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J. J. Charatonik

W. J. Charatonik
Missouri University of Science and Technology, wjcharat@mst.edu

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A DEGREE OF NONLOCAL CONNECTEDNESS

JANUSZ J. CHARATONIK AND WŁODZIMIERZ J. CHARATONIK

ABSTRACT. To any continuum X we assign an ordinal number (or the symbol ∞) $s(X)$, called the degree of nonlocal connectedness of X. We show that (1) the degree cannot be increased under continuous surjections; (2) for hereditarily unicoherent continua X, the degree of a subcontinuum of X is less than or equal to $s(X)$; (3) $s(C(X)) \leq s(X)$, where $C(X)$ denotes the hyperspace of subcontinua of a continuum X. We also investigate the degrees of Cartesian products and inverse limits. As an application we construct an uncountable family of metric continua X homeomorphic to $C(X)$.

Introduction. The idea of using ordinal numbers as a "measure" of some local or global properties of (compact) spaces is not new. Usually these properties are related to (non-)connectedness, and the defined "measure" can be used as a tool in studying various other properties of investigated spaces, both structural (internal) and mapping (external) ones. For example, Iliadis in [**14**] defines the notion of a normal sequence for hereditarily decomposable and hereditarily unicoherent metric continua (i.e., for λ-dendroids) as follows. Let X be such a continuum. A continuum $H \subset X$ is said to be in $\mathcal{I}(X)$ if, given any decomposition of X into finitely many subcontinua, H is contained in one element of the decomposition. Let $\Sigma = \{H_\alpha : \alpha < \lambda\}$ be a transfinite sequence of subcontinua of X, where λ is some countable ordinal number. Then Σ is called a *normal sequence* if (i) $H_0 = X$, (ii) $H_\beta = \mathcal{I}(H_\alpha)$ for ordinals $\beta = \alpha + 1$, (iii) $H_\beta = \cap\{H_\alpha : \alpha < \beta\}$ for limit ordinals β, and (iv) for each $\alpha < \lambda$ the continuum H_α is nondegenerate. The least upper bound (or minimum) $k(X)$ of the lengths of all normal sequences in X is called the *depth* of X. The concept was used to study various phenomena in λ-dendroids. For its modification, see [**31**].

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A similar idea was applied by Mohler [**24**] to define the notion of the depth of tranches in λ-dendroids, and earlier by the first-named author in [**3**] to establish the degree of nonlocal connectedness (for hereditarily unicoherent metric continua) and by Lelek in defining the nonconnectivity index of a space at a point [**20**, p. 360]. Note that the concepts introduced in [**3**], [**14**], [**24**] or in [**31**] can be applied not to all continua, but to hereditarily unicoherent ones or to λ-dendroids only, and these in [**14**], [**24**] and in [**31**] are global ones in the sense that they describe the structure of a continuum in the whole, not locally. The concept introduced by Lelek in [**20**] can be applied to all separable metric spaces, but again its nature is rather global than local. For compact metric spaces, a very interesting approach to define a suitable concept of a "measure" of nonlocal connectedness was made by Prajs in [**28**]. Unfortunately, no proofs of its properties are given in that paper and, until now, the author has not published his results in full. We would like to stress that some ideas of the present paper are taken from the mentioned paper [**28**] or from its unpublished full version.

The same concept of a degree of nonlocal connectedness was also studied by Katsuura in [**16**], who used the notion to show the nonexistence of Peano mappings (i.e., continuous surjections from a continuum onto its square) for some continua.

The aim of this paper is to present a concept which is an extension of the degree of nonlocal connectedness defined in [**3**]; it applies to the much wider class of all Hausdorff continua, has many nice properties, and seems to be a good tool in studying various properties of these spaces.

Preliminaries. A *space* means a topological Hausdorff space, and a *mapping* means a continuous transformation.

We denote by Ord the class of all ordinal numbers, by Lim the subclass of Ord composed of all limit ordinals and by $\mathbf{N} \subset \mathrm{Ord}$ the set of all finite ordinals. The symbol ω_1 denotes the first uncountable ordinal. For a given ordinal number β, we define its predecessor $\mathrm{pred}\,(\beta)$ by

$$\mathrm{pred}\,(\beta) = \begin{cases} \alpha & \text{if } \beta = \alpha + 1, \\ \beta & \text{if } \beta \in \mathrm{Lim}. \end{cases}$$

Any ordinal number α is understood as the set of all ordinals less than

α, equipped with the order topology, if we consider α as a topological space. Then α is a nondegenerate compact space if and only if $\alpha \notin \mathrm{Lim}$.

Let X and Y be two disjoint spaces, $A \subset X$ and $f : A \to Y$ a mapping. In the disjoint union $X \oplus Y$ generate an equivalence relation $\simeq$ by $a \simeq f(a)$ for each $a \in A$. The quotient space $(X \oplus Y)/\simeq$ is denoted by $X \cup_f Y$ (see [**6**, p. 127]).

A *continuum* denotes a compact connected space. A continuum X containing two points a and b is called an *arc* (from a to b, or with endpoints a and b), provided that each point of $X \setminus \{a, b\}$ separates a and b in X. A continuum X is said to be *arcwise connected* provided that for every two points a and b of X there is an arc from a to b contained in X. A continuum X is said to be *hereditarily unicoherent* provided that the intersection of every two of its subcontinua is connected. A hereditarily unicoherent and arcwise connected continuum is called an *arboroid*. A metrizable arboroid is named a *dendroid*.

Two special dendroids will be used in the paper. We denote by F_H the *harmonic fan*, i.e., the cone over the closure of the harmonic sequence $\{1/n : n \in \mathbf{N}\}$ (or, equivalently, the cone over $\omega + 1$), and by F_C the *Cantor fan*, i.e., the cone over the Cantor ternary set $C \subset [0, 1]$.

The concept of local connectedness of a space at a given point is a crucial one in further considerations. Since this term is not uniquely defined, and some authors use this name in various ways, it is necessary to start with some definitions concerning the concept. Following Engelking [**8**, p. 373], Kuratowski [**18**, p. 227] and Whyburn [**33**, p. 18], a space X is said to be *locally connected at a point* $p \in X$ provided that every neighborhood of p contains a connected neighborhood of p. Note that the same concept is called "connectedness im kleinen" at p by some authors, e.g., in [**12**, p. 113], [**27**, p. 75] and in [**11**], or "weak local connectedness" in [**34**], while the term "local connectedness at a point" is defined in another way (by having a local basis consisting of connected (open) sets; see, e.g., [**6**, p.113]). The reader is referred to [**4**], where relations between these concepts are discussed and further references are given. In particular, it is well known that if the condition holds at each point of a continuum, then the two concepts coincide (see, e.g., [**27**, p. 84]).

1. General properties. For a given compact space X we denote by $L(X)$ the set of all points of X at which the space X is locally connected, and we put $N(X) = X \setminus L(X)$. Note that, in general, $N(X)$ need not be closed. The following result is known (see [**9**, p. 28]).

Fact 1.1. *For each compact space X and mapping f defined on X, the inclusion $N(f(X)) \subset f(N(X))$ holds.*

In the next fact the first conclusion is an easy observation, the second one is shown in [**9**, p. 28], and the third one is in [**33**, p. 19].

Fact 1.2. *If A and B are compact spaces, then*

(1.2.1) $N(A \cup B) \subset N(A) \cup N(B)$;

(1.2.2) $N(A \times B) = (N(A) \times B) \cup (A \times N(B))$;

(1.2.3) *if A is a nonlocally connected continuum, then $N(A)$ is nondegenerate.*

Definition 1.3. Let a continuum X and an ordinal number λ be given. A sequence of continua $\{X_\alpha : \alpha \leq \lambda\}$ is said to be *good*, and λ is called the *length* of the sequence, if the following conditions are satisfied.

(1.3.1) $X_0 = X$;

(1.3.2) $X_{\alpha+1} \subset X_\alpha$ for each ordinal $\alpha < \lambda$;

(1.3.3) $N(X_\alpha) \subset X_{\alpha+1}$ for each ordinal $\alpha < \lambda$;

(1.3.4) $X_\beta = \cap\{X_\alpha : \alpha < \beta\}$ for each limit ordinal $\beta \leq \lambda$;

(1.3.5) X_λ is a one-point set if $\lambda \in \mathrm{Lim}$ or the empty set if $\lambda \notin \mathrm{Lim}$.

For a given continuum X define $s(X)$ as the smallest ordinal number λ such that there is a good sequence of subcontinua of X of length λ if such a sequence exists, and $s(X) = \infty$ otherwise. We agree that any ordinal number is less than the symbol ∞ and that $\mathrm{pred}\,(\infty) = \infty$.

Some properties of the number $s(X)$ will be stated.

Proposition 1.4. *For each continuum X and for each mapping $f : X \to f(X)$ if $\{X_\alpha : \alpha \le \lambda\}$ is a good sequence in X, then $\{f(X_\alpha) : \alpha \le \lambda\}$ is a good sequence in $f(X)$.*

Proof. Conditions (1.3.1), (1.3.2), (1.3.4) and (1.3.5) are evidently satisfied, and (1.3.3) is just Fact 1.1. ◻

As an immediate consequence of the above proposition, we get the next result.

Theorem 1.5. *For each continuum X and for each mapping $f : X \to f(X)$ the inequality $s(f(X)) \le s(X)$ holds.*

In [**3**] the degree of nonlocal connectedness $\tau(H)$ is defined of a hereditarily unicoherent metric continuum H, and a number of its properties are proved. Later Mohler in [**24**, p. 345] extended the definition and basic properties of the notion to arbitrary Hausdorff (not necessarily metric) hereditarily unicoherent continua. To compare the two concepts, $\tau(H)$ and $s(H)$, we recall the definition of the former one.

For a subset S of a continuum X, let $I(S)$ denote a subcontinuum of X which is *irreducible about* S, i.e., such that $S \subset I(S)$ and no proper subcontinuum of $I(S)$ contains S. It is known that the operation I is unique in hereditarily unicoherent continua, and vice versa, see [**3**, p. 187] and [**23**, p. 346] and that the following implications hold, see [**3**, p. 188] and compare [**23**, p. 346].

Fact 1.6. (a) *If a continuum X is hereditarily unicoherent, then the condition $S_1 \subset S_2 \subset X$ implies that $I(S_1) \subset I(S_2)$.*

(b) *If continua S_1 and S_2 are hereditarily unicoherent, then the condition $S_1 \subset S_2$ implies that $N(S_1) \subset N(S_2)$.*

For a hereditarily unicoherent continuum X, put $J(X) = I(N(X))$

and define, for an ordinal number $\beta > 0$, $J^0(X) = X$ and

$$J^\beta(X) = \begin{cases} J(J^\alpha(X)) & \text{if } \beta = \alpha + 1, \\ \cap\{J^\alpha(X) : \alpha < \beta\} & \text{if } \beta = \lim_{\alpha<\beta} \alpha. \end{cases}$$

The *degree of nonlocal connectedness* $\tau(X)$ of a hereditarily unicoherent continuum X is defined as

$$\tau(X) = \begin{cases} \min\{\alpha : J^{\alpha+1}(X) = \varnothing\} & \text{if } \{\alpha : J^{\alpha+1}(X) = \varnothing\} \neq \varnothing, \\ \infty & \text{otherwise.} \end{cases}$$

For properties of the degree τ, see [**3**, pp. 190–193] and [**23**, pp. 345–347].

One can observe that, for a hereditarily unicoherent continuum X, the sequence $\{J^\alpha(X) : \alpha \leq s(X)\}$ is a good one in X. This leads to the following definition; the next lemma justifies the name used.

Definition 1.7. For a hereditarily unicoherent continuum X, the sequence

$$\{J^\alpha(X) : \alpha \leq s(X)\}$$

is called the *best* sequence in X.

Lemma 1.8. *If a continuum X is hereditarily unicoherent, then for each good sequence $\{X_\alpha : \alpha \leq \beta\}$ in X, we have $J^\alpha(X) \subset X_\alpha$.*

Proof. We will proceed by induction. For $\alpha = 0$, we have $J^0(X) = X_0 = X$ by the definitions. Assume that $J^\alpha(X) \subset X_\alpha$ for each $\alpha < \gamma$. If $\gamma \in \mathrm{Lim}$, then

$$J^\gamma(X) = \bigcap\{J^\alpha(X) : \alpha < \gamma\} \subset \bigcap\{X_\alpha : \alpha < \gamma\} = X_\gamma,$$

and we are done. If $\gamma \notin \mathrm{Lim}$, then put $\delta = \mathrm{pred}\,(\gamma)$. Therefore, $J^\delta(X) \subset X_\delta$ by the inductive assumption, and Fact 1.6(b) implies that $N(J^\delta(X)) \subset N(X_\delta)$, whence by Fact 1.6(a) and by (1.3.3) we get $J^{\delta+1}(X) = I(N(J^\delta(X))) \subset I(N(X_\delta)) \subset X_{\delta+1}$. Since $\gamma = \delta + 1$, we have $J^\gamma(X) \subset X_\gamma$ as needed. $\square$

Corollary 1.9. *For a hereditarily unicoherent continuum X, the degree $s(X)$ is the length of the best sequence.*

In the next theorem, connections between the two degrees, τ and s, are shown.

Theorem 1.10. *If a continuum X is hereditarily unicoherent, then*

$$\tau(X) = \operatorname{pred}(s(X)).$$

Proof. Let $\tau(X) = \beta$, i.e., $J^{\beta+1}(X) = \varnothing$. If $J^{\beta}(X)$ is a one-point set, then by (1.2.3), β is a limit ordinal, $\{J^{\alpha}(X) : \alpha \le \beta\}$ is the best sequence in X, and consequently $s(X) = \beta$ according to Corollary 1.9. If $J^{\beta}(X)$ is nondegenerate, then $\{J^{\alpha}(X) : \alpha \le \beta + 1\}$ is the best sequence in X, whence $s(X) = \beta + 1$, as required. So the proof is complete. □

Proposition 1.11. *If a continuum X is hereditarily unicoherent and Y is a subcontinuum of X, then $s(Y) \le s(X)$.*

Proof. Let $\{X_{\alpha} : \alpha \le la\}$ be a good sequence in X. Consider the sequence $\{Y \cap X_{\alpha} : \alpha \le \lambda\}$. We will show that it is good in Y. Indeed, conditions (1.3.1), (1.3.2), (1.3.4) and (1.3.5) are obviously satisfied and condition (1.3.3) holds by Fact 1.6(b). Applying the definition, we complete the proof. □

Remark 1.12. If we define $\nu(X)$ for an arbitrary continuum X by the formula $\nu(X) = \operatorname{pred}(s(X))$, then ν generalizes the function τ to arbitrary (not necessarily hereditarily unicoherent) continua, and therefore it solves the problem posed in the second paragraph of Remarks in [**3**, p. 192]. However, the next example shows that the function s distinguishes more situations than the function ν.

Example 1.13. There are two hereditarily unicoherent continua X and Y such that $\tau(X) = \tau(Y) = \omega$, while $s(X) = \omega + 1$ and $s(Y) = \omega$.

Proof. Let X be the cone over $\omega^\omega + 1$. One can verify that $J^\omega(X)$ is an arc and thus $\tau(X) = \omega$ and $s(X) = \omega + 1$. To obtain the continuum Y, shrink the arc $J^\omega(X) \subset X$ to a point. Then $J^\omega(Y)$ is degenerate, and therefore $\tau(Y) = s(Y) = \omega$. □

2. Hyperspaces. Now we will study certain properties of the degree s related to hyperspaces. Some definitions are in order first.

Given a space X, we denote by 2^X the family of all nonempty compact subsets of X and by $C(X)$ the family of all nonempty compact connected subsets of X. Thus $C(X) \subset 2^X$. The families 2^X and $C(X)$ equipped with the *Vietoris topology* (see, e.g., [**25**, p. 10] for the definition) are called *hyperspaces* of X. The reader is referred to [**15**] and to [**25**] for needed information on hyperspaces.

An *order arc* in the hyperspace either 2^X or $C(X)$ means an arc which is also a chain with respect to the partial order on the hyperspace induced by set inclusion. The following fact (for the metric case, compare [**25**, p. 59]) is known. Its standard proof using the Kuratowski-Zorn lemma is left to the reader.

Fact 2.1. *Let a continuum X and sets $A, B \in 2^X$ be given. Then the following two statements are equivalent.*

(2.1.1) *There is an order arc from A to B in 2^X;*

(2.1.2) *$A \subset B$ and each component of B intersects A.*

In particular, if $A, B \in C(X)$, then there is an order arc from A to B in $C(X)$ if and only if $A \subset B$.

Theorem 2.2. *For each continuum X, we have*

$$s(2^X) = \begin{cases} 0 & \text{if } X \text{ is a singleton;} \\ 1 & \text{if } X \text{ is nondegenerate and locally connected;} \\ \infty & \text{if } X \text{ is not locally connected.} \end{cases}$$

Proof. If X is a singleton, the equality follows from the definition. In the second case, since the local connectedness of X is equivalent to the one of the hyperspace 2^X, see [**22**, p. 166], the conclusion follows as well. If the continuum X is not locally connected, then there is a surjective mapping $f : X \to F_H$, where F_H is the harmonic fan, see [**1**, p. 107]. Then the mapping $2^f : 2^X \to 2^{F_H}$ defined by $2^f(A) = f(A)$ for $A \in 2^X$ is a (continuous) surjection from 2^X onto 2^{F_H} (compare [**22**, p. 170]). Therefore, $s(2^X) \geq s(2^{F_H})$ according to Theorem 1.5. Since F_H is a metric nonlocally connected continuum, the hyperspace 2^{F_H} can be mapped onto the Cantor fan F_C, see [**25**, p. 94], and thus we have $s(2^{F_H}) \geq s(F_C)$ again by Theorem 1.5. Then the conclusion follows since $s(F_C) = \infty$. □

Lemma 2.3. *For each continuum X we have $N(C(X)) \subset C(N(X))$.*

Proof. Let P be a nonempty subcontinuum of X which is not an element of $C(N(X))$. Thus P has a point p of local connectedness of X. It is known that if X is locally connected at p and a subcontinuum P of X contains p, then the hyperspace $C(X)$ is locally connected at P, see [**7**, p. 170]. Thereby, P is a point of local connectedness of $C(X)$, so $P \notin N(C(X))$. □

Proposition 2.4. *Let X be a continuum. If $\{X_\alpha : \alpha \leq \lambda\}$ is a good sequence in X, then $\{C(X_\alpha) : \alpha \leq \lambda\}$ is a good sequence in $C(X)$.*

Proof. Conditions (1.3.1), (1.3.2), (1.3.4) and (1.3.5) obviously hold for $\{C(X_\alpha) : \alpha \leq \lambda\}$, and (1.3.3) is a consequence of Lemma 2.3. Therefore, the conclusion holds. □

As a consequence of Proposition 2.4, we obtain the following result.

Theorem 2.5. *For each continuum X, we have $s(C(X)) \leq s(X)$.*

It is known that a continuum X is locally connected if and only if the hyperspace $C(X)$ is locally connected, i.e., $s(X) = 1$ if and only if $s(C(X)) = 1$, see [**25**, p. 134]. Of course, we also have that $s(X) = 0$ if and only if $s(C(X)) = 0$. In the next results we will show that the above statements and Theorem 2.5 are the only restrictions for possible relations between $s(X)$ and $s(C(X))$. To formulate the first of them, we recall two definitions and a statement.

An arboroid X is said to be *smooth at the point* $p \in X$ provided that, for each convergent net $\{x_n : n \in D\}$ of points of X, where D is a set directed by a relation $\leq$, with $x_0 = \lim x_n$, the net of the arcs $\{px_n : n \in D\}$ converges to the arc px_0 (see [**30**, p. 564]).

Given a continuum X, a (continuous) *selection* on the hyperspace $C(X)$ is a mapping $f : C(X) \to X$ such that $f(A) \in A$ for each $A \in c(X)$. Let an arboroid X be given which is smooth at a point $p \in X$, and let $f : C(X) \to X$ be the least element mapping, see [**10**, p. 217], i.e., a mapping that assigns to each subcontinuum A of X the point $f(A) = y \in A$ of X such that $py \cap A = \{y\}$. Then f is a continuous selection. Thus the following statement is true (for the metric case, compare [**32**, p. 1043]).

Statement 2.6. *Each smooth arboroid X admits a selection on $C(X)$.*

Define

$$\mathbf{G} = \{0\} \cup \left\{\frac{1}{2^k} : k \in \mathbf{N}\right\}, \tag{2.7}$$

$$G = (\mathbf{G} \times [0,1]) \cup ([0,1] \times \{1\}). \tag{2.8}$$

The continuum G is called the *geometric comb*.

Proposition 2.9. *For each ordinal number α there is a smooth arboroid $X(\alpha)$, metrizable if $\alpha < \omega_1$, such that $s(X(\alpha)) = s(C(X(\alpha))) = \alpha$.*

Proof. We will proceed by transfinite induction. Let $X(0)$ be a one-point set $\{p(0)\}$, and assume that we have defined, for an ordinal α, an arboroid $X(\alpha)$ smooth at a point $p(\alpha)$. To define $X(\alpha+1)$, consider two cases.

Case 1. $\alpha \notin \mathrm{Lim}$. Let $A = (\{p(\alpha)\} \times \mathbf{G}) \cup (X(\alpha) \times \{0\}) \subset X(\alpha) \times \mathbf{G}$. Define a mapping $f : A \to G$ by $f(\langle p(\alpha), t\rangle) = \langle t, 0\rangle$ for $t \in \mathbf{G}$, and $f(\langle x, 0\rangle) = \langle 0, 0\rangle$ for $x \in X(\alpha)$. Finally, put $X(\alpha+1) = (X(\alpha) \times \mathbf{G}) \cup_f G$ and $p(\alpha+1) = \langle 0, 1\rangle \in G \subset X(\alpha+1)$, and note that $X(\alpha+1)$ is an arboroid which is smooth at $p(\alpha+1)$.

Assuming $s(X(\alpha)) = \alpha$, we will show that $s(X(\alpha+1)) = \alpha+1$. Let $\{X_\gamma(\alpha) : \gamma \in \{0, \dots, \alpha\}\}$ be the best sequence in the continuum $X(\alpha)$. Denote by Y_γ the (unique) continuum irreducible with respect to containing $(X_\gamma(\alpha) \times \mathbf{G}) \cup_f G$ for each $\lambda \in \{0, \dots, \mathrm{pred}\,(\alpha)\}$. Put $Y_\alpha = \{0\} \times [0,1] \subset G \subset X(\alpha+1)$ and $Y_{\alpha+1} = \varnothing$. One can verify that $\{Y_\gamma : \gamma \in \{0, \dots, \alpha+1\}\}$ is the best sequence in $X(\alpha+1)$, whence it follows by Corollary 1.9 that $s(X(\alpha+1)) = \alpha+1$.

Case 2. $\alpha \in \mathrm{Lim}$. In the space $X(\alpha) \times \{0,1\}$, take $A = \{p(\alpha)\} \times \{0,1\}$ and define $f : A \to [0,1]$ by $f(\langle p(\alpha), t\rangle) = t$ for $t \in \{0,1\}$. Then put $X(\alpha+1) = (X(\alpha) \times \{0,1\}) \cup_f [0,1]$. In other words, $X(\alpha+1)$ is the disjoint union of two copies of $X(\alpha)$ joined by the segment $[0,1]$ attached at endpoints of $[0,1]$ to the corresponding points $p(\alpha)$. It follows that the constructed continuum is an arboroid which is smooth at the point $p(\alpha+1) = \langle p(\alpha), 0\rangle$.

As previously, let $\{X_\gamma(\alpha) : \gamma \in \{0, \dots, \alpha\}\}$ be the best sequence in $X(\alpha)$, and define Y_γ as the (unique) continuum irreducible with respect to containing $(X_\gamma(\alpha) \times \{0,1\}$ for $\gamma \in \{0, \dots, \alpha\}$. Since α is a limit ordinal, $Y_\alpha = [0,1] \subset X(\alpha+1)$, and we may put $Y_{\alpha+1} = \varnothing$ to obtain the best sequence $\{Y_\gamma : \gamma \in \{0, \dots, \alpha+1\}\}$. This shows, again by Corollary 1.9, that $s(X(\alpha+1)) = \alpha+1$.

Assume $\beta \in \mathrm{Lim}$ and that, for each $\alpha < \beta$, we have defined an arboroid $X(\alpha)$ smooth at a point $p(\alpha)$. We will construct $X(\beta)$. To this aim, consider the union $X(\beta) = \cup\{X(\alpha) : \alpha < \beta\}$ with $X(\alpha) \cap X(\alpha') = \{p(\alpha)\} = \{p(\alpha')\} = \{p(\beta)\}$. We will define a compact topology on $X(\beta)$. Let U be a subset of $X(\beta)$. Then U is said to be open in $X(\beta)$ if $U \cap X(\alpha)$ is open in $X(\alpha)$ for all $\alpha < \beta$ and one of the two conditions holds for all but finitely many ordinals $\alpha < \beta$:

(2.9.1) $X(\alpha) \subset U$;

(2.9.2) $p(\beta) \notin U$ and $X(\alpha) \cap U = \varnothing$.

In other words, $X(\beta)$ is a compact one-point union of spaces $X(\alpha)$ for $\alpha < \beta$. In particular, it follows that $X(\beta)$ is an arboroid which is smooth at $p(\beta)$.

If, for each $\alpha < \beta$, the sequence $\{X_\gamma(\alpha) : \gamma \in \{0, \dots, \alpha\}\}$ is the best one in $X(\alpha)$, then putting $Y_\gamma = \cup\{X_\gamma(\alpha) : \gamma \in \{0, \dots, \alpha\}\}$ and

$Y_\beta = \{p(\beta)\}$ we get the best sequence in $X(\beta)$ of length β. According to Corollary 1.9, this shows that $s(X(\beta)) = \beta$. This finishes the proof that $s(X(\alpha)) = \alpha$.

To see that $s(X(\alpha)) = s(C(X(\alpha)))$, note that the inequality $s(X) \leq s(C(X))$ is a consequence of Statement 2.6 and Theorem 1.5. The reverse one is just Theorem 2.5.

To see that the continuum $X(\alpha)$ is metrizable for $\alpha < \omega_1$, it is enough to observe that then it is second countable by its construction. Thus its metrizability follows from its compactness, see [**8**, p. 260]. The proof is finished. □

Admit the following notation. In the plane $\mathbf{R}^2$, let $I = \{\langle 0, y\rangle : y \in [-1,1]\}$, $p = \langle 0, -1\rangle$ and $q = \langle 1, \sin 1\rangle$, and let S be the $\sin(1/x)$-curve, i.e.,

$$S = I \cup \{\langle x, \sin(1/x)\rangle \in \mathbf{R}^2 : x \in (0,1]\}. \tag{2.10}$$

Thus S is a continuum irreducible from p to q.

The next example is related to Theorem 2.5 and Proposition 2.9. It shows that the inequality in Theorem 2.5 cannot be replaced by the equality, and therefore the equality in the conclusion of Proposition 2.9 need not be true in general.

Example 2.11. There exists a continuum X such that $s(C(X)) = 2$, while $s(X) = \infty$.

Proof. Let $X_0 = S$ be the $\sin(1/x)$-curve, and let X_2 be the image of X_0 under the symmetry with respect to the straight line $x = 1$. Put $X = X_0 \cup X_2$, and denote by I_0 and I_2 the two limit segments of X, i.e., $I_k = \{k\} \times [-1,1]$ for $k \in \{0,2\}$. Then, by the definition, we have $s(X) = \infty$. Further, $N(C(X)) \subset C(I_0 \cup I_2) = C(I_0) \cup C(I_2)$ according to Lemma 2.3. For $k \in \{0,2\}$, let $\mathcal{J}_k$ be an order arc from I_k to X in $C(X)$. Put $\mathcal{P} = C(I_0) \cup \mathcal{J}_0 \cup \mathcal{J}_2 \cup C(I_2)$. Since $C(I_0)$ and $C(I_2)$ are locally connected, the continuum P is locally connected, and thus the sequence $\{C(X), \mathcal{P}, \varnothing\}$ is a good one in $C(X)$, whence it follows that $s(C(X)) \leq 2$. Since $C(X)$ is not locally connected, we have $s(C(X)) = 2$. □

Proposition 2.12. *For each ordinal number $\beta \geq 2$, there is a hereditarily unicoherent continuum $X(\beta)$, metrizable if $\beta < \omega_1$, such that $s(X(\beta)) = \beta$ and $s(C(X(\beta))) = 2$.*

Proof. We proceed by transfinite induction. Let $X(2) = S$ be the $\sin(1/x)$-curve. Assume that we have defined, for some $\alpha \geq 2$, a hereditarily unicoherent continuum $X(\alpha)$ satisfying $s(X(\alpha)) = \alpha$ and $s(C(X(\alpha))) = 2$. Let $\{X_\gamma(\alpha) : \gamma \in \{0, \dots, \alpha\}\}$ be the best sequence in $X(\alpha)$. Then $X_{\text{pred}\,(\alpha)}(a)$ is a locally connected (possibly degenerate) continuum. Let $p(\alpha) \in X_{\text{pred}\,(\alpha)}(\alpha)$. To define $X(\alpha+1)$ we consider two cases.

Case 1. $\alpha \notin \text{Lim}$. Define $X(\alpha+1) = S \cup X(\alpha)$ with the points $q \in S$ and $p(\alpha) \in X(\alpha)$ identified, i.e., with $S \cap X(\alpha) = \{q\} = \{p(\alpha)\}$. Then $X(\alpha+1)$ is a hereditarily unicoherent continuum. Putting

$$Y_\gamma = S \cup X_\gamma(\alpha) \quad \text{for } \gamma \leq \text{pred}\,(\alpha),$$
$$Y_\alpha = I \subset S \quad \text{and} \quad Y_{\alpha+1} = \varnothing$$

we get the best sequence in $X(\alpha+1)$ of length $\alpha+1$, whence by Corollary 1.9 it follows that $s(X(\alpha+1)) = \alpha+1$.

Now we will show that $s(C(X(\alpha+1))) = 2$. Let $\{C(X(\alpha)), \mathcal{P}, \varnothing\}$ be a good sequence in $C(X(\alpha))$, where $\mathcal{P}$ is a locally connected subcontinuum of $C(X(\alpha))$. Choose $P \in \mathcal{P}$ and let $\mathcal{A}$ and $\mathcal{B}$ be two order arcs in $C(X(\alpha+1))$: the former from P to $X(\alpha+1)$ and the latter from I also to $X(\alpha+1)$. Thus the continuum $\mathcal{Q} = \mathcal{P} \cup \mathcal{A} \cup \mathcal{B} \cup C(I)$ is a locally connected subcontinuum of $C(X(\alpha+1))$ containing the set $N(C(X(\alpha+1))) = N(C(X(\alpha))) \cup (C(I) \setminus \{I\})$. Thus the sequence $\{C(X(\alpha+1)), \mathcal{Q}, \varnothing\}$ is good in $C(X(\alpha+1))$. This shows that $s(C(X(\alpha+1))) \leq 2$. Since $C(X(\alpha+1))$ is not locally connected, we have $s(C(X(\alpha+1))) \geq 2$, whence $s(C(X(\alpha+1))) = 2$, as required.

Case 2. $\alpha \in \text{Lim}$. We proceed exactly as in Case 2 in the proof of Proposition 2.9. We need only to show that $s(C(X(\alpha+1))) = 2$. To this end, let $\{C(X(\alpha)), \mathcal{P}, \varnothing\}$ be a good sequence in $C(X(\alpha))$. Choose $P \in \mathcal{P}$ and consider, for $i \in \{0,1\}$ an order arc $\mathcal{A}_i$ from $P \times \{i\}$ to $X(\alpha+1)$ in $C(X(\alpha+1))$. Then the sequence

$$\big\{C(X(\alpha+1)), \{P \times \{0\} : P \in \mathcal{P}\} \cup \mathcal{A}_0 \cup \mathcal{A}_1 \cup \{P \times \{1\} : P \in \mathcal{P}\}, \varnothing\big\}$$

is good in $C(X(\alpha+1))$. This shows that $s(C(X(\alpha+1)))=2$.

Now assume that $\beta \in \mathrm{Lim}$ and that, for each $\alpha<\beta$, we have defined a hereditarily unicoherent continuum $X(\alpha)$ satisfying $s(X(\alpha))=\alpha$ and $s(C(X(\alpha)))=2$. Let $\{X_\gamma(\alpha):\gamma\in\{0,\dots,\alpha\}\}$ be the best sequence in $X(\alpha)$, and let $p(\alpha)\in X_{\mathrm{pred}\,(\alpha)}(\alpha)$. Define, like in the corresponding part of the proof of Proposition 2.9, the space $X(\beta)$ as the compact one-point union of the continua $X(\alpha)$ for $\alpha<\beta$. Thus it is a hereditarily unicoherent continuum. Further, $s(X(\beta))=\beta$ by an argument similar to that used in the proof of Proposition 2.9.

To show $s(C(X(\beta)))=2$, take, for each $\alpha<\beta$, a good sequence $\{C(X(\alpha)),\mathcal{P}_\alpha,\varnothing\}$ in $C(X(\alpha))$. Thus $\mathcal{P}_\alpha$ is a locally connected subcontinuum of $C(X(\alpha))$ containing the set $N(C(X(\alpha)))$. For each $\alpha<\beta$, let P_α be any element of $\mathcal{P}_\alpha$, and let $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ be two order arcs in $C(X(\alpha))$, the former from P_α, and the latter from the singleton $\{p(\alpha)\}=\{p(\beta)\}$, both to $X(\alpha)$. Put $\mathcal{P}=\cup\{\mathcal{P}_\alpha\cup\mathcal{A}_\alpha\cup\mathcal{B}_\alpha:\alpha<\beta\}$. By the definition of the topology in $X(\beta)$ the set $\mathcal{P}$ is compact. Since $p(\beta)\in\mathcal{B}_\alpha$ for each $\alpha<\beta$, the set $\mathcal{P}$ is connected, so it is a continuum. Observe further that $\mathcal{P}$ is locally connected at $\{p(\beta)\}$ and, by the inductive assumption, at any other point. Thus it is locally connected. Finally note that if $p(\beta)\in Q\in C(X(\beta))$, then (by [**11**, p. 391]) Q does not belong to $N(C(X(\beta)))$, whence $N(C(X(\beta)))=\cup\{N(C(X(\alpha))):\alpha<\beta\}$. Thus we have $N(C(X(\beta)))\subset\mathcal{P}$, and thereby $\{C(X(\beta)),\mathcal{P},\varnothing\}$ is a good sequence in $C(X(\beta))$. The proof is complete. □

Proposition 2.13. *For every two ordinals α and β satisfying $2\le\alpha\le\beta$, there is a hereditarily unicoherent continuum $X(\alpha,\beta)$, metrizable if $\beta<\omega_1$, such that $s(X(\alpha,\beta))=\beta$ and $s(C(X(\alpha,\beta)))=\alpha$.*

Proof. Let $X(\alpha)$ be an arboroid as in Proposition 2.9, i.e., smooth at a point $p(\alpha)$ and such that $s(X(\alpha))=s(C(X(a)))=\alpha$. Further, let $Y(\beta)$ be a hereditarily unicoherent continuum as in Proposition 2.12, i.e., such that $s(Y(\beta))=\beta$ and $s(C(Y(\beta)))=2$, and let $q(\beta)\in Y_{\mathrm{pred}\,(\beta)}(\beta)$. Define $X(\alpha,\beta)=X(\alpha)\cup Y(\beta)$ with the points $p(\alpha)$ and $q(\beta)$ identified, i.e., with $X(\alpha)\cap Y(\beta)=\{p(\alpha)\}=\{q(\beta)\}$.

If $\{X_\gamma(\alpha):\gamma\in\{0,\dots,\alpha\}\}$ and $\{Y_\gamma(\beta):\gamma\in\{0,\dots,\beta\}\}$ are the best sequences in $X(\alpha)$ and in $Y(\beta)$, respectively, then the sequence

$\{Z_\gamma : \gamma \in \{0, \dots, \beta\}\}$ defined by

$$Z_\gamma = \begin{cases} I(X_\gamma(\alpha) \cup Y_\gamma(\beta)) & \text{if } \gamma \le \alpha, \\ Y(\beta) & \text{if } \alpha < \gamma \le \beta, \end{cases}$$

is the best one of length β in $X(\alpha, \beta)$. Therefore $s(X(\alpha, \beta)) = \beta$ according to Corollary 1.9.

Define a retraction $R : C(X(\alpha, \beta)) \to C(X(\alpha))$ by $R(P) = P \cap X(\alpha)$ for each $P \in C(X(\alpha, \beta))$. One can verify that R is continuous. Therefore, $s(C(X(\alpha, \beta))) \ge s(C(X(\alpha))) = \alpha$ by Theorem 1.5. To show the other inequality, let $\{\mathcal{X}_\gamma : \gamma \in \{0, \dots, \alpha\}\}$ be a good sequence in $C(X(\alpha))$, and let $\{C(Y(\beta)), \mathcal{P}, \varnothing\}$ be a good sequence in $C(Y(\beta))$. Arguing as in the corresponding part of the proof of Proposition 2.12, we can show that

$$N(C(X(\alpha, \beta))) = N(C(X(\alpha))) \cup N(C(Y(\beta))). \tag{2.13.1}$$

Let $P \in \mathcal{P}$. Choose three order arcs: $\mathcal{A}$ from P to $Y(\beta)$, $\mathcal{B}$ from $\{q(\beta)\} = \{p(\alpha)\}$ also to $Y(\beta)$, both in $C(Y(\beta))$, and $\mathcal{C}$ from $\{p(\alpha)\} = \{q(\beta)\}$ to an element of $\mathcal{X}_1$ in $C(X(\alpha))$. Define

$$\mathcal{Y}_0 = C(X(\alpha, \beta)), \qquad \mathcal{Y}_1 = \mathcal{P} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{X}_1$$
$$\text{and } \mathcal{Y}_\gamma = \mathcal{X}_\gamma \quad \text{for } \gamma \in \{2, \dots, \alpha\}\}.$$

Since $\mathcal{P}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are locally connected continua, we infer from (2.13.1) that the sequence $\{\mathcal{Y}_\gamma : \gamma \in \{0, \dots, \alpha\}\}$ is a good one in $C(X(\alpha, \beta))$. This shows that $s(C(X(\alpha, \beta))) \le \alpha$, and consequently it equals α, as required. The proof is finished. □

3. Cartesian products. Now we will discuss the degree of nonlocal connectedness for the Cartesian products of continua. To do this let us recall the concept of the natural sum $\alpha(+)\beta$ of ordinal numbers α and β.

Let

$$\alpha = \omega^{\xi_1} p_1 + \omega^{\xi_2} p_2 + \dots + \omega^{\xi_h} p_h,$$
$$\beta = \omega^{\xi_1} q_1 + \omega^{\xi_2} q_2 + \dots + \omega^{\xi_h} q_h,$$

where $\xi_1 > \xi_2 > \cdots > \xi_h$ and where p_i, q_i, for $i \in \{1, 2, \dots, h\}$, are natural numbers, some of them possibly zeros. Then the *natural sum* is defined by (see [**19**, p. 253])

$$\alpha(+)\beta = \omega^{\xi_1}(p_1 + q_1) + \omega^{\xi_2}(p_2 + q_2) + \cdots + \omega^{\xi_h}(p_h + q_h).$$

The following two lemmas will be useful in the sequel.

Lemma 3.1. *Let* $\beta \in \mathrm{Lim}$, *and to each ordinal number* $\alpha < \beta$ *let two ordinal numbers* α_1 *and* α_2 *be assigned such that* $\alpha = \alpha_1(+)\alpha_2$. *Then*

$$\sup\{\alpha_1 : \alpha < \beta\}(+)\sup\{\alpha_2 : \alpha < \beta\} \geq \beta.$$

Proof. Put $\gamma = \sup\{\alpha_1 : \alpha < \beta\}(+)\sup\{\alpha_2 : \alpha < \beta\}$ and suppose on the contrary that $\gamma < \beta$. Let $\delta = \gamma + 1 < \beta$. Then $\gamma \geq \delta_1(+)\delta_2 = \delta$, a contradiction. $\square$

Lemma 3.2. *Let ordinal numbers* α *and* β *be given with* $\beta = \beta_1(+)\beta_2$ *and* $\alpha < \beta$. *Then there exist* α_1 *and* α_2 *such that*

$$\alpha_1 \leq \beta_1, \qquad \alpha_2 \leq \beta_2, \quad \text{and} \quad \alpha = \alpha_1(+)\alpha_2. \tag{3.2.1}$$

Proof. Consider three cases.

Case 1. $\alpha < \beta_1$. Define $\alpha_1 = \alpha$ and $\alpha_2 = 0$. Then conditions (3.2.1) obviously hold.

Case 2. $\alpha < \beta_2$. This case can be treated analogously.

Case 3. $\alpha \geq \beta_1$ and $\alpha \geq \beta_2$. Put

$$\beta_1 = \omega^{\xi_1}p_1 + \omega^{\xi_2}p_2 + \cdots + \omega^{\xi_h}p_h,$$
$$\beta_2 = \omega^{\xi_1}q_1 + \omega^{\xi_2}q_2 + \cdots + \omega^{\xi_h}q_h,$$
$$\alpha = \omega^{\xi_1}r_1 + \omega^{\xi_2}r_2 + \cdots + \omega^{\xi_h}r_h,$$

where $\xi_1 > \xi_2 > \cdots > \xi_h$ and $p_i, q_i, r_i \in \mathbf{N}$ for $i \in \{1, 2, \ldots, h\}$, with $p_i > 0$ or $q_i > 0$. Let $k \in \{1, 2, \ldots, h\}$ be the smallest index such that $r_k < p_k + q_k$. Then, by the case considered, we have $r_k \geq p_k$ and $r_k \geq q_k$, and we may assume $p_k > 0$. Define

$$\alpha_1 = \omega^{\xi_1} p_1 \cdots + \omega^{\xi_{k-1}} p_{k-1} + \omega^{\xi_k}(p_k - 1) + \omega^{\xi_{k+1}} r_{k+1} + \cdots + \omega^{\xi_h} r_h$$

and

$$\alpha_2 = \omega^{\xi_k}(r_k - p_k + 1).$$

Thus $r_k - p_k > 0$, so α_1 and α_2 are well defined, and conditions (3.2.1) obviously hold.

The proof is complete. □

Proposition 3.3. *Let*

$$\mathcal{X} = \{X_\alpha : \alpha \leq \lambda_X\} \quad \textit{and} \quad \mathcal{Y} = \{Y_\alpha : \alpha \leq \lambda_Y\}$$

be good sequences for continua X and Y, respectively, and let

$$\lambda = \begin{cases} \lambda_X(+)\lambda_Y & \textit{if } X_{\lambda_X} \neq \varnothing \neq Y_{\lambda_Y}, \\ \operatorname{pred}(\lambda_X)(+)\operatorname{pred}(\lambda_Y)(+)1 & \textit{otherwise.} \end{cases}$$

Then the sequence $\mathcal{Z} = \{Z_\alpha : \alpha \leq \lambda\}$ defined by

(3.3.1)

$$Z_\alpha = \cup\{X_{\alpha_X} \times Y_{\alpha_Y} : \alpha = \alpha_X(+)\alpha_Y \textit{ with } \alpha_X \leq \lambda_X \textit{ and } \alpha_Y \leq \lambda_Y\}$$

is good in the product $X \times Y$.

Proof. First we have to show that each of the sets Z_α defined by (3.3.1) is a continuum. To this goal it is enough to note that if $X_{\alpha_X} \times Y_{\alpha_Y}$ and $X_{\alpha'_X} \times Y_{\alpha'_Y}$ are nonempty uniands of the union in the right member of (3.3.1), then $X_{\max\{\alpha_X, \alpha'_X\}} \times Y_{\max\{\alpha_Y, \alpha'_Y\}}$ is a nonempty subset of $(X_{\alpha_X} \times Y_{\alpha_Y}) \cap (X_{\alpha'_X} \times Y_{\alpha'_Y})$.

Second, note (see [**19**, p. 253]) that

(3.3.2) the union in the right member of the formula (3.3.1) has finitely many uniands.

Assuming that the conditions (1.3.1)–(1.3.5) hold for the sequences $\mathcal{X}$ and $\mathcal{Y}$, we will verify them for the sequence $\mathcal{Z}$.

Putting $\alpha = 0$ in (3.3.1), we see that $Z_0 = X_0 \times Y_0$, so (1.3.1) is satisfied.

To show (1.3.2) for $\mathcal{Z}$, take a point $\langle x, y\rangle \in Z_{\alpha+1}$. Then there are ordinal numbers $\alpha_X \le \lambda_X$ and $\alpha_Y \le \lambda_Y$ such that $\alpha_X(+)\alpha_Y = \alpha + 1$ and that $\langle x, y\rangle \in X_{\alpha_X} \times Y_{\alpha_Y}$. Thus either α_X or α_Y, say α_X, is not a limit ordinal. Put $\alpha_X = \alpha'_X + 1$. Then $X_{\alpha_X} \subset X_{\alpha'_X}$ by (1.3.2) for $\mathcal{X}$, so $\langle x, y\rangle \in X_{\alpha'_X} \times Y_{\alpha_Y}$. Note that $\alpha'_X(+)\alpha_Y = \alpha$, whence it follows that $X_{\alpha'_X} \times Y_{\alpha_Y} \subset Z_\alpha$. Thus, $\langle x, y\rangle \in Z_\alpha$ as needed, and so (1.3.2) for $\mathcal{Z}$ is shown.

To verify (1.3.3) for $\mathcal{Z}$, take a point $\langle x, y\rangle \in N(Z_\alpha)$. Then by Fact 1.2.1 and (3.3.1), there are ordinal numbers $\alpha_X \le \lambda_X$ and $\alpha_Y \le \lambda_Y$ such that $\alpha_X(+)\alpha_Y = \alpha$ and $\langle x, y\rangle \in N(X_{\alpha_X} \times Y_{\alpha_Y})$. Further, $N(X_{\alpha_X} \times Y_{\alpha_Y}) = (N(X_{\alpha_X}) \times Y_{\alpha_Y}) \cup (X_{\alpha_X} \times N(Y_{\alpha_Y}))$ by Fact 1.2.2. Since $(\alpha_X + 1)(+)\alpha_Y = \alpha_X(+)(\alpha_Y + 1) = (\alpha_X(+)\alpha_Y)(+)1 = \alpha + 1$, the conclusion follows.

Now we will show that (1.3.4) holds for $\mathcal{Z}$, i.e., that

$$Z_\beta = \bigcap\{Z_\alpha : \alpha < \beta\} \quad \text{for each } \beta \in \mathrm{Lim} \quad \text{with } \beta \le \lambda. \tag{3.3.3}$$

First, let $\langle x, y\rangle \in Z_\beta$. Then there are ordinal numbers $\beta_X \le \lambda_X$ and $\beta_Y \le \lambda_Y$ such that $\beta_X(+)\beta_Y = \beta$ and $\langle x, y\rangle \in X_{\beta_X} \times Y_{\beta_Y}$. Let $\alpha < \beta$. By Lemma 3.2 there are ordinal numbers α_X and α_Y such that

$$\alpha_X \le \beta_X, \qquad \alpha_Y \le \beta_Y \quad \text{and} \quad \alpha_X(+)\alpha_Y = \alpha.$$

Thus $X_{\beta_X} \times Y_{\beta_Y} \subset X_{\alpha_X} \times Y_{\alpha_Y} \subset Z_\alpha$, whence $\langle x, y\rangle \in Z_\beta$, as needed. So one inclusion in the equality (3.3.3) is shown.

To prove the other one, let $\langle x, y\rangle$ belong to the intersection in the right member of (3.3.3). Then, by the definition (3.3.1) of Z_α, for each $\alpha < \beta$ there are ordinal numbers α_X and α_Y such that $\alpha_X(+)\alpha_Y = \alpha$ and $\langle x, y\rangle \in X_{\alpha_X} \times Y_{\alpha_Y}$. Put

$$\beta_X = \sup\{\alpha_X : \alpha < \beta\} \quad \text{and} \quad \beta_Y = \sup\{\alpha_Y : \alpha < \beta\}.$$

By Lemma 3.1 we have $\beta_X(+)\beta_Y \ge \beta$, and by Lemma 3.2 there are ordinal numbers β'_X and β'_Y such that

$$\beta'_X \le \beta_X, \qquad \beta'_Y \le \beta_Y \quad \text{and} \quad \beta'_X(+)\beta'_Y = \beta.$$

Therefore, $\langle x, y\rangle \in X_{\beta_X} \times Y_{\beta_Y} \subset X_{\beta'_X} \times Y_{\beta'_Y} \subset Z_\beta$. So the other inclusion is shown and thus the equality (3.3.3) holds.

Finally we have to verify condition (1.3.5) for $\mathcal{Z}$. If $\lambda_X \in \mathrm{Lim}$ and $\lambda_Y \in \mathrm{Lim}$, then $Z_\lambda = X_{\lambda_X} \times Y_{\lambda_Y}$ is a one-point set, and $\lambda = \lambda_X(+)\lambda_Y$ is a limit ordinal.

So assume that one of λ_X and λ_Y, say λ_X, is not in Lim. Then $\lambda_X = \mathrm{pred}\,(\lambda_X)(+)1$. Let ordinal numbers α and β be such that $\alpha \leq \lambda_X$, $\beta \leq \lambda_Y$ and $\alpha(+)\beta = \lambda = \mathrm{pred}\,(\lambda_X)(+)\mathrm{pred}\,(\lambda_Y)(+)1 = \lambda_X(+)\mathrm{pred}\,(\lambda_Y)$. Since $\alpha \leq \lambda_X$, we have $\lambda_X(+)\mathrm{pred}\,(\lambda_Y) \leq \lambda_X(+)\beta$. Further, since the natural sum is an increasing function with respect to each of the summands (see [**19**, p. 253]), we get $\mathrm{pred}\,(\lambda_Y) \leq \beta$, whence either $\beta = \mathrm{pred}\,(\lambda_Y)$ or $\beta = \mathrm{pred}\,(\lambda_Y)(+)1$. Thus we have either $Z_\lambda = X_{\mathrm{pred}\,(\lambda_X)} \times Y_{\mathrm{pred}\,(\lambda_Y)(+)1} \cup X_{\mathrm{pred}\,(\lambda_X)(+)1} \times Y_{\mathrm{pred}\,(\lambda_Y)}$ if $\lambda_Y \notin \mathrm{Lim}$ or $Z_\lambda = X_{\mathrm{pred}\,(\lambda_X)(+)1} \times Y_{\mathrm{pred}\,(\lambda_Y)}$ if $\lambda_Y \in \mathrm{Lim}$. Since $X_{\mathrm{pred}\,(\lambda_X)(+)1} = Y_{\mathrm{pred}\,(\lambda_Y)(+)1} = \varnothing$, we have $Z_\lambda = \varnothing$. The proof is complete. □

Proposition 3.3 implies the following results.

Theorem 3.4. *For every two continua X and Y, we have*

$$s(X \times Y) \leq \begin{cases} s(X)(+)s(Y) & \textit{if } s(X), s(Y) \in \mathrm{Lim}, \\ \mathrm{pred}\,(s(X))(+)\mathrm{pred}\,(s(Y))(+)1 & \textit{otherwise.} \end{cases}$$

Corollary 3.5. *For every two continua X and Y, we have*

$$\nu(X \times Y) \leq \nu(X)(+)\nu(Y).$$

It would be interesting to know whether Theorem 3.4 is the best possible. In other words, we have the following question.

Question 3.6. For arbitrary ordinal numbers α and β, do continua X and Y exist satisfying $s(X) = \alpha$, $s(Y) = \beta$ and

$$s(X \times Y) = \begin{cases} \alpha(+)\beta & \text{if } \alpha, \beta \in \mathrm{Lim}, \\ \mathrm{pred}\,(\alpha)(+)\mathrm{pred}\,(\beta)(+)1 & \text{otherwise?} \end{cases}$$

Proposition 3.7. *For every two continua X and Y with $s(X), s(Y) \in \mathbf{N} \setminus \{0,1\}$ we have $s(X \times Y) \geq s(X) + 1$.*

Proof. Let $k = s(X \times Y)$, and let $\{X \times Y = Z_0, \dots, Z_k\}$ be a good sequence in $X \times Y$. Denote by $\pi : X \times Y \to X$ the projection onto the first factor. Note that, by Proposition 1.4, the sequence $\{\pi(Z_0), \dots, \pi(Z_k)\}$ is a good one in X. Further, observe that $Z_1 \supset N(Z_0) = (N(X) \times Y) \cup (X \times N(Y))$ according to Fact 1.2.2, so $\pi(Z_1) = X$. Thus the sequence $\{\pi(Z_1), \dots, \pi(Z_k)\}$ is a good one in X, and therefore $s(X) \leq k - 1$, whence $k \geq s(X) + 1$, as required. □

It is shown in [**9**, p. 29] that if a continuum X is not locally connected and the set $N(X)$ is contained in a locally connected subcontinuum of X, i.e., if $s(X) = 2$, then $X \times X$ is not a continuous image of X. The following corollary, which follows from Proposition 3.7 and Theorem 1.5, generalizes this result (compare also [**16**, p. 1140]).

Corollary 3.8. *For each continuum X, if $s(X) = n \in \mathbf{N} \setminus \{0,1\}$, then there is no mapping from X onto $X \times X$.*

One can ask if continua X and Y exist such that the inequality in Theorem 3.4 is strict. In the next proposition we present a result from which a positive answer to this question follows.

For a space X, we denote by $\Delta(X)$ the diagonal of its Cartesian square, i.e., $\Delta(X) = \{\langle x, x\rangle : x \in X\}$.

Proposition 3.9. *For each natural number $n \in \mathbf{N} \setminus \{0,1\}$, there is a metric hereditarily unicoherent continuum X such that $s(X) = n$ and $s(X \times X) = n + 1$.*

Proof. For $k \in \mathbf{N}$, put

$$I_k = \{k\} \times [-1,1] \quad \text{and} \quad L_k = \left\{\left\langle x, \sin\frac{1}{x-k}\right\rangle \in \mathbf{R}^2 : x \in (k, k+1]\right\},$$

and note that $I_k \cup L_k$ is homeomorphic to the $\sin(1/x)$-curve S. Define

$$X = I_n \cup \bigcup \{I_k \cup L_k : k \in \{1, \dots, n-1\}\}.$$

Thus X is a hereditarily unicoherent (plane) continuum. Note that the sequence $\{X_0, X_1, \dots, X_n\}$ defined by

$$X_j = I_{n-j} \cup \bigcup \{I_k \cup L_k : k \in \{1, \dots, n-j-1\}\} \quad \text{for } j \in \{0, \dots, n-2\},$$

$X_{n-1} = I_1$ and $X_n = \varnothing$, is the best one in X. Thus by Corollary 1.9 we get $s(X) = n$.

For $j \in \{0, \dots, n-2\}$ define $M_j = I_1 \cup \dots \cup I_{n-j}$ and $M_n = \varnothing$ and note that

$$M_j = M_{j+1} \cup I_{n-j}; \tag{3.9.1}$$

$$N(X_j) = M_{j+1}. \tag{3.9.2}$$

Now we will define a good sequence $\{Y_0, \dots, Y_{n+1}\}$ in $X \times X$. Put $Y_0 = X \times X$, $Y_{n+1} = \varnothing$, and for $j \in \{1, \dots, n\}$, let

$$Y_j = (X_{j-1} \times M_j) \cup (M_j \times X_{j-1}) \cup \Delta(X_{j-1}) \cup (I_{n-j+1} \times I_{n-j+1}). \tag{3.9.3}$$

Then conditions (1.3.1), (1.3.2) and (1.3.5) are obviously satisfied, and (1.3.4) does not apply. We will verify (1.3.3), which means that $N(Y_k) \subset Y_{k+1}$ for $k \in \{0, \dots, n\}$. If $k = 0$, the inclusion is a consequence of the definitions and of Fact 1.2.2. So assume $k \in \{1, \dots, n-1\}$. Then

$$\begin{aligned} N(Y_k) &\subset (N(X_{k-1}) \times M_k) \cup (M_k \times N(X_{k-1}) \cup N(\Delta(X_{k-1})) && \text{by Fact 1.2} \\ &= M_k \times M_k && \text{by (3.9.2)} \\ &\subset (M_{k+1} \cup I_{n-k}) \times (M_{k+1} \cup I_{n-k}) && \text{by (3.9.1)} \\ &\subset (X_k \times M_{k+1}) \cup (M_{k+1} \times X_k) \cup \Delta(X_k) \cup (I_{n-k} \times I_{n-k}) \\ &= Y_{k+1} && \text{by (3.9.3).} \end{aligned}$$

For $k = n$, we have $N(Y_n) = N(I_1 \times I_1) = \varnothing = Y_{n+1}$. So (1.3.3) holds, and we have proved that the sequence $\{Y_0, \dots, Y_{n+1}\}$ is a good one in $X \times X$. Therefore $s(X \times X) \le n + 1$. The opposite inequality is just Proposition 3.7. The proof is complete. □

Now we will pass to infinite Cartesian products. We denote by $\mathcal{J}$ an arbitrary set of indices, by $X = \prod\{X_i : i \in \mathcal{J}\}$ the Cartesian product of the spaces X_i, and by $\pi_i : X \to X_i$ the natural projection of the product onto the $i$the factor X_i for each $i \in \mathcal{J}$. For shortness we put $x_i = \pi_i(x)$.

The following lemma is an extension of statement (4) in [**9**, p. 28] for continua. Recall that $L(X)$ denotes the set of all points of a continuum X at which X is locally connected.

Lemma 3.10. *Let an arbitrary family of continua X_i be given, where $i \in \mathcal{J}$. Then*

$$L\Big(\prod\{X_i : i \in \mathcal{J}\}\Big) = \prod\{L(X_i) : i \in \mathcal{J}\}. \tag{3.10.1}$$

Proof. Let x be a point of the left member of equation (3.10.1). For any fixed $j \in \mathcal{J}$ we will show that $x_j \in L(X_j)$. Let U_j be an open set in X_j which contains x_j. For each $i \in \mathcal{J}$, define

$$U_i = \begin{cases} U_j & \text{for } i = j, \\ X_i & \text{for } i \neq j, \end{cases}$$

and put $\mathcal{U} = \prod\{U_i : i \in \mathcal{J}\}$. Then $\mathcal{U}$ is open in X and $x \in \mathcal{U}$. Since x is in the left member of equality (3.10.1), a connected set $\mathcal{V}$ exists such that $x \in \operatorname{int} \mathcal{V} \subset \mathcal{U}$. Put $V = \pi_j(\mathcal{V})$. Since π_j is open, we have $x_j \in \operatorname{int} V \subset U$, whence $x_j \in L(X_j)$. Therefore, x is in the right member of equality (3.10.1), and one inclusion is shown. The other one is proved in [**18**, p. 229]. The proof is complete. □

Theorem 3.11. *Let $\mathcal{J}$ be an infinite set and, for each $i \in \mathcal{J}$, let X_i be a nonlocally connected continuum. Then $s(\prod\{X_i : i \in \mathcal{J}\}) = \infty$.*

Proof. Put $X = \prod\{X_i : i \in \mathcal{J}\}$. It is enough to prove that the set $N(X)$ is dense in X. To this aim for each $i \in \mathcal{J}$, fix a point $a_i \in N(X_i)$. Let $\mathcal{U}$ be a basic open subset of X, i.e., $\mathcal{U} = \prod\{U_i : i \in \mathcal{J}\}$, where $U_i = X_i$ for all but finitely many indices $i \in \mathcal{J}$. Fix $j \in \mathcal{J}$ such that $U_j = X_j$. Then a point $x = \{x_i : i \in \mathcal{J}\} \in \mathcal{U}$ exists such that $x_j \in N(X_j)$. By Lemma 3.10 we have $x \in \mathcal{U} \cap N(X)$ as needed. □

Corollary 3.12. *For each continuum X, we have*

$$s(X^{\aleph_0}) = \begin{cases} 0 & \text{if } X \text{ is a singleton;} \\ 1 & \text{if } X \text{ is nondegenerate and locally connected;} \\ \infty & \text{if } X \text{ is not locally connected.} \end{cases}$$

4. Inverse limits. Suppose that for every $\sigma \in \Sigma$, where Σ is a set directed by a relation $\leq$, we have a topological space X_σ, and for every $\sigma, \tau \in \Sigma$ with $\sigma \leq \tau$, a mapping $f_\sigma^\tau : X_\tau \to X_\sigma$ is defined such that the following two conditions are satisfied:

(i) $f_\sigma^\tau \circ f_\tau^\upsilon = f_\sigma^\upsilon$ for any $\sigma, \tau, \upsilon \in \Sigma$ satisfying $\sigma \leq \tau \leq \upsilon$,

(ii) f_σ^σ is the identity on X_σ for each $\sigma \in \Sigma$.

Then the family $\mathbf{S} = \{X_\sigma, f_\sigma^\tau, \Sigma\}$ is called an *inverse system of spaces* X_σ with *bonding mappings* f_σ^τ. An inverse system $\mathbf{S} = \{X_n, f_n^m, \mathbf{N}\}$, where $\mathbf{N}$ is the set of all positive integers directed by its natural order, is called an *inverse sequence*.

Let $\mathbf{S} = \{X_\sigma, f_\sigma^\tau, \Sigma\}$ be an inverse system. An element $p = \langle p_\sigma \rangle$ of the Cartesian product $\Pi\{X_\sigma : \sigma \in \Sigma\}$ such that $f_\sigma^\tau(p_\tau) = p_\sigma$ for any $\sigma, \tau \in \Sigma$ with $\sigma \leq \tau$ is called a *thread* of $\mathbf{S}$, and the subspace of $\Pi\{X_\sigma : \sigma \in \Sigma\}$ consisting of all threads of $\mathbf{S}$ is called the *limit* of the inverse system $\mathbf{S}$ and is denoted by $X = \varprojlim\{X_\sigma, f_\sigma^\tau, \Sigma\}$. Further, we denote by $f_\sigma : X \to X_\sigma$ the projection from the inverse limit space into the sth factor space. Then $p_\sigma = f_\sigma(p) \in X_\sigma$ for each $s \in \Sigma$.

The sets of the form $f_\sigma^{-1}(U_\sigma)$, where U_σ is an open subset of X_σ, called *basic open sets*, constitute a base in X. The reader is referred to Engelking's monograph [**8**] for more information on inverse systems.

We start with the following statement which is a consequence of Theorem 1.5.

Statement 4.1. *For a given inverse system* $\mathbf{S} = \{X_\sigma, f_\sigma^\tau, \Sigma\}$ *of continua* X_σ *with surjective bonding mappings* f_σ^τ, *we have*

$$s(\varprojlim \mathbf{S}) \geq \sup\{s(X_\sigma) : \sigma \in \Sigma\}.$$

It is known that the inverse limit of locally connected continua with monotone surjective bonding mappings is a locally connected continuum, see [**2**, p. 241]. The next two examples show that the degree s can increase under the inverse limit operation with monotone or even monotone and open surjective bonding mappings.

Example 4.2. There is an inverse sequence $\mathbf{S} = \{X_n, f_n^m, \mathbf{N}\}$ of continua X_n with monotone surjective bonding mappings f_n^m satisfying $s(X_n) = 2$ for each $n \in \mathbf{N}$, while $s(\varprojlim \mathbf{S}) = 3$.

Proof. Let X_0 be the geometric comb, i.e., $X_0 = G$, see (2.8). Given $m \in \mathbf{N} \setminus \{0\}$, define

$$\begin{aligned} A_m = &\Big(\Big[\frac{1}{2^{m-1}}, \frac{1}{2^{m-1}} + \frac{1}{2^m}\Big] \times \{0\}\Big) \\ &\cup \Big(\Big\{\frac{1}{2^{m-1}}\Big\} \times \Big[-\frac{1}{2^m}, 0\Big]\Big) \\ &\cup \bigcup \Big\{\Big(\Big\{\frac{1}{2^{m-1}} + \frac{1}{2^{j+m-1}}\Big\} \times \Big[-\frac{1}{2^m}, 0\Big]\Big) : j \in \mathbf{N} \setminus \{0\}\Big\}. \end{aligned}$$

Note that A_m is homeomorphic to X_0 for each $m \in \mathbf{N} \setminus \{0\}$, and put, for each $n \in \mathbf{N} \setminus \{0\}$,

$$X_n = X_0 \cup \bigcup \{A_m : m \in \{1, \dots, n\}\}.$$

Observe that X_n is a continuum for each $n \in \mathbf{N}$, and that

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots.$$

For each $N \in \mathbf{N}$, define $f_n^{n+1} : X_{n+1} \to X_n$ as the retraction satisfying $f_n^{n+1}(A_{n+1}) = \{\langle(1/2^{m-1}), 0\rangle\}$. Then f_n^{n+1} is a monotone surjection. Applying the Anderson-Choquet theorem (see [**27**, p. 23]) to the inverse sequence $\mathbf{S} = \{X_n, f_n^m, \mathbf{N}\}$, we see that $X = \varprojlim \mathbf{S}$ is homeomorphic to $\cup\{X_n : n \in \mathbf{N}\}$. One can observe that $s(X_N) = 2$ for each $n \in \mathbf{N}$, and that $s(\varprojlim \mathbf{S}) = 3$, as required. □

Example 4.3. There is an inverse sequence $\mathbf{S} = \{X_n, f_n^m, \mathbf{N}\}$ of continua X_n with monotone and open surjective bonding mappings f_n^m satisfying $s(X_n) < \omega$ for each $n \in \mathbf{N}$, while $s(\varprojlim \mathbf{S}) = \infty$.

Proof. Let G be the geometric comb defined by (2.8) and put $X_n = G^n$ for each $n \in \mathbf{N}$. Let $f_n^m : X_m \to X_n$ for $n \leq m$ be the natural projection. Thus f_n^m is monotone and open. Note that $s(G) = 2$, whence $s(X_n) < \omega$ for each $n \in \mathbf{N}$ according to Theorem 3.4. On the other hand, $X = \varprojlim \mathbf{S}$ is homeomorphic to $\prod\{G_n : n \in \mathbf{N}\}$, where each $G_n = G$ and thus $s(X) = \infty$ by Corollary 3.12. □

5. Applications. Note that we already have shown some applications of the degree s to the nonexistence of so-called Peano mappings for some continua in Corollary 3.8. Two other applications are presented below. The first concerns continuously homogeneous continua; we present an answer to a question asked by Cook in [**5**, p. 388]. In the second we construct an uncountable family of distinct metric continua X having the property that the hyperspace $C(X)$ is homeomorphic to X.

5a. Continuously homogeneous continua. A continuum X is said to be *continuously homogeneous* provided that, for every two points $x, y \in X$, a surjective mapping $f : X \to X$ exists such that $f(x) = y$. If f is a homeomorphism, then X is said to be *homogeneous*. Two continua X and Y are said to be *continuously equivalent* provided that there are surjective mappings from X onto Y and from Y onto X.

Proposition 5.1. *For each homogeneous continuum X, we have*

$$s(X) = \begin{cases} 0 & \text{if } X \text{ is a singleton;} \\ 1 & \text{if } X \text{ is nondegenerate and locally connected;} \\ \infty & \text{if } X \text{ is not locally connected.} \end{cases}$$

Proof. If $N(X) \neq \varnothing$, then $N(X) = X$ by homogeneity of X, and thus there is no good sequence in X. So $s(X) = \infty$. □

Proposition 5.2. *Every two continuously equivalent continua have the same degree of nonlocal connectedness.*

Proof. Indeed, let continua X and Y be continuously equivalent. According to Theorem 1.5, since there is a mapping from X onto Y, we have $s(Y) \leq s(X)$, and since there is a mapping from Y onto X, we have $s(X) \leq s(Y)$. Thus $s(X) = s(Y)$ as required. □

Cook asks in [**5**, p. 388] whether for each continuously homogeneous continuum X there exists a homogeneous continuum Y such that X and Y are continuously equivalent. Prajs in his comments to the above question states that the answer is negative, using a result of Krupski on continuous homogeneity of the harmonic fan (see [**17**, p. 346]). Below we present a full argument.

Theorem 5.3. *Let a nonlocally connected continuum X be continuously homogeneous, and let $s(X) < \infty$. Then X is not continuously equivalent to any homogeneous continuum.*

Proof. Since X is not locally connected, we have $s(X) > 1$. Assume on the contrary that X is continuously equivalent to a homogeneous continuum Y. Therefore, by Proposition 5.2, we have $1 < s(X) = s(Y)$, and $s(Y) \leq 1$ according to Proposition 5.1. The contradiction finishes the proof. □

Let $\mathbf{G}$ be the set defined by (2.7), and let F be the cone over $\mathbf{G}$, i.e., $F = (\mathbf{G} \times [0,1])/(\mathbf{G} \times \{0\})$. Thus F is homeomorphic to the harmonic fan. It is known that F is continuously homogeneous, see [**17**, p. 346]. Since $s(F) = 2$, all assumptions of Theorem 5.3 are satisfied with $X = F$, and we get the following corollary.

Corollary 5.4. *The harmonic fan is not continuously equivalent to any homogeneous continuum.*

5b. Continua homeomorphic to their hyperspaces. Given a continuum X, the concept of the hyperspace $C(X)$ can be extended in the following way. Let $C^0(X) = X$, and for each $n \in \mathbf{N}$, put $C^{n+1}(X) = C(C^n(X))$. Further, let $\cup : C^2(X) \to C(X)$ be the *union mapping*, i.e., for each element $\mathcal{E} \in C^2(X)$ the value $\cup(\mathcal{E})$ is defined as the union of all members of $\mathcal{E}$. See [**25**, pp. 102, 100] for correctness and continuity of $\cup$, respectively.

Proposition 5.5. *For an arbitrary continuum X, the union mapping $\cup : C^2(X) \to C(X)$ is monotone.*

Proof. Let $P \in C(X)$. We have to show that $\cup^{-1}(P)$ is connected. We will prove more, namely that it is arcwise connected, by showing that any element $\mathcal{P} \in \cup^{-1}(P)$ can be joined to $C(P) \in \cup^{-1}(P)$ by an order arc contained in $\cup^{-1}(P)$. Indeed, since $\mathcal{P} \subset C(P)$, there is an order arc from $\mathcal{P}$ to $C(P)$ in $C^2(X)$, see Fact 2.1. Since $\cup(\mathcal{P}) = \cup(C(P)) = P$, the arc is contained in $\cup^{-1}(P)$, as needed. The proof is complete. □

It is known that the inverse limit of locally connected continua with monotone bonding mappings is locally connected, see [**2**, p. 241]. In the sequel we will need a local version of this result. Although we will use the version for inverse sequences only, we will show it in a more general form for inverse systems. Recall that $L(X)$ denotes the set of all points of a continuum X at which X is locally connected.

Proposition 5.6. *Let $\mathbf{S} = \{X_\sigma, f_\sigma^\tau, \Sigma\}$ be an inverse system of continua X_σ with surjective monotone bonding mappings f_σ^τ, and let $X = \varprojlim \mathbf{S}$ be its limit. If $x = \{x_\sigma : \sigma \in \Sigma\}$ is a thread satisfying $x_\sigma \in L(X_\sigma)$ for each $\sigma \in \Sigma$, then $x \in L(X)$.*

Proof. Let $U \subset X$ be a basic open set with $x \in U$. Thus there is an index $\sigma_0 \in \Sigma$ such that $U = f_{\sigma_0}^{-1}(U_{\sigma_0})$, where U_{σ_0} is an open subset of X_{σ_0} such that $x_{\sigma_0} \in \operatorname{int} V_{\sigma_0} \subset U_{\sigma_0}$, whence taking preimages under the projection f_{σ_0} we have

$$x \in f_{\sigma_0}^{-1}(x_{\sigma_0}) \subset f_{\sigma_0}^{-1}(\operatorname{int} V_{\sigma_0}) \subset f_{\sigma_0}^{-1}(V_{\sigma_0}) \subset f_{\sigma_0}^{-1}(U_{\sigma_0}) = U.$$

Since the projections f_σ are monotone, see [**2**, p. 241], the set $V = f_{\sigma_0}^{-1}(V_{\sigma_0})$ is a continuum containing the point x in its interior (because $f_{\sigma_0}^{-1}(\operatorname{int} V_{\sigma_0})$ is open), and contained in U, as needed. Thus $x \in L(X)$. □

In the next proposition we shall use the concept of an induced mapping. We recall its definition. Given two continua X and Y and a mapping $f : X \to Y$, we denote by $C(f) : C(X) \to C(Y)$ the mapping *induced* by f, i.e., defined by the condition $C(f)(A) = f(A)$ for each $A \in C(X)$. More generally, for $n \in \mathbf{N}$, we define inductively $C^n(f) : C^n(X) \to C^n(Y)$ by: $C^0(f) = f$ and $C^{n+1}(f) = C(C^n(f))$.

If the mapping f is a surjection, then $C(f)$ does not have to be surjective, unless f is *weakly confluent*, that is, provided that for each subcontinuum Q of Y some component of $f^{-1}(Q)$ is mapped onto Q under f, see [**25**, p. 24].

The idea of the proof of the next proposition is taken from [**21**, p. 325], where the hyperspaces of all nonempty closed subsets and the corresponding induced mappings were discussed.

Proposition 5.7. *For an arbitrary continuum X, consider an inverse sequence $\mathbf{S} = \{C^n(X), f_n^m, \mathbf{N} \setminus \{0\}\}$, where for each $n \in \mathbf{N} \setminus \{0\}$ the bonding mapping $f_n^{n+1} : C^{n+1}(X) = C(C^n(X)) \to C^n(X)$ is the induced union mapping, i.e., $f_n^{n+1} = C^{n-1}(\cup)$. If $Y = \varprojlim \mathbf{S}$, then $C(Y)$ is homeomorphic to Y.*

Proof. The mapping $f_1^2 = \cup$ is monotone according to Proposition 5.5. Since monotoneity of a mapping between continua implies (even is equivalent to) monotoneity of the induced mapping between hyperspaces of subcontinua (see [**25**, p. 204]; compare [**26**, p. 750] and [**13**, p. 241]), the bonding mappings in $\mathbf{S}$ are monotone, thus obviously weakly confluent, hence onto by the above-mentioned argument. Applying the functor C both to the spaces and to the bonding mappings of the sequence $\mathbf{S}$, we get a new inverse sequence $C(\mathbf{S})$ of continua with monotone onto (induced) bonding mappings. Using a homeomorphism between $C(Y)$ and $\varprojlim C(\mathbf{S})$, see [**25**, pp. 171, 174], we see that $C(Y)$ is homeomorphic to Y, as required. $\square$

Theorem 5.8. *For each ordinal number β, there exists a continuum Y, metrizable if $\beta < \omega_1$, such that $s(Y) = \beta$ and that $C(Y)$ is homeomorphic to Y.*

Proof. Let $X = X(\beta)$ be the smooth arboroid defined in Proposition 2.9, and let $Y = \varprojlim \mathbf{S}$, where $\mathbf{S}$ is the inverse sequence as in Proposition 5.7. Then $C(Y)$ is homeomorphic to Y. Using the projection from the inverse limit Y onto the first factor $C(X)$ of $\mathbf{S}$, by Theorem 1.5 we have $\beta = s(C(X)) \leq s(Y)$.

To show the inequality $s(Y) \leq \beta$, we need to construct a good sequence of length β in Y. Let $\{X_\alpha : \alpha \leq \beta\}$ be a good sequence in X. For each $\alpha \leq \beta$, consider the inverse sequence $\mathbf{S}(\alpha) = \{C^n(X_\alpha), f_n^m|C^m(X_\alpha), \mathbf{N} \setminus \{0\}\}$ and let $Y_\alpha = \varprojlim \mathbf{S}(\alpha)$. We will show that the sequence $\mathcal{Y} = \{Y_\alpha : \alpha \leq \beta\}$ is good in Y, i.e., that conditions (1.3.1)–(1.3.5) are satisfied for $\mathcal{Y}$.

Conditions (1.3.1) and (1.3.2) are obvious. (1.3.4) follows from commutativity of the inverse limit operation with the intersection of a decreasing transfinite sequence of continua. Condition (1.3.5) follows from the fact that $C(\varnothing) = \varnothing$ and that $C(\{x\}) = \{\{x\}\}$. It remains to show (1.3.3). Fix $\gamma < \beta$. Take a thread $y = \{y_n : n \in \mathbf{N}\setminus\{0\}\} \in N(Y_\gamma)$. By Proposition 5.6 and Lemma 2.3, we have $y_1 \in N(C(X_\gamma))$. Since the sequence $\mathcal{Z} = \{C(X_\alpha) : \alpha \leq \beta\}$ is good in $C(X)$ by Proposition 2.4, we infer from condition (1.3.3) for $\mathcal{Z}$ that $y_1 \in C(X_{\gamma+1})$ and, consequently, $y \in Y_{\gamma+1}$, as needed.

This shows that $s(Y) \leq \beta$. Finally, since the limit of a countable inverse system of metric spaces is metric, the metrizability of Y in the case $\beta < \omega_1$ follows. The proof is then complete. □

Corollary 5.9. *There exists an uncountable family of metrizable continua Y having the property that $C(Y)$ is homeomorphic to Y.*

5c. Continuous images of the harmonic fan. Recall once more that local connectedness of continua is preserved under the inverse limit operation provided that the bonding mappings are monotone surjections, [**2**, p. 241]. In the metric case being locally connected is equivalent to being a continuous image of the closed unit interval. Therefore, it is natural to ask if this theorem can be generalized to the statement that the property of being a continuous image of a continuum X is preserved by the inverse limit operation (for inverse sequences), provided that the bonding mappings are monotone surjections. In the hierarachy of continua with the quasi-order $\leq$ defined by

$$X \leq Y \quad \text{if there is a surjective mapping of } Y \text{ onto } X,$$

the class of locally connected continua is the least element. By a theorem of Bellamy [**1**, p. 107] there is exactly one class $\mathcal{H}$ located in this hierarchy just above the class of locally connected metric continua. This class is composed of elements *continuously equivalent to* the harmonic fan, i.e., of such continua which are images and preimages of the harmonic fan. Another interesting property of the class $\mathcal{H}$ is shown in [**29**].

We will show that being in the class $\mathcal{H}$ is not preserved by limits of inverse sequences of continua with monotone surjective bonding mappings.

Example 5.10. There is an inverse sequence $\mathbf{S} = \{X_n, f_n^m, \mathbf{N}\}$ of continua X_n with monotone surjective bonding mappings f_n^m such that X_n is a continuous image of the harmonic fan F_H for each positive integer n, while $\varprojlim \mathbf{S}$ is not.

Proof. Let continua X_n and mappings f_n^m be as in Example 4.2. Since $s(\varprojlim \mathbf{S}) = 3$ and $s(F_H) = 2$, we see by Theorem 1.5 that $\varprojlim \mathbf{S} \notin \mathcal{H}$.

To see that each X_n is a continuous image of F_H, we need an auxiliary construction. Denote by F_{n_H} the one-point union of n copies of F_H with all the vertices identified. Let H be the closure of the harmonic sequence $\{1/n : n \in \mathbf{N}\}$. Note that F_H is homeomorphic to $(H \times [0,1])/(H \times \{0\})$ and that F_{2H} is homeomorphic to $(H \times [-1,1])/(H \times \{0\})$. For $\langle c, t\rangle \in F_H$, define

$$g(\langle c,t\rangle) = \begin{cases} \langle c, 2t\rangle & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ \langle c, 3-4t\rangle & \text{if } t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Thus, g maps F_H onto F_{2H}. Repeating this procedure finitely many times, we can find a surjective mapping from F_H onto F_{nH} for each $n \in \mathbf{N} \setminus \{0\}$.

Now we will describe, for each $n \in \mathbf{N}$, a surjective mapping $f : F_{(n+1)H} \to X_n$. Map homeomorphically the set of end points of the first harmonic fan in $F_{(n+1)H}$ onto the set of end points of X_0. Next map, also homeomorphically, the set of end points of the ith harmonic fan in $F_{(n+1)H}$ onto the set of end points $A_{i-1} \subset X_n$ for $i \in \{2, \dots, n+1\}$. If v is the vertex of $F_{(n+1)H}$, let $f(v) = \langle 0, 1\rangle \in X_0 \subset X_n$. Thus f is defined on the set E of end points of $F_{(n+1)H}$ and on the vertex v. Using the coordinates in $F_{(n+1)H} = (E\times[0,1])/(E\times\{0\})$ we define $f(\langle c,t\rangle)\in X_n$ as the point in the arc $f(v)f(\langle c,1\rangle)$ such that the length of the subarc $f(v)f(\langle c,t\rangle)$ equals t times the length of the arc $f(v)f(\langle c,1\rangle)$. One can verify that f is continuous.

The composition of g and f maps F_H onto X_n. This finishes the proof. □

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Mathematical Institute, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384, Wrocław, Poland

Current address: Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D.F., México
E-mail address: `jjc@.matem.unam.mx`

Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409-0020, U.S.A.
E-mail address: `wjcharat@umr.edu`