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CONFLUENT MAPPINGS AND ARC KELLEY CONTINUA

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ABSTRACT. A Kelley continuum X, also called a continuum with the property of Kelley, such that, for each $p \in X$, each subcontinuum K containing p is approximated by arcwise connected continua containing p, is called an *arc Kelley continuum.* A continuum homeomorphic to the inverse limit of locally connected continua with confluent bonding maps is said to be *confluently $\mathcal{LC}$-representable.* The main subject of the paper is a study of deep connections between the arc Kelley continua and confluent mappings. It is shown that if a continuum X admits, for each $\varepsilon > 0$, a confluent ε-mapping onto a(n) (arc) Kelley continuum, then X itself is a(n) (arc) Kelley continuum. In particular each confluently $\mathcal{LC}$-representable continuum is arc Kelley. It is also proved that if continua X and Y are confluently $\mathcal{LC}$-representable, then also are their product $X \times Y$ and the hyperspaces 2^X and $C(X)$.

1. Introduction. The arc Kelley continua form a natural subclass of Kelley continua, known also as continua with the property of Kelley, or continua with property κ. In a recent study the authors proved that each absolute retract for any of the classes of: hereditarily unicoherent continua, tree-like continua, λ-dendroids and dendroids, is an arc Kelley continuum, [**9**]. All absolute retracts mentioned above share this property with all members of another significant class, the class $\mathcal{LC}$ of locally connected continua. One of the basic and essential results of the previous study says that every confluently $\mathcal{T}$-representable continuum (i.e., the inverse limit of trees with confluent bonding mappings, $\mathcal{T}$ stands for the class of trees) is an absolute retract for hereditarily unicoherent (tree-like) continua, [**11**]. These results led to the question whether there is some deep connection between confluent mappings and arc Kelley continua. For instance, is the inverse limit of arc

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Kelley continua with confluent bonding maps an arc Kelley continuum? A similar question for Kelley continua is much easier to answer, see Theorem 2.2 below. In the case of arc Kelley continua a real difficulty is to prove that there is a sufficiently numerous family of arcwise connected subsets in a space where the existence of a single arc is far from being obvious. The study of this question, in particular, and of connections between arc Kelley continua and confluent mappings in general, is the main subject of this paper.

One of the most important results of the paper, and the main result of Section 2, provides an answer to the above question in the affirmative. More generally, we prove that the class of arc Kelley continua is confluently whole (a class $\mathcal{K}$ of compacta is said to be confluently whole if each confluently $\mathcal{K}$-like compactum belongs to $\mathcal{K}$). Special attention is paid in the paper to the classes of confluently $\mathcal{LC}$-like and confluently $\mathcal{LC}$-representable continua. As an application of results obtained in Section 2, we investigate in the next section a particular but important case of atriodic confluently $\mathcal{LC}$-like continua. Section 4 is devoted to studying properties of products and of hyperspaces of the considered continua. For instance, we show that if a continuum is in any of these classes, then its hyperspaces of subcontinua and of closed subsets also are in the respective classes.

By a *space* we mean a topological space, and a *mapping* means a continuous function. Let X be a metric space with a metric d. For a point $p \in X$ and a subset $A \subset X$ define $d(p, A) = \inf\{d(p, a) : a \in A\}$; we denote by $B(p, \varepsilon)$ the (open) ball in X centered at a point $p \in X$ and having the radius ε, and we put $N(A, \varepsilon) = \cup\{B(a, \varepsilon) : a \in A\}$. The symbol $\mathbf{N}$ stands for the set of positive integers, and $\mathbf{R}_+$ denotes the set of positive real numbers. The symbol $\operatorname{Lim} A_n$ stands for the limit of a sequence of sets A_n as defined for example in [**31**, 4.9, page 56].

A *compactum* means a compact metric space and by a *continuum* we mean a connected compactum. A *tree* is a graph containing no simple closed curve. A continuum is said to be *tree-like* (*arc-like, circle-like*) provided that for each $\varepsilon > 0$ it admits an ε-mapping onto a tree (an arc, a circle, respectively); equivalently, provided that it is the inverse limit of an inverse sequence of trees (arcs, circles, respectively).

Given a metric continuum X, we let $C(X)$ denote the hyperspace of nonempty subcontinua of X equipped with the Hausdorff distance H, see e.g., [**30**, (0.1), page 1, (0.12), page 10]. For a point $p \in X$ the symbol $C(p, X)$ denotes the hyperspace of subcontinua of X that contain p.

The following notation will be used. Given an inverse sequence $\mathbf{S} = \{X_n, f_m^n\}$ of compact spaces X_n with bonding mappings $f_m^n : X_n \to X_m$, where the set of positive integers $\mathbf{N}$ is taken as the directed set of indices, we denote by $X = \varprojlim \mathbf{S}$ its inverse limit and by $f_n : X \to X_n$ the projections.

A mapping $f : X \to Y$ between continua is said to be:

(i) *open* if it maps open subsets of the domain onto open subsets of the range;

(ii) *monotone* if it has connected point inverses;

(iii) *confluent* provided that for each subcontinuum Q of Y and for each component K of $f^{-1}(Q)$ the equality $f(K) = Q$ holds.

(iv) *OM-mapping* provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$ such that f_1 is monotone and f_2 is open.

Obviously each monotone mapping is confluent, and each open one is also, see [**35**, Theorem 7.5, page 148]. For relations between these classes of mappings and for their properties, see e.g., [**6, 27**].

Let an $\varepsilon > 0$ be given. A mapping $f : X \to Y$ from a metric continuum X is called an *ε-mapping* provided that $\operatorname{diam} f^{-1}(f(x)) < \varepsilon$ for each point $x \in X$.

Let $\mathcal{K}$ be a class of continua. The following definition is patterned after the well-known concepts of arc-like, tree-like, circle-like, and, in general, $\mathcal{P}$-like spaces, where $\mathcal{P}$ is a given collection of compacta, see [**31**, 2.12, 2.13, page 24; Proposition 12.18, page 243; Theorem 12.19, page 246]; compare also [**28**, page 71 and Lemma 1, page 73].

Definition 1.1. A continuum X is said to be *confluently $\mathcal{K}$-like* provided that for each $\varepsilon > 0$ there is a confluent ε-mapping from X onto a continuum Y belonging to $\mathcal{K}$.

We will also consider classes $\mathcal{K}$ of continua that are closed in the topology generated by confluent ε-mappings. The next definition says this in a more precise way.

Definition 1.2. A class $\mathcal{K}$ of continua is said to be *confluently whole* provided that the class of confluently $\mathcal{K}$-like continua coincides with $\mathcal{K}$.

Following the term of a "representation by inverse limits" in [**31**, page 241], let us accept the following definition.

Definition 1.3. A continuum X is said to be *confluently $\mathcal{K}$-representable* provided that it can be represented as the inverse limit of continua belonging to $\mathcal{K}$, with confluent bonding mappings, i.e., if there exist a sequence of continua $X_n \in \mathcal{K}$ and a sequence of confluent mappings $f_m^n : X_n \to X_m$ such that X is homeomorphic to the inverse limit $\varprojlim \{X_n, f_m^n\}$. In the sequel we will neglect the mentioned homeomorphism, and we will simply write $X = \varprojlim \{X_n, f_m^n\}$.

Let us recall that some results concerning confluently $\mathcal{LC}$-representable continua have already appeared in the literature. For example, Hosokawa has shown in [**18**, Theorem 2.7, page 775] that if a mapping $f : X \to Y$ from a continuum X onto a $\mathcal{LC}$-representable continuum Y is confluent, then the induced mapping $C(f)$ is also confluent.

Since the projections in the inverse sequence of compacta with confluent bonding mappings are confluent, [**7**, Corollary 4, page 5], the following statement is a consequence of [**30**, Lemma 1.162, page 167].

Statement 1.4. For each class $\mathcal{K}$ of continua, if a continuum is confluently $\mathcal{K}$-representable, then it is confluently $\mathcal{K}$-like.

Question 1.5. Let $\mathcal{K}$ be: (a) an arbitrary class of continua, (b) the class $\mathcal{LC}$ of locally connected continua, (c) the class of compact connected pohyhedra. Is it true that if a continuum is confluently $\mathcal{K}$-like then it is confluently $\mathcal{K}$-representable?

The first version of this article was written before the results of the paper [**32**] were obtained. In that paper a substantial progress was obtained in the study of Question 1.5. Indeed, it is proved in [**32**, Theorem 3.2] that if X is a one-dimensional continuum, then for any collection $\mathcal{K}$ of graphs, X is confluently $\mathcal{K}$-like if and only if X is confluently $\mathcal{K}$-representable. Unfortunately, the methods of [**32**], which are very technical, cannot be directly applied to higher dimensions. In this paper, in Section 3, we present a simpler proof of results of [**32**] in the particular case $\mathcal{K} = \{\text{an arc, a circle}\}$.

2. Arc Kelley continua. A metric continuum X is called a *Kelley continuum* provided that for each point $p \in X$, for each subcontinuum K of X containing p and for each sequence of points p_n converging to p, there exists a sequence of subcontinua K_n of X containing p_n and converging to the continuum K (see e.g., [**30**, Definition 16.10, page 538]).

The property, introduced by J.L. Kelley as Property 3.2 in [**20**, page 26], has been used there to study hyperspaces, in particular their contractibility (see, e.g., [**30**, Chapter 16], where references for further results in this area are given). Now Kelley continua play an important role in the investigation of various properties of continua, they are interesting in their own right, and have numerous applications to continuum theory; many of them are not related to hyperspaces.

The following condition equivalent to being Kelley continuum will be used in the sequel, see [**8**, Observation 1.1, page 258].

Proposition 2.1. *A metric continuum X is Kelley if and only if the following condition holds.*

(2.1.1) *For each $\varepsilon > 0$, for each $K \in C(X)$, for each point $p \in K$ and for each sequence of points p_n converging to p there is a sequence of continua $K_n \in C(p_n, X)$ such that if a subsequence K_{n_j} is convergent to a continuum L, then $H(K, L) < \varepsilon$.*

In the next result we use the term of a confluently whole class of continua as introduced in Definition 1.2. Its proof is a straightforward application of Proposition 2.1.

Theorem 2.2. *The class of metric Kelley continua is confluently whole.*

Statement 1.4 and Theorem 2.2 imply the following.

Corollary 2.3. *Each confluently $\mathcal{LC}$-representable continuum is Kelley.*

Remark 2.4. (a) The above corollary is also a consequence of Theorem 2 of [**14**, page 190] saying that an inverse sequence of Kelley continua with confluent bonding mappings is a Kelley continuum. However, stronger results than Corollary 2.3 are shown below, see Corollary 2.22.

(b) It is shown in Remark 3.8 below that the implication in Corollary 2.3 cannot be reversed.

Remark 2.5. The definition of a Kelley continuum has been localized in [**34**, Section II, page 291] as follows. A continuum X is *Kelley at a point* $p \in X$ provided that for each $K \in C(p, X)$ and for each sequence p_n converging to p there exists a sequence $K_n \in C(p_n, X)$ converging to K. Using this concept and repeating arguments as in the proof of Theorem 2.2, one can prove the following localized version of that theorem.

Theorem 2.6. *Let p be a point of a continuum X. If, for each $\varepsilon > 0$, the continuum X admits a confluent ε-mapping $f : X \to Y$ onto some continuum Y such that Y is Kelley at $f(p)$, then X is Kelley at p.*

A continuum X is said to have the *the arc approximation property* provided that for each point $p \in X$, for each subcontinuum K of X containing p there exists a sequence of arcwise connected subcontinua K_n of X containing p and converging to the continuum K, see [**16**, Section 3, page 113]. The following proposition is known, see [**16**, Proposition 3.10, page 116].

Proposition 2.7. *If a continuum has the arc approximation property, then each arc component of the continuum is dense.*

Investigating absolute retracts for some classes of continua, we have found that the following concept of arc Kelley continua that joins the arc approximation property and Kelley continua turns out to be both natural and useful. See [**9**, Definition 3.3].

A continuum X is called *arc Kelley* provided that for each point $p \in X$, for each subcontinuum K of X containing p and for each sequence of points $p_n \in X$ converging to p, there exists a sequence of arcwise connected subcontinua K_n of X containing p_n and converging to the continuum K. Therefore, the following equivalence holds, see [**9**, Proposition 3.4].

Proposition 2.8. *A continuum is an arc Kelley continuum if and only if it is a Kelley continuum with the arc approximation property.*

Remark 2.9. In the definition of arc Kelley continua, the arcwise connected continua K_n can be approximated from inside by locally connected continua. Thus, we can assume in this definition that the continua K_n are locally connected.

Since arc approximation property is invariant under weakly confluent mappings (thus under confluent ones), see [**16**, Theorem 3.5, page 114], and since Kelley continua are preserved under confluent mappings, see [**34**, Theorem 4.3, page 296], we have the next result.

Proposition 2.10. *The confluent image of an arc Kelley continuum is an arc Kelley continuum, i.e., if a continuum X is arc Kelley and a surjection $f : X \to Y$ is confluent, then Y is arc Kelley.*

Since the projections in an inverse sequence of continua with confluent bonding mappings are confluent, [**7**, Corollary 4, page 5], Proposition 2.10 leads to the following corollary.

Corollary 2.11. *Let* $X = \varprojlim \{X_n, f_m^n\}$, *where* X_n *are continua and* f_m^n *are confluent. If* X *is an arc Kelley continuum, then each factor space* X_n *also is an arc Kelley continuum.*

However, the opposite implication, from each factor continuum being arc Kelley to the inverse limit being arc Kelley (if the bonding mappings are confluent) is much more interesting for us. The most natural way of showing this implication is to use Proposition 2.8 and prove the implication separately for Kelley continua and for the arc approximation property. For Kelley continua the implication is already known, i.e., Kelley continua are preserved under the inverse limit operation with confluent bonding mappings, [**14**, Theorem 2, page 190]. But we do not know whether for arc approximation property the implication is true or not, and thus this "natural way" cannot be applied. The authors are obliged to P. Krupski for his contribution to the discussion about the following question.

Question 2.12. Let $\{X_n, f_m^n\}$ be an inverse sequence of arc Kelley continua X_n with confluent bonding mappings f_m^n. Is the inverse limit of this sequence an arc Kelley continuum?

Let us come back to the obstacles discussed before Question 2.12. To overcome them some auxiliary concepts will be used.

A finite indexed collection $\mathcal{M} = (M_1, \dots, M_k)$ of continua is called a *weak chain of continua* provided that $M_i \cap M_{i+1} \neq \varnothing$ for each $i \in \{1, \dots, k-1\}$. A weak chain is called a *weak* ε*-chain* if $\operatorname{diam} M_i < \varepsilon$ for each $i \in \{1, \dots, k\}$, see [**31**, 8.11, page 125]. If a point $p \in M_1$ is fixed, then we say that the weak ε-chain *starts at* p. If, moreover, a point q is in M_k, then $\mathcal{M}$ is said to be *from* p *to* q. Let $\cup\mathcal{M}$ stand for $M_1 \cup \dots \cup M_k$. Recall the following observation, see [**31**, Lemma 8.13, page 125].

Observation 2.13. If a continuum C is the union of a finite collection of its subcontinua, then for every two points $c_1, c_2 \in C$ the elements of the collection can be indexed so as to form a weak chain from c_1 to c_2.

Lemma 2.14. *Let a continuum X satisfy the following condition.*

(2.15) *For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each continuum K in X, for every three points $p, q \in K$ and $r \in X$ with $d(p,r) < \delta$ there exists a weak ε-chain $(M_1, \dots, M_k)$ of continua in X such that $r \in M_1$, $H(K, M_1 \cup \dots \cup M_k) < \varepsilon$, and $q \in N(M_k, \varepsilon)$.*

Let $\mathcal{M} = (M_1, \dots, M_k)$ be a weak chain of continua in X starting at some point $p_0 \in X$. Then for each $\varepsilon > 0$ there are a weak ε-chain $\mathcal{L} = (L_1, \dots, L_m)$ of continua in X and a function $\alpha : \{1, \dots, m\} \to \{1, \dots, k\}$ such that:

(a) $p_0 \in L_1$;

(b) *if $i \leq j$, then $\alpha(i) \leq \alpha(j)$;*

(c) $H(\cup\{L_i : \alpha(i) = j\}, M_j) < \varepsilon$ *for each $j \in \{1, \dots, k\}$.*

Proof. We will use the function $\delta : \mathbf{R}_+ \to \mathbf{R}_+$ as defined in condition (2.15) for the continuum X such that $\delta(\varepsilon) < \varepsilon$. For a given $\varepsilon > 0$ we define $\delta_0 = \varepsilon$ and $\delta_{i+1} = \delta(\delta_i)$.

For each $i \in \{1, \dots, k\}$, choose a point $q_i \in M_i \cap M_{i+1}$. The construction will be in k steps.

In the first step apply (2.15) for $K = M_1$, $p = r = p_0$, $q = q_1$ and $\varepsilon = \delta_{k-1}$. Observe that in this step we have $d(p,r) = 0 < \delta(\delta_{k-1})$. According to (2.15) we get a weak δ_{k-1}-chain of continua $(L_1, \dots, L_{m_1})$ from p_0 to some point r_1 such that $d(r_1, q_1) < \delta_{k-1}$ and that $H(L_1 \cup \dots \cup L_{m_1}, M_1) < \delta_{k-1}$.

In the second step apply (2.15) for $K = M_2$, $p = q_1$, $r = r_1$ $q = q_2$ and $\varepsilon = \delta_{k-2}$. Note that in this step we have $d(p,r) = d(r_1, q_1) < \delta_{k-1} = \delta(\delta_{k-2})$. By (2.15) we obtain a weak δ_{k-2}-chain of continua $(L_{m_1+1}, \dots, L_{m_2})$ from the point r_1 to some point r_2 such that $d(r_2, q_2) < \delta_{k-2}$ and that $H(L_{m_1+1} \cup \dots \cup L_{m_2}, M_2) < \delta_{k-2}$.

Repeating this construction k times we obtain a finite sequence of weak chains of continua

$$(L_{m_i+1}, \dots, L_{m_{i+1}}) \quad \text{for} \quad i \in \{0, \dots, k-1\} \quad \text{with} \quad m_0 = 0$$

such that $H(L_{m_i+1} \cup \dots \cup L_{m_{i+1}}, M_{i+1}) < \delta_{k-i-1} \leq \varepsilon$ and $L_{m_i} \cap L_{m_i+1} \neq \varnothing$. Define $m = m_k$ and $\alpha(j) = i$ for $m_{i-1}+1 \leq j \leq m_i$, where $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, m\}$. It follows from the construction that

$(L_1, \ldots, L_m)$ is a weak ε-chain of continua that satisfies the needed conditions (a), (b) and (c). □

For an arbitrary continuum X, define

$$\tau(X) = \inf\{\varepsilon > 0 : \text{there exists a finite collection of subcontinua } M_i \text{ of } X \text{ with } \operatorname{diam} M_i < \varepsilon \text{ that covers } X\}.$$

Then $\tau(X) \geq 0$. Note that $\tau(X)$ is a kind of a measure of nonlocal connectedness of X. Namely, it is related to the property S of Sierpiński, and thereby the following is true (compare [**31**, Theorem 8.4, page 120]).

Statement 2.16. A continuum X is locally connected if and only if $\tau(X) = 0$.

The next result is a fundamental intermediate step in the proof of one of the main results of the paper, viz. Theorem 2.21.

Theorem 2.17. *Let X be a continuum. Then X is an arc Kelley continuum if and only if the following condition holds.*

(2.18) *For each continuum K in X, for each point $p \in K$ and for each sequence of points $p_n \in X$ converging to p, there is a sequence of continua K_n converging to K with $p_n \in K_n$ and $\lim \tau(K_n) = 0$.*

Proof. It follows from Remark 2.9 and Statement 2.16 that if X is an arc Kelley continuum, then condition (2.18) holds. So, one implication is proved.

To show the other one, assume (2.18). By Observation 2.13 condition (2.18) implies the following one.

(2.19) *For each continuum K in X, for each point $p \in K$ and for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for each point $r \in X$ with $d(p, r) < \delta$ there exists a weak ε-chain $\mathcal{M}$ of continua in X that starts at the point r and such that $H(K, \cup\mathcal{M}) < \varepsilon$.*

We will show that condition (2.19) is equivalent to its uniform variant (2.15). Indeed, note that the set

$$Z = \{(K, p, q, r) : K \in C(X); p, q \in K; \text{ and } r \in X\}$$

is a closed subset of $C(X) \times X^3$, so we can apply a standard method of proof. Suppose on the contrary that there is an $\varepsilon_0 > 0$ such that for each $n \in \mathbf{N}$ there is a quadruple $(K_n, p_n, q_n, r_n) \in Z$ with $d(p_n, r_n) < 1/n$ satisfying the following condition:

$(*)$ there is no weak ε_0-chain $\mathcal{M}_n$ of continua in X from r_n to q_n such that $H(K_n, \cup\mathcal{M}_n) < \varepsilon_0$ and q_n is in an ε_0-neighborhood of the last link of $\mathcal{M}_n$.

By compactness of the set Z, we can assume that the sequence of quadruples (K_n, p_n, q_n, r_n) converges to (K_0, p_0, q_0, r_0) (note that $p_0 = r_0$). By (2.19) there are weak $1/n$-chains $\mathcal{M}'_n$ starting at r_n and such that $H(K_n, \cup\mathcal{M}'_n) < 1/n$. Again, by compactness, the unions $\cup\mathcal{M}'_n$ converge to K_0. Thus, they are ε_0-close to continua K_n for almost all $n \in \mathbf{N}$, contrary to $(*)$. Therefore, (2.15) holds, and thus Lemma 2.14 can be applied to the continuum X.

Note that (2.18) implies by definition that X is a Kelley continuum. To prove that it is an arc Kelley continuum, it remains to show, according to Proposition 2.7, that it has the arc approximation property. To this aim, take a subcontinuum K of X, choose a point $p \in K$ and fix some $\varepsilon > 0$. To complete the proof, it is enough to find a locally connected continuum $L \subset X$ such that $p \in L$ and $H(K, L) < \varepsilon$.

Applying Lemma 2.14, we construct, by induction, a sequence of weak chains $(L_1^n, \dots, L_{m_n}^n)$ and functions $\alpha_n : \{1, \dots, m_{n+1}\} \to \{1, \dots, m_n\}$, where $n \in \mathbf{N}$, such that putting

$$(**) \qquad L_n = L_1^n \cup \dots \cup L_{m_n}^n$$

we have

(i) $m_1 = 1$ and $L_1^1 = K$;

(ii) $(L_1^n, \dots, L_{m_n}^n)$ is a weak $\varepsilon/(2^{n+1})$-chain of continua for each $n > 1$;

(iii) $p \in L_1^n$ for each $n \in \mathbf{N}$;

(iv) if $i \le j$, then $\alpha_n(i) \le \alpha_n(j)$ for each $n \in \mathbf{N}$;

(v) $H(L_j^n, \cup\{L_i^{n+1} : \alpha_n(i) = j\}) < \varepsilon/2^{n+1}$.

For each L_n defined by $(**)$ by condition (v) we get

(vi) $H(L_n, L_{n+1}) < \varepsilon/2^{n+1}$.

Thus, the sequence of continua L_n converges to some continuum L and $H(K, L) < \varepsilon$. We will prove that L is locally connected by showing that $\tau(L) = 0$, see Statement 2.16.

For a given number $\xi > 0$, choose $k > 1$ such that $\varepsilon/2^k < \xi/3$. Let, for $n \in \{0, 1, 2, \dots\}$, a function $\beta_k^{k+n} : \{1, \dots, m_{k+n}\} \to \{1, \dots, m_k\}$ be the identity if $n = 0$, and if $n > 0$, put $\beta_k^{k+n} = \alpha_{k+1} \circ \cdots \circ \alpha_{k+n}$. Define a sequence of weak chains $(M_1^{k+n}, \dots, M_{m_k}^{k+n})$ for $n \in \{0, 1, 2, \dots\}$, by

$$M_i^{k+n} = \bigcup \{L_j^{k+n} : \beta_k^{k+n}(j) = i\} \quad \text{for} \quad i \in \{1, \dots, m_k\}.$$

Thus, we have $H(M_i^{k+n}, M_i^{k+n+1}) < \varepsilon/2^{k+n+1}$. Therefore, if n tends to infinity, the sequence of continua M_i^{k+n} converges to a continuum $M_i \subset L$, and $H(L_i^k, M_i) = H(M_i^k, M_i) < \varepsilon/2^k$. Since $\operatorname{diam}(L_i^k) < \varepsilon/2^{k+1}$, we have $\operatorname{diam}(M_i) \le \operatorname{diam}(L_i^k) + 2H(L_i^k, M_i) < 3\varepsilon/2^k < \xi$. To finish the proof, it is enough to observe that, by construction, $L = M_1 \cup M_2 \cup \cdots \cup M_{m_k}$. □

To show the next result we need the following lemma.

Lemma 2.20. *Given $\varepsilon > 0$, let $f : X \to Y$ be a confluent ε-mapping between compacta X and Y, and let $M \subset Y$ be a locally connected continuum. Then there exists a subcontinuum $K \subset X$ such that $f(K) = M$ and $\tau(K) < \varepsilon$.*

Proof. According to the Whyburn factorization theorem applied to confluent mappings, see [**35**, (2.3), page 297] and [**3**, VII, page 215], the mapping f can be uniquely factorized as the composition $f = f_2 \circ f_1$ such that $f_1 : X \to Z$ is monotone and $f_2 : Z \to Y$ is light and confluent. By the Hahn-Mazurkiewicz theorem [**31**, 8.14, page 126], there exists a path $g : [0, 1] \to M$ such that $g([0, 1]) = M$. By the path-lifting property for light confluent mappings (see [**13**, Theorem 16], compare [**22**, Main Theorem, page 357 and Corollaries 2.1 and 2.3, page 364]) there is a path $g_1 : [0, 1] \to Z$ such that $g = f_2 \circ g_1$. Define $K' = g_1([0, 1]) \subset Z$. Thus K' is a locally connected continuum such that $f_2(K') = M$. Let $K = f_1^{-1}(K')$. Then $f(K) = (f_2 \circ f_1)(f_1^{-1}(K')) = f_2(K') = M$. To finish the proof, observe that the partial mapping $f_1|K : K \to K'$ is monotone and has point-inverses of diameters less than ε. Therefore, for sufficiently small subcontinua of K' their preimages under f_1 are continua of diameters less than ε. Since the continuum K' is locally connected, it has the property S of Sierpiński, see [**31**, Theorem 8.4, page 120], and thus it can be represented as the union $K' = K'_1 \cup \cdots \cup K'_m$ of finitely many continua K'_i having diameters so small that for each $i \in \{1, \ldots, m\}$ we have $\operatorname{diam}(f_1^{-1}(K'_i)) < \varepsilon$. Since $K = f_1^{-1}(K'_1) \cup \cdots \cup f_1^{-1}(K'_m)$, we get $\tau(K) < \varepsilon$ as needed. The proof is complete. □

The following theorem is the most interesting and one of the main results of the paper.

Theorem 2.21. *The class of metric arc Kelley continua is confluently whole.*

Proof. Let $\mathcal{K}$ denote the class of metric Kelley continua, and let a metric continuum X be confluently $\mathcal{K}$-like. In view of Theorem 2.17, it suffices to prove that X satisfies condition (2.18).

Take a subcontinuum K of X, a point $p \in K$ and a sequence of points $p_n \in X$ converging to p. For each $n \in \mathbf{N}$, let $f_n : X \to Y_n$ be a confluent $1/n$-mapping from X onto an arc Kelley continuum Y_n. Thus, each Y_n is Kelley continuum according to Proposition 2.8. By Theorem 2.2 X is a Kelley continuum. Therefore, there are continua $L_n \subset X$ such that $p_n \in L_n$ and $\operatorname{Lim} L_n = K$. Since, for each $n \in \mathbf{N}$, Y_n is an arc Kelley continuum, the continuum $f_n(L_n) \subset Y_n$ is the limit of some locally connected, see Remark 2.9, continua $M_{n,m} \subset Y_n$ such that $q_n = f_n(p_n) \in M_{n,m}$ for each $m \in \mathbf{N}$. For each pair of indices n, m denote by $K_{n,m}$ a continuum obtained from the continuum $M_{n,m}$ according to Lemma 2.20; thus, $\tau(K_{n,m}) < \varepsilon$. Since the continua $M_{n,m}$ are locally connected, Statement 2.16 implies $\tau(M_{n,m}) = 0$, and since f_n are $1/n$-mappings, we conclude that $\tau(K_{n,m}) < 1/n$. Again, since f_n are confluent and since they are $1/n$-mappings, it follows that $H(K_{n,m}, L_n) < 1/n$ for almost all $m \in \mathbf{N}$. Applying the well-known diagonal procedure to the double indexed sequence $\{K_{n,m} : n, m \in \mathbf{N}\}$, one can choose a sequence of continua $K_n = K_{n,m(n)}$ converging to the continuum K. The sequence satisfies the conditions required in (2.18). The proof is complete. □

Since the projections in an inverse sequence of continua with confluent bonding mappings are confluent, [**7**, Corollary 4, page 5], Theorem 2.21 and Corollary 2.11 lead to the following result.

Corollary 2.22. *Let $\{X_n, f_m^n\}$ be an inverse sequence of continua X_n with confluent bonding mappings f_m^n. Then the inverse limit space $X = \varprojlim \{X_n, f_m^n\}$ is an arc Kelley continuum if and only if each continuum X_n is such.*

Remark 2.23. It is natural to ask whether confluent mappings in the above results can be replaced by more general ones. A mapping $f : X \to Y$ is said to be *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$. The class of semi-confluent mappings, see [**27**, (vi), page 12 and Table II, page 28], appears as one of the smallest among all classes studied in the literature that are wider than the one of confluent mappings. However, the reader can easily verify that the following continuum X is semi-confluently $\mathcal{LC}$-representable and semi-confluently $\mathcal{LC}$-like, but it is not a Kelley continuum. Indeed, consider an inverse sequence $\{X_n, f_m^n\}$, where for each $n \in \mathbf{N}$ the continuum X_n is a simple n-od, i.e., it is homeomorphic to the cone over an n-element set, with the vertex v and with its arms being straight line segments ve_i for $i \in \{1, \dots, n\}$, and the bonding mapping $f_n^{n+1} : X_{n+1} \to X_n$ is such that if c is the mid point of ve_1, then

$$f_n^{n+1}(v) = v,$$
$$f_n^{n+1}|ve_{n+1} : ve_{n+1} \to vc \subset ve_1 \text{ is linear},$$
$$f_n^{n+1}|ve_i : ve_i \to ve_i \text{ is the identity mapping for } i \in \{1, \dots, n\}.$$

Thus, each X_n is a Kelley continuum, each f_m^n is a semi-confluent mapping, while $X = \varprojlim \{X_n, f_m^n\}$ is not a Kelley continuum.

Remark 2.24. Note that if $X = \varprojlim \{X_n, f^n_m\}$, where X_n are confluently $\mathcal{LC}$-like continua and f^n_m are confluent mappings, then the continuum X is confluently $\mathcal{LC}$-like. If, additionally, X is one-dimensional, then X is confluently $\mathcal{LC}$-representable, according to [**32**, Corollary 3.14]; actually, X is confluently graph-like in this case. However, the question whether X is confluently $\mathcal{LC}$-representable for higher dimensional X remains open.

Question 2.25. Let $X = \varprojlim \{X_n, f^n_m\}$, where X_n are confluently $\mathcal{LC}$-representable continua and f^n_m are confluent. Is then the continuum X confluently $\mathcal{LC}$-representable?

Since each locally connected continuum is a Kelley continuum, [**19**, 20.4, page 167] and has the arc approximation property, [**16**, Corollary 3.7, page 115], it is an arc Kelley continuum. Thus, we have the following corollary, which is a stronger form of Corollary 2.3.

Corollary 2.26. *Each confluently $\mathcal{LC}$-like* (*confluently $\mathcal{LC}$-representable*) *continuum is an arc Kelley continuum. In particular, each such a continuum X has the arc approximation property and each of its arc components is dense in X.*

It will be shown in Remark 3.9 below that the implication in the above corollary cannot be reversed.

3. Atriodic confluently $\mathcal{LC}$-like continua.

In this section we discuss confluently $\mathcal{LC}$-like continua in a specific but important atriodic case. It was proved in [**32**, Corollary 3.5] that a nondegenerate confluently $\mathcal{LC}$-like atriodic continuum is either a solenoid or a Knaster type continuum. This result is a consequence of a more general fact proved there [**32**, Theorem 3.2]. Its proof is technical and depends on a long collection of auxiliary results developed in [**32**]. Here we prove that a substantial part of the above result about atriodic continua can also be obtained by simple speculation based on the study from this paper as well as on other known facts.

A continuum Y is called a *triod* provided that it contains a subcontinuum Z such that $Y \setminus Z = E_1 \cup E_2 \cup E_3$, where the set E_i are nonempty and mutually separated, see e.g., [**19**, 12.20, page 105]. If a continuum does not contain any triod, it is said to be *atriodic*.

In [**10**, Theorem 3.1 and Corollary 3.3], the authors proved the following.

Proposition 3.1. *An arc Kelley continuum is atriodic if and only if it contains no simple triod.*

Proposition 3.2. *If an arc Kelley continuum X contains no simple triod, then each proper subcontinuum of X is an arc.*

As a consequence of Proposition 3.2 and Theorem 2.21 (or Corollary 2.22) we have a corollary.

Corollary 3.3. *A confluently $\mathcal{LC}$-like continuum X is atriodic if and only if it contains no simple triod.*

Corollary 3.4. *If a confluently $\mathcal{LC}$-like continuum X contains no simple triod, then each proper subcontinuum of X is an arc.*

Proposition 3.5. *If a confluently $\mathcal{LC}$-like continuum X contains no simple triod, then X is either confluently arc-like or confluently circle-like.*

Proof. Indeed, by Corollary 3.4, the continuum X is atriodic. Recall that a confluent image of an atriodic continuum is atriodic, see [**26**, Proposition 5.19, page 147]; compare [**27**, 8.4, page 71]. Therefore, for each $\varepsilon > 0$ the continuum X admits a confluent ε-mapping onto an atriodic locally connected continuum. But an arc and a circle are the only nondegenerate atriodic locally connected continua, so the conclusion follows. $\square$

Similarly, we can prove the next proposition.

Proposition 3.6. *If a confluently $\mathcal{LC}$-representable continuum X contains no simple triod, then X is either confluently arc-representable or confluently circle-representable.*

Proof. Let $X = \varprojlim \{X_n, f_m^n\}$, where X_n are locally connected continua and f_m^n are confluent. Using a similar argument to the one in the proof of Proposition 3.6, we see that X_n is an arc or a circle for almost all n. Since there is no confluent mapping from an arc onto a circle, see [**31**, Theorem 13.31, page 292], it follows that either almost all continua X_n are circles or almost all of them are arcs. This completes the proof. □

As a consequence of Proposition 3.6 and of [**5**, Theorems 3.3 and 3.4, page 224], we obtain the following result.

Corollary 3.7. *Let a continuum X be confluently $\mathcal{LC}$-representable. Then the following conditions are equivalent*:

(a) *X contains no simple triod*;

(b) *X is atriodic*;

(c) *X is either a Knaster type continuum or a solenoid.*

Corollary 3.7 justifies the following remark.

Remark 3.8. The $\sin(1/x)$-curve is an arc-like Kelley continuum, but it is not confluently $\mathcal{LC}$-representable. Thus, the implication in Corollary 2.3 cannot be reversed.

Remark 3.9. Also the implication in Corollary 2.26 (from the condition that X is confluently $\mathcal{LC}$-like to X is an arc Kelley continuum) cannot be reversed. Indeed, let X be the simplest indecomposable continuum with three end points (in the sense that for any two continua containing the point, one of them contains the other, [**1**, pages 660 and 661]; see [**31**, 1.5, page 5 and 1.10, page 7]). According to [**11**, Example 5.4], X is an arc Kelley continuum. Suppose that X admits a confluent ε-mapping onto a locally connected continuum Y, where ε is less than the minimal distance between any two end points of X. Being a confluent image of X, the space Y is atriodic. Since confluent mappings preserve end points, [**4**, Lemma, page 172], and f cannot identify any two end points of X, the continuum Y must have at least three end points. There is no such locally connected atriodic continuum, a contradiction.

Theorem 3.10. *Each confluently circle-like continuum is a solenoid.*

Proof. Let a continuum X be confluently circle-like. Thus, it is atriodic, see [**19**, Exercise 39.7, page 260], and by Corollary 3.5 each proper subcontinuum of X is an arc. Further, by Corollary 2.26 and Proposition 2.8, the continuum X has property of Kelley. In [**23**, Theorem 1, page 379], solenoids are characterized as circle-like continua X having property of Kelley and such that each point $x \in X$ belongs to an arc $ab \subset X$ with $a \neq x \neq b$ (thus X has no end point). Therefore, only this last condition has to be proved.

Suppose on the contrary that there exists a point $x \in X$ such that x is an end point of every arc containing it. Since each proper subcontinuum of X is an arc, it follows that x is an end point of X. Since confluent mappings preserve end points, [**4**, Lemma, page 172], there is no confluent mapping from X onto a circle, so X is not confluently circle-like, a contradiction. □

The question whether each confluently arc-like continuum is a Knaster type one was included as open in the original version of this paper. It gave initial motivation to the entire study presented in [**32**]. A positive answer to this question is shown in [**32**, Corollary 3.4]. Combining the above investigation with this last result from [**32**], we have the following.

Corollary 3.11. *For each nondegenerate continuum X, the following conditions are equivalent*:

(a) *X is confluently $\mathcal{LC}$-like and it contains no simple triod*;

(b) *X is atriodic and confluently $\mathcal{LC}$-like*;

(c) *X is atriodic and confluently $\mathcal{LC}$-representable*;

(d) *X is either a Knaster type continuum or a solenoid.*

4. Products and hyperspaces of confluently $\mathcal{LC}$-like continua. We start with a result that is related to Cartesian products.

Theorem 4.1. *If continua X and Y are confluently $\mathcal{LC}$-representable* (*are confluently $\mathcal{LC}$-like*), *then the product $X \times Y$ is confluently $\mathcal{LC}$-representable* (*is confluently $\mathcal{LC}$-like, respectively*), *too.*

Proof. We prove this for the version of "confluently $\mathcal{LC}$-representable." For "confluently $\mathcal{LC}$-like" the argument is very similar.

Let $X = \varprojlim \{X_n, f_m^n\}$ and $Y = \varprojlim \{Y_n, g_m^n\}$, where X_n and Y_n are locally connected continua and f_m^n and g_m^n are confluent. Then $X \times Y$ is homeomorphic to $\varprojlim \{X_n \times Y_n, f_m^n \times g_m^n\}$, see [**17**, 2.5.D.(b), page 105]. Note that confluent mappings onto locally connected continua coincide with OM-mappings, [**27**, (6.2), page 51], and the class of OM-mappings has the product property (i.e., the product of two OM-mappings is also an OM-mapping), see [**27**, (5.33), page 36]. Therefore, the product mappings $f_m^n \times g_m^n$ are confluent (compare also [**33**, Corollary, page 234]). The proof is complete. □

Questions 4.2. Are the converse implications to Theorem 4.1 true? In other words, if X and Y are continua, does the condition that $X \times Y$ is confluently $\mathcal{LC}$-representable (or confluently $\mathcal{LC}$-like) imply that X is such? In particular, do the implications hold if $Y = X$?

Note that the answers to these questions are positive if we know that projections (more general open mappings, more general confluent ones) preserve the class of confluently $\mathcal{LC}$-representable continua. So, the next question can be asked.

Question 4.3. Let a continuum X be confluently $\mathcal{LC}$-representable (or confluently $\mathcal{LC}$-like), and let a mapping $f : X \to f(X)$ be confluent. Is then $f(X)$ confluently $\mathcal{LC}$-representable (or confluently $\mathcal{LC}$-like, correspondingly)?

Given a metric continuum X, we let 2^X denote the hyperspace of nonempty closed subsets of X equipped with the Hausdorff distance H, see e.g., [**30**, (0.1), page 1 and (0.12), page 10], and $C(X)$ stands for the hyperspace of subcontinua of X, i.e., of connected members of 2^X. If $k \in \mathbf{N}$ is fixed, then $C_k(X)$ and $F_k(X)$ mean the hyperspaces of nonempty closed subsets of X with at most k components, and consisting of at most k points, respectively. Thus, $C_1(X) = C(X)$, and $F_1(X)$ is homeomorphic to X. It is known that, for each $k \in \mathbf{N}$, the space $C_k(X)$ is an arcwise connected continuum, see [**25**, Theorem 3.1, page 240], and that $F_k(X)$ also is a continuum, see [**25**, (a), page 877] and compare [**29**, 2.4.2, page 156, and Theorem 4.10, page 165].

Given a mapping $f : X \to Y$ between continua X and Y, we let $2^f : 2^X \to 2^Y$ and $C_k(f) : C_k(X) \to C_k(Y)$ denote the corresponding induced mappings defined by $2^f(A) = f(A)$ and $C_k(A) = f(A)$. The reader is referred to [**12, 16, 19, 30**] and the references therein for properties of the induced mappings between hyperspaces.

Proposition 4.4. *If a continuum X is confluently $\mathcal{LC}$-representable, then the hyperspaces 2^X and $C(X)$ are also confluently $\mathcal{LC}$-representable.*

Proof. Let $X = \varprojlim \{X_n, f_m^n\}$, where X_n are locally connected continua and f_m^n are confluent. Then $2^X = \varprojlim \{2^{X_n}, 2^{f_m^n}\}$ and $C(X) = \varprojlim \{C(X_n), C(f_m^n)\}$, see [**17**, 3.12.27 (f), page 245]; see also [**30**, (1.169), page 171]. The induced mappings $2^{f_m^n}$ and $C(f_m^n)$ are confluent, see [**16**, Corollary 4.5, page 134], and thus the result follows. □

Similarly, using [**19**, Proposition 22.4, page 189], one can prove the next result.

Proposition 4.5. *If a continuum X is confluently $\mathcal{LC}$-like, then the hyperspaces 2^X and $C(X)$ are also confluently $\mathcal{LC}$-like.*

It would be interesting to know when the inverse implications to that of Propositions 4.4 and 4.5 are true.

Questions 4.6. Let X be a continuum such that the hyperspace (a) 2^X, (b) $C(X)$ is confluently $\mathcal{LC}$-representable (or is confluently $\mathcal{LC}$-like). Under what conditions is X confluently $\mathcal{LC}$-representable (or is it confluently $\mathcal{LC}$-like, respectively)?

It is natural to ask if analogs of Propositions 4.4 and 4.5 are true for the hyperspaces $C_k(X)$ with $k > 1$. A full answer to these questions is presented below, see Proposition 4.9 and Corollary 4.14. Surprisingly, the answer is positive for $k = 2$ and negative for $k > 2$. The following results will be used in the proof, see [**25**, Corollary 4.5, page 244] and [**12**, Theorem 14, p. 788].

Proposition 4.7. *Let* $X = \varprojlim \{X_n, f_m^n\}$, *where* X_n *are continua and* f_m^n *are surjections. For each* $k \in \mathbf{N}$ *the hyperspace* $C_k(X)$ *is homeomorphic to the inverse limit* $\varprojlim \{C_k(X_n), C_k(f_m^n)\}$.

Theorem 4.8. *Let* $f : X \to Y$ *be an* OM-*mapping between continua. Then the induced mapping* $C_2(f) : C_2(X) \to C_2(Y)$ *is also an* OM-*mapping.*

Proposition 4.9. *If a continuum* X *is confluently* $\mathcal{LC}$*-representable, then the hyperspace* $C_2(X)$ *is also confluently* $\mathcal{LC}$*-representable.*

Proof. Let $X = \varprojlim \{X_n, f_m^n\}$, where X_n are locally connected continua and f_m^n are confluent. Then $C_2(X) = \varprojlim \{C_2(X_n), C_2(f_n)\}$ according to Theorem 4.7. The coordinate spaces $C_2(X_n)$ are known to be locally connected continua, see [**36**, Théorème II_m, page 191] and compare [**25**, Theorem 3.2, page 240]. Further, f_m^n are OM-mappings, see [**27**, (6.2), page 51], whence the induced bonding mappings $C_2(f_m^n)$ are also OM-mappings by Theorem 4.8. Since OM-mappings are confluent, the conclusion follows. □

Analogously, we have the following proposition.

Proposition 4.10. *If a continuum* X *is confluently* $\mathcal{LC}$*-like, then the hyperspace* $C_2(X)$ *is also confluently* $\mathcal{LC}$*-like.*

The next questions are analogs of Questions 4.6.

Questions 4.11. Let X be a continuum such that the hyperspace $C_2(X)$ is confluently $\mathcal{LC}$-representable (or is confluently $\mathcal{LC}$-like). Under what conditions is X confluently $\mathcal{LC}$-representable (or is it confluently $\mathcal{LC}$-like, respectively)?

The next theorem and the obtained corollary show connections between the property that some hyperspaces of a given continuum are Kelley and local connectedness of the considered continuum. The main idea of the proof of the theorem below is due to Illanes, see [**12**, Example 15, page 790].

Theorem 4.12. *Let a continuum X be given. If the hyperspace $C_3(X)$ is a Kelley continuum, then X is locally connected.*

Proof. Suppose on the contrary that there is a point $p \in X$ at which X is not locally connected. Let $q \in X$ with $q \neq p$, and let R be a nondegenerate subcontinuum of X containing q and not containing p. Choose $\varepsilon > 0$ such that:

(4.12.1) $d(q, r) > 2\varepsilon$ for some $r \in R$;

(4.12.2) $B(p, \varepsilon) \cap N(R, \varepsilon) = \varnothing$;

(4.12.3) there exists a sequence of points p_n tending to p such that each continuum containing p and p_n has diameter greater than ε.

For this ε there is a $\delta > 0$ as in the definition of a Kelley continuum for $C_3(X)$. Let

$$\mathcal{K} = \{\{p\} \cup K : K \in C_2(R)\}.$$

Then $\mathcal{K}$ is a subcontinuum of $C_3(X)$ containing the sets $\{p, q\}$ and $\{p, q, r\}$. Take $n \in \mathbf{N}$ such that $d(p, p_n) < \delta$. Then $H(\{p, q\}, \{p, p_n, q\}) < \delta$, and therefore there is a subcontinuum $\mathcal{L}$ of $C_3(X)$ containing $\{p, p_n, q\}$ which is ε-near to $\mathcal{K}$. Let $U = \mathrm{cl}_X(B(p, \varepsilon))$ and $V = \mathrm{cl}_X(N(R, \varepsilon))$. Denote by K^p, K^{p_n} and K^q the components of $U \cup V$ containing the points p, p_n and q, respectively.

It is known that two compact subsets of a compactum Z are in the same component of 2^Z if and only if they intersect the same components of Z, see [**15**, Lemma 23, page 214]. Taking $Z = U \cup V$, it follows that each element of $\mathcal{L}$ intersects each of the sets K^p, K^{p_n} and K^q. Thus each element of $\mathcal{L}$ has exactly three components. Denote by L an element of $\mathcal{L}$ which is ε-near to $\{p, q, r\}$. Then the component of L contained in K^q must be ε-near to $\{q, r\}$, a contradiction with (4.12.1). □

Remarks 4.13. a) Note that in Theorem 4.12 one can replace $C_3(X)$ by $C_k(X)$ for an arbitrary $k \geq 3$, with a similar proof.

b) The same implication is true if we substitute, in Theorem 4.12, $F_k(X)$ in place of $C_k(X)$.

c) The converse is also true because the local connectedness of the continuum X is equivalent to the local connectedness of the hyperspaces $C_k(X)$ and $F_k(X)$, see [**25**, Theorem 3.2, page 240] and [**24**, Lemma 2, page 286]; compare also [**2**, (a), page 877], and therefore each of them is a Kelley continuum, [**30**, 16.11, page 539].

Theorem 4.12 and Remarks 4.13 imply the following corollary.

Corollary 4.14. *The following conditions are equivalent for a continuum X and an integer $k \geq 3$:*

(4.14.1) *the hyperspace $C_k(X)$ is a Kelley continuum for each (for some) k;*

(4.14.2) *the hyperspace $F_k(X)$ is a Kelley continuum for each (for some) k;*

(4.14.3) *the hyperspace $C_k(X)$ is confluently $\mathcal{LC}$-representable;*

(4.14.4) *the hyperspace $F_k(X)$ is confluently $\mathcal{LC}$-representable;*

(4.14.5) *the hyperspace $C_k(X)$ is confluently $\mathcal{LC}$-like;*

(4.14.6) *the hyperspace $F_k(X)$ is confluently $\mathcal{LC}$-like;*

(4.14.7) *the hyperspace $C_k(X)$ is locally connected;*

(4.14.8) *the hyperspace $F_k(X)$ is locally connected;*

(4.14.9) *X is locally connected.*

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