.4). Specific application is made of this work to the area of taper identification of distributed RC networks.

Chapter II develops and proves some sufficiency theorems for taper identification of the RC distributed network involving driving-point information. Chapter III extends the theorems in Chapter II so that transfer-function type information can be utilized. Chapter IV presents a practical parameter identification routine using numerical optimization techniques. The concluding chapter extends the work of Chapters II and III to cover other 2-element-kind distributed networks. Also included in Chapter V are detailed suggestions for further research.

CHAPTER II: UNIQUENESS THEOREMS ON TAPER IDENTIFICATION OF RC DISTRIBUTED NETWORKS

A. Introduction

This chapter defines taper identification of a distributed RC network and contains theorems which provide for the uniqueness of tapers in such networks. Theorems in this chapter involve knowledge of a circuit driving-point impedance; thus, both voltage and current at a port must be known or measurable. Proofs are included with the theorems.

B. Taper Identification of a Distributed RC Network

Definition 2.1: Taper is the set of coefficients r(x) and c(x) in the distributed RC network governing equations (1.1) through (1.4).

Definition 2.1: A taper is said to be identified if r(x) and c(x) are uniquely determined.

In light of definitions 2.1. and 2.2., the taper identification problem is that of finding the coefficients r(x) and c(x) given certain circuit terminal electrical measurements and other circuit properties. Chapters II and III discuss sufficiency conditions to guarantee a unique taper.

C. Equivalence and Uniqueness

Several authors, [1,pp.129-133] and [5], have discussed methods by which equivalent networks (networks with the same terminal electrical properties) can be generated. These methods are based on spatial transformations which generate new taper functions but still satisfy the same basic differential equations. The topic of equivalence provides a good starting point when looking for uniqueness properties. Characteristics may be found which differentiate a given network from all other equivalent networks. As an example, [5] demonstrates that three networks, a) $r(x)c(x) = K$, b) $r(x) = r_o$ and $c(x) = c_o e^{-ax}$, and c) $r(x) = r_o e^{ax}$ and $c(x) = c_o$ have the same terminal characteristics including the same total resistance and capacitance. A characteristic which differentiates each of these networks from the other two is found to be the physical length. From the same work, a second example may be drawn using Bessel-tapered networks which produce equivalent terminal characteristics:
a) $c = c_o[1 + \sqrt{r_o c_o}\ x/z_o]$ and $r = r_o/[1 + \sqrt{r_o c_o}\ x/z_o]$,
b) $c = c_o \exp(z\sqrt{r_o c_o}\ x/z_o)$ and $r = r_o$, and c) $c = c_o$ and $r = r_o/[1 + 2\sqrt{r_o c_o}\ x/z_o]$. As in the first example, a distinguishing characteristic of the three equivalent networks is found to be the physical length. The same result is also concluded from examples in [1,pp.131-133]. These examples suggest that the physical length of the distributed RC network is a parameter to be considered in the taper identification problem; however, it should be

noted that $r(x)c(x)$ and $\frac{r(x)}{c(x)}$ also provide characteristics which distinguish the three networks.

The following example demonstrates that identical terminal characteristics, physical length, and the ratio $r(x)/c(x)$ do not provide sufficient information for taper uniqueness. Consider two tapers, (1) $r(x) = c(x) = 3/2$ and (2) $r(x) = c(x) = 1 + x$, where $d = 1$ for both lines. It can be shown that both tapers have identical terminal characteristics; they also have identical physical lengths and $r(x)/c(x)$ ratios. The [y] parameters are as follows:

$$y_{11} = y_{22} = \sqrt{s}\ \coth(\tfrac{3}{2}\sqrt{s})$$

$$y_{12} = y_{21} = -\sqrt{s}\ \operatorname{csch}(\tfrac{3}{2}\sqrt{s})$$

Thus without further information concerning the networks it is impossible to distinguish one network from the other by just a knowledge of the terminal characteristics, physical length, and the ratio $r(x)/c(x)$. If, however, the $r(x)c(x)$ product is constrained to be constant, then networks of the same length which exhibit identical terminal characteristics consist of identical taper functions as proven in Theorem 2.1.

D. Taper Uniqueness Theorems

In the lumped-network case, a Driving-Point (D.P.) function does not imply a unique circuit configuration. For uniqueness other constraints must be invoked. The same holds true for the distributed RC network. An RC D.P. function is not sufficient to totally characterize all properties of a given network as demonstrated in the following theorem.

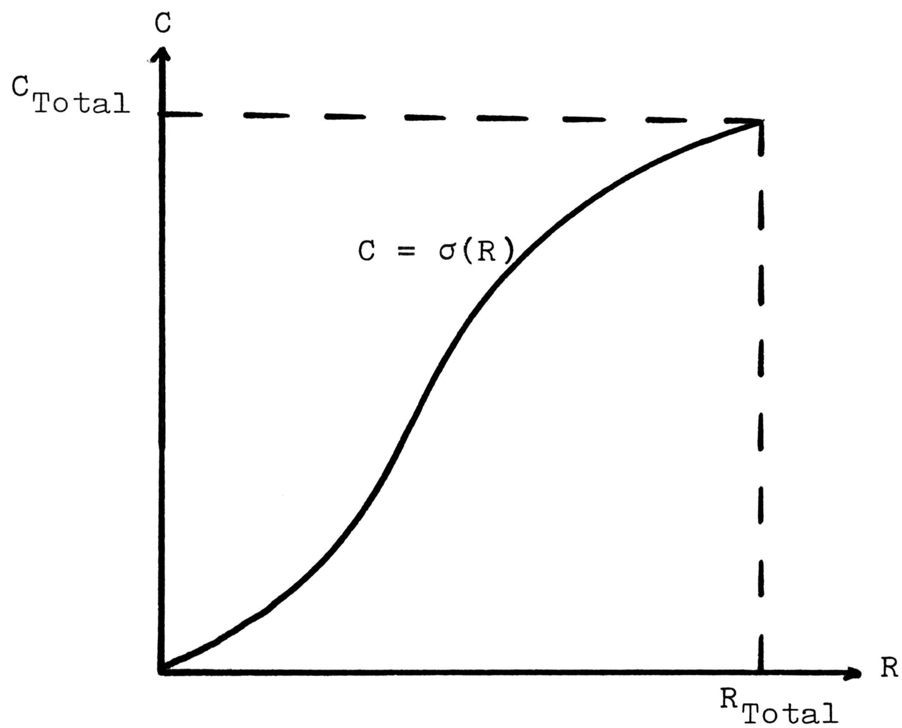
Theorem 2.1: Given the driving-point function from a continuous taper, the physical length, d, and that the r(x)c(x) product remains constant; then the taper, r(x) and c(x), for a distributed network is unique.

Proof: [8] proved that for a given RC D.P. function, the $\sigma(R)$ function is unique. $\sigma(R)$ is a function relating total accumulated resistance to total accumulated capacitance at some point, x, along the line as shown in Figure 2.1.

Since negative and zero resistances and capacitances are not allowed, $\sigma(R)$ is an increasing differentiable function, and $\frac{d\sigma(R)}{dR} > 0$. Therefore, $\sigma(R)$ has a unique inverse, $\theta(C) = R$.

Let $\sigma(R) = C$ and $\theta(C) = R$ be the unique functions associated with the given RC D.P. function. Assume that two different tapers exist which satisfy the hypothesis: $r_1(x)$, $c_1(x)$ and $r_2(x)$, $c_2(x)$.

The total resistance and capacitance associated with each taper must be equal since both tapers produce the same $\sigma(R)$ function.



$$R = \text{Total Resistance at Point } x = \int_0^x r(y)dy$$

$$C = \text{Total Capacitance at Point } x = \int_0^x c(y)dy$$

Figure 2.1. Accumulated Capacitance
vs.
Accumulated Resistance

$$R_{1_{Total}} = R_{2_{Total}} = \int_o^d r_1(y)dy = \int_o^d r_2(y)dy \tag{2.1}$$

$$C_{1_{Total}} = C_{2_{Total}} = \int_o^d c_1(y)dy = \int_o^d c_2(y)dy \tag{2.2}$$

Both tapers have the same $\sigma(R)$ and $\theta(C)$ functions but do not necessarily occur concurrently on the x axis; this can be formalized by

$$R = \int_o^{x_1} r_1(y)dy = \int_o^{x_2} r_2(y)dy \tag{2.3}$$

$$C = \int_o^{x_1} c_1(y)dy = \int_o^{x_2} c_2(y)dy \tag{2.4}$$

where the functional relationship between x_2 and x_1 may be defined as $x_2 = x_2(x_1)$. Differentiating (2.3) and (2.4) with respect to x_1 yields (The validity of this differentiation is proven in Appendix A)

$$r_1(x_1) = r_2(x_2) \frac{dx_2}{dx_1} \tag{2.5}$$

$$c_1(x_1) = c_2(x_2) \frac{dx_2}{dx_1} \tag{2.6}$$

Dividing (2.6) by (2.5) yields

$$\frac{c_1(x_1)}{r_1(x_1)} = \frac{c_2(x_2)}{r_2(x_2)} \tag{2.7}$$

Recalling that $r(x)c(x) = K$, where K is unspecified, (2.7) can be manipulated to yield

$$r_1(x_1) = K_a r_2(x_2) \tag{2.8}$$

$$c_1(x_1) = K_b c_2(x_2) \tag{2.9}$$

where K_a and K_b are constants. Substituting (2.8) and (2.9) into (2.5) and (2.6) yields

$$K_a = \frac{dx_2}{dx_1} \tag{2.10}$$

$$K_b = \frac{dx_2}{dx_1} \tag{2.11}$$

with boundary conditions $x_2(0) = 0$ and $x_2(d) = d$. From the solution of either (2.10) or (2.11), $x_2 = x_1$. Therefore $r_1(x) = r_2(x)$, $c_1(x) = c_2(x)$, and the taper is unique. This completes the proof of Theorem 2.1 by contradiction.

The $r(x)c(x) = K$ restriction in the hypothesis of Theorem 1.1 is theoretically appealing. [1,pp.115-129] devotes several pages in consideration of networks with such configurations. Analysis is simplified under this restriction, resulting in closed-form solutions and network parameters in some cases. From a construction point of view this restriction calls for a uniform dielectric-layer thickness, an assumption frequently made. However, in cases where the dielectric is made as thin as possible to yield a relatively large capacitance, measurement and control of this layer's thickness present the most problems of the three layers. For these reasons holding $r(x)c(x) = K$ may be too restrictive, and Theorem 2.2, which follows, may present a more practical hypothesis where the uniform thickness requirement is not required. The proof of Theorem 2.2 is an amended version of that used for Theorem 2.1.

Theorem 2.2: Given a distributed RC D.P. function, $r(x)$, and d; then $c(x)$ is unique.

The proof is very similar to that of Theorem 2.1. Since $r_1(x) = r_2(x)$, (2.3) implies

$$x_1 = x_2 \tag{2.12}$$

Substitution of (2.12) into (2.7) yields

$$c_1(x) = c_2(x) \tag{2.13}$$

which proves the uniqueness of $c(x)$.

The utility of the preceding theorem lies in the knowledge of $r(x)$. In a fabrication process the engineer may have confidence in the accuracy of $r(x)$ or may be able to make point measurements for its determination. In any case $c(x)$ is the parameter for which the lesser information is available.

An obvious corollary can be made by simply interchanging $r(x)$ and $c(x)$.

Corollary 2.1: Given a distributed RC D.P. function, $c(x)$, and d; then $r(x)$ is unique.

CHAPTER III: UNIQUENESS IN TERMS OF A CHAIN PARAMETER

A. Introduction

This chapter presents a uniqueness theorem which is less restrictive than those in Chapter II. Preliminary to this uniqueness theorem is another theorem and a set of corollaries which prove that one of the chain parameters and R_T and C_T are sufficient to specify a driving-point immittance of a distributed RC network. From a practical point of view, the transfer function required when specifying one of the chain parameters may be more easily obtainable than the driving-point function. Before statement of the uniqueness theorem the chain parameters as given in [8] are stated for further reference in this chapter.

$$A(s) = \prod_{i=1}^{\infty} (1 + s/\alpha_i) \tag{3.1}$$

$$B(s) = R_T \prod_{i=1}^{\infty} (1 + s/\beta_i) \tag{3.2}$$

$$C(s) = C_T s \prod_{i=1}^{\infty} (1 + s/\gamma_i) \tag{3.3}$$

$$D(s) = \prod_{i=1}^{\infty} (1 + s/\delta_i) \tag{3.4}$$

where

$$0 < (\alpha_i, \delta_i) < \gamma_i \tag{3.5a}$$

and $\qquad i = 1,2....$

$$0 < (\delta_i, \alpha_i) < \beta_i \tag{3.5b}$$

B. A Theorem Relating A(s), R_T, and C_T to a D.P. Impedance

The theorem in this section specifies the relationship of A(s), R_T, and C_T to a driving-point impedance of a distributed RC network. In particular A(s), R_T, and C_T provide sufficient information to completely specify the short-circuit output impedance, $\frac{1}{y_{22}}$.

Theorem 3.1: Given A(s) from a distributed RC network and the total resistance and capacitance, R_T and C_T; then the short-circuit output impedance is completely specified.

Proof: Let A(s) and B(s) take the form of (3.1) and (3.2) where the α_i's are known and the β_i's are unspecified but bounded by $\alpha_i < \beta_i < \alpha_{i+1}$ for $i = 1,2,3....$

Protonotarios and Wing [8] established the relationship between C_T, A(s), and B(s) as:

$$C_T = \sum_{i=1}^{\infty} \frac{1}{\alpha_i{}^2 A'(-\alpha_i) B(-\alpha_i)} \tag{3.6}$$

where the prime denotes differentiation with respect to s.

Substitution of $B_1(s) = B(s)/R_T$ in (3.6) produces

$$R_TC_T = \sum_{i=1}^{\infty} \frac{1}{\alpha_i^2 A'(-\alpha_i)B_1(-\alpha_i)} \tag{3.7}$$

Now let the β_i's vary within the given bounds implying that C_T is variable. The β_i's which give $R_TC_{T_{MIN}}$ can be found by taking

$$\frac{\partial(R_TC_T)}{\partial\beta_k} = 0 \tag{3.8}$$

Substituting (3.7) into (3.8) and interchanging the order of differentiation and summation (as justified in Appendix B) results in

$$\frac{\partial(R_TC_T)}{\partial\beta_k} = \frac{1}{\beta_k} \sum_{i=1}^{\infty} \frac{1}{\alpha_i A'(-\alpha_i)B_1(-\alpha_i)(\alpha_i-\beta_k)} = 0 \tag{3.9}$$

The specific values of $-\beta_k$, $k = 1,2,3....$ which satisfy (3.8) are the zeros of

$$K(s) = \sum_{i=1}^{\infty} \frac{1}{\alpha_i A'(-\alpha_i)B_1(-\alpha_i)(s+\alpha_i)} \tag{3.10}$$

The poles of (3.10) are the $-\alpha_i$, $i = 1,2,3....$; therefore, (3.10) can be expanded using the Weierstrass factor theorem [1] yielding

$$K(s) = \lambda(s) \prod_{i=1}^{\infty} \frac{(1 + s/\beta_i)}{(1 + s/\alpha_i)} = \lambda(s) \frac{B_1(s)}{A(s)} \tag{3.11}$$

where $\lambda(s)$ is an entire function with no zeros.

Invoking the Mittag-Leffler theorem [11,p.157-162] on (3.11) (as justified in Appendix C) results in

$$K(s) = \lambda(s) \sum_{i=1}^{\infty} \frac{B_1(-\alpha_i)}{A'(-\alpha_i)(s+\alpha_i)} \tag{3.12}$$

It can be shown that $\lambda(s)$ is a positive constant, λ_o (Appendix D); thus, comparison of (3.10) and (3.12) yields

$$\left| B_1(-\alpha_i) \right| = \frac{1}{\sqrt{\alpha_i \lambda_o}} \tag{3.13}$$

The absolute value is required, because the interlace property of α_i and β_i causes

$$B_1(-\alpha_i) > 0 \quad \text{for i odd}$$

$$B_1(-\alpha_i) < 0 \quad \text{for i even}$$

It can be shown that $A'(-\alpha_i)B(-\alpha_i) > 0$; therefore, substitution of (3.13) into (3.12) results in

$$K(s) = \sum_{i=1}^{\infty} \frac{\sqrt{\alpha_i \lambda_o}}{\alpha_i \left| A'(-\alpha_i) \right| (s + \alpha_i)} = \frac{\lambda_o B_1(s)}{A(s)} \tag{3.14}$$

Manipulation of (3.14) results in

$$\frac{B_1(s)}{A(s)} = \frac{1}{\sqrt{\lambda_o}} \sum_{i=1}^{\infty} \frac{1}{\sqrt{\alpha_i} \left| A'(-\alpha_i) \right| (s + \alpha_i)} \tag{3.15}$$

where λ_o can be evaluated by letting s = 0 in (3.15) resulting in

$$\sqrt{\lambda_o} = \sum_{i=1}^{\infty} \frac{1}{\alpha_i^{3/2} \left| A'(-\alpha_i) \right|} \tag{3.16}$$

To completely specify the short-circuit output impedance with $C_{T_{MIN}}$ (3.15) must be multiplied by R_T resulting in

$$\frac{1}{y_{22}} = \frac{B(s)}{A(s)} = \frac{R_T}{\sqrt{\lambda_o}} \sum_{i=1}^{\infty} \frac{1}{\sqrt{\alpha_i} \left| A'(-\alpha_i) \right| (s + \alpha_i)} \tag{3.17}$$

To obtain $1/y_{22}$ for the actual C_T, s is replaced by $sC_T/C_{T_{MIN}}$. Thus the knowledge of the open-circuit voltage transfer function, $1/A(s)$, R_T, and C_T is sufficient to completely specify a distributed RC network output impedance.

C. Resulting Corollaries

A series of three corollaries are now stated with partial proofs which require one of the three remaining chain parameters as part of the hypothesis.

Theorem 3.1 is easily modified to form a corollary requiring knowledge of the short-circuit current transfer function rather than the voltage transfer function.

Corollary 3.1: Given D(s) from a distributed RC network and the total resistance and capacitance, R_T and C_T; then the short-circuit input impedance $1/y_{11}$ is completely specified.

The proof follows the same pattern as Theorem 3.1 except that an equivalent expression for (3.6) [8] is used involving δ_i, D(s), and B(s).

$$C_T = \sum_{i=1}^{\infty} \frac{1}{\delta_i^{\,2} D'(-\delta_i) B(-\delta_i)} \tag{3.18}$$

The proof is not given, but the short-circuit input impedance is

$$1/y_{11} = \frac{B(s)}{D(s)} = \frac{R_T}{\sqrt{\lambda_o}} \sum_{i=1}^{\infty} \frac{1}{\sqrt{\delta_i}\,\left|D'(-\delta_i)\right|(s/k + \gamma_i)} \tag{3.19}$$

where $k = C_{T_{MIN}}/C_T$.

where

$$\sqrt{\lambda_o} = \sum_{i=1}^{\infty} \frac{1}{\delta_i^{3/2} \left| D'(-\delta_i) \right|}$$

Corollary 3.2: Given B(s) from a distributed RC network and the total capacitance C_T; then the short-circuit output admittance y_{22} is completely specified.

The proof of Corollary 3.2 differs slightly from that of Theorem 3.1. The differing points are mentioned, but a complete proof is not given.

From [8,(56)]

$$C_T = - \sum_{i=1}^{\infty} \frac{1}{\beta_i^{2} B'(-\beta_i) A(-\beta_i)} + \frac{1}{R_T} \left(a_1 + \frac{b_1}{R_T}\right) \qquad (3.20)$$

where a_1 and b_1 are Taylor series coefficients of A(s) and B(s), respectively. After substitution of $B_1'(-\beta_i) = B'(-\beta_i)/R_T$ and $\sum_{i=1}^{\infty} \frac{1}{\alpha_i} = a_1$ in (3.20), differentiation with respect to α_k and simplification yields

$$\sum_{i=1}^{\infty} \frac{-1}{\beta_i B_1'(-\beta_i) A(-\beta_i)(\beta_i - \alpha_k)} - \frac{1}{\alpha_k} = 0 \qquad (3.21)$$

The values of $-\alpha_k$ which satisfy (3.21) are zeros of

$$H(s) = \sum_{i=1}^{\infty} \frac{-1}{\beta_i B_1'(-\beta_i) A(-\beta_i)(s + \beta_i)} + \frac{1}{s} \qquad (3.22)$$

The poles of (3.22) are the $-\beta$'s and the origin. H(s) can be written in infinite product form using the Weierstrass factor theorem and then expanded by the Mittag-Leffler Theorem yielding

$$H(s) = \frac{\lambda_1(s)}{s} \prod_{i=1}^{\infty} \frac{(1+s/\alpha_i)}{(1+s/\beta_i)} = \frac{\lambda_1(s)}{s} \frac{A(s)}{B_1(s)}$$

$$= \lambda_1(s) \sum_{i=1}^{\infty} \frac{-A(-\beta_i)}{\beta_i B_1'(-\beta_i)(s+\beta_i)} + \frac{1}{s} \qquad (3.23)$$

where $\lambda_1(s)$ is an entire function with no zeros. Comparing (3.22) and (3.23) yields $\lambda_1(s) = \lambda_1 = 1$ and $A^2(-\beta_i) = 1$. Thus $A(-\beta_i) = 1$; therefore, since $A(-\beta_i)B_1'(-\beta_i) < 0$ for $i = 1,2....$, (3.23) simplifies to

$$\frac{A(s)}{sB_1(s)} = \sum_{i=1}^{\infty} \frac{1}{\beta_i \; B_1'(-\beta_i)(s+\beta_i)} + \frac{1}{s} \qquad (3.24)$$

After multiplication by s substitution of B(s), (3.24) can be recognized as the following short-circuit output admittance:

$$y_{22} = \frac{A(s)}{B(s)} = \frac{1}{R_T} + \frac{s}{k} \sum_{i=1}^{\infty} \frac{1}{\beta_i \left| B'(-\beta_i) \right| (s/k+\beta_i)} \qquad (3.25)$$

By substitution of γ_i and C(s) for β_i and B(s), respectively, Corollary 3.2 can be modified to form Corollary 3.3.

Corollary 3.3: Given C(s) and the total resistance of a distributed RC network; the open-circuit input impedance z_{11} is completely specified.

The proof follows directly from Corollary 3.2. and only the results are indicated as follows:

$$z_{11} = \frac{A(s)}{C(s)} = \frac{k}{sC_T} + \sum_{i=1}^{\infty} \frac{1}{\sqrt{\gamma_i}\,|C'(-\gamma_i)|\,(s/k+\gamma_i)} \qquad (3.26)$$

Although the preceding theorems and corollaries are proven for specific driving-point immittances, similar expressions for other driving-point immittances can be derived with little trouble.

D. Another Uniqueness Theorem

The collection of Theorem 3.1 and Corollaries 3.1, 3.2, and 3.3 states that one of the chain parameters and R_T and/or C_T are sufficient to specify an RC driving-point immittance. These results coupled with the theorems and corollaries of Chapter II result in the following theorem.

Theorem 3.2: Given one of the chain parameters of a distributed RC network, R_T and C_T, the condition that the r(x)c(x) product remains constant, and the physical length, then the taper is unique.

The proof requires the application of the preceding theorems in this chapter and Theorem 2.1. One of the chain parameters and R_T and C_T provide for a driving-point immittance of the network. Theorem 2.1 requires such a driving-point immittance and the remainder of the hypothesis, thus yielding the uniqueness of the taper.

Obviously Corollaries 2.1 and 2.2 may also be invoked to form this variation of Theorem 3.2.

Corollary 3.4: Given one of the chain parameters, R_T and C_T, r(x) (or c(x)), and the physical length of a distributed network; then the taper is unique.

CHAPTER IV: A PRACTICAL TAPER IDENTIFICATION ROUTINE

A. Introduction

This chapter presents a practical approach to taper identification by utilization of a numerical minimization routine. A discussion of the program is included, as well as a table containing the results of a number of variations on the problem. Preceding the presentation of the routine is a discussion of the elements involved in taper identification.

B. Elements of Taper Identification

The preceding chapters have been devoted to theorems and corollaries involving the uniqueness of the taper. A uniqueness condition only fulfills part of the requirements of a good identification scheme. Also included in a good scheme are the utilization of all known properties of the network and an iterative algorithm which converges to the unique taper function.

It seems reasonable that the identification routine include as much known information as possible if for no other reason than a better model fit. Many of the basic properties, such as the interlace property, which are known about distributed networks are derived from the general form of the defining equations. Thus from an identification point of view, the best model to choose is one based on the defining equations; consequently these basic properties are incorporated implicitly. Also, in the taper identification problem the coefficients to be identified appear explicitly in the defining equations.

A high-order ordinary differential equation can also be used for the model; however, several problems are inherent with this model. For example, suppose that a high-order differential equation is chosen as the model for the open-circuit voltage transfer function of an unknown RC distributed network. If the poles in the transfer function model are to remain on the negative real axis, as in the physical device, then one of two techniques is required. The first is to implement the modeling process in a constrained optimization routine in which the only feasible poles are those on the negative real axis. Unfortunately, the technique is quite formidable for high-order systems. The second technique is to use classical identification techniques implementing interval analysis for the pole selection. Such a method requires an interval factorization routine, which is again quite formidable for high-order systems.

The finite-order differential equation model does not lend itself to the taper identification problem because the taper coefficients do not appear in the model. However, a crude taper identification can be accomplished by identifying coefficients which form a driving-point impedance.

From this function a continued-fraction expansion yields the ladder-line element values. Taper identification by this method is quite poor because of the high-order system required for marginal results; thus, this method can only provide theoretical interest rather than a practical identification scheme.

This discussion does not mean to imply that a finite-dimension model can not be used to advantage. Relatively small-order models have been constructed quite effectively for low-frequency applications; however, for taper identification purposes, the model chosen is one based on the defining equations because basic properties are implicitly incorporated and the coefficients r(x) and c(x) are incorporated explicitly.

C. Preliminary Observations

Before presenting a taper identification routine, some preliminary considerations are discussed concerning the theorems of the previous chapters and also the domain in which the problem is solved. The objective is to find a routine which not only incorporates the hypotheses of the theorems in the previous chapters but also finds the taper function. It should be noted that two examples are tabulated which do not fulfill all the hypotheses of the theorems. These examples are explained in section D-3. Recall the result of Chapter III: given one of the chain parameters, physical length, R_T and C_T, and the condition that the $r(x)c(x)$ product remain constant; then the taper is unique. This implies that the implementation of taper identification can be done in the frequency domain. Other advantages exist for identification in the frequency domain. For example, laboratory equipment can also impose a constraint on the domain in which to work. Most laboratories are better equipped to measure frequency response data as opposed to the time domain impulse response data.

Consider the time-domain representation of one of the chain parameters or even a driving-point impedance. If the taper identification is to be done in the time domain, then the routine is forced to solve equations such as (1.1) or (1.3). Suppose driving-point impulse response data is available; then the routine must solve either (1.1) or (1.3) from knowledge gained at one boundary. If the time-domain equivalent of one of the chain parameters is specified, then data is specified at both boundaries which necessitates the solution of a two-point boundary-value problem. Thus two programs are required to evaluate the conditions on the transmission line. In the frequency domain, the problem is simpler. The only program required is one to evaluate the specified transfer function which can be accomplished by solving a simple matrix ordinary differential equation with variable coefficients at several frequencies.

Another problem present in a practical identification routine in either the time or frequency domains is the approximation of an infinite number of samples by a finite number. In the frequency domain, for example, the α_i's of

$$A(s) = \prod_{i=1}^{\infty} (1 + s/\alpha_i)$$

can not possibly be determined be a finite number of data points of A(s). A similar statement can be made regarding the time domain. The approach used in the following routine is to take a large number of samples in the frequency range of interest; first use a portion of the data points noting the results; then increase the number of data points, and compare the results. Ten samples have been found to yield good results for simulated data; in the case of actual laboratory data, more measurements produce a smoothing effect on the data. In the measured data cases, samples were taken over one decade and the range of the generated samples were one to two decades. This procedure produces good results, and the cost of additional evaluations is relatively insignificant.

From the data-measurement point of view, several checks are available concerning the accuracy of the data. The low-frequency short-circuit driving-point impedance is shown to approach R_T by simply taking the ratio of $B(j\omega)/A(j\omega)$ and letting $\omega \to 0$. In a similar manner, the low-frequency open-circuit driving-point impedance is shown to approach that of a capacitor whose value is C_T.

The driving-point response for large ω is most easily considered by using the asymptotic expressions for the chain parameters as given in [8]:

$$A(s) \sim Ke^{k\sqrt{s}}.$$

$$B(s) \sim (K_1 e^{k\sqrt{s}})/\sqrt{s}$$

$$C(s) \sim K_2\sqrt{s}e^{k\sqrt{s}}$$

$$D(s) \sim K_3 e^{k\sqrt{s}}$$

when k, K, K_1, K_2, and K_3 are positive constants and the asymptotic expressions are valid for large s and $|\arg(s)| \leq \frac{\pi}{2}$. From the limit of the short-cricuit input impedance results the following:

$$\lim_{j\omega \to \infty} \frac{1}{Y_{11}} = \lim_{j\omega \to \infty} \frac{B(j\omega)}{D(j\omega)} =$$

$$\lim_{j\omega \to \infty} \frac{(K_1 3^{k\sqrt{j\omega}})/\sqrt{j\omega}}{K_3 e^{k\sqrt{j\omega}}} = 0\angle{-45^\circ}$$

In a similar manner the open-circuit case can also be treated.

$$\lim_{j\omega \to \infty} z_{11} = \lim_{j\omega \to \infty} \frac{A(j\omega)}{C(j\omega)} =$$

$$\lim_{j\omega \to \infty} \frac{Ke^{k\sqrt{j\omega}}}{K_2\sqrt{j\omega}e^{k\sqrt{j\omega}}} = 0\angle{-45^\circ}$$

The magnitude result of the high-frequency response is intuitively expected. The phase-response result serves as a convenient check for laboratory data.

A number of synthesis routines have been published in the literature which minimize a given objective function by finding optimal values for the given parameters. These programs with proper modification could be applied to the distributed RC network taper identification problem if, for example, a program written for a general line is converted to the RC case. Typical programs have picked a convenient value for the physical length, run the program, and then scaled the results to the desired length. To apply such programs to the RC taper identification problem requires that the actual length be supplied before running the program. The theorems of Chapters II and III preclude the possibility of more than one global minimum; however, one must still deal with problems of relative minima.

D. A Taper Identification Routine

1. Introduction

A practical taper identification routine is presented in this section. The routine minimizes an error function involving actual driving-point (or transfer) data and calculated data by varying expansion coefficients of $r(x)$ and $c(x)$. The minimization portion of the program is implemented by a version of the Davidon method as modified by Fletcher and Powell [17]. The RC transmission-line equations are solved by a program written by Peirson [16]. Both programs are well documented in the literature; therefore, little space is devoted to their explanation, although a flow chart (Figure 4.1) of the overall routine is given to aid explanation. Program results appear in tabular form and are generated by a number of variations on the basic identification problem.

2. Program Explanation

The Davidon-Fletcher-Powell algorithm is an unconstrained minimizational technique using a modified gradient method. For further information on the topic, [7] or [8] may be consulted. The specific program [17], is chosen not only for its convergence properties, but also for its utilization of difference approximations for derivatives.

$$J = \sum_{i=1}^{n} \left| Z(\omega_i)_{\text{Actual}} - Z(\omega_i)_{\text{Calc}} \right|^2 \qquad (4.1)$$

The values of $Z(\omega_i)_{\text{Calc}}$ are found by solving an ordinary differential equation with variable coefficients (see(1.3)). In general, closed form solutions do not exist for such equations, and thus they must be solved numerically. Therefore, a closed-form expression for the derivative of J with respect to a coefficient is not possible, and an approximation for the derivative is required.

The subroutine for solving the transmission-line equation is given in [16]. The program solves an ordinary differential equation with variable coefficients by breaking the line into a finite number of sections and finding a Taylor series approximation of the chain parameters of each section. Then the overall chain matrix is simply the product of all incremental chain matrices. Care should be exercised that the number of sections and the number of terms in the Taylor series approximation of each section are large enough to insure the desired accuracy. For further details [16] should be consulted.

A flow chart of the overall program is given in Figure 4.1. The flexibility of the program allows for its utilization in a variety of circuit configurations and tapers; however, this flexibility requires the user to be knowledgeable of every part of the program, rather than merely supplying data and observing the output.

The basic scheme of the program is to identify the coefficients r(x) and c(x) by driving the error expression, such as (4.1), to a minimum. In general expansions of r(x) and c(x) take these forms:

$$r(x) = \sum_{i=1}^{n} f_1(r_i, x) \tag{4.2}$$

$$c(x) = \sum_{i=1}^{n} f_2(c_i, x) \tag{4.3}$$

The user is at liberty to choose the exact expansion which he feels best fits the taper being identified. Naturally, if the user suspects a fairly uniform taper, then a small-degree power series may suffice; however, if the taper is one of significant variation, then a trigonometric expansion may be in order. As examples, the following expansions were found successful.

$$r(x) = r_0 + \sum_{i=1}^{n} r_i (x-d/2)^n \qquad (4.4)$$

$$r(x) = r_0 + \sum_{i=1}^{n} r_i \cos(a_i x + b_i) \qquad (4.5)$$

Referring to (4.4), initial results were obtained with an approximating function taking the form of a Taylor series expanded about x = 0; however, less error was achieved by using (4.4). It should be noted that within the subroutine, which solves the transmission-line equation, is a subroutine which returns to a calling program the values of r(x) and c(x) evaluated at some point on the line, as well as the first n derivatives of r(x) and c(x) with respect to x evaluated at the same point. The value of n is determined by the degree of the Taylor series expansion of the incremental transition matrix. Normally the value of n is small: two or three. Therefore, the expansions chosen must be differentiable n times in the interval x = 0 to x = length of line; furthermore, the user must program the derivatives in the subroutine. Care must be exercised in choosing an approximating expansion.

Referring to Figure 4.1, the identification routine is terminated within the Davidon-Fletcher-Powell routine by checking each component of two vectors: $H\nabla f(\bar{x})$ and $\lambda H\nabla f(\bar{x})$, where H is a positive definite matrix, $\nabla f(\bar{x})$

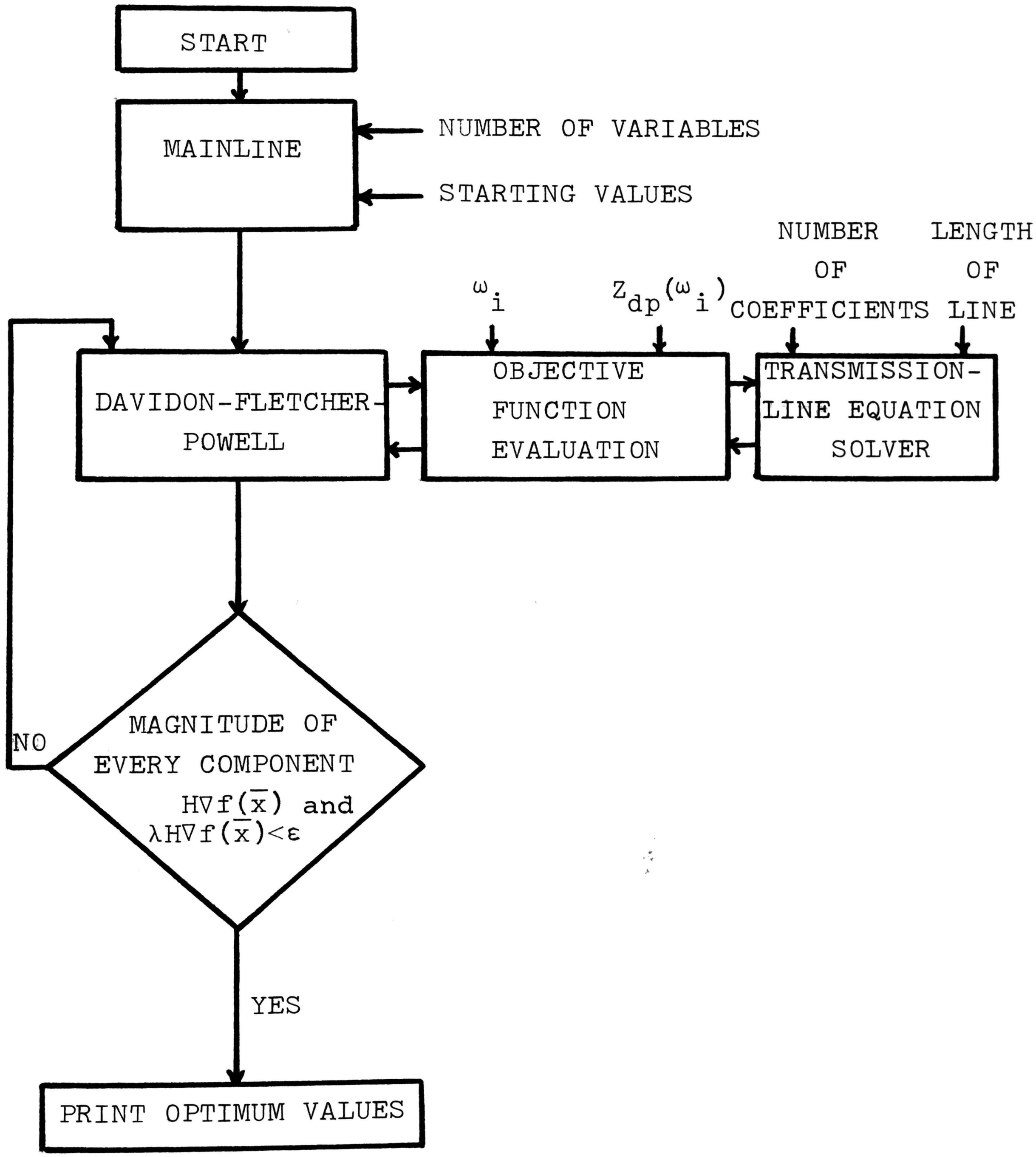


Figure 4.1. Flow Chart of Taper Identification Program

is the gradient of f, and λ is a constant chosen by quadratic interpolation during each iteration to minimize $f(\bar{x}-\lambda H\nabla f(\bar{x}))$. $H\nabla f(\bar{x})$ is referred to by [18] as a directional vector and $\lambda H\nabla f(\bar{x})$ is a vector containing the step-size information. For this specific program, each component of both vectors must be less than 10^{-6} before termination occurs.

3. Results

Program results presented in Table I are discussed in this section. The first column refers to the source of the data. Two distributed RC networks (Figure 4.2 and Figure 4.3) were available for testing from which actual laboratory data were obtained. Magnitude and phase information was measured from analog outputs of an impedance bridge for one decade in frequency. The generated data was simulated by programming the given network response for various tapers over the given range. For the first case, the taper used to generate a short-circuit transfer admittance was $r(x) = e^{2x}$ and $c(x) = e^{-2x}$. For the short-circuit current-transfer function, the taper $r(x) = e^{x}$ and $c(x) = e^{-x}$ was chosen. Finally, for the open-circuit voltage transfer function the taper was $c(x) = \cos^{2} 5\pi x$ and $r(x) = 1/c(x)$. The second column of Table I refers to the type of data used. Driving-point data can be impedance or admittance, and the transfer data can be a voltage ratio, current ratio, impedance, or admittance. The program is

Table I. Identification Program Results

SOURCE OF DATA	TYPE OF DATA	FREQUENCY RANGE	NUMBER OF DATA POINTS	PHYSICAL LENGTH	COEFFICIENTS IDENTIFIED
MEASURED	OPEN-CKT. D.P. IMPEDANCE	100-1kHz	21	.184m	$r(x)=[16.19+(x-.092)]10^{4}$ $c(x)=[1.51+.162(x-.092)]10^{-7}$
MEASURED	OPEN-CKT. D.P. IMPEDANCE	100-1kHz	21	.512m	$r(x)=[3.9+.999(x-.206)]10^{5}$ $c(x)=[3.09+1.00(x-.206)]10^{-7}$
GENERATED	SHORT-CKT. TRANSFER ADMITTANCE	16-130Hz	10	.1	$r(x)=1.1069+1.9949(x-.05)$ $+.96752(x-.05)^{2}+.9991(x-.05)^{3}$ $c(x)=e^{-2x}$=GIVEN
GENERATED	SHORT-CKT. CURRENT TRANSFER	16-1kHz	10	.1	$r(x)=e^{x}$=GIVEN $c(x)=.95214-.9982(x-.05)$ $-2.5908(x-.05)^{2}+1.0648(x-.05)^{3}$
GENERATED	OPEN-CKT. VOLTAGE TRANSFER	2.5-20Hz	10	.09	$r(x)=1/c(x)$ $c(x)=.49977+.50031\cos(31.416x-.0021531)$

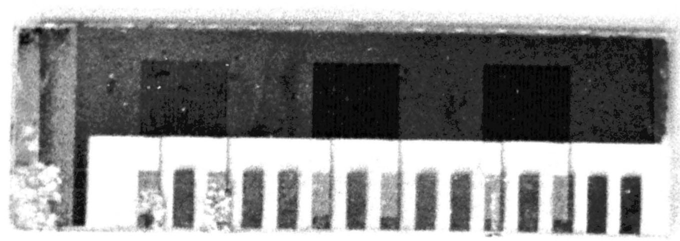


Figure 4.2. Distributed RC Network
Fabricated at Bell Telephone Laboratory

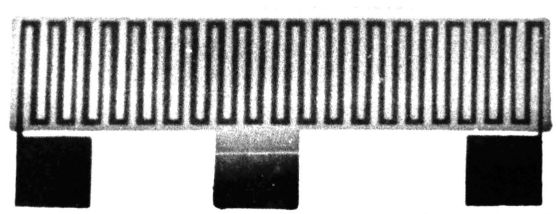


Figure 4.3. Distributed RC Network
Fabricated at University of Missouri - Rolla

written to accept data as an ordered pair, (Real, Imaginary). The following three columns contain information concerning the frequency range tested, the number of points sampled, and the physical length of the network. The frequency range of the data was chosen such that the real part of the data was within an order of magnitude of the imaginary part, with data points spaced logrithmically. All data was generated by use of the [4] parameters given in [1,p.120]. The last column displays the type of expansion as well as the optimized coefficients. The program can be run in several modes depending on the identification requirement. Coefficients for both r(x) and c(x) can be found, r(x) can be given and coefficient for c(x) can be found, or c(x) can be given and coefficients of r(x) can be found.

It should be noted that in the measured data the r(x)c(x) product was not constrained to be constant. The reason for relaxing this constraint was to allow for a more accurate model. Since all of the hypotheses of the sufficiency Theorem 2.1 were not satisfied, a single global minimum could not necessarily be expected; therefore, several starting values were utilized to insure the given results represented the physical network.

The optimum coefficients for r(x) in the short-circuit admittance transfer case are given in Table I. The percentage deviation from the desired r(x), shown in Figure 4.4, is always within ± 1.33 per cent of the desired r(x). The short-

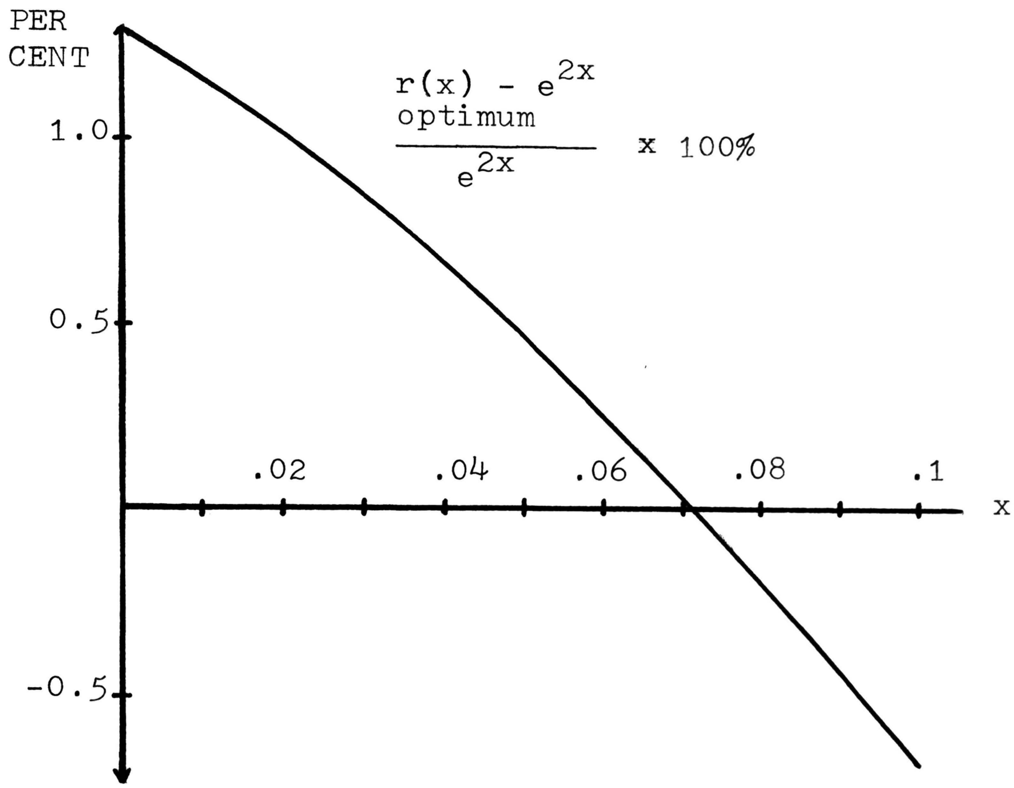


Figure 4.4. Per Cent Deviation of Optimum r(x) From $r(x) = e^{2x}$ vs. Distance

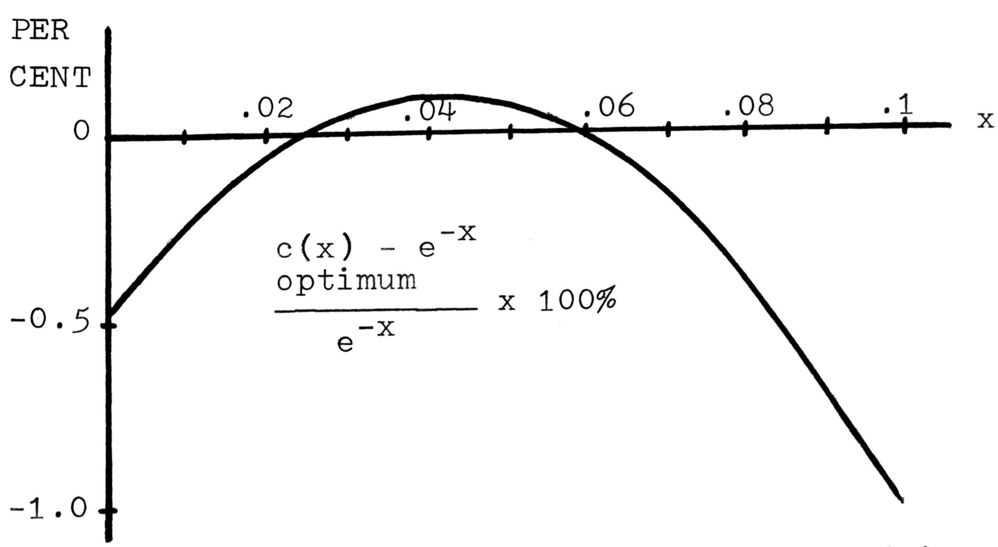


Figure 4.5. Per Cent Deviation of Optimum c(x) From $c(x) = e^{-x}$ vs. Distance

circuit current-transfer case provides a similar result. The percentage deviation from the desired c(x), shown in Figure 4.5, is always within ±.9 per cent of the desired c(x). The slightly larger deviation of the r(x) function as compared to the c(x) function is attributed to the larger variation of $r(x) = e^{2x}$ as compared to $c(x) = e^{-x}$ over the distance 0 to 0.1. A polynomial was chosen for the tapers for investigative reasons. Naturally, the deviation can be reduced by choosing an approximating function of the same form as the original taper as shown in the open-circuit voltage-transfer case.

For the open-circuit voltage-transfer case, the generated data was that of $c(x) = \cos^2(5\pi x)$ and $r(x) = 1/c(x)$. The approximating function chosen was $c(x) = c_1 + c_2 \cos(c_3 x + c_4)$, $r(x) = 1/c(x)$. Note that this function is capable of exactly reproducing the initial taper. Two sets of data were used to examine the effects of data accuracy on the routine. The initial set was data of seven-digit accuracy. The second set of data was formed by rounding the first set of data to two digits. The exact c(x) and the two optimum functions are given below.

$$c(x) = 0.5 + 0.5 \cos(10\pi x) \tag{4.6}$$

Generated

$$c(x) = 0.49977 + 0.50031 \cos(31.416x - .0021531) \tag{4.7}$$

7 Digit

$$c(x) = 0.51058 + 0.49952 \cos(31.422x + .057268) \tag{4.8}$$

2 Digit

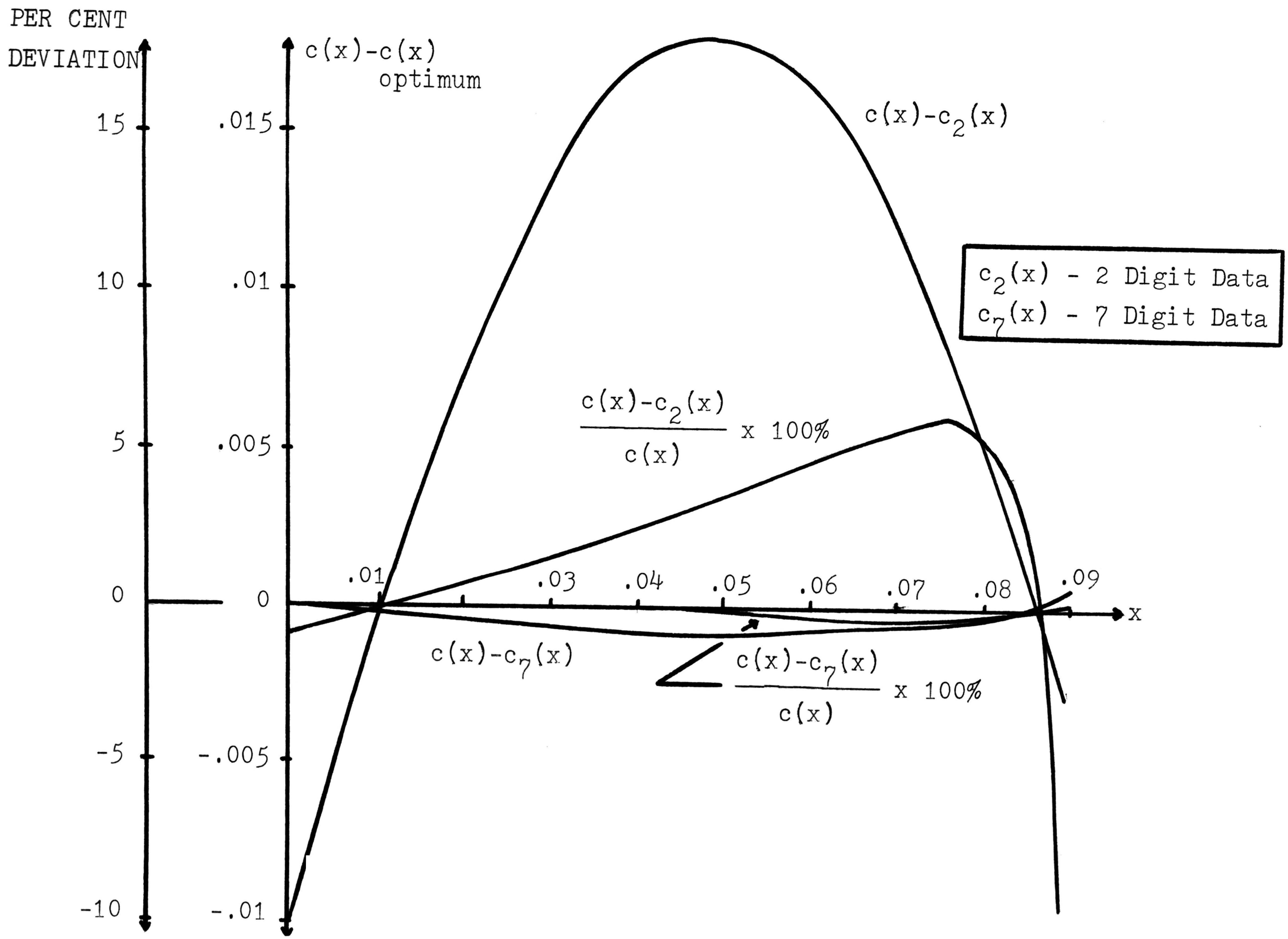


Figure 4.6. Percentage Deviation and Actual Error of Capacitance For Two Degrees of Data Accuracy

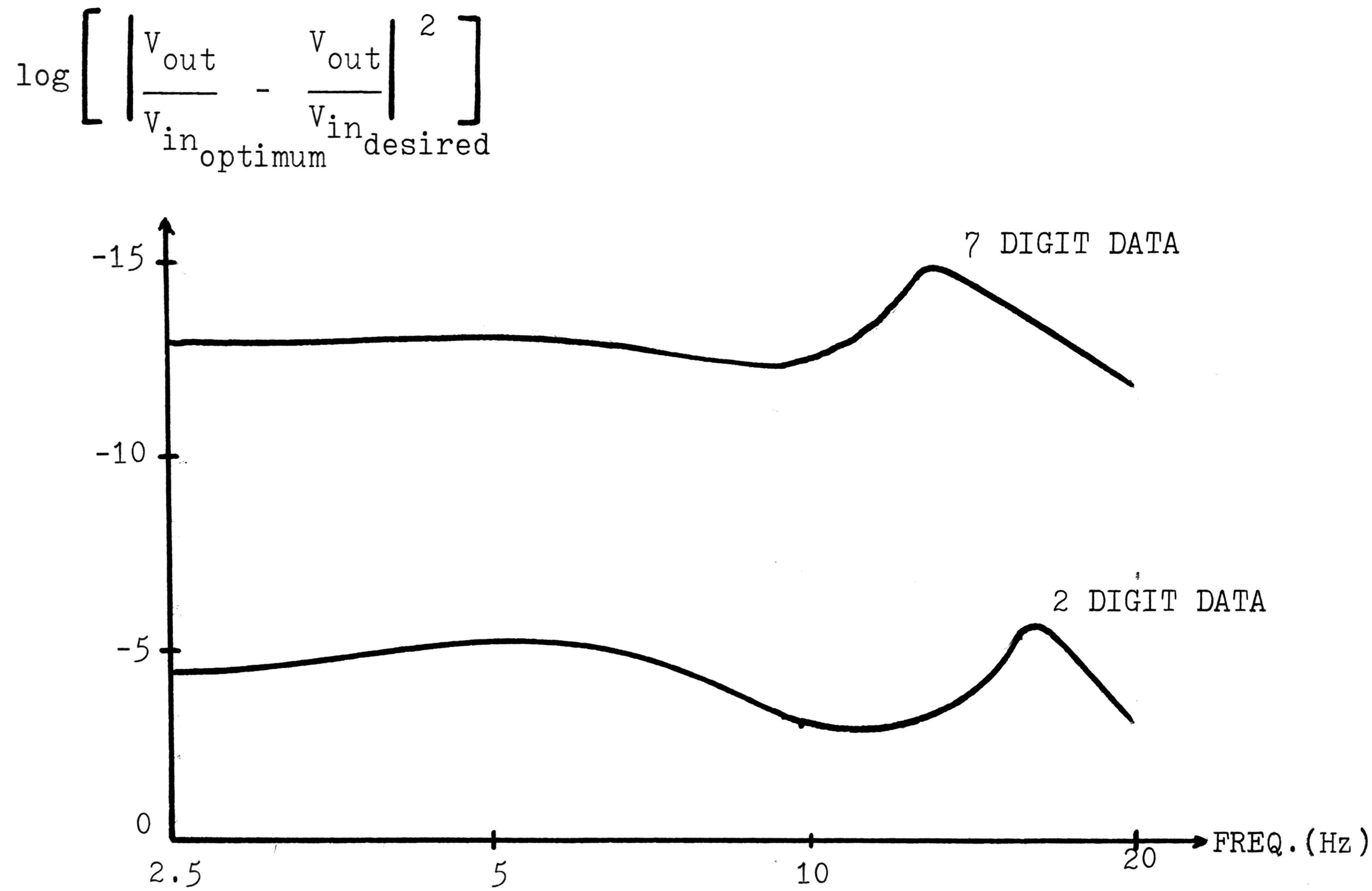


Figure 4.7. Objective Function Error for Two Degrees of Data Accuracy

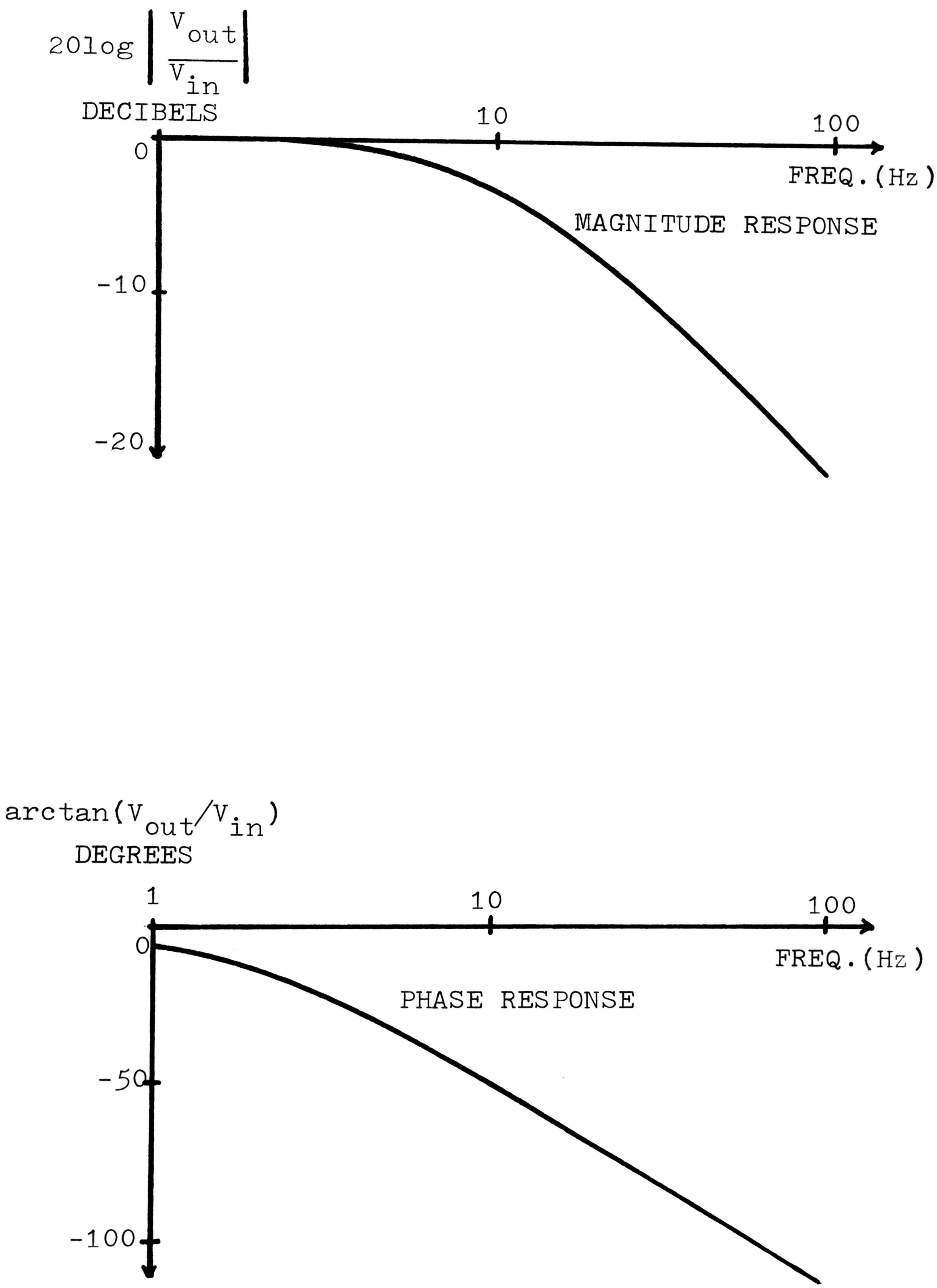


Figure 4.8. Frequency Responses of Three Tapers Whose Capacitances are (4.6) - (4.8)

The results of these two examples are displayed in Figure 4.6 and Figure 4.7. As can be seen from Figure 4.6, by increasing the data accuracy from two digits to seven digits, the maximum percentage deviation is reduced from 12 per cent to .8 per cent and the maximum actual error is reduced from .018 to .0089. A significant change occurred in the objective function errors.

As shown in Figure 4.7, by increasing data accuracy by 5 digits, the objective function error was reduced by approximately nine orders of magnitude. Figure 4.8 displays the conventional frequency characteristic of tapers generated by c(x)'s of (4.6) - (4.8). The three tapers have frequency responses which are coincident; this is consistent with the fact that frequency response is relatively insensitive to small variations in the taper [8,p.11]. Thus an increase in the accuracy of the data has produced a significant reduction in the per-unit capacitance. It is the conclusion of this section that, although frequency responses are relatively taper insensitive, if the data accuracy is sufficient a small taper variation can be detected.

Data scaling is required in most cases. Good results were obtained by scaling magnitudes to the range of one to ten and scaling the frequencies to the same range. All runs were made on an IBM 370-168 machine with compile and execution time running from 2.06 minutes to 3.8 minutes depending on the number of variables and starting values.

4. Conclusion

The purpose of the taper identification routine presented is to show the feasibility of such a method, rather than presenting a routine which optimally identifies any type of taper.

Although the Davidon-Fletcher-Powell routine was used as the search routine, similar results were obtained using a Hook-Jeeves pattern search routine in the small-number-of-variables cases. The Hook-Jeeves method becomes increasingly inefficient compared to the Davidon-Fletcher-Powell as the number of variables increases.

Although the routine is written for taper identification of distributed RC networks, the program is easily modified to handle any general RLGC line or any subset thereof.

CHAPTER V: CONCLUSIONS

A. Introduction

This concluding chapter contains information on how work in the preceding chapters may be extended to cover more general distributed networks; also included is a summary of the dissertation, as well as some promising areas for further research. More specifically, the work done on RC distributed networks is shown to be applicable to the GL and LC distributed lines also. Suggestions for further research include work in the areas of two-dimensional RC taper identification and the general RLGC distributed networks, as well as more general transform methods for the treatment of distributed networks.

B. Application to Analogous Two-element Distributed Networks

The results of the corollaries and theorems of previous chapters can also be applied to analogous two-element distributed networks with only minor modifications in the hypotheses and proofs. For example, consider the series inductance-shunt conductance case where the incremental model is given in Figure 5.1.

A second-order Laplace transformed equation relating the transformed voltage to the distance variable is given by

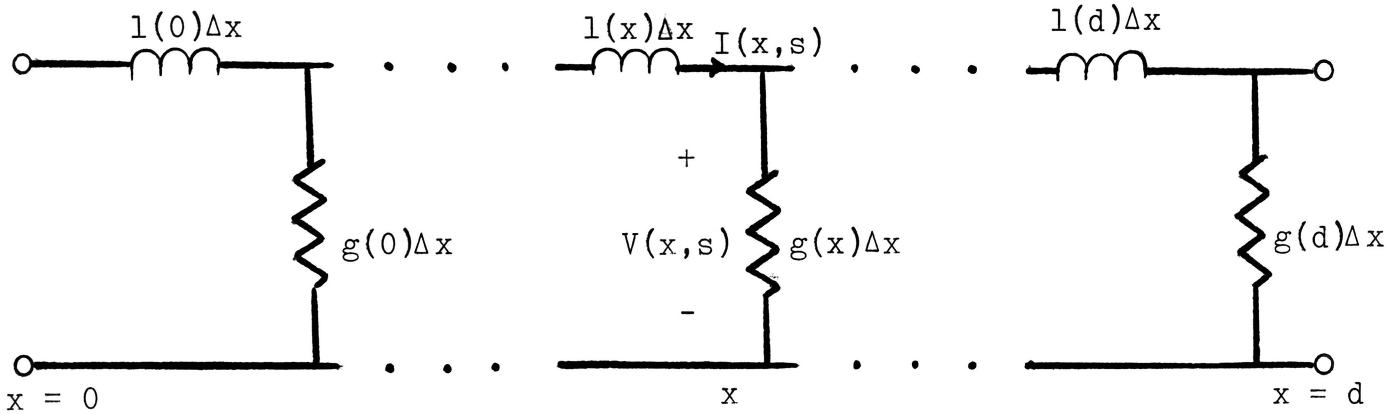


Figure 5.1. Incremental Model of an L-G line

$$\frac{d^2V(x,s)}{dx^2} - \frac{1}{l(x)}\frac{dl(x)}{dx}\frac{dV(x,s)}{dx} - sl(x)g(x)V(x,s) = 0 \quad (5.1)$$

Note the duality between (5.1) and (1.4b); the voltage is analogous to current, g(x) analogous to r(x), and l(x) analogous to c(x). This duality of lines implies that impedances of RC lines are of the same form as the admittances of LG lines. The reverse statement is also true. The networks do not possess identical chain parameters, because the critical frequencies nearest the origin differ; however, the forms are the same as established by duality.

The duality that exists between the RC and GL lines is the following:

A_{RC} ——— D_{LG}

B_{RC} ——— C_{LG}

C_{RC} ——— B_{LG}

D_{RC} ——— A_{LG}

where "———" is interpreted as "is the dual of".

The series-inductance shunt-capacitance distributed network may be treated by using appropriate transformations suggested by [9,p.164] to reduce the equation to an RC case. The transformations are

$$\sqrt{p} = s \tag{5.2}$$

$$\overline{I}(x,p) = sI(x,s) \tag{5.3}$$

Substitutions of (5.2) and (5.3) into the defining LC equations

$$\frac{dV(x,s)}{dx} = - sl(x)I(x,s) \tag{5.4}$$

$$\frac{dI(x,s)}{dx} = - sc(x)V(x,s) \tag{5.5}$$

yields two first-order differential equations of the form (1.2), the RC case.

Thus it has been shown that the governing equations of GL and LC distributed networks can be transformed to equations of the same form as the RC case, which has been investigated in this dissertation. Therefore, corollaries and theorems of Chapter II and III can be applied with only slight modifications to the hypotheses.

C. Summary

The purpose of this dissertation is to study the conditions under which RC distributed networks can be identified as well as to provide a practical means by which these networks can be identified. Taper identification is defined in terms of the knowledge of certain coefficients in the governing differential equations. Sufficiency conditions for taper identification are examined and found to be knowledge of $Z_{dp}(s)$, $r(x)c(x) = K$, and d (physical length). These results are generalized to the less restrictive sufficiency conditions: knowledge of R_T, C_T, $r(x)c(x) = K$, d, and one of the chain parameters. The actual identification is done with either a Hooke-Jeeves direct-search algorithm or a Fletcher-Powell modified gradient algorithm. Although a major portion of this dissertation is devoted to RC distributed networks, the previous section shows that the results of the paper also apply to the LG and LC cases as well.

D. Suggestions for Further Research

1. Two-Dimensional Taper Identification

If two-dimensional variations in r and c are allowed, then the defining equation relating voltage to the two-dimensional space-coordinate system in the time domain is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{1}{r}\frac{\partial r}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial v}{\partial y} = rc\frac{\partial v}{\partial t} \qquad (5.6)$$

where $v = v(x,y,t)$, $r = r(x,y)$, and $c = c(x,y)$ [1,p.242].

An equation such as (5.6) may be required when studying networks where the problem can not be cast into one of the four coordinate systems suggested by [1,p.247] which allows for a solution by separation of variables and one-dimensional current flow. Another case where an equation of the form of (5.6) may be required is in networks which have an abrupt taper transition without the utilization of an equipotential strip. In such cases one-dimensional current flow is no longer valid. A physical motivation for studying such an equation is to obtain a more accurate model of the network; or, from a fabrication point of view, predict more closely the device physically constructed.

The identification problem remains the same as in the one-dimensional case, that of identifying the taper. The solution of the two-dimensional identification problem requires more than modification of work presented in this dissertation. Beyond the modification of definition of taper, impedance, etc. for two-dimensional utilization, the question of the uniqueness of the equivalent of the $C = \sigma(R)$ function must be addressed. Little research has been done in this area; however, a good starting point for such work would be [12,pp.39-50]. The problem treated there is not that of (5.6), but of the heat equation.

2. Taper Identification of a General RLGC Distributed Network

The motivation for the identification problem associated with the general RLGC distributed network again lies in the modeling process. [4] addressed the problem of synthesis of a general line but did not consider the uniqueness properties required in an identification problem.

The uniqueness property associated with a general RLGC line is closely related to the inverse Sturm-Liouville problem. The following is the defining equation for a general RLGC distributed network in the transformed domain [1,p.28]:

$$\frac{d^2V(x,s)}{dx^2} - \frac{1}{r(x)+sl(x)} \frac{d}{dx}[r(x)+sl(x)] \frac{dV(x,s)}{dx} - [g(x)+sc(x)][r(x)+sl(x)]\, V(x,s) = 0 \qquad (5.7)$$

(5.7) can be simplified using the following transformation

$$Z = \int_0^x [r(y) + sl(y)]dy \qquad (5.8)$$

$$Y = \int_0^x [g(y) + sc(y)]dy \qquad (5.9)$$

(5,8) and (5.9) transform (5.7) into

$$\frac{d^2V}{dZ^2} - \frac{dY}{dZ} V = 0 \tag{5.10}$$

where

$$\frac{dY}{dZ} = \frac{g(x) + sc(x)}{r(x) + sl(x)} \tag{5.11}$$

[13,pp.424-436] and [14] have considered the problem

$$\frac{d^2y}{dx^2} - q(x)y + \lambda^2 y = 0 \tag{5.12a}$$

with boundary conditions

$$\frac{d}{dx} y(o) - hy(o) = 0 \text{ and } \frac{dy(1)}{dx} + h_1 y(1) = 0 \tag{5.12b}$$

where x is a real variable and g(x), h, and h_1 can be complex. The set of values of λ for which (5.12) has a nontrival solution is called the spectrum of the operator $L[y] = \frac{d^2y}{dx^2} - q(x)y$. The inverse Sturm-Liouville problem is that of finding L[y] given two spectra corresponding to different boundary conditions. From the identification viewpoint, this requires determining q(x) from these spectra. For the RC case the work by [14] provides for the uniqueness of $\sigma'(R)$ in the following equation:

$$\frac{d^2V}{dR^2} - s\sigma'(R)V = 0 \qquad (5.13)$$

For the general RLGC line, the transformed equation is (5.10). The major difference between (5.10) and (5.13) is that the independent variable is a complex variable. Thus the work to be done in this area is to extend the work by [14] to complex differential equations of a complex variable.

3. More General Transform Techniques

The defining partial differential equations for a distributed RC line given in Chapter 1 are functions of two variables, x and t. When Laplace-transformed with respect to t, these equations become ordinary differential equations in x. If the coefficients r(x) and c(x) are specified, then, in general, the equations can be solved numerically [15], [16], and for a small class of tapers, closed-form solutions are possible. Another approach, if r(x) and c(x) are specified, is to take a second Laplace transform with respect to x. The second transform may or may not yield an algebraic equation. The work to be done in this area would be to find another transform which would convert the ordinary differential equation in x into an algebraic equation. Although this might not be possible for a general taper, series expansions, such as Taylor or Fourier, for $r(x)$ and $c(x)$ might be fruitful. Associated with this problem is the inverse transform. Obviously, the inverse transform must exist and be tractable for the method to succeed.

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VITA

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APPENDICES

APPENDIX A: PROOF OF THE DIFFERENTIABILITY OF (2.3) AND (2.4)

Let $r_1(x_1)$ and $r_2(x_2)$ be positive continuous function, and let

$$\alpha = \int_o^{x_1} r_1(y)dy = \int_o^{x_2} r_2(y)dy \tag{A.1}$$

The inverse functions associated with (A.1) are guaranteed to exist and are defined as follows:

$$x_1 = f_1(\alpha) \tag{A.2}$$

$$x_2 = f_2(\alpha) \tag{A.3}$$

By differentiating (A.2) and (A.3) with respect to α, the following may be formed:

$$\frac{dx_2}{d\alpha} \bigg/ \frac{dx_1}{d\alpha} = dx_2/dx_1 \tag{A.4}$$

(A.4) exists because (A.2) and (A.3) are differentiable and do not take on values of 0 or ∞ . It should be noted that if $r(x)$ is sectionally continuous (A.4) does not necessarily exist. In a practical case where the taper is not continuous, such as a pair of cascaded lines, the connection made between the lines is an equipotential conductive strip. An external connection is easily made to the strip so that the cascaded lines may be treated as two separate lines. Any loading problems may be surmounted by a short circuit.

APPENDIX B: INTERCHANGE OF OPERATIONS IN EQUATION (3.15)

Uniform convergence of the differentiated sum allows the interchange of the operation of differentiation and summation. The approach is to apply the Weierstrass M-test to (3.15).

Rewriting (3.15) yields

$$\frac{\partial(R_T C_T)}{\partial \beta_j} = \frac{1}{\beta_j} \sum_{i=1}^{\infty} \frac{1}{\alpha_i A'(-\alpha_i) B_1(-\alpha_i)(\alpha_i - \beta_j)} \tag{B.1}$$

Consider the k^{th} term of the sum (B.1)

$$\frac{1}{\alpha_k A'(-\alpha_k) B_1(-\alpha_k)(\alpha_k - \beta_j)} < \frac{1}{\alpha_k A'(-\alpha_k) B_1(-\alpha_k)(\alpha_k - \alpha_{j+1})} = N_k \tag{B.2}$$

for $k > j + 1$.

Rewriting the k^{th} term of (3.7) yields

$$\frac{1}{\alpha_k^2 A'(-\alpha_k) B_1(-\alpha_k)} = M_k \tag{B.3}$$

The α_k have no upper bound.

Invoking the limit form of the comparison test to (B.2) and (B.3) yields

$$\lim_{k \to \infty} \frac{M_k}{N_k} = \lim_{k \to \infty} \frac{\alpha_k - \alpha_{j+1}}{\alpha_k} = 1$$

Thus ΣN_k converges; therefore, (B.1) converges uniformly by the Weierstrass M-test.

APPENDIX C: PROOF OF EQUATION (3.18)

The purpose of this appendix is to show that (3.17) can be expanded in the partial-fraction expansion shown in (3.18) by use of the Mittag-Leffler theorem. The approach is to prove that (3.17) is uniformly bounded on a set of contours where no contour passes through a pole of (3.17).

Let the contours be the circles, C_n, such that

$$s_n = \beta_n e^{i\theta} \qquad \text{(C.1)}$$

where β_n is a zero of (3.17) and $0 \leq \theta < 2\pi$.

It is sufficient to show that

$$\left| \prod_{i=1}^{\infty} \frac{(1 + s_n/\beta_i)}{(1 + s_n/\alpha_i)} \right| = \left| \prod_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} \frac{(s_n + \beta_i)}{(s_n + \alpha_i)} \right| < M \qquad \text{(C.2)}$$

where M is independent of n.

The magnitude of each factor will now be shown to be less than one. Consider the following factor:

$$\left| \frac{s_n + \beta_i}{s_n + \alpha_i} \right| \qquad \text{(C.3)}$$

By expanding s_n using Euler's relationship and substituting into (C.3), the extrema of (C.3) can be found by differentiation with respect to θ. The extrema occur on the real axis, minima at $\theta = \pi$ and maxima at $\theta = 0$. Thus it is sufficient to consider (C.3) at its maximum. Evaluating (C.3) at $\theta = 0$ yields

$$\frac{\beta_n + \beta_i}{\beta_n + \alpha_i} \tag{C.4}$$

It is easily shown that

$$\frac{\alpha_i}{\beta_i} \frac{\beta_n + \beta_i}{\beta_n + \alpha_i} < 1 \tag{C.5}$$

therefore

$$\left| \prod_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} \frac{(s_n + \beta_i)}{(s_n + \alpha_i)} \right| < 1$$

thus satisfying the hypothesis of the Mittag-Leffler theorem [11,p.159] and justifying the following expansion:

$$\prod_{i=1}^{\infty} \frac{(1 + s/\beta_i)}{(1 + s/\alpha_i)} = 1 + \sum_{i=1}^{\infty} b_i \left[\frac{1}{s + \alpha_i} - \frac{1}{\alpha_i} \right] \tag{C.6}$$

where

$$b_i = \frac{B_1(-\alpha_i)}{A'(-\alpha_i)}$$

If the infinite sums

$$\sum_{i=1}^{\infty} \frac{b_i}{s + \alpha_i} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{b_i}{\alpha_i}$$

converge then (C.6) can be simplified. The b_i's are bounded; therefore, $\sum_{i=1}^{\infty} \frac{b_i}{\alpha_i}$ converges and further, $\sum_{i=1}^{\infty} \frac{b_i}{s + \alpha_i}$ converges by the comparison test. Thus (C.6) simplifies to

$$\prod_{i=1}^{\infty} \frac{(1 + s/\beta_i)}{(1 + s/\alpha_i)} = \sum_{i=1}^{\infty} \frac{b_i}{s + \alpha_i} = \sum_{i=1}^{\infty} \frac{B_1(-\alpha_i)}{A'(-\alpha_i)(s + \alpha_i)} \tag{C.7}$$

APPENDIX D: JUSTIFICATION OF $\lambda(s) = \lambda_o$

By (3.16) and (3.18)

$$\sum_{i=1}^{\infty} \frac{1}{\alpha_i A'(-\alpha_i) B_1(-\alpha_i)(s+\alpha_i)} = \lambda(s) \sum_{i=1}^{\infty} \frac{B_1(-\alpha_i)}{A'(-\alpha_i)(s+\alpha_i)} \tag{D.1}$$

where $\lambda(s)$ is an entire function with no zeros in the finite complex plane. Comparing residues of like poles of (D.1) yields

$$\frac{1}{\alpha_i B_1(-\alpha_i)} = \lambda(s) B_1(-\alpha_i) \qquad i = 1,2.... \tag{D.2}$$

Solving for $\lambda(s)$ yields

$$\lambda(s) = \frac{1}{\alpha_i B_1^{\,2}(-\alpha_i)} \tag{D.3}$$

therefore

$$\lambda(s) = \lambda_o = \frac{1}{\alpha_i B_1^{\,2}(-\alpha_i)}$$