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Oscillation Theory for Second Order Dynamic Equations [Book Review]

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Book Reviews

Edited by Robert E. O'Malley, Jr.

Featured Review: Applied Mathematics: Body and Soul. Volumes 1–3. *By K. Eriksson, D. Estep, and C. Johnson.* Springer-Verlag, Berlin, 2004. $149.85 ($49.95/volume). xliv+1213 pp., hardcover. ISBN 3-540-00890-X; 3-540-00889-6; 3-540-00891-8.

Prologue. One of the attractions for visitors to Moscow is the gravestone of Nikita Khrushchev in the Novedevichy cemetery. It is composed of very white and very black sculpted marble, which are supposed to represent the brilliant and the dark sides of his character and his achievements. Similarly, the subsequent review will be in two parts.

The First Project. The original title of the underlying project was *Engineering Mathematics 2000*, and this title expresses precisely the contents and style of the treatise: it grew out of lectures of mathematics at Chalmers Technical University in Göteborg, Sweden, addressed to students of chemical engineering, who "enthusiastically participated in the development of the reform project of this book." And as such, the book deserves a lot of white marble: it covers *analysis* from the introduction of numbers to multiple integrals and Stokes' theorem, the basic notation and results of *linear algebra*, and some techniques of *numerical analysis* (bisection algorithm, Newton's method, ordinary differential equations, optimization strategies) and culminates in an explanation of the solution of partial differential equations by the finite element method. The elaboration of many interesting examples, often taken from concrete applications, and many historical notes (the most beautiful being the account on Reynolds' teaching style on page 119) bring a lot of motivation for students. Many new concepts to be introduced are very carefully prepared for with numerous figures, numerical examples, and situations from real life (paying taxes, driving a car, etc.) and make many parts of the book very readable and easily understandable. The project is accompanied by a homepage: http://www.phi.chalmers.se/bodysoul/.

The Actual Treatise. But now the work comes with a more pretentious title, *Applied Mathematics: Body and Soul*, in three volumes, totaling 1213 pages, 87 chapters, and 839 subsections. Each volume contains 26 pages of table of contents for all volumes and a preface of 9 pages announcing "The Need of Reform of Mathematics Education," "...as we now pass into the new millennium," and telling us that the "program is based on a synthesis of mathematics" (what is meant by "soul") and "computation and application" (what is meant by "body"), and that it "is based on new literature ... giving the student a solid understanding of basic mathematical concepts such as real numbers, Cauchy sequences, Lipschitz continuity and constructive tools for solving algebraic/differential equations, together with an ability to utilize these tools in advanced applications such as molecular dynamics."

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *University City Science Center, Philadelphia, PA* 19104-2688.

Such an ambitious program is now addressed to a much wider class of students "in engineering and science," which may have higher requirements of mathematical precision—and call for a more critical referee. After closer inspection, we discover a lot of "black marble."

The "new literature," on which the book is said to be based, is never cited, and the book is full of phrases like "one can prove that the number of iterations..." (p. 675), "It can be shown that..." (p. 538), "... is known *not* to be an elementary function..." (p. 521), and "One can prove..." (p. 518). None of these statements is accompanied either by a proof or by a hint, nor by a reference. And *if* occasionally other "Calculus text books" are mentioned—"in your nearest library or on your book shelf" with their "pathological side effects," "... quite meaningless circular reasoning, and some Calculus books completely fall into this trap" (p. 436)—it is then just to criticize them.

Volume I. Already in the first chapters of Volume 1 (*Derivatives and Geometry in* R^3) we find many inaccuracies: we read in Chapter 5 that by an argument like

$$(1+1)+(1+1+1) = 1+1+1+1+1 = (1+1+1)+(1+1)$$

the commutative rule for addition $m+n = n+m$ can be proved. However, this line just proves that $2+3 = 3+2$; the general formula contains an infinity of statements which must either be taken as an axiom or proved by induction, say, from Peano's system of axioms. But the authors think that "the question is if we get a more clear idea of the natural numbers from the Peano axiom system than from our intuitive feeling" (p. 229).

In Chapter 8, on Euclid, the proof of Pythagoras' theorem on page 88 uses the properties of similar triangles, which are explained on page 91 ("Euclid says ...") without proof, which is left as exercise 8.4, while on page 99 we read that Euclid needed six books (!) to arrive at a rigorous proof of this result. Cartesian coordinates are introduced on page 100, a second time on page 268, but are already used on page 93 for obtaining a "magic formula" for orthogonality which is proved again on page 281. What is declared to be "Euclid's famous fifth axiom" is actually Euclid's Proposition I.29. "Euclidean geometry" is declared to concern "plane geometry" (as if the hyperbolic plane would not be flat) and "non-Euclidean geometry" would be a geometry "in which parallel lines *can* meet," which contradicts the definition of parallels. Finally the statement "For Euclid, the diagonal of a square was just a geometric entity" (p. 99) ignores the fact that two entire books of Euclid (books V and X) develop complete theories of irrational numbers not very different from Dedekind cuts.

In Chapter 12 we encounter a major *credo* of the book, namely, to banish all usual definitions of continuous and uniformly continuous functions "(attractive to many pure mathematicians), but suffer[ing] from an (often confusing) use of limits" (p. 211). All this is put into one sack by considering only the uniform Lipschitz continuity over an interval I,

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \quad \text{for all } x_1, x_2 \in I,$$

and *all* subsequent results of the whole book are constantly formulated in terms of this definition ("Lipschitz continuity gives a quantitative precise formulation, while the connection in the standard definition is more farfetched. Right?"). This simplifies perhaps some of the proofs, but also lowers the value of the whole treatise.

In Chapter 13, on limits, we learn that "the risk is thus that using the $\epsilon - N$-jargon, we may get confused and believe that something vague, in fact is very precise.

So be cautious and don't get fooled by simple tricks: the $\epsilon - N$ definition is vague to the extent the dependence of N on ϵ is vague." A very strange characterization of Weierstrass' analysis.

With Chapter 15 we arrive at the real numbers. This book explains them as the set of infinite decimal expansions. It is true that this approach is very attractive at first sight, and many theorems—for example, the convergence theorem of Cauchy sequences—become nearly evident. But the price to pay is that algebraic operations (and the proof of their laws) are not at all trivial. Another difficulty is that decimal expansions are not free of traps. For example, the statement on page 190 that "when $|x_i - x_j| \leq 10^{-N-1}$, then the first N decimals of x_j are the same as the first N decimals in x_i" is wrong (counterexample: $x_i = 1.00002$, $x_j = 0.99998$, and $N = 3$). The extension of functions to irrational values of x leads the authors, being so attached to decimal expansions, to the conclusion that "if some association of x-values to values $f(x)$ is not Lipschitz continuous, this association should not deserve to be called a function. We are thus led to the conclusion that *all functions are Lipschitz continuous* (more or less). This statement would be shocking to many mathematicians...." Indeed, it is! In Chapter 17 it is then declared (on page 230): "We have defined R as the set of all possible infinite decimal expansions.... We may say that we use a constructivist/intuitionist definition of R. The formalist/logistic would rather like to define R as the set of all infinite decimal expansions ... in what we called a universal Big Brother style above." This distinction is difficult to follow.

The definition of a *differentiable* function in Chapter 23 introduces the requirement that the error term should satisfy an estimation $|E_f(x, \overline{x})| \leq K_f(\overline{x})|x - \overline{x}|^2$, which is much stronger than the usual definition in mathematics and "which pleases many mathematicians because of its maximal generality," and "using a more demanding definition we focus on normality rather than the extreme or degenerate." This definition does *not* simplify the proofs of the differentiation rules in Chapter 24, however, because $K_f(\overline{x})|x - \overline{x}|$ is a more complicated expression than ϵ. Furthermore, this is a curious standpoint in a book which takes two pages (pp. 14–15) to emphasize "Languages" and "Mathematics as the Language of Science." Students who might sometime want to read *other* mathematical literature will have to adapt their vocabulary. But the first confusion arrives already in *this* work, in Volume 3, section 83.13, where Weierstrass' famous "nowhere differentiable function"

$$f(x) = \sum_{n=1}^{\infty} a^{-n} \sin(b^n x), \qquad a > 1, \quad b > a,$$

is presented. First, the authors boldly replace Weierstrass' "continuous" by "Lipschitz continuous" and turn a true statement into a wrong one; second, it is stated that "the series obtained by termwise differentiation does not converge, which indicates that $f(x)$ is nowhere differentiable" (a wrong conclusion—counterexample: the series $\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots$, when differentiated term by term, does not converge, which does not hinder its limit function from being infinitely differentiable); third, it is not said for *which* definition this function is nowhere "differentiable." Furthermore, Weierstrass' condition $a^{-1}b > 1 + 3\pi/2$ is not mentioned and is not satisfied by the function drawn in Figure 83.7.

Chapter 26 prepares for the main tool of Volume 2, namely, various differential equations, here nicely motivated with practical problems: Newton's and Galileo's laws of motion, Hooke's law, Newton's (not Fourier's!) law for heat flow, and population growth.

Volume 2. Volume 2 (*Integrals and Geometry in* R^n) starts by treating *integrals* as the solution of the differential equation $u'(x) = f(x)$. The whole theory is formulated only for Lipschitz continuous functions, so that, for example, the mean value theorem on page 464 receives a very strange formulation: "If $u(x)$ is uniformly differentiable on $[a, b]$ with Lipschitz continuous derivative $u'(x)$, then...." Later the exponential function is defined by $u'(x) = u(x)$, and trigonometric functions by $u''(x)+u(x) = 0$. Accordingly, we assist four times at always the same existence proof by Euler polygons (pp. 438–441, 493–498, 563–565, 579–580). All proofs use constant stepsize sequences $h_n = 2^{-n}$; the fact that arbitrary step-size sequences converge to the same result is mentioned in only one of these cases (on page 483). The last of these proofs, for systems, uses vector notation, which, however, is introduced only two chapters later. To sum up, the trigonometric functions are "defined" on page 505 by a definition which is based on a proof on page 580, which in turn uses notation introduced on page 600, in order to verify the trigonometric identities (on page 509), which are already known from page 97.

Chapter 30 on *numerical quadrature* considers only Riemann sum approximations and the midpoint rule, both of which were already used by Archimedes. It thus brings us back to pre-Newtonian times. Not even Simpson's rule is mentioned.

Deliberately poor is also Chapter 37 on *series*, because "there are limitations to both Fourier and power series and the role of such series is today largely being taken over by computational methods. We therefore do not go into any excessive treatment of series." Absolutely convergent series are introduced, but their fundamental property (independence of reordering) is not mentioned. Leibniz's criterion is proved and applied to the alternating harmonic series, but the (quite interesting) fact that this sum is $\ln 2$ is not mentioned, nor is Leibniz's (still more interesting) series for $\frac{\pi}{4}$ mentioned.

Contrary to the pretention of the book, the treatment of *linear algebra* in Chapters 42, 43, and the beginning of Chapter 44 is very "pure mathematics." Householder reflections, the most important numerical tool, are never mentioned and the QR-decomposition is obtained from the Gram–Schmidt orthogonalization, a numerically unstable algorithm.

The *method of steepest descent* for minimization algorithms is explained in three different places (pp. 662, 802, and 878) without cross-referencing. The drawing on page 664 is wrong: the iteration points overshoot the optimal α values, perhaps to make the zig-zag more drastic.

Towards the end of Volume 2, we observe a dramatic increase in difficulty of the subjects. Differential equations of mechanical systems are derived from Lagrange's variational principle, which is explained by many examples but not justified. Then come reaction-diffusion-convection initial-boundary value problems, whose stationary versions are solved by Galerkin's finite element method. Many of these equations are just written down (on pp. 774–775), without any explanation and without references.

Volume 3. Volume 3 (*Calculus in Several Dimensions*) starts by defining the differentiability of functions of *several* variables. As we may expect, the definition requires a quadratic error estimation $\|E_f(x, \overline{x})\| \leq K_f(\overline{x})\|x - \overline{x}\|^2$. It is then shown that the existence of the partial derivatives is necessary. However, in none of the subsequent examples is the original criterion for differentiability ever checked. If the mean value theorem is then written in the form $f(x) - f(\overline{x}) = f'(y)(x - \overline{x})$, it is (rightly) mentioned that the y value "may be different for different rows of $f'(y)$," but it is *not* mentioned that this precaution is not necessary in the subsequent norm-estimation ("We may then estimate") if it is cleverly done.

Further theorems are the *inverse function theorem* (p. 810), the *implicit function theorem* (p. 811), and a theorem on the existence of *level curves* (p. 817). Each of these theorems concludes with a statement saying that "the equation ... has a unique solution." As was pointed out by an anonymous referee already four years ago, this is not the correct formulation. A counterexample follows right after Theorem 55.1, because the equation $x^2 + y^2 = 1$ has, of course, for fixed values of y with $|y| < 1$, *two* solutions for x.

Chapter 56 is on the *stability of initial value problems* and has again very sloppy formulations: it is said that "if all eigenvalues $\lambda_i \leq 0$ then ... we say that the solution $\overline{u}$ is stable" (p. 826). It has been known since Poincaré that the limiting case $\Re\lambda_i = 0$ is dangerous; the easiest counterexample is $\dot{u} = u^3,\ u(0) = 0$.

Chapter 57, on the solution of *IVPs for ordinary differential equations*, covers a subject in which the reviewer is a specialist at heart. This chapter is entirely written in the "Galerkin-jargon." As the main workhorse for all applications is proposed the cG(1) method of order 2, which when discretized becomes the implicit midpoint rule. The quite interesting question, how the corresponding implicit equation should *best* be solved, say, for nonstiff problems, is not clearly discussed. We also learn the existence of dG(0), which corresponds to the implicit Euler method. Both methods appear already in Cauchy's work from 1824. The entire development of high quality methods by Adams, Runge, or Kutta is not mentioned. At least G. Dahlquist, a Swedish mathematician, could have been honored. For *stiff* problems, the authors discuss in detail and with many examples the explicit Euler method combined with some small stabilizing Euler steps (section 57.8). Thus this chapter leaves the impression that *nonstiff* problems should be solved by an *implicit* method and that *stiff* problems should be attacked by an *explicit* method—the world upside-down.

Chapter 64 treats *double integrals*. The access of Chapter 27, by solving a differential equation $u' = f$, is here useless and is quickly forgotten (on p. 448: "The Fundamental Theorem of Calculus states that the integral of $f(x)$ over the interval $[a, b]$ is equal to a limit of Riemann sums..."; on p. 912: "Recall that we define the one dimensional integral as $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^N f(x_i^n)h_n \ldots$"). The chapter culminates in a proof of the formula for the change of variables in a double integral. In the middle of this proof we read, "If $g'(y)$ were constant over $\tilde{\Omega}_i$, and so $g(y)$ were linear on $\tilde{\Omega}_i$, then ..." and the proof then finishes after some lines. Nowhere is it remarked what happens if this "If" is *not* satisfied. It is true that a rigorous proof of this formula is a nasty undertaking and that the presented explanation is the best way to explain the formula, but it is not a proof.

The subsequent chapters treat surface integrals, multiple integrals, and Gauss' and Stokes' theorems and enter the kingdom of partial differential equations and their numerical solution by the finite element method, which is the true specialty of the authors. This part is also very well supported by numerical software from the Internet homepage.

Chapter 82, on *analytic functions*, starts with a nice explanation of complex differentiability, characterized locally by translation, rotation, and change of modulus. Then comes the following statement (p. 1108): "We shall shortly prove the surprising fact that if $f : \Omega \to C$ is analytic, then also $f' : \Omega \to C$ is analytic with derivative $f'' : \Omega \to C$, which is also analytic, and so on." However, an inspection of the following pages never shows any sign of such a proof. On the contrary, on page 1113 we read "if we assume that $u(x, y)$ and $v(x, y)$ are twice differentiable..." and later "Now, one can show that the solutions of the Cauchy–Riemann equations indeed must be twice differentiable...." No sign of the duck? Next, on page 1114, "one may ask if $f'(z)$ itself has a derivative in $\Omega, \ldots$. The plain answer is YES, which we prove below." A few lines later: "To answer the question posed, it is sufficient to notice that if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations, then so do all derivatives...." But the *existence* of these derivatives is still not settled. The next hope comes from Cauchy's integral formula ("We prove that if $f(z)$ is analytic..."). However, in the given proof appears a sentence "and K a bound for $|f^{(3)}(z)|$, which proves the desired result" (p. 1126). But no hypothesis on the existence of a third derivative is formulated in Theorem 82.6.

Chapters 83 and 84 compare nicely *Fourier series* with *Fourier transforms*. We read in the beginning that Fourier series arose from the solution of the heat equation and "has influenced the development of mathematical analysis profoundly.... However, as any highly specialized tool or organism, these techniques have not been able to adapt to the needs of a changing world with computational methods for nonlinear differential equations taking over as work-horse in applications." So, Fourier series are here introduced without the beautiful motivation from the heat equation or the wave equation by the idea of separating variables. Later, in section 83.14, the heat equation is nevertheless mentioned, but the solution is just written down ("We observe that ... satisfies...") and not derived. The Fourier coefficients are derived from the orthogonality by integrating an infinite series term-by-term, without justification. Many examples then illuminate the theory. A clever student may perhaps discover the fact that in all these examples the Fourier coefficients $c_m(f)$ tend to zero with one power of m^{-1} higher than stated on page 1154. The *inversion formula* (i.e., the convergence of the Fourier series to f) is stated and proved only for periodic differentiable functions "with piecewise Lipschitz continuous derivative." Without justification, it is then mentioned that the assumption on f could be relaxed to "piecewise differentiable with piecewise Lipschitz continuous derivative." It is not mentioned that even Lipschitz continuity of f is sufficient (more precisely, bounded variation) and that this has been known since 1829.

Summary. The authors present a rich amount of material, much of which belongs to the standard mathematics education, in an at times unorthodox style. The book suffers from a discrepancy between the high ambitions set out in the preface and the actual realization of the program, which lacks care and attention in many places.

Perhaps we may summarize our overall impression of the book by the sentence: the "body" is strong, but the "soul" is weak.

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A Multigrid Tutorial. Second Edition. *By Williams L. Briggs, Van Emden Henson, and Steve F. McCormick*. SIAM, Philadelphia, 2000. $43.00. xii+193 pp., softcover. ISBN 0-89871-462-1.

This is the ultimate textbook for providing a basic grounding in the subject. Students will find it very readable with just the right amount of material necessary to reflect a growing field. Completion of this book and its exercises will serve as a launching platform for more in-depth treatments of the subject found in the references.

Chapters 1–5 are essentially the same as in the first edition [1], with the following notable improvements:

- Key concepts are nicely boxed in the exposition.
- More exercises are included for each chapter. All exercises are carefully designed to ensure the student understands and can carry out his/her own experiments to validate the concepts presented.
- Exercises are organized by concept and a related keyword to the concept appears in a title to each exercise.

I have used Chapters 1–5 to comprise half of a quarter course on iterative methods at the graduate level and as a self-paced reading course for graduate students prior to tackling more advanced treatments. In both situations, students like the book and continue to keep it on their shelves.

Chapters 6–10 have been added since the first edition to reflect the growing field of multilevel methods. The same exceptional care has been taken with this more advanced material to make it accessible to students. A nice feature is that Chapters 6–10 can be studied in any order once students have mastered Chapters 1–5. I have found that these chapters provide excellent coverage of important topics for numerical analysis seminars that can be given by either graduate students or faculty.

Chapter 6 explains the full approximation scheme (FAS) for nonlinear problems. Since the material in Chapters 1–5 deals entirely with linear problems, this addition is very important. The exposition follows naturally from that used in the first five chapters.

Chapter 7 deals with selected applications that every graduate student studying the field should encounter. Topics include the treatment of Neumann boundary conditions, anisotropic problems, semicoarsening, line relaxation, full coarsening, variable mesh problems, and variable coefficient problems.

Chapter 8 is an introduction to the algebraic multigrid method. The treatment is a beginning one that focuses mainly on symmetric positive definite M-matrices, but the book references more advanced literature.

Chapter 9 walks the reader through the tedious details of the fast adaptive composite grid (FAC) method with very simple one- and two-dimensional examples.

Chapter 10 develops a multigrid framework appropriate for self-adjoint problems that have been discretized by finite elements. A very clear development derives the multigrid interpolation and restriction operators and the coarse grid problem matrix directly from the point of view of the minimization of a functional. The first seven chapters relied on finite difference discretizations, so the inclusion of Chapter 10 gives the student a starting point to understanding the vast literature described from this significantly different point of view.

The set of topics that advanced students would not find in this book include, but are not limited to, the following:

- A detailed treatment of nonsymmetric problems, including choices for restriction and prolongation operators.
- Local mode analysis that leads to an amplification matrix rather than the amplification factor of the elementary examples.
- Issues that arise in algebraic multigrid for problems that do not have symmetric positive definite M-matrices.

The recent book by Shapira [2] gives a matrix-based treatment of these topics for the more advanced reader.

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[1] W. Briggs, *A Multigrid Tutorial*, SIAM, Philadelphia, 1987.

[2] Y. Shapira, *Matrix-Based Multigrid: Theory and Applications*, Kluwer Academic, London, 2003.

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Higher-Order Finite Element Methods. *By Pavel Solin, Karel Segeth, and Ivo Dolezel.* Chapman & Hall/CRC, Boca Raton, FL, 2003. $89.95. xxii+382 pp., hardcover. ISBN 1-58488-438-X.

hp-finite elements refer to an optimized finite element method where the mesh size h and the polynomial approximation degree p are selected to minimize the discretization error. Since optimal *hp*-finite element methods yield exponential convergence rates, they have gained much popularity over the last two decades. Solin, Segeth, and Dolezel have succeeded in achieving their stated goal of providing the reader with a set of tools to solve partial differential equations using higher order finite element approximations.

Although the authors assume that the reader is familiar with the basic finite element concepts, they include an introduction to the method in Chapter 1. The authors define the function spaces H^1, $H(\text{curl})$, and $H(\text{div})$ as well as other basic concepts. In the last section they illustrate the finite element method on a two-point boundary-value problem.

Chapter 2 shows several hierarchic master elements and their corresponding shape functions. The authors show popular master elements in two and three dimensions and use the De Rham diagram to construct the shape functions in $H(\text{div})$ and $H(\text{curl})$ needed to solve Maxwell's equations.

Chapter 3 shows how to construct H^1-, $H(\text{curl})$-, and $H(\text{div})$-conforming finite element approximations on uniform meshes, locally refined meshes allowing for irregular nodes and edges, and hybrid meshes. The authors discuss orientation of edges and faces and assembling procedures in addition to other implementation issues.

Chapter 4 is devoted to high-precision numerical quadratures for one-, two- and three-dimensional elements. The book provides tables of nodes and weights listing 10 significant digits or more (the accompanying CD has quadratures tabulated to 15 significant digits or more). The chapter begins with a helpful discussion on how to select a numerical quadrature.

Chapter 5 deals with methods for solving the discrete finite element equations. Since the emphasis is not on solving algebraic problems, the book contains a brief discussion of several methods to solve algebraic problems such as direct and iterative (MINRES, GMRES, and multigrid) methods for sparse matrices. In the last section the authors discuss methods for solving initial value problems.

Chapter 6 is dedicated to *hp*-adaptive strategies guided by global a posteriori error estimates. The book describes several *h*-, *p*-, and *hp*-refinement procedures for one- and two-dimensional problems. Goal-oriented adaptive algorithms are also discussed in this chapter.

This book is a valuable addition to the existing literature on adaptive *hp*-finite element methods [1, 2, 3] and provides the practitioner with the necessary tools and techniques for understanding and implementing high-order hierarchic finite element methods. The book contains many interpolation and finite element error results without proofs.

Overall this is a useful and well-written text. However, the book is not appropriate for learning the basics of the finite element method nor for learning the convergence proofs of finite element approximations. I recommend the book for applied mathematicians and practitioners using the finite element method.

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[2] I. Babuska and T. Strouboulis, *The Finite Element Method and Its Reliability*, Oxford University Press, New York, 2001.

[3] C. Schwab, *p- and hp-Finite Element Methods*, Oxford University Press, New York, 1998.

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Oscillation Theory for Second Order Dynamic Equations. *By Ravi P. Agarwal, Said R. Grace, and Donal O'Regan.* Taylor & Francis, London, 2003. $80.00. viii+404 pp., hardcover. ISBN 0-415-30074-6.

There are numerous books on oscillation theory for differential equations such as [1, 2, 5, 6], to name but a few. This monograph by Agarwal, Grace, and O'Regan is an excellent addition to the existing literature. It covers topics related to oscillation theory for differential equations with deviating arguments, neutral functional differential equations, second order ordinary differential equations, and impulsive differential equations.

This book is very well organized; being in the classical mathematical style, the presented material is divided into definitions, theorems, proofs, lemmas, remarks, and examples. It is self-contained and consistent, and it is easy to follow the presentation. Only some background in calculus and differential equations is required. One definitely does not have to be a differential equations specialist in order to follow this well-written, compact, but thorough treatment.

In an effort to make this book available to a wide audience of scientists, a first chapter collects some basic results for second order linear ordinary differential equations such as Sturm's comparison theorem and Picone's identity, and several fixed point theorems are presented that are used in the remainder of the book, e.g., Brower's, Schauder's, Banach's, Krasnosel′skiĭ's, and Knaster's fixed point theorems. The remaining four chapters deal with criteria for oscillation, nonoscillation, and various other asymptotic behavior for solutions of the studied second order differential equations.

This book can easily be used as a textbook for a special topics course at the graduate level, and it can also serve as an encyclopedia and reference book on oscillation theory for researchers in various fields. Finally, it can be an inspiring source for advisors and graduate students who are seeking topics for their theses. Even though some of the material has been studied for a long time, there are still many open problems, and research in the area is continuing and even rapidly growing. A search on MathSciNet using the word "oscillation" in the title produces 8707 hits. The three authors of this monograph, Professors Agarwal, Grace, and O'Regan, are well known and very active experts in this and many other areas.

The title of the book can be interpreted in several ways. The authors use the expression "dynamic equation" in the sense of a collection of differential equations such as delay, neutral, functional, ordinary, and impulsive differential equations. On the other hand, the study of "dynamic equations" is indeed a mathematical subject, not very well known among the mathematical community due to its recent origin, but which is also connected to the material presented in the monograph under review. In fact, dynamic equations on so-called "time scales" have been introduced in order to unify the theories of differential and difference equations and to extend them to cases "in between."

A time scale $\mathbb{T}$ is an arbitrary closed subset of the real numbers $\mathbb{R}$, and for functions $f : \mathbb{T} \to \mathbb{R}$ a delta derivative f^Δ may be defined that has the following properties: When $\mathbb{T} = \mathbb{R}$, f^Δ equals f', the ordinary derivative. When $\mathbb{T} = \mathbb{Z}$ (the set of integers), f^Δ equals Δf, the ordinary forward difference. Two important objects in the study of time scales that are unbounded above are the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the graininess operator $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$. Now one can study equations that involve unknown functions and their delta derivatives, i.e., so-called dynamic equations. For example, one of the main equations that is studied in the book under review, in particular in Chapter 4, namely,

$$(a(t)x')' + b(t)x = 0, \tag{1}$$

could be studied in the form of a dynamic equation as

$$(a(t)x^\Delta)^\Delta + b(t)x = 0. \tag{2}$$

However, with such an equation one would be unable to prove classical analoga for the differential equation (1) in the dynamic equations setting. It turns out that even the well-studied discrete version of (1),

$$a_{k+1}x_{k+2} - (a_k + a_{k+1} - b_k)\, x_{k+1} + a_k x_k = 0, \tag{3}$$

does not fit into the form of (2). Instead one should consider the equation

$$(a(t)x^\Delta)^\Delta + b(t)x^\sigma = 0, \tag{4}$$

where x^σ is defined as $x \circ \sigma$. For dynamic equations of the form (4), it is in fact possible to obtain standard results and oscillation criteria that have results for differential equations (such as those presented in the book under review) and also results for corresponding difference equations as special cases. Furthermore, such dynamic equations contain not only differential and difference equations as special cases but also very different kinds of equations because a time scale is allowed to be any closed subset of the real numbers. For example, so-called q-difference equations are included, and when $\mathbb{T}$ is the collection of nonnegative integer powers of a number $q > 1$, (4) takes the form

$$\begin{aligned} &a(qt)x(q^2t) \\ (5)\ &- \left(qa(t) + a(qt) - q(q-1)^2t^2b(t)\right)x(qt) \\ &+ qa(t)x(t) = 0, \end{aligned}$$

where $t \in \mathbb{T}$.

Oscillation of the dynamic equation (4) has been studied intensively in the recent literature (see [3, 4]), but the book under review does not contain any of those results; in particular, it does not deal with difference equations. For difference equations, see the monograph [1] by the same three authors, Agarwal, Grace, and O'Regan. The book under review deals with differential equations solely, and hence the connection to dynamic equations is only a remote one in the sense that differential equations are special cases of dynamic equations for $\mathbb{T} = \mathbb{R}$.

Now, returning to the actual topic of the monograph by Agarwal, Grace, and O'Regan, the book covers oscillation results for differential equations of the following forms. In Chapter 2, the second order functional differential equation

$$(6)\qquad (a(t)x')' + q(t)f(x(g(t))) = 0$$

is considered, where the functions occurring are assumed to satisfy certain conditions; f may be a nonlinear function, a is positive, and q is either positive or negative. In later sections of this chapter, damping and forcing terms are added so that the most general form appears as

$$(7)\quad (a(t)x')' + p(t)x' + F(t, x(g(t))) = e(t).$$

Special cases such as the Emden–Fowler equation or the Klein–Gordon equation are treated as well. Chapter 3 covers nonlinear neutral differential equations such as

$$\begin{aligned} (8)\qquad &\frac{\mathrm{d}}{\mathrm{d}t}\left(a(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) + p(t)x(\tau(t))\right)\right) \\ &+ q(t)F(x(g(t))) = 0, \end{aligned}$$

and again, in later sections of this chapter, damping and forcing terms are added to (8). Equations with mixed type, i.e., in which both advanced and retarded arguments occur, are also studied. Chapter 4 is concerned with the classical self-adjoint differential equation (1). Standard results of this equation are already contained in Chapter 1, so Chapter 4 offers some results connected to conjugacy (rather than disconjugacy) that are not easily accessible in the literature. The Emden–Fowler equation in both the sublinear and superlinear cases is studied as well. Finally, Chapter 5 offers a treatment of impulsive delay differential equations in the form

$$(9)\qquad \begin{aligned} &x'(t) + q(t)x(t-\tau) = 0,\ t \neq t_k; \\ &x(t_k^+) - x(t_k) = p_k x(t_k), \end{aligned}$$

where the numbers t_k are the impulse points.

Each chapter concludes with a very nice section on "Notes and General Discussions," followed by the bibliography for that particular chapter (i.e., this book has five different lists of references). An extremely important and nice feature is the inclusion of 82 examples throughout the book that serve well to illustrate the presented theory. This monograph opens research directions in various ways: One could try to find analogues of the results presented for difference equations or q-difference equations, or, in more generality, for dynamic equations on time scales. The discrete case has already been well studied, but although first steps and progress have been made for dynamic equations, it will be quite a while until time scale results corresponding to the continuous theory presented in this book are really investigated and an "oscillation theory for second order dynamic equations" is fully developed.

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Normal Forms and Unfoldings for Local Dynamical Systems. *By James Murdock.* Springer-Verlag, New York, 2003. $69.95. xx+494 pp., hardcover. ISBN 0-387-95464-3.

Normal form theory is of fundamental importance in the study of dynamical systems. The idea of simplifying a system in a vicinity of a fixed point through a sequence of near identity changes goes back at least to Poincaré. It has been of fundamental importance in applications and has been developed into a rich theory. Most introductory textbooks on dynamical systems provide an overview of the basic ideas of normal form theory. The book under review aims to introduce both the algebraic structure of the coordinate transformations that are used in the normalization and (to a lesser extent) the geometric structure of the vector fields that are thus obtained.

The book is written with a newcomer to the field in mind. All sections start with a careful motivation and conclude with remarks and pointers to the literature. As in his earlier book [4], the author takes the reader through all the details of the arguments, frequently pointing out obvious, and not so obvious, pitfalls. This results in a relatively long book, which, of necessity, covers only a fraction of the results in the field. While a novice will likely find this approach helpful, the expert may find the book prolix.

After several examples that introduce the main ideas of the monograph, certain facts about finite-dimensional linear operators are considered. The discussion of the Jordan canonical in Chapter 2, using the idea of a chain basis important later in the book, is the most lucid I have found to date. Although the ideas are straightforward, this was usually a topic I dreaded teaching in an advanced linear algebra course. This chapter solves that problem.

The heart of the book is contained in Chapters 3 and 4, which introduce linear and nonlinear normal forms, respectively. Several "formats" for the computation of normal forms are presented. "Direct" formats involve the direct application of either a single or a sequence of near identity changes of coordinates, while "generated" formats rely on ideas from the theory of Lie groups. The direct formats require less thought, but it is the generated formats that give more insight into the structure of the normalization algorithms and the resulting normal form equations, and are therefore emphasized.

At the heart of the normal form method, regardless of format, is the analysis of the operator $\mathrm{ad}_A = [\cdot, A]$. Roughly speaking, monomials can be removed via near identity coordinate changes if and only if they are in the range of this operator. However, since in general one can choose different bases for the complement of the range of ad_A, normal forms are not uniquely determined by the linear part of the vector field at a point. The author refers to the different choices that can be made as normal form "styles." The different styles discussed in the second half of Chapter 4, including the semisimple and $sl(2)$ styles introduced in [3] and [2], respectively, lead to unique normal forms, each with its own advantages. Chapter 4 concludes with an introduction of "hypernormal forms."

A reader interested in applications will find the last two chapters of the book of most value. The first part of Chapter 5 is devoted to the description of various geometrical structures that are preserved under the flow of the truncated normal form (obtained by removing terms above a certain degree). As a consequence, the flow of the truncated normal form is frequently easier to analyze than the flow of the full equation. In the second part of this chapter it is shown that, under certain conditions, the truncated system provides a good approximation to the full equation over certain timescales. While these results are probably known to most specialists, I have not seen them collected in this way before.

The last chapter is an introduction to bifurcation theory and provides another example of the utility of normal form theory. Given the author's careful exposition a full treatment of even part of the theory would have resulted in a far longer monograph. This chapter is therefore limited to the development of the theory that is sufficient to treat several standard examples, with the goal of motivating interested readers to continue their study using other sources.

The reader who expects to learn the basic ideas and techniques of normal form theory will find this book rewarding. Its algebraic approach is well suited to readers interested in automated computations of normal forms. Unfortunately, several interesting topics have not been considered in the book. For instance, normal forms for systems with symmetry or diffeomorphisms are hardly mentioned. For a different perspective on this monograph written by an expert in the field, see [1].

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Difference Schemes with Operator Factors. *By A. A. Samarskii, P. P. Matus, and P. N. Vabishchevich.* Kluwer Academic, Dordrecht, The Netherlands, 2002. \$118.00. x+384 pp., hardcover. ISBN 1-4020-0856-2.

There are two main branches of stability theory for discretizations of partial differential equations: One is based on estimates of the Fourier symbol of frozen coefficient problems and is often called normal mode analysis or GKS stability theory. See, for example, [1, 2]. The other, which is the subject of the book under review, is often called the energy method. Normal mode analysis applies to more general problems, but is often very difficult to use. The energy method applies to more specific equations, which are, however, most important in applications.

When discretizing a time-dependent PDE, it is often useful to proceed (conceptually or actually) in two steps: First, discretizing in space, one obtains a method-of-lines system, typically an ODE system such as $du/dt + Au = f(t)$ or $d^2u/dt^2 + Au = f(t)$. Here, for every time t, the unknown $u(t)$ is an element of a finite-dimensional Hilbert space H of functions on a spatial grid, and $A : H \to H$ is a linear operator. In a second step, one discretizes time and replaces the ODE system by difference equations such as

$$\frac{1}{\tau} B(y_{n+1} - y_n) + Ay_n = \phi_n \tag{1}$$

or

$$\begin{aligned} &\frac{1}{\tau^2} D(y_{n+1} - 2y_n + y_{n+1}) \\ &\quad + \frac{1}{2\tau} B(y_{n+1} - y_{n-1}) + Ay_n = \phi_n. \end{aligned} \tag{2}$$

In general, the operators A, B, and D will depend on time and on the time step τ. Of course, they will also depend on the space discretization that leads to the method-of-lines system in the first place.

Difference schemes are only useful if they are stable, i.e., if one can estimate the solution $y = (y_n)$ by its initial data and the right-hand side $\phi = (\phi_n)$, and the estimate does not deteriorate as $\tau \to 0$. The energy technique makes systematic use of symmetry and definiteness properties of the operators A, B, and D, as inherited from the PDE, to establish such stability estimates.

The book under review contains eight chapters and consists of two parts. (The division into two parts is due to the reviewer.) After the introduction, Chapters 2 to 5 deal rather abstractly with two-level and three-level difference equations such as (1) and (2) and give conditions on the operators involved that lead to stability estimates. Here the presentation is rigorous and relentlessly abstract. One example appears on page 52, and only in a short sentence is it mentioned that the assumed form $A = T^*T$ is often used for discretizing $\operatorname{div}(k \operatorname{grad} u)$. Otherwise, the reader has to wait until Chapter 6, starting on page 149, for any motivation and all the different assumptions of the stability theorems.

Beginning with the second part in Chapter 6, one sees stability theory at work and the presentation is directly oriented towards the numerical solution of PDEs. In Chapter 6, the reader not only finds standard applications to diffusion, convection-diffusion, and wave equations, but can also read about diffusion equations with discontinuous coefficients, equations with rough right-hand sides and rough solutions, and hyperbolic and parabolic equations that are coupled at an interface. The general abstract formulation of the stability theory presented in Chapters 2 to 5 pays off. There is also a treatment of conservative discretizations of the KdV equation, but the main emphasis is on linear problems.

In Chapter 7, called *Schemes on Adaptive Grids*, the authors consider the equations $u_t = (ku_x)_x + f(x,t)$ and $u_{tt} = (ku_x)_x + f(x,t)$ and assume that the solution behaves badly in some (known) region of space-time, possibly a moving singularity. This calls for local grid refinements. The stability issues for the resulting difference schemes are investigated, again relying on the results of Chapters 2 to 5.

The final Chapter 8 is on domain decomposition for nonstationary problems. A diffusion equation and a wave equation in a rectangle are considered as model problems. The issues addressed here are of current research interest. Unfortunately, Figures 8.1 and 8.2 are wrong and, at times, the presentation loses focus.

Nevertheless, in summary, the book gives a rather comprehensive treatment of the energy method to investigate stability of difference methods applied to diffusion and wave equations. It covers an important topic of numerical analysis, complementing [1, 2].

My main criticism regards the use of the English language, which is frequently inadequate. Quite a few sentences only make sense after one substitutes words like restriction for contraction, jump condition for conjugation condition, etc. The book deserved better editing.

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Matrix Riccati Equations in Control and Systems Theory. *By Hisham Abou-Kandil, Gerhard Freiling, Vlad Ionescu, and Gerhard Jank.* Birkhäuser-Verlag, Basel, 2003. $149.00. xx+ 571 pp., hardcover. ISBN 3-7643–085-X.

Matrix Riccati equations arise naturally as one of the major mathematical equations in many applications from control, filtering, game theory and others. They are nonlinear (quadratic) matrix equations but they can nevertheless be analyzed and solved using methods from linear algebra. These facts, their great importance in many applications, and the availability of well-studied tools has led to an immense interest and an explosion in the number of publications on this topic in the last 50 years. Moreover, since the applications come from many different areas of engineering, and the mathematical theories and methods used range from elementary matrix theory, analysis of differential equations, geometric theory, and operator theory to the explicit construction of numerical methods, it is nowadays very difficult to keep track of the available results of this topic.

In such a situation a monograph that covers a wide range of the mathematics of matrix Riccati equations is urgently needed, but to write it is a very difficult task. Since the fundamental work of Reid [6] on Riccati differential equations in 1972, not many substantial monographs devoted to this topic have appeared. Although Riccati equations are discussed in many books on systems and control, I would say that the book by Lancaster and Rodman [4] (see also *SIAM Review*, 38 (1996), p. 694) has so far been the most fundamental recent work on this topic from the theoretical (mostly algebraic and geometric) point of view. It, however, mostly discusses symmetric (Hermitian) algebraic Riccati equations.

Since [4] is relatively recent, the question may be: Was there really a need for another book on this topic? Looking at the new book by Abou-Kandil, Freiling, Ionescu, and Jank, the answer to this question must certainly be *yes*! First, the new book covers a much different range of topics than [4]. The new book really complements [4] in many ways, but not only that, it also adds a different, more analytically oriented point of view.

A short overview of the content may be appropriate. After two introductory chapters, global aspects of Riccati differential and difference equations are discussed. Then the particular case of Hermitian Riccati differential equations is followed by a chapter on periodic Riccati equations. Nonsymmetric and generalized Riccati equations, as well as coupled systems of Riccati equations, are also discussed in detail. Finally two sections are devoted to applications in robust control and Nash equilibria in differential games.

For everyone who is interested in the topic of Riccati equations and their applications to problems from control, signals, and systems I suggest obtaining this monograph. Together with [4] it covers the state-of-the-art in the theoretical analysis of finite-dimensional matrix Riccati equations of all flavors. But, although there is little overlap with [4] and together they have more than 1000 pages, still important topics like numerical methods (see, e.g., [1, 5]), infinite-dimensional Riccati equations (see, e.g., [3]), and perturbation theory (see [2]) are covered only tangentially. This shows the broadness of this topic and why this book was really needed.

My only point of criticism with this book is that a few more examples and illustrations would have improved the usefulness of the book even further.

In summary, this is a very well written and extremely useful book. It should be included in the library of everybody working on Riccati equations, and in all areas of control, signals, and systems. Together with [4], it will be the standard reference book in this area in the future.

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Capture Dynamics and Chaotic Motions in Celestial Mechanics: With Applications to the Construction of Low Energy Transfers. *By Edward Belbruno.* Princeton University Press, Princeton, NJ, 2004. $49.95. xx+211 pp., hardcover. ISBN 0-691-09480-2.

In January of 1990 the Japanese ISAS Institute launched a linked pair of small spacecraft into a nearly elliptical orbit about the earth. The larger one, MUSES-A, was supposed to remain in an earth orbit while conducting scientific experiments and commutating with the smaller, MUSES-B. The small craft B was to unlink from the larger craft A and then go into lunar orbit. For some reason contact with the lunar probe B was lost and its mission failed.

In order to partially save the mission the ISAS Institute decided to try to get the larger MUSES-A into a lunar orbit. The problem was that A did not have enough thrust to be put into a lunar orbit that was computed by the traditional methods of celestial mechanics. It was never designed to go to the moon.

By June 1990, the author, Ed Belbruno, and his collaborator, J. Miller, at the Jet Propulsion Laboratory designed a low-energy transfer orbit which would take MUSES-A from its near earth orbit to a near lunar orbit. Thus, they rescued the Japanese lunar mission. Their solution used the tools of modern dynamical systems theory. This book guides the reader to an understanding of their solution.

The first chapter is an introduction to the N-body problem and classical results: the Kepler problem, the restricted three-body problem, regularization, and collision. Most of this chapter is well-known background material. The exception is the last section, where the author gives a nice presentation of the equivalence of geodesic flow on a sphere and the Kepler problem. This is the regularization of the Kepler problem of Moser, Belbruno, and Osipov.

The second chapter presents the basic theory of Hamiltonian dynamical systems: fixed and periodic points, hyperbolicity, KAM theory, and Aubrey–Mather theory. These are some of the big tools in the theory which require lengthy proofs, but the reader will have to go elsewhere to find them.

The final chapter, which takes up nearly half the book, is the raison d'être for the book. The classical approach to transferring a space craft from a near earth orbit to a near moon orbit starts by piecing together Kepler orbits of the earth-craft system and Kepler orbits of the moon-craft system. This is the first approximation to what is called the Hohmann transfer orbit. It was used to send men to the moon and back. It is a simple method to understand and easy to implement, but it does not yield a low-energy orbit.

The simplest model for a low-energy transfer is the circular restricted three-body problem, which was studied theoretically by Conley and his students in the 1960s and 1970s. The restricted three-body problem has two bodies of finite mass (the earth and moon) moving on circular Kepler orbits about their center of mass and a third body of zero mass (the space craft) moving under the gravitational attraction of the two finite bodies. In a rotating coordinate system the finite bodies are at rest and the system admits an integral (which we can think of as energy). For very negative values of the integral the craft is trapped in the potential well about one or the other of the finite masses. As the value of the integral increases, the potential wells increase and then a neck develops between them. For values of the integral just above where the neck is formed, a low-energy transfer is possible. In the neck there is an unstable periodic orbit with a stable and an unstable manifold. There are orbits which enter the neck near the stable manifold and exit near the unstable manifold. Such an orbit moves from the region around the earth to the region around the moon with near minimal energy. The hard part is finding such an orbit with the desired limit as time tends to plus/minus infinity. To that end the author defines what is called the "weak stability boundary" and devotes a considerable number of pages to its definition, properties, and computation.

Finally, a more realistic model is considered which is a restricted four-body problem consisting of four point masses: P_1, the earth; P_2, the moon; P_3, the space craft; and P_4, the sun in an earth-based coordinate system. Assume $m_4 \gg m_1 \gg m_2$ and $m_3 = 0$. The sun moves on a slightly elliptical but large orbit about the center of mass of the earth–moon system, and the earth–moon system moves on slightly elliptical orbits about their center of mass. The actual trajectories of the sun, earth, and moon are to be taken from their ephemeris.

This is a how-to book on how to construct a low energy orbit from the earth to the moon by the man who did just that to save the Japanese space mission in 1990.

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Advanced Arithmetic for the Digital Computer. Design of Arithmetic Units. *By Ulrich W. Kulisch.* Springer-Verlag, Vienna, 2002. $34.95. xii+141 pp., softcover. ISBN 3-211-83870-8.

In spite of phenomenal advances in speed and memory size of computers during the past several decades, the basic arithmetic operations provided in hardware have not changed much. Standard floating-point arithmetic in hardware still has the same shortcomings it had 40 years ago. Some changes are overdue. Ulrich Kulisch has designed new arithmetic units for more advanced computer arithmetic with improvements in both accuracy and speed. With only a modest increase in the complexity of hardware design, his new design of arithmetic units would speed up computations while also eliminating the most serious sources of rounding error such as catastrophic loss of significance in subtraction of nearly equal numbers, especially during the computation of inner products of vectors. This is done by having a long adder in hardware similar to the way old electromechanical calculators did, so that the multiply and add instruction is effectively carried out in fixed-point arithmetic without rounding until the final result is obtained. Details are clearly explained in the first of three chapters.

The special problem of rounding near zero is dealt with rigorously in a short second chapter.

A major part of the exposition, Chapter 3, is devoted to the question of how to design hardware for interval arithmetic as fast as floating-point arithmetic, with two operation units and a few multiplexers and comparators. Rounding modes have been added in recent years, but current hardware design requires setting the rounding mode before each interval operation, which greatly slows the process.

The proposal is to provide, for example, multiply and round away from zero as a single hardware operation. In all, 15 basic operations are required instead of the usual 12. As in the rest of the volume, details are given clearly and rigorously.

Kulisch has shown how to design this kind of advanced computer arithmetic in hardware. At small cost, the benefit would be to greatly improve the reliability of numerical computation.

Manufacturing such arithmetic units is an idea whose time has come. It remains only to see who does it first.

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Tools for Computational Finance. *By Rudiger Seydel.* Springer-Verlag, Berlin, 2003. $44.95. xiv+224 pp., softcover. ISBN 3-540-40604-2.

Computational techniques for valuing financial derivatives are increasingly popular among financial practitioners and academics alike, as efficiency and speed become more and more important. Many financial engineering books to date focus on simulation techniques, and very few on other computational aspects of finance, such as finite difference or finite element methods. This book attempts to fill this gap by providing a broader practical introductory exposition of financial mathematics with minimal mathematical formalism and a focus on readability. This is the first book I am aware of that has devoted an entire chapter to using finite element methods in finance.

The book is divided into four main sections: financial and stochastic background, tools for simulation, partial differential equations for options, and further requisites and additional material. Each chapter is concluded with very helpful comments and numerous exercises. The book is written from an applied mathematics standpoint. Where possible, the author provides a corresponding numerical algorithm. Many mathematical results are laid out without derivation or a sense of how one gets there. The first few chapters are very straightforward, easy to read, and cover topics that are standard in most financial engineering books.

In Chapter 4 on finite difference methods, the author discusses three ways of numerically solving PDEs, the explicit, implicit, and Crank–Nicolson methods. The discussion on stability or accuracy is very brief, making the book very much a "tool kit." This chapter is very well illustrated by the valuation of American options as a free boundary-value problem accompanied by several figures and several numerical algorithms.

Chapter 5 on the finite element method is very attractive in the sense that not many books discuss this method. Yet, I found this chapter disappointingly short and without any extensive examples of applying the method. In concluding this chapter the author writes, "finite-element methods are frequently used in the area of computational finance," but he provides only one reference. This chapter actually leaves the unsophisticated reader wondering about the usefulness of finite element methods in finance.

The last chapter, on pricing exotic options, is really an introduction to the upward schemes and higher resolution schemes in order to cope with the complexity of valuing path-dependent options. Like the previous chapter, one gets the feeling that this chapter would have been better served with a few more examples.

This book is very easy to read and one can gain a quick snapshot of computational issues arising in financial mathematics. Researchers or students of the mathematical sciences with an interest in finance will find this book a very helpful and gentle guide to the world of financial engineering. We recommend that you read this book alongside Baxter and Rennie [1] for the mathematical formalism and Tavella and Randall [2] for a more extensive discussion on finite difference methods.

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A Primer of Analytic Number Theory. *By J. Stopple.* Cambridge University Press, Cambridge, UK, 2003. $35.00. xiv+383 pp., softcover. ISBN 0-521-01253-8.

This book is an elementary introduction to the subject of analytic number theory. The subject of number theory is as old as recorded history. Emerging in the Babylonian civilization, then meandering through Egyptian, Greek, Indian, Chinese, and Arabic civilizations, it re-emerged in the European Renaissance period with an exotic and sinuous history. It would be no exaggeration to say that it played a seminal role in the development of modern mathematics and, on a larger scale, the modern scientific tradition. The book under review tries to capture some of this theme in its historical interludes.

The distribution of prime numbers is still not completely understood. The celebrated prime number theorem tells us that the number of prime numbers $\leq x$, often denoted $\pi(x)$, is asymptotically $x/\log x$ as x tends to infinity. This theorem was first conjectured by Gauss in 1792 (during his teenage years) when he wrote to the astronomer Encke, predicting that

$$\operatorname{li} x := \int_2^x \frac{dt}{\log t} \sim \pi(x).$$

If we integrate by parts, we find that the first term in the asymptotic expansion is indeed $x/\log x$. Almost a century later, in 1896, Hadamard and de la Vallée Poussin proved the prime number theorem, essentially developing a program Riemann had outlined in 1860 and supplementing it with some ingenious new ideas.

In his 1860 paper, Riemann defined his famous ζ-function as a function of a complex variable that has the property that for $\Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

As Riemann emphasized, its connection with prime numbers is on account of the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the latter product is over prime numbers p. This identity is an analytic formulation of the fact that every natural number can be written as a product of prime numbers uniquely. The celebrated Riemann hypothesis (still unresolved) predicts that if $\zeta(s) = 0$ for some s in the region $0 < \Re(s) < 1$, then $\Re(s) = 1/2$. This is equivalent to the following stronger form of the conjecture of Gauss:

$$\pi(x) = \operatorname{li} x + O(x^{1/2} \log x).$$

This is listed as a million dollar prize problem by the Clay Mathematics Institute (see http://www.claymath.org/prizeproblems).

Stopple's book takes us literally from scratch to the point where we can understand the Hadamard and de la Vallée Poussin proof of the prime number theorem. The second half of the book discusses Dirichlet's theorem about primes in arithmetic progressions. Still, the presentation is accessible to undergraduates and sprinkled with interesting historical anecdotes. The book culminates with a discussion of the theorem by Goldfeld, Gross, and Zagier giving an effective lower bound for the class number of an imaginary quadratic field. This theorem, certainly one of the Himalayan achievements of 20th century number theory, requires for its solution an extensive treatment of the theory of elliptic curves and modular forms. It would be difficult to give such an exhaustive treatment, and so the author takes a brief excursion into the theory of elliptic curves. He motivates this topic by first considering the "simpler" equation

$$x^2 - Dy^2 = 1,$$

whose history goes back to the work of Brahmagupta in 6th century A.D. India. It is erroneously referred to as the Pell equation, even though scholars know that Pell had little to do with it. The theory of elliptic curves would include the study of cognate equations such as $x^3 - Dy^2 = 1$, and such equations played a fundamental role in the Goldfeld–Gross–Zagier theorem referred to above.

In summary, this is a well-written book at the level of senior undergraduates. My only criticism is that in many places the historical treatments are incomplete. For example, in Chapter 5, where the prime number theorem is first discussed, Gauss is nowhere mentioned. Pell's equation is surveyed in Chapter 11, but nowhere is the work of Brahmagupta and others even alluded to. To deflate some of this criticism, the reader would do well to augment this study with André Weil's excellent book, *Number Theory, An Approach through History* (Birkhäuser Boston, Cambridge, MA, 1984).

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Basic Classes of Linear Operators. *By I. Gohberg, Seymour Goldberg, and Marinus A. Kaashoek.* Birkhäuser-Verlag, Basel, 2004. $89.95. xviii+423 pp., softcover. ISBN 3-7643-6930-2.

With a name like Gohberg–Goldberg–Kaashoek, it has got to be good. But let me count the ways. If you are interested in learning the basic theories of Hilbert and Banach spaces together with the well-known operators that act on them, this book is for you. It is intended for advanced undergraduates and beginning graduate students in mathematics. It begins with inner product and Hilbert spaces and eventually covers general Banach spaces. If you are looking for the latest results in operator theory, you will be disappointed.

The volume is actually a second edition of *Basic Operator Theory* by Gohberg and Goldberg. It is intended to be an introduction to *Classes of Linear Operators, Vols.* 1 *and* 2 by the three authors. It differs from the first edition in several ways. In particular, several types of operator are added, such as Laurent, Toeplitz, Fredholm, and singular integral operators. Exercises and examples have been added. Almost all of the text is devoted to linear operators. There is a small chapter at the end that discusses fixed-point theorems for nonlinear operators. The main tool used there is the contraction mapping principle.

In previous generations, integral operators were considered fundamental tools of analysis. In fact, their study led to the formulation of spectral theory for bounded operators. Researchers devoted careers to the analysis of integral operators, and practically every university offered a course, at the graduate and undergraduate level, devoted to integral equations. Those days are gone, and differential equations have won, even though many of them are solved by first converting to integral equations and then solving. This situation is lamented by the authors (and the reviewer), especially since integral operators make perfect examples of bounded operators on a Hilbert or Banach space. The authors (and the reviewer) were determined to correct this situation by illustrating the theories through studying many examples of integral operators.

Since the reviewer has written a book [1] on the same topic, it might be instructive to indicate the similarities and differences between the two texts. Both are written for advanced undergraduate and beginning graduate students. Both assume only a background in linear algebra and advanced calculus. Both limit the presentation to Banach and Hilbert spaces. There are good reasons for this latter choice. First, there is much useful information that can be learned concerning these spaces and operators acting on them. Second, the vast majority of mathematical problems arising in analysis and applications can be analyzed within the framework of these spaces. Third, it would be very confusing to students at this stage to introduce more types of spaces. We did not want to overwhelm the student with highly technical and advanced methods.

The types of operators studied are similar to those considered in [1]. The approach of the present authors is to present each topic in an orderly fashion and expand upon it as needed. The approach in [1] is not so orderly. Rather, problems are introduced in [1], and it is shown how functional analysis can be used to solve them. New concepts and methods are introduced only as needed to solve the problems. The present authors study Hilbert space operators throughout most of their book. Then Banach spaces are introduced to cover operators that do not require Hilbert space structure or cannot be dealt with in a Hilbert space setting. In [1], everything is studied within a general Banach space setting until we reach a point when Hilbert space structure is needed.

The two books do not overlap completely. There are topics that are covered in one, but not in the other. But the level of presentation is almost identical, although the book under review requires a bit more background in linear algebra and calculus than [1].

In both cases Lebesgue integration and the L^p spaces cause an awkward situation. On the one hand, these spaces are ideal examples of Banach spaces (with L^2 an ideal example of a Hilbert space). On the other hand, the student is not expected to know about Lebesgue integration. Under the circumstances, there is no ideal way of handling the situation. The authors chose to confront the situation head on. The reader is told, "This is the definition of the space L^p, and if you are unfamiliar with it, you can go to a reference on mathematical analysis for the proofs of the theorems that are described in an appendix." In [1], the reviewer chose another approach without using measure theory.

The text is well written, clear, and readable. (However, it is not free of typos.) A student at the specified level will have no difficulty understanding the presentation.

Of course, the reviewer is prejudiced in favor of his own text [1], but he can give nothing but praise for this book.

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Regularity Theory for Mean Curvature Flow. *By Klaus Ecker.* Birkhäuser Boston, Boston, 2004. $129.00. xiv+165 pp., hardcover. ISBN 0-8176-3243-3.

It is a rare occurrence when new breakthroughs in pure and applied mathematics occur simultaneously and generate enormous excitement. For the last 20 years, the computational and theoretical study and application of generalized motion by mean curvature and more general curvature flows have had enormous impact in diverse areas of pure and applied mathematics. Klaus Ecker's new book provides an attractive, elegant, and largely self-contained introduction to the study of classical mean curvature flow (initiated by Huisken [29, 30]), developing some fundamental ideas from minimal surface theory (maximum principles and local geometric estimates, monotonicity formulas, blow-ups and self-similar solutions, etc.), all with the aim of proving a version of Brakke's regularity theorem [10] and estimating the size of the "singular set." In order to minimize technicalities, the discussion is basically limited to classical flows up until a first singularity develops. This makes the book very readable and suitable for students and applied mathematicians who want to gain more insight into the subtleties of the subject. Let us backtrack now and define the classical flow.

A smooth embedded n-dimensional hypersurface $M^n \subset R^{n+1}$ is said to be moving by (classical) mean curvature flow if there is a family of smooth embeddings $F_t = F(\cdot, t) : M^n \to R^{n+1}$ with velocity $\frac{\partial}{\partial t}F(p,t) = \vec{H}(F(p,t))$ for $p \in M^n$ and $0 < t < t*$, where $t*$ is either the first time a singularity develops or the extinction time when M^n disappears. If ν is (say) the inner unit normal of an oriented hypersurface M, then $\vec{H} = H\nu$ is the mean curvature vector and H is the mean curvature of M with respect to ν (positive for a mean convex hypersurface). If Δ_t is the Laplace–Beltrami operator on the moving hypersurface $M_t = F_t(M^n)$, then the mean curvature flow can be rewritten (up to diffeomorphisms of M^n) as

$$\frac{\partial}{\partial t}F = \Delta_t F,$$

which formally resembles the heat equation, but is really a degenerate (because of the diffeomorphism group of M^n) quasi-linear parabolic system of PDEs. It is well known (see [22, 16]) that for a reasonably smooth initial hypersurface M_0, there is short-time smooth existence of the flow. This is the smoothing property of parabolic equations and explains why the mean curvature flow is used in image reconstruction. However, as we explain below, singularities do develop if $n \geq 2$.

The mean curvature vector points to the "inside" of M and, according to the classical first variation of area formula (see [1] for a very general version), is the direction in which area decreases most rapidly. Thus the mean curvature flow is the gradient flow for the area functional and so is an extremely natural and important geometric flow related to the motion of phase interfaces and (surprisingly) the asymptotics of reaction diffusion equations such as the Allen–Cahn equation (see [18, 8, 34] and the references therein).

The one-dimensional mean curvature flow or curve-shortening flow was introduced by Mullins [38] in his 1956 paper to model the movement of grain boundaries. As far as I can tell, it received scant attention in both the pure and applied math communities. It was not until the early 1980s that Gage and Hamilton [20, 21, 22] studied the classical curve-shortening flow for a smooth strictly convex curve and proved that the solution remains smooth and strictly convex and shrinks to a "round point"; that is, after rescaling to a fixed area, it converges smoothly to a circle. This is a beautiful and difficult result but is very plausible. In a remarkable paper, Grayson [25] proved the highly unintuitive result that an arbitrary embedded smooth curve remains smoothly embedded and after finite time becomes strictly convex and so (by Gage and Hamilton) shrinks to a round point.

Inspired by the work of Hamilton on the Ricci flow (see [27, 28]), Huisken [29], in a groundbreaking paper, introduced differential geometric methods to study the classical mean curvature flow in higher dimensions and proved that the Gage–Hamilton result for strictly convex hypersurfaces remains valid. However, Grayson's result is definitely false, as one can show that a suitably thin axisymmetric "dumbbell" must develop a neck pinch. How, then, can we continue the flow and define a generalized mean curvature flow past the onset of singularities? Even before the systematic study of the classical mean curvature flow was undertaken, Brakke [10] initiated the study of generalized mean curvature flows using methods of geometric measure theory and proved "almost everywhere regularity" under certain conditions. Because Brakke's definition was quite difficult to work with (as it suffers from both nonuniqueness and is nonconstructive), his seminal work languished from inattention.

Many applied mathematicians are now well-informed about the level set formulation of the generalized mean curvature flow introduced by Osher and Sethian [45, 42] (see also [40]) as a numerical method which could handle the formation of singularities and changes of topology. Motivated by this work, Evans and Spruck [15, 16, 17, 19] and Chen, Giga, and Goto [11] gave a theoretical justification of the level set method using the notion of viscosity solutions of PDEs [13]. Later Ilmanen [35] clarified and modified Brakke's construction and related it to the level set approach (see also Evans and Spruck [19]). In doing so, Evans and Spruck and Ilmanen showed that for almost every initial M_0 and for almost every t, the generalized flow M_t is smooth almost everywhere. Recent work of White [52, 53] and Huisken and Sinestrari [31] makes substantial improvements in this regularity in the case of mean convex M_0. Stimulated by these works, many other approaches to defining generalized flows have been developed, such as set theoretic motion (see [48, 5, 23]), DeGiorgi's definition of minimal barriers (see [9]), and the time step variational method of Almgren, Taylor, and Wang [3], to name a few. Roughly speaking, all of these definitions yield the same weak flow (which agrees with the smooth classical mean curvature flow as long as it exists) unless the level set flow "fattens" (develops an interior); see [15] for examples and [50, 23] for a discussion of various definitions. The fattening of the level set flow represents nonuniqueness of possible continuations of the flow past a singularity. In this regard, Ilmanen and White have announced the construction of a smooth embedded initial surface M_0 (of very high genus) for which fattening occurs. In this case, computations may randomly choose one possible flow.

An important remaining task for mathematicians, then, is to understand when fattening occurs, the nature of the possible singularities, and a better estimation of the size (Hausdorff dimension) of the singular set. For the applied mathematician, the task is to use these new mathematical tools to help solve impossible (ill-posed) problems. It is clear that new generations of engineers and applied mathematicians need to understand more and more geometric analysis (differential geometry, geometric measure theory, partial differential equations) in order to better carry out their research. Expository research monographs, such as Klaus Ecker's book, are an invaluable aid in this endeavor.

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Noncommutative Dynamics and *E*-Semigroups. *By William Arveson*. Springer-Verlag, New York, 2003. $79.95. x+434 pp., hardcover. ISBN 0-387-00151-4.

This book is a self-contained research monograph based on research on the theory of E_0-semigroups of the author and some others during the last decade or so. As the author himself is a dominant leading figure of the field, this book serves as a good introduction to the currently very active field of E_0-semigroups. The book assumes basic knowledge of the theory of operator algebras, C^*-algebras, and von Neumann algebras and the general theory of functional analysis. The author cites the following books on operator algebras as basic references: [Arv], [Dxm], and [Pdsn].

This reviewer thinks that this is a timely publication in an area that has advanced rapidly in recent years, and it is becoming increasingly difficult for a newcomer to join the research front without the guidance of an expert; at the same time the field has shown a certain maturity, so that a monograph evaluating the current status is highly desirable. The book under review fulfills this need. Nevertheless the reader is cautioned that the book is not a neutral treatise of the subject, but rather the research monograph of an author summarizing his own research with some new results and/or new approaches.

The theory of E-semigroups was first introduced by Powers in articles in 1987 [Pws1] and 1988 [Pws2], in conjunction with the theory of unbounded densely defined closed $*$-derivation on an operator algebra which was developed in the late 1970s and throughout the 1980s. A densely defined closed $*$-derivation does not necessarily generate a one-parameter automorphism group of the algebra in question unless the derivation satisfies a condition similar to the self-adjointness condition for a closed symmetric operator. It generates a one-parameter semigroup of $*$-endomorphisms instead. So Powers thought that there should be an analogy between the theory of symmetric operators and the theory of derivations and/or an analogy between the theory of one-parameter semigroups of isometries on a Hilbert space and the theory of one-parameter semigroups of endomorphisms of a von Neumann algebra. So he first introduced the index of the E-semigroup analogous to the deficiency index of a densely defined, closed symmetric operator. At any rate, the theory of the E-semigroup is far more complex and deep than Powers originally expected. Inspired by the above-mentioned papers by Powers, a small number of specialists started to work on the subject, including Arveson (the author of the book under review), Powers, Price, and Robinson. They mostly concentrated on the theory of E_0-semigroups on a factor $B(H)$ of type I, which is unexpectedly deep and complex. For instance, it allows each specialist to take different motivations and approaches to the subject. The author of this book took the point of view that an E_0-semigroup represents a continuous nonreversible noncommutative, or quantum, stochastic process. In fact, such a point of view brought about a radical development of the subject in recent years. The field is now rich enough that there are many ways to attack the subject. Before making further comment, let us take a closer look at the book.

Arveson devotes Chapter 1 to introductory discussions of motivations of E_0-semigroups. Then he divides the book into five parts:

Part 1. Index and Perturbation Theory, Chapters 2–4;

Part 2. Classification: Type I Cases, Chapters 5 and 6;

Part 3. Noncommutative Laplacians, Chapters 7–10;

Part 4. Causality and Dynamics, Chapters 11–12;

Part 5. Type III Examples, Chapters 13–14.

An E-semigroup is by definition a one-parameter family $\{\alpha_t : t \geq 0\}$ of $*$-endomorphisms of a von Neumann algebra $\mathcal{M}$:

(i) $\alpha_0 = \mathrm{id}_{\mathcal{M}}$;

(ii) $\alpha_s \circ \alpha_t = \alpha_{s+t}, \quad s, t \geq 0$;

(iii) for every $a \in \mathcal{M}$ and $\rho \in \mathcal{M}_*$, the function: $t \in \mathbb{R}_+ \mapsto \rho(\alpha_t(a)) \in \mathbb{C}$ is continuous.

If each α_t is unital, i.e., if $\alpha_t(1_{\mathcal{M}}) = 1_{\mathcal{M}}$, then it is called an E_0-semigroup. The book is almost exclusively concentrated on the theory of E_0-semigroups of a factor $B(H)$ of type I with the exception of the canonical E-semigroup associated with a product system E. A cocycle means a strongly continuous family $\{U_t : t \geq 0\} \subset \mathcal{M}$ such that $U_{s+t} = U_s \alpha_s(U_t), s, t \geq 0$. The cocycle perturbation α by a cocycle $U = \{U_t\}$ means the new E-semigroup ${}_U\alpha$ given by ${}_U\alpha_t(A) = \mathrm{Ad}(U_t)\alpha_t(A), t > 0$. Two E-semigroups α and β are called *cocycle conjugate* if α is conjugate to a cocycle perturbation of β. The theory of E-semigroups is mainly concerned with their cocycle conjugacy classification by computable invariants.

One quickly observes that if α is a $*$-endomorphism[1] of the factor $B(H)$ of all bounded operators on a Hilbert space H, then the set

$$E(\alpha) = \{A \in B(H) : AT = \alpha(A)T, T \in B(H)\}$$

carries an intrinsic Hilbert space structure as the product $T^*S, S, T \in E(\alpha)$, is a scalar multiple of the identity operator 1_H since it commutes with every operator of $B(H)$ as seen below:

$$\begin{aligned}(T^*S)A = T^*\alpha(A)S &= (\alpha(A^*)T)^*S \\ &= (TA^*)^*S = A(T^*S), \\ &\qquad S, T \in E(\alpha), A \in B(H).\end{aligned}$$

Conversely, a Hilbert space E of elements of $B(H)$ whose inner product is given by $T^*S, T, S \in E$, gives an endomorphism α^E determined by

$$\alpha^E(A) = \sum_i S_i A S_i^*, \quad A \in B(H),$$

with an orthonormal basis $\{S_i\}$ of E. Thus one has the bijective correspondence

$$\alpha = \alpha^{E(\alpha)}, \qquad E = E(\alpha^E).$$

The endomorphism α^E is unital in the sense that $\alpha^E(1) = 1$ if and only if the ranges of members of E span the entire space H. This is a fact discovered by mathematical physicist John Roberts [Rbt]. One should note here that every element of a Hilbert space E in $B(H)$ is a scalar multiple of an isometry; in particular, every vector of norm 1 is indeed an isometry. Also, the product of endomorphisms α and β corresponds to the product $E(\alpha)E(\beta)$ Hilbert space, i.e., the closed linear span of products of vectors of $E(\alpha)$ and $E(\beta)$ that behaves precisely like the tensor product as seen below:

$$\begin{aligned}&STA = S\beta(A)T = \alpha(\beta(A))ST, \\ &\qquad\qquad S \in E(\alpha), T \in E(\beta), A \in B(H); \\ &(S_2T_2)^*(S_1T_1) = T_2^*S_2^*S_1T_1 = T_2^*\langle S_1, S_2\rangle 1_H T_1 \\ &= \langle S_2, S_1\rangle T_1^*T_2 = \langle S_1, S_1\rangle\langle T_1, T_2\rangle 1_H, \\ &\qquad\qquad S_1, S_2 \in E(\alpha), T_1, T_2 \in E(\beta).\end{aligned}$$

So Arveson replaces a given E_0-semigroup $\alpha = \{a_t : t \geq 0\}$ of $B(H)$ by a one-parameter family $\mathcal{E}_\alpha = \{E(t) : t > 0\}$ of Hilbert spaces in $B(H)$. He calls this a concrete product system. The product of two intertwining Hilbert spaces $E(s)$ and $E(t)$ gives $E(s)E(t) = E(t)E(s) = E(s+t), s, t > 0$. This is the source of the terminology *product system* and also quickly relates to the theory of the continuous tensor product of Hilbert spaces, which is still a mystery despite its strong relevance to quantum field theory. The product system $\mathcal{E}_\alpha$ is easily seen to be a complete cocycle conjugacy invariant. From the beginning of the theory of E_0-semigroups, the existence of intertwining one-parameter semigroups $S = \{S(t) : t \geq 0\}$ such that $S(t)A = \alpha_t(A)S(t),\ t \geq 0,\ A \in B(H)$, was a great concern. He calls such a semigroup of isometries a *unit* and denotes the set of units by $\mathcal{U} = \mathcal{U}_\alpha$. In order to deal with this problem, he gives an abstract characterization of product systems by a natural set of postulates and calls an abstract product system E. He then associates the Hilbert space $L^2(E)$ of square-integrable cross sections of the abstract product system E and defines the left and right regular representation of

[1] In what follows, when we refer to an endomorphism, we always mean a $*$-endomorphism.

E and the spectral C^*-algebra $C^*(E)$, which behaves like the reduced group C^*-algebra of a locally compact group, and proves that the regular representation of $C^*(E)$ is faithful, a kind of amenability. Through the study of $C^*(E)$, he establishes that every abstract product system is uniquely isomorphic to a concrete product system $\mathcal{E}_\alpha$; i.e., it corresponds to an E_0-semigroup. In doing so, he makes a detailed analysis of states on $C^*(E)$ which gives rise to an E_0-semigroup by means of local behavior of the product system near the origin, a counterpart of the Powers approach based on derivations.

Arveson relates a unit to a decomposable element of the product system and proves the second $\mathbb{C}^\times$-valued cohomology vanishing on the additive semigroup $\mathbb{R}_+$ of nonnegative reals using his operator algebraic technique. Also, the product system approach allows him to introduce an inner product on the space of finitely supported complex-valued functions with total sum 0 over the set $\mathcal{U}_\alpha$ of units. Then he proves that the dimension of the Hilbert space obtained by the completion of this "semi-inner" product space is precisely the index of α defined by Powers and establishes the tensor product formula:

$$\mathrm{ind}(\alpha \otimes \beta) = \mathrm{ind}(\alpha) + \mathrm{ind}(\beta)$$

for E_0-semigroups α and β on the factor $B(H)$ of type I.

Given an E_0-semigroup α, an operator $T \in E_\alpha(t), t > 0$, is said to be *decomposable* if for any $0 < s < t$, there exists a pair $A \in E_\alpha(s)$ and $B \in E_\alpha(t-s)$ with $T = AB$. Let $D_\alpha(t)$ be the set of all decomposable operators in $E_\alpha(t)$. The E_0-semigroup α is said to be *decomposable* if the closed span of $D_\alpha(t)$ is $E_\alpha(t)$ for some $t > 0$, and equivalently for every $t > 0$.

The class of E_0-semigroups is divided into three groups: those of type I, of type II, and of type III, where an E_0-semigroup α is said to be of type I if it is decomposable; of type II if it is not decomposable but for some $t_0 > 0$, $D_\alpha(t_0) \neq \{0\}$; and of type III if it has no nonzero decomposable operators.

The cocycle conjugacy classification of E_0-semigroups of type I is completed by means of an index, and their models are given as CCR/CAR flows in Chapter 6.

This reviewer found interesting the part dealing with the C^*-dilation of a one-parameter semigroup of CP-contractions on a C^*-algebra and a von Neumann algebra, i.e., Part 3. There have been a number of attempts to develop a theory of dilations of a one-parameter semigroup of CP-contractions on a C^*-algebra into a one-parameter semigroup of endomorphisms on a larger C^*-algebra, so that the original semigroup is given as the corner of the semigroup of endomorphisms. Arveson has succeeded by constructing the universal dilation and then restricting it to the relevant representation of the system, restoring the minimality and/or normality in the case when the original system is a semigroup of normal CP-contractions of a von Neumann algebra. The construction does not appear to work for von Neumann algebras; in particular, the question about the normality does not appear to be easy, but it does work, surprisingly, in the end. Since many second-order elliptic differential operators give rise to semigroups of CP-contractions on the C^*-algebra of functions on the manifold in question, one could expect some applications of Arveson's dilation theory to differential operators, which remains to be seen at this point.

Part 4 can be summarized as Arveson's adaptation of scattering theory to the context of E_0-semigroups. In this part, he treats the problem in which an E_0-semigroup can be viewed as a subsystem of a reversible system, i.e., a corner of a one-parameter automorphism group.

Arveson concludes the monograph in Part 4 with some very mysterious examples of type III due to Powers and Tsirelson. This area is deeply related to the study of stochastic processes. The reviewer found the very last part of the book, section 14.5, interesting, where the author discusses the intrinsic Hilbert space—exactly the intrinsic Hilbert space $L^2(\mathcal{M})$ associated with a von Neumann algebra $\mathcal{M}$ that was discussed in the reviewer's book [Tks, Exercises XII.6.6 and 6.7] as it appears from different approaches and motivations.

Finally, the reviewer is happy to recommend this book.

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Mathematical Analysis. Volume I. *By V. A. Zorich*. Springer-Verlag, Berlin, 2004. $69.95. xviii+574 pp., hardcover. ISBN 3-540-40386-8.

Mathematical Analysis. Volume II. *By V. A. Zorich*. Springer-Verlag, Berlin, 2004. $69.95. xvi+681 pp., hardcover. ISBN 3-540-40633-6.

Let's get one thing straight from the very beginning. I like this two-volume set. It will make an excellent reference for students and provides a vast reservoir of interesting exercises and exam questions for analysis teachers. Get your library to order a copy as soon as possible.

I am not so keen on it as a text for an undergraduate course in real analysis. It certainly has its good points. It offers a panoramic view of analysis, beginning with an axiomatic introduction to the real line, followed by a complete development of all the standard material on differentiation and (Riemann) integration, both in one variable and in several variables. It also contains chapters on vector calculus, metric and topological spaces, differentiable manifolds, normed linear spaces, Banach spaces, distributions, Fourier series and transforms, and asymptotic expansions. However, covering the entire two-volume set would take my students three to four years. Yes, it could serve as an entire undergraduate curriculum in analysis, but the engineers would never stand for all that abstraction, and I doubt that many of my first-year undergraduates could handle the sophistication of the opening chapters. There is, of course, a huge difference between the average Russian undergraduate and the average American undergraduate, and that difference is amply demonstrated by this two-volume set.

What special features, beside enormous breadth, distinguish these volumes from other introductory analysis texts? I shall answer this question with seven declarative sentences (in boldface below), followed by detailed examples from the work to back up my claims.

1. The Foundations Are Carefully Laid. Volume I begins with material on rules and notation of logic, including truth tables. The writing is very dense. By Vol. I, p. 6, Russell's paradox has been discussed and a proof has been given that the "set of all sets" is contradictory. The supplemental material at the end of Chapter 1 (Vol. I, pp. 25–33) even discusses the cardinality of sets and the axioms of set theory, including the Axiom of Choice and the Axiom of Replacement.

Nothing is left to chance. For example, when the order axioms are introduced in Chapter 2, the transitive properties are stated not only for $\leq$, but also for the various combinations of $<$ and $\leq$; e.g., $x < y$, $y \leq z$ imply $x < z$, and $x \leq y$, $y < z$ imply $x < z$ (Vol. I, p. 41). Another example: The proof that a function is differentiable if and only if it has a tangent line is followed by a frank discussion (Vol. I, p. 184) which begins: "the analytic definition of tangent may cause vague uneasiness . . . " and then lays bare all of the difficulties that occur when making a rigorous definition of limits of secant lines, including what it means for points to converge to points along curves.

The author wants to banish imprecision and vagueness. Even the smallest points are discussed. Thus we find a proof that $n \in \mathbf{N}$ and $n \neq 1$ imply $n-1 \in \mathbf{N}$ (Vol. I, p. 47), a proof of the existence of $\sqrt{2}$ (Vol. I, p. 51), a complete description of positional notation with respect to an arbitrary base, not just base 10 (Vol. I, pp. 61–62), and a precise definition of exponents starting with integer powers and ending with irrational powers (Vol. I, pp. 118–121). He also uses the definition of logarithms to prove the usual laws of logarithms (Vol. I, p. 118). This work is complete and absolutely rigorous.

2. It Is Comprehensive and Encyclopedic. This is a text that the author expects students to use later for reference. Thus each subject is thoroughly covered before moving on to the next. For example, the product rule for differentiation is accompanied by Leibniz's rule for taking the nth derivative of a product. Tests for extrema include not only the standard second derivative test, but also the nth derivative test as well. Both versions of the change-of-variable formula on an interval are included, the one for $\mathcal{C}^\infty$ changes φ as well as the one for strictly monotone φ. The material on improper integrals includes the Cauchy principle value version as well as the usual "asymmetric" definition. There is even a discussion of Lusin's conjecture (Fourier series of continuous functions converge almost everywhere) and a reference to Carleson's magnificent solution to it.

Standard results are sometimes recast to milk every last bit of information from them. For example, to prove that a 1–1 continuous function has a continuous inverse, the author begins with: A continuous function on $[a, b]$ is 1–1 if and only if f is strictly monotone on $[a, b]$. His version of the Intermediate Value Theorem is: A monotonic real-valued function f on $[a, b]$ is continuous if and only if $f([a, b])$ is a closed interval with end points $f(a)$ and $f(b)$.

The treatment of topics is quite general. The discussion of uniform convergence includes not only sequences, but also parameter families, e.g., families indexed by a continuum. Fubini's theorem is proved not only for Riemann integrals, but also for improper integrals.

The author also includes some results that are usually reserved for a later course in Lebesgue integration, e.g., the inequalities of Young, Hölder, Minkowski, and Jensen. These are cleverly introduced first for finite sums (Vol. I, p. 248), with the integral versions left as exercises (Vol. I, pp. 358–359). Other "advanced topics" covered include elliptic integrals, complex infinite series, normed linear spaces, Banach spaces, and proofs of the Lebesgue criterion for Riemann integrability, the Baire category theorem, an existence theorem for the first-order initial value problem, the Ascoli–Arzelà theorem, and the Stone–Weierstrass theorem. The discussion of trigonometric series even includes a couple of results on uniqueness. The author also defines the Fréchet derivative (i.e., differentiation of functions with normed linear space domains), develops enough calculus of variations to solve the brachistochrome problem, and gives a physical interpretation of the n-dimensional curl via potential fields. He also includes material usually reserved for algebra courses, e.g., a discussion of the exponential of an operator with applications to matrices, and a description of homology and cohomology groups culminating with a proof of De Rham's theorem.

3. Material Is Carefully Motivated by Practical Considerations. You can tell that the department at Moscow State University is called the department of "Mathematics and Mechanics." The author derives Snell's law from Fermat's principle about minimal paths and includes discussions of barometric pressure, of motion of bodies with variable mass (e.g., rockets), of falling bodies in atmosphere, of pendulums, and of Buffon's needle problem.

One expects the derivative to be motivated by velocity and surface integrals to be motivated by work in a force field or flux of a liquid. But here we also find limits motivated by measurement of physical quantities (Vol. I, p. 79) and differentials motivated by the two-body problem (Vol. I, p. 173).

Even very abstract concepts are motivated by real-world problems, e.g., metric spaces by coding theory, distributions by a point mass attached to the end of an elastic spring, and differentiation of linear operators by angular velocity of a rigid body with a fixed point (e.g., a top). No student can come away without a profound appreciation for the applicability of analysis.

4. Important Ideas Are Introduced More Than Once. Early versions of a definition tend to be more concrete and less formal.

Later iterations of the same concept tend to be more abstract and, ultimately, very general. For example, a function is first defined as a "correspondence" that is single-valued (Vol. I, p. 11), and later defined as a relation which satisfies "$(x\mathcal{R}y_1) \wedge (x\mathcal{R}y_2) \implies (y_1 = y_2)$" (Vol. I, p. 21). The derivative is first defined for functions with real domains, then vector domains, and finally with normed linear space domains.

Major results are sometimes introduced informally first, and the theory is developed later. I found this especially effective for Taylor series. These are first introduced in a table that is used to obtain results such as $(x - \sin x)/x^3 \to 1/3!$ as $x \to 0$. Later, Taylor series are developed rigorously. By the time proofs about convergence of power series are presented, the student is already aware of how important Taylor series are, and ready, if not eager, to find out what limitations these wonderful tools have. The author is obviously an excellent teacher as well as a fully competent mathematician.

5. The Pace Accelerates as the Text Progresses. Concepts which appear more than once appear with increasingly greater abstraction and with increasingly less detail. For example, limits of functions are first treated thoroughly in great detail in the one-dimensional case, next (with less detail) in $\mathbf{R}^n$, then in metric spaces, and finally in topological spaces.

By presenting complete details in the more concrete cases, much of the theory for the more abstract cases is left as exercises. For example, the inverse function theorem is proved for the one-dimensional case and left as an exercise for the normed linear space-valued case (Vol. II, p. 105).

By the second volume, after his students have grown more sophisticated, the author begins to state without proof results which are either easy or tedious to verify. For example, he lets the reader verify that $\partial(E_1 \cup E_2)$, $\partial(E_1 \cap E_2)$, and $\partial(E_1 \setminus E_2)$ are subsets of $\partial E_1 \cup \partial E_2$ (Vol. II, p. 117), and that the definition of boundary point is independent of the choice of a local chart (Vol. II, p. 178).

Still, when a really tough result comes up, one whose proof has not been covered in a previous discussion, the author slows down and gives his usual careful, thorough presentation. Nowhere is this care more evident than in the section on change of variables in $\mathbf{R}^n$ (Vol. II, section 11.5). He begins with a heuristic argument that explains why $\det \phi'$ appears in the change-of-variables formula and why it should be nonzero, then breaks the proof into half a dozen steps, the first of which is that sets of measure zero are preserved by diffeomorphisms. When he starts looking at concrete cases (e.g., polar coordinates) he is careful to note that the hypotheses about the determinant of ϕ' not being zero do not hold, but shows how one can get around this difficulty by opening up the boundary. Not many texts even mention these difficulties. Indeed, one usually applies the change-of-variables formula to polar, cylindrical, and spherical coordinates without even considering whether the hypotheses hold or not.

6. This Two-Volume Set Contains Plenty of Good Examples. One place this work really shines is in its examples. Every definition is followed by nearly a dozen examples, some of them non-standard. These examples are often carefully laid out so that there is a gradual revelation of the nuances of the concept being illustrated. For example, the fourth example of a topological space is the set of germs of continuous functions (Vol. II, p. 11). Moreover, a significant portion of these examples are designed to demonstrate to the student that these ideas are useful; e.g., the third example of a function (Vol. I, p. 13) is the Galilean transition from one inertial coordinate system (x, t) to another (x', t') given by $x' = x - vt$, $t' = t$.

7. It Also Contains Plenty of Exercises. Nearly every section is followed by a massive set of exercises. Early exercises in a typical section tend to be standard and routine. This essential type of exercise is designed to "make friends with the concepts." Later exercises in a given section take more creativity and skill to solve.

Many exercises deal with applied mathematics. These range from modeling grinding lathes (Vol. I, p. 478) to the Heisenberg uncertainty principle (Vol. II, pp. 590–591).

Other exercises are devoted to extending the theory. This, of course, is a good idea. It gives the students a chance to master the material at a deeper level and keeps the text from being overburdened with too many details. The level of these exercises can be rather high for a beginning student; e.g., prove that $f(A \cap B) = f(A) \cap f(B)$ if and only if f is 1–1 (Vol. I, p. 23), construct the Hausdorff metric on closed subsets of a metric space (Vol. II, p. 9), verify the one-point compactification of a topological space (Vol. II, p. 18), or prove that there exist $\mathcal{C}^\infty$ partitions of unity (Vol. II, p. 148). Perhaps some of these exercises are designed to be given as projects. There is no doubt that students trained by such exercises are ready to do some serious work by the time they enter graduate school. The range of subject matter in these "projects" is impressive. They include Chebyshev polynomials, Lie algebras, Bessel functions, the Gamma function, Tauberian theorems, Legendre polynomials, Hermite polynomials, Haar polynomials, spherical harmonics, and Sturm–Liouville theory.

This two-volume set also contains a number of "thinking exercises," i.e., exercises which require several steps to solve. Although these exercises are a little too deep for the solution to be obvious, they are often easier to solve than they seem at first. All a student needs to do is look carefully at what is being asked, and then be willing to try the three or four tools that apply to the situation at hand. Here are some of my favorites.

(a) Find (in the context of Riemann sums) the limit, as $n \to \infty$, of

$$\frac{n}{(n+1)^2} + \cdots + \frac{n}{(2n)^2}.$$

(b) Prove that the algebraic numbers $a + b\sqrt{n}$ satisfy the Archimedean principle but not the completeness axiom.

(c) If $b_n/b_{n+1} = 1 + \beta_n$ and if $\sum \beta_n$ converges absolutely, prove that b_n converges to some finite real number, as $n \to \infty$.

(d) Find the infimum of

$$\int_a^b f(x)\,dx \int_a^b f^{-1}(x)\,dx$$

as f ranges over all real-valued, continuous functions that do not vanish on a closed interval $[a, b]$.

(e) If f is integrable, prove that

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} f(x_n)\,dx_n \cdots dx_2$$
$$= \frac{1}{n!} \int_0^x (x-y)^n f(y)\,dy.$$

(f) If f is continuous, prove that

$$\lim_{h \to 0} \int_{-1}^1 \frac{h}{h^2 + x^2} f(x)\,dx = f(0).$$

8. Unusual Touches. It is clear that although this text is very much in the Cauchy–Weierstrass tradition, the author has gone to the trouble of thinking out the curriculum with an eye toward simplification. This results in some unusual presentations. Examples follow.

One of the order axioms is that $x \le y$, $y \le x$ imply $x = y$. The trichotomy property is proved, not introduced as an axiom.

Zorich's version of the completeness axiom is that for each pair of real numbers $x \le y$, there is a real number c such that $x \le c \le y$.

He uses Bernoulli's inequality to prove that $(1 + 1/n)^n$ has a limit, as $n \to \infty$ (Vol. I, p. 89), by definition the natural base e, and derives from it the usual series representation of e (Vol. I, p. 103).

Zorich shows that if a_k is decreasing and nonnegative, then $\sum a_k$ converges if and only if $\sum 2^k a_{2^k}$ converges, and uses this result to prove the p-series test (Vol. I, p. 101).

He uses frames (essentially local coordinate systems) to define orientation of a surface.

He introduces the exterior product axiomatically, then shows that an arbitrary differential k-form is a "linear" combination of elementary k-forms, i.e., a finite sum of products of functions and basis elements $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

9. Other Features. The text is further enhanced by the historic notes that are sprinkled throughout. These add both a human and an international dimension to the text. I was especially intrigued by an implied ranking that the descriptions of individual mathematicians indicated; e.g., Johann Bernoulli is "from the distinguished family of Swiss mathematicians," Riemann is "the outstanding German mathematician," Lobachevskii is "a great Russian scholar," Kantorovich is "an eminent Soviet mathematician," but Darboux is only "a French mathematician."

10. What's NOT to Like? I found the excessive use of logical notation to be severe in spots. For example, the proof of DeMorgan's law (Vol. I, p. 9) contains practically no words at all, just symbols. This is easy for a trained mathematician to read, but I wonder what a beginning student would make of it. Even some definitions are given in this style; e.g., the "definition" of the *image* of a set X under a function f (Vol. I, p. 12) is stated as

$$f(X) := \{y \in Y \bigm| \exists x\,((x \in X) \wedge (y = f(x)))\},$$

and that of *infimum* (Vol. I, p. 45) as

$$i = \inf X := \forall x \\ \in X((i \le x) \wedge (\forall i' > i\,\exists x' \in X(x' < i'))).$$

Granted, this happens only occasionally after the first chapter, so it's easy to ignore. Moreover, this habit practically disappears by the middle of Volume II.

I also found the differences between Russian and American nomenclature annoying after a while. We find Borel–Lebesgue for Heine–Borel, Schwarz–Bunyakovsky for Cauchy–Schwarz, generalized functions for distributions, fundamental sequence for Cauchy sequence, the Lagrange finite-increment theorem for the mean value theorem (at least "Cauchy's finite-increment theorem" is called the generalized mean value theorem in a footnote), the Gauss–Ostrogradskii theorem for the divergence theorem, and rapidly decreasing functions for Schwartz functions. Before you hand this two-volume set to one of your students, you might consider explaining that standard American usage is different, and provide a dictionary so that the student can learn our terminology (hence, be able to communicate with the rest of us).

11. Misprints. I read nearly every page of both volumes and found very few misprints.

1. Exercise 3a in Vol. I (p. 11) asks for a proof of the same version of DeMorgan's law which was proved in the text on p. 9.
2. The M is missing from the Weierstrass M-Test (Vol. I, p. 99).
3. The set $\{x \in X : d(a, x) \ge r\}$ is supposedly open (Vol. II, p. 5).
4. The function $\chi_{E_1 \cap E_2}$ should be multiplied by f (Vol. II, p. 123, line 11).
5. The symbol $\int$ limits$Y\,dy$ should be $\int_Y\, dy$ (Vol. II, p. 127, line -3).
6. "Orhogan" (Vol. II, p. 522) should be "orthogonal."

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Solving Nonlinear Equations with Newton's Method. *By C. T. Kelley.* SIAM, Philadelphia, 2003. $39.00. xiv+104 pp., softcover. ISBN 0-89871-546-6.

This excellent short book is a practical guide to the solution of nonlinear equations by Newton's method, where "Newton's method" is to be understood in a rather broad sense. The subject matter is a class of iterative methods that solve a system of linear equations at each iteration to determine a correction. For safety, a simple line search is used to determine whether or not to take a "full" step. At each iteration the norm of the residual should be reduced. The matrix in the linear system to be solved is the Jacobian or an approximation to the Jacobian, and the system may be solved "exactly" by a direct method or approximately by an iterative method.

The book has four chapters. The first is an introduction and overview. Chapter 2 discusses direct methods (Gaussian elimination) for solving the linear equations exactly. In this situation there is pressure to avoid frequent updates of the Jacobian, since each update requires a new LU factorization. When an update is made, the Jacobian can be computed exactly or approximately. Chapter 3 covers iterative (Krylov subspace) methods for solving the linear systems. Here we have iterations within iterations; that is, we have "inner" and "outer" iterations. On each outer iteration we hope to obtain a sufficiently good approximation in only a few inner iterations, so preconditioners play a big role. The final chapter discusses Broyden's method, which builds increasingly good approximations to the Jacobian by making a rank-one update at each (outer) iteration. Here, too, preconditioners are important. The objective is to precondition the nonlinear system in such a way that the identity matrix is a good initial approximation to the Jacobian.

Each chapter includes a brief discussion of theory and practice, advice about which methods can be expected to work well in which situations, and a list of things that can go wrong. Each of the methods is illustrated by a variety of examples coded in MATLAB. The largest examples are from discretizations of nonlinear partial differential equations, both steady state and time dependent. All of the MATLAB codes can be downloaded from the SIAM web site (http://www.siam.org/books/fa01/) so readers can try them out, play with them, modify them, and use them as templates for solving their own problems.

In the preface the author states that he assumes the reader has a good understanding of numerical analysis at the level of Atkinson's *An Introduction to Numerical Analysis* and of numerical linear algebra at the level of Demmel or Trefethen and Bau. Although these prerequisites are technically correct, I hope that students will not be intimidated by them. Whether they have the prerequisites or not, I would encourage them to dive right in and learn something.

This book promises to be a useful supplement to a variety of numerical analysis courses and a helpful guide for practitioners who need to solve nonlinear systems in their research.

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